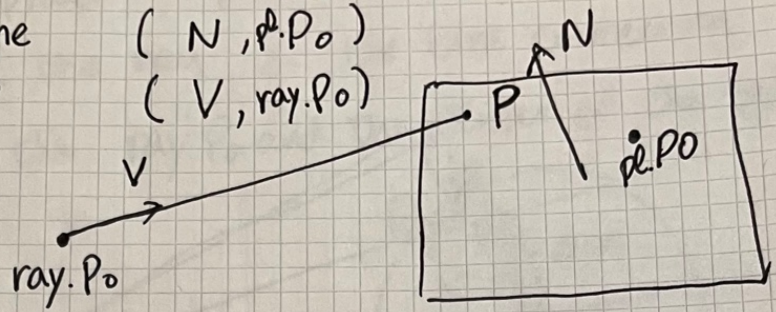
Objective : Looking for intersection of the Ray & a geometric object

Case 1 : Plane $(N, pl.P_0)$
Ray $(V, ray.P_0)$



Check if Plane // Ray :

$$|N \times V| == 0 \quad \text{if yes , return } (-1)$$

otherwise , look for P , n, d

$$\begin{cases} (P - pl.P_0) \cdot N = 0 & (1) \\ P = ray.P_0 + tV \quad , \quad t \geq 0 & (2) \end{cases}$$

Substitute (2) into (1) , we get

$$(ray.P_0 + tV - pl.P_0) \cdot N = 0$$

Solve for t :

$$t(V \cdot N) + (ray.P_0 - pl.P_0) \cdot N = 0$$

$$t^* = \frac{(pl.P_0 - ray.P_0) \cdot N}{V \cdot N}$$

$$\frac{\begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}0\\0\\0\end{bmatrix}}{\begin{bmatrix}2\\1\\1\end{bmatrix} \cdot \begin{bmatrix}0\\1\\0\end{bmatrix}}$$

---

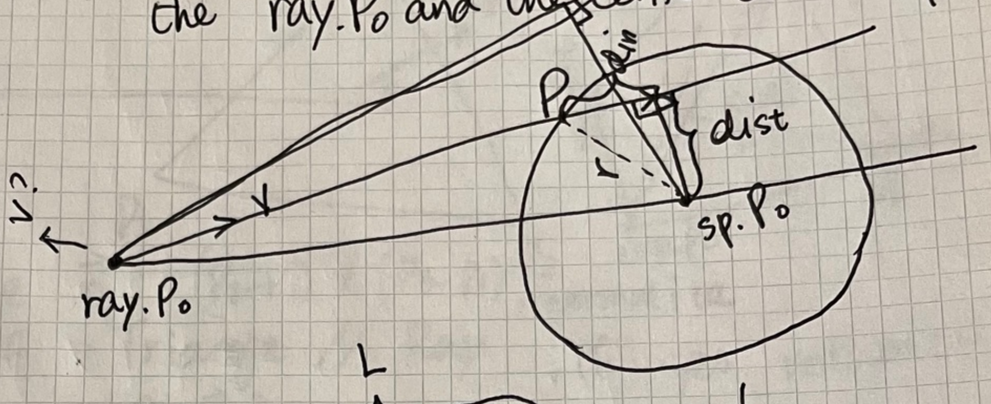
return $P = ray.P_0 + t^* V$

$n = N$ or $-N$ Check

$d = t^*$

Case 2 : Sphere (sp. P_0 , d) , Ray(V, ray. P_0)

idea: Geometric Approach : build a plane with 2 line (the ray & the line connecting the ray.P_0 and the center of the sphere)



$$dist = \left| L - proj(\overbrace{sp.P_0 - ray.P_0}^{L}, V) \right|$$

$$\cancel{dist = \sqrt{|sp.P_0 - ray.P_0|^2 - (\underbrace{comp(sp.P_0 - ray.P_0, V)}_{\text{scalar projection}})^2}}$$

compar dist & r to decide if hit

$$din = \sqrt{r^2 - (dist)^2}$$

$$t = comp(sp.P_0 - ray.P_0, V) - din$$

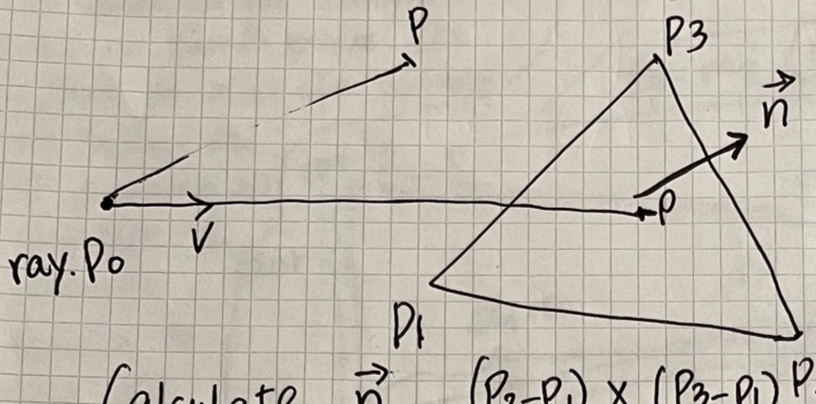
if dist > r , return (-1)

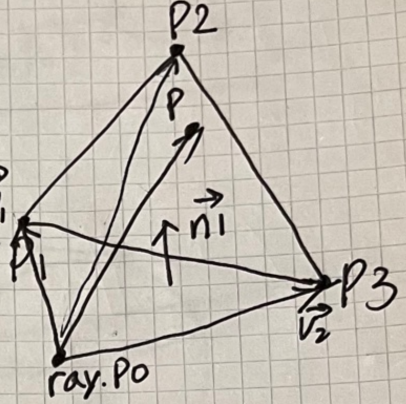
else return t ,

$$P = ray.P_0 + tV$$

$$n = \frac{P - sp.P_0}{|P - sp.P_0|}$$

Case 3: Triangle (P1, P2, P3)
Ray (ray. Po, V)





Calculate $\vec{n}$ $(P_2-P_1) \times (P_3-P_1)$, normalize

Check if Triangle // Ray, if yes return (-1)

plane$(P1, \vec{n})$

else find $P_{(a, \vec{n})}$, the intersection of ray & plane.

We need to use geometric method (inequalities istead of solving equation) to figure out if P is inside △.

$\vec{V_2} \times \vec{V_1}$, normalize $\Rightarrow \vec{n_1}$

Check: ~~(inequality) & same side ($\vec{V_1}, \vec{V_2}$, P, P2)~~

If P_2 and P are on different sides of the plane defined by vectors $\vec{V_2}$ and $\vec{V_1}$,

where $\vec{V_2}$ = P3 - ray.Po , $\vec{V_1}$ = P1 - ray.Po

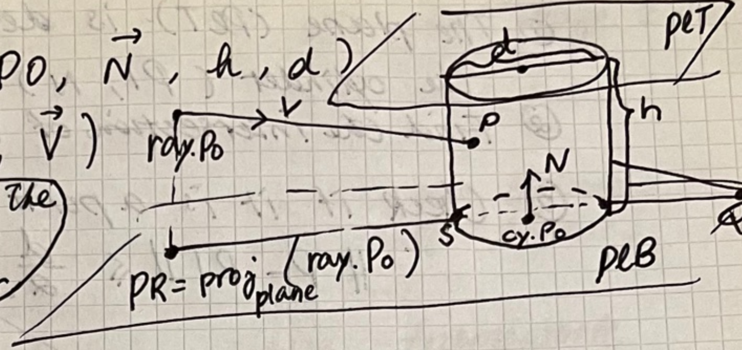
$(P - ray.P_0) \cdot \vec{n_1} \cdot (P_2 - ray.P_0) \cdot \vec{n_1} > 0 \Rightarrow$ same side

Do this for all three sides of the triangle.

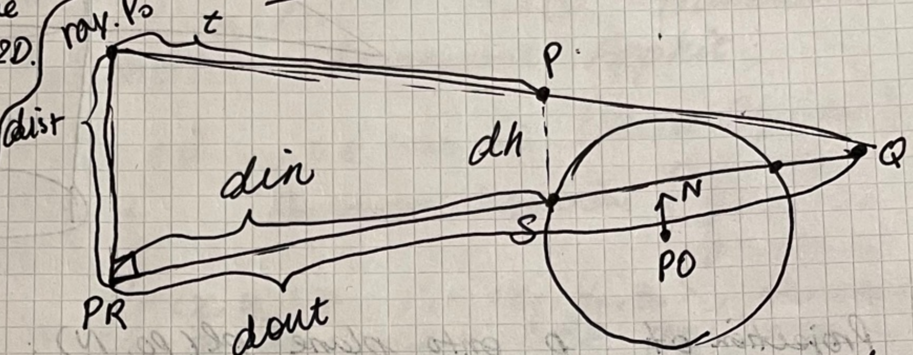
Case 4: Square (same idea).

Case 5: Cylinder (cy.PO, $\vec{N}$, h, d)

Ray (ray.Po, $\vec{V}$)



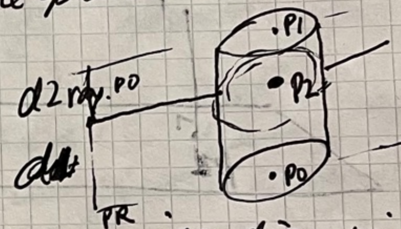
The idea for the solution is the same as Case 2: sphere. Use a plane to slice the cylinder and reduce the 3D problem to 2D. Then use the geometric method to find the solution.



If the ray (ray.V) is parallel to the planes of the cylinder face, then

If ~~dist~~ ≤ h, and Ray hit sphere ✓

otherwise return (−1)



Otherwise,

1. Make sure V and N is in the opposite direction. Otherwise, redefine the top and bottom of the cylinder and change direction of the norm. ⇒ (PO, $d\vec{N}$) bottom → P1 top.
2. The plane (pℓB) is defined by the bottom (PO, pℓB.N).
3. Project ray.Po to the plane and find point PR. (dist)
4. Find the intersection of the Ray (ray.Po V) and the plane, call it Q.
5. Find the intersection of the Ray$\left(PR, \dfrac{\overrightarrow{(Q-PR)}}{\|Q-PR\|}\right)$, $\|Q-PR\| = dout$ and the sphere (PO, d), and call it S and the distance din. If no intersection, return (−1).
6. Use similar triangles, find $dh \left(\dfrac{dh}{dist} = \dfrac{dout - din}{dout}\right)$.
7. If $dh < h$ (hight of the cylinder),

$$t = \sqrt{(din)^2 + (dist - dh)^2}, \quad P = S + dh\,\vec{N}.$$

$$n^* = \frac{S - PO}{\|S - PO\|}$$

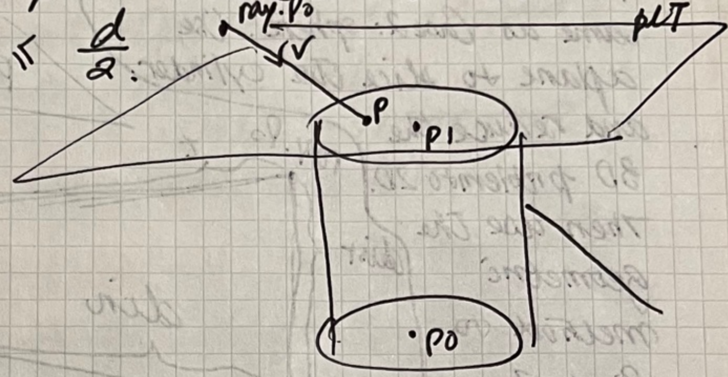
Else

① The plane (PlT) is defined by the top of the cylinder (P1, N).

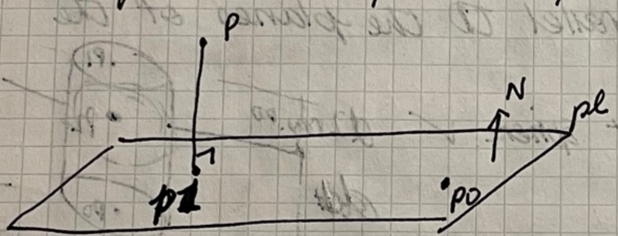
② Find the intersection of the Ray and the Plane.
(ray.P0, V) (PlT).

③ Check if it is a point inside the disk.

$$\| P - P1 \| \leq \frac{d}{2}.$$



Projection of p onto plane $pl(p_0, N)$:



$$(p + tN - p_0) \cdot N = 0$$

$$\Rightarrow t = \frac{P_0 \cdot N - p \cdot N}{N \cdot N}$$

return (sqrt(t*t));

$\boxed{\text{miniRT}}$

mlx_new_image.1
mlx_loop.1
mlx_pixel_put.1

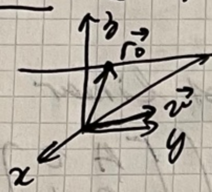
man /usr/share/man/man3/mlx.1

man /usr/share/man/man3/mlx_new_window.1

To compile: (Add the framework -OpenGL and -Appkit)

```
cc -I /usr/local/include main.c -L
    /usr/local/lib/ -lmlx -framework
    OpenGL -framework Appkit
```

Line: $\vec{r} = \vec{r}_0 + t\vec{v}$, $t \in (-\infty, \infty)$ (Vector Equation)



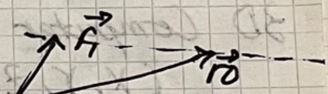
Parametric Equation: $\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$

$\vec{r} = \langle x, y, z \rangle$, $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$, $\vec{v} = \langle a, b, c \rangle$

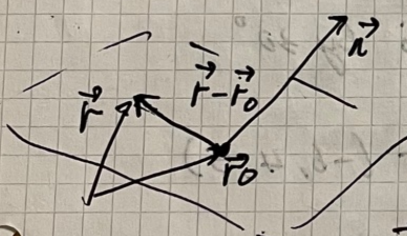
? Find the vector equation & P.E. pass through point $(5, 1, 3)$ & parallel to vector $\vec{i} + 4\vec{j} - 2\vec{k}$.

- Eliminate t: $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$ Symmetric Equations
- The line segment from $\vec{r}_0$ to $\vec{r}_1$ is

$\vec{r} = (1-t)\vec{r}_0 + t\vec{r}_1$, $0 \le t \le 1$



Plane: determined by a point in the plane & a normal vector



$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$ Vector Equation

$(\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0)$

Scala Equation: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$\vec{n} = \langle a, b, c \rangle$, $\vec{r} = (x, y, z)$, $\vec{r}_0 = (x_0, y_0, z_0)$

$(ax + by + cz = d$, $d = -(ax_0 + by_0 + cz_0))$

? Find the equation that passes through the points $(1, 3, 2), (3, -1, 6)$, $(5, 2, 0)$

$6x + 10y + 7z = 50$

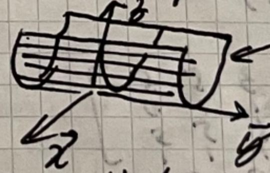
? Find the angle between planes $x + y + z = 1$, $x - 2y + 3z = 1$

Find the line of intersection $\frac{x-1}{5} = \frac{y}{-2} = \frac{z}{3}$. $\theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right) = 72°$

? distance between a point & plane, between two lines.

Cylinder: is a surface that consists of all lines (called rulings) that are parallel to a given line and pass through a given plane curve.

? $x^2 + y^2 = 1$, $y^2 + z^2 = 1$



← $z = x^2$ (~~plane curve~~)

parabolic cylinder

(any vertical plane $y = k$ intersects graph in a curve with equation $z = x^2$)

⇒ vertical traces are parabolas

Sphere: Center $C(h, k, l)$, radius r

$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$

Further topic: Cylindrical & Spherical Coordinates.

Google search: linear algebra computer graphics

2D Transformation: linear transformation $T(\vec{x}) = A\vec{x}$

where $A = [T(\vec{e}_1)\ \ T(\vec{e}_2)]$ $\quad \vec{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$ $\vec{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$

↑ standard matrix

$A = \begin{bmatrix}\cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi\end{bmatrix}$ rotation

$\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ $\begin{bmatrix}-1&0\\0&1\end{bmatrix}$ $\begin{bmatrix}0&1\\1&0\end{bmatrix}$ reflection $\begin{bmatrix}0&-1\\-1&0\end{bmatrix}$ $\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$

$\begin{bmatrix}k&0\\0&1\end{bmatrix}$ $\begin{bmatrix}1&0\\0&k\end{bmatrix}$, contraction / expansion

$\begin{bmatrix}1&k\\0&1\end{bmatrix}$ $\begin{bmatrix}1&0\\k&1\end{bmatrix}$ horizontal / vertical shear

$\begin{bmatrix}1&0\\0&0\end{bmatrix}$ $\begin{bmatrix}0&0\\0&1\end{bmatrix}$ projection

$(x, y) \mapsto (x+h, y+k)$ translation is not linear

$$\begin{bmatrix}1&0&h\\0&1&k\\0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\1\end{bmatrix} = \begin{bmatrix}x+h\\y+k\\1\end{bmatrix} \qquad \left(\begin{bmatrix}A&0\\0&1\end{bmatrix}\right)$$

homogeneous coordinates

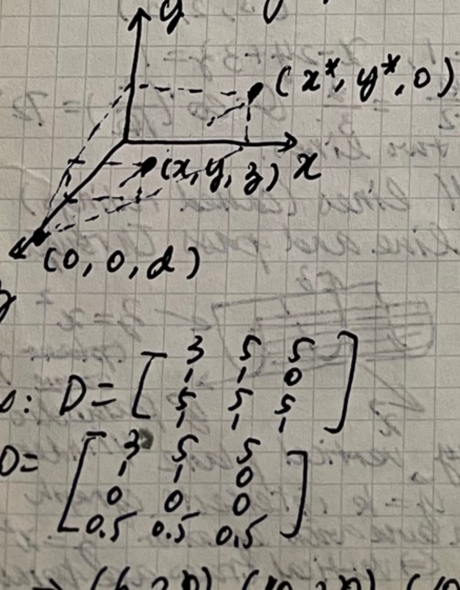
3D Computer Graphics:

(X, Y, Z, H) are homogeneous coordinates for (x, y, z) if $H \neq 0$ and $x = \frac{X}{H}$, $y = \frac{Y}{H}$, $z = \frac{Z}{H}$.

$\begin{bmatrix}\cos 30° & 0 & \sin 30° \\ 0 & 1 & 0 \\ -\sin 30° & 0 & \cos 30°\end{bmatrix}$ rotation about y axis by 30°

$\begin{bmatrix}1&0&0&-6\\0&1&0&4\\0&0&1&5\\0&0&0&1\end{bmatrix}$ translation by vector $(-6, 4, 5)$

Perspective Projection: A 3-D object is represented on the 2D computer screen by projecting the object onto a viewing plane.



$\frac{x}{x^*} = \frac{d-z}{d} \Rightarrow x^* = \frac{x}{1-\frac{z}{d}}$

$\frac{y}{y^*} = \frac{d-z}{d} \Rightarrow y^* = \frac{y}{1-\frac{z}{d}}$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&-\frac{1}{d}&1\end{bmatrix}\begin{bmatrix}x\\y\\z\\1\end{bmatrix} = \begin{bmatrix}x\\y\\0\\1-\frac{z}{d}\end{bmatrix}$$

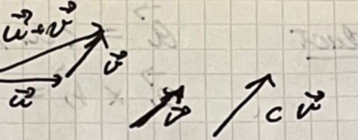
↑ homogeneous coordinate for $(x^*, y^*, 0)$

ex: $D = \begin{bmatrix}3&5&5\\1&1&0\\5&5&5\\1&1&1\end{bmatrix}$

$PD = \begin{bmatrix}3&5&5\\1&1&0\\0&0&0\\0.5&0.5&0.5\end{bmatrix}$

$\Rightarrow (6, 2, 0), (10, 2, 0), (10, 0, 0)$

vector addition: $\vec{u}+\vec{v}$



Scala multiplication: $c\vec{v}$

? $\vec{u}-\vec{v}$

? $-2\vec{v}$

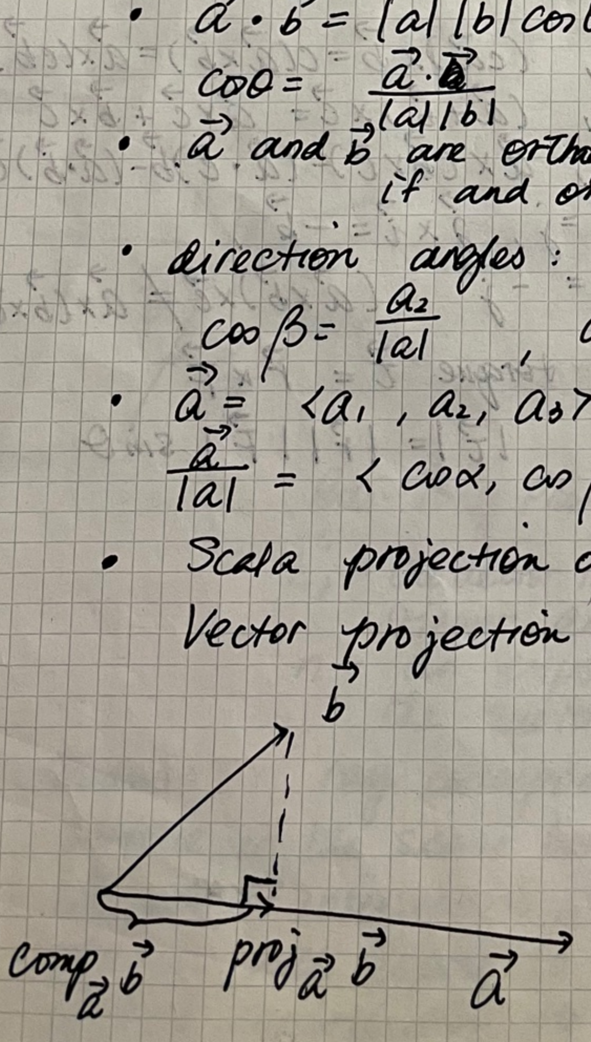
3D: $\vec{a} = \langle a_1, a_2, a_3 \rangle$ (refers to a vector) (a_1, a_2, a_3) a point

- Given points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\vec{a}$ with representation $\vec{AB}$ is $\vec{a} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$
- The length of a vector is $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, $\vec{a} = (a_1, a_2, a_3)$
- $\vec{a} = \langle a_1, a_2, a_3 \rangle$ $\vec{b} = \langle b_1, b_2, b_3 \rangle$
  $\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$
  $c\vec{a} = \langle ca_1, ca_2, ca_3 \rangle$
- Properties: $\vec{a}+\vec{b} = \vec{b}+\vec{a}$, $\vec{a}+(\vec{b}+\vec{c}) = (\vec{a}+\vec{b})+\vec{c}$,
  $\vec{a}+\vec{0} = \vec{a}$, $\vec{a}+(-\vec{a}) = \vec{0}$, $c(\vec{a}+\vec{b}) = c\vec{a}+c\vec{b}$,
  $(c+d)\vec{a} = c\vec{a}+d\vec{a}$, $(cd)\vec{a} = c(d\vec{a})$, $1\vec{a} = \vec{a}$.
- Standard basis: $\vec{i} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $\vec{j} = \begin{bmatrix}0\\1\\0\end{bmatrix}$ $\vec{k} = \begin{bmatrix}0\\0\\1\end{bmatrix}$
  $\vec{a} = \langle a_1, a_2, a_3 \rangle = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$
- Unit vector $\vec{u} = \frac{1}{|a|}\vec{a}$ $|\vec{u}| = 1$? $2\vec{i} - \vec{j} - 2\vec{k}$ find unit vector in this direction

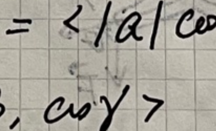
dot product $\vec{a} = \langle a_1, a_2, a_3 \rangle$ $\vec{b} = \langle \vec{b}_1, \vec{b}_2, \vec{b}_3 \rangle$

$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

- Properties: $\vec{a}\cdot\vec{a} = |a|^2$, $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$, $\vec{0}\cdot\vec{a} = 0$
  $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$, $(c\vec{a})\cdot\vec{b} = c(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(c\vec{b})$
- $\vec{a}\cdot\vec{b} = |a||b|\cos\theta$ $\vec{b}$ θ $\vec{a}$? 4 6 $\theta = \frac{\pi}{3}$
  $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|a||b|}$? $a = \langle 2, 2, -1 \rangle$ $b = \langle 5, -3, 2 \rangle$
- $\vec{a}$ and $\vec{b}$ are orthogonal ($\theta = \cos^{-1}(\frac{2}{3\sqrt{38}}) = 1.46 = 84°$)
  if and only if $\vec{a}\cdot\vec{b} = 0$
- direction angles: $\cos\alpha = \frac{\vec{a}\cdot\vec{i}}{|a||\vec{i}|} = \frac{a_1}{|\vec{a}|}$
  $\cos\beta = \frac{a_2}{|a|}$, $\cos\gamma = \frac{a_3}{|a|}$ $(\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1)$
- $\vec{a} = \langle a_1, a_2, a_3 \rangle = \langle |a|\cos\alpha, |a|\cos\beta, |a|\cos\gamma \rangle$
  $\frac{\vec{a}}{|a|} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$
- Scala projection of $\vec{b}$ ont $\vec{a}$: $\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$
  Vector projection of $\vec{b}$ onto $\vec{a}$: $\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}$



(Physics Example:



hauled 8m with force of 200N angle 25°

Work $W = F \cdot D$

$= |F||D|\cos 25°$

$= 200 \cdot 8 \cdot \cos 25°$

≈ 1450 J

Cross Product

$\vec{a} = \langle a_1, a_2, a_3 \rangle \quad \vec{b} = \langle b_1, b_2, b_3 \rangle$

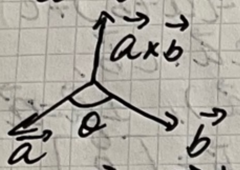
$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

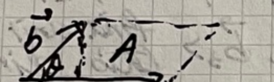
$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

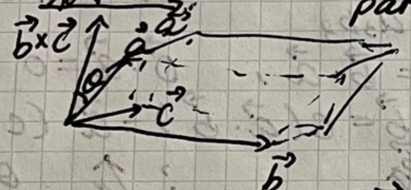
? $\vec{a} = \langle 1, 3, 4 \rangle \quad \vec{b} = \langle 2, 7, -5 \rangle \quad \vec{a} \times \vec{b} = \langle -43, 13, 1 \rangle$

- $\vec{a} \times \vec{a} = \vec{0}$
- $\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$ and $\vec{b}$, i.e., $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$
  $(\vec{a} \times \vec{b}) \cdot \vec{b} = 0$



right hand rule

- $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta \quad (0 \le \theta \le \pi)$
- $\vec{a} \times \vec{b} = \vec{0} \iff \vec{a}, \vec{b}$ are parallel ($\vec{a}, \vec{b}$ nonzero vectors.)
- 

  area parallelogram $A = |\vec{a} \times \vec{b}|$
- 

  Volume parallelepiped $V = |\vec{a} \cdot (\vec{b} \times \vec{c})|$
- Properties: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}, \quad (c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$

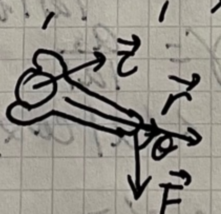
  $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}, \quad (\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

  $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}, \quad \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

  $\vec{i} \times \vec{j} = \vec{k}, \quad \vec{j} \times \vec{k} = \vec{i}, \quad \vec{k} \times \vec{i} = \vec{j}, \quad \vec{j} \times \vec{i} = -\vec{k},$

  $\vec{k} \times \vec{j} = -\vec{i}, \quad \vec{i} \times \vec{k} = -\vec{j} \quad (\vec{a} \times \vec{b}) \times \vec{c} \ne \vec{a} \times (\vec{b} \times \vec{c})$

(Physics Example:



torque $\vec{\tau} = \vec{r} \times \vec{F}$

$|\vec{\tau}| = |\vec{r}|\,|\vec{F}| \sin\theta$

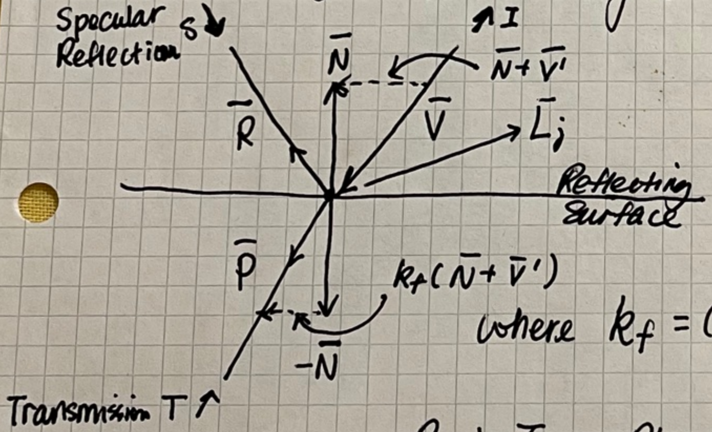
Transparency
Shadow
Reflection

$\xrightarrow{\text{Solved}}$ Turner Whitted, An Improved Illumination Model For Shaded Display
Graphic and Image Processing, 1980

$$I = I_a + k_d \underbrace{\sum_{j=1}^{j=l_s} (\bar{N} \cdot \bar{L}_j)} + k_s S + k_t T$$ (Apply linear attenuation with distance)

Lambert's cosine law: The intensity of the reflected light is proportional to the dot product of the surface normal and light source direction, simulating a perfect diffuser and yielding a reasonable looking to a dull, matte surface.

I: the reflected intensity; I_a: reflection due to ambient light
k_d: diffuse reflection constant
$\bar{N}$: unit surface normal
$\bar{L}_j$: the vector in the j-th light source direction
k_s: the specular reflection constant
S: the intensity of light incident from the $\bar{R}$ direction
k_t: the transmission coefficient
T: the intensity of light from $\bar{P}$ direction



$$\bar{V}' = \frac{\bar{V}}{|\bar{V} \cdot \bar{N}|}$$

$$\bar{R} = \bar{V}' + 2\bar{N}$$

$$\bar{P} = k_f(\bar{N} + \bar{V}') - \bar{N}$$

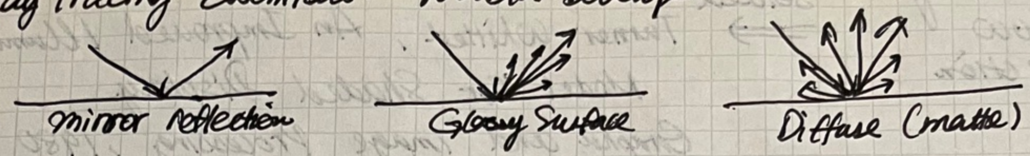
where $k_f = (k_n^2 |\bar{V}'|^2 - |\bar{V}' + \bar{N}|^2)^{-\frac{1}{2}}$, k_n = the index of refraction

Bui-Tong Phong Model: $S = \sum_{j=1}^{j=l_s} (\bar{N} \cdot \bar{L}'_j)^n$ (easier to understand Princeton Notes!)

$\bar{L}'_j$ = the vector in the direction halfway between the viewer and the j-th light source
n = an exponent depends on the glossiness of the surface

Shadow: If ray corresponding to $\bar{L}_j$ term intersects some surface in the scene before it reaches the light source, the point of intersection represented by the node lies in shadow with respect to that light source. That light source's contribution to the diffuse reflection from the point is then attenuated.

Ray Tracing Essentials: NVIDIA Developer



$$\underbrace{L_o(X,\hat{\omega}_o)}_{\text{outgoing light}} = \underbrace{L_e(X,\hat{\omega}_o)}_{\text{emitting light}} + \int_{S^2} \underbrace{L_i(X,\hat{\omega}_i)}_{\text{Incoming light}} \underbrace{f_X(\hat{\omega}_i,\hat{\omega}_o)}_{\text{material}}$$

$$\underbrace{|\hat{\omega}_i \cdot \hat{n}|}_{\text{lambert}} \, d\hat{\omega}_i$$

$(X,\hat{\omega}_o)$: a point in the scene

$\hat{S}$: all incoming directions

$\hat{n}$: surface normal

$\hat{\omega}_i$: incoming direction

$\boxed{\text{The Rendering Equation}}$