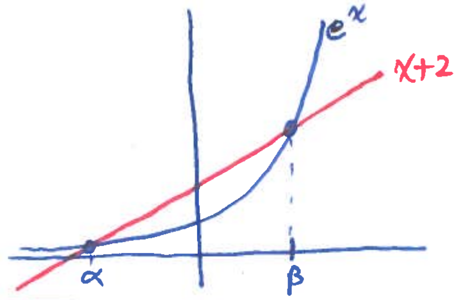
2018 2 (2016 2) (see thm 1.9 Suli)

Show $e^x = x+2$ has 2 real sol^{ns} $\alpha < 0$ & $\beta > 0$:

graphically,



$f(x) = e^x - x - 2$, NR $\Rightarrow$ $\phi(x_n) = x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ $(*)$

$x_0 > 0$:

- Consider $x_0 \in [\beta, \infty)$

$f'(x) = e^x - 1 > 0$ for $x \in [\beta, \infty)$ $\Rightarrow$ f inc.

& since $f(\beta) = 0$, have $f(x) > 0$ for $x \in (\beta, \infty)$

$\Rightarrow \beta < x_{n+1} < x_n \Rightarrow \{x_n\}$ bdd. & dec. $\Rightarrow$ conv.

$\Rightarrow$ let $z = \lim_{n\to\infty} x_n$

clearly $z \in [\beta, \infty)$ & taking $n \to \infty$ in $(*)$ we get $f(z) = 0$

but in $[\beta, \infty)$, the only zero is β, so $z = \beta$

- Consider $x_0 \in (0, \beta]$

$f'(x) = e^x - 1 > 0$ for $x \in (0, \beta] \Rightarrow f$ inc.

since $f(\beta) = 0$, have $f(x) \leq 0$ for $x \in (0, \beta]$

$\Rightarrow 0 < x_n < x_{n+1} < \beta \Rightarrow \{x_n\}$ bdd & inc. $\Rightarrow$ conv.

so as before, have $z = \beta = \lim_{n\to\infty} x_n$

[So, have if $x_0 > 0$ then $x_n \to \beta$ as $n \to \infty$]

$\longrightarrow$

I $x_0 < 0$:

- consider $x_0 \in [\alpha, 0)$

$f'(x) = e^x - 1 < 0 \Rightarrow f$ dec.

& since $f(\alpha) = 0$, have $f(x) \leq 0$ for $x \in [\alpha, 0)$

$\Rightarrow \alpha < x_{n+1} < x_n < 0 \Rightarrow \{x_n\}$ bdd & dec $\Rightarrow$ conv.

So as before, have $z = \alpha = \lim_{n \to \infty} x_n$

- consider $x_0 \in (-\infty, \alpha]$

$f'(x) = e^x - 1 < 0 \Rightarrow f$ dec.

& since $f(\alpha) = 0$, have $f(x) > 0$ for $x \in (-\infty, \alpha)$

$\Rightarrow x_n < x_{n+1} < \alpha \Rightarrow \{x_n\}$ bdd & inc. $\Rightarrow$ conv.

So as before, have $z = \alpha = \lim_{n \to \infty} x_n$

[So we have if $x_0 < 0$ then $x_n \to \alpha$ as $n \to \infty$]