Math 387 Lab Final Exam

Honours Numerical Analysis

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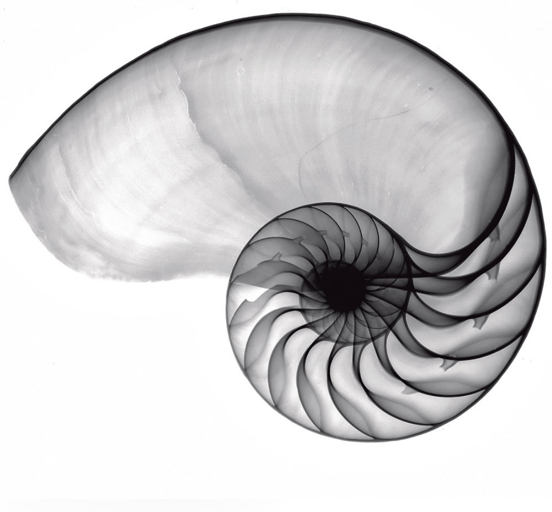

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1 Exercise 1

Recall that $f \in C^{0,\alpha}([-1,1])$ for $0 < \alpha \leq 1$ iff

$$\sup_{-1 \leq x < y \leq 1} \frac{\mid f(x) - f(y) \mid}{\mid x - y \mid^{\alpha}} < \infty$$

and that $f \in C^{k,\alpha}([-1,1])$ for $k \in \mathbb{N}$ and $0 < \alpha \leq 1$ iff $f \in C^{k}([-1,1])$ and $f^{(k)} \in C^{0,\alpha}([-1,1])$. Our goal is to find the largest $k+\alpha$ such that $f_a \in C^{k,\alpha}([-1,1])$, depending on $a > 0$.

1.1 Largest Possible k

1.1.1 Case 1

First, consider $a \in \mathbb{N}$, $(a > 0)$.

If we take $k = a$, then

$$f_a{}^{(a)}(x) = \begin{cases} a! & x > 0 \\ 0 & x \leq 0 \end{cases}$$

which has discontinuity at $x = 0$ since $a > 0$.

If we take $k = a - 1$, then

$$f_a{}^{(a-1)}(x) = \begin{cases} (a-1)!x & x > 0 \\ 0 & x \leq 0 \end{cases}$$

which is continuous.

1.1.2 Case 2

Now, consider $a \in \mathbb{R} \setminus \mathbb{N}$, $(a > 0)$.

From Case 1, we know that we can take at least $k = a - 1$. But in this case, the fractional part of a $(frac(a))$ is nonzero, so we can take $k = a - frac(a)$ and still retain continuity.

1.1.3 Conclusion

So, in conclusion, the maximum possible k is:

$$k = \begin{cases} (a-1) & a \in \mathbb{N} \\ a - frac(a) & a \in \mathbb{R} \setminus \mathbb{N} \end{cases} \tag{1.1}$$

1.2 Largest Possible Alpha

We are given that $0 < \alpha \leq 1$. If we decrease $k \in \mathbb{N}$ by one, to maximize $k + \alpha$ we would need to increase α by one, which is not possible. Thus, to determine the largest possible α, we set k to be given by (1.1) and find the corresponding values of α such that

$$\sup \frac{|f_a(x) - f_a(y)|}{|x-y|^\alpha} < \infty$$

Remark. We check with f_a and not ${f_a}^{(k)}$ because with the maximum possible k, checking our condition is equivalent up to a constant, which does not affect the analysis in the regime of infinity.

Note that for $x, y \in [-1, 0]$, we have that $f_a(x) = f_a(y) = 0$. Thus, $f_a \in C^k([-1, 0])$ and ${f_a}^{(k)} \in C^{0,\alpha}([-1, 0])$ trivially ($\forall k \in \mathbb{N}$, $\forall \alpha$).

So, we just need to consider x or y OR both x and y in $(0, 1]$. WLOG, take $x, y \in (0, 1]$ with $x > y$.

1.2.1 Case 1

First consider $\alpha \leq a$ ($a > 0$). We have that:

$$\begin{aligned} \sup_{-1 \leq x,y \leq 1} \frac{|f_a(x) - f_a(y)|}{|x-y|^\alpha} &= \sup_{-1 \leq x,y \leq 1} \frac{|x^a - y^a|}{|x-y|^\alpha} \\ &\leq \sup_{-1 \leq x,y \leq 1} \frac{|x-y|^\alpha}{|x-y|^\alpha} \\ &= 1 \\ &< \infty \end{aligned}$$

1.2.2 Case 2

Now consider $\alpha > a$ ($a > 0$). We have that:

$$\begin{aligned} \sup_{-1 \leq x,y \leq 1} \frac{|f_a(x) - f_a(y)|}{|x-y|^\alpha} &\geq \sup_{-1 \leq x \leq 1} \frac{|(2x)^a - x^a|}{|2x - x|^\alpha} \\ &= (2^a - 1) \sup_{-1 \leq x \leq 1} x^{a-\alpha} \\ &\to \infty \quad (x = 0) \end{aligned}$$

1.2.3 Conclusion

So, in conclusion, we have that $\alpha \leq a$.

1.3 Result

From the above ($\alpha \leq a$) and also (1.1), we conclude that the largest $k + \alpha$ such that $f_a \in C^{k,\alpha}([-1, 1])$ is:

$$max\ k + \alpha = \begin{cases} 2a - 1 & a \in \mathbb{N} \\ 2a - frac(a) & a \in \mathbb{R} \setminus \mathbb{N} \end{cases}$$