Math 254 Tutorial 7

Honours Analysis 1

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Assignment 5 Problem 1

In this section, we shall examine how the first part of Assignment 5 Problem 1 may be solved using Assignment 4 Problem 3. To that end, first let us recall the result of Assignment 4 Problem 3:

Theorem 1.0.1 (Assign4#3)**.** *Let P_{nk} , $n, k \in \mathbb{N}$ be real numbers satisfying the following:*

1. $P_{nk} \geq 0$ *for all* n, k
2. $\sum_{k=1}^{n} P_{nk} = 1$ *for all* n
3. $\lim_{n\to\infty} P_{nk} = 0$ *for all* k

Let (x_n) *be a convergent sequence and let a sequence* (y_n) *be defined by*

$$y_n = \sum_{k=1}^{n} P_{nk} x_k$$

Then, (y_n) *is a convergent sequence and* $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n$*.*

Now we shall prove the following:

Theorem 1.0.2 (Assign5#1)**.** *Let* (y_n) *be an unbounded sequence of positive numbers satisfying* $y_{n+1} > y_n$ *for all* $n \in \mathbb{N}$*. Let* (x_n) *be another sequence, and suppose that the limit*

$$\lim \frac{x_{n+1} - x_n}{y_{n+1} - y_n}$$

exists. Then

$$\lim \frac{x_n}{y_n} = \lim \frac{x_{n+1} - x_n}{y_{n+1} - y_n}$$

Proof. Let $P_{nk} := \frac{y_{k+1} - y_k}{y_{n+1} - y_1}$. First we note that $P_{nk} \geq 0$ for all n, k, since the y_n are positive and strictly increasing. Now,

$$\begin{aligned}
\sum_{k=1}^{n} P_{nk} &= \sum_{k=1}^{n} \frac{y_{k+1} - y_k}{y_{n+1} - y_1} \\
&= \frac{1}{y_{n+1} - y_1} \sum_{k=1}^{n} y_{k+1} - y_k \\
&= \frac{1}{y_{n+1} - y_1} (y_2 - y_1 + y_3 - y_2 + y_4 - y_3 + \ldots + y_{n+1} - y_n) \\
&= \frac{1}{y_{n+1} - y_1} [-y_1 + y_{n+1}] \quad \text{(Telescoping sum)} \\
&= 1
\end{aligned}$$

Also, since the y_n are strictly increasing, positive, and unbounded (hence $y_n \to \infty$) we have that

$$\lim_{n\to\infty} P_{nk} = \lim_{n\to\infty} \frac{y_{k+1} - y_k}{y_{n+1} - y_1} = 0$$

So, we have that P_{nk} satisfies 1,2,3 of theorem 1.0.1. Now, let $z_n := \frac{x_{n+1}-x_n}{y_{n+1}-y_n}$, and note that (z_n) is convergent (given). We have that:

$$\begin{aligned}
\sum_{k=1}^{n} P_{nk} z_k &= \sum_{k=1}^{n} \frac{y_{k+1} - y_k}{y_{n+1} - y_1} \frac{x_{k+1} - x_k}{y_{k+1} - y_k} \\
&= \sum_{k=1}^{n} \frac{x_{k+1} - x_k}{y_{n+1} - y_1} \\
&= \frac{1}{y_{n+1} - y_1} \sum_{k=1}^{n} (x_{k+1} - x_k) \\
&= \frac{1}{y_{n+1} - y_1} [x_{n+1} - x_1] \\
&= \frac{x_{n+1} - x_1}{y_{n+1} - y_1}
\end{aligned}$$

So by theorem 1.0.1 we know that $\lim_{n\to\infty} \frac{x_{n+1}-x_1}{y_{n+1}-y_1} = \lim_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n}$. But,

$$\frac{x_{n+1} - x_1}{y_{n+1} - y_1} = \frac{\frac{x_{n+1}-x_1}{y_{n+1}}}{\frac{y_{n+1}-y_1}{y_{n+1}}} = \frac{\frac{x_{n+1}}{y_{n+1}} - \frac{x_1}{y_{n+1}}}{1 - \frac{y_1}{y_{n+1}}}$$

and since $y_n \to \infty$, we conclude that $\lim_{n\to\infty} \frac{x_{n+1}-x_1}{y_{n+1}-y_1} = \lim_{n\to\infty} \frac{x_{n+1}}{y_{n+1}} = \lim_{n\to\infty} \frac{x_n}{y_n}$. So, we have shown that

$$\lim \frac{x_n}{y_n} = \lim \frac{x_{n+1} - x_n}{y_{n+1} - y_n}$$

□

Remark. The above result theorem 1.0.2 is a discrete analog of L'Hopitals rule. I had actually mentioned this theorem by name (it is called the Stolz-Cesaro Theorem) in an announcement post (MyCourses) on October 14th, because someone had brought up L'Hopitals rule during my tutorial.

2 The divergence of sin(n)

Recall the following Proposition, which was proved in my Tutorial-Oct14 as well as the Lecture-Oct19:

Proposition 2.0.1 (Lecture Oct 19)**.** *Subsequences of a convergent sequence converge to the same limit as the original sequence.*

We will need to use the following definition:

Definition 2.0.1. For an open interval $(a, b) \subset \mathbb{R}$ let us define the length of (a, b) to be

$$l((a, b)) := b - a > 0$$

Proposition 2.0.2. *Let $I = (a, b)$ with $0 \leq a < b$ and $l(I) = b - a > 2$. There exist two natural numbers contained in I.*

Proof. We know that there exists $m \in \mathbb{Z}$ such that $m - 1 \leq a < m$ (see my Tutorial on Sep30). In particular $m \geq 1$ (since otherwise $m \leq 0$, but we have $a \geq 0$), and so $m \in \mathbb{N}$. We have that $m \leq a + 1 < a + 2 < b$ and hence $m \in I = (a, b)$. Also, $m + 1 \in \mathbb{N}$ and $m + 1 \leq a + 2 < b$, so $(m + 1) \in I = (a, b)$ as well. $\square$

We now prove the following:

Theorem 2.0.1. *The sequence $(\sin(n))$ is divergent.*

Proof. The idea of the proof is to construct two subsequences that do not converge to the same limit.

First, let us recall some properties of sin. We have that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} = \sin\left(\frac{5\pi}{6}\right)$. We have that $\sin(x) > \frac{1}{2}$ for all $x \in (\frac{\pi}{6}, \frac{5\pi}{6})$. Since sin has a period of 2π, it follows that for each $k \in \mathbb{N}$, $\sin(x) > \frac{1}{2}$ for all x in the interval

$$I_k := \left(\frac{\pi}{6} + 2\pi(k - 1), \frac{5\pi}{6} + 2\pi(k - 1)\right)$$

We note that for any $k \in \mathbb{N}$, $l(I_k) = \frac{5\pi}{6} - \frac{\pi}{6} = \frac{2\pi}{3} > 2$.

Now, since I_1 is an interval of length > 2, by the previous Proposition, we know that there exist two natural numbers contained in I_1. We let n_1 be the smallest natural number in I_1.

Similarly, I_2 contains at least 2 natural numbers, and we let n_2 be the smallest natural number in I_2. So in general, we let n_k be the smallest natural number contained in I_k.

Since the right-endpoint of the interval I_k is less than the left-endpoint of I_{k+1}, we have that (n_k) is a strictly increasing sequence of natural numbers. This means that $(\sin(n_k))$ is a subsequence of $(\sin(n))$. And by our previous remarks, for any $k \in \mathbb{N}$ we have that $\sin(x) > \frac{1}{2}$ on I_k. So since $n_k \in I_k$ for all k, we have that $\sin(n_k) > \frac{1}{2}$ for all $k \in \mathbb{N}$.

On the other hand, we can do a similar construction to get another subsequence $(\sin(n_j))$ of $(\sin(n))$ satisfying the following:

$$n_j \in \left(\frac{7\pi}{6} + 2\pi(j-1), \frac{11\pi}{6} + 2\pi(j-1)\right)$$

for all $j \in \mathbb{N}$. We will then have $\sin(n_j) < -\frac{1}{2}$ for all $j \in \mathbb{N}$. So, these two subsequences do not converge to the same limit.

$\square$

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Example 1

Theorem 3.0.1. *Let (x_n) , $x_n > 0$ be a convergent sequence. Then,*

$$\lim_{n\to\infty} (x_1 x_2 x_3 \cdots x_n)^{\frac{1}{n}} = \lim_{n\to\infty} x_n$$

Proof. Let $L := \lim_{n\to\infty} x_n$ and $y_n := (x_1 x_2 x_3 \cdots x_n)^{\frac{1}{n}}$. We shall deal with the two possible cases: $L = 0$ or $L > 0$. (Recall that by the Order Limit Theorem, we cannot have $L < 0$.)

Lets start with the case that $L = 0$. Let $\epsilon > 0$, then $\exists N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have that $|x_n - 0| = |x_n| = x_n < \frac{\epsilon}{2}$. So (for $n \geq N_1$):

$$\begin{aligned}
y_n &= (x_1 x_2 x_3 \cdots x_n)^{\frac{1}{n}} \\
&= c^{\frac{1}{n}} (x_{N_1+1} \cdots x_n)^{\frac{1}{n}} \\
&< c^{\frac{1}{n}} \left(\frac{\epsilon}{2}\right)^{\frac{n-N_1}{n}} \\
&= c^{\frac{1}{n}} \left(\frac{\epsilon}{2}\right)^{1-\frac{N_1}{n}} \\
&= c^{\frac{1}{n}} \left(\frac{\epsilon}{2}\right) \left(\frac{\epsilon}{2}\right)^{-\frac{N_1}{n}} \\
&= c^{\frac{1}{n}} \left(\frac{2}{\epsilon}\right)^{\frac{N_1}{n}} \left(\frac{\epsilon}{2}\right) \\
&= \left(\frac{2^{N_1} c}{\epsilon^{N_1}}\right)^{\frac{1}{n}} \frac{\epsilon}{2} \\
&= a^{\frac{1}{n}} \frac{\epsilon}{2}
\end{aligned}$$

where we have defined $c := (x_1 \cdots x_{N_1})$ and $a := \frac{2^{N_1} c}{\epsilon^{N_1}}$. Now, since $a > 0$ we have that $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$ (for a partial proof see the Oct 21 Tutorial by Edward). Hence there exists $N_2 \in \mathbb{N}$ such that for $n \geq N_2$, we have that $a^{\frac{1}{n}} < 2$. Define $N := \max\{N_1, N_2\}$. Then for $n \geq N$ we have that:

$$y_n < a^{\frac{1}{n}} \frac{\epsilon}{2} < 2\frac{\epsilon}{2} = \epsilon$$

and thus $\lim_{n\to\infty} y_n = 0$ as required.

Now we shall deal with the more difficult case $L > 0$. Note that we need to equivalently show that $\lim_{n\to\infty} \frac{y_n}{L} = 1$. Letting $z_n := \frac{y_n}{L}$, or goal is then to show that $\lim_{n\to\infty} z_n = 1$.

Well, first note that we can rewrite z_n:

$$\begin{aligned}
z_n &= \frac{(x_1 x_2 \cdots x_n)^{\frac{1}{n}}}{L} \\
&= \left(\frac{x_1 x_2 \cdots x_n}{L^n}\right)^{\frac{1}{n}} \\
&= \left[\left(\frac{x_1}{L}\right)\left(\frac{x_2}{L}\right) \cdots \left(\frac{x_n}{L}\right)\right]^{\frac{1}{n}}
\end{aligned}$$

Now let $\epsilon > 0$ (WLOG take $\epsilon < 1$). Then $\exists N_1 \in \mathbb{N}$ large enough such that for $n \geq N_1$, we have $|x_n - L| < \frac{\epsilon}{2} L$. In particular, we have that $L - x_n < \frac{\epsilon}{2} L$ and so $x_n > L - \frac{\epsilon}{2} L = L(1 - \frac{\epsilon}{2})$ for $n \geq N_1$. Therefore:

$$\begin{aligned}
z_n &> \left[\left(\frac{x_1}{L}\right)\left(\frac{x_2}{L}\right) \cdots \left(\frac{x_{N_1}}{L}\right)\right]^{\frac{1}{n}} \left(\frac{L(1-\frac{\epsilon}{2})}{L}\right)^{\frac{n-N_1}{n}} \\
&= \left(\prod_{i=1}^{N_1} \frac{x_i}{L}\right)^{\frac{1}{n}} \left(1 - \frac{\epsilon}{2}\right) \left(\frac{2-\epsilon}{2}\right)^{-\frac{N_1}{n}} \\
&= c^{\frac{1}{n}} \left(1 - \frac{\epsilon}{2}\right)
\end{aligned}$$

where we have defined

$$c := \frac{2^{N_1} \prod_{i=1}^{N_1} \frac{x_i}{L}}{(2-\epsilon)^{N_1}}$$

We note that $c > 0$ (since $x_n > 0$, $L > 0$, and $0 < \epsilon < 1$), hence we have $\lim_{n\to\infty} c^{\frac{1}{n}} = 1$. So by definition, there exists $N_2 \in \mathbb{N}$ such that whenever $n \geq N_2$ we have that $|c^{\frac{1}{n}} - 1| < \frac{\epsilon}{2}$. In particular, $1 - c^{\frac{1}{n}} < \frac{\epsilon}{2}$ and so $c^{\frac{1}{n}} > 1 - \frac{\epsilon}{2}$. Define $N := \max\{N_1, N_2\}$, then for $n \geq N$ we have that:

$$z_n > c^{\frac{1}{n}} \left(1 - \frac{\epsilon}{2}\right) > \left(1 - \frac{\epsilon}{2}\right)^2 = 1 - \epsilon + \frac{\epsilon^2}{4} > 1 - \epsilon$$

In an analogous manner, following the previous arguments one may ensure that N is sufficiently large so that for $n \geq N$ we have $z_n < 1 + \epsilon$. Hence combining both, we arrive at $|z_n - 1| < \epsilon$ for $n \geq N$. I.e. $\lim_{n\to\infty} z_n = 1$, as required.

□

4 Example 2

Recall the definition of a Cauchy sequence:

Definition 4.0.1. A sequence (x_n) is called Cauchy if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that whenever $n, m \geq N$ we have that $|x_n - x_m| < \epsilon$.

In addition, recall the following Proposition, which was proved in the Lecture:

Proposition 4.0.1 (Lecture Oct 21)**.** *Let (x_n) be a Cauchy sequence. Then the sequence (x_n) is bounded.*

We now prove the following:

Theorem 4.0.1. *If (x_n) and (y_n) are Cauchy sequences, then $(x_n + y_n)$ and $(x_n y_n)$ are Cauchy sequences.*

Proof. Let $\epsilon > 0$. Since (x_n) is Cauchy, $\exists N_1 \in \mathbb{N}$ such that whenever $n, m \geq N_1$, then $|x_n - x_m| < \frac{\epsilon}{2}$. Similarly, since (y_n) is Cauchy, $\exists N_2 \in \mathbb{N}$ such that if $n, m \geq N_2$, then $|y_n - y_m| < \frac{\epsilon}{2}$. Define $N := \max\{N_1, N_2\}$. Then for $n, m \geq N$ we have that:

$$\begin{aligned}
|(x_n + y_n) - (x_m + y_m)| &= |(x_n - x_m) + (y_n - y_m)| \\
&\leq |x_n - x_m| + |y_n - y_m| \quad \text{(Triangle Inequality)} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

Hence $(x_n + y_n)$ is Cauchy.

For the second part: Since (x_n) is Cauchy, by the above Proposition we have that $|x_n| \leq M_1$ for all $n \in \mathbb{N}$ $(M_1 > 0)$. Also since (y_n) is Cauchy, by the above Proposition we have that $|y_n| \leq M_2$ for all $n \in \mathbb{N}$ $(M_2 > 0)$. Let $M := \max\{M_1, M_2\}$. (hence (x_n) and (y_n) are bounded by M)

Let $\epsilon > 0$. Since (x_n) is Cauchy, $\exists N_1 \in \mathbb{N}$ such that whenever $n, m \geq N_1$, then $|x_n - x_m| < \frac{\epsilon}{2M}$. Similarly, since (y_n) is Cauchy, $\exists N_2 \in \mathbb{N}$ such that if $n, m \geq N_2$, then $|y_n - y_m| < \frac{\epsilon}{2M}$. Define $N := \max\{N_1, N_2\}$. Then for $n, m \geq N$ we have that:

$$\begin{aligned}
|x_n y_n - x_m y_m| &= |x_n y_n - x_n y_m + x_n y_m - x_m y_m| \\
&\leq |x_n y_n - x_n y_m| + |x_n y_m - x_m y_m| \quad \text{(Triangle Inequality)} \\
&= |x_n||y_n - y_m| + |y_m||x_n - x_m| \\
&< M\frac{\epsilon}{2M} + M\frac{\epsilon}{2M} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

Hence $(x_n y_n)$ is Cauchy.

□

5 Example 3

Theorem 5.0.1. *Let (x_n) be a Cauchy sequence such that x_n is an integer for every $n \in \mathbb{N}$. Then (x_n) is ultimately constant.*

Proof. Since (x_n) is Cauchy, $\exists N \in \mathbb{N}$ such that whenever $n, m \geq N$, then $|x_n - x_m| < \frac{1}{2}$ (taking $\epsilon = \frac{1}{2}$). In particular, for all $n \geq N$ we have that $|x_n - x_N| < \frac{1}{2}$. Now let $n \in \mathbb{N}$ such that $n \geq N$. We note that $x_n - x_N \in \mathbb{Z}$. So $x_n - x_N = 0$ or $|x_n - x_N| \geq 1$. The latter leads to a contradiction. Hence we must have $x_n - x_N = 0$. Since $n \geq N$ was arbitrary, we thus have that $x_n = x_N$ for all $n \geq N$. I.e. (x_n) is ultimately constant.

□