Math 254 Tutorial 11

Honours Analysis 1

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1 Continuous Functions

We begin with the $\epsilon - \delta$ definition of continuity:

Definition 1.0.1. A function $f : A \to \mathbb{R}$ is *continuous at a point* $c \in A$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that $\forall x \in A$ satisfying $|x - c| < \delta$, one has $|f(x) - f(c)| < \epsilon$.

Definition 1.0.2. If f is continuous at every point in A, we say that f is *continuous on* A.

Definition 1.0.3. If f is not continuous at a point $c \in A$, then we say that f is *discontinuous* at c.

Remark. If $c \in A$ is a cluster point of A, then f is continuous at c iff $\lim_{x\to c} f(x) = f(c)$.

Remark. If $c \in A$ is an isolated point of A (i.e. is not a cluster point), then $\lim_{x\to c} f(x)$ is not defined. But, we can still apply the definition of continuity to get: any function $f : A \to \mathbb{R}$ is continuous at isolated points of A.

Now, we introduce some tools which we shall use in a particular example:

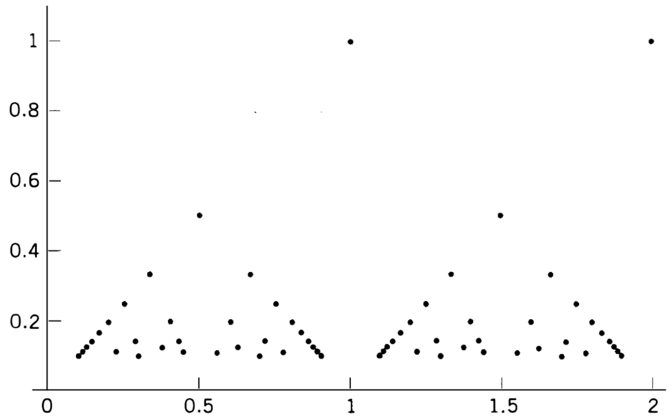
Theorem 1.0.1 (Sequential Characterization of Continuity)**.** *Let $f : A \to \mathbb{R}$ and $c \in A$. The following are equivalent:*

1. *f is continuous at c*
2. *For any sequence (x_n) in A such that $\lim_{n\to\infty} x_n = c$, we have that $\lim_{n\to\infty} f(x_n) = f(c)$.*

Corollary 1.0.1 (Criterion for Discontinuity)**.** *Let $f : A \to \mathbb{R}$ and $c \in A$ be a cluster point of A. If there exists a sequence (x_n) in A such that $(x_n) \to c$ but $f(x_n)$ does not converge to $f(c)$, we may conclude that f is not continuous at c.*

1.1 Example - Thomae's Function

Let $A := \{x \in \mathbb{R} : x > 0\}$. For any irrational $x > 0$, let $h(x) := 0$. For rational numbers in A of form $\frac{m}{n}$ (where $m, n \in \mathbb{N}$ have no common factors except 1), we define $h(\frac{m}{n}) := \frac{1}{n}$. This function is referred to as *Thomae's Function.*



Lemma 1.1.1. *Given $n_0 \in \mathbb{N}$ and $a, b \in \mathbb{R}$ with $0 \leq a < b < \infty$, there are only finitely many rationals $x \in (a, b)$ where $x = \frac{m}{n}$ ($gcd(m, n) = 1$) such that $0 < n \leq n_0$.*

Proof. We provide a brief sketch. We have that $0 < n \leq n_0$, and only finitely many natural numbers n satisfy this (namely n_0 of them). For each of these denominators n, the numerator m must satisfy $an \leq m \leq bn$ (since $\frac{m}{n} \in (a, b)$). There are only a finite number of natural numbers m that can exist in that range. □

Theorem 1.1.1. *h is continuous at every irrational number in A.*

Proof. Let $\epsilon > 0$ and $b \in A$ be irrational. By the Archimedean Property, there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \epsilon$. Now by lemma 1.1.1, we know that the interval $(b-1, b+1)$ contains at most finitely many rationals with denominator less than n_0. Hence, we can choose $0 < \delta < 1$ small enough so that $(b - \delta, b + \delta)$ contains no rationals with denominator less than n_0.

So for $|x - b| < \delta$ (and $x \in A$):

1. If x is irrational: $|h(x) - h(b)| = |0 - 0| = 0 < \epsilon$.
2. If $x = \frac{m}{n}$ is rational, then we know by the choice of δ that $n \geq n_0$, and $|h(x) - h(b)| = |h(x)| = |\frac{1}{n}| = \frac{1}{n} \leq \frac{1}{n_0} < \epsilon$

Either way, we have that h is continuous at b. □

Theorem 1.1.2. *h is discontinuous at every rational number in A.*

Proof. Let $a > 0$ be rational. Let (x_n) be a sequence of irrational numbers in A that converges to a. (for example could take $x_n := a(\sqrt{2})^{\frac{1}{n}}$). We have that $h(x_n) = 0$ for all n and hence $\lim_{n\to\infty} h(x_n) = 0$. But, $h(a) > 0$. So by the Criterion for Discontinuity we are done. □

2 Practice Homework

Here we provide solutions to selected problems from the Practice Homeworks. We remark that for Practice Assignment 1, we may use (without proof) that $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$ and $\sin x < x$ for all $x > 0$. We may also use all trigonometric identities covered in standard Calculus courses.

Remark. In particular, the sum-to-product formula will be useful:

$$\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) \tag{2.1}$$

2.1 Practice1 Problem 8

Let (x_n) be defined recursively as $x_1 = 1$, $x_{n+1} = \sin x_n$, $n \geq 1$. Prove that (x_n) is convergent and find $\lim x_n$.

First, we shall show that $0 < x_n < \frac{\pi}{2}$ for all $n \in \mathbb{N}$. The base case is trivial because $0 < x_1 = 1 < \frac{\pi}{2}$. Now, suppose that $0 < x_n < \frac{\pi}{2}$ holds. Then, we have that $x_{n+1} = \sin x_n > 0$ (as $0 < x_n < \frac{\pi}{2}$). On the other hand, we have that $x_{n+1} = \sin x_n < x_n < \frac{\pi}{2}$ (where we have used that $\sin x < x$ for all $x > 0$). Combining, we arrive at $0 < x_{n+1} < \frac{\pi}{2}$, and hence by AI we are done.

Now, for all n we have that $x_{n+1} = \sin x_n < x_n$ (using what we have just shown). So (x_n) is decreasing and we know it is bounded from below by 0. By the Monotone Convergence Theorem, the sequence (x_n) is convergent (say $\lim_{n\to\infty} x_n = x$). It remains to determine its limit x.

To that end, recall that $\lim_{n\to\infty} x_{n+1} = x$. We have that:

$$x = \lim x_n = \lim x_{n+1} = \lim \sin(x_n) = \sin x$$

where the last equality follows from a previous problem, which Edward did in his last Tutorial. Hence $x = \sin x$. But, by the Order Limit Theorem, we know that $x \geq 0$. And recalling that $\sin x < x$ for all $x > 0$ we must have that $x = 0$.

2.2 Practice1 Problem 9

Let

$$x_n = \sin\left(\pi\sqrt{n^2+1}\right) \quad , \quad n \in \mathbb{N}$$

Prove that (x_n) is convergent and find $\lim x_n$

For all $n \in \mathbb{N}$:

$$\begin{aligned}
|x_n| &= \left|\sin\left(\pi\sqrt{n^2+1}\right)\right| \\
&= \left|\sin\left(\pi\sqrt{n^2+1}\right) - \sin(\pi n)\right| \quad (\sin(\pi n) = 0) \\
&= 2\left|\sin\left(\frac{\pi}{2}(\sqrt{n^2+1}-n)\right)\right|\left|\cos\left(\frac{\pi}{2}(\sqrt{n^2+1}+n)\right)\right| \quad (\text{using } (2.1)) \\
&\leq 2\left|\sin\left(\frac{\pi}{2}(\sqrt{n^2+1}-n)\right)\right| \\
&\leq 2\left|\frac{\pi}{2}(\sqrt{n^2+1}-n)\right| \\
&= \pi\left|\sqrt{n^2+1}-n\right| \\
&= \pi\left|\frac{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)}{(\sqrt{n^2+1}+n)}\right| \\
&= \frac{\pi}{\sqrt{n^2+1}+n} \\
&< \frac{\pi}{n}
\end{aligned}$$

So by Squeeze Theorem, $\lim x_n = 0$.

2.3 Practice2 Problem 3

2.3.1 A

Using the $\epsilon - \delta$ definition of the limit of a function, prove that

$$\lim_{x \to a} \frac{x}{1+x} = \frac{a}{1+a}$$

for all $a \in \mathbb{R}$, $a \neq -1$

Let $\epsilon > 0$ be arbitrary and take $0 < \delta < \frac{1}{2}|1+a|$ (note that $|1+a| > 0$ since $a \neq -1$). For $|x-a| < \delta$:

$$\begin{aligned}
|1+x| &= |(1+a) - (a-x)| \\
&\geq |1+a| - |a-x| \quad \text{(Reverse Triangle Inequality)} \\
&= |1+a| - |x-a| \\
&\geq |1+a| - \delta \\
&> \frac{1}{2}|1+a|
\end{aligned}$$

(note that this implies that $f(x) = \frac{x}{1+x}$ is defined for $|x-a| < \delta$). So, for $|x-a| < \delta$, we have that:

$$\begin{aligned}
\left|\frac{x}{1+x} - \frac{a}{1+a}\right| &= \left|\frac{x(1+a) - a(1+x)}{(1+x)(1+a)}\right| \\
&= \left|\frac{x-a}{(1+x)(1+a)}\right| \\
&= \frac{|x-a|}{|1+x||1+a|} \\
&< \frac{\delta}{|1+x||1+a|} \\
&< \frac{\delta}{\frac{1}{2}|1+a||1+a|} \\
&= \frac{2\delta}{(1+a)^2}
\end{aligned}$$

But, $\frac{2\delta}{(1+a)^2} < \epsilon \iff \delta < \frac{\epsilon}{2}(1+a)^2$. So taking $\delta < \min\{\frac{1}{2}|1+a|, \frac{\epsilon}{2}(1+a)^2\}$, we have that

$$\left|\frac{x}{1+x} - \frac{a}{1+a}\right| < \epsilon$$

for all x satisfying $|x-a| < \delta$. In conclusion, we have that $\lim_{x\to a} \frac{x}{1+x} = \frac{a}{1+a}$.

2.3.2 B

Using the $\epsilon - \delta$ definition of the limit of a function, prove that

$$\lim_{x\to -1} \frac{x}{1+x}$$

does not exist.

We proceed via contradiction, hence assume $\lim_{x\to -1} \frac{x}{1+x} = L$ for some $L \in \mathbb{R}$. Let $\epsilon := 1$ and let $\delta > 0$ be arbitrary. Choose $\tilde{\delta} < \min\{\delta, \frac{1}{2+|L|}\}$. Then for all x satisfying $|x-(-1)| < \tilde{\delta} < \delta$ we have that:

$$\begin{aligned}
\left|\frac{x}{1+x} - L\right| &= \left|\frac{1+x-1}{1+x} - L\right| \\
&= \left|(1-L) - \frac{1}{1+x}\right| \\
&\geq \frac{1}{|1+x|} - |1-L| \quad \text{(Reverse Triangle Inequality)} \\
&\geq \frac{1}{|1+x|} - (1+|L|) \quad \text{(Triangle Inequality)} \\
&\geq \frac{1}{\tilde{\delta}} - 1 - |L| \\
&> 2 + |L| - 1 - |L| \\
&= 1 \\
&= \epsilon
\end{aligned}$$

Hence we arrive at a contradiction.

2.3.3 Remarks

Recall the following (we used it in my last tutorial):

Theorem 2.3.1 (Divergence Criterion for Functional Limits)**.** *Let $f : A \to \mathbb{R}$ and c be a cluster point of A. If there exist two sequences (x_n) and (y_n) in A with $x_n \neq c$ and $y_n \neq c$ and*

$$\lim x_n = \lim y_n = c \quad \textit{but} \quad \lim f(x_n) \neq \lim f(y_n)$$

then we can conclude that the functional limit $\lim_{x\to c} f(x)$ does not exist.

In the lecture notes, there is an additional divergence criterion:

Theorem 2.3.2 (Divergence Criterion for Functional Limits II)**.** *Let $f : A \to \mathbb{R}$ and c be a cluster point of A. If there exists a sequence (x_n) in A with $x_n \neq c$ and $\lim x_n = c$ but $(f(x_n))$ is not convergent, then $\lim_{x\to c} f(x)$ does not exist.*

We can use this criterion to provide an easier solution to the above part B (although of course the prompt specifically asks to use the $\epsilon - \delta$ definition, which is why we had solved it that way.).

Let $x_n := -1 - \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $x_n \neq -1$ and $\lim x_n = -1$. But,

$$\begin{aligned} f(x_n) &= \frac{x_n}{1+x_n} \\ &= \frac{-1-\frac{1}{n}}{1-1-\frac{1}{n}} \\ &= \frac{-1-\frac{1}{n}}{-\frac{1}{n}} \\ &= \frac{1+\frac{1}{n}}{\frac{1}{n}} \\ &= n+1 \end{aligned}$$

which diverges to $+\infty$. So by the divergence criterion we are done.

2.4 Practice2 Problem 4

Use the $\epsilon - \delta$ definition of the limit of a function to prove that

$$\lim_{x\to a} x^n = a^n$$

for all $n \in \mathbb{N}$ and all $a \in \mathbb{R}$.

First, we shall take note of the factorization:

$$x^n - a^n = (x-a)\sum_{k=0}^{n-1} a^k x^{n-1-k} \tag{2.2}$$

(can prove it easily directly or via induction). Now, let $\epsilon > 0$ be arbitrary and take $0 < \delta < 1$.

For $|x-a|<\delta$:

$$\begin{aligned}
|x^n - a^n| &= \left|(x-a)\sum_{k=0}^{n-1} a^k x^{n-1-k}\right| \\
&= |x-a|\left|\sum_{k=0}^{n-1} a^k x^{n-1-k}\right| \\
&\leq |x-a|\sum_{k=0}^{n-1} \left|a^k x^{n-1-k}\right| \quad \text{(Triangle Inequality)} \\
&= |x-a|\sum_{k=0}^{n-1} |a|^k |x|^{n-1-k} \\
&< \delta \sum_{k=0}^{n-1} |a|^k |x|^{n-1-k}
\end{aligned}$$

But we have that

$$|x| = |x-a+a| \leq |x-a| + |a| < \delta + |a| < 1 + |a|$$

and $|a| < 1 + |a|$. Hence,

$$|x^n - a^n| < \delta \sum_{k=0}^{n-1} (1+|a|)^k (1+|a|)^{n-1-k} = \delta \sum_{k=0}^{n-1} (1+|a|)^{n-1} = \delta n (1+|a|)^{n-1}$$

But, $\delta n(1+|a|)^{n-1} < \epsilon \iff \delta < \frac{\epsilon}{n(1+|a|)^{n-1}}$. So taking $\delta < \min\{1, \frac{\epsilon}{n(1+|a|)^{n-1}}\}$, we have that $|x^n - a^n| < \epsilon$ for all x satisfying $|x-a| < \delta$. In conclusion, we have that $\lim_{x\to a} x^n = a^n$.

2.5 Practice2 Problem 6

Let $A \subset \mathbb{R}$, let a be a cluster point of A and let $f : A \to \mathbb{R}$ be a function.

2.5.1 A

Prove that $\lim_{x\to a} f(x) = L$ iff $\lim_{x\to a} |f(x) - L| = 0$.

By the $\epsilon - \delta$ definition for functional limits:

$$\begin{aligned}
\lim_{x\to a} f(x) = L &\iff \forall \epsilon > 0\ \exists \delta > 0 : |f(x) - L| < \epsilon\ \forall x \in A \text{ with } 0 < |x-a| < \delta \\
&\iff \forall \epsilon > 0\ \exists \delta > 0 : ||f(x) - L| - 0| < \epsilon\ \forall x \in A \text{ with } 0 < |x-a| < \delta \\
&\iff \lim_{x\to a} |f(x) - L| = 0
\end{aligned}$$

2.5.2 B

Prove that $\lim_{x\to a} f(x) = L$ iff $\lim_{x\to 0} f(x+a) = L$.

Let $B := \{x \in \mathbb{R} : x + a \in A\}$, and for any $x \in \mathbb{R}$ define $\tilde{x} := x - a$. Then we have that $x \in A$ iff $\tilde{x} \in B$, and $x \neq a \iff \tilde{x} \neq 0$. Furthermore, if for a sequence (x_n) we define a sequence $(\tilde{x}_n)$ by $\tilde{x}_n := x_n - a$ for all $n \in \mathbb{N}$, then $\lim(x_n) = a \iff \lim(\tilde{x}_n) = 0$ (apply the Algebraic Limit Theorem). Thus, by the Sequential Criterion for Functional Limits, we have that:

$$\begin{aligned}\lim_{x\to a} f(x) = L &\iff \lim f(x_n) = L \; \forall (x_n) : x_n \in A, x_n \neq a \; \forall n \text{ and } \lim(x_n) = a \\ &\iff \lim f(\tilde{x}_n + a) = L \; \forall (\tilde{x}_n) : \tilde{x}_n \in B, \tilde{x}_n \neq 0 \; \forall n \text{ and } \lim(\tilde{x}_n) = 0 \\ &\iff \lim_{x\to 0} f(x+a) = L\end{aligned}$$

2.6 Practice2 Problem 7

Prove $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$

Since $-1 \le \sin(z) \le 1$ for all $z \in \mathbb{R}$, we have that $-|x| \le x \sin\left(\frac{1}{x}\right) \le |x|$ for all $x \in \mathbb{R}$, $x \neq 0$. From a previous example in a Tutorial, we showed that $\lim_{x\to 0} |x| = 0$, and thus we also know by the Algebraic Limit Theorem that $\lim_{x\to 0} -|x| = 0$. Hence by the Squeeze Theorem, we have that $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$.