# APM346 Tutorial 5

Partial Differential Equations


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Recall that the Full Fourier series of $f(x)$ on $(-l, l)$ is given by

$$f(x) = C_0 + \sum_{n=1}^{\infty} \left( C_n \cos\left(\frac{n\pi x}{l}\right) + D_n \sin\left(\frac{n\pi x}{l}\right) \right)$$

where

$$C_0 = \frac{1}{2l} \int_{-l}^{l} f(x)dx \quad C_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx \quad D_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

For a differentiable function $f(x)$ defined on $(-l, l)$ with $f(-l) = f(l)$, we can consider its derivative $f'(x)$. The derivative of the Fourier Series would be exactly equal to the derivative $f'(x)$.

Lastly, recall that if we have Fourier series $f(x) = \sum_{k=0}^{\infty} a_k X_k(x)$, $a < x < b$ and if $\int_a^b |f(x)|^2 dx$ is finite, then we have Parseval's Equality:

$$\sum_{k=0}^{\infty} |a_k|^2 \int_a^b |X_k(x)|^2 = \int_a^b |f(x)|^2$$

# Example 1

Find the Full Fourier series of $x$ on $(-l, l)$.

First,

$$C_0 = \frac{1}{2l} \int_{-l}^{l} x dx = \frac{1}{2l}(0) = 0$$

Now,

$$\begin{aligned}
C_n &= \frac{1}{l} \int_{-l}^{l} x \cos\left(\frac{n\pi x}{l}\right) dx \\
&= \frac{1}{l} \left[ x \frac{l}{n\pi} \sin\left(\frac{n\pi x}{l}\right) \Big|_{-l}^{l} - \int_{-l}^{l} \frac{l}{n\pi} \sin\left(\frac{n\pi x}{l}\right) dx \right] \quad \text{(IBP)} \\
&= \frac{-1}{l} \frac{l}{n\pi} (-1) \frac{l}{n\pi} \cos\left(\frac{n\pi x}{l}\right) \Big|_{-l}^{l} \\
&= \frac{l}{n^2\pi^2} [\cos(n\pi) - \cos(-n\pi)] \\
&= 0
\end{aligned}$$

Lastly,

$$\begin{aligned}
D_n &= \frac{1}{l}\int_{-l}^{l} x\sin\Big(\frac{n\pi x}{l}\Big)dx \\
&= \frac{1}{l}\left[-x\frac{l}{n\pi}\cos\Big(\frac{n\pi x}{l}\Big)\Big|_{-l}^{l} + \int_{-l}^{l}\frac{l}{n\pi}\cos\Big(\frac{n\pi x}{l}\Big)dx\right] \quad \text{(IBP)} \\
&= \frac{-x}{n\pi}\cos\Big(\frac{n\pi x}{l}\Big)\Big|_{-l}^{l} + \frac{l}{n^2\pi^2}\sin\Big(\frac{n\pi x}{l}\Big)\Big|_{-l}^{l} \\
&= \frac{-l}{n\pi}\cos(n\pi) + \frac{-l}{n\pi}\cos(-n\pi) + 0 \\
&= \frac{-2l}{n\pi}\cos(n\pi) \\
&= \frac{-2l}{n\pi}(-1)^n \\
&= (-1)^{n+1}\frac{2l}{n\pi}
\end{aligned}$$

Hence we conclude that

$$x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2l}{n\pi}\sin\Big(\frac{n\pi x}{l}\Big)$$

# Example 2

Find the Full Fourier series of $|x|$ on $(-\pi,\pi)$.

First,

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|x|dx = \frac{1}{\pi}\int_0^{\pi}|x|dx = \frac{1}{\pi}\int_0^{\pi}xdx = \frac{1}{\pi}\frac{\pi^2}{2} = \frac{\pi}{2}$$

Now,

$$\begin{aligned}
C_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos(nx)dx \\
&= \frac{2}{\pi}\int_0^{\pi}x\cos(nx)dx \quad \text{(Both } |x| \text{ and } \cos \text{ are even)} \\
&= \frac{2}{\pi}\left[\frac{x}{n}\sin(nx)\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nx)dx\right] \quad \text{(IBP)} \\
&= \frac{2}{\pi}\left[0 + \frac{1}{n^2}\cos(nx)\Big|_0^{\pi}\right] \\
&= \frac{2}{n^2\pi}[\cos(n\pi) - \cos(0)] \\
&= \frac{2}{n^2\pi}[(-1)^n - 1]
\end{aligned}$$

so that $C_n = 0$ for $n$ even and $\frac{-4}{n^2\pi}$ for $n$ odd. Lastly, we note that since $|x|$ is even and $\sin(nx)$ is odd,

$$D_n = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\sin(nx)dx = 0$$

Hence, we conclude that

$$|x| = \frac{\pi}{2} + \sum_{k=0}^{\infty} \frac{-4}{\pi(2k+1)^2} \cos((2k+1)x)$$

where $n = 2k+1$.

# Example 3

Recall the Full Fourier series of $f(x) = |x|$ on $(-\pi, \pi)$ from the previous example.

1. With $f(x) = |x|$, what is $f'(x)$?
2. Find the Full Fourier series of $f'(x)$ on $(-\pi, \pi)$ with justification.
3. Use Parseval's Equality on $f(x)$ to find the value to the series $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4}$

For the first part, since

$$|x| = \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x, & x < 0 \end{cases}$$

we conclude that

$$f'(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

which is called the "sign function". We write $f'(x) = \text{sgn } x$.

For the second part, first note that $f(-\pi) = \pi = f(\pi)$, so we can indeed differentiate the Fourier series. Recalling the series we found in the previous example, we have that

$$f'(x) = \sum_{k=0}^{\infty} \frac{4}{\pi(2k+1)} \sin((2k+1)x)$$

For the last part, by Parseval's Equality we have

$$\int_{-\pi}^{\pi} |x|^2 dx = \left|\frac{\pi}{2}\right|^2 \int_{-\pi}^{\pi} |1|^2 dx + \sum_{k=0}^{\infty} \left|\frac{-4}{\pi(2k+1)^2}\right|^2 \int_{-\pi}^{\pi} |\cos((2k+1)x)|^2 dx$$

$$\to \frac{x^3}{3}\Big|_{-\pi}^{\pi} = \frac{\pi^2}{4}(2\pi) + \sum_{k=0}^{\infty} \frac{16}{\pi^2(2k+1)^4} [\pi] \quad \text{(using orthogonality relation)}$$

$$\to \frac{2\pi^3}{3} = \frac{\pi^3}{2} + \sum_{k=0}^{\infty} \frac{16}{\pi(2k+1)^4}$$

$$\to \frac{\pi^3}{6} = \sum_{k=0}^{\infty} \frac{16}{\pi(2k+1)^4}$$

$$\to \frac{\pi^4}{96} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^4}$$

# Bibliography

[1] Xiao Jie, Instructor's course notes (Quercus)

[2] W. Strauss, *Partial Differential Equations: An Introduction*, 2nd edition, Wiley