Math 254 Tutorial 3

Honours Analysis 1

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Diagonal Counting Argument

First, we recall some definitions:

Definition 1.0.1. Let $f : A \to B$ be a function from A to B.

1. f is said to be injective if whenever $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. If f is an injective function, we also say that f is an injection.

2. f is said to be surjective if $f(A) = B$ (for any $b \in B$, there exists at least one $x \in A$ such that $f(x) = b$). If f is a surjective function, we also say that f is a surjection.

3. If f is both injective and surjective, then f is said to be bijective. If f is bijective, we also say that f is a bijection.

Definition 1.0.2. 1. A set S is said to be denumerable (or countably infinite) if there exists a bijection of $\mathbb{N}$ onto S.

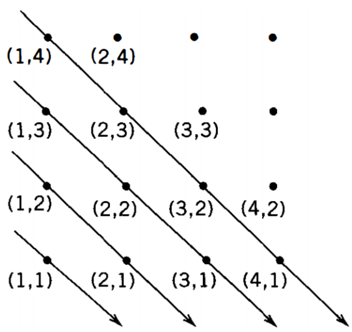
2. A set S is said to be countable if it is either finite or denumerable.

3. A set S is said to be uncountable if it is not countable.

With the above definitions in hand, we now proceed to the main theorem of this section:

Theorem 1.0.1. *The set* $\mathbb{N} \times \mathbb{N}$ *is denumerable (countably infinite).*

Proof. First, we shall define a counting function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ (our end goal will then be to prove that this counting function is a bijection). To start, we shall view $\mathbb{N} \times \mathbb{N}$ as a collection of diagonals:



This picture will be very important throughout the entire proof, so please refer to it as needed. Note that the first diagonal (bottom left) has one point, the second diagonal has 2 points, and the k-th diagonal has k points. Let $\psi(k)$ denote the total number of points in diagonals 1 through k. Then recalling the first lecture (an induction example), we have that:

$$\psi(k) = 1 + 2 + ... + k = \frac{1}{2}k(k+1)$$

We remark that ψ is strictly increasing, which we shall use later. We can show this via:

$$\psi(k+1) = \psi(k) + (k+1) \tag{1.1}$$

Now, the point (m, n) in $\mathbb{N} \times \mathbb{N}$ lies on the k-th diagonal when $k = m + n - 1$, and it is the m-th point in that diagonal (moving downward from left to right). Thus we count the point (m, n) by first counting the points in the first $k - 1 = m + n - 2$ diagonals and then adding m. This defines our counting function $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by:

$$h(m, n) := \psi(m + n - 2) + m \tag{1.2}$$

for $(m, n) \in \mathbb{N} \times \mathbb{N}$.

We shall now prove that the counting function h is indeed a bijection. We start with injectivity. Note that if $(m, n) \neq (m', n')$ then either $m + n \neq m' + n'$ OR $m + n = m' + n'$ with $m \neq m'$ (and also $n \neq n'$).

For the first case, without loss of generality (from now on I shall say WLOG), we may suppose that $m + n < m' + n'$. Then we have that:

$$\begin{aligned} h(m, n) &:= \psi(m + n - 2) + m \\ &\leq \psi(m + n - 2) + (m + n - 1) \quad (n \geq 1) \\ &= \psi(m + n - 1) \quad (\text{using } (1.1)) \\ &\leq \psi(m' + n' - 2) \quad (\psi \text{ is strictly increasing}) \\ &< \psi(m' + n' - 2) + m' \quad (m' > 0) \\ &= h(m', n') \end{aligned}$$

and hence $h(m, n) \neq h(m', n')$.

For the second case ($m + n = m' + n'$ with $m \neq m'$), we have that:

$$\begin{aligned} h(m, n) - m &= \psi(m + n - 2) \\ &= \psi(m' + n' - 2) \quad (m + n = m' + n') \\ &= h(m', n') - m' \end{aligned}$$

Thus since $m' \neq m$ we conclude that $h(m, n) \neq h(m', n')$. So we have proved injectivity for h.

It remains to show that h is surjective. First notice that $h(1, 1) = \psi(0) + 1 = 1$. Now, let $p \in \mathbb{N}$ with $p \geq 2$ (so we need to find a pair $(m_p, n_p) \in \mathbb{N} \times \mathbb{N}$ with $h(m_p, n_p) = p$). Well, since $p < \psi(p)$ (by the definition of ψ and since $p \geq 2$), the set

$$E_p := \{k \in \mathbb{N} : p \leq \psi(k)\}$$

is nonempty. Thus, applying the Well-Ordering Property, we can let $k_p > 1$ be the least element in E_p (note that $k_p > 1$ since $\psi(1) = 1$). Now,

$$\begin{aligned}\psi(k_p - 1) &< p \quad (k_p \text{ is the least element})\\ &\leq \psi(k_p) \quad (k_p \in E_p)\\ &= \psi(k_p - 1) + k_p \quad (\text{using } (1.1))\end{aligned}$$

Hence, we can let

$$m_p := p - \psi(k_p - 1)$$

so that $1 \leq m_p \leq k_p$. And we can let

$$n_p := k_p - m_p + 1$$

such that $1 \leq n_p \leq k_p$ (rearrange and thus also $m_p + n_p - 1 = k_p$). Therefore:

$$\begin{aligned}h(m_p, n_p) &= \psi(m_p + n_p - 2) + m_p\\ &= \psi(k_p - 1) + m_p\\ &= p\end{aligned}$$

□

2 Exercises with Solutions

In this section, we solve problems from §1.4 of the textbook.

Example 2.0.1. Let $A \subset \mathbb{R}$ be nonempty and bounded above, and let $s \in \mathbb{R}$ have the property that for all $n \in \mathbb{N}$, $s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A. Here we shall show that $s = \sup(A)$.

First we need to show that s is an upper bound for A. We proceed via contradiction, so suppose that $s < x$ for some $x \in A$. Then, $x - s > 0$ and we can use the Archimedean Property to get an $n \in \mathbb{N}$ such that $\frac{1}{n} < x - s$. But then $x > s + \frac{1}{n}$, contradicting that $s + \frac{1}{n}$ is an upper bound for A.

Now, let $\epsilon > 0$. By the Archimedean Property, there exists $n \in \mathbb{N}$ such that $\epsilon > \frac{1}{n}$. In addition, since $s - \frac{1}{n}$ is not an upper bound for A, there exists $x \in A$ with $x > s - \frac{1}{n}$. Putting these together we have:

$$x > s - \frac{1}{n} > s - \epsilon$$

So we are finished (we have used the Proposition from Lec-Sep21).

Example 2.0.2. Here we shall prove that

$$\cap_{n=1}^{\infty} \left(0, \frac{1}{n}\right) = \emptyset$$

Let $x \in \mathbb{R}$. We just need to show that $x \notin (0, \frac{1}{n})$ for some $n \in \mathbb{N}$. Well, for $x \leq 0$, simply take $n = 1$; $x \notin (0, 1)$. So any $x \leq 0$ is not in $\cap_{n=1}^{\infty}(0, \frac{1}{n})$.

For $x > 0$, by the Archimedean Property, there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < x$. So $x \notin (0, \frac{1}{N})$ and hence any $x > 0$ cannot be in $\cap_{n=1}^{\infty}(0, \frac{1}{n})$.

Remark. The above example demonstrates that the intervals in the Nested Interval Property must be closed for the theorem to hold.

Example 2.0.3. Let $a < b$ be real numbers and consider the set $T = \mathbb{Q} \cap [a, b]$. Here we show that $\sup(T) = b$.

First, we note that b is trivially an upper bound for $T = \mathbb{Q} \cap [a, b]$. Now, let φ be an upper bound for T. We proceed via contradiction, hence suppose that $\varphi < b$. Then by the Density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $r \in \mathbb{Q}$ such that $\varphi < r < b$. So $r \in T$ and φ is not an upper bound. Contradiction.

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Bonus Examples from Office Hours

Example 3.0.1. Consider $A = [0, 1)$. We shall prove that $\sup(A) = 1$.

First, note that $\forall x \in [0, 1)$, we have that $x \leq 1$ and hence 1 is an upper bound for A.

Now, let b be an upper bound for A. Suppose that $b < 1$. Then we have that $b < \frac{b+1}{2} \in [0, 1)$. Hence b is not an upper bound for A, which is a contradiction. So $b \geq 1$, and 1 is the least upper bound.

Remark. The above example may be proven in multiple ways (for example using the Archimedean Property). Also, in the method above, note that we assume $b \geq 0$, since otherwise it is trivial that b is not an upper bound for A.

Example 3.0.2. We shall prove that given a real number z, there exists $m \in \mathbb{Z}$ such that $m \leq z < m + 1$.

First, let $A := \{n \in \mathbb{Z} : n \leq z\}$. By the Archimedean property, there exists $N \in \mathbb{N}$ such that $N > z$. Also, applying the Archimedean property again (to $-z$), there exists $M \in \mathbb{N}$ such that $M > -z$. In other words, we have that $-M < z < N$, where $M, N \in \mathbb{N}$. Thus, we have that A is non-empty. So, since A is also trivially bounded from above (z is an upper bound), by the Axiom of Completeness, we have a supremum, call it $s = \sup(A)$. Since s is a supremum, there exists $m \in A$ such that $s - 1 < m \leq s$.

First, since $m \in A$, we trivially have that $m \leq z$. Then from $s - 1 < m$, we have that $s < m + 1$ and thus $(m + 1) \notin A$. So $m + 1 > z$. Combining, we have that $m \leq z < m + 1$.

Remark. As is with many theorems in Analysis, the above example may be proven in multiple ways (for example using the Well Ordering Property).

4 Results Used

Axiom of Completeness - Every non-empty set of real numbers that is bounded from above has a least upper bound (supremum).
Proposition - Let s be an upper bound for a set $A \subset \mathbb{R}$. Then $s = \sup(A)$ iff for every $\epsilon > 0$, there exists $x \in A$ such that $s - \epsilon < x$.
Archimedean Property -

1. for any $x \in \mathbb{R}$, $\exists n \in \mathbb{N}$ such that $n > x$
2. for any real number $y > 0$, $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < y$

Density of $\mathbb{Q}$ in $\mathbb{R}$ - For any two real numbers a and b with $a < b$, there exists a rational number r such that $a < r < b$.
Well-Ordering Property - Every non-empty subset S of $\mathbb{N}$ has a least element.