MAT351 Tutorial 7

Partial Differential Equations

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In the previous tutorial (Tutorial 6), we finished up/clarified aspects of Chapter 3 of Strauss [1]. Now, we move to Chapter 4 where we consider boundary value problems on a finite interval $0 < x < l$ (you may perhaps consider this as more physically realistic). The method employed here is *Separation of Variables*, which has been covered in the class lectures. In this tutorial specifically we shall consider the case of Robin boundary conditions.

1 Separation of Variables: The Robin Boundary Condition

For the Robin condition, the method of separation of variables yields the following eigenvalue problem for $X(x)$:

$$\begin{cases} -X'' = \lambda X \\ X' - a_0 X = 0 \quad \text{at } x = 0 \\ X' + a_l X = 0 \quad \text{at } x = l \end{cases} \tag{1.1}$$

where the two constants a_0 and a_l are considered as given.

Remark. Recall that an *eigenfunction* is a solution $X \not\equiv 0$, and an *eigenvalue* is a number λ for which there exists a solution $X \not\equiv 0$

Remark. If $a_0 = a_l = 0$, then we are simply considering the Neumann problem.

Zero Eigenvalue

In this section, let us examine the possibility of a zero eigenvalue (this is left as an exercise in Strauss). If $\lambda = 0$, then our ODE is simply $X'' = 0$ and hence $X(x) = Cx + D$. The boundary conditions give:

$$\begin{cases} X'(0) - a_0 X(0) = C - a_0 D = 0 \\ X'(l) + a_l X(l) = C + a_l(Cl + D) = 0 \end{cases}$$

so that $C = a_0 D$ and $a_0 D + a_l(a_0 Dl + D) = 0$ (so $D[a_0 + a_l a_0 l + a_l] = 0$). If $D = 0$ then $C = 0$, and X is the zero function and therefore not an eigenfunction. Therefore $\lambda = 0$ is an eigenvalue if and only if $a_0 + a_l a_0 l + a_l = 0$. That is:

$$\lambda = 0 \text{ is an eigenvalue iff } a_0 + a_l = -a_0 a_l l$$

Negative Eigenvalue

In this section, let us examine the possibility of a negative (real) eigenvalue $\lambda < 0$, which we shall write as $\lambda = -\gamma^2$. Then our ODE is $X'' = \gamma^2 X$ which has solution

$$X(x) = C\cosh(\gamma x) + D\sinh(\gamma x)$$

The boundary conditions give:

$$\begin{cases} X'(0) - a_0 X(0) = \gamma C \sinh(0) + \gamma D \cosh(0) - a_0 (C \cosh(0) + D \sinh(0)) = \gamma D - a_0 C = 0 \\ X'(l) + a_l X(l) = \gamma C \sinh(\gamma l) + \gamma D \cosh(\gamma l) + a_l (C \cosh(\gamma l) + D \sinh(\gamma l)) = 0 \end{cases}$$

so that $D = \frac{a_0}{\gamma} C$ and

$$\gamma C \sinh(\gamma l) + a_0 C \cosh(\gamma l) + a_l C \cosh(\gamma l) + \frac{a_0 a_l}{\gamma} C \sinh(\gamma l) = 0 \tag{1.2}$$

Then, (1.2) gives

$$\sinh(\gamma l)\left(\gamma + \frac{a_0 a_l}{\gamma}\right) + \cosh(\gamma l)\,(a_0 + a_l) = 0$$
$$\implies \tanh(\gamma l) = \frac{-(a_0 + a_l)}{\gamma + \frac{a_0 a_l}{\gamma}}$$

Hence, we arrive at precisely the eigenvalue equation for negative eigenvalues given in Strauss (but note that Strauss left the derivation we have done as an exercise):

$$\tanh(\gamma l) = -\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_0 a_l} \tag{1.3}$$

Solving (1.3) for $\gamma > 0$ would give us a negative eigenvalue $\lambda = -\gamma^2$. Of course, solving (1.3) is difficult - this is the general issue with Robin Boundary condition. Although the method of separation of variables still works, unlike the cases of Neumann and Dirichlet conditions the eigenvalues have no simple formula - one must resort to numerical methods or graphical analysis for instance. See the problem below for an example.

Strauss 4.3.4

> Consider the Robin eigenvalue problem. If $a_0 < 0$, $a_l < 0$, and $-a_0 - a_l < a_0 a_l l$, show that there are two negative eigenvalues.

We need to look for intersections of the graphs on each side of (1.3) for $\gamma > 0$. To that end, let $y(\gamma) = -(a_0 + a_l)\gamma/(\gamma^2 + a_0 a_l)$ and $f(\gamma) = \tanh(\gamma l)$.

Figure 1.1: Example of $f(\gamma)$ and $y(\gamma)$ for $l = 1, a_0 = -5.5, a_l = -2.3$.

Then,

$$y'(\gamma) = \frac{-(a_0 + a_l)(a_0 a_l - \gamma^2)}{(a_0 a_l + \gamma^2)^2}$$

which vanishes at $\gamma = \sqrt{a_0 a_l}$ (note since $a_0, a_l < 0$ then $\gamma > 0$), and

$$\begin{aligned} y(\sqrt{a_0 a_l}) &= \frac{-(a_0 + a_l)\sqrt{a_0 a_l}}{a_0 a_l + a_0 a_l} \\ &= \frac{-(a_0 + a_l)}{2\sqrt{a_0 a_l}} \\ &= \frac{|a_0| + |a_l|}{2\sqrt{|a_0||a_l|}} \quad (a_0, a_l < 0) \\ &\geq \frac{|a_0| + |a_l|}{|a_0| + |a_l|} \\ &= 1 \end{aligned}$$

where we have used the inequality $a^2 + b^2 \geq 2ab$ with $a := \sqrt{|a_0|}, b := \sqrt{|a_l|}$. Since $f(\gamma) = \tanh(\gamma l) < 1$ for all $\gamma \in [0, \infty)$ we have that $y(\sqrt{a_0 a_l}) > f(\sqrt{a_0 a_l})$. Now, $f'(0) = l \operatorname{sech}^2(0) = l > \frac{-(a_0+a_l)}{a_0 a_l}$ and $y'(0) = \frac{-(a_0+a_l)}{a_0 a_l}$ so for γ in a small interval of the form $(0, \delta)$ we have $f(\gamma) > y(\gamma)$. Putting it together, we conclude that there exists some $\gamma_1 \in (0, \sqrt{a_0 a_l})$ such that $f(\gamma_1) = y(\gamma_1)$. That is, we have one negative eigenvalue $-\gamma_1^2$.

Now, we have that $\lim_{\gamma \to \infty} f(\gamma) = 1$ and $\lim_{\gamma \to \infty} y(\gamma) = 0$. Hence, for sufficiently large γ we have that $f(\gamma) > y(\gamma)$. So, there is another crossing point: some $\gamma_2 > \sqrt{a_0 a_l}$ such that $f(\gamma_2) = y(\gamma_2)$.

To conclude, both $-\gamma_1^2$ and $-\gamma_2^2$ are negative eigenvalues.

Example with Robin condition on one end

In this tutorial I have not addressed positive eigenvalues - you should have seen this in class (also refer to the beginning of section 4.3 of [1]). Here I just provide a simple example (following [2]) which has positive eigenvalues. We consider

$$\begin{cases} -X'' = \lambda X \\ X(0) = 0 \\ X(1) + X'(1) = 0 \end{cases}$$

and look for positive eigenvalues $\lambda = \beta^2$ where $\beta > 0$. The general solution is

$$X(x) = C\cos(\beta x) + D\sin(\beta x)$$

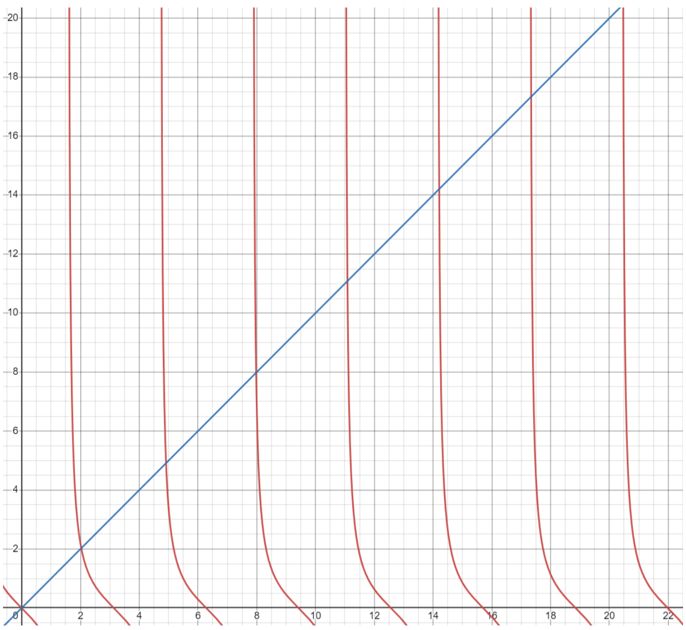
The first boundary condition gives $X(0) = C = 0$ and therefore the second boundary condition yields

$$X(1) + X'(1) = D\sin(\beta) + \beta D\cos(\beta) = 0$$

or simply $\sin(\beta) + \beta\cos(\beta) = 0$ so that

$$\beta = -\tan\beta$$

As usual, solving the above for $\beta > 0$ is not super simple as it has no closed form solution, but we can look at the intersections of the graphs on each side:



and we see that we have a sequence of such intersections $\beta_n, n = 1, 2, ..$ and hence a sequence of positive eigenvalues $\lambda_n = \beta_n^2$.

Bibliography

[1] W. Strauss, *Partial Differential Equations: An Introduction*, 2nd edition, Wiley

[2] R. Choksi, *Partial Differential Equations: A First Course*, AMS 2022