APM346 Tutorial 8

Partial Differential Equations

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Example 1

> Consider $u = u(x, y)$. Find the general solution to the PDE $u_x + u_y = x$.

Let $s = x + y$ and $t = x - y$. By the chain rule,

$$u_x = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} = u_s + u_t$$

and

$$u_y = \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} = u_s - u_t$$

Substituting into the PDE we arrive at $2u_s = x = \frac{s+t}{2}$ (since $s + t = (x + y) + (x - y) = 2x$). That is, $u_s = \frac{s+t}{4}$ and the general solution is $u(s, t) = \frac{s^2}{8} + \frac{st}{4} + f(t)$. In terms of the original variables, we conclude that

$$u(x, y) = \frac{(x + y)^2}{8} + \frac{(x + y)(x - y)}{4} + f(x - y)$$

Example 2 - Choksi

> Find the particular solution $u(x, y)$ to $3u_x + 2u_y = 0$, $u(x, 0) = x^3$.

Let $s = 3x + 2y$ and $t = 2x - 3y$. By the chain rule,

$$u_x = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} = 3u_s + 2u_t$$

and

$$u_y = \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} = 2u_s - 3u_t$$

Substituting into the PDE, we have that $3(3u_s+2u_t)+2(2u_s-3u_t) = 0$ or $9u_s+6u_t+4u_s-6u_t = 0$. Thus $13u_s = 0$ and so $u_s = 0$. The general solution is $u(s, t) = f(t)$, and in terms of the original variables this is $u(x, y) = f(2x - 3y)$. Now, $u(x, 0) = f(2x) = x^3$ and so $f(w) = \frac{w^3}{8}$. In conclusion,

$$u(x, y) = \frac{(2x - 3y)^3}{8}$$

Example 3 - Strauss 2.1.9

> Solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0$, $u(x, 0) = x^2$, $u_t(x, 0) = e^x$.

Factoring the operator, we have that $(\partial_x + \partial_t)(\partial_x - 4\partial_t)u = 0$. Introduce the characteristic coordinates

$$\eta = x - t \quad \text{and} \quad \zeta = 4x + t$$

Then by the chain rule, $\partial_x = \partial_\eta + 4\partial_\zeta$ and $\partial_t = -\partial_\eta + \partial_\zeta$. Hence,

$$(\partial_x + \partial_t) = 5\partial_\zeta \quad \text{and} \quad (\partial_x - 4\partial_t) = 5\partial_\eta$$

So our equation takes form $25u_{\zeta\eta} = 0$, or simplified $u_{\zeta\eta} = 0$. This has general solution $u = f(\eta) + g(\zeta)$, or in terms of the original variables we have that:

$$u(x,t) = f(x-t) + g(4x+t) \tag{1}$$

Now, initial conditions give that $u(x,0) = f(x)+g(4x) = x^2$ and $u_t(x,0) = -f'(x)+g'(4x) = e^x$. Integrating, we obtain

$$\begin{cases} f(x) + g(4x) = x^2 \\ -f(x) + \frac{1}{4}g(4x) = e^x \end{cases}$$

Now just solve for f, g. In particular, we have that $\frac{5}{4}g(4x) = x^2+e^x$ so that $g(4x) = \frac{4}{5}(x^2+e^x)$ and we conclude $g(x) = \frac{4}{5}(\frac{x^2}{16} + \exp(x/4))$. Then,

$$f(x) = x^2 - g(4x) = x^2 - \frac{4}{5}(x^2+e^x) = \frac{x^2}{5} - \frac{4}{5}e^x$$

Finally, recalling (1) we have our solution:

$$u(x,t) = \frac{(x-t)^2}{5} - \frac{4}{5}e^{x-t} + \frac{4}{5}\left(\frac{(4x+t)^2}{16} + e^{\frac{4x+t}{4}}\right)$$

Example 4

Find the particular solution $u(x,y)$ to $u_x + xu_y = 0$, $u(x,0) = x^2$.

To get the characteristic lines, $\frac{dy}{dx} = \frac{x}{1}$, $xdx = dy$ so $\frac{1}{2}x^2 = y + C$ or $-C = y - \frac{x^2}{2}$. Thus, we set $s = x$ and $t = y - \frac{x^2}{2}$. By the chain rule,

$$u_x = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} = u_s - xu_t$$

and

$$u_y = \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} = u_t$$

Substituting into the PDE we arrive at $u_s = 0$ and the general solution is $u(s,t) = f(t)$. In terms of the original variables, we have that $u(x,y) = f(y - \frac{x^2}{2})$. Now, $u(x,0) = f(-\frac{x^2}{2}) = x^2$ and so $f(w) = -2w$ (let $w = -\frac{x^2}{2}$ then $x^2 = -2w$). In conclusion,

$$u(x,y) = f(y - \frac{x^2}{2}) = -2(y - \frac{x^2}{2}) = -2y + x^2$$

Example 5 - Strauss 1.2.7

Solve the equation $yu_x + xu_y = 0$ with $u(0,y) = e^{-y^2}$. Sketch some of the characteristics.

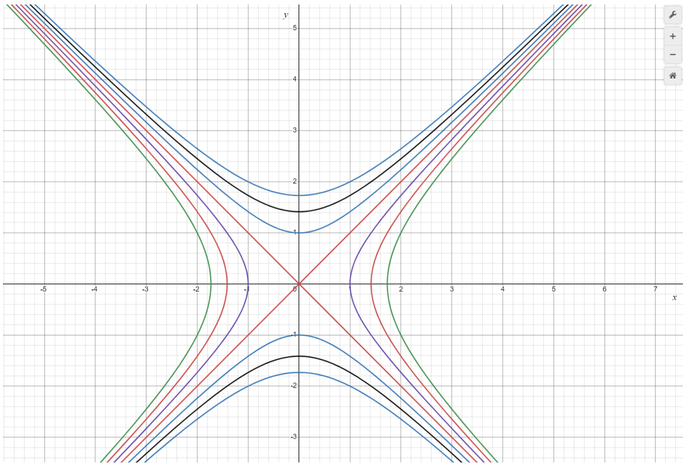
The characteristics must solve $\frac{dy}{dx} = \frac{x}{y}$. So $ydy = xdx$ and we conclude that $\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$ or $y^2 = x^2 + \tilde{C}$. The general solution is thus $u(x,y) = f(y^2 - x^2)$. However, we are given that $u(0,y) = e^{-y^2}$ and so:

$$u(0,y) = f(y^2) = e^{-y^2}$$

i.e. $f(w) = e^{-w}$. In conclusion, we have a solution

$$u(x, y) = e^{x^2 - y^2}$$

Here are some characteristic curves:



Example 6

Find the general solution $u(x, t)$ to the PDE $u_{xx} - u_{xt} - 6u_{tt} = 0$.

Factoring the operator, we have that $(\partial_x + 2\partial_t)(\partial_x - 3\partial_t)u = 0$. Introduce the characteristic coordinates

$$\eta = 3x + t \quad \text{and} \quad \zeta = 2x - t$$

Then by the chain rule, $\partial_x = 3\partial_\eta + 2\partial_\zeta$ and $\partial_t = \partial_\eta - \partial_\zeta$. Hence,

$$(\partial_x + 2\partial_t) = 5\partial_\eta \quad \text{and} \quad (\partial_x - 3\partial_t) = 5\partial_\zeta$$

So our equation takes form $25u_{\zeta\eta} = 0$, or simplified $u_{\zeta\eta} = 0$. This has general solution $u = f(\eta) + g(\zeta)$, or in terms of the original variables we have that:

$$u(x, t) = f(3x + t) + g(2x - t)$$

Bibliography

[1] Xiao Jie, Instructor's course notes (Quercus)

[2] W. Strauss, *Partial Differential Equations: An Introduction*, 2nd edition, Wiley

[3] R. Choksi, *Partial Differential Equations: A First Course*, AMS 2022