APM346 Tutorial 9

Partial Differential Equations

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Example 1 - Strauss 1.5.5

Consider the equation $u_x + yu_y = 0$ with the boundary condition $u(x, 0) = \phi(x)$.

(A) For $\phi(x) \equiv x$, show that no solution exists.

The characteristics must solve $\frac{dy}{dx} = \frac{y}{1} = y$. So $\frac{dy}{y} = dx$ and we have $\ln y = x + C$ or $y = \tilde{C}e^x$. So, set $s = x$ and $t = ye^{-x}$. By the chain rule,

$$u_x = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} = u_s - ye^{-x}u_t$$

and

$$u_y = \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} = e^{-x}u_t$$

Substituting into the PDE, we have that $u_s - ye^{-x}u_t + ye^{-x}u_t = 0$, or $u_s = 0$. The general solution is $u(s, t) = f(t)$, and in terms of the original variables this is $u(x, y) = f(ye^{-x})$.

For $\phi(x) \equiv x$ we would have $u(x, 0) = f(0) = x$, a contradiction. Note that when $y = 0$ the equation reads $u_x = 0$ so $u(x, 0)$ should be constant.

(B) For $\phi(x) \equiv 1$, show that there are many solutions.

For $\phi(x) \equiv 1$ we have $u(x, 0) = f(0) = 1$, and $u = 1$ satisfies the equation. So , all we can say about u is that

$$u(x, y) = \begin{cases} 1, & y = 0 \\ f(ye^{-x}), & \text{else} \end{cases}$$

Example 2 - Strauss 2.1.1

Solve $u_{tt} = c^2 u_{xx}$, $u(x, 0) = e^x$, $u_t(x, 0) = \sin(x)$

By D'Alembert's formula:

$$\begin{aligned} u(x, t) &= \frac{1}{2}\left(\phi(x + ct) + \phi(x - ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds \\ &= \frac{1}{2}\left(e^{x+ct} + e^{x-ct}\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin(s)ds \\ &= \frac{1}{2}e^x\left(e^{ct} + e^{-ct}\right) - \frac{1}{2c}(\cos(x + ct) - \cos(x - ct)) \\ &= e^x \cosh(ct) + \frac{1}{c}\sin(ct)\sin(x) \end{aligned}$$

Example 3 - Strauss 5.6.9

Use the method of subtraction to solve $u_{tt} = 9u_{xx}$ for $0 \le x \le 1$, with $u(0, t) = h$, $u(1, t) = k$, where h and k are given constants, and $u(x, 0) = 0$, $u_t(x, 0) = 0$.

Earlier in the course when we considered the wave equation on a finite domain $0 < x < l$, we only had homogeneous boundary conditions (for example $u(0,t) = u(l,t) = 0$). Here it is important to note that we have inhomogeneous boundary conditions.

To combat this, by subtraction the data can be shifted from the boundary to another spot in the problem. We let $w(x,t) = w(x) = (1 - \frac{x}{1})h + \frac{x}{1}k$ and $v(x,t) = u(x,t) - w(x,t)$. Then, $v(x,t)$ satisfies:

$$\begin{cases} v_{tt} = 9v_{xx} \\ v(0,t) = v(1,t) = 0 \\ v(x,0) = -w(x), \quad v_t(x,0) = 0 \end{cases} \tag{1}$$

Recall that the solution of problem (1) is

$$v(x,t) = \sum_{n=1}^{\infty} (A_n \cos(n\pi 3t) + B_n \sin(n\pi 3t)) \sin(n\pi x)$$

with

$$A_n = \frac{2}{1}\int_0^1 (-w(x)) \sin(n\pi x) dx = 2\int_0^1 \sin(n\pi x)((h-k)x - h) dx = \frac{2}{n\pi}(k(-1)^n - h)$$

where the integral is evaluated via IBP, and $B_n = \frac{2}{3n\pi}\int_0^1 0 dx = 0$. So, $v(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(k(-1)^n - h)\cos(n\pi 3t)\sin(n\pi x)$ and we have our solution:

$$u(x,t) = (1-x)h + xk + \sum_{n=1}^{\infty} \frac{2}{n\pi}(k(-1)^n - h)\cos(n\pi 3t)\sin(n\pi x)$$

Duhamel's Principle - Wave equation with a source

Please note that for this section, I follow [3]. We now look at the (1D) wave equation with an external force given by $f(x,t)$. The corresponding IVP with inhomogeneous right hand side source term is:

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), \quad -\infty < x < \infty, t > 0 \\ u(x,0) = \phi(x), \quad -\infty < x < \infty \\ u_t(x,0) = \psi(x), \quad -\infty < x < \infty \end{cases} \tag{2}$$

The claim is that the solution is the extension of D'Alembert's formula:

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s) ds + \frac{1}{2c}\int\int_D f(y,\tau) dy d\tau \tag{3}$$

where D is the "characteristic triangle" or "domain of dependence" associated with (x, t) (i.e. the triangle in the $x - t$ plane with top point (x, t) and base points $(x - ct, 0)$, $(x + ct, 0)$).

Remark. The double integral in (3) is simply equal to $\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(y, \tau) dy d\tau$

Remark. Naturally, if $f(x, t) \equiv 0$ then we recover the regular D'Alembert's formula.

Duhamel's principle (named after Jean-Marie Duhamel B:1797-D:1872) is a general method for solving linear PDEs with inhomogeneous terms. Here we demonstrate its use with (2).

First we note that by superposition, we need only to solve

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x, t), & -\infty < x < \infty, t > 0 \\ u(x, 0) = 0, & -\infty < x < \infty \\ u_t(x, 0) = 0, & -\infty < x < \infty \end{cases} \tag{4}$$

Adding this solution to D'Alembert's formula for the (homogenous) wave equation with initial data $\phi(x)$ and $\psi(x)$ then yields the solution for (2). Introducing the temporal parameter $s \in [0, \infty)$, Duhamel's principle states that the solution to (4) is given by $u(x, t) = \int_0^t w(x, t; s) ds$ where for each fixed s, $w(x, t; s)$ is solution to

$$\begin{cases} w_{tt}(x, t; s) = c^2 w_{xx}(x, t; s), & -\infty < x < \infty, t > s \\ w(x, s; s) = 0, & -\infty < x < \infty \\ w_t(x, s; s) = f(x, s), & -\infty < x < \infty \end{cases}$$

Check

Let us verify Duhamel's principle here. Recall:

Theorem 0.0.1 (Leibniz Rule)**.** *Let $f(t, s)$ smooth, then*

$$\frac{d}{dt} \left(\int_0^t f(t, s) ds \right) = f(t, t) + \int_0^t f_t(t, s) ds$$

By Leibniz rule we have that

$$u_t(x, t) = w(x, t; t) + \int_0^t w_t(x, t; s) ds = \int_0^t w_t(x, t; s) ds \tag{5}$$

since $w(x, t; t) = 0$. Then by Leibniz rule again

$$u_{tt}(x, t) = w_t(x, t; t) + \int_0^t w_{tt}(x, t; s) ds = f(x, t) + \int_0^t w_{tt}(x, t; s) ds$$

since $w_t(x, t; t) = f(x, t)$. On the other hand, by differentiating under the integral, we have that $u_{xx}(x, t) = \int_0^t w_{xx}(x, t; s) ds$. Hence:

$$u_{tt} - c^2 u_{xx} = \int_0^t \left[w_{tt}(x, t; s) - c^2 w_{xx}(x, t; s) \right] ds + f(x, t) = f(x, t)$$

where we have used that $w_{tt}(x, t; s) = c^2 w_{xx}(x, t; s)$.

Now, $u(x, 0) = 0$ trivially and $u_t(x, 0) = 0$ by (5).

Last Comment

So far we have shown that the solution to (2) is

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds + \int_0^t w(x,t;s)ds$$

But, we can solve for $w(x,t;s)$ for each fixed s as it satisfies homogenous wave equation, just now starting at $t = s$ instead of $t = 0$ (and now the source is incorporated into the initial data). One finds $w(x,t;s) = \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dy$ so that

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds$$

which is precisely the extension of D'Alembert's formula which was proposed in the beginning of this section.

Bibliography

[1] Xiao Jie, Instructor's course notes (Quercus)

[2] W. Strauss, *Partial Differential Equations: An Introduction*, 2nd edition, Wiley

[3] R. Choksi, *Partial Differential Equations: A First Course*, AMS 2022