Math 254 Tutorial - 10/14/2020

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[Sequences]

$\boxed{\text{Ex. } 2.3.1}$ Let $x_n \geq 0 \; \forall n \in \mathbb{N}$

a) if $(x_n) \to 0$, show that $(\sqrt{x_n}) \to 0$

<u>proof</u> - Let $\varepsilon > 0$. Since $(x_n) \to 0$, by def^n $\exists N \in \mathbb{N}$ such that whenever $n \geq N$ we have that $|x_n - 0| = |x_n| = x_n < \varepsilon^2$

Thus for $n \geq N$: $|\sqrt{x_n} - 0| = \sqrt{x_n} < \varepsilon$

$$\Rightarrow (\sqrt{x_n}) \to 0 \qquad \square$$

b) If $(x_n) \to x$, show that $(\sqrt{x_n}) \to \sqrt{x}$

<u>proof</u> - by part a) we can take $x > 0$ (& since $x_n \geq 0$)

(<u>Order Limit Theorem</u> - Let (x_n) & (y_n) be 2 sequences s.t. $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Then:

1) if $x_n \geq 0 \; \forall n \in \mathbb{N}$, then $x \geq 0$
2) if $x_n \leq y_n \; \forall n \in \mathbb{N}$, then $x \leq y$
3) if $\exists c \in \mathbb{R}$ s.t. $c \leq x_n \; \forall n$, then $c \leq x$. Similarly, if $\exists c \in \mathbb{R}$ s.t. $c \geq x_n \; \forall n$, then $c \geq x$)

Let $\varepsilon > 0$. Since $(x_n) \to x$, by def^n $\exists N \in \mathbb{N}$ s.t. whenever $n \geq N$ we have that $|x_n - x| < \varepsilon\sqrt{x}$

Thus for $n \geq N$: $|\sqrt{x_n} - \sqrt{x}| = |\sqrt{x_n} - \sqrt{x}| \left(\dfrac{\sqrt{x_n} + \sqrt{x}}{\sqrt{x_n} + \sqrt{x}} \right)$

$$= \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}}$$

$$\leq \frac{|x_n - x|}{\sqrt{x}} \qquad (x_n \geq 0)$$

$$< \frac{\varepsilon\sqrt{x}}{\sqrt{x}}$$

$$= \varepsilon$$

$$\Rightarrow (\sqrt{x_n}) \to \sqrt{x}$$

$\square$

(Brief aside on Subsequences)

definition - Let (x_n) be a sequence of real numbers, and let $n_1 < n_2 < n_3 < n_4 < \dots$ be an increasing sequence of natural numbers. Then the sequence $(x_{n_1}, x_{n_2}, x_{n_3}, \dots)$ is called a subsequence of (x_n) and is denoted by (x_{n_k}), where $k \in \mathbb{N}$ indexes the subsequence.

Theorem - Subsequences of a convergent sequence converge to the same limit as the original sequence.

proof - Let $(x_n) \to x$ and (x_{n_k}) be a subsequence. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ whenever $n \geq N$.

Since $n_1 < n_2 < n_3 < n_4 < \dots$ is an increasing sequence of natural numbers, we have that $n_k \geq k$ for all k (can show via induction)

So if $k \geq N$ then $n_k \geq k \geq N$
and thus $|x_{n_k} - x| < \varepsilon$

(so the same N worked). $\Rightarrow (x_{n_k}) \to x$

$\square$

$\boxed{\text{ex } 2.3.5}$ Let (x_n) & (y_n) be given, and define (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, \ldots)$. Prove that (z_n) is convergent iff (x_n) & (y_n) are both convergent with $\lim x_n = \lim y_n$

($\Rightarrow$) Let (z_n) converge to some z

Note $z_{2k-1} = x_k$ and $z_{2k} = y_k$

So, (x_n) & (y_n) are subsequences of (z_n)

& hence by previous theorem, $\lim x_n = \lim y_n = z$

($\Leftarrow$) Let $\lim x_n = \lim y_n = z$

So given any $\varepsilon > 0$, $\exists\, N_1, N_2 \in \mathbb{N}$ such that

$$|x_k - z| < \varepsilon \quad \text{for} \quad k \geq N_1$$

$$\& \quad |y_k - z| < \varepsilon \quad \text{for} \quad k \geq N_2$$

Define $N := 2 \max\{N_1, N_2\}$

- if $n = 2k-1 \geq N$ is odd, then $k \geq \frac{N+1}{2} \geq N_1$

  So $|z_n - z| = |x_k - z| < \varepsilon$

- if $n = 2k \geq N$ is even, then $k \geq \frac{N}{2} \geq N_2$

  So $|z_n - z| = |y_k - z| < \varepsilon$

Either way, we have that $n \geq N$ gives $|z_n - z| < \varepsilon$

$\Rightarrow (z_n)$ is convergent

$\square$

Algebraic Limit Theorem - Let (x_n) & (y_n) be two sequences such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Then:

1) for any $c \in \mathbb{R}$, $\lim_{n\to\infty} (c x_n) = cx$
2) $\lim_{n\to\infty} (x_n + y_n) = x + y$
3) $\lim_{n\to\infty} (x_n y_n) = xy$
4) suppose that $y \neq 0$ and that $y_n \neq 0 \ \forall n$, Then $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y}$

*Warning: Need both $(x_n) \to x$ and $(y_n) \to y$ *

Ex 2.3.9 a) Let (a_n) be a bounded (not necessarily convergent) sequence, and assume $\lim b_n = 0$. Show that $\lim(a_n b_n) = 0$

proof - (a_n) is bounded so $\exists M > 0$ such that $|a_n| \leq M \;\; \forall n$

Let $\varepsilon > 0$. Since $\lim b_n = 0$, by def^n $\exists N \in \mathbb{N}$ such that $|b_n - 0| = |b_n| < \frac{\varepsilon}{M}$ whenever $n \geq N$.

Thus for $n \geq N$: $|a_n b_n - 0| = |a_n b_n| = |a_n|\,|b_n|$

$$< M \frac{\varepsilon}{M}$$

$$= \varepsilon$$

$$\Rightarrow \lim(a_n b_n) = 0$$

$\square$

Remark - We cannot use the Algebraic Limit Theorem since (a_n) does not necessarily converge.

Ex 2.3.9 b) Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some nonzero limit b?

No! Let $a_n = (-1)^{n+1}$ & $b_n = 1$

- (a_n) is bounded, doesn't converge
- $\lim b_n = 1$

but $(a_n b_n) = (-1)^{n+1}$, doesn't converge.

$\boxed{\text{Ex 2.3.13 (Iterated Limits)}}$ Given a doubly indexed array a_{mn} where $m,n \in \mathbb{N}$, what should $\lim_{m,n\to\infty} a_{mn}$ represent?

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a) Let $a_{mn} = \frac{m}{m+n}$ and compute the iterated limits
$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) \quad \text{and} \quad \lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right)$$

Well

$$\lim_{n\to\infty}\left[\lim_{m\to\infty} a_{mn}\right] = \lim_{n\to\infty}\left[\lim_{m\to\infty} \frac{m}{m+n}\right]$$

$$= \lim_{n\to\infty}\left[\lim_{m\to\infty} \frac{1}{1+\frac{n}{m}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{1+\lim_{m\to\infty}\left(\frac{n}{m}\right)}\right]$$

(Algebraic Limit Theorem. Check!! Recall we proved $\lim\left(\frac{1}{n}\right)=0$ in Lecture)

$$= \lim_{n\to\infty}\left[\frac{1}{1+0}\right]$$

$$= 1$$

$$\lim_{m\to\infty}\left[\lim_{n\to\infty} a_{mn}\right] = \lim_{m\to\infty}\left[\lim_{n\to\infty} \frac{m}{m+n}\right]$$

$$= \lim_{m\to\infty}(0)$$

$$= 0$$

(* easy to check that $\lim_{n\to\infty} \frac{m}{m+n} = 0$ *)

$\boxed{\text{Ex 2.3.13 b)}}$ We define $\lim_{m,n\to\infty} a_{mn} = a$ to mean that $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. if both $m, n \geq N$, then $|a_{mn} - a| < \varepsilon$.

Let $a_{mn} = \frac{1}{m+n}$. Does $\lim_{m,n\to\infty} a_{mn}$ exist in this case? Do the two iterated limits exist? How do these 3 values compare?

☐ Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ s.t. $N > \frac{1}{2\varepsilon}$ (exists by Archimedean Property). Then for $m, n \geq N$ we have that:

$$\left| \frac{1}{m+n} - 0 \right| = \frac{1}{m+n} \leq \frac{1}{N+N} = \frac{1}{2N} < \varepsilon$$

$\Rightarrow \lim_{m,n\to\infty} a_{mn}$ exists and is 0

☐ $$\lim_{n\to\infty}\left[\lim_{m\to\infty} a_{mn}\right] = \lim_{n\to\infty}\left[\lim_{m\to\infty} \frac{1}{m+n}\right]$$

$$= \lim_{n\to\infty}[0] \quad \text{(check!)}$$

$$= 0$$

☐ $$\lim_{m\to\infty}\left[\lim_{n\to\infty} a_{mn}\right] = \lim_{m\to\infty}\left[\lim_{n\to\infty} \frac{1}{m+n}\right]$$

$$= \lim_{m\to\infty}[0] \quad \text{(similar to above)}$$

$$= 0$$

$\therefore$ all three values are 0

$\boxed{2.3.13\ c)}$ Produce an example where $\lim_{m,n\to\infty} a_{mn}$ exists, but where neither iterated limits can be computed.

Take $a_{mn} = \dfrac{(-1)^n}{m} + \dfrac{(-1)^m}{n}$

Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ s.t. $N > \frac{2}{\varepsilon}$ (Archimedean Property)

Then for $n, m \geq N$ we have that:

$$\left| a_{mn} - 0 \right| = \left| a_{mn} \right| = \left| \frac{(-1)^n}{m} + \frac{(-1)^m}{n} \right|$$

$$\leq \left| \frac{(-1)^n}{m} \right| + \left| \frac{(-1)^m}{n} \right| \quad \text{(triangle inequality)}$$

$$\leq \frac{1}{m} + \frac{1}{n}$$

$$\leq \frac{2}{N}$$

$$< \varepsilon$$

$\Rightarrow \lim_{m,n\to\infty} a_{mn}$ exists and is 0

(BUT: neither iterated limit can be computed, as a consequence of $\lim_{n\to\infty} \frac{(-1)^n}{m}$ & $\lim_{m\to\infty} \frac{(-1)^m}{n}$ do not exist.)

$\boxed{\text{Ex. 2.3.4}}$ Let $(a_n) \to 0$, & use the Algebraic Limit Theorem to compute each of the following limits (assuming the fractions are always defined):

a) $\lim_{n\to\infty} \frac{1+2a_n}{1+3a_n-4a_n^2} = \frac{\lim_{n\to\infty} 1 + 2\lim_{n\to\infty} a_n}{\lim_{n\to\infty} 1 + 3\lim_{n\to\infty} a_n - 4(\lim_{n\to\infty} a_n)^2}$

$$= \frac{1+2[0]}{1+3[0]-4[0]^2}$$

$$= 1$$

b) $\lim \left(\frac{(a_n+2)^2-4}{a_n}\right)$

First, $\frac{(a_n+2)^2-4}{a_n} = \frac{a_n^2+4a_n+4-4}{a_n} = \frac{a_n^2+4a_n}{a_n} = a_n+4$

So $\lim_{n\to\infty} \frac{(a_n+2)^2-4}{a_n} = \lim_{n\to\infty}(a_n+4) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} 4$

$$= 0+4 = 4$$

c) $\lim \left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right)$

First, $\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5} = \frac{2+3a_n}{1+5a_n}$

So $\lim_{n\to\infty} \frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5} = \lim_{n\to\infty} \frac{2+3a_n}{1+5a_n}$

$$= \frac{\lim_{n\to\infty} 2 + 3\lim_{n\to\infty} a_n}{\lim_{n\to\infty} 1 + 5\lim_{n\to\infty} a_n} = \frac{2+3[0]}{1+5[0]} = 2$$