APM346 Tutorial 2

Partial Differential Equations

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In this tutorial, we look at the *Heat Equation*

$$\begin{cases} u_t = ku_{xx} \quad (0 < x < l,\ t > 0) \\ u(x,0) = \phi(x) \end{cases} \tag{1}$$

Example 1

Solve the heat equation (1) on $0 < x < l$ with boundary conditions $u_x(0,t) = u_x(l,t) = 0$ and initial condition $\phi(x) = 4 + 2\cos\left(\frac{3\pi x}{l}\right)$.

From the class notes ([1]), via separation of variables we have the following:

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right) e^{-(n\pi/l)^2 kt} \tag{2}$$

where the constants are given by $A_0 = \frac{1}{l}\int_0^l \phi(x)dx$ and $A_m = \frac{2}{l}\int_0^l \phi(x)\cos\left(\frac{m\pi x}{l}\right)dx,\ m \neq 0$. Here, we have that

$$A_0 = \frac{1}{l}\int_0^l \left(4 + 2\cos\left(\frac{3\pi x}{l}\right)\right) dx = \frac{1}{l}\left[4x + 2\frac{l}{3\pi}\sin\left(\frac{3\pi x}{l}\right)\right]_0^l = 4$$

Now, for $n \neq 0$ we have:

$$\begin{aligned} A_n &= \frac{2}{l}\int_0^l \left[4 + 2\cos\left(\frac{3\pi x}{l}\right)\right]\cos\left(\frac{n\pi x}{l}\right)dx \\ &= \frac{8}{l}\int_0^l \cos\left(\frac{n\pi x}{l}\right)dx + \frac{4}{l}\int_0^l \cos\left(\frac{3\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx \\ &:= \frac{8}{l}\mathcal{I} + \frac{4}{l}\mathcal{J} \end{aligned}$$

But, the integral $\mathcal{I} = \int_0^l \cos\left(\frac{n\pi x}{l}\right)dx$ is 0, and by orthogonality relations we know that the second integral is $\mathcal{J} = \int_0^l \cos\left(\frac{3\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{l}{2}$ when $n = 3$ (and 0 otherwise). Hence, we conclude that:

$$A_n = \begin{cases} 4, & n = 0 \\ 0, & n \neq 3 \\ 2, & n = 3 \end{cases}$$

In conclusion, the solution is

$$u(x,t) = 4 + 2\cos\left(\frac{3\pi x}{l}\right)e^{-(3\pi/l)^2 kt}$$

Example 2

Solve the heat equation (1) on $0 < x < l$ with boundary conditions $u(0,t) = u(l,t) = 0$ and initial condition $\phi(x) = 1$.

In the class notes ([1]), for Question 3 you are asked to find the solution to the heat equation with these (homogeneous Dirichlet) boundary conditions ($u(0,t) = u(l,t) = 0$). Please do this problem and review the HW2 Solutions posted on Quercus for the full derivation via separation of variables. There you will find that:

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) e^{-(n\pi/l)^2 kt} \tag{3}$$

where the constants are given by $A_m = \frac{2}{l}\int_0^l \phi(x)\sin\left(\frac{n\pi x}{l}\right)dx$. Here, we have that

$$A_n = \frac{2}{l}\int_0^l \sin\left(\frac{n\pi x}{l}\right)dx = \frac{2}{l}\left[\frac{-l}{n\pi}\cos\left(\frac{n\pi x}{l}\right)\right]_0^l = \frac{-2}{n\pi}(\cos(n\pi) - 1)$$

Noting that $\cos(n\pi)$ is 1 for even n and -1 for odd n, we have that

$$A_n = \begin{cases} 0, & n \text{ even} \\ \frac{4}{n\pi}, & n \text{ odd} \end{cases}$$

In conclusion, the solution is

$$u(x,t) = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{1}{n}\sin\left(\frac{n\pi x}{l}\right)e^{-(n\pi/l)^2 kt}$$

Example 3

Solve the heat equation (1) on $0 < x < l$ with boundary conditions $u_x(0,t) = u_x(l,t) = 0$ and initial condition $\phi(x) = x$.

Remark. Before we do this problem, I just want to note that we write the Fourier cosine series of $\phi(x)$ as $\phi(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right)$. Other references (for example [2]) may use different conventions such as $\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right)$. In this case, their A_0 will have a factor of 2; that is $A_0 = \frac{2}{l}\int_0^l \phi(x)dx$. Recalling Example 1, this then allows us to write the coefficients simply as $A_m = \frac{2}{l}\int_0^l \phi(x)\cos\left(\frac{m\pi x}{l}\right)dx$ without splitting off the $m = 0$ case (note this checks out since $\cos(0) = 1$). Naturally, either way gives the same answer and it is purely convention. In order to avoid confusion, I will choose to follow the convention of the course instructor.

In this problem, as $\phi(x) = x$ we have that:

$$A_0 = \frac{1}{l}\int_0^l x\,dx = \frac{1}{l}\left(\frac{l^2}{2}\right) = \frac{l}{2}$$

Now, for $n \neq 0$ we have:

$$\begin{aligned}
A_n &= \frac{2}{l}\int_0^l x\cos\Big(\frac{n\pi x}{l}\Big)dx \\
&= \frac{2}{l}\frac{l}{n\pi}x\sin\Big(\frac{n\pi x}{l}\Big)|_0^l - \frac{2}{l}\int_0^l \frac{l}{n\pi}\sin\Big(\frac{n\pi x}{l}\Big)dx \text{ (IBP)} \\
&= \frac{2l}{n\pi}\sin(n\pi) + \frac{2}{l}\frac{l}{n\pi}\frac{l}{n\pi}\cos\Big(\frac{n\pi x}{l}\Big)|_0^l \\
&= \frac{2l}{n\pi}\sin(n\pi) + \frac{2l}{m^2\pi^2}(\cos(m\pi) - 1) \\
&= \frac{2l}{n^2\pi^2}(\cos(n\pi) - 1) \quad \text{(first term is 0)}
\end{aligned}$$

Recalling again that $\cos(n\pi)$ is 1 for even n and -1 for odd n, we have that

$$A_n = \begin{cases} 0, & n \text{ even} \\ \frac{-4l}{n^2\pi^2}, & n \text{ odd} \end{cases}$$

In conclusion, the solution is

$$u(x,t) = \frac{l}{2} + \frac{-4l}{\pi^2}\sum_{n \text{ odd}} \frac{1}{n^2}\cos\Big(\frac{n\pi x}{l}\Big)e^{-(n\pi/l)^2kt}$$

Example 4

Let $u(x,t)$ be the solution of the the heat equation (1) on $0 < x < l$ with boundary conditions $u_x(0,t) = u_x(l,t) = 0$ and initial condition $\phi(x) = 8 + \cos(\frac{3\pi x}{l})$. Find $\lim_{t\to\infty} u(x,t)$.

Here, we have that

$$A_0 = \frac{1}{l}\int_0^l \left(8 + \cos\Big(\frac{3\pi x}{l}\Big)\right)dx = \frac{1}{l}\left[8x + \frac{l}{3\pi}\sin\Big(\frac{3\pi x}{l}\Big)\right]_0^l = 8$$

Now, for $n \neq 0$ we have:

$$\begin{aligned}
A_n &= \frac{2}{l}\int_0^l \left[8 + \cos\Big(\frac{3\pi x}{l}\Big)\right]\cos\Big(\frac{n\pi x}{l}\Big)dx \\
&= \frac{16}{l}\int_0^l \cos\Big(\frac{n\pi x}{l}\Big)dx + \frac{2}{l}\int_0^l \cos\Big(\frac{3\pi x}{l}\Big)\cos\Big(\frac{n\pi x}{l}\Big)dx \\
&:= \frac{16}{l}\mathcal{I} + \frac{2}{l}\mathcal{J}
\end{aligned}$$

But, the integral $\mathcal{I} = \int_0^l \cos(\frac{n\pi x}{l})dx$ is 0, and by orthogonality relations we know that the second integral is $\mathcal{J} = \int_0^l \cos(\frac{3\pi x}{l})\cos(\frac{n\pi x}{l})dx = \frac{l}{2}$ when $n = 3$ (and 0 otherwise). Hence, we conclude that:

$$A_n = \begin{cases} 8, & n = 0 \\ 0, & n \neq 3 \\ 1, & n = 3 \end{cases}$$

So, the solution is

$$u(x,t) = 8 + \cos\left(\frac{3\pi x}{l}\right)e^{-(3\pi/l)^2 kt}$$

and due to the decaying exponential in time, we have that $\lim_{t\to\infty} u(x,t) = 8 = A_0$.

Example 5 - Strauss 4.1.3

A quantum-mechanical particle on the line with an infinite potential outside the interval $(0,l)$ ("particle in a box") is given by Schrödinger's equation $u_t = iu_{xx}$ on $(0,l)$ with Dirichlet conditions at the ends (i.e. $u(0,t) = u(l,t) = 0$). Separate the variables to find its representation as a series.

I am not actually going to separate variables and derive the solution here (indeed it is done **exactly** like Question 3 in the class notes but with $k = i$. That is (recalling (3)):

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)e^{-(n\pi/l)^2 it}$$

The reason I introduce this problem is to note that here k is no longer a real constant. Thus, the exponential in the solution is now of the form e^{ix} (complex exponential) which we can write in terms of sines and cosines (waves!) by Euler's formula. The next topic covered in this course is Schrödinger's equation, and so I thought this problem is a neat connection to that (indeed we will see a similar solution).

Bibliography

[1] Xiao Jie, Instructor's course notes (Quercus)

[2] W. Strauss, *Partial Differential Equations: An Introduction*, 2nd edition, Wiley