MAT244 Tutorial 2

Introduction to Ordinary Differential Equations

Written By

David Knapik

University of Toronto
david.knapik@mail.utoronto.ca

September 28, 2022

2.1.10

> Find the solution of the initial value problem:
>
> $$y' + 2y = te^{-2t}, \quad y(1) = 0$$

The differential equation is in the standard form $\frac{dy}{dt} + p(t)y = g(t)$ where $p(t) = 2$ and $g(t) = te^{-2t}$. The integrating factor is $\mu(t) = \exp\left(\int p(t)dt\right) = \exp(2t)$ and thus the general solution is

$$\begin{aligned}
y &= \frac{1}{\mu(t)}\left(\int \mu(s)g(s)ds + c\right) \\
&= \exp(-2t)\left(\int e^{2s}se^{-2s}ds + c\right) \\
&= \exp(-2t)\left(\int sds + c\right) \\
&= \exp(-2t)(t^2/2 + c)
\end{aligned}$$

Now use the initial condition, so we have that $0 = e^{-2}(1/2 + c)$ and conclude that $c = \frac{-1}{2}$. So the solution of the IVP is

$$y(t) = \exp(-2t)(t^2/2 - 1/2)$$

Lets quickly do a sanity check. We first check the initial condition: $y(1) = \exp(-2)(1/2 - 1/2) = 0$. Now,

$$\begin{aligned}
y' + 2y &= e^{-2t}(t - t^2 + 1) + 2\left(\exp(-2t)(t^2/2 - 1/2)\right) \\
&= te^{-2t} - t^2e^{-2t} + e^{-2t} + t^2e^{-2t} - e^{-2t} \\
&= te^{-2t}
\end{aligned}$$

as desired.

2.1.18

> Consider the initial value problem
>
> $$y' + \frac{2}{3}y = 1 - \frac{1}{2}t, \quad y(0) = y_0$$
>
> Find the value of y_0 for which the solution touches, but does not cross, the t-axis.

The differential equation is in the standard form $\frac{dy}{dt} + p(t)y = g(t)$ where $p(t) = \frac{2}{3}$ and $g(t) = 1 - \frac{1}{2}t$. The integrating factor is $\mu(t) = \exp\left(\int p(t)dt\right) = \exp\left(\frac{2}{3}t\right)$ and thus the general solution is

$$\begin{aligned} y &= \frac{1}{\mu(t)}\left(\int \mu(s)g(s)ds + c\right) \\ &= \exp\left(-\frac{2}{3}t\right)\left(\int e^{\frac{2}{3}s}(1-\frac{1}{2}s)ds + c\right) \\ &= \exp\left(-\frac{2}{3}t\right)\left(\frac{-3}{8}e^{\frac{2t}{3}}(2t-7) + c\right) \\ &= \frac{(21-6t)}{8} + ce^{\frac{-2t}{3}} \end{aligned}$$

Since we have initial condition $y(0) = y_0$, then $y_0 = \frac{21}{8} + c$ and we conclude that

$$y = \frac{(21-6t)}{8} + \left(y_0 - \frac{21}{8}\right)e^{\frac{-2t}{3}}$$

Now, we would like the solution to touch, **but not cross**, the t-axis. Hence we need $y'(t_0) = y(t_0) = 0$ for some t_0. Taking the derivative, we have that

$$y'(t_0) = \frac{-3}{4} + \left(\frac{7}{4} - \frac{2y_0}{3}\right)e^{\frac{-2t_0}{3}}$$

Setting it equal to 0, we then have that $(\frac{7}{4} - \frac{2y_0}{3})e^{\frac{-2t_0}{3}} = 3/4$. So, $\frac{-2}{3}(y_0 - \frac{21}{8})e^{\frac{-2t_0}{3}} = 3/4$ and then we conclude that

$$\left(y_0 - \frac{21}{8}\right)e^{\frac{-2t_0}{3}} = -\frac{9}{8}$$

Plug this into $y(t_0) = 0$, and we arrive at:

$$\frac{(21-6t_0)}{8} - \frac{9}{8} = 0$$

and hence $t_0 = 2$. Use $t_0 = 2$ in $y'(t_0) = 0$, and we have that

$$\frac{-3}{4} + \left(\frac{7}{4} - \frac{2y_0}{3}\right)e^{\frac{-4}{3}} = 0 \tag{1}$$

We can now use some rearranging/algebra to solve the above equation (1) for y_0. One should arrive at our solution:

$$y_0 = \frac{3}{8}\left(7 - 3e^{\frac{4}{3}}\right)$$

2.2.1

Solve the given differential equation:

$$y' = \frac{x^2}{y}$$

The equation is separable, so write it as $ydy = x^2dx$. Integrating the left hand side w.r.t. y

and the right-hand side w.r.t. x yields the implicit solution

$$\frac{y^2}{2} = \frac{x^3}{3} + c$$

Since we are not given an intial condition, we can stop here.

2.2.3

Solve the given differential equation:

$$y' = \cos^2(x)\cos^2(2y)$$

The equation is separable, so write it as $\sec^2(2y)dy = \cos^2(x)dx$. Integrating the left hand side w.r.t. y and the right-hand side w.r.t. x yields the implicit solution

$$\frac{1}{2}\tan(2y) = \frac{1}{2}(x + \sin x \cos x) + c$$

or $\tan(2y) = x + \sin x \cos x + C$. Since we are not given an initial condition, we can stop here.

2.2.5

Solve the given differential equation:

$$\frac{dy}{dx} = \frac{x - e^{-x}}{y + e^y}$$

This equation is separable, so write it as $(y + e^y)dy = (x - e^{-x})dx$. Integrating the left hand side w.r.t. y and the right-hand side w.r.t. x yields the implicit solution

$$\frac{y^2}{2} + e^y = \frac{x^2}{2} + e^{-x} + c$$

Since we are not given an initial condition, we can stop here.

2.2.19

Solve the initial value problem

$$y' = 2y^2 + xy^2, \quad y(0) = 1$$

and determine where the solution attains its minimum value.

The equation $y' = y^2(2 + x)$ is separable, write it as $(\frac{1}{y^2})dy = (2 + x)dx$. Integrating the left hand side w.r.t. y and the right-hand side w.r.t. x yields the implicit solution

$$\frac{-1}{y} = 2x + \frac{x^2}{2} + c$$

Now, use the initial condition: $-1/1 = 0 + 0 + c$, so that $c = -1$. We thus have our solution:

$$y = \frac{1}{1 - 2x - \frac{x^2}{2}} \tag{2}$$

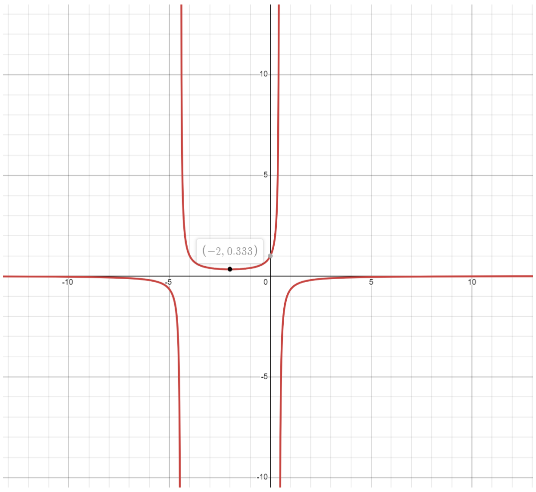

Figure 1: Plot of (2) using Desmos.

Referring back to the differential equation, we have that $y' = 0$ when $x = -2$ and corresponding value $y(-2) = 1/(1 + 4 - 2) = \frac{1}{3}$. The solution attains its minimum value at $x = -2$, as seen in the above figure.

2.3.6

A young person with no initial capital invests k dollars per year at an annual rate of return r. Assume that investments are made continuously and that the return is compounded continuously.

Part A - Determine the sum $S(t)$ accumulated at any time t.

We have two contributions to the change in the sum $S(t)$:

- continuous investments k
- continuous compounding rS

i.e. $\frac{dS}{dt} = rS + k$. Rewrite this in the standard form $\frac{dS}{dt} + p(t)S = g(t)$ where $p(t) = -r$ and $g(t) = k$. The integrating factor is $\mu(t) = \exp\big(\int p(t)dt\big) = \exp(-rt)$ and thus the general solution is

$$\begin{aligned} S &= \frac{1}{\mu(t)}\left(\int \mu(s)g(s)ds + c\right) \\ &= e^{rt}\left(\int e^{-rs}kds + c\right) \\ &= e^{rt}\left(-\frac{k}{r}e^{-rt} + c\right) \\ &= ce^{rt} - \frac{k}{r} \end{aligned}$$

Now we use the initial condition $S(0) = 0$, so $0 = c - \frac{k}{r}$ and hence $c = \frac{k}{r}$. In conclusion,

$$S(t) = \frac{k}{r}(e^{rt} - 1)$$

> Part B - If $r = 7.5\%$, determine k so that \$1 million will be available for retirement in 40 years.

Plug in the relevant quantities, using our solution from part A:

$$1000000 = \frac{k}{0.075}\left(e^{(0.075)(40)} - 1\right)$$

Solving for k, we get $k \approx \$3929.68$

> Part C - If $k = \$2000/\text{year}$, determine the return rate r that must be obtained to have \$1 million available in 40 years.

Similarly to Part B, we plug in the relevant quantities using our solution for $S(t)$. Here we know k and would like to determine r. We have that

$$1000000 = \frac{2000}{r}(e^{(40)r} - 1)$$

Divide by 2000 and multiply by r (implicitly assuming $r \neq 0$), we have that $500r = e^{40r} - 1$. Actually solving this equation for r is rather difficult. One may consider solving it graphically, and shall find that $r \approx 0.0977$ or $r \approx 9.77\%$.