APM346 Tutorial 10

Partial Differential Equations

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1 Inhomogeneous Heat Equation

Recall the inhomogeneous heat equation over an infinite domain:

$$\begin{cases} u_t = cu_{xx} + f(x,t) \\ u(x,0) = \phi(x) \end{cases} \tag{1}$$

for $-\infty < x < \infty$ and $c > 0$.

Define the solution operator, S^t, as the solution to the homogeneous equation:

$$\left[\partial_t - c\frac{\partial^2}{\partial x^2}\right]\left[S^t \cdot (u_0(x))\right] = 0$$

$$u(x,t) = S^t[u_0(x)], \quad u(x,0) = u_0(x)$$

Verify that the solution to the inhomogeneous equation is the following:

$$u(x,t) = S^t(\phi(x)) + \int_{s=0}^{s=t} S^{t-s} \cdot (f(x,s))\mathrm{d}s$$

By superposition, we can separate the term of $u(x,t)$ into $u(x,t) = u_1(x,t) + u_2(x,t)$.

Let $u_1(x,t) = S^t(\phi(x))$ and let $u_2(x,t) = I(x,t) = \int_{s=0}^{s=t} S^{t-s} \cdot (f(x,s))\mathrm{d}s$

Consider u_2,

$$\frac{\partial I}{\partial t} = \frac{\partial}{\partial t}\int_{s=0}^{s=t} S^{t-s} \cdot (f(x,s))\mathrm{d}s$$

Now, let $v = t$ be the upper limit of integration and $w = t$ inside the integral

$$\frac{\partial I}{\partial t} = \frac{\partial I}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial I}{\partial w}\frac{\partial w}{\partial t} = I_v + I_w \tag{2}$$

$$= \frac{\partial}{\partial v}\left[\int_{s=0}^{s=v} S^{w-s} \cdot (f(x,s))\mathrm{d}s\right] + \frac{\partial}{\partial w}\left[\int_{s=0}^{s=v} S^{w-s} \cdot (f(x,s))\mathrm{d}s\right] \tag{3}$$

Using the Fundamental Theorem of Calculus

$$= S^{w-v} \cdot (f(x,v)) + \frac{\partial}{\partial w}\left[\int_{s=0}^{s=v} S^{w-s} \cdot (f(x,s))\mathrm{d}s\right] \tag{4}$$

Using Fubini's Theorem

$$= S^{w-v} \cdot (f(x,v)) + \int_{s=0}^{s=v} \frac{\partial}{\partial w}\left[S^{w-s} \cdot (f(x,s))\right]\mathrm{d}s \tag{5}$$

Substituting t back into the expression

$$= S^{t-t} \cdot (f(x,t)) + \int_{s=0}^{s=t} \frac{\partial}{\partial t}\left[S^{t-s} \cdot (f(x,s))\right]\mathrm{d}s \tag{6}$$

Recall that S^0 does not evolve the function it acts on, so $S^0 \cdot (f(x,t)) = f(x,t)$.

Evaluating the differential operator on u_2 we show that it satisfies the following equation:

$$\begin{cases} (u_2)_t = c(u_2)_{xx} + f(x,t) \\ u_2(x,0) = 0 \end{cases} \tag{7}$$

$$\left[\partial_t - c\frac{\partial^2}{\partial x^2}\right] I = f(x,t) + \int_{s=0}^{s=t} \frac{\partial}{\partial t}\left[S^{t-s} \cdot (f(x,s))\right]\mathrm{d}s - c\frac{\partial^2}{\partial x^2}\left[\int_{s=0}^{s=t} S^{t-s} \cdot (f(x,s))\mathrm{d}s\right] \tag{8}$$

$$= f(x,t) + \int_{s=0}^{s=t} \frac{\partial}{\partial t}\left[S^{t-s} \cdot (f(x,s))\right]\mathrm{d}s - c\int_{s=0}^{s=t} \frac{\partial^2}{\partial x^2}\left[S^{t-s} \cdot (f(x,s))\right]\mathrm{d}s \tag{9}$$

$$= f(x,t) + \int_{s=0}^{s=t} \left[\frac{\partial}{\partial t} - c\frac{\partial^2}{\partial x^2}\right]\left[S^{t-s} \cdot (f(x,s))\right]\mathrm{d}s \tag{10}$$

Since the solution operator solves the homogeneous equation

$$= f(x,t) + \int_{s=0}^{s=t} 0 \mathrm{d}s \quad (11)$$

$$= f(x,t) \quad (12)$$

Plugging in initial conditions

$$u_2(x,0) = I(x,0) = \int_{s=0}^{s=0} S^{t-s} \cdot (f(x,s))\mathrm{d}s = 0$$

We now verify that $u(x,t) = u_1(x,t) + u_2(x,t)$ satisfies the original PDE

$$\left[\frac{\partial}{\partial t} - c\frac{\partial^2}{\partial x^2}\right] u(x,t) = \left[\frac{\partial}{\partial t} - c\frac{\partial^2}{\partial x^2}\right](u_1(x,t) + u_2(x,t)) \quad (13)$$

$$= 0 + f(x,t) \quad (14)$$

$$= f(x,t) \quad (15)$$

and the initial condition

$$u(x,0) = u_1(x,0) + u_2(x,0) \quad (16)$$

$$= \phi(x) + 0 \quad (17)$$

$$= \phi(x) \quad (18)$$

Therefore,

$$u(x,t) = u_1(x,t) + u_2(x,t)$$

solves the inhomogeneous equation

$$\begin{cases} u_t = cu_{xx} + f(x,t) \\ u(x,0) = \phi(x) \end{cases} \quad (19)$$

2 Transport Equation

2.1 Homogeneous Case

Solve the homogeneous transport equation:

$$\begin{cases} u_t + cu_x = 0 \\ u(x,0) = u_0(x) \end{cases}$$

where $c > 0$ is given

Recall this PDE can be solved using a change of variable: Let $s = cx + t$ and $w = x - ct$.

$$u_x = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial x} = cu_s + u_w \quad (20)$$

$$u_t = \frac{\partial u}{\partial s}\frac{\partial s}{\partial t} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial t} = u_s - cu_w \quad (21)$$

Substituting, we get

$$u_s - cu_w + c^2u_s + cu_w = 0 \quad (22)$$

$$(1 + c^2)u_s = 0 \quad (23)$$

$$u(s,w) = f(w) \implies u(x,t) = f(x - ct) \quad (24)$$

Plugging in the initial condition we further get

$$u(x,0) = f(x - c \cdot 0) = u_0(x) \quad (25)$$

$$u(x,t) = u_0(x - ct) \quad (26)$$

$$(27)$$

2.2 Inhomogeneous Case

Find the solution operator and solve the transport equation in the inhomogeneous case:

$$\begin{cases} u_t + cu_x = f(x,t) \\ u(x,0) = \phi(x) \end{cases}$$

where $c > 0$ is again given

Recall that the solution operator is the solution to the homogeneous equation acting on the initial condition. This is computed above:

$$S^t \cdot (u_0(x))) = u_0(x - ct)$$

We can employ Duhamel's principle to solve the inhomogeneous equation

$$u(x,t) = S^t \cdot (\phi(x)) + \int_{s=0}^{s=t} S^{t-s} \cdot (f(x,s))\mathrm{d}s \tag{28}$$

$$= \phi(x - ct) + \int_{s=0}^{s=t} f(x - c(t-s), s)\mathrm{d}s \tag{29}$$

$$\tag{30}$$