Math 254 Tutorial 9

Honours Analysis 1

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1 Functional Limits

We begin with the $\epsilon - \delta$ definition for functional limits:

Definition 1.0.1. Let $A \subset \mathbb{R}$ and let $f : A \to \mathbb{R}$. Let $c \in \mathbb{R}$ be a cluster point of A. We say that a real number L is a limit of f at c (denoted $\lim_{x\to c} f(x) = L$) if for all $\epsilon > 0$ there exists $\delta > 0$ such that $\forall x \in A$ satisfying $0 < |x - c| < \delta$, one has $|f(x) - L| < \epsilon$.

Now, we introduce some tools which we shall use in examples:

Theorem 1.0.1 (Sequential Criterion for Functional Limits)**.** *Given a function $f : A \to \mathbb{R}$ and a cluster point c of A, the following two statements are equivalent:*

1. $\lim_{x\to c} f(x) = L$
2. *For all sequences (x_n) in A satisfying $x_n \neq c$ and $(x_n) \to c$, it follows that $f(x_n) \to L$.*

Corollary 1.0.1 (Divergence Criterion for Functional Limits)**.** *Let $f : A \to \mathbb{R}$ and c be a cluster point of A. If there exist two sequences (x_n) and (y_n) in A with $x_n \neq c$ and $y_n \neq c$ and*

$$\lim x_n = \lim y_n = c \quad \textit{but} \quad \lim f(x_n) \neq \lim f(y_n)$$

then we can conclude that the functional limit $\lim_{x\to c} f(x)$ does not exist.

Theorem 1.0.2 (Algebraic Limit Theorem for Functional Limits)**.** *Let $A \subset \mathbb{R}$ and let c be a cluster point of A. Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ such that*

$$\lim_{x\to c} f(x) = L \quad \textit{and} \quad \lim_{x\to c} g(x) = M$$

Then the following holds:

1. $\lim_{x\to c} kf(x) = kL$ *for all* $k \in \mathbb{R}$
2. $\lim_{x\to c}(f(x) + g(x)) = L + M$
3. $\lim_{x\to c}(f(x)g(x)) = LM$
4. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ *provided* $M \neq 0$ *and* $g(x) \neq 0$ *for all* $x \in A$

1.1 Example 1

Compute the limit or state that it does not exist: $\lim_{x\to 2}\frac{|x-2|}{x-2}$.

Let $x_n = 2-\frac{1}{n}$ and $y_n = 2+\frac{1}{n}$ (note that $x_n \neq 2$ and $y_n \neq 2$). We have that $\lim(x_n) = 2 = \lim(y_n)$. However:

$$\lim f(x_n) = \lim \frac{\left|\frac{-1}{n}\right|}{\frac{-1}{n}} = \lim \frac{\frac{1}{n}}{\frac{-1}{n}} = -1$$

and

$$\lim f(y_n) = \lim \frac{\left|\frac{1}{n}\right|}{\frac{1}{n}} = \lim \frac{\frac{1}{n}}{\frac{1}{n}} = 1$$

So by the Corollary (Divergence Criterion), we conclude that the functional limit $\lim_{x\to 2}\frac{|x-2|}{x-2}$ does not exist.

1.2 Example 2

Compute the limit or state that it does not exist: $\lim_{x\to\frac{7}{4}}\frac{|x-2|}{x-2}$.

We shall show (via the $\epsilon-\delta$ definition) that the limit is -1. To that end, let $\epsilon > 0$ be arbitrary and take $0<\delta<\frac{1}{4}$.

We note that $0<|x-\frac{7}{4}|<\delta<\frac{1}{4}$ gives $x-\frac{7}{4}<\frac{1}{4}$ (so $x<2$) and $\frac{7}{4}-x<\frac{1}{4}$ (so $x>\frac{3}{2}$). Hence, x satisfying $0<|x-\frac{7}{4}|<\delta$ here means $x\in(\frac{3}{2},2)$.

So, noting that $f(x)=\frac{|x-2|}{x-2}=-1$ for all $x\in(\frac{3}{2},2)$, we have the following: for all x satisfying $0<|x-\frac{7}{4}|<\delta$,

$$|f(x)-(-1)| = |-1+1| = 0 < \epsilon$$

So we have that $\lim_{x\to\frac{7}{4}}\frac{|x-2|}{x-2} = -1$.

1.3 Example 3

Compute the limit or state that it does not exist: $\lim_{x\to 0}(-1)^{\frac{1}{x}}$.

Let $x_n = \frac{1}{2n+1}$ and $y_n = \frac{1}{2n}$ (note that $x_n \neq 0$ and $y_n \neq 0$). We have that $\lim(x_n) = 0 = \lim(y_n)$. However:

$$\lim f(x_n) = \lim(-1)^{2n+1} = -1$$

and

$$\lim f(y_n) = \lim(-1)^{2n} = 1$$

So by the Corollary (Divergence Criterion), we conclude that the functional limit $\lim_{x\to 0}(-1)^{\frac{1}{x}}$ does not exist.

1.4 Example 4

Compute the limit or state that it does not exist: $\lim_{x\to 0} x^{\frac{1}{3}}(-1)^{\frac{1}{x}}$.

We shall show (via the $\epsilon-\delta$ definition) that the limit is 0. To that end, let $\epsilon > 0$ be arbitrary and take $0 < \delta = \epsilon^3$. Then for all x satisfying $0 < |x| < \delta$,

$$|f(x) - 0| = |x^{\frac{1}{3}}(-1)^{\frac{1}{x}}| = |x^{\frac{1}{3}}| < \epsilon$$

So we have that $\lim_{x\to 0} x^{\frac{1}{3}}(-1)^{\frac{1}{x}} = 0$.

1.5 Example 5

Compute the limit or state that it does not exist: $\lim_{x\to 0} x^2$.

We shall show (via the $\epsilon-\delta$ definition) that the limit is 0. To that end, let $\epsilon > 0$ be arbitrary and take $0 < \delta = \min\{\epsilon, 1\}$.

We note that if $0 < |x| < \delta$, then since $\delta \leq 1$ we have that $|x| < 1$. So $|x^2| = |x|^2 < |x|$. And since $\delta \leq \epsilon$, we have $|x| < \epsilon$.

So, for all x satisfying $0 < |x| < \delta$,

$$|f(x) - 0| = |x^2| = |x|^2 < |x| < \epsilon$$

In conclusion, we have that $\lim_{x\to 0} x^2 = 0$.

1.6 Example 6

Compute the limit or state that it does not exist: $\lim_{x\to 0} |x|$.

We shall show (via the $\epsilon-\delta$ definition) that the limit is 0. To that end, let $\epsilon > 0$ be arbitrary and take $\delta = \epsilon$.

So for all x satisfying $0 < |x| < \delta$,

$$|f(x) - 0| = ||x|| = |x| < \epsilon$$

In conclusion, we have that $\lim_{x\to 0} |x| = 0$.

1.7 Example 7

Compute the limit or state that it does not exist: $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$.

Let $x_n = \frac{1}{\pi n}$ and $y_n = \frac{1}{2\pi n + \frac{\pi}{2}}$ (note that $x_n \neq 0$ and $y_n \neq 0$). We have that $\lim(x_n) = 0 = \lim(y_n)$. However:

$$\lim f(x_n) = \lim \sin(\pi n) = \lim 0 = 0$$

and

$$\lim f(y_n) = \lim \sin\left(2\pi n + \frac{\pi}{2}\right) = \lim 1 = 1$$

So by the Corollary (Divergence Criterion), we conclude that the functional limit $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$ does not exist.

2 Squeeze Theorem

Theorem 2.0.1 (Squeeze Theorem). *Let $A \subset \mathbb{R}$ and c be a cluster point of A. Let $f : A \to \mathbb{R}$, $g : A \to \mathbb{R}$, and $h : A \to \mathbb{R}$ be functions such that $f(x) \le g(x) \le h(x)$ for all $x \in A$. If*

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$$

then we have that $\lim_{x\to c} g(x) = L$.

2.1 Example 1

Prove $\lim_{x\to 0} x^{\frac{3}{2}} = 0 \quad (x > 0)$

Proof. Let $g(x) = x^{\frac{3}{2}}$ for $x > 0$. We note that $x < x^{\frac{1}{2}} \le 1$ for $0 < x \le 1$. Hence $x^2 \le x^{\frac{3}{2}} \le x$ for $0 < x \le 1$. But we know that $\lim_{x\to 0} x^2 = \lim_{x\to 0} x = 0$. Hence by the Squeeze Theorem we have that $\lim_{x\to 0} x^{\frac{3}{2}} = 0$.

□

2.2 Example 2

Prove $\lim_{x\to 0}\cos x = 1$

Proof. (Most likely later in the course), we will show that $1 - \frac{1}{2}x^2 \le \cos(x) \le 1$ for all $x \in \mathbb{R}$. But by the Algebraic Limit Theorem for Functional Limits, we know that

$$\lim_{x\to 0}\left(1 - \frac{1}{2}x^2\right) = \lim_{x\to 0} 1 - \frac{1}{2}\lim_{x\to 0} x^2 = 1 - 0 = 1 \tag{2.1}$$

where we have used our previous example that $\lim_{x\to 0} x^2 = 0$. Hence by the Squeeze Theorem we have that $\lim_{x\to 0}\cos x = 1$.

□

2.3 Example 3

Prove $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$

Proof. First note that we cannot use the Algebraic Limit Theorem!! However, by the previous problem, we know that

1. $\frac{-1}{2}x \leq \frac{\cos(x)-1}{x} \leq 0$ for $x > 0$
2. $0 \leq \frac{\cos(x)-1}{x} \leq -\frac{1}{2}x$ for $x < 0$

So let $f(x) := \frac{-x}{2}$ for $x \geq 0$ and $f(x) := 0$ for $x < 0$. Let $h(x) := 0$ for $x \geq 0$ and $h(x) := -\frac{x}{2}$ for $x < 0$. Then we have that:

$$f(x) \leq \frac{\cos(x) - 1}{x} \leq h(x)$$

for $x \neq 0$. But it is easy to see that $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 0$. Hence by the Squeeze Theorem, we have that $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$.

□

2.4 Example 4

Prove $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$

Proof. Since $-1 \leq \sin(z) \leq 1$ for all $z \in \mathbb{R}$, we have that $-|x| \leq x\sin\left(\frac{1}{x}\right) \leq |x|$ for all $x \in \mathbb{R}$, $x \neq 0$. From a previous example we showed that $\lim_{x\to 0} |x| = 0$, and thus we also know by the Algebraic Limit Theorem that $\lim_{x\to 0} -|x| = 0$. Hence by the Squeeze Theorem, we have that $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$.

□

3 Bonus Proof

Theorem 3.0.1. *Let $A \subset \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. If $\lim_{x\to c} f$ exists, and if $|f|$ denotes the function defined for $x \in A$ by $|f|(x) := |f(x)|$, then*

$$\lim_{x\to c} |f| = \left|\lim_{x\to c} f\right|$$

Proof. Since $\lim_{x\to c} f$ exists, lets say $\lim_{x\to c} f(x) = L$. So by definition, for all $\epsilon > 0$, there exists $\delta > 0$ such that $\forall x \in A$ satisfying $0 < |x - c| < \delta$, we have $|f(x) - L| < \epsilon$. Also recall the Reverse Triangle Inequality : $||a| - |b|| \leq |a - b|$ for $a, b \in \mathbb{R}$.

So, let $\epsilon > 0$ be arbitrary. For all x satisfying $0 < |x - c| < \delta$, we have:

$$\begin{aligned} ||f(x)| - |L|| &\leq |f(x) - L| \quad \text{(by the Reverse Triangle Inequality)} \\ &< \epsilon \end{aligned}$$

Hence, $\lim_{x\to c} |f(x)| = |L| = |\lim_{x\to c} f(x)|$.

$\square$

Remark. To prove the Reverse Triangle Inequality: note that $|b| = |a + b - a| \leq |a| + |b - a|$ and hence

$$|b| - |a| \leq |b - a| = |a - b|$$

Similarly, $|a| = |b + a - b| \leq |b| + |a - b|$ and hence

$$|a| - |b| \leq |a - b|$$

In conclusion, we have that $||a| - |b|| \leq |a - b|$.