MAT351 Tutorial 1

Partial Differential Equations

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1 First-order linear equations

1.1 Strauss 1.2.8

Solve $au_x + bu_y + cu = 0$.

We shall proceed by changing variables; to that end let:

$$\zeta = ax + by \quad \text{and} \quad \eta = bx - ay$$

Then by the chain rule

$$u_x = \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = au_\zeta + bu_\eta$$

and

$$u_y = \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = bu_\zeta - au_\eta$$

Substituting back into the PDE we have that $(a^2+b^2)u_\zeta + cu = 0$. The general solution is $u(\zeta,\eta) = f(\eta)\exp\{\frac{-c\zeta}{a^2+b^2}\}$ (indeed $\frac{u_\zeta}{u} = \frac{-c}{a^2+b^2}$, then $\ln u = \frac{-c\zeta}{a^2+b^2} + g(\eta)$, $\implies$ $u = f(\eta)\exp\{\frac{-c\zeta}{a^2+b^2}\}$). In terms of the original variables, we have that

$$u(x,y) = f(bx-ay)e^{\frac{-c(ax+by)}{a^2+b^2}}$$

1.2 Strauss 1.2.9

Solve the equation $u_x + u_y = 1$.

We shall proceed by changing variables; to that end let:

$$\zeta = x + y \quad \text{and} \quad \eta = x - y$$

Then by the chain rule

$$u_x = \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = u_\zeta + u_\eta$$

and

$$u_y = \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = u_\zeta - u_\eta$$

Hence substituting into the PDE we arrive at $2u_\zeta = 1$ or $u_\zeta = \frac{1}{2}$. The general solution is $u(\zeta, \eta) = \frac{1}{2}\zeta + f(\eta)$. In terms of the original variables, we conclude that

$$u(x, y) = \frac{1}{2}(x + y) + f(x - y)$$

Remark. We could also solve the equation using the geometric method/characteristics. As an exercise, please try this.

2 Types of second-order equations

During the tutorial, I briefly mentioned another reference for this course -

- R. Choksi, *Partial Differential Equations: A First Course*, AMS 2022

Although I may be a little biased as I helped with proofreading and figures, I truly think this is a worthwhile textbook to look into. In fact, Strauss's textbook (the textbook for this course) is acknowledged in Prof. Choksi's textbook.

In particular for this tutorial, see section 13.1 for the standard classification.

2.1 Aside - Conic sections

In cartesian coordinates, curves of form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Recall that after excluding degenerate conics (lines) then:

- if $B^2 - 4AC < 0$, the equation represents an ellipse
- if $B^2 - 4AC = 0$, the equation represents a parabola
- if $B^2 - 4AC > 0$, the equation represents a hyperbola

2.2 Classification of (linear) second-order equations

Consider the PDE

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = 0 \tag{2.1}$$

Remark. In Strauss, a factor of 2 is included with b "for convenience". Personally, I don't see the point of this; if we don't include the 2 then we have the familiar "discriminant" $b^2 - 4ac$. At the end of the day this factor of 2 is a matter of preference, but I just wanted to take note of this as it was brought up during the tutorial.

We have the following:

- if $b^2 - 4ac < 0$, PDE (2.1) is called elliptic
- if $b^2 - 4ac = 0$, PDE (2.1) is called parabolic
- if $b^2 - 4ac > 0$, PDE (2.1) is called hyperbolic

The respective canonical examples are:

- Laplace's equation $u_{xx} + u_{yy} = 0$ ($a = 1, b = 0, c = 1$, so elliptic)
- Diffusion equation $u_y - u_{xx} = 0$ ($a = -1, b = 0, c = 0$, so parabolic)
- Wave equation $u_{yy} - u_{xx} = 0$ ($a = -1, b = 0, c = 1$, so hyperbolic)

In fact, one can prove that for any PDE of the form (2.1) (where a, b, c are not all 0), there exists a linear change of the independent variables to new coordinates which reduces the PDE into either the Laplace, diffusion, or wave equation with lower-order terms.

2.3 Strauss 1.6.2

Find the regions in the xy plane where the equation

$$(1 + x)u_{xx} + 2xyu_{xy} - y^2u_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Sketch them.

Here the discriminant is

$$\mathcal{D} = (2xy)^2 - 4(1 + x)(-y^2) = 4x^2y^2 + 4y^2 + 4xy^2 = 4y^2(x^2 + x + 1)$$

Noting that $x^2 + x + 1 > 0$ for all x, we have that the PDE is parabolic along the line $y = 0$, and hyperbolic elsewhere.

2.4 Strauss 1.6.4

What is the type of the equation $u_{xx} - 4u_{xy} + 4u_{yy} = 0$? Show by direct substitution that $u(x, y) = f(y + 2x) + xg(y + 2x)$ is a solution for arbitrary functions f and g.

Here the discriminant is

$$\mathcal{D} = (-4)^2 - 4(1)(4) = 0$$

and hence the PDE is parabolic.

Remark. The second part of the question is straightforward direct substitution. As an exercise, please try it.