MAT351 Tutorial 3

Partial Differential Equations

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1 Diffusion on the whole line

Recall that the IVP for the diffusion equation,

$$\begin{cases} u_t = ku_{xx} & \text{on } \mathbb{R} \times (0, \infty) \\ u(x,0) = g(x) & \text{on } \mathbb{R} \end{cases}$$

has solution formula

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4kt}} g(y)dy = (\Phi(\cdot, t) * g)(x) = \int_{-\infty}^{\infty} \Phi(x-y,t)g(y)dy \tag{1.1}$$

where

$$\Phi(x,t) := \frac{1}{\sqrt{4\pi kt}} e^{\frac{-x^2}{4kt}}, \quad x \in \mathbb{R}, t > 0$$

is the *fundamental solution of the diffusion equation* (also called the *heat kernel*).

Error function

Often, one cannot evaluate the integral (1.1) completely in terms of elementary functions. Sometimes one may encounter integrals of the form $\int_0^x e^{-p^2} dp$. To that end, it is fruitful to recall the error function:

$$\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2} dp$$

Strauss 2.4.1

Solve the diffusion equation with the initial condition

$$\phi(x) = 1 \text{ for } |x| < l \quad \text{and} \quad \phi(x) = 0 \text{ for } |x| > l$$

Write your answer in terms of $\operatorname{erf}(x)$.

Making the following substitution $p = \frac{(x-y)}{\sqrt{4kt}}$, by the solution formula we have that:

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4kt}} \phi(y)dy \\
&= \frac{1}{\sqrt{4\pi kt}} \int_{-l}^{l} e^{\frac{-(x-y)^2}{4kt}} dy \\
&= -\frac{1}{\sqrt{\pi}} \int_{(x+l)/(\sqrt{4kt})}^{(x-l)/(\sqrt{4kt})} e^{-p^2} dp \\
&= \frac{1}{\sqrt{\pi}} \int_{(x-l)/(\sqrt{4kt})}^{(x+l)/(\sqrt{4kt})} e^{-p^2} dp \\
&= \frac{1}{\sqrt{\pi}} \int_{(x-l)/(\sqrt{4kt})}^{0} e^{-p^2} dp + \frac{1}{\sqrt{\pi}} \int_{0}^{(x+l)/(\sqrt{4kt})} e^{-p^2} dp \\
&= -\frac{1}{\sqrt{\pi}} \int_{0}^{(x-l)/(\sqrt{4kt})} e^{-p^2} dp + \frac{1}{\sqrt{\pi}} \int_{0}^{(x+l)/(\sqrt{4kt})} e^{-p^2} dp \\
&= -\frac{1}{2}\text{erf}\left(\frac{x-l}{\sqrt{4kt}}\right) + \frac{1}{2}\text{erf}\left(\frac{x+l}{\sqrt{4kt}}\right)
\end{aligned}$$

Strauss 2.4.6

Compute $\int_0^\infty e^{-x^2} dx$.

You should have seen such an integral before perhaps in a calculus class. The reason I do it here is for the next exercise, which will then be used for Strauss 2.4.3. The standard trick is to let $I = \int_0^\infty e^{-x^2} dx$ then

$$\begin{aligned}
I^2 &= \left(\int_0^\infty e^{-x^2} dx\right)\left(\int_0^\infty e^{-y^2} dy\right) \\
&= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy \\
&= \int_0^{\pi/2} \int_0^\infty e^{-r^2} r dr d\theta \\
&= \int_0^{\frac{\pi}{2}} \left[\frac{-e^{-r^2}}{2} |_0^\infty\right] d\theta \\
&= \int_0^{\pi/2} \frac{1}{2} d\theta \\
&= \frac{\pi}{4}
\end{aligned}$$

so we conclude that $I = \frac{\sqrt{\pi}}{2}$.

Strauss 2.4.7

Using the previous exercise, show that $\int_{-\infty}^{\infty} e^{-p^2} dp = \sqrt{\pi}$.

Since e^{-x^2} is even, via symmetry we simply have that

$$\int_{-\infty}^{\infty} e^{-p^2} dp = 2 \int_{0}^{\infty} e^{-p^2} dp = \sqrt{\pi}$$

Strauss 2.4.3

Solve the diffusion equation if $\phi(x) = e^{3x}$.

By the solution formula we have that:

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4kt}} \phi(y) dy \\
&= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4kt}} e^{3y} dy \\
&= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\left(\frac{-(x-y)^2 + 12kty}{4kt}\right) dy \\
&= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\left(\frac{-(y-x-6kt)^2 + 36k^2t^2 + 12kxt}{4kt}\right) dy \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} e^{9kt+3x} dp \\
&= e^{9kt+3x}
\end{aligned}$$

where we let $p = \frac{y-x-6kt}{\sqrt{4kt}}$ and we have used the result for the integral from the previous exercise.

Strauss 2.4.9

Solve the diffusion equation $u_t = ku_{xx}$ with the initial condition $u(x,0) = x^2$ by the following special method. First show that u_{xxx} satisfies the diffusion equation with zero initial condition. Therefore, by uniqueness, $u_{xxx} \equiv 0$. Integrating this result thrice, obtain $u(x,t) = A(t)x^2 + B(t)x + C(t)$. Finally, it's easy to solve for A, B, C by plugging into the original problem.

Let u be the solution to the diffusion equation. Then $(u_t)_{xxx} = (ku_{xx})_{xxx}$ and hence $(u_{xxx})_t = k(u_{xxx})_{xx}$ so that u_{xxx} satisfies the diffusion equation with initial condition $u_{xxx}(x,0) = \frac{d^3}{dx^3}x^2 = 0$. So by uniqueness we have that $u_{xxx} \equiv 0$. Now, integrate with respect to x three times:

$$\begin{cases} u_{xx} = a(t) \\ u_x = a(t)x + B(t) \\ u = \frac{1}{2}a(t)x^2 + B(t)x + C(t) = A(t)x^2 + B(t)x + C(t) \end{cases}$$

where we let $A(t) := \frac{1}{2}a(t)$.

Now, substituting into $u_t = ku_{xx}$ we have that $A'(t)x^2 + B'(t)x + C'(t) = 2kA(t)$. Hence

$A'(t) = B'(t) = 0$ and $C'(t) = 2kA(t)$. Therefore:

$$\begin{cases} A(t) = A_0 \\ B(t) = B_0 \\ C(t) = 2kA_0 t + C_0 \end{cases}$$

so that

$$u(x,t) = A_0 x^2 + B_0 x + 2kA_0 t + C_0$$

Now, by the initial condition $u(x,0) = x^2$ we have that $x^2 = A_0 x^2 + B_0 x + C_0$. So $A_0 = 1$, $B_0 = 0$, and $C_0 = 0$. We conclude:

$$u(x,t) = x^2 + 2kt$$

Indeed, for a sanity check: $u_t = 2k$ and $ku_{xx} = k(2)$.

Strauss 2.4.18

Solve the heat equation with convection:

$$u_t - ku_{xx} + Vu_x = 0 \quad \text{for } -\infty < x < \infty \quad \text{with } u(x,0) = \phi(x)$$

where V is a constant. Hint: go to a moving frame of reference by substituting $y = x - Vt$.

Let $y = x - Vt$ and $v(y,t) = u(y + Vt, t)$. Then

$$\begin{cases} v_t(y,t) = Vu_x(y+Vt,t) + u_t(y+Vt,t) \\ v_y(y,t) = u_x(y+Vt,t) \\ v_{yy}(y,t) = u_{xx}(y+Vt,t) \end{cases}$$

Hence,

$$v_t(y,t) - kv_{yy}(y,t) = Vu_x(y+Vt,t) + u_t(y+Vt,t) - ku_{xx}(y+Vt,t) = 0$$

and so v solves the diffusion equation with initial data $v(y,0) = u(y,0) = \phi(y)$. Therefore,

$$v(y,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-(y-w)^2}{4kt}} \phi(w)dw$$

And then our solution is:

$$u(x,t) = v(x - Vt, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-(x-Vt-w)^2}{4kt}} \phi(w)dw$$

Bibliography

[1] R. Choksi, *Partial Differential Equations: A First Course*, AMS 2022

[2] W. Strauss, *Partial Differential Equations: An Introduction*, 2nd edition, Wiley