MAT351 Tutorial 2

Partial Differential Equations

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1 The Wave Equation (in one space dimension)

Recall that D'Alembert's formula

$$u(x,t) = \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds$$

is the solution formula for the IVP for 1D wave equation:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & -\infty < x < \infty \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

1.1 The plucked string

For this example, I shall follow [1]. For simplicity, let $c = 1$. Take

$$\phi(x) = \begin{cases} 1 - |x|, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

and initial velocity $\psi \equiv 0$. By D'Alembert's, we know that $u(x,t) = \frac{1}{2}\left(\phi(x+t) + \phi(x-t)\right)$. Hence, for any time t, the solution is the sum of two waves with the same initial shape but half the amplitude. One is moved to the right by t and one to the left by t.

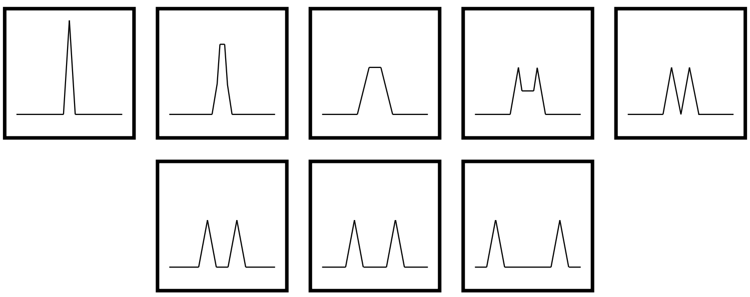

Figure 1.1: Profiles at increasing times, showing the propagation of the plucked string.

1.2 Strauss 2.1.1

Solve $u_{tt} = c^2 u_{xx}$, $u(x,0) = e^x$, $u_t(x,0) = \sin(x)$

By D'Alembert's :

$$\begin{aligned}
u(x,t) &= \frac{1}{2}\left(e^{x+ct}+e^{x-ct}\right)+\frac{1}{2c}\int_{x-ct}^{x+ct}\sin(s)ds \\
&= \frac{1}{2}e^{x}\left(e^{ct}+e^{-ct}\right)-\frac{1}{2c}(\cos(x+ct)-\cos(x-ct)) \\
&= e^{x}\cosh(ct)+\frac{1}{c}\sin(ct)\sin(x)
\end{aligned}$$

1.3 Strauss 2.1.3

The midpoint of a piano string of tension T, density ρ, and length l is hit by a hammer whose head diameter is $2a$. A flea is sitting at a distance $l/4$ from one end (assume that $a < l/4$; otherwise RIP flea) How long does it take for the disturbance to reach the flea?

The speed of the wave is $c = \sqrt{T/\rho}$. The left hand edge of the disturbance begins at $l/2 - a$. So the time to reach the flea (at $l/4$) is

$$t_{flea} = \frac{l/2 - a - l/4}{c} = \sqrt{\rho/T}(l/4 - a)$$

1.4 Strauss 2.1.9

Solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0$, $u(x,0) = x^2$, $u_t(x,0) = e^x$. (Hint: factor the operator as we did for the wave equation).

Factoring the operator we have that $(\partial_x + \partial_t)(\partial_x - 4\partial_t)u = 0$. Introduce the characteristic coordinates

$$\eta = x - t \quad \text{and} \quad \zeta = 4x + t$$

Then by the chain rule, $\partial_x = \partial_\eta + 4\partial_\zeta$ and $\partial_t = -\partial_\eta + \partial_\zeta$. Hence,

$$(\partial_x + \partial_t) = 5\partial_\zeta \quad \text{and} \quad (\partial_x - 4\partial_t) = 5\partial_\eta$$

So our equation takes form $25u_{\zeta\eta} = 0$, or simplified $u_{\zeta\eta} = 0$. This has general solution $u = f(\eta) + g(\zeta)$, or in terms of the original variables we have that:

$$u(x,t) = f(x-t) + g(4x+t) \tag{1.1}$$

Now, initial conditions give that $u(x,0) = f(x)+g(4x) = x^2$ and $u_t(x,0) = -f'(x)+g'(4x) = e^x$. Integrating, we obtain

$$\begin{cases} f(x) + g(4x) = x^2 \\ -f(x) + \frac{1}{4}g(4x) = e^x \end{cases}$$

Now just solve for f, g. In particular, we have that $\frac{5}{4}g(4x) = x^2+e^x$ so that $g(4x) = \frac{4}{5}(x^2+e^x)$ and we conclude $g(x) = \frac{4}{5}(\frac{x^2}{16} + \exp(x/4))$. Then,

$$f(x) = x^2 - \frac{4}{5}(x^2+e^x) = \frac{x^2}{5} - \frac{4}{5}e^x$$

Finally, recalling (1.1) we have our solution:

$$u(x,t) = \frac{(x-t)^2}{5} - \frac{4}{5}e^{x-t} + \frac{4}{5}\left(\frac{(4x+t)^2}{16} + e^{\frac{4x+t}{4}}\right)$$

2

Causality and Energy

2.1 Strauss 2.2.1

Use the energy conservation of the wave equation to prove that the only solution with $\phi \equiv 0$ and $\psi \equiv 0$ is $u \equiv 0$.

Theorem 2.1.1 (Strauss - Vanishing Theorem -1D). *Let $f(x)$ be a continuous function in a finite closed interval $[a,b]$. Assume that $f(x) \geq 0$ in the interval and that $\int_a^b f(x)dx = 0$. Then $f(x)$ is identically zero.*

Theorem 2.1.2 (Strauss -"First" Vanishing Theorem). *Let $f(\mathbf{x})$ be a continuous function in $\bar{D}$ where D is a bounded domain. Assume that $f(\mathbf{x}) \geq 0$ in $\bar{D}$ and that $\int_D f(\mathbf{x})d\mathbf{x} = 0$. Then $f(\mathbf{x})$ is identically zero.*

Since the initial displacement is 0 and the initial velocity is 0, we can conclude that $E(0) = 0$ (or just plug in and you will find this!). Then by energy conservation, we have that $E(t) = E(0) = 0$ for all t. So

$$0 = \frac{1}{2}\int_{-\infty}^{\infty}(\rho u_t^2 + T u_x^2)dx$$

Now applying the vanishing theorem (why can we apply it here? we have integral from $-\infty$ to ∞. Discussion during tutorial!) we conclude that $(\rho u_t^2 + T u_x^2) = 0$, or rewriting $\frac{\rho}{T}u_t^2 = -u_x^2$. Hence we must have that $u_t^2 = u_x^2 = 0$. So, $u = f(x)$ for some f and $u = g(t)$ for some g. Thus it must be that $f = g = K$ for some constant K. However, we are given that $u(x,0) = 0$ and therefore $K = 0$. We have shown that $u \equiv 0$.

Bibliography

[1] R. Choksi, *Partial Differential Equations: A First Course*, AMS 2022