MAT351 Tutorial 4

Partial Differential Equations

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Properties of (one-dimensional) diffusions

Recall that last week we dealt with the solution formula for the diffusion equation $u_t = ku_{xx}$ on the whole real line. This week, we shall not use such formula, and rather consider properties that can be derived without an explicit solution formula.

Theorem 1.0.1 (Maximum Principle, Strauss)**.** *If $u(x,t)$ satisfies the diffusion equation in a rectangle (say, $0 \le x \le l$, $0 \le t \le T$) in space-time, then the maximum value of $u(x,t)$ is assumed either initially $(t = 0)$ or on the lateral sides ($x = 0$ or $x = l$).*

Remark. The minimum value has the same property; it too can be attained only on the bottom or the lateral sides.

The Maximum Principle implies the following:

Theorem 1.0.2 (Uniqueness)**.** *The solution to*

$$\begin{cases} u_t - ku_{xx} = f(x,t) \quad \text{for } \ 0 < x < l, t > 0 \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \quad u(l,t) = h(t) \end{cases}$$

is unique (if it exists).

Strauss 2.3.2

Consider a solution of the diffusion equation $u_t = u_{xx}$ in $\{0 \le x \le l, 0 \le t < \infty\}$.

(A) Let $M(T)$ = the maximum of $u(x,t)$ in the closed rectangle $\{0 \le x \le l, 0 \le t \le T\}$. Does $M(T)$ increase or decrease as a function of T?

$M(T)$ increases as a function of T. Suppose $T_1 < T_2$. Since by max principle the max is on the bottom or lateral sides, if the values of u on the extended lateral portion of the rectangle for T_2 are less than the max on the rectangle $\{0 \le x \le l, 0 \le t \le T_1\}$, then there is no change. If the value on the extended lateral portion is greater than the max on the T_1 rectangle, then the max has increased.

(B) Let $m(T)$ = the minimum of $u(x,t)$ in the closed rectangle $\{0 \le x \le l, 0 \le t \le T\}$. Does $m(T)$ increase or decrease as a function of T?

$m(T)$ decreases as a function of T. Suppose $T_1 < T_2$. Since by the minimum principle the min is on the bottom or lateral sides, if the values of u on the extended lateral portion of the rectangle for T_2 are greater than the min on the rectangle $\{0 \le x \le l, 0 \le t \le T_1\}$, then there is no change. If the value on the extended lateral portion is less than the min on the T_1 rectangle, then the min has decreased.

Strauss 2.3.4

Consider the diffusion equation $u_t = u_{xx}$ in $\{0 < x < 1, 0 < t < \infty\}$ with $u(0,t) = u(1,t) = 0$ and $u(x,0) = 4x(1-x)$.

(A) Show that $0 < u(x,t) < 1$ for all $t > 0$ and $0 < x < 1$.

By the Maximum Principle, the max value of u must occur either initially ($t = 0$) or on the lateral sides ($x = 0$ or $x = 1$). Well, $u(x,0) = 4x(1-x)$ which has greatest value $u(1/2, 0) = 1$. On the other hand, $u(0,t) = u(1,t) = 0$ (so u is zero on the lateral sides). Hence we conclude that $u(x,t) < 1$ for all $t > 0$ and $0 < x < 1$.

Similarly, since $u = 0$ on the lateral sides and minimum value at time $t = 0$ is 0, by the Minimum Principle, we have that $u(x,t) > 0$ for all $t > 0$ and $0 < x < 1$.

(B) Show that $u(x,t) = u(1-x,t)$ for all $t \geq 0$ and $0 \leq x \leq 1$.

Let $v(x,t) := u(1-x,t)$. Note that $0 < x < 1 \to 0 < 1-x < 1 \to -1 < -x < 0 \to 0 < x < 1$. Now, by the chain rule:

$$\begin{aligned}
\frac{\partial}{\partial t}v(x,t) &= \frac{\partial}{\partial t}u(1-x,t) = u_t \\
\frac{\partial}{\partial x}v(x,t) &= \frac{\partial}{\partial x}u(1-x,t) = -u_x \\
\frac{\partial^2}{\partial x^2}v(x,t) &= \frac{\partial^2}{\partial x^2}u(1-x,t) = u_{xx}
\end{aligned}$$

Hence we have that $v_t = v_{xx}$ for $0 < x < 1, t > 0$. Moreover:

$$\begin{cases}
v(x,0) = u(1-x,0) = 4(1-x)(1-(1-x)) = 4x(1-x) \\
v(0,t) = u(1,t) = 0 \\
v(1,t) = u(0,t) = 0
\end{cases}$$

So v is a solution to the diffusion equation with the same initial data and boundary conditions as u. By uniqueness we are done.

(C) Use the energy method to show that $\int_0^1 u^2 dx$ is a strictly decreasing function of t.

We have that

$$\begin{aligned}
\frac{d}{dt}\int_0^1 u^2 dx &= 2\int_0^1 u(x,t)u_t(x,t)dx \\
&= 2\int_0^1 u(x,t)u_{xx}(x,t)dx \quad (u_t = u_{xx}) \\
&= 2u(x,t)u_x(x,t)|_{x=0}^{x=1} - 2\int_0^1 u_x(x,t)u_x(x,t)dx \quad \text{(IBP)} \\
&= -2\int_0^1 (u_x(x,t))^2 dx \quad (u(0,t) = u(1,t) = 0) \\
&:= -2\mathcal{I}
\end{aligned}$$

I claim that $\mathcal{I} > 0$. Indeed, if $\mathcal{I} = 0$ then by the Vanishing theorem we have that $u_x(x,t) = 0$. So then for each t, $u(x,t)$ is a constant (say k) in x. Since $u(0,t) = 0$, k must be 0. This contradicts part A (that $0 < u(x,t) < 1$ for all $t > 0$ and $0 < x < 1$). In conclusion, $\int_0^1 u^2 dx$ is a strictly decreasing function of t.

Strauss 2.3.5

The purpose of this exercise is to show that the maximum principle is not true for the equation $u_t = xu_{xx}$, which has a variable coefficient. Verify that $u = -2xt - x^2$ is a solution. Find the location of its maximum in the closed rectangle $\{-2 \le x \le 2, 0 \le t \le 1\}$

First since $u_t = -2x$, $u_x = -2t - 2x$, $u_{xx} = -2$, we have that $u_t = xu_{xx}$.

Now, let $R := \{-2 \le x \le 2, 0 \le t \le 1\}$. To find the maximum first check critical point; so want $u_t = u_x = 0$ i.e. the point $(0,0)$ with value $u(0,0) = 0$. Now we check each of the sides of R:

- $u(-2,t) = 4t - 4$, which has highest value at $t = 1$, $u(-2,1) = 0$.
- $u(2,t) = -4t - 4$, which has highest value at $t = 0$, $u(2,0) = -4$.
- $u(x,0) = -x^2$, which has highest value at $x = 0$, $u(0,0) = 0$.
- $u(x,1) = -2x - x^2$, which has highest value at $x = -1$, $u(-1,1) = 1$.

So, the location of the maximum in closed rectangle R is at $(-1,1)$, which is on the top of R. Hence the maximum principle does not hold.

Bibliography

[1] W. Strauss, *Partial Differential Equations: An Introduction*, 2nd edition, Wiley