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# APM346 TUTORIAL 5

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PARTIAL DIFFERENTIAL EQUATIONS

WRITTEN BY

DAVID KNAPIK

*University of Toronto*  
*david.knapik@mail.utoronto.ca*



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Recall that the Full Fourier series of  $f(x)$  on  $(-l, l)$  is given by

$$f(x) = C_0 + \sum_{n=1}^{\infty} \left( C_n \cos\left(\frac{n\pi x}{l}\right) + D_n \sin\left(\frac{n\pi x}{l}\right) \right)$$

where

$$C_0 = \frac{1}{2l} \int_{-l}^l f(x) dx \quad C_n = \frac{1}{l} \int_{-l}^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx \quad D_n = \frac{1}{l} \int_{-l}^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

For a differentiable function  $f(x)$  defined on  $(-l, l)$  with  $f(-l) = f(l)$ , we can consider its derivative  $f'(x)$ . The derivative of the Fourier Series would be exactly equal to the derivative  $f'(x)$ .

Lastly, recall that if we have Fourier series  $f(x) = \sum_{k=0}^{\infty} a_k X_k(x)$ ,  $a < x < b$  and if  $\int_a^b |f(x)|^2 dx$  is finite, then we have Parseval's Equality:

$$\sum_{k=0}^{\infty} |a_k|^2 \int_a^b |X_k(x)|^2 = \int_a^b |f(x)|^2$$

## Example 1

Find the Full Fourier series of  $x$  on  $(-l, l)$ .

First,

$$C_0 = \frac{1}{2l} \int_{-l}^l x dx = \frac{1}{2l} (0) = 0$$

Now,

$$\begin{aligned} C_n &= \frac{1}{l} \int_{-l}^l x \cos\left(\frac{n\pi x}{l}\right) dx \\ &= \frac{1}{l} \left[ x \frac{l}{n\pi} \sin\left(\frac{n\pi x}{l}\right) \Big|_{-l}^l - \int_{-l}^l \frac{l}{n\pi} \sin\left(\frac{n\pi x}{l}\right) dx \right] \quad (\text{IBP}) \\ &= \frac{-1}{l} \frac{l}{n\pi} (-1) \frac{l}{n\pi} \cos\left(\frac{n\pi x}{l}\right) \Big|_{-l}^l \\ &= \frac{l}{n^2 \pi^2} [\cos(n\pi) - \cos(-n\pi)] \\ &= 0 \end{aligned}$$

Lastly,

$$\begin{aligned}
 D_n &= \frac{1}{l} \int_{-l}^l x \sin\left(\frac{n\pi x}{l}\right) dx \\
 &= \frac{1}{l} \left[ -x \frac{l}{n\pi} \cos\left(\frac{n\pi x}{l}\right) \Big|_{-l}^l + \int_{-l}^l \frac{l}{n\pi} \cos\left(\frac{n\pi x}{l}\right) dx \right] \quad (\text{IBP}) \\
 &= \frac{-x}{n\pi} \cos\left(\frac{n\pi x}{l}\right) \Big|_{-l}^l + \frac{l}{n^2\pi^2} \sin\left(\frac{n\pi x}{l}\right) \Big|_{-l}^l \\
 &= \frac{-l}{n\pi} \cos(n\pi) + \frac{-l}{n\pi} \cos(-n\pi) + 0 \\
 &= \frac{-2l}{n\pi} \cos(n\pi) \\
 &= \frac{-2l}{n\pi} (-1)^n \\
 &= (-1)^{n+1} \frac{2l}{n\pi}
 \end{aligned}$$

Hence we conclude that

$$x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2l}{n\pi} \sin\left(\frac{n\pi x}{l}\right)$$

## Example 2

Find the Full Fourier series of  $|x|$  on  $(-\pi, \pi)$ .

First,

$$C_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| dx = \frac{1}{\pi} \int_0^{\pi} |x| dx = \frac{1}{\pi} \int_0^{\pi} x dx = \frac{1}{\pi} \frac{\pi^2}{2} = \frac{\pi}{2}$$

Now,

$$\begin{aligned}
 C_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos(nx) dx \\
 &= \frac{2}{\pi} \int_0^{\pi} x \cos(nx) dx \quad (\text{Both } |x| \text{ and } \cos \text{ are even}) \\
 &= \frac{2}{\pi} \left[ \frac{x}{n} \sin(nx) \Big|_0^{\pi} - \frac{1}{n} \int_0^{\pi} \sin(nx) dx \right] \quad (\text{IBP}) \\
 &= \frac{2}{\pi} \left[ 0 + \frac{1}{n^2} \cos(nx) \Big|_0^{\pi} \right] \\
 &= \frac{2}{n^2\pi} [\cos(n\pi) - \cos(0)] \\
 &= \frac{2}{n^2\pi} [(-1)^n - 1]
 \end{aligned}$$

so that  $C_n = 0$  for  $n$  even and  $\frac{-4}{n^2\pi}$  for  $n$  odd. Lastly, we note that since  $|x|$  is even and  $\sin(nx)$  is odd,

$$D_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \sin(nx) dx = 0$$

Hence, we conclude that

$$|x| = \frac{\pi}{2} + \sum_{k=0}^{\infty} \frac{-4}{\pi(2k+1)^2} \cos((2k+1)x)$$

where  $n = 2k + 1$ .

### Example 3

Recall the Full Fourier series of  $f(x) = |x|$  on  $(-\pi, \pi)$  from the previous example.

1. With  $f(x) = |x|$ , what is  $f'(x)$ ?
2. Find the Full Fourier series of  $f'(x)$  on  $(-\pi, \pi)$  with justification.
3. Use Parseval's Equality on  $f(x)$  to find the value to the series  $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4}$

For the first part, since

$$|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$$

we conclude that

$$f'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

which we can write as the "sign function"  $f'(x) = \operatorname{sgn} x$  (note that we have restricted the domain to  $\mathbb{R} \setminus \{0\}$ ).

For the second part, first note that  $f(-\pi) = \pi = f(\pi)$ , so we can indeed differentiate the Fourier series. Recalling the series we found in the previous example, we have that

$$f'(x) = \sum_{k=0}^{\infty} \frac{4}{\pi(2k+1)} \sin((2k+1)x)$$

For the last part, by Parseval's Equality we have

$$\begin{aligned} \int_{-\pi}^{\pi} |x|^2 dx &= \left| \frac{\pi}{2} \right|^2 \int_{-\pi}^{\pi} |1|^2 dx + \sum_{k=0}^{\infty} \left| \frac{-4}{\pi(2k+1)^2} \right|^2 \int_{-\pi}^{\pi} |\cos((2k+1)x)|^2 dx \\ &\rightarrow \frac{x^3}{3} \Big|_{-\pi}^{\pi} = \frac{\pi^2}{4} (2\pi) + \sum_{k=0}^{\infty} \frac{16}{\pi^2(2k+1)^4} [\pi] \quad (\text{using orthogonality relation}) \\ &\rightarrow \frac{2\pi^3}{3} = \frac{\pi^3}{2} + \sum_{k=0}^{\infty} \frac{16}{\pi(2k+1)^4} \\ &\rightarrow \frac{\pi^3}{6} = \sum_{k=0}^{\infty} \frac{16}{\pi(2k+1)^4} \\ &\rightarrow \frac{\pi^4}{96} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} \end{aligned}$$

**Bibliography**

- [1] Xiao Jie, Instructor's course notes (Quercus)
- [2] W. Strauss, *Partial Differential Equations: An Introduction*, 2nd edition, Wiley