APM346 Tutorial 11

Partial Differential Equations

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1 Definitions

Recall the definition for the Fourier transform,

$$\mathcal{F}\{f\}(k) = \hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-\imath kx}\mathrm{d}x \tag{1}$$

and for the convolution operation,

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-\tau)g(\tau)\mathrm{d}\tau \tag{2}$$

Show that the Fourier transform is bounded.

$$|\hat{f}(k)| \leq \int_{-\infty}^{\infty} |f(x)||\exp(-\imath kx)|\mathrm{d}x = \int_{-\infty}^{\infty} |f(x)|\mathrm{d}x < \infty \tag{3}$$

Show that the Fourier transform of a convolution is the product of the Fourier transforms.

$$
\begin{aligned}
(f \hat{*} g)(k) &= \int_{-\infty}^{\infty} (f * g)(x)e^{-\imath kx}\mathrm{d}x && (4)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\tau)g(\tau)\mathrm{d}\tau e^{-\imath kx}\mathrm{d}x && (5)\\
&\text{let } z = x-\tau \quad \mathrm{d}z = \mathrm{d}x && (6)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z)g(\tau)e^{-\imath k(z+\tau)}\mathrm{d}z\mathrm{d}\tau && (7)\\
&= \int_{-\infty}^{\infty} f(z)e^{-\imath kz}\mathrm{d}z \int_{-\infty}^{\infty} g(\tau)e^{-\imath k\tau}\mathrm{d}\tau && (8)\\
&= \hat{f}(k)\hat{g}(k) && (9)
\end{aligned}
$$

Aside we can show that convolution is commutative by making the substitution $z = x - \tau$ and obtaining $f(z)$, $g(x-z)$.

2 Examples

Find the Fourier transform of

$$f(x) = e^{-a|x|}$$

with $a > 0$

$$
\begin{aligned}
\mathcal{F}(e^{-a|x|}) &= && (10)\\
&= \int_{-\infty}^{\infty} e^{-a|x|}e^{-\imath kx}\mathrm{d}x && (11)\\
&= \int_{-\infty}^{0} e^{ax}e^{-\imath kx}\mathrm{d}x + \int_{0}^{\infty} e^{-ax}e^{-\imath kx}\mathrm{d}x && (12)\\
&= \lim_{b\to-\infty} \frac{e^{(a-\imath k)x}}{a-\imath k}\bigg|_b^0 + \lim_{b\to\infty} \frac{e^{(-a-\imath k)x}}{-a-\imath k}\bigg|_0^b && (13)\\
&= \frac{1}{a-\imath k} - \frac{1}{-a-\imath k} && (14)\\
&= \frac{2a}{a^2+k^2} && (15)
\end{aligned}
$$

Show that the Fourier transform of a derivative is equal to multiplication by $\imath k$

$$\mathcal{F}(f'(x)) = \int_{-\infty}^{\infty} f'(x)e^{-\imath kx}\mathrm{d}x \tag{16}$$

$$= f(x)e^{-\imath kx}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(-\imath k)e^{-\imath kx}\mathrm{d}x \tag{17}$$

At this point, we require that f be integrable and that it vanishes in the limit of $|x| \to \infty$ to make the first term vanish. Then,

$$\mathcal{F}(f'(x)) = (\imath k)\int_{-\infty}^{\infty} f(x)e^{-\imath kx}\mathrm{d}x \tag{18}$$

$$= \imath k\hat{f}(k) \tag{19}$$

or more generally

$$\mathcal{F}\left(\frac{\mathrm{d}^n f}{\mathrm{d}x^n}\right) = (\imath k)^n \hat{f}(k) \tag{20}$$

Find the Fourier transform for a function shifted horizontally, $f(x-a)$.

$$\mathcal{F}(f(x-a)) = \int_{-\infty}^{\infty} f(x-a)e^{-\imath kx}\mathrm{d}x \tag{21}$$

$$= \int_{-\infty}^{\infty} f(z)e^{-\imath k(z+a)}\mathrm{d}z \tag{22}$$

$$= e^{-\imath ka}\int_{-\infty}^{\infty} f(z)e^{-\imath kz}\mathrm{d}z \tag{23}$$

$$= e^{-\imath ka}\hat{f}(k) \tag{24}$$

Find the Fourier transform for a function stretched horizontally, $f(ax)$ with nonzero a

<u>Case $a > 0$</u>

$$\mathcal{F}(f(ax)) = \int_{-\infty}^{\infty} f(ax)e^{-\imath kx}\mathrm{d}x \tag{25}$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} f(z)e^{-\imath kz/a}\mathrm{d}z \tag{26}$$

$$= \frac{1}{a}\hat{f}\left(\frac{k}{a}\right) \tag{27}$$

<u>Case $a < 0$</u>

$$\mathcal{F}(f(ax)) = \int_{-\infty}^{\infty} f(ax)e^{-\imath kx}\mathrm{d}x \tag{28}$$

$$= \frac{1}{a}\int_{\infty}^{-\infty} f(z)e^{-\imath kz/a}\mathrm{d}z \tag{29}$$

$$= -\frac{1}{a}\int_{\infty}^{\infty} f(z)e^{-\imath kz/a}\mathrm{d}z \tag{30}$$

$$= -\frac{1}{a}\hat{f}\left(\frac{k}{a}\right) \tag{31}$$

Both can be expressed as

$$\mathcal{F}(f(ax)) = \frac{1}{|a|}\hat{f}\left(\frac{k}{a}\right) \tag{32}$$

Find the Fourier transform for the following function,

$$f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| \geq a \end{cases} \tag{33}$$

for $a \neq 0$

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-\imath kx} \mathrm{d}x \tag{34}$$

$$= \int_{-a}^{a} e^{-\imath kx} \mathrm{d}x \tag{35}$$

$$= \left. \frac{e^{-\imath kx}}{-\imath k} \right|_{-a}^{a} \tag{36}$$

$$= \frac{e^{-\imath ka} - e^{\imath ka}}{-\imath k} \tag{37}$$

$$= \sin(ka) \cdot \frac{2}{k} \tag{38}$$

3 ODEs

Let $f(x)$ be integrable, find a solution $y(x)$ to

$$y''(x) - y(x) = f(x) \tag{39}$$

Taking the Fourier transform on both sides, applying linearity and derivative rule

$$(ik)^2 \hat{y}(k) - \hat{y}(k) = \hat{f}(k) \tag{40}$$

$$\hat{y}(k) = -\frac{\hat{f}(k)}{1 + k^2} \tag{41}$$

The next step requires knowledge of inverse Fourier transforms (recall this one from above results),

$$\hat{g}(k) = \frac{1}{1 + k^2} \tag{42}$$

$$g(x) = \frac{1}{2} e^{-|x|} \tag{43}$$

Now we can express $\hat{y}$ as a product of two transforms, which can be inverted into a convolution by the convolution theorem,

$$\hat{y}(k) = -\hat{f} \cdot \hat{g} \tag{44}$$

$$y = -\frac{1}{2} \int_{-\infty}^{\infty} f(x - \tau) e^{-|\tau|} \mathrm{d}\tau \tag{45}$$