MAT351 Tutorial 8

Partial Differential Equations

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November 15, 2022

This tutorial is a review tutorial to help prepare for next week's test.

Strauss 1.2.7

Solve the equation $yu_x + xu_y = 0$ with $u(0, y) = e^{-y^2}$. Sketch some of the characteristics.

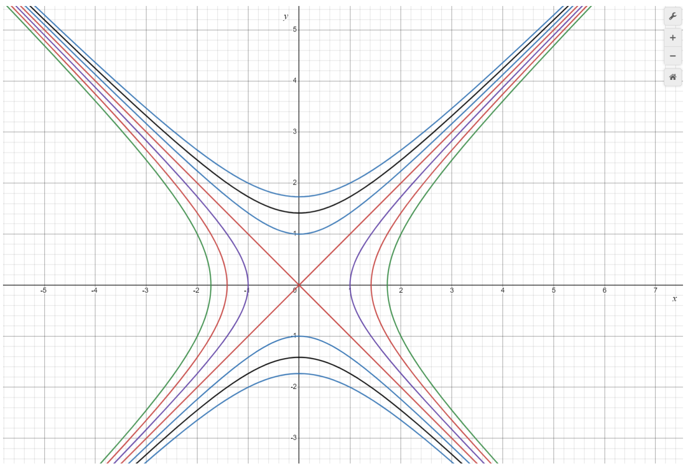
The characteristics must solve $\frac{dy}{dx} = \frac{x}{y}$. So $ydy = xdx$ and we conclude that $\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$ or $y^2 = x^2 + \tilde{C}$. The general solution is thus $u(x, y) = f(y^2 - x^2)$. However, we are given that $u(0, y) = e^{-y^2}$ and so:

$$u(0, y) = f(y^2) = e^{-y^2}$$

i.e. $f(w) = e^{-w}$. In conclusion, we have a solution

$$u(x, y) = e^{x^2 - y^2}$$

Here are some characteristic curves:



Strauss 1.5.5

Consider the equation $u_x + yu_y = 0$ with the boundary condition $u(x, 0) = \phi(x)$.

(A) For $\phi(x) \equiv x$, show that no solution exists.

The characteristics must solve $\frac{dy}{dx} = y$. So $\frac{dy}{y} = dx$ and we have $\ln y = x + C$ or $y = \tilde{C}e^x$. The general solution is thus $u(x, y) = f(ye^{-x})$.

For $\phi(x) \equiv x$ we would have $u(x, 0) = f(0) = x$, a contradiction. Note that when $y = 0$ the equation reads $u_x = 0$ so $u(x, 0)$ should be constant.

(B) For $\phi(x) \equiv 1$, show that there are many solutions.

For $\phi(x) \equiv 1$ we have $u(x, 0) = f(0) = 1$, and $u = 1$ satisfies the equation. So , all we can say about u is that

$$u(x, y) = \begin{cases} 1, & y = 0 \\ f(ye^{-x}), & \text{else} \end{cases}$$

Strauss 2.3.4

Consider the diffusion equation $u_t = u_{xx}$ in $\{0 < x < 1, 0 < t < \infty\}$ with $u(0,t) = u(1,t) = 0$ and $u(x,0) = 4x(1-x)$. Please note that we did this problem in Tutorial 4!

(A) Show that $0 < u(x,t) < 1$ for all $t > 0$ and $0 < x < 1$.

By the Maximum Principle, the max value of u must occur either initially ($t = 0$) or on the lateral sides ($x = 0$ or $x = 1$). Well, $u(x,0) = 4x(1-x)$ which has greatest value $u(1/2, 0) = 1$. On the other hand, $u(0,t) = u(1,t) = 0$ (so u is zero on the lateral sides). Hence we conclude that $u(x,t) < 1$ for all $t > 0$ and $0 < x < 1$.

Similarly, since $u = 0$ on the lateral sides and minimum value at time $t = 0$ is 0, by the Minimum Principle, we have that $u(x,t) > 0$ for all $t > 0$ and $0 < x < 1$.

(B) Show that $u(x,t) = u(1-x,t)$ for all $t \geq 0$ and $0 \leq x \leq 1$.

Let $v(x,t) := u(1-x,t)$. Note that $0 < x < 1 \to 0 < 1-x < 1 \to -1 < -x < 0 \to 0 < x < 1$. Now, by the chain rule:

$$\begin{aligned}
\frac{\partial}{\partial t}v(x,t) &= \frac{\partial}{\partial t}u(1-x,t) = u_t \\
\frac{\partial}{\partial x}v(x,t) &= \frac{\partial}{\partial x}u(1-x,t) = -u_x \\
\frac{\partial^2}{\partial x^2}v(x,t) &= \frac{\partial^2}{\partial x^2}u(1-x,t) = u_{xx}
\end{aligned}$$

Hence we have that $v_t = v_{xx}$ for $0 < x < 1, t > 0$. Moreover:

$$\begin{cases}
v(x,0) = u(1-x,0) = 4(1-x)(1-(1-x)) = 4x(1-x) \\
v(0,t) = u(1,t) = 0 \\
v(1,t) = u(0,t) = 0
\end{cases}$$

So v is a solution to the diffusion equation with the same initial data and boundary conditions as u. By uniqueness we are done.

(C) Use the energy method to show that $\int_0^1 u^2 dx$ is a strictly decreasing function of t.

We have that

$$\begin{aligned}
\frac{d}{dt}\int_0^1 u^2 dx &= 2\int_0^1 u(x,t)u_t(x,t)dx \\
&= 2\int_0^1 u(x,t)u_{xx}(x,t)dx \quad (u_t = u_{xx}) \\
&= 2u(x,t)u_x(x,t)|_{x=0}^{x=1} - 2\int_0^1 u_x(x,t)u_x(x,t)dx \quad \text{(IBP)} \\
&= -2\int_0^1 (u_x(x,t))^2 dx \quad (u(0,t) = u(1,t) = 0) \\
&:= -2\mathcal{I}
\end{aligned}$$

I claim that $\mathcal{I} > 0$. Indeed, if $\mathcal{I} = 0$ then by the Vanishing theorem we have that $u_x(x,t) = 0$. So then for each t, $u(x,t)$ is a constant (say k) in x. Since $u(0,t) = 0$, k must be 0. This contradicts part A (that $0 < u(x,t) < 1$ for all $t > 0$ and $0 < x < 1$). In conclusion, $\int_0^1 u^2 dx$ is a strictly decreasing function of t.

Strauss 4.2.1

Solve the diffusion problem $u_t = ku_{xx}$ in $0 < x < l$, with the mixed boundary conditions $u(0,t) = u_x(l,t) = 0$.

Via separation of variables we have the following eigenvalue problem for $X(x)$:

$$\begin{cases} -X'' = \lambda X \\ X(0) = 0 \\ X'(l) = 0 \end{cases}$$

Remark. Recall that an *eigenfunction* is a solution $X \not\equiv 0$, and an *eigenvalue* is a number λ for which there exists a solution $X \not\equiv 0$.

Let's look for positive eigenvalues first; write $\lambda = \beta^2$ where $\beta > 0$. Then our ODE is $X'' = -\beta^2 X$ which has solution

$$X(x) = C\cos(\beta x) + D\sin(\beta x)$$

The boundary conditions give:

$$\begin{cases} X(0) = C = 0 \\ X'(l) = -(0)\beta\sin(\beta l) + \beta D\cos(\beta l) = 0 \end{cases}$$

so that $D\cos(\beta l) = 0$. Since we do not want $D = 0$ as well, we conclude that $\cos(\beta l) = 0$ and so $\beta = \frac{1}{l}(n + 1/2)\pi$. Therefore, we have the eigenvalues $\lambda_n = (\frac{1}{l}(n + 1/2)\pi)^2$ and eigenfunctions $X_n(x) = \sin\big(\frac{1}{l}(n + 1/2)\pi x\big)$.

Now let us examine the possibility of a negative eigenvalue $\lambda < 0$ which we write as $\lambda = -\gamma^2$. Then our ODE is $X'' = \gamma^2 X$ which has solution

$$X(x) = C\cosh(\gamma x) + D\sinh(\gamma x)$$

The boundary conditions give:

$$\begin{cases} X(0) = C = 0 \\ X'(l) = \gamma(0)\sinh(\gamma l) + \gamma D\cosh(\gamma l) = 0 \end{cases}$$

so that $D\cosh(\gamma l) = 0$. Hence $D = 0$ since $\cosh(\gamma l) \neq 0$. So we have $C = D = 0$ and therefore no negative eigenvalues.

Lastly, let us check if there is a zero eigenvalue. If $\lambda = 0$, then our ODE is simply $X'' = 0$ and hence $X(x) = Cx + D$. The boundary conditions give $X(0) = D = 0$ and $X'(l) = C = 0$. Therefore, zero is not an eigenvalue and we conclude that all the eigenvalues are positive.

As usual, solving $T_n' = -\lambda_n k T_n$ gives $T_n(t) = C_n e^{-\lambda_n k t}$ and our solution is

$$u(x,t) = \sum_{n=0}^{\infty} C_n e^{-(\frac{1}{l}(n+1/2)\pi)^2 kt} \sin\left(\frac{(n+1/2)\pi x}{l}\right)$$

Bibliography

[1] W. Strauss, *Partial Differential Equations: An Introduction*, 2nd edition, Wiley

[2] R. Choksi, *Partial Differential Equations: A First Course*, AMS 2022