Math 254 Tutorial 1

Honours Analysis 1

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1 Axiom of Mathematical Induction

In this section, we shall consider the 5th Peano axiom - the Axiom of Mathematical Induction (AI) (also called the Principle of Mathematical Induction).

Axiom 1.0.1 (AI1)**.** *Let $S \subset \mathbb{N}$ such that the following holds:*

1. $1 \in S$
2. *If a natural number n is in S, then the natural number $n+1$ is also in S.*

Then we have that $S = \mathbb{N}$.

Example 1.0.1. We shall prove the class exercise: for every natural number n, we have that

$$1^3 + 2^3 + ... + n^3 = \left(\frac{n(n+1)}{2}\right)^2 \tag{1.1}$$

To that end, first we let S denote the set of natural numbers for which (1.1) holds. The base case is trivial since:

$$1^3 = \left(\frac{1(1+1)}{2}\right)^2 = 1$$

Now suppose that $(n-1) \in S$. We have:

$$\begin{aligned}
1^3 + 2^3 + ... + (n-1)^3 + n^3 &= \left(\frac{(n-1)n}{2}\right)^2 + n^3 \\
&= \frac{n^4 - 2n^3 + n^2}{4} + n^3 \\
&= \frac{n^4 + 2n^3 + n^2}{4} \\
&= \frac{n^2(n^2 + 2n + 1)}{4} \\
&= \frac{n^2(n+1)^2}{4}
\end{aligned}$$

So by AI1, we have shown that $S = \mathbb{N}$, and we are done.

Now, as noted in class, there are many equivalent reformulations of AI.

Axiom 1.0.2 (AI2)**.** *Let $S \subset \mathbb{N}$ such that the following holds:*

1. *A natural number m is in S*

2. *If a natural number n is in S, then $n+1$ is also in S.*

Then we have that $\{m, m+1, m+2, ...\} \subset S$.

Example 1.0.2. We shall prove the class exercise: for any natural number $n \geq 2$, we have that

$$n^2 > n+1 \tag{1.2}$$

To that end, first we let S denote the set of natural numbers for which (1.2) holds. The base case ($m = 2$) is trivial since $4 > 3$. Now suppose that $n \in S$ (thus, $n^2 > n+1$). Hence $n^2 + 2n > n+1$, which implies that $n^2 + 2n + 1 > (n+1) + 1$. Simplifying, we have that $(n+1)^2 > (n+1) + 1$. So by AI2, we have shown that $\{2, 3, 4, ...\} \subset S$, so we are done.

Axiom 1.0.3 (AI3 - Principle of Complete Mathematical Induction)**.** *Let $S \subset \mathbb{N}$ such that the following holds:*

1. $1 \in S$

2. *If the natural numbers $1, 2, ..., n-1$ are in S, then n is also in S*

Then we have that $S = \mathbb{N}$.

Example 1.0.3. Consider the sequence a_n given by $a_1 = 1$, $a_2 = 8$, and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$ (a recurrence relation). In this example, we shall prove that for every natural number n, we have that

$$a_n = 3 \cdot 2^{n-1} + 2(-1)^n \tag{1.3}$$

To that end, we first let S denote the set of natural numbers for which (1.3) holds. We have that $1 \in S$ since $a_1 = 1$ and $3 \cdot 1 + 2(-1) = 1$. In addition, $2 \in S$ since $a_2 = 8$ and $3 \cdot 2 + 2 = 8$. Now take $n \geq 2$ (with $n \in \mathbb{N}$) such that $1, 2, ..., n$ are in S. We have:

$$\begin{aligned} a_{n+1} &= a_n + 2a_{n-1} \\ &= 3 \cdot 2^{n-1} + 2(-1)^n + 2\left(3 \cdot 2^{n-2} + 2(-1)^{n-1}\right) \\ &= 3 \cdot 2^n + 2(-1)^{n+1} \quad \text{(with some algebra)} \end{aligned}$$

Hence $(n+1) \in S$, and by AI3 we are done.

Lastly, we can combine version 2 and version 3 to arrive at:

Axiom 1.0.4 (AI4)**.** *Let $S \subset \mathbb{N}$ such that the following holds:*

1. *A natural number m is in S*

2. *If $m, m+1, ..., m+n$ are in S, then also $m+n+1$ is in S*

Then we have that $\{m, m+1, m+2, ...\} \subset S$.

Example 1.0.4. In this example, we shall prove a property of the Fibonacci sequence F_n via AI4. We first begin with a brief introduction to the Fibonacci numbers and the golden ratio.

Recall the definition: $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The Fibonacci numbers are strongly related to the golden ratio φ, namely we can express F_n in terms of n and the golden ratio. We say that two quantities $a > b > 0$ are in the golden ratio φ if

$$\frac{a+b}{a} = \frac{a}{b} = \varphi \tag{1.4}$$

To calculate the golden ratio, we have that:

$$\frac{a+b}{a} = \frac{a}{a} + \frac{b}{a} = 1 + \frac{b}{a} = 1 + \frac{1}{\varphi}$$

and hence $1 + \frac{1}{\varphi} = \varphi$. Thus $\varphi + 1 = \varphi^2$, and by the quadratic formula we arrive at $\varphi = \frac{1+\sqrt{5}}{2}$ (the solution gives $\varphi = \frac{1\pm\sqrt{5}}{2}$, but we take the positive quantity since φ must be positive).

With the preliminaries out of the way, we now shall prove that for any natural number $n \geq 3$, we have that

$$F_n > \varphi^{n-2} \tag{1.5}$$

To that end, we start with our base cases - note that $F_3 = 2$ and $\varphi < 2$ hence $F_3 > \varphi$. Also, $F_4 = 3$ and $\varphi^2 \approx 2.62$ so $F_4 > \varphi^2$. Now, assume that $F_i > \varphi^{i-2}$ for $3 \leq i \leq n$ (our induction hypothesis). So we need to show that $F_{n+1} > \varphi^{n-1}$ for $n+1 > 4$.

Well, recall that we had $\varphi + 1 = \varphi^2$. Hence,

$$\varphi^{n-1} = (\varphi + 1)(\varphi^{n-3}) = \varphi^{n-2} + \varphi^{n-3}$$

Now, by the definition of the Fibonacci sequence, we have that $F_{n+1} = F_n + F_{n-1}$. In addition, by the induction hypothesis, $F_n > \varphi^{n-2}$ and $F_{n-1} > \varphi^{n-3}$. Bringing it all together:

$$F_{n+1} = F_n + F_{n-1} > \varphi^{n-2} + \varphi^{n-3} = \varphi^{n-1}$$

2 Real Numbers

The absolute value function is extremely important, and hence has its own special notation $|x|$. It is defined for every real number via:

$$|x| = \begin{cases} x \ , \ x \geq 0 \\ -x \ , \ x < 0 \end{cases}$$

For all $a, b \in \mathbb{R}$, it satisfies:

1. $|ab| = |a||b|$
2. $|a - b| = |b - a|$
3. $|a + b| \leq |a| + |b|$

The third property above is very important, and is called the *Triangle Inequality*. It will prove to be extremely useful, and in particular we often introduce another real number c:

$$\begin{aligned}|a-b| &= |a-c+c-b| \\ &\leq |a-c|+|c-b|\end{aligned}$$

Example 2.0.1. Here we prove Exercise 1.2.6(c) of the textbook. We want to show that for all $a, b, c, d \in \mathbb{R}$,

$$|a-b| \leq |a-c|+|c-d|+|d-b|$$

Well, we just need to apply the Triangle Inequality twice:

$$\begin{aligned}|a-b| &= |(a-c)+(c-d+d-b)| \\ &\leq |a-c|+|(c-d)+(d-b)| \\ &\leq |a-c|+|c-d|+|d-b|\end{aligned}$$

Example 2.0.2. Here we shall prove Exercise 1.2.6(d) of the textbook. We want to show that $||a|-|b|| \leq |a-b|$. To do so, first note that this is equivalent to showing

$$\begin{aligned}&1.\ |a|-|b| \leq |a-b| \\ \text{and}\ \ &2.\ |b|-|a| \leq |a-b|\end{aligned}$$

For the first part, we have that

$$|a| = |(a-b)+b| \leq |a-b|+|b|$$

and hence rearranging we arrive at $|a|-|b| \leq |a-b|$. Similarly, for the second part, we have that

$$|b| = |(b-a)+a| \leq |b-a|+|a|$$

and hence rearranging we arrive at $|b|-|a| \leq |b-a| = |a-b|$.

3 Extras

Here we connect the previous two sections by providing one example that uses results from both of the sections. We shall prove that for every natural number n and real number x, we have that

$$|\sin(nx)| \leq n|\sin(x)| \tag{3.1}$$

To that end, first we let S denote the set of natural numbers for which (3.1) holds. The base case is trivial since $|\sin(x)| \leq |\sin(x)|$. Now, suppose that n is in S. We have:

$$\begin{aligned}
|\sin((n+1)x)| &= |\sin(nx)\cos(x) + \sin(x)\cos(nx)| \quad \text{(Trig. Identity)} \\
&\leq |\sin(nx)\cos(x)| + |\sin(x)\cos(nx)| \quad \text{(Triangle Inequality)} \\
&= |\sin(nx)||\cos(x)| + |\sin(x)||\cos(nx)| \\
&\leq |\sin(nx)| + |\sin(x)| \\
&\leq n|\sin(x)| + |\sin(x)| \\
&= (n+1)|\sin(x)|
\end{aligned}$$

So by AI1, we have shown that $S = \mathbb{N}$, and we are done.