MAT351 Tutorial 6

Partial Differential Equations

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1 Diffusion with a source on a half-line

Recall the *inhomogeneous* IVP for the diffusion equation (on the whole line):

$$\begin{cases} u_t - ku_{xx} = f(x,t), & \text{on } \mathbb{R} \times (0,\infty) \\ u(x,0) = \phi(x), & \text{on } \mathbb{R} \end{cases} \tag{1.1}$$

Via Duhamel's principle (the textbook also calls it "the operator method"), you should have seen in class that the solution of (1.1) is

$$u(x,t) = \int_{-\infty}^{\infty} \Phi(x-y,t)\phi(y)dy + \int_0^t \int_{-\infty}^{\infty} \Phi(x-y,t-s)f(y,s)dyds$$

where as usual Φ is the heat kernel. Now, lets work on the half-line. Let us solve the inhomogeneous diffusion equation on the half-line with Dirichlet boundary condition:

$$\begin{cases} u_t - ku_{xx} = f(x,t), & \text{on } \{0 < x < \infty, 0 < t < \infty\} \\ u(x,0) = \phi(x) \\ u(0,t) = 0 \end{cases} \tag{1.2}$$

We proceed via the method of reflection; to that end let $\phi_{\text{odd}}(x)$ and $f_{\text{odd}}(x,t)$ be the odd extensions of ϕ and f with respect to $x=0$. Then, the solution to

$$\begin{cases} v_t - kv_{xx} = f_{\text{odd}}(x,t), & \text{on } \{-\infty < x < \infty, 0 < t < \infty\} \\ v(x,0) = \phi_{\text{odd}}(x) \end{cases}$$

is given by (1.1); that is:

$$v(x,t) = \int_{-\infty}^{\infty} \Phi(x-y,t)\phi_{\text{odd}}(y)dy + \int_0^t \int_{-\infty}^{\infty} \Phi(x-y,t-s)f_{\text{odd}}(y,s)dyds$$

But, note that since $\Phi(y,t)$ is even w.r.t. $y=0$ and the product of an even function and an odd function is an odd function, we have that $v(0,t)=0$. Hence, our solution to (1.2) is the restriction $u(x,t) = v(x,t)$ for $x>0$. We now just need to simplify the formula - it is very similar to what we did in a previous tutorial. Simply split each integral over $(-\infty,0)$ and $(0,\infty)$, use the definition of the odd extensions in these intervals, and change of variables $z=-y$ for integrals over $(-\infty,0)$. I spare you the details here as we have done this a few times before; in any case you should arrive at

$$
\begin{aligned}
u(x,t) = &\int_0^\infty \left(\Phi(x-y,t) - \Phi(x+y,t)\right)\phi(y)dy \\
&+ \int_0^t \int_0^\infty \left(\Phi(x-y,t-s) - \Phi(x+y,t-s)\right) f(y,s)dyds, \quad x > 0
\end{aligned}
$$

Let's take it a step further and consider the *completely inhomogeneous* diffusion problem on the half-line

$$
\begin{cases}
u_t - ku_{xx} = f(x,t), & \text{on } \{0 < x < \infty, 0 < t < \infty\} \\
u(x,0) = \phi(x) \\
u(0,t) = h(t)
\end{cases}
$$

We can actually just reduce this to the previous simpler case by letting $V(x,t) = u(x,t) - h(t)$. Then $V_t = u_t - h'(t)$, $V_{xx} = u_{xx}$ and so $(V_t + h'(t)) - kV_{xx} = f(x,t)$, that is:

$$
\begin{cases}
V_t - kV_{xx} = f(x,t) - h'(t) \ , & 0 < x < \infty, t > 0 \\
V(x,0) = u(x,0) - h(0) = \phi(x) - h(0) \\
V(0,t) = u(0,t) - h(t) = h(t) - h(t) = 0
\end{cases}
$$

At this point, we can solve for V as before with the method of reflection. Our solution then is simply $u(x,t) = h(t) + V(x,t)$.

2 Waves with a source on whole line - Duhamel's Principle

Please note that for this section, I follow [2].

We have already looked at the 1D wave equation - now we introduce an external force given by $f(x,t)$. The corresponding IVP with inhomogeneous right hand side source term is:

$$
\begin{cases}
u_{tt} - c^2 u_{xx} = f(x,t), & -\infty < x < \infty, t > 0 \\
u(x,0) = \phi(x), & -\infty < x < \infty \\
u_t(x,0) = \psi(x), & -\infty < x < \infty
\end{cases} \tag{2.1}
$$

The claim is that the solution is the extension of D'Alembert's formula:

$$
u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds + \frac{1}{2c}\int\int_D f(y,\tau)dyd\tau \tag{2.2}
$$

where D is the "characteristic triangle" or "domain of dependence" associated with (x,t) (i.e. the triangle in the $x-t$ plane with top point (x,t) and base points $(x-ct,0)$, $(x+ct,0)$).

Remark. The double integral in (2.2) is simply equal to $\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(y,\tau)dyd\tau$

Remark. Naturally, if $f(x,t) \equiv 0$ then we recover the regular D'Alembert's formula.

Duhamel's principle (named after Jean-Marie Duhamel B:1797-D:1872) is a general method for solving linear PDEs with inhomogeneous terms. Here we demonstrate its use with (2.1).

First we note that by superposition, we need only to solve

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), & -\infty < x < \infty, t > 0 \\ u(x,0) = 0, & -\infty < x < \infty \\ u_t(x,0) = 0, & -\infty < x < \infty \end{cases} \tag{2.3}$$

Adding this solution to D'Alembert's formula for the (homogenous) wave equation with initial data $\phi(x)$ and $\psi(x)$ then yields the solution for (2.1). Introducing the temporal parameter $s \in [0,\infty)$, Duhamel's principle states that the solution to (2.3) is given by $u(x,t) = \int_0^t w(x,t;s)ds$ where for each fixed s, $w(x,t;s)$ is solution to

$$\begin{cases} w_{tt}(x,t;s) = c^2 w_{xx}(x,t;s), & -\infty < x < \infty, t > s \\ w(x,s;s) = 0, & -\infty < x < \infty \\ w_t(x,s;s) = f(x,s), & -\infty < x < \infty \end{cases}$$

Check

Let us verify Duhamel's principle here. Recall:

Theorem 2.0.1 (Leibniz Rule)**.** *Let $f(t,s)$ smooth, then*

$$\frac{d}{dt}\left(\int_0^t f(t,s)ds\right) = f(t,t) + \int_0^t f_t(t,s)ds$$

By Leibniz rule we have that

$$u_t(x,t) = w(x,t;t) + \int_0^t w_t(x,t;s)ds = \int_0^t w_t(x,t;s)ds \tag{2.4}$$

since $w(x,t;t) = 0$. Then by Leibniz rule again

$$u_{tt}(x,t) = w_t(x,t;t) + \int_0^t w_{tt}(x,t;s)ds = f(x,t) + \int_0^t w_{tt}(x,t;s)ds$$

since $w_t(x,t;t) = f(x,t)$. On the other hand, by differentiating under the integral, we have that $u_{xx}(x,t) = \int_0^t w_{xx}(x,t;s)ds$. Hence:

$$u_{tt} - c^2 u_{xx} = \int_0^t \left[w_{tt}(x,t;s) - c^2 w_{xx}(x,t;s)\right] ds + f(x,t) = f(x,t)$$

where we have used that $w_{tt}(x,t;s) = c^2 w_{xx}(x,t;s)$.

Now, $u(x,0) = 0$ trivially and $u_t(x,0) = 0$ by (2.4).

Last Comment

So far we have shown that the solution to (2.1) is

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds + \int_0^t w(x,t;s)ds$$

But, we can solve for $w(x,t;s)$ for each fixed s as it satisfies homogenous wave equation, just now starting at $t=s$ instead of $t=0$ (and now the source is incorporated into the initial data). One finds $w(x,t;s) = \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dy$ so that

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds$$

which is precisely the extension of D'Alembert's formula which was proposed in the beginning of this section.

Bibliography

[1] W. Strauss, *Partial Differential Equations: An Introduction*, 2nd edition, Wiley

[2] R. Choksi, *Partial Differential Equations: A First Course*, AMS 2022