MAT351 Tutorial 11

Partial Differential Equations

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Strauss 6.3.3

Solve the Dirichlet problem for a disk

$$\begin{cases} u_{xx} + u_{yy} = 0 \quad \text{for} \quad x^2 + y^2 < a^2 \\ u = h(\theta) \quad \text{for} \quad x^2 + y^2 = a^2 \end{cases}$$

in the case where the boundary condition is $h(\theta) = \sin^3(\theta)$. Hint: use the identity $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

As derived in section 6.3 of [1] by separating variables in polar coordinates, our solution is

$$u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

So, on one hand we have that

$$u(a,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta + B_n \sin n\theta)$$

On the other hand, using the hint we have that

$$u(a,\theta) = \sin^3(\theta) = \frac{-1}{4}\sin 3\theta + \frac{3}{4}\sin\theta$$

So by inspection we can match the coefficients above. Take $A_n = 0$ for all n, $B_1 = \frac{3}{4a}$, $B_2 = 0$, $B_3 = \frac{-1}{4a^3}$, and $B_n = 0$ for $n \geq 4$. We conclude that

$$u(r,\theta) = r\frac{3}{4a}\sin\theta - r^3\frac{1}{4a^3}\sin(3\theta)$$

The Annulus

Consider the Dirichlet problem for an annulus:

$$\begin{cases} u_{xx} + u_{yy} = 0 \quad \text{in } 0 < a^2 < x^2 + y^2 < b^2 \\ u = g(\theta) \quad \text{for } x^2 + y^2 = a^2 \\ u = h(\theta) \quad \text{for } x^2 + y^2 = b^2 \end{cases}$$

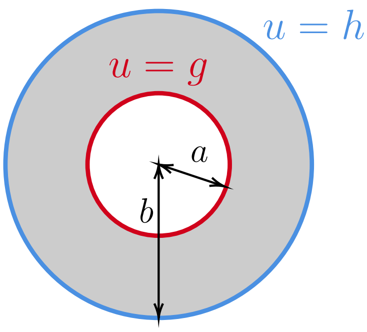

Figure 1: The annulus; the region between two concentric circles of radius a and b. Figure created using TikZ.

The separated solutions are the same as for the disk, except now we do not throw out the functions r^{-n} and $\log r$ as they are finite within the annulus. Hence, the solution is

$$u(r,\theta) = \frac{1}{2}(C_0 + D_0 \log r) + \sum_{n=1}^{\infty} (C_n r^n + D_n r^{-n}) \cos n\theta + (A_n r^n + B_n r^{-n}) \sin n\theta$$

It remains to determine the coefficients by setting $r = a$ and $r = b$. This is exercise 6.4.3 in [1]. Let's give the idea here; we have that

$$g(\theta) = u(a,\theta) = \frac{1}{2}(C_0 + D_0 \log a) + \sum_{n=1}^{\infty} (C_n a^n + D_n a^{-n}) \cos n\theta + (A_n a^n + B_n a^{-n}) \sin n\theta$$
$$h(\theta) = u(b,\theta) = \frac{1}{2}(C_0 + D_0 \log b) + \sum_{n=1}^{\infty} (C_n b^n + D_n b^{-n}) \cos n\theta + (A_n b^n + B_n b^{-n}) \sin n\theta$$

and hence for $n \geq 1$,

$$\begin{cases} A_n a^n + B_n a^{-n} = \frac{1}{\pi} \int_0^{2\pi} g(\theta) \sin n\theta d\theta \\ C_n a^n + D_n a^{-n} = \frac{1}{\pi} \int_0^{2\pi} g(\theta) \cos n\theta d\theta \\ A_n b^n + B_n b^{-n} = \frac{1}{\pi} \int_0^{2\pi} h(\theta) \sin n\theta d\theta \\ C_n b^n + D_n b^{-n} = \frac{1}{\pi} \int_0^{2\pi} h(\theta) \cos n\theta d\theta \end{cases}$$

The above system can then be solved for A_n, B_n, C_n, D_n $(n \geq 1)$. For $n = 0$,

$$C_0 + D_0 \log a = \frac{1}{\pi} \int_0^{2\pi} g(\theta) d\theta, \quad C_0 + D_0 \log b = \frac{1}{\pi} \int_0^{2\pi} h(\theta) d\theta$$

which is a (much) easier system, that can be solved for C_0 and D_0.

The Exterior of a Disk

Consider the Dirichlet problem for the exterior of a disk:

$$\begin{cases} u_{xx} + u_{yy} = 0 \quad \text{for} \quad x^2 + y^2 > a^2 \\ u = h(\theta) \quad \text{for} \quad x^2 + y^2 = a^2 \\ u \text{ bounded as } x^2 + y^2 \to \infty \end{cases}$$

We follow the same reasoning as for inside the disk, but now we have boundedness at infinity instead of finiteness at the origin. Hence, only r^{-n} is retained. The resulting solution is

$$u(r,\theta) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} r^{-n} (A_n \cos n\theta + B_n \sin n\theta)$$

Setting $r = a$, we require that

$$h(\theta) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} a^{-n} (A_n \cos n\theta + B_n \sin n\theta)$$

Thus,

$$A_n = \frac{a^n}{\pi} \int_0^{2\pi} h(\theta) \cos n\theta d\theta, \quad B_n = \frac{a^n}{\pi} \int_0^{2\pi} h(\theta) \sin n\theta d\theta$$

Please note that the remainder of the tutorial time was spent handing back the term tests and addressing any grading issues/questions. I also talked about some tangentially related questions concerning generalizations of the tutorial material. Have a great holiday season!

Bibliography

[1] W. Strauss, *Partial Differential Equations: An Introduction*, 2nd edition, Wiley

[2] R. Choksi, *Partial Differential Equations: A First Course*, AMS 2022