MAT351 Tutorial 10

Partial Differential Equations

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Fourier Series - Inhomogeneous boundary conditions

Strauss 5.6.9 (an inhomogeneous wave problem)

Use the method of subtraction to solve $u_{tt} = 9u_{xx}$ for $0 \leq x \leq 1$, with $u(0,t) = h$, $u(1,t) = k$, where h and k are given constants, and $u(x,0) = 0$, $u_t(x,0) = 0$.

By subtraction, the data can be shifted from the boundary to another spot in the problem. We let $w(x,t) = w(x) = (1 - \frac{x}{1})h + \frac{x}{1}k$ and $v(x,t) = u(x,t) - w(x,t)$. Then, $v(x,t)$ satisfies:

$$\begin{cases} v_{tt} = 9v_{xx} \\ v(0,t) = v(1,t) = 0 \\ v(x,0) = -w(x), \quad v_t(x,0) = 0 \end{cases} \tag{1.1}$$

Recalling sections 4.1 and 5.1 of [1], the solution of problem (1.1) is

$$v(x,t) = \sum_{n=1}^{\infty} (A_n \cos(n\pi 3t) + B_n \sin(n\pi 3t)) \sin(n\pi x)$$

with

$$A_n = \frac{2}{1}\int_0^1 (-w(x)) \sin(n\pi x) dx = 2\int_0^1 \sin(n\pi x)((h-k)x - h) dx = \frac{2}{n\pi}(k(-1)^n - h)$$

where the integral is evaluated via IBP, and $B_n = \frac{2}{3n\pi}\int_0^1 0 dx = 0$. So, $v(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(k(-1)^n - h) \cos(n\pi 3t) \sin(n\pi x)$ and we have our solution:

$$u(x,t) = (1-x)h + xk + \sum_{n=1}^{\infty} \frac{2}{n\pi}(k(-1)^n - h) \cos(n\pi 3t) \sin(n\pi x)$$

2
Harmonic functions

Strauss 6.1.2

Find the solutions that depend only on r of the equation $u_{xx} + u_{yy} + u_{zz} = k^2 u$, where k is a positive constant. (Hint: substitute $u = v/r$.)

First, let us recall the Laplacian in spherical coordinates:

$$\Delta u = u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2}\left(u_{\theta\theta} + (\cot\theta) u_\theta + \frac{1}{\sin^2\theta} u_{\phi\phi} \right) \tag{2.1}$$

We seek solution $u(r, \theta, \phi) = u(r)$ which depends only on r, hence using (2.1) we have reduced our equation to $u_{rr} + \frac{2}{r} u_r = k^2 u$. As suggested in the hint, let $v = ru$ then we have that $v_r = ru_r + u$ and $v_{rr} = ru_{rr} + 2u_r$ so that

$$u_r = \frac{v_r}{r} - \frac{u}{r} = \frac{v_r}{r} - \frac{v}{r^2}, \quad u_{rr} = \frac{v_{rr}}{r} - \frac{2u_r}{r} = \frac{v_{rr}}{r} - \frac{2v_r}{r^2} + \frac{2v}{r^3}$$

Hence, $u_{rr} + \frac{2}{r} u_r = \frac{v_{rr}}{r} - \frac{2v_r}{r^2} + \frac{2v}{r^3} + \frac{2v_r}{r^2} - \frac{2v}{r^3} = \frac{v_{rr}}{r}$ so that $\frac{v_{rr}}{r} = k^2 \frac{v}{r}$. Solving $v_{rr} = k^2 v$ we have that $v = A\cosh(kr) + B\sinh(kr)$. Thus:

$$u = \frac{1}{r}\left(A\cosh(kr) + B\sinh(kr) \right)$$

Strauss 6.1.6

Solve $u_{xx} + u_{yy} = 1$ in the annulus $a < r < b$ with $u(x, y)$ vanishing on both parts of the boundary $r = a$ and $r = b$.

First, let us recall the Laplacian in polar coordinates:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{2.2}$$

We assume a radial solution (note that u is not a function of θ on the boundary), hence using (2.2) we have reduced our equation to the ODE $u_{rr} + \frac{1}{r} u_r = 1$, which we rewrite as

$(ru_r)_r = r$. Integrating, we have that $u_r = \frac{r}{2} + \frac{c_1}{r}$. Again integrating, we have that

$$u(r) = \frac{r^2}{4} + c_1 \ln r + c_2 \tag{2.3}$$

It only remains now to use the boundary conditions - we have that

$$\begin{cases} u(a) = \frac{a^2}{4} + c_1 \ln a + c_2 = 0 \\ u(b) = \frac{b^2}{4} + c_1 \ln b + c_2 = 0 \end{cases}$$

The above is a system of 2 equations with two unknowns; solving we have that $c_1 = \frac{a^2 - b^2}{4\ln(b/a)}$ and $c_2 = \frac{b^2 \ln a - a^2 \ln b}{4 \ln(b/a)}$. So our solution then is simply (2.3) with the constants as determined. Of course we can lastly convert our solution back to Cartesian coordinates via $r = \sqrt{x^2 + y^2}$.

Strauss 6.2.1

Solve $u_{xx} + u_{yy} = 0$ in the rectangle $0 < x < a, 0 < y < b$ with the following boundary conditions:

$$\begin{cases} u_x(0, y) = -a \\ u_x(a, y) = 0 \\ u_y(x, 0) = b \\ u_y(x, b) = 0 \end{cases}$$

Hint: a shortcut is to guess that the solution might be a quadratic polynomial in x and y.

The technique of section 6.2 is separation of variables. This problem has a specific hint for the solution form, so we wont be separating variables. For an example using the separation of variables method, please check out the next problem in this document.

Following the hint, guess $u(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$. Then:

$$\begin{cases} u_x(0, y) = -a \implies By + D = -a \\ u_x(a, y) = 0 \implies 2Aa + By + D = 0 \\ u_y(x, 0) = b \implies Bx + E = b \\ u_y(x, b) = 0 \implies By + 2Cb + E = 0 \end{cases}$$

Immediately, we see that B must be 0 and then $D = -a$ and $E = b$. Then with these values we have

$$\begin{cases} 2Aa - a = 0 \\ 2Cb + b = 0 \end{cases}$$

so we have $A = \frac{1}{2}$ and $C = -\frac{1}{2}$. Note that $u_{xx} + u_{yy} = 2A + 2C$ is 0 for $C = -A$. In conclusion,

$$u(x, y) = \frac{1}{2}x^2 - \frac{1}{2}y^2 - ax + by + F$$

where F arbitrary.

Strauss 6.2.3

Find the harmonic function $u(x,y)$ in the square $D = \{0 < x < \pi, 0 < y < \pi\}$ with boundary conditions

$$\begin{cases} u_y(x,0) = u_y(x,\pi) = 0 \\ u(0,y) = 0 \\ u(\pi,y) = \cos^2 y = \frac{1}{2}(1+\cos 2y) \end{cases}$$

We separate variables $u(x,y) = X(x)Y(y)$ and get $\frac{X''}{X} + \frac{Y''}{Y} = 0$, so $X'' = \lambda X$ and $Y'' = -\lambda Y$ for some constant λ. Note that $u_y(x,0) = u_y(x,\pi) = 0$ yields $Y'(0) = Y'(\pi) = 0$ and hence we have eigenvalue problem

$$\begin{cases} Y'' = -\lambda Y \\ Y'(0) = Y'(\pi) = 0 \end{cases}$$

Recall from section 4.2 of [1] that the list of all the eigenvalues is $\lambda_n = (\frac{n\pi}{\pi})^2 = n^2$ for $n = 0, 1, 2, \ldots.$ The eigenfunctions are $Y_n(y) = \cos(ny)$ for $n = 1, 2, ..$ and any constant function for the zero eigenvalue.

The related ODE for X will now be solved. Since we have that $X'' = n^2 X$ we have $X(x) = A\cosh(nx) + B\sinh(nx)$. But the boundary condition $u(0,y) = 0$ yields $X(0) = 0$ and hence $X(0) = A = 0$. Thus, $X_n(x) = \sinh(nx)$ for $n = 1, 2, \ldots.$ For the zero eigenvalue, we have that $X''(x) = 0$ and hence $X(x) = Ax + B$. The boundary condition then gives $B = 0$ so $X_0 = x$. At this point, our expansion is

$$u(x,y) = A_0 x + \sum_{n=1}^{\infty} A_n \sinh(nx)\cos(ny)$$

To determine the coefficients, we shall use the boundary condition at $x = \pi$. That is,

$$u(\pi,y) = A_0\pi + \sum_{n=1}^{\infty} A_n \sinh(n\pi)\cos(ny) = \frac{1}{2} + \frac{1}{2}\cos(2y)$$

Given the above form, we can match coefficients on both sides. We have that $A_0\pi = \frac{1}{2}$ so that $A_0 = \frac{1}{2\pi}$, we have $A_2\sinh(2\pi) = \frac{1}{2}$ so that $A_2 = \frac{1}{2\sinh(2\pi)}$, and $A_n = 0$ for all other n. In conclusion the desired harmonic function is

$$u(x,y) = \frac{1}{2\pi}x + \frac{1}{2\sinh(2\pi)}\sinh(2x)\cos(2y)$$

Bibliography

[1] W. Strauss, *Partial Differential Equations: An Introduction*, 2nd edition, Wiley

[2] R. Choksi, *Partial Differential Equations: A First Course*, AMS 2022