APM346 Tutorial 4

Partial Differential Equations

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Example 1

I received a few questions about the integral $\int_0^l x\sin^2(\frac{n\pi x}{l})dx$, which is used both in the notes [1], as well as the previous tutorial. Here I will show one way to evaluate it.

We have:

$$\begin{aligned}
\int_0^l x\sin^2(\frac{n\pi x}{l})dx &= \frac{1}{2}\int_0^l xdx - \frac{1}{2}\int_0^l x\cos\left(\frac{2\pi nx}{l}\right)dx \quad (\text{using } \sin^2(x) = \frac{1-\cos(2x)}{2})\\
&= \frac{1}{2}\frac{l^2}{2} - \frac{1}{2}\left[x\sin\left(\frac{2\pi nx}{l}\right)\frac{l}{2\pi n}\Big|_0^l - \int_0^l \frac{l}{2\pi n}\sin\left(\frac{2\pi nx}{l}\right)dx\right] \quad (\text{IBP})\\
&= \frac{l^2}{4} - \frac{1}{2}\left[0 - \frac{l}{2\pi n}\int_0^l \sin\left(\frac{2\pi nx}{l}\right)dx\right]\\
&= \frac{l^2}{4} + \frac{l}{4\pi n}\left[-\frac{l\cos(\frac{2\pi nx}{l})}{2n\pi}\right]_0^l\\
&= \frac{l^2}{4} + \frac{l}{4\pi n}\left[-\frac{l}{2n\pi}(1-1)\right]\\
&= \frac{l^2}{4}
\end{aligned}$$

Remark. In the first line, by using the trigonometric identity we encounter an integral of the form $\int_0^l x\cos(ax/l)dx$. Following the IBP I did above, you can then easily determine the other commonly used integral from the notes and last tutorial: $\int_0^l x\cos(\frac{n\pi x}{l})dx = \frac{l^2(\cos(n\pi)-1)}{n^2\pi^2}$. Please try this if you are not comfortable with such integration as this is not the focus of the course and you will be required to do such integrals.

Example 2

Find the Fourier Sine series of x^2 on $(0,1)$.

We have that

$$\begin{aligned}
A_n &= \frac{2}{1}\int_0^1 x^2 \sin(n\pi x)dx \\
&= 2x^2\left[-\cos(n\pi x)\frac{1}{n\pi}\right]_0^1 - 2\int_0^1 2x\left(-\cos(n\pi x)\frac{1}{n\pi}\right)dx \\
&= -\frac{2}{n\pi}\cos(n\pi) + \frac{4}{n\pi}\int_0^1 x\cos(n\pi x)dx \\
&= -\frac{2}{n\pi}\cos(n\pi) + \frac{4}{n\pi}\left[x\sin(n\pi x)\frac{1}{n\pi}\Big|_0^1 - \int_0^1 \sin(n\pi x)\frac{1}{n\pi}dx\right] \\
&= -\frac{2}{n\pi}(-1)^n + \frac{4}{n^2\pi^2}(0) - \frac{4}{n^2\pi^2}\int_0^1 \sin(n\pi x)dx \\
&= -\frac{2}{n\pi}(-1)^n - \frac{4}{n^2\pi^2}\left(\frac{1-\cos(n\pi)}{n\pi}\right) \\
&= -\frac{2}{n\pi}(-1)^n + \frac{4}{n^3\pi^3}((-1)^n - 1)
\end{aligned}$$

Hence we conclude

$$x^2 = \sum_{n=1}^{\infty}\left[-\frac{2}{n\pi}(-1)^n + \frac{4}{n^3\pi^3}((-1)^n - 1)\right]\sin(n\pi x)$$

Example 3

Find the Fourier Cosine series of $f(x) = x$ on $(0, l)$. Integrate the Fourier series (as an indefinite integral), and using $x = \frac{l}{2}$, find the value to the following series:

$$\sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+1)^3}$$

For the Cosine series of x on $(0, l)$, we have seen this in Tutorial 2 when solving the heat equation. We had that

$$A_0 = \frac{1}{l}\int_0^l x dx = \frac{1}{l}\left(\frac{l^2}{2}\right) = \frac{l}{2}$$

And for $n \neq 0$ we have:

$$\begin{aligned}
A_n &= \frac{2}{l}\int_0^l x\cos\Big(\frac{n\pi x}{l}\Big)dx \\
&= \frac{2}{l}\frac{l}{n\pi}x\sin\Big(\frac{n\pi x}{l}\Big)|_0^l - \frac{2}{l}\int_0^l \frac{l}{n\pi}\sin\Big(\frac{n\pi x}{l}\Big)dx \text{ (IBP)} \\
&= \frac{2l}{n\pi}\sin(n\pi) + \frac{2}{l}\frac{l}{n\pi}\frac{l}{n\pi}\cos\Big(\frac{n\pi x}{l}\Big)|_0^l \\
&= \frac{2l}{n\pi}\sin(n\pi) + \frac{2l}{m^2\pi^2}(\cos(m\pi) - 1) \\
&= \frac{2l}{n^2\pi^2}(\cos(n\pi) - 1) \quad \text{(first term is 0)}
\end{aligned}$$

Then since $\cos(n\pi)$ is 1 for even n and -1 for odd n, we have that

$$A_n = \begin{cases} 0, & n \text{ even} \\ \frac{-4l}{n^2\pi^2}, & n \text{ odd} \end{cases}$$

and $x = \frac{l}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right)$.

Now, integrating the Fourier Cosine series term by term,

$$\int x dx = \int \frac{l}{2} dx + \sum_{n=1}^{\infty} A_n \int \cos\left(\frac{n\pi x}{l}\right) dx$$
$$\frac{x^2}{2} = C + \frac{l}{2}x + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)\frac{l}{n\pi}$$

Note that $C = 0$ (let $x = 0$). So we have that $\frac{x^2}{2} = \frac{l}{2}x + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)\frac{l}{n\pi}$. Let $x = \frac{l}{2}$, and thus we have:

$$\frac{l^2}{8} = \frac{l^2}{4} + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{2}\right)\frac{l}{n\pi}$$

Since $A_n = 0$ for n even, we set $n = 2k+1$ where k ranges from 0 to ∞:

$$-\frac{l^2}{8} = \sum_{k=0}^{\infty} \frac{-4l}{(2k+1)^2\pi^2} \sin\left(\frac{(2k+1)\pi}{2}\right)\frac{l}{(2k+1)\pi}$$
$$\to -\frac{l^2}{8} = \sum_{k=0}^{\infty} \frac{-4l^2}{(2k+1)^3\pi^3}(-1)^k$$
$$\to \frac{\pi^3}{32} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)^3}$$

Example 4 - Strauss 5.1.5

Consider the Fourier Sine series of $f(x) = x$ on $(0, l)$,

$$x = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)$$

where $A_n = \frac{(-1)^{n+1}2l}{n\pi}$.

Integrate the Fourier series (as an indefinite integral), and using $x = 0$, find the value to the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

Integrating term by term,

$$\int x dx = \sum_{n=1}^{\infty} A_n \int \sin\left(\frac{n\pi x}{l}\right) dx$$

$$\to \frac{x^2}{2} = C + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2l}{n\pi}(-1)\cos\left(\frac{n\pi x}{l}\right)\frac{l}{n\pi}$$

$$\to \frac{x^2}{2} = C + \sum_{n=1}^{\infty} \frac{(-1)^{n} 2l^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{l}\right)$$

Here the constant C must be $C = A_0 = \frac{1}{l}\int_0^l \frac{x^2}{2} dx = \frac{l^2}{6}$. So we have that $\frac{x^2}{2} = \frac{l^2}{6} + \sum_{n=1}^{\infty} \frac{(-1)^n 2l^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{l}\right)$. Now, let $x = 0$:

$$0 = \frac{l^2}{6} + \sum_{n=1}^{\infty} \frac{(-1)^n 2l^2}{n^2\pi^2}$$

$$\to -\frac{l^2}{6} = \sum_{n=1}^{\infty} \frac{(-1)^n 2l^2}{n^2\pi^2}$$

$$\to -\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\to \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

Bibliography

[1] Xiao Jie, Instructor's course notes (Quercus)

[2] W. Strauss, *Partial Differential Equations: An Introduction*, 2nd edition, Wiley