APM346 Tutorial 11

Partial Differential Equations

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1 Definitions

Recall the definition for the Fourier transform,

$$\mathcal{F} = \hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-\imath kx}\mathrm{d}x \tag{1}$$

and for the convolution operation,

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(x-\tau)g(\tau)\mathrm{d}\tau \tag{2}$$

Show that the Fourier transform is bounded.

$$|\hat{f}(k)| \leq \int_{-\infty}^{\infty} |f(x)||\exp(-\imath kx)|\mathrm{d}x \tag{3}$$

$$\leq \int_{-\infty}^{\infty} |f(x)|\mathrm{d}x \tag{4}$$

Show that the Fourier transform of a convolution is the product of the Fourier transforms.

$$(f \hat{\star} g)(k) = \int_{-\infty}^{\infty} (f \star g)(x)e^{-\imath kx}\mathrm{d}x \tag{5}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\tau)g(\tau)\mathrm{d}\tau e^{-\imath kx}\mathrm{d}x \tag{6}$$

$$\text{let } z = x - \tau \quad \mathrm{d}z = \mathrm{d}x \tag{7}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z)g(\tau)e^{-\imath k(z+y)}\mathrm{d}z\mathrm{d}\tau \tag{8}$$

$$= \int_{-\infty}^{\infty} f(z)e^{-\imath kz}\mathrm{d}z \int_{-\infty}^{\infty} g(\tau)e^{-\imath k\tau}\mathrm{d}\tau \tag{9}$$

$$= \hat{f}(k)\hat{g}(k) \tag{10}$$

Aside we can show that convolution is commutative by making the substitution $z = x - \tau$ and obtaining $f(z)$, $g(x-z)$.

2 Examples

Find the Fourier transform of

$$f(x) = e^{-a|x|}$$

with $a > 0$

$$\mathcal{F}(e^{-a|x|}) = \tag{11}$$

$$= \int_{-\infty}^{\infty} e^{-a|x|}e^{-\imath kx}\mathrm{d}x \tag{12}$$

$$= \int_{-\infty}^{0} e^{ax}e^{-\imath kx}\mathrm{d}x + \int_{0}^{\infty} e^{-ax}e^{-\imath kx}\mathrm{d}x \tag{13}$$

$$= \lim_{b\to-\infty} \left.\frac{e^{(a-\imath k)x}}{a-\imath k}\right|_{b}^{0} + \lim_{b\to\infty} \left.\frac{e^{(-a-\imath k)x}}{-a-\imath k}\right|_{0}^{b} \tag{14}$$

$$= \frac{1}{a-\imath k} + \frac{1}{-a-\imath k} \tag{15}$$

$$= \frac{2a}{a^2+k^2} \tag{16}$$

Show that the Fourier transform of a derivative is equal to multiplication by $\imath k$

$$\mathcal{F}(f'(x)) = \int_{-\infty}^{\infty} f'(x)e^{-\imath kx}\mathrm{d}x \tag{17}$$

$$= f(x)e^{-\imath kx}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(-\imath k)e^{-\imath kx}\mathrm{d}x \tag{18}$$

At this point, we require that f be integrable and that it vanishes in the limit of $|x| \to \infty$ to make the first term vanish. Then,

$$\mathcal{F}(f'(x)) = (\imath k)\int_{-\infty}^{\infty} f(x)e^{-\imath kx}\mathrm{d}x \tag{19}$$

$$= \imath k\hat{f}(k) \tag{20}$$

or more generally

$$\mathcal{F}\left(\frac{\mathrm{d}^n f}{\mathrm{d}x^n}\right) = (\imath k)^n\hat{f}(k) \tag{21}$$

Find the Fourier transform for a function shifted horizontally, $f(x-a)$.

$$\mathcal{F}(f(x-a)) = \int_{-\infty}^{\infty} f(x-a)e^{-\imath kx}\mathrm{d}x \tag{22}$$

$$= \int_{-\infty}^{\infty} f(z)e^{-\imath k(z+a)}\mathrm{d}z \tag{23}$$

$$= e^{-\imath ka}\int_{-\infty}^{\infty} f(z)e^{-\imath kz}\mathrm{d}z \tag{24}$$

$$= e^{-\imath ka}\hat{f}(k) \tag{25}$$

Find the Fourier transform for a function stretched horizontally, $f(ax)$ with nonzero a

Case $a > 0$

$$\mathcal{F}(f(ax)) = \int_{-\infty}^{\infty} f(ax)e^{-\imath kx}\mathrm{d}x \tag{26}$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} f(z)e^{-\imath kz/a}\mathrm{d}z \tag{27}$$

$$= \frac{1}{a}\hat{f}\left(\frac{k}{a}\right) \tag{28}$$

Case $a < 0$

$$\mathcal{F}(f(ax)) = \int_{-\infty}^{\infty} f(ax)e^{-\imath kx}\mathrm{d}x \tag{29}$$

$$= \frac{1}{a}\int_{\infty}^{-\infty} f(z)e^{-\imath kz/a}\mathrm{d}z \tag{30}$$

$$= -\frac{1}{a}\int_{\infty}^{\infty} f(z)e^{-\imath kz/a}\mathrm{d}z \tag{31}$$

$$= -\frac{1}{a}\hat{f}\left(\frac{k}{a}\right) \tag{32}$$

Both can be expressed as

$$\mathcal{F}(f(ax)) = \frac{1}{|a|}\hat{f}\left(\frac{k}{a}\right) \tag{33}$$

Find the Fourier transform for the door function,

$$\begin{cases} 1 & |x| < a \\ 0 & |x| \geq a \end{cases} \tag{34}$$

for $a \neq 0$

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-\imath kx} \mathrm{d}x \tag{35}$$

$$= \int_{-a}^{a} e^{-\imath kx} \mathrm{d}x \tag{36}$$

$$= \left. \frac{e^{-\imath kx}}{-\imath k} \right|_{-a}^{a} \tag{37}$$

$$= \frac{e^{-\imath ka} - e^{\imath ka}}{-\imath k} \tag{38}$$

$$= \sin(ka) \cdot \frac{2}{k} \tag{39}$$

3 ODEs

Let $f(x)$ be integrable, find a solution $y(x)$ to

$$y''(x) - y(x) = f(x) \tag{40}$$

Taking the Fourier transform on both sides, applying linearity and derivative rule

$$(ik)^2 \hat{y} - \hat{y} = \hat{f}(k) \tag{41}$$

$$\hat{y} = -\frac{\hat{f}(k)}{1 + k^2} \tag{42}$$

The next step requires knowledge of inverse Fourier transforms (recall this one from above results),

$$\hat{g}(k) = \frac{1}{1 + k^2} \tag{43}$$

$$g(x) = \frac{1}{2} e^{-|x|} \tag{44}$$

Now we can express $\hat{y}$ as a product of two transforms, which can be inverted into a convolution by the convolution theorem,

$$\hat{y} = -\hat{f} \cdot \hat{g} \tag{45}$$

$$y = -\frac{1}{2} \int_{-\infty}^{\infty} f(x - \tau) e^{-|\tau|} \mathrm{d}\tau \tag{46}$$