APM346 Tutorial 6

Partial Differential Equations

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We finish *Chapter 5 - Fourier Series* with a question from the course notes. At this point we have seen significantly more complicated examples with Fourier series; I chose to include this problem only for completeness since the solution was not provided.

Example 1

Find the Fourier Sine series of $f(x) = 1$ on $(0, \pi)$.

We have that

$$\begin{aligned} A_n &= \frac{2}{\pi}\int_0^\pi \sin(nx)dx \\ &= \frac{-2}{n\pi}\cos(nx)|_0^\pi \\ &= -\frac{2}{n\pi}(\cos(n\pi) - \cos(0)) \\ &= -\frac{2}{n\pi}((-1)^n - 1) \\ &= \frac{2}{n\pi}(1 - (-1)^n) \end{aligned}$$

which is 0 for n even and $\frac{4}{n\pi}$ for n odd. We conclude that

$$1 = \sum_{n \text{ odd}} \frac{4}{n\pi}\sin(nx)$$

ODE Review

The remainder of the tutorial consists of examples for the next two chapters of the course notes: *Chapter 6 - Separable Equation* and *Chapter 7 - First Order Linear Equation.*

Example 2

Solve the initial value problem

$$y' = 2y^2 + xy^2, \quad y(0) = 1$$

The equation $y' = y^2(2 + x)$ is separable, write it as $(\frac{1}{y^2})dy = (2 + x)dx$. Integrating the left hand side w.r.t. y and the right-hand side w.r.t. x yields the implicit solution

$$\frac{-1}{y} = 2x + \frac{x^2}{2} + c$$

Now, use the initial condition: $-1/1 = 0 + 0 + c$, so that $c = -1$. We thus have our solution:

$$y = \frac{1}{1 - 2x - \frac{x^2}{2}}$$

Example 3

Solve the given differential equation:

$$\frac{dy}{dx} = \frac{x - e^{-x}}{y + e^y}$$

This equation is separable, so write it as $(y + e^y)dy = (x - e^{-x})dx$. Integrating the left hand side w.r.t. y and the right-hand side w.r.t. x yields the implicit solution

$$\frac{y^2}{2} + e^y = \frac{x^2}{2} + e^{-x} + c$$

Since we are not given an initial condition, we can stop here.

Example 4

Find the solution of the initial value problem:

$$y' + 2y = xe^{-2x}, \quad y(1) = 0$$

The differential equation is in the standard form $\frac{dy}{dx} + p(x)y = q(x)$ where $p(x) = 2$ and $q(x) = xe^{-2x}$. The integrating factor is $\mu(x) = \exp\big(\int p(x)dx\big) = \exp(2x)$ and thus the general solution is

$$\begin{aligned} y &= \frac{1}{\mu}\left(\int \mu \cdot q + c\right) \\ &= \exp(-2x)\left(\int e^{2x}xe^{-2x}dx + c\right) \\ &= \exp(-2x)\left(\int xdx + c\right) \\ &= \exp(-2x)(x^2/2 + c) \end{aligned}$$

Now use the initial condition, so we have that $0 = e^{-2}(1/2 + c)$ and conclude that $c = \frac{-1}{2}$. So the solution of the IVP is

$$y(x) = \exp(-2x)(x^2/2 - 1/2)$$

Lets quickly do a sanity check. We first check the initial condition: $y(1) = \exp(-2)(1/2 -$

$1/2) = 0$. Now,

$$\begin{aligned}
y' + 2y &= e^{-2x}(x - x^2 + 1) + 2\left(\exp(-2x)(x^2/2 - 1/2)\right) \\
&= xe^{-2x} - x^2e^{-2x} + e^{-2x} + x^2e^{-2x} - e^{-2x} \\
&= xe^{-2x}
\end{aligned}$$

as desired.

Example 5

Solve the initial value problem

$$y' + \frac{2}{3}y = 1 - \frac{1}{2}x, \quad y(0) = y_0$$

The differential equation is in the standard form $\frac{dy}{dx} + p(x)y = q(x)$ where $p(x) = \frac{2}{3}$ and $q(x) = 1 - \frac{1}{2}x$. The integrating factor is $\mu(x) = \exp\left(\int p(x)dx\right) = \exp\left(\frac{2}{3}x\right)$ and thus the general solution is

$$\begin{aligned}
y &= \frac{1}{\mu}\left(\int \mu \cdot q + c\right) \\
&= \exp\left(-\frac{2}{3}x\right)\left(\int e^{\frac{2}{3}x}(1 - \frac{1}{2}x)dx + c\right) \\
&= \exp\left(-\frac{2}{3}x\right)\left(\frac{-3}{8}e^{\frac{2x}{3}}(2x - 7) + c\right) \\
&= \frac{(21 - 6x)}{8} + ce^{\frac{-2x}{3}}
\end{aligned}$$

Since we have initial condition $y(0) = y_0$, then $y_0 = \frac{21}{8} + c$ and we conclude that

$$y = \frac{(21 - 6x)}{8} + \left(y_0 - \frac{21}{8}\right)e^{\frac{-2x}{3}}$$

Example 6

Solve the given differential equation:

$$y' = \cos^2(x)\cos^2(2y)$$

The equation is separable, so write it as $\sec^2(2y)dy = \cos^2(x)dx$. Integrating the left hand side w.r.t. y and the right-hand side w.r.t. x yields the implicit solution

$$\frac{1}{2}\tan(2y) = \frac{1}{2}(x + \sin x \cos x) + c$$

or $\tan(2y) = x + \sin x \cos x + C$. Since we are not given an initial condition, we can stop here.

Example 7

Solve the given differential equation:

$$y' = \frac{x^2}{y}$$

The equation is separable, so write it as $ydy = x^2dx$. Integrating the left hand side w.r.t. y and the right-hand side w.r.t. x yields the implicit solution

$$\frac{y^2}{2} = \frac{x^3}{3} + c$$

Since we are not given an initial condition, we can stop here.

Bibliography

[1] Xiao Jie, Instructor's course notes (Quercus)

[2] W. Strauss, *Partial Differential Equations: An Introduction*, 2nd edition, Wiley

[3] W. Boyce, R.C. DiPrima, D.B Meade, *Elementary Differential Equations and Boundary Value Problems*, 11th edition, Wiley