MAT351 Tutorial 5

Partial Differential Equations

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1 Revisiting a question from Homework 2

Recall that in your second homework you were asked to do the following problem:

> [Strauss 2.1.7] If both ϕ and ψ are odd functions of x, show that the solution $u(x,t)$ of the wave equation is also odd in x for all t.

Every student who did this problem correctly followed a similar method - use d'Alembert's formula. To that end, I want to provide an alternate solution which does not make use of such a solution formula.

We have that $u(x,t)$ is solution of wave equation ($u_{tt} = c^2 u_{xx}$, $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$). The IVP for $u(-x,t)$ is

$$\begin{cases} u_{tt}(-x,t) = c^2 u_{xx}(-x,t) \\ u(-x,0) = \phi(-x) = -\phi(x) \\ u_t(-x,0) = \psi(-x) = -\psi(x) \end{cases}$$

We consider $u(x,t) + u(-x,t)$, and since the sum of solutions to wave equation is a solution, we have $u(x,t) + u(-x,t)$ solves

$$\begin{cases} [u(x,t) + u(-x,t)]_{tt} = c^2 [u(x,t) + u(-x,t)]_{xx} \\ u(x,0) + u(-x,0) = \phi(x) - \phi(x) = 0 \\ u_t(x,0) + u_t(-x,0) = \psi(x) - \psi(x) = 0 \end{cases}$$

A solution satisfying these initial conditions is $u(x,t) + u(-x,t) = 0$. Then by uniqueness, it is the only solution and we have shown that $u(-x,t) = -u(x,t)$ as desired.

Remark. One can use a similar argument and consider $u(-x,t) - u(x,t)$ to prove that if both ϕ and ψ are even functions of x, then $u(x,t)$ is also even in x for all t.

2 Diffusion on the half-line

Recall that the half-line Dirichlet problem

$$\begin{cases} v_t - kv_{xx} = 0 & \text{on } \{0 < x < \infty, 0 < t < \infty\} \\ v(x,0) = \phi(x) \\ v(0,t) = 0 \end{cases}$$

has solution formula (via method of odd extension)

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt} \right] \phi(y) dy$$

Now, let's try and do similar with the half-line Neumann problem

$$\begin{cases} w_t - kw_{xx} = 0 & \text{on } \{0 < x < \infty, 0 < t < \infty\} \\ w(x,0) = \phi(x) \\ w_x(0,t) = 0 \end{cases} \tag{2.1}$$

Remark. The Neumann boundary condition $w_x(0,t) = 0$ can be viewed as insulating the side end for $t > 0$ - there is no transfer of heat at $x = 0$.

Let $\phi_{\text{even}}(x)$ be the even extension of $\phi(x)$ to the whole real line, that is

$$\phi_{\text{even}}(x) = \begin{cases} \phi(x), & x \geq 0 \\ \phi(-x), & x \leq 0 \end{cases}$$

Then, the solution to

$$\begin{cases} u_t - ku_{xx} = 0 \\ u(x,0) = \phi_{\text{even}}(x) \end{cases}$$

on the whole line is given (recall Tutorial 3) by $u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp(-(x-y)^2/4kt) \phi_{\text{even}}(y) dy$. Then, our solution to (2.1) is the restriction $w(x,t) = u(x,t)$ for $x > 0$. We now just need to simplify the formula:

$$\begin{aligned}
w(x,t) &= \frac{1}{\sqrt{4\pi kt}}\left[\int_{-\infty}^{0} e^{-(x-y)^2/4kt}\phi(-y)dy + \int_0^\infty e^{-(x-y)^2/4kt}\phi(y)dy\right] \\
&= \frac{1}{\sqrt{4\pi kt}}\left[-\int_{\infty}^{0} e^{-(x+z)^2/4kt}\phi(z)dz + \int_0^\infty e^{-(x-y)^2/4kt}\phi(y)dy\right] \\
&= \frac{1}{\sqrt{4\pi kt}}\left[\int_0^\infty e^{-(x+z)^2/4kt}\phi(z)dz + \int_0^\infty e^{-(x-y)^2/4kt}\phi(y)dy\right] \\
&= \frac{1}{\sqrt{4\pi kt}}\int_0^\infty \left[e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt}\right]\phi(y)dy, \quad x > 0
\end{aligned}$$

Note that it only differs from the solution to the half-line Dirichlet problem by a minus sign. As an example, let's use this newly derived formula to solve (2.1) in the case where $\phi(x) = 1$. We have that

$$\begin{aligned}
w(x,t) &= \frac{1}{\sqrt{4\pi kt}}\int_0^\infty e^{-(x-y)^2/4kt}dy + \frac{1}{\sqrt{4\pi kt}}\int_0^\infty e^{-(x+y)^2/4kt}dy \\
&= \frac{-1}{\sqrt{\pi}}\int_{x/\sqrt{4kt}}^{-\infty} e^{-p^2}dp + \frac{1}{\sqrt{\pi}}\int_{x/\sqrt{4kt}}^{\infty} e^{-q^2}dq \\
&= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{x/\sqrt{4kt}} e^{-p^2}dp + \frac{1}{\sqrt{\pi}}\int_{x/\sqrt{4kt}}^{\infty} e^{-q^2}dq \\
&= \frac{1}{\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2}\left(\operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)+1\right)\right] + \frac{1}{\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2}\left(1-\operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)\right)\right] \\
&= 1
\end{aligned}$$

where $p = (x-y)/\sqrt{4kt}$ and $q = (x+y)/\sqrt{4kt}$.

Strauss 3.1.2

Solve $u_t = ku_{xx}$, $u(x,0) = 0$, $u(0,t) = 1$ on the half-line $0 < x < \infty$.

Let $v(x,t) := u(x,t) - 1$, then v satisfies $v_t = kv_{xx}, v(x,0) = -1, v(0,t) = 0$ on the half line. By the solution formula, we thus have:

$$\begin{aligned}
v(x,t) &= \frac{1}{\sqrt{4\pi kt}}\int_0^\infty \left[e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt}\right](-1)dy \\
&= \frac{-1}{\sqrt{\pi}}\int_{-x/\sqrt{4kt}}^{\infty} e^{-p^2}dp + \frac{1}{\sqrt{\pi}}\int_{x/\sqrt{4kt}}^{\infty} e^{-q^2}dq \\
&= \frac{-1}{\sqrt{\pi}}\int_{-x/\sqrt{4kt}}^{x/\sqrt{4kt}} e^{-p^2}dp \\
&= \frac{-2}{\sqrt{\pi}}\int_0^{x/\sqrt{4kt}} e^{-p^2}dp \\
&= -\operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)
\end{aligned}$$

where $p = (y-x)/\sqrt{4kt}$ and $q = (x+y)/\sqrt{4kt}$. Thus $u(x,t) = 1 - \operatorname{erf}(x/\sqrt{4kt})$.

Bibliography

[1] W. Strauss, *Partial Differential Equations: An Introduction*, 2nd edition, Wiley