APM346 Tutorial 7

Partial Differential Equations

Written By

David Knapik

University of Toronto
david.knapik@mail.utoronto.ca

July 11, 2023

1 Homogeneous Second Order Linear Equation with Constant Coefficients

General form

$$ay'' + by' + cy = 0$$

Recall from the course notes that there are three types of solutions. In this section I provide one example for each case (these examples can be found in [3] Chapter 3).

Example 1

Find the solution of the IVP $4y'' - 8y' + 3y = 0$, $y(0) = 2$, $y'(0) = \frac{1}{2}$.

The characteristic equation here is $4r^2 - 8r + 3 = 0$. Hence,

$$r = \frac{8 \pm \sqrt{64 - 16(3)}}{8} = \frac{8 \pm \sqrt{16}}{8} = \frac{8 \pm 4}{8}$$

so we have two distinct roots $r_1 = \frac{3}{2}$ and $r_2 = \frac{1}{2}$. The general solution is $y = c_1 e^{\frac{3}{2}t} + c_2 e^{\frac{1}{2}t}$. From the initial conditions,

$$\begin{cases} y(0) = c_1 + c_2 = 2 \\ y'(0) = \frac{3}{2}c_1 + \frac{1}{2}c_2 = \frac{1}{2} \end{cases}$$

and so we have that $c_1 = \frac{-1}{2}$ and $c_2 = \frac{5}{2}$. In conclusion,

$$y = -\frac{1}{2}e^{\frac{3}{2}t} + \frac{5}{2}e^{\frac{1}{2}t}$$

Example 2

Find the solution of the IVP $y'' - y' + \frac{y}{4} = 0$, $y(0) = 2$, $y'(0) = \frac{1}{3}$.

The characteristic equation here is $r^2 - r + \frac{1}{4} = 0$. Hence, $r = \frac{1 \pm \sqrt{1-1}}{2} = \frac{1}{2}$, a repeated root. The general solution is $y = c_1 e^{\frac{1}{2}t} + c_2 t e^{\frac{1}{2}t}$. From the initial conditions,

$$\begin{cases} y(0) = c_1 = 2 \\ y'(0) = \frac{1}{2}c_1 + c_2 = \frac{1}{3} \end{cases}$$

and so we have that $c_1 = 2$ and $c_2 = -\frac{2}{3}$. In conclusion,

$$y = 2e^{\frac{1}{2}t} - \frac{2}{3}te^{\frac{1}{2}t}$$

Example 3

Find the solution of the IVP $16y'' - 8y' + 145y = 0$, $y(0) = -2$, $y'(0) = 1$.

The characteristic equation here is $16r^2 - 8r + 145 = 0$. Hence,

$$r = \frac{8 \pm \sqrt{64 - 4(16)(145)}}{32} = \frac{8 \pm i\sqrt{9216}}{32} = \frac{8 \pm 96i}{32} = \frac{1}{4} \pm 3i$$

so we have complex conjugate roots. The general solution is $y = c_1 e^{\frac{1}{4}t}\cos(3t) + c_2 e^{\frac{1}{4}t}\sin(3t)$. From the initial conditions,

$$\begin{cases} y(0) = c_1 = -2 \\ y'(0) = \frac{1}{4}c_1 + 3c_2 = 1 \end{cases}$$

and so we have that $c_1 = -2$ and $c_2 = \frac{1}{2}$. In conclusion,

$$y = -2e^{\frac{1}{4}t}\cos(3t) + \frac{1}{2}e^{\frac{1}{4}t}\sin(3t)$$

2 Second Order Euler ODE

Consider $y = y(x)$. General form

$$ax^2y'' + bxy' + cy = 0$$

Note here that we have non-constant coefficients. To solve, we first guess solution $y(x) = x^n$ and then plug into the ODE and solve for n. There are three cases of interest:

1. Two distinct roots n_1 and n_2. Then general solution is $y = c_1x^{n_1} + c_2x^{n_2}$
2. One real repeated root n. Then general solution is $y = c_1x^n + c_2\ln(x)x^n$
3. Complex roots $\alpha \pm \beta i$. Then general solution is $y = c_1x^{\alpha}\cos(\beta\ln(x)) + c_2x^{\alpha}\sin(\beta\ln x)$

Example 4

Consider $y = y(x)$. Find the general solution of $x^2y'' + 7xy' + 9y = 0$.

Taking $y = x^n$, we have that:

$$\begin{aligned}
&x^2 n(n-1)x^{n-2} + 7xnx^{n-1} + 9x^n = 0\\
&n(n-1)x^n + 7nx^n + 9x^n = 0\\
&n(n-1) + 7n + 9 = 0 \quad \text{(divide by } x^n\text{)}\\
&n^2 - n + 7n + 9 = 0\\
&n^2 + 6n + 9 = 0\\
&(n+3)^2 = 0
\end{aligned}$$

and so we are in case 2; that is we have one real repeated root $n = -3$. We conclude that the general solution is

$$y = c_1 x^{-3} + c_2 \ln(x) x^{-3}$$

3

First Order Linear PDEs with Constant Coefficients

The primary tool of this section is the Chain Rule. In the course notes, the instructor often uses s and t for the change of variables. It does not matter what you choose to use as long as you are consistent and do not get lost with your notation. Paraphrasing a quote of McGill Professor Rustum Choksi (author of [4]), *you must stay in control of the notation and not let the notation take control of you.*

Example 5 - Strauss 1.2.8

Solve $au_x + bu_y + cu = 0$.

We shall proceed by changing variables; to that end let:

$$\zeta = ax + by \quad \text{and} \quad \eta = bx - ay$$

Then by the chain rule

$$u_x = \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = au_\zeta + bu_\eta$$

and

$$u_y = \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = bu_\zeta - au_\eta$$

Substituting back into the PDE we have that $(a^2+b^2)u_\zeta + cu = 0$. The general solution is $u(\zeta,\eta) = f(\eta)\exp\{\frac{-c\zeta}{a^2+b^2}\}$ (indeed $\frac{u_\zeta}{u} = \frac{-c}{a^2+b^2}$, then $\ln u = \frac{-c\zeta}{a^2+b^2} + g(\eta)$, $\implies$ $u =$

$f(\eta)\exp\{\frac{-c\zeta}{a^2+b^2}\})$. In terms of the original variables, we have that

$$u(x,y) = f(bx-ay)e^{\frac{-c(ax+by)}{a^2+b^2}}$$

Example 6 - Strauss 1.2.9

Solve the equation $u_x + u_y = 1$.

We shall proceed by changing variables; to that end let:

$$\zeta = x + y \quad \text{and} \quad \eta = x - y$$

Then by the chain rule

$$u_x = \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = u_\zeta + u_\eta$$

and

$$u_y = \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = u_\zeta - u_\eta$$

Hence substituting into the PDE we arrive at $2u_\zeta = 1$ or $u_\zeta = \frac{1}{2}$. The general solution is $u(\zeta,\eta) = \frac{1}{2}\zeta + f(\eta)$. In terms of the original variables, we conclude that

$$u(x,y) = \frac{1}{2}(x+y) + f(x-y)$$

Bibliography

[1] Xiao Jie, Instructor's course notes (Quercus)

[2] W. Strauss, *Partial Differential Equations: An Introduction*, 2nd edition, Wiley

[3] W. Boyce, R.C. DiPrima, D.B Meade, *Elementary Differential Equations and Boundary Value Problems*, 11th edition, Wiley

[4] R. Choksi, *Partial Differential Equations: A First Course*, AMS 2022