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Triangular fuzzification of random variables and power of distribution tests: Empirical discussion

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Abstract

A fuzzifying process of finitely valued random variables by means of triangular fuzzy sets is analyzed. Empirical studies show that if the random variable takes on a small number of different values, the one-sample test about the (fuzzy) mean of the fuzzified random variable is frequently more powerful than the classical test about the mean of the original random variable. This empirical conclusion is theoretically supported as follows: whenever the number of different values of a random variable X is up to 4, the mean of the fuzzified random variable captures the whole information on its distribution. As a consequence, the statistical test about the mean of the fuzzified random variable can be considered in fact as a goodness-of-fit test for the original random variable and, analogously, the J-sample test becomes a test for the equality of J distributions. Comparative simulation studies of these procedures with respect to other well-known methods are carried out. A real-life example illustrates the introduced methodology.

Keywords: Fuzzy representation; Finite distributions; Goodness-of-fit; Equality of distributions; Hypothesis testing

1. Introduction

Triangular and trapezoidal fuzzy numbers are often used to model many imprecise data (see, for instance, García et al., 2001; Di Lascio et al., 2002; Montenegro et al., 2004b; Coppi et al., 2006a). Statistics with imprecise data are frequently approximated by coding these data in terms of real numbers. However, it is usually much more realistic to identify them with fuzzy numbers (see Coppi et al., 2006b; Zadeh, 2006; Näther, 2006 for discussions about the integration of Fuzzy Sets, Probability and Statistics).

A real-life illustration of this assertion is found in the Forestry context. The forest fire risk index depends on the wind, temperature, and other factors, and it is usually quantified within the scale of the integer numbers from 0 to 4 (0 corresponding to the minimum risk, and 4 to the maximum one). In practice, each of the values 0 to 4 summarizes in a sense the "surrounding risks". In this respect, the extreme values 0 and 4 are essentially different from the intermediate ones. To reveal these differences, the intermediate index values could be described as symmetrical triangular functional data whereas the extreme ones could be identified with truncated triangular functions (see Fig. 1).

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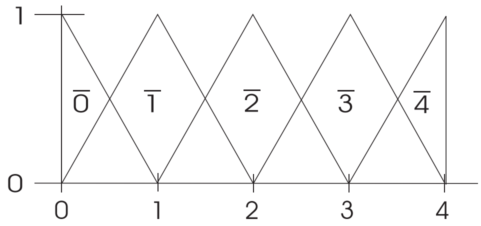


Fig. 1. Triangular fuzzy description of the forest risk index values.

The fuzzy values in Fig. 1 could be labelled as "the risk index is around i" ($i = 0, \dots, 4$). This representation can be viewed as a fuzzification of the integer values from 0 to 4. Consequently, the quantification process means a transformation of a real-valued random variable into a fuzzy-valued random element.

The fuzzy representation above can be immediately applied to finitely valued random variables. In this respect, it would be interesting to analyze the effect of the triangular fuzzy representation above when some recent tests about the means of fuzzy random variables are applied (see, for instance, Montenegro et al., 2000, 2004a,b; Körner, 2000; Gil et al., 2006; González-Rodríguez et al., 2006a).

For such an analysis, after recalling some preliminaries, the expected value of the triangular fuzzy representation of a random variable taking on up to four different values is proved to characterize the full distribution of the variable. As a consequence, the one-sample test for the mean of the fuzzy representation becomes a goodness-of-fit test for the original random variable, and the J-sample test of the equality of means becomes a test for the equality of J distributions. In order to compare the empirical behavior of the new testing methods with some of the most usual ones, some simulations are shown. Furthermore, an introductory discussion concerning the effect of the choice of the distance between fuzzy values into the power of the tests is made. An illustrative real-life example and some conclusions are finally included.

2. Preliminaries

Let $\mathscr{K}_c(\mathbb{R})$ be the class of the non-empty compact intervals and $\mathscr{F}_c(\mathbb{R})$ be the class of *fuzzy numbers* $V : \mathbb{R} \to [0, 1]$ so that the α-levels $V_\alpha \in \mathscr{K}_c(\mathbb{R})$ for all $\alpha \in [0, 1]\}$, where $V_\alpha = \{x \in \mathbb{R} \mid V(x) \geqslant \alpha\}$ if $\alpha \in (0, 1]$ and $V_0 = \mathrm{cl}\{x \in \mathbb{R} \mid V(x) > 0\}$. The space $\mathscr{F}_c(\mathbb{R})$ has a cone structure when it is endowed with the extensions of the Minkowski sum $U + V$ and the interval product by a scalar λV, which are defined so that

$$(U + V)_\alpha = U_\alpha + V_\alpha = \{u + v \mid u \in U_\alpha, v \in V_\alpha\}, \quad (\lambda V)_\alpha = \lambda V_\alpha = \{\lambda v \mid v \in V_\alpha\}$$

for all $U, V \in \mathscr{F}_c(\mathbb{R})$ and $\alpha \in [0, 1]$. This arithmetic agrees with the one based on Zadeh's extension principle (Zadeh, 1975).

For short, $\mathrm{Tri}(a, b, c)$ will denote a *triangular fuzzy number* with left-spread a, center b and right-spread c (that is, $[\mathrm{Tri}(a, b, c)]_0 = [b - a, b + c]$ and $[\mathrm{Tri}(a, b, c)]_1 = \{b\}$).

There are many different metrics defined on $\mathscr{F}_c(\mathbb{R})$ (see, for instance, Diamond and Kloeden, 1994; Körner and Näther, 2002). One of the most operative and versatile distances for statistical purposes is the (W, φ)-*generalized metric* (Bertoluzza et al., 1995), which is defined as

$$D_W^\varphi(U, V) = \sqrt{\int_{[0,1]} \left[\int_{[0,1]} [f_{U_\alpha}(\lambda) - f_{V_\alpha}(\lambda)]^2 \, \mathrm{d}W(\lambda) \right]^2 \mathrm{d}\varphi(\alpha)}$$

for all $U, V \in \mathscr{F}_c(\mathbb{R})$, where $f_K(\lambda) = \lambda \sup K + (1 - \lambda) \inf K$ for all $K \in \mathscr{K}_c(\mathbb{R})$. The normalized weighted measure W on $([0, 1], \mathscr{B}_{[0,1]})$ is assumed to be associated with a non-degenerate distribution. The normalized measure φ on $([0, 1], \mathscr{B}_{[0,1]})$ is assumed to be associated with an increasing distribution function, and plays a decisive role. For instance, φ could assign different weights to low and high values of the level α, which in a sense is equivalent to weight the level of "imprecision" or "consensus".

Let $(\Omega, \mathscr{A}, P)$ be a probability space. An $\mathscr{F}_c(\mathbb{R})$-valued *fuzzy random variable* in Puri and Ralescu's sense (1986) is a mapping $\mathscr{X} : \Omega \to \mathscr{F}_c(\mathbb{R})$ so that the α-level mappings $\mathscr{X}_\alpha : \Omega \to \mathscr{K}_c(\mathbb{R})$ (with $\mathscr{X}_\alpha(\omega) = (\mathscr{X}(\omega))_\alpha$) are random sets (or, equivalently, $\inf \mathscr{X}_\alpha$ and $\sup \mathscr{X}_\alpha$ are real-valued random variables) for all $\alpha \in [0, 1]$. If $\sup_{x \in \mathscr{X}_0} |x| \in L^1$, the

Aumann-type expected value (see Puri and Ralescu, 1986) is the unique fuzzy set $\widetilde{E}(\mathscr{X}) \in \mathscr{F}_c(\mathbb{R})$ satisfying that for all $\alpha \in [0, 1]$

$$(\widetilde{E}(\mathscr{X}))_\alpha = \text{Aumann's integralof } \mathscr{X}_\alpha = [E(\inf \mathscr{X}_\alpha), E(\sup \mathscr{X}_\alpha)].$$

On the basis of the results by Körner and Näther (2002) and Colubi et al. (2001, 2002), fuzzy random variables in Puri and Ralescu's sense (1986) can also be defined as Borel-measurable mappings with respect to the metric D_W^φ (see also López-Díaz and Ralescu, 2006). Thus, notions like the (induced) probability distribution of the fuzzy random variable, the independence of fuzzy random variables, can be formalized as usual in metric spaces.

Recently, asymptotic and bootstrap testing procedures about means of fuzzy random variables have been developed (see, for instance, Montenegro et al., 2000, 2001, 2004a,b; Gil et al., 2006; González-Rodríguez et al., 2006a). The null hypotheses in these tests are given by

- $H_0 : \widetilde{E}(\mathscr{X}) = V$ for a given $V \in \mathscr{F}_c(\mathbb{R})$ in the one-sample test;
- $H_0 : \widetilde{E}(\mathscr{X}) = \widetilde{E}(\mathscr{Y})$ in the two-sample test for two independent fuzzy random variables $\mathscr{X}$ and $\mathscr{Y}$;
- $H_0 : \widetilde{E}(\mathscr{X}_1) = \cdots = \widetilde{E}(\mathscr{X}_J)$ in the J-sample case of J independent fuzzy random variables $\mathscr{X}_1, \ldots, \mathscr{X}_J$.

The equalities in the null hypotheses above can be trivially expressed as equalities of real values by using the (W, φ)-metric.

3. Triangular fuzzy representation of a random variable

To illustrate the effects of the triangular fuzzy representation, the example introduced in Section 1 is again considered. The analysis of these effects will be focussed on the empirical power of the one-sample tests for both the real-valued risk index X and its triangular fuzzy representation, that will be denoted by $\widetilde{X}$.

Specifically, the interest is centered on comparing, by means of simulations, the power of the usual t-test $H_0 : E(X) = a \in \mathbb{R}$ and the analogous for FRVs (see, for instance, Montenegro et al., 2004b) $H_0 : \widetilde{E}(\widetilde{X}) = \widetilde{a}$ (where $\widetilde{a} = \text{Tri}(1, a, 1)$ is the corresponding triangular fuzzy representation of a).

In most of the simulations, the empirical power of the second test has been substantially greater than the first one. Consider that the real-valued random variable (forest risk index) is determined by the population vector of probabilities $\overrightarrow{p} = (0.01, 0.08, 0.35, 0.55, 0.01)$, whence the true population mean equals $E(X) = 2.47$. The percentages of rejections in 10,000 simulations of samples of size 1000 at the nominal significance level 0.05 have been computed by using different 'deviations' from the null hypothesis $H_0 : E(X) = 2.47$. These deviations have been determined by the parameter λ in the linear combinations $\lambda\overrightarrow{p} + (1 - \lambda)\overrightarrow{p}'$, where the vector $\overrightarrow{p}' = (0.15, 0.2, 0.2, 0.2, 0.25)$ determines a distribution with mean value $a' = 2.2$ (which is far enough from the true population mean). The empirical power associated with the fuzzy representation is clearly greater than the original one (see Fig. 2).

This empirical conclusion motivates the analysis of the triangular fuzzy representation of random variables, which is formalized in general as follows:

Definition 3.1. Let X be a finitely valued random variable with mass function $\{(x_i, p_i)\}_{i=1}^k$ with $x_i \uparrow$. The *triangular fuzzy representation of* X is the fuzzy random variable $\gamma_{\text{Tri}}(X)$ with mass function $\{(\widetilde{x}_i, p_i)\}_{i=1}^k$, where $\widetilde{x}_1 = \text{Tri}(0, x_1, 1)$, $\widetilde{x}_i = \text{Tri}(1, x_i, 1)$ if $1 < i < k$ and $\widetilde{x}_k = \text{Tri}(1, x_k, 0)$.

One of the uses of this representation is inspired by the following result:

Theorem 3.1. *Let X be a real-valued random variable with mass function $\{(x_i, p_i)\}_{i=1}^k$, $x_i \uparrow$, and $k \leqslant 4$, and let $\gamma_{\text{Tri}}(X)$ be its triangular fuzzy representation. Then, $\widetilde{E}(\gamma_{\text{Tri}}(X))$ characterizes the distribution of X.*

Proof. Assume that $k = 4$,and suppose that there is another random variable X^* with mass function $\{(x_i, p_i^*)\}_{i=1}^k$ so that $\widetilde{E}(\gamma_{\text{Tri}}(X)) = \widetilde{E}(\gamma_{\text{Tri}}(X^*))$. Since the involved outcomes are triangular fuzzy numbers, the expected values are also triangular fuzzy numbers determined by the left-spread, the center and the right-spread. Thus, the equality of these 'parameters' for the expected values $\widetilde{E}(\gamma_{\text{Tri}}(X))$ and $\widetilde{E}(\gamma_{\text{Tri}}(X^*))$ is derived, and since $\sum_{i=1}^4 p_1 = \sum_{i=1}^4 p_i^* = 1$, the

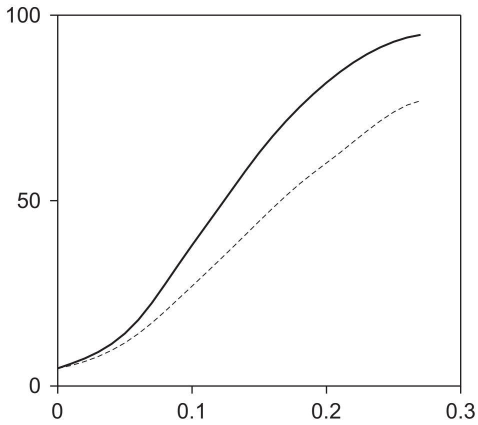


Fig. 2. Empirical power functions. Real case (dashed line)/fuzzy case (dark line). Percentage of rejections as a function of the 'deviation' from the null hypothesis.

following system is obtained:

$$\begin{cases} (p_2 - p_2^*) + (p_3 - p_3^*) + (p_4 - p_4^*) = 0, \\ x_1(p_1 - p_1^*) + x_2(p_2 - p_2^*) + x_3(p_3 - p_3^*) + x_4(p_4 - p_4^*) = 0, \\ (p_1 - p_1^*) + (p_2 - p_2^*) + (p_3 - p_3^*) = 0, \\ (p_1 - p_1^*) + (p_2 - p_2^*) + (p_3 - p_3^*) + (p_4 - p_4^*) = 0. \end{cases}$$

This system is determined compatible, and its unique solution is $p_i = p_i^*$ for $i \in \{1, 2, 3, 4\}$. The same arguments could be used to prove the result in case that $k < 4$. $\square$

The arguments in Theorem 3.1 could be also considered to prove similar results for many other fuzzy representations, as well as to check that, in case $k \geqslant 5$, $\widetilde{E}(\gamma_{\mathrm{Tri}}(X))$ does not characterize the distribution, because the system would be compatible but undetermined. In Definition 3.1 the non-null left and right-spreads of the triangular fuzzy representation were chosen to be equal to 1 for the sake of simplicity. Nevertheless, any other positive real number could be considered for these spreads. Moreover, these spreads could be different depending on the values of the considered variable, although certain constraints should be satisfied to ensure the preceding system is determined compatible.

Theorem 3.1 implies that if $k \leqslant 4$, $\widetilde{E}(\gamma_{\mathrm{Tri}}(\cdot))$ means a $[0, 1]$-valued characteristic function (instead of a complex-valued one) with an easy and intuitive interpretation. The result can be extended to characterize any real-valued distribution by means of other much more complex fuzzy representations although, in some sense, less versatile than the triangular one (see González-Rodríguez et al., 2006b).

On the basis of Theorem 3.1 one can conclude that for $k \leqslant 4$ the null hypotheses $\mathrm{H}_0 : E(X) = a$ and $\mathrm{H}_0 : \widetilde{E}(\gamma_{\mathrm{Tri}}(X)) = \mathrm{Tri}(1, a, 1))$ do not coincide. In fact, testing about the expected value of the triangular fuzzy representation of a random variable is equivalent to testing about the distribution of X. In this respect, the goodness-of-fit null hypothesis

$$\mathrm{H}_0 : \text{the distribution of } X \text{ is } \{(x_i, p_i)\}_{i=1}^4$$

is equivalent to $\mathrm{H}_0 : \widetilde{E}(\gamma_{\mathrm{Tri}}(X)) = \sum_{i=1}^4 p_i \widetilde{x}_i$.

In the same way, the null hypothesis corresponding to the equality of J independent distributions

$$\mathrm{H}_0 : X_1 \overset{\mathrm{d}}{\sim} \cdots \overset{\mathrm{d}}{\sim} X_J$$

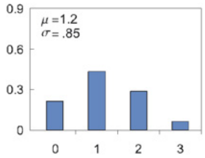
is equivalent to $\mathrm{H}_0 : \widetilde{E}(\gamma_{\mathrm{Tri}}(X_1)) = \cdots = \widetilde{E}(\gamma_{\mathrm{Tri}}(X_J))$.

The last assertions justify the empirical studies gathered in next two sections.

4. Goodness-of-fit test based on the triangular fuzzy representation. Empirical results

In this section the empirical behavior of the goodness-of-fit test based on the triangular fuzzy representation in contrast to some of the most usual 'omnibus' tests is discussed.

Table 1
Distribution $B(3, .4)$ and empirical behavior under H_0

	KS	CVM	χ^2	LR	T_λ	T_β
A_0^X	4.98	4.98	4.97	4.98	5.00	4.98

In this way, the null hypothesis to be tested will be $H_0 : \widetilde{E}(\gamma_{\text{Tri}}(X)) = \sum_{i=1}^{4} p_i \widetilde{x}_i$, which is equivalent to $H_0 : D_W^\varphi(\widetilde{E}(\gamma_{\text{Tri}}(X)), \sum_{i=1}^{4} p_i \widetilde{x}_i) = 0$. The test statistic is based on D_W^φ; since the choice of W is not too relevant in testing about means, it has been supposed to be the Lebesgue measure λ on [0, 1]. However, the choice of φ can be crucial, because this measure allows to weight different characteristics of the distribution. To illustrate the effect of the choice of φ, two options have been considered, namely, $\varphi = \lambda$ (that is, α-levels are equally weighted for all $\alpha \in [0, 1]$) and $\varphi = \beta =$ beta distribution $\beta(1, 10)$ (that is, the lower the α-level, the more strongly weighted). In the first case, the metric is more sensitive to "location changes", and hence the suggested statistic will be "more sensitive" to differences between the original mean values. In the second case, the metric is more sensitive to "shape" differences, and the statistic allocates much less importance to the original mean value than to other characteristics.

For the simulations, several kinds of random variables X have been considered to test H_0: 'the distribution A^X of X is A_0' for different distributions A_0 and A^X. We have developed the tests at a nominal significance level $\alpha = 0.05$ and we have simulated 10,000 samples of size $n = 30$. The new test statistic is given by

$$T_\varphi = D_\lambda^\varphi(\widetilde{E}(\gamma_{\text{Tri}}(X)), \widetilde{E}(\gamma_{\text{Tri}}(A_0)))$$

with $\varphi = \lambda$ and β. The comparison has been made with several classical statistics: Kolmogorov–Smirnov (KS), Cramer–Von Mises (CVM), χ^2 and likelihood ratio (LR). The distribution of all the statistics under H_0 has been approximated by Monte Carlo method.

To analyze the *tests sizes*, the true distribution A_0^X has been chosen to be equal to $A_0 = B(3, .4)$ (B denoting the binomial distribution). The considered distribution and the percentages of rejections in the simulations are shown in Table 1. Since the distributions of the test statistics have been approximated by Monte Carlo method, the empirical percentages of rejections are always similar to the nominal one.

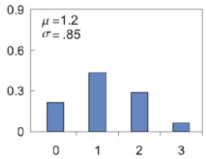
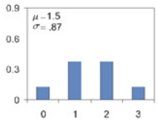
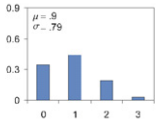
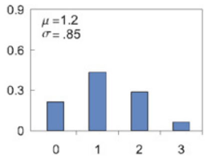
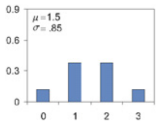
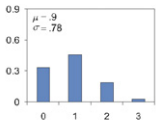
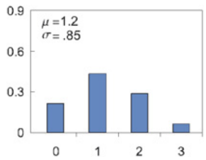
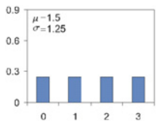
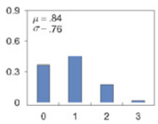
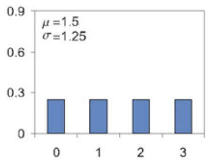
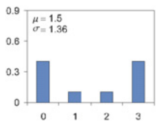
To analyze the *tests power*, different situations have been examined, all of them collected in Table 2. For the six first cases, the hypothetical distribution of X has been assumed to be a binomial (more precisely, $A_0 = B(3, .4)$), and different possibilities have been considered for the true distributions.

Firstly, binomial distributions with different success probabilities, for which new statistic based on $\varphi = \lambda$ seems to be quite convenient. This is because, in this case, differences between distributions are completely determined by the difference between the mean values. To illustrate this assertion, the results for the distributions $A_0 = B(3, .4)$, $A_1^X = B(3, .5)$ and $A_2^X = B(3, .3)$ are shown. This first comparison indicates that the best empirical results in both cases are obtained for T_λ, and also CVM shows a close behavior.

Secondly, combinations of independent Bernoulli distributions with different success probabilities have been considered. The conclusions are quite similar, because the mean value still preserves part of its importance. To illustrate this assertion, the results for $A_3^X = B(1, .4) + B(1, .5) + B(1, .6)$, and $A_4^X = B(1, .4) + B(1, .3) + B(1, .2)$ are shown.

Thirdly, the conclusions vary if we keep the hypothetical distribution A_0 above, but the true one A^X is another discrete distribution. In particular, in Table 2 the selected distributions have been $A_0 = B(3, .4)$, $A_5^X =$ uniform distribution over the same range, and A_6^X (see graphic in Table 2). For the uniform distribution A_5^X (in which the variability is much greater than that for the hypothetical distribution) the best results are obtained for the statistics which do not take into account the scale of the variable, namely, χ^2 and LR, and T_β shows a behavior better than T_λ. This is because T_λ is mainly focussed on the difference between the mean values, and the mean values are less representative when variability is high, whereas T_β pays attention to other characteristics. When the true distribution is A_6^X, T_λ shows the highest empirical power, because the mean is more representative for this distribution. Although A_6^X is visually quite similar to A_4^X and A_2^X, the probabilities of the last value are, in relative terms, rather different (for instance, the

Table 2
Empirical goodness-of-fit tests power

Hypothetical distribution of X (A_0)	True distribution of X (A^X)		Empirical percentage of rejections					
			KS	CVM	χ^2	LR	T_λ	T_β
		A_1^X	30.66	45.08	37.64	32.84	49.25	23.20
		A_2^X	42.92	43.68	32.08	35.25	47.96	42.13
		A_3^X	30.97	44.60	37.63	33.67	48.76	25.13
		A_4^X	43.89	46.33	29.20	34.74	49.85	37.42
		A_5^X	57.25	65.33	87.92	80.43	50.38	55.67
		A_6^X	59.55	62.87	44.79	50.84	66.59	51.99
		A_7^X	56.75	47.00	84.85	83.96	14.20	86.81

probability of this value in A_6^X is near to a half that in A_4^X), and this fact contributes to the remarkable differences in power of all the tests.

Finally, if the distributions do not belong to the usual parametric classes, the results depend on the importance of the mean value in comparing the hypothetical and the true distributions. For instance, as an extreme case the true distribution of X has been assumed to be A_7^X, and the hypothetical one has been assumed to be the uniform distribution A_0^* (see the last row in Table 2). Both distributions have the same mean value, whence the mean is irrelevant to detect differences between the distributions. In this case, the behavior of T_λ is quite wrong, because (as we have pointed out repeatedly) it is mainly focussed on the mean value, whereas the best empirical results are obtained for T_β, which is centered on other characteristics of the distributions.

5. Test of equality of two distributions based on the triangular fuzzy representation. Empirical results

Let X and Y be two real-valued random variables with distributions B^X and B^Y, respectively. The aim is now to compare empirically several methods to test $\mathrm{H}_0 : B^X = B^Y$ (i.e., $\mathrm{H}_0 : D_W^\varphi(\widetilde{E}(\gamma_{\mathrm{Tri}}(B^X)), \widetilde{E}(\gamma_{\mathrm{Tri}}(B^Y))) = 0$) at a nominal significance level $\alpha = 0.05$. For this purpose, 10,000 samples of sizes $n_1 = n_2 = 30$ have been simulated.

Table 3
Test sizes for distributions B_0 (on the left) and B_1 (on the center)

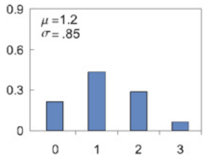
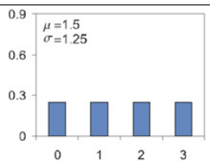
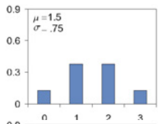
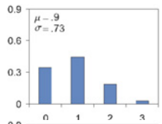
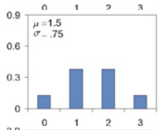
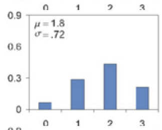
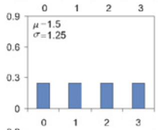
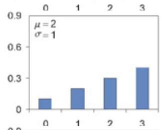
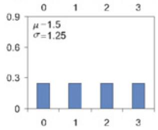
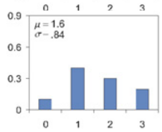
			χ^2	LR	T_λ	T_β
						
		$B_0^X = B_0^Y = B_0$	4.58	6.69	4.67	5.24
		$B_1^X = B_1^Y = B_1$	5.25	6.09	5.02	4.76

Table 4
Empirical equality of two distributions tests power

Distribution of X (B^X)	Distribution of Y (B^Y)		Empirical percentage of rejections χ^2	LR	T_λ	T_β
		B_2^X and B_2^Y	58.50	64.08	76.56	55.57
		B_3^X and B_3^Y	15.63	19.74	26.27	11.34
		B_4^X and B_4^Y	28.87	31.93	43.91	31.60
		B_5^X and B_5^Y	29.25	32.17	9.21	35.06

In this case, the new statistic is given by

$$T_\varphi = D_\lambda^\varphi(\widetilde{E}(\gamma_{\text{Tri}}(X)), \widetilde{E}(\gamma_{\text{Tri}}(Y))),$$

where φ is either λ or β. In this comparison the Monte Carlo method cannot be applied to approximate the test distributions under H_0. Thus, the well-known asymptotic limit distributions of χ^2 and the LR test for finitely valued variables will be used, and the bootstrap method will be applied to approximate the test distribution based on the new statistic.

To analyze the *tests sizes* two cases have been examined, namely, the one in which $B_0^X = B_0^Y = B_0 = B(3, .4)$ (graphic on the left in Table 3) and the one in which $B_1^X = B_1^Y = B_1$, B_1 being a uniform distribution (graphic on the center in Table 3). The worst results in Table 3 correspond to the LR test, especially in the first case, due to the limitations regarding the number of observations per cell.

To analyze the *tests power*, different situations have been examined, all of them collected in Table 4. As to be expected, the results for binomial cases are good for T_λ and depend on the difference between the mean values, while for other finite distributions, the results depend on the importance of the mean on the difference of the distributions.

Table 5
Frequency table of the erosion risk caused by the forest fires

	Before 2000	After 2000
Low	5519	6836
Medium	2646	4752
High	523	1221

6. Real-life example

The forest fires were especially severe in Asturias (Spain) in the 90s. For this reason the policy concerning the protection against the fire have changed since 2000. In Marquínez et al. (2006) a deep comparative study of the forest fires before and after 2000 was carried out in order to verify whether the changes have been or not appropriate. One of the variables in the study has been the erosion risk caused by the fire. The data collected by the fire brigades from 1988 to 2004 are gathered in Table 5.

The aim is now to analyze the erosion risk before and after 2000. This has been studied from a descriptive approach in Marquínez et al. (2006). However, if we assume that the forest fires arise at random, data in Table 5 can be viewed as samples and managed from an inferential perspective.

The values of the erosion risk can be identified with its order, namely, 1="low", 2="medium", 3="high", and represented by the triangular fuzzy numbers in Definition 3.1. If $\varphi = \lambda$, the two-sample test analyzed in Section 5 leads to a p-value $= 0.000$. This can be interpreted in two ways. On one hand, since the fuzzified data seem a suitable representation of the original categories, the test of equality of means allows to conclude that the fuzzy mean risk was different before and after 2000. On the other hand, since the number of values of the variable is $k = 3$, Theorem 3.1 assures that the preceding test is also a test of the *equality of the distributions*, and from this point of view, it can be concluded that the distribution of the risk was different before and after 2000. Thus, from both viewpoints a significant difference has been detected.

7. Conclusions

The fuzzy representation of random variables introduced in this paper leads to an integral methodology in testing about distributions which is quite easy to use. Although this representation can be modified to characterize any kind of random variable, the one considered here is very intuitive and straightforward to use; it allows also to illustrate the behavior of the statistics with respect to the weighted distance in a very simple way. In this respect, the distance between fuzzy sets has been shown to induce a versatile distance between distributions, since the derived test statistics can focus either on the mean or on other parameters. The introductory empirical studies show a good practical behavior, although further theoretical/empirical comparisons to other relevant techniques depending on concrete problems should be developed.

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References

Bertoluzza, C., Corral, N., Salas, A., 1995. On a new class of distances between fuzzy numbers. Math. Soft Comput. 2, 71–84.

Colubi, A., Domínguez-Menchero, J.S., López-Díaz, M., Ralescu, D.A., 2001. On the formalization of fuzzy random variables. Inform. Sci. 133, 3–6.

Colubi, A., Domínguez-Menchero, J.S., López-Díaz, M., Ralescu, D.A., 2002. A $D_E[0, 1]$-representation of random upper semicontinuous functions. Proc. Amer. Math. Soc. 130, 3237–3242.

Coppi, R., D'Urso, P., Giordani, P., Santoro, A., 2006a. Least squares estimation of a linear regression model with LR fuzzy response. Comput. Statist. Data Anal. 51, 267–286.

Coppi, R., Gil, M.A., Kiers, H.A.L., 2006b. The fuzzy approach to statistical analysis. Comput. Statist. Data Anal. 51, 1–14.

Diamond, P., Kloeden, P., 1994. Metric Spaces of Fuzzy Sets. World Scientific, Singapore.

DiLascio, L., Gisolfi, A., Albunia, A., Galarti, G., Meschi, F., 2002. A fuzzy-based methodology for the analysis of diabetic neuropathy. Fuzzy Sets and Systems 129, 203–228.

García, D., Lubiano, M.A., Alonso, M.C., 2001. Estimating the expected value of fuzzy random variables in the stratified random sampling from finite populations. Inform. Sci. 138, 165–184.

Gil, M.A., Montenegro, M., González-Rodríguez, G., Colubi, A., Casals, M.R., 2006. Bootstrap approach to the multi-sample test of means with imprecise data. Comput. Statist. Data Anal. 51, 148–162.

González-Rodríguez, G., Montenegro, M., Colubi, A., Gil, M.A., 2006a. Bootstrap techniques and fuzzy random variables: synergy in hypothesis testing with fuzzy data. Fuzzy Sets and Systems 157, 2608–2613.

González-Rodríguez, G., Colubi, A., Gil, M.A., 2006b. A fuzzy representation of random variables: an operational tool in exploratory analysis and hypothesis testing. Comput. Statist. Data Anal. 51, 163–176.

Körner, R., 2000. An asymptotic α-test for the expectation of random fuzzy variables. J. Statist. Plann. Inference 83, 331–346.

Körner, R., Näther, W., 2002. On the variance of random fuzzy variables. In: Bertoluzza, C., Gil, M.A., Ralescu, D.A. (Eds.), Statistical Modeling, Analysis and Management of Fuzzy Data. Physica-Verlag, Heidelberg, pp. 22–39.

López-Díaz, M., Ralescu, D.A., 2006. Tools for fuzzy random variables: embeddings and measurabilities. Comput. Statist. Data Anal. 51, 109–114.

Marquínez, J., Álvarez-García, M.A., Lobo, T., Recondo, C., Colubi, A., González-Rodríguez, G., Menéndez-Duarte, R., Fernández-Menéndez, S., Suárez-Seoane, S., Woznia, E., 2006. Incendios en Asturias: análisis de los últimos 20 años. KRK. Oviedo.

Montenegro, M., Casals, M.R., Gil, M.A., 2000. Asymptotic comparison of two fuzzy expected values. In: Proceedings of the JCIS 2000—Seventh FT&T Conference, pp. 150–153.

Montenegro, M., Casals, M.R., Lubiano, M.A., Gil, M.A., 2001. Two-sample hypothesis tests of means of a fuzzy random variable. Inform. Sci. 133, 89–100.

Montenegro, M., González-Rodríguez, G., Gil, M.A., Colubi, A., Casals, M.R., 2004a. Introduction to ANOVA with fuzzy random variables. In: López-Díaz, M., Gil, M.A., Grzegorzewski, P., Hryniewicz, O., Lawry, J. (Eds.), Soft Methodology and Random Information Systems. Springer, Berlin, pp. 487–494.

Montenegro, M., Colubi, A., Casals, M.R., Gil, M.A., 2004b. Asymptotic and bootstrap techniques for testing the expected value of a fuzzy random variable. Metrika 59, 31–49.

Näther, W., 2006. Regression with fuzzy random data. Comput. Statist. Data Anal. 51, 235–252.

Puri, M.L., Ralescu, D.A., 1986. Fuzzy random variables. J. Math. Anal. Appl. 114, 409–422.

Zadeh, L.A., 1975. The concept of a linguistic variable and its application to approximate reasoning, Part 1. Inform. Sci. 8, 199–249; Part 2. Inform. Sci. 8, 301–353; Part 3. Inform. Sci. 9, 43–80.

Zadeh, L.A., 2006. Generalized theory of uncertainty (GTU)-principal concepts and ideas. Comput. Statist. Data Anal. 51, 15–46.