- Angles are sometimes θ_1, θ_2 OR ρ_1, ρ_2 in various text books.

Let us consider three tasks

- T1: Given an arbitrary trajectory of the end effector (given (x, y) as a fn of time), make the robot follow the trajectory.

- T2: Given a location on a wall, make the robot touch the wall at that location and apply a pre-specified (constant) force at that location.

- T3: Make the robot behave like a virtual spring connected from ~~the~~ E to a given pt. (x_0, y_0).

Now,

$$x = l_1 \cos q_1 + l_2 \cos q_2$$
$$y = l_1 \sin q_1 + l_2 \sin q_2$$

OR

$$\left.\begin{aligned} x &= l_1 cq_1 + l_2 cq_2 \\ y &= l_1 sq_1 + l_2 sq_2 \end{aligned}\right\} \quad (1)$$

Differentiating (1), we get

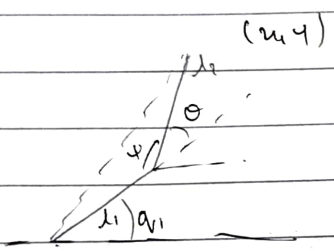
$$\dot{x} = -l_1 sq_1 \cdot \dot{q}_1 - l_2 sq_2 \dot{q}_2$$
$$\dot{y} = l_1 cq_1 \cdot \dot{q}_1 + l_2 cq_2 \dot{q}_2$$

$$\text{End effector velocity} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1 sq_1 & -l_2 sq_2 \\ l_1 cq_1 & l_2 cq_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \quad (2)$$

We will really need the ~~se~~ inverse relationship, Given x, y we need to be able to find q_1, q_2.

- Option 1: Solve numerically.

- Option 2: Derive a closed-form expression.
  - Hard in general.
  - Multiple solutions.



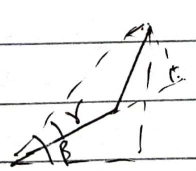
$$x^2 + y^2 = l_1^2 + l_2^2 + 2l_1 l_2 \cos\theta$$ (Using cosine rule + switching to acute angle).

$$\theta = \cos^{-1}\left(\frac{x^2+y^2-l_1^2-l_2^2}{2l_1 l_2}\right)$$

$$q_1 = \beta - \gamma = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{l_2 \sin\theta}{l_1 + l_2 \cos\theta}\right)$$

$$q_2 = q_1 + \theta.$$

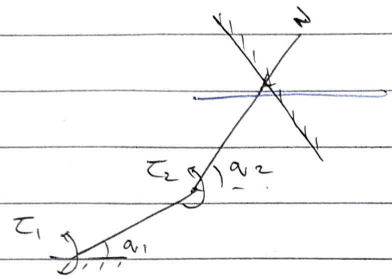
(3)



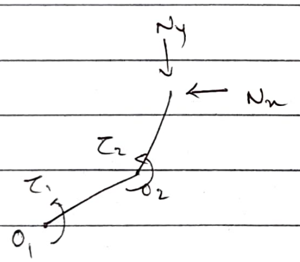
This is first level answer to T_1.

- We will later start using the notation x_d and y_d. (and q_{1d} & q_{2d}) here for desired values. (they are not necessarily actual values).

Task 2:



FBD of entire robot.



Forces applied by the manipulator.

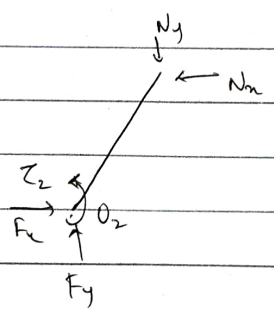
$F_x = -N_x$

$F_y = -N_y$

Neglect gravity.

Static eqb^m.

FBD of each link separately.

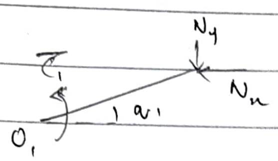


$\Sigma M_{O_2} = 0$

$\Rightarrow$ C.C.W is +ve

$$+N_y l_2 \cos q_2 - N_x l_2 sq_2 = +\tau_2$$

FBD of link 1



$$\left.\begin{aligned} N_y l_1 cq_1 - N_x l_1 sq_1 &= \tau_1 \\ N_y l_2 cq_2 - N_x l_2 sq_{12} &= \tau_2 \end{aligned}\right] - (4)$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} l_1 sq_1 & l_1 cq_1 \\ -l_2 sq_2 & l_2 cq_2 \end{bmatrix} \begin{bmatrix} N_x \\ N_y \end{bmatrix}$$

(3) along with (4) solves T_2.

T_3 and next level answer to T_1:

Need to understand dynamics.

Lagrange's Equation.

Lagrangian

$L = K - V$

k = kinetic energy
V - potential energy.

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) - \frac{\partial L}{\partial q_i} = Q_i'} \quad (5).$$

Q_i' are generalised forces derived using principle of virtual work.

$$K = \underbrace{\frac{1}{2}\left(\frac{1}{3} m_1 l_1^2\right)\dot q_1^2}_{\text{pure rotation of } L_1} + \underbrace{\frac{1}{2}\left(\frac{1}{12} m_2 l_2^2\right)\dot q_2^2 + \frac{1}{2} m_2 v_{c_2}^2}$$

where, : $v_c^2 = (l_1 \dot q_1)^2 + \left(\frac{l_2}{2}\dot q_2\right)^2 + 2 l_1 \dot q_1 \frac{l_2}{2}\dot q_2 \cos(q_2 - q_1)$

$$V = m_1 g \frac{l_1}{2} sq_1 + m_2 g \left(l_1 sq_1 + \frac{l_2}{2} sq_2\right).$$

$$\frac{1}{3} m_1 l_1^2 \ddot q_1 + m_2 l_1^2 \ddot q_1 + \frac{m_2 l_1 l_2}{2} \ddot q_2 \cos(q_2 - q_1) - \frac{m_2 l_1 l_2}{2} \dot q_2 (\dot q_2 - \dot q_1)$$
$$\sin(q_2 - q_1) + m_1 g \frac{l_1}{2} cq_1 + m_2 g l_1 cq_2 = \tau_1 - (i)$$

$$\frac{1}{3} m_2 l_2^2 \ddot q_2 + m_2 \frac{l_2^2}{4} \ddot q_2 + \frac{m_2 l_1 l_2}{2} \ddot q_1 \cos(q_2 - q_1) - \frac{m_2 l_1 l_2}{2} \dot q_1 (\dot q_2 - \dot q_1)$$
$$\sin(q_2 - q_1) + m_2 g \frac{l_2}{2} sq_2 = \tau_2.$$

Save this, need later.

Next, we note that (4) is valid for any forces F_x, F_y at end effector. (not just wall forces).

want,

$$F_x = kx \quad = \quad \begin{bmatrix} F_x = k_x\,(x - x_0) \\ F_y = k_y\,(y - y_0). \end{bmatrix}$$
$$F_y = ky$$

from (1),

$$F_x = k\,(l_1 cq_1 + l_2 cq_2)$$
$$F_y = k\,(l_1 sq_1 + l_2 sq_2).$$

from (4)

$\Rightarrow$ • $k\,(l_1 sq_1 + l_2 sq_2)\, l_2 cq_2 - k\,(l_1 cq_1 + l_2 cq_2)\, l_2 sq_2 = \tau_{2s}$

• $k\,(l_1 sq_1 + l_2 sq_2)\, l_1 cq_1 - k\,(l_1 cq_1 + l_2 cq_2)\, l_1 sq_1 = \tau_{1s}$

Set motor torques to be $\tau_1 + \tau_{1s}$ and $\tau_2 + \tau_{2s}$, respectively!
Answer to T3.

• Another way to tackle T1, is to solve for $q_1 d$ & $q_2 d$ from (3)

$$\Downarrow$$

$$\dot{q}_{1d},\ \dot{q}_{2d},\ \ddot{q}_{1d},\ \ddot{q}_{2d}$$

$$\downarrow$$

$$\tau_1,\ \tau_2 \text{ from (6).}$$

• work better when dynamic effects are significant.
• Still need feedback control.