Lecture 1 Basic Concepts of Probability and Inference

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E6100 Introduction to Bayesian Statistics

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What Is the Course About?

- Introduce Bayesian inferential methods & develop hands-on skills for Python data science
- Why Bayesian paradigm? Handle sophisticated models & uncertainty in decision making
- Main references
  - required: Greenberg (2008), "*Introduction to Bayesian Econometrics*"
  - optional: Geweke (2005), "*Contemporary Bayesian Econometrics and Statistics*"
- Homework production
  - LaTeX typesetting: Overleaf
  - Python programming: GitHub Codespaces

The Road Ahead...

1. Probability
2. Prior, Likelihood, and Posterior

Frequentist v.s. Bayesian

Probability axioms

1. $0 \leq \mathbb{P}(A) \leq 1 \quad$ for any event A
2. $\mathbb{P}(A) = 1 \quad$ if event A represents logical truth
3. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) \quad$ for disjoint events A and B
4. $\mathbb{P}(A|B) = \mathbb{P}(A \cap B)/\mathbb{P}(B) \quad$ (conditional probability)

- Satisfied by any assignment of probabilities
  - frequentists assign probabilities to events describing outcome of *repeated* experiment
  - Bayesians assign 'subjective' probabilities to uncertain events [de Finetti's (1990) coherency principle]
- How likely it rains tomorrow?

The Road Ahead...

1. Probability
2. Prior, Likelihood, and Posterior

Prior, Likelihood, and Posterior

Bayes theorem

$$\pi(\theta|y) = \frac{f(y|\theta)\pi(\theta)}{m(y)} \propto f(y|\theta)\pi(\theta)$$

- Bayesians treat parameters θ as random variables & data $y = [y_1, \ldots, y_n]'$ as given
  - start with prior density $\pi(\theta)$
  - update by likelihood function $f(y|\theta)$
  - posterior density $\pi(\theta|y)$ proportional to prior $\times$ likelihood
  - marginal likelihood $m(y) = \int f(y|\theta)\pi(\theta)d\theta$

Coin-Tossing Example

- Likelihood function
  - one toss (Bernoulli): $\mathbb{P}(Y_i = 1) = \theta = 1 - \mathbb{P}(Y_i = 0)$

$$f(y_i|\theta) = \theta^{y_i}(1-\theta)^{1-y_i}$$

  - n independent tosses

$$f(y_1, \ldots, y_n|\theta) = \theta^{\sum y_i}(1-\theta)^{n-\sum y_i}$$

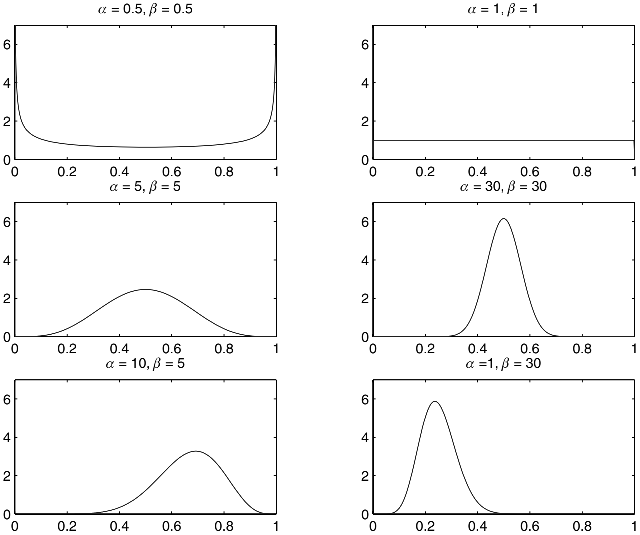
- (Conjugate) beta prior: $\theta \sim \mathcal{B}(\alpha, \beta)$

$$\pi(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}, \quad 0 \le \theta \le 1, \quad \alpha, \beta > 0$$

- Beta posterior: $\theta|y \sim \mathcal{B}(\alpha + \sum y_i, \beta + n - \sum y_i)$

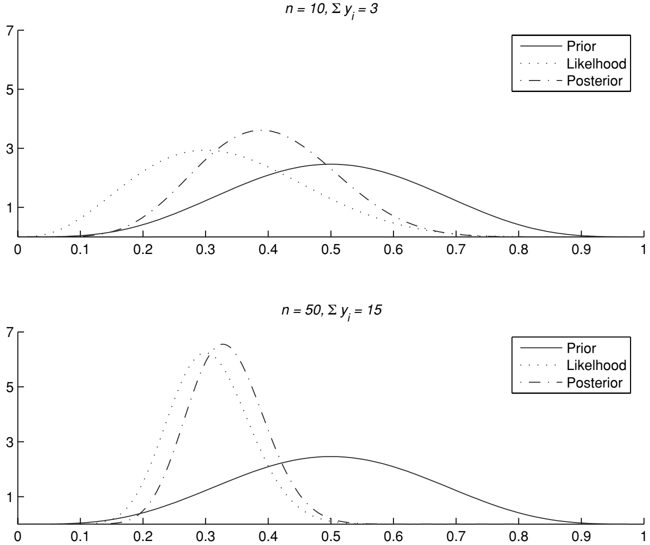
$$\pi(\theta|y) \propto \theta^{\alpha+\sum y_i - 1}(1-\theta)^{\beta+n-\sum y_i - 1}$$

Hyperparameters



- Shape of beta: $\mathbb{E}(\theta) = \frac{\alpha}{\alpha+\beta}$, $\mathbb{V}(\theta) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Sample Size



- $\mathbb{E}(\theta|y) = \frac{\alpha+\beta}{\alpha+\beta+n}\mathbb{E}(\theta) + \frac{n}{\alpha+\beta+n}\bar{y} \rightarrow_{n\rightarrow\infty} \bar{y}$ (MLE)

References

- de Finetti (1990), "Theory of Probability", John Wiley & Sons