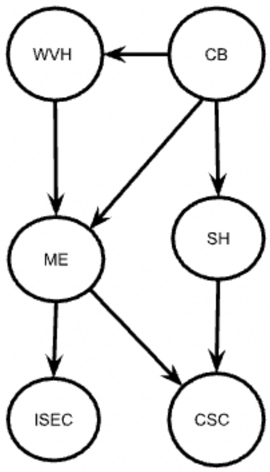
CS1800 Fall 2021 Exam 1 Solutions

Problem 1

The graph below is Northeastern's campus. This is a *dependency graph*. An edge from a to b indicates that a must be visited before b.

Nate wants to visit all the buildings in the graph, but Laney has secretly gone all over campus putting in force fields that prevent him from wandering freely around. These force fields apply only to Nate, and they impose the following constraints (represented by the edges in the graph):



PART A Nate needs to start at a building that has no dependencies. What is the only valid place for him to start his campus tour?

Solution: Cabot Center. There's a cafe nearby for his morning coffee!

PART B For the paths listed below, identify which Nate can take and which he can't. For invalid paths, identify the first pair of vertices that is broken.

- CB, SH, WVH, ME, CSC, ISEC
  Solution: valid
- CB, ME, SH, WVH, CSC, ISEC
  Solution: invalid, can't visit ME before WVH
- CB, WVH, ME, ISEC, SH, CSC
  Solution: valid
- CB, SH, CSC, WVH, ME, ISEC
  Solution: invalid, can't visit CSC before ME

Problem 2

Simplify your solution to a single value for each subquestion.

E (ze/hir) is coordinating hir outfit for an upcoming interview. The outfit will comprise a shirt, pants, and shoes. Hir closet has the following:

- 3 red shirts, 2 blue shirts, 2 black shirts
- 2 blue jeans, 1 black jeans, 2 white slacks
- 2 pairs of black shoes, 3 pairs of white shoes

E considers all items of clothing distinct, even if they are the same type and color. For instance, ze can tell all hir red shirts apart from each other.

PART A) How many outfits can E make?

Solution: E must choose 1 shirt AND 1 pair of pants/jeans AND 1 pair of shoes.

- Shirts: 3 red + 2 blue + 2 black = 7 shirts
- Pants: 2 blue jeans + 1 black jeans + 2 white slacks = 5 pants
- Shoes: 2 black + 3 white = 5 shoes

These are combined by the product rule to give us 7*5*5= 175 outfits.

PART B) How many outfits can ze make with a black shirt, or with black jeans, or with black shoes (inclusively)?

Solution:

Subtraction Rule

We know the total number of outfits, so let's find the invalid ones. Those invalid ones are the ones without any black clothing items. That is a grand total of (3+2)*(2+2)*3 = 60 outfits. The final answer is 175 - 60 = 115 total outfits.

Problem 3

The game of Mastermind requires a "target" consisting of a sequence of four pegs, each of which may be of any of six colors: Red, Orange, Yellow, Green, Blue, and Violet. Order matters and repetition of colors is allowed.

Laney is playing a version of this game with evil genius Nate Derbinsky. The target is constructed randomly, by a neutral third party. Nate thinks he can take all of Laney's money and offers her the following terms:

- Laney wins \$10 if the target has exactly two pegs of the same color and exactly two pegs of different colors (e.g., the target might be RRYG or RYRG but it cannot be RRRY or RRYY).
- She loses \$30 if the target has at least three pegs of same color.

After the target is revealed, Laney collects her money or pays up to Nate. Then everything is re-set so that each round begins anew.

Let X be a random variable associated with the amount of money Laney has after five rounds of play. What's the expected value of X?

Solution: We'll use our expected value formula here, where we want for each round $E[X_i] =$ (probability of exactly two of the same color) * \$10 + (probability of at least three of the same color) * -\$30.

What are those probabilities? They'll have the same sample space, so let's start there. The sample space is all possible codes when repetition is allowed and order matters, so we get: $|S| = 6^4 = 1296$. Event spaces? Let's break them down:

Exactly two of the same color: What's $|E|$?

- Choose the color that gets repeated: 6 ways to do that
- Choose the two positions where that color goes: $\binom{4}{2} = 6$ ways to do that.
- Choose the colors for the remaining positions: 5 options for the first one, 4 for the second because they are required to be two different colors. 5 * 4 = 20
- Total: 6 * 6 * 20 =720

$$Pr(E) = \frac{|E|}{|S|} = \frac{720}{1296} = .5556$$

At least three of the same color: What's $|E|$?

Let's break this into two subcases: exactly three of the same color and exactly four. The exactly four is easiest, so tackle that first.

- Exactly four of the same color: 6 ways to choose the color and we're done.

- Exactly three: 6 ways to choose the three-peat color
- Choose the positions where that color goes: $\binom{4}{3} = 4$
- Choose the color of the fourth peg: 5 ways
- Total: 6 + (6 * 4 * 5) = 126

$$Pr(E) = \frac{|E|}{|S|} = \frac{126}{1296} = .0972$$

Putting it all together

Finally, plug these in to get the expected value of the game. The expected value of each round is exactly the same, so we'll compute it once and multiply by five.

$$E[X_i] = (.5556) * (10) + (.0972) * (-30) = 2.64 \text{ for } i = 1, 2, 3, 4, 5 \cdots$$
$$E[X] = 5 * E[X_i] = 5 * 2.64 = \$13.20$$

Laney comes out ahead, obviously! Nate was wrong to offer her these terms.

Problem 4

Ye Olde Donut Shoppe sells two types of donuts: Funfetti and Vegan Fluff. An order can be placed online or in-person, but not both. An order is placed for only ONE donut at a time.

65% of orders are made online, and the rest are in-person. 47% of the donuts sold are Funfetti, and the rest are Vegan Fluff. Furthermore, there is a 31% chance an order was made online and was for a vegan fluff donut.

Keisha (she/her) works at the counter and you see her put a Vegan Fluff donut in a bag. What is the probability the order was made online, given that the order was for a Vegan Fluff donut?

Solution: We must find $Pr(online|vegan)$. We know the following probabilities:

- $Pr(online) = 0.65$
- $Pr(funfetti) = 0.47$
- $Pr(online \wedge vegan) = 0.31$

Conditional probability

We know $Pr(online|vegan) = Pr(online \wedge vegan)/Pr(vegan)$. We know the probability of both occurring. We also know since the only two types of donuts are vegan and funfetti, $Pr(vegan) = 1 - Pr(funfetti)$. Altogether, we can calculate the conditional.

$$\begin{aligned} Pr(online \mid vegan) &= \frac{Pr(online \wedge vegan)}{Pr(vegan)} \\ &= 0.31/(1 - 0.47) \\ &= 0.31/0.53 \end{aligned}$$

Joint Distribution Table

We can actually fill out a joint distribution table with this information.

	Funfetti	Vegan	
Online		0.31	0.65
In-person			
	0.47	0.53	

We can calculate Pr(vegan) as needed (penciled in blue). From there, we know we want $Pr(online|vegan)$. That is, a vegan fluff donut was taken! So we live in a world that looks like the following table:

	Vegan
Online	0.31/0.53
In-person	
	0.53/0.53

Since there are only 2 possible outcomes in this world, we must make sure they add up to 1. We do so by dividing all the cells by 0.53. What remains in the cell for Online and Vegan is our conditional probability, $Pr(online|vegan)$! For reference, here is the entire distribution table for the whole world with both events. We use black to show every extra probability we calculated to fill the table.

	Funfetti	Vegan	
Online	0.34	0.31	0.65
In-person	0.13	0.22	0.35
	0.47	0.53	

All these approaches lead to $0.31/0.53 \approx 0.5849$, or 58.49%.

Problem 5

Lucia wants to bring 1 plush for her trip to the Berkshires. She owns two sets of plushies, one in Boston and one in New York.

Boston Set

- 4 blue ones
- 2 pink ones
- 4 white ones
- 2 brown ones

New York Set

- 4 white ones
- 3 gray one
- 2 blue ones
- 4 pink ones

Lucia will choose from the Boston set 7/11 times, and from the New York set 4/11 times. Once she decides on a set, she picks a plush from the set at random.

PART A) What is the probability that she will choose a pink plush and choose from the New York set?

Solution: We are looking for $Pr(pink \wedge NY)$. This means we need to choose the New York set and then choose a pink plush from the New York set. That gives us $4/11 * 4/13 = 16/143 \approx 0.112$.

PART B) What is the probability that she will choose a blue plush for the trip?

Solution: We are looking for $Pr(blue)$. Any blue plush must come from either the Boston set or the New York set.

$$\begin{aligned}
Pr(blue) &= Pr(blue \wedge Boston) + Pr(blue \wedge NY) \\
&= Pr(Boston) * Pr(blue|Boston) + Pr(NY) * Pr(blue \wedge NY) \\
&= 7/11 * 4/12 + 4/11 * 2/13 \\
&= 115/429 \approx 0.268
\end{aligned}$$

PART C) Lucia went on her trip with a blue plush. What is the probability Lucia chose the plush from the New York set given it was blue?

Solution: We are looking for $Pr(NY|blue)$. We can use the definition of conditional probability and solve $Pr(NY \wedge blue)/Pr(blue)$. We solved all of these probabilities in PART B!

$$
\begin{aligned}
Pr(NY|blue) &= \frac{Pr(NY \wedge blue)}{Pr(blue)} \\
&= \frac{4/11 * 2/13}{115/429} \\
&= \frac{24}{115} \approx 0.209
\end{aligned}
$$

Problem 6

Two sorted sequences, lengths 11 and 8, are given: $(a, b, c, d, e, f, g, h, i, j, k)$ and $(1, 2, 3, 4, 5, 6, 7, 8)$.
We want to interleave them into one sequence of length 19 such that:

- The letters $a...k$ remain in relative order, and
- The numbers 1...8 remain in relative order.

For example $a, b, c, 1, 2, d, 3, e, 4, 5, f, g, h, 6, i, 7, 8, j, k$ is a valid interleaving.
How many valid interleavings exist?

Solution: Choose one sequence and place it down in its relative order. Let's say we put down the letters, like so:

__ a __ b __ c __ d __ e __ f __ g __ h __ i __ j __ k __

Then we can place the numbers in the following spaces:

- Between all the letters for 11-1 = 10 places.
- Before all the letters or after all the letters for 2 more places.

So we are putting 8 numbers in 12 places. The order we place the numbers is *fixed.* This means we cannot use a new order to get a new outcome. Therefore order does not matter. We can choose the same place multiple times when putting the numbers down. This is a balls and bins/stars and bars problem with 8 balls and 12 bins. This gives us

$$\binom{8+12-1}{8} = \binom{19}{8} = 75582 \text{ interleavings}$$

If we instead chose to put down the numbers, we would get $\binom{19}{11}$ which is the same number!