A New Perspective on the Divergence of the Harmonic Series

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Introduction

In this note, we will prove the well-known divergence of the harmonic series using an alternative approach. Although there are existing tests for series divergence of the given series, the aim of this paper is to show that it can be rigorously proven from a new perspective using only high school-level mathematical knowledge.

1 If the harmonic series converges

Theorem 1. *If a series X is absolutely convergent, i.e., if the series Y converges, then the series will converge regardless of the rearrangement of its terms. Furthermore, the sum of the series will remain constant, independent of the order in which the terms are rearranged.(Let $X = \sum_{n=1}^{\infty} a_n$ and $Y = \sum_{n=1}^{\infty} |a_n|$.)*

Proof. Assume that the harmonic series converges absolutely to a certain real number B. Then, the following statement holds true.

$$\sum_{n=1}^{\infty} \frac{1}{n} = B$$

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} + \sum_{n=1}^{\infty} \frac{1}{2n} = B$$

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n} = B$$

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} = \frac{B}{2}$$

2 The extraction of a series

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}=\sum_{n=1}^{\infty}\frac{1}{2n-1}$$

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots=\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\ldots$$

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots=(\frac{1}{2}+\frac{1}{2})+(\frac{1}{4}+\frac{1}{12})+(\frac{1}{6}+\frac{1}{30})+\ldots$$

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots=(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots)+(\frac{1}{2}+\frac{1}{12}+\frac{1}{30}+\ldots)$$

$$\frac{B}{2}=\frac{B}{2}+(\frac{1}{2}+\frac{1}{12}+\frac{1}{30}+\ldots)$$

Let $\{a_n\}$ denote the sequence of terms in $(1/2+1/12+1/30+\ldots)$. Then $a_n=\frac{1}{2n-1}-\frac{1}{2n}$, since $\{a_n\}$ is derived from the difference between the series consisting of terms with odd denominatores. Since it is trivial that both $(2n-1)>0$ and $(2n)>0$ hold for all natural numbers 'n', multiplying both sides of $(2n)>(2n-1)$ by $\frac{1}{(2n-1)(2n)}$ preserves the direction of the inequality. This yields the inequality $\frac{1}{(2n-1)}>\frac{1}{(2n)}$, which is derived from an inequality proven to hold for all natural numbers 'n' by mathematical induction. Therefore, $\frac{1}{(2n-1)}>\frac{1}{(2n)}$ must also be valid, and it follows that every term of $\{a_n\}$ is positive.

Moreover, if the preceding conclusion holds, then the following equation must be satisfied as well.

$$\sum_{n=1}^{\infty}(\frac{1}{2n-1}-\frac{1}{2n})=0$$

$$\sum_{n=1}^{\infty}a_n=0$$

$$\frac{1}{2}+\sum_{n=2}^{\infty}a_n=0$$

$$\sum_{n=2}^{\infty}a_n=-\frac{1}{2}$$

3 Absolute Convergence and Conclusion

We now aim to prove the absolute convergence of the harmonic series under the assumption that it converges, by showing that such convergence cannot be merely conditional.

$$\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \left| \frac{1}{n} \right| = B$$

In conclusion, since the sequence $\{\frac{1}{n}\}$ remains positive for all natural numbers 'n', the identity above holds, and the series formed by taking the absolute value of each term converges to B rather than diverging. Therefore, if the given series is assumed to converge, it must converge absolutely.

However, under the assumption of absolute convergence, the conclusion reached in Section 2 follows—namely, that the infinite sum of a series consisting entirely of positive terms is negative—which leads to a mathematical contradiction. Hence, the harmonic series cannot converge to any particular real value; in other words, it diverges.

$\square$