

# MCIT 515

## Fundamentals of Linear Algebra and Optimization Jean Gallier and Jocelyn Quaintance

### Project 3: Curve Interpolation

The purpose of this project is to solve a curve interpolation problem using cubic splines. This type of problem arises frequently in computer graphics and in robotics (path planning).

Recall from Project 1A that polynomial curves of degree  $\leq m$  can be defined in terms of control points and the Bézier polynomials. Cubic Bézier curves are often used because they are cheap to implement and give more flexibility than quadratic Bézier curves.

A *cubic Bézier curve*  $C(t)$  (in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ ) is specified by a list of four *control points*  $(b_0, b_1, b_2, b_3)$  and is given parametrically by the equation

$$C(t) = (1-t)^3 b_0 + 3(1-t)^2 t b_1 + 3(1-t) t^2 b_2 + t^3 b_3.$$

Clearly,  $C(0) = b_0$ ,  $C(1) = b_3$ , and for  $t \in [0, 1]$ , the point  $C(t)$  belongs to the convex hull of the control points  $b_0, b_1, b_2, b_3$ .

*Interpolation problems* require finding curves passing through some given data points and possibly satisfying some extra constraints.

It is known that Lagrange interpolation is not very satisfactory when  $N \geq 5$  since Lagrange interpolants tend to oscillate in an undesirable manner. Thus, we turn to Bézier spline curves.

A *Bézier spline curve*  $F$  is a curve which is made up of curve segments which are Bézier curves, say  $C_1, \dots, C_m$  ( $m \geq 2$ ). We will assume that  $F$  is defined on  $[0, m]$ , so that for  $i = 1, \dots, m$ ,

$$F(t) = C_i(t - i + 1), \quad i - 1 \leq t \leq i.$$

Typically, some smoothness is required between any two junction points, that is, between any two points  $C_i(1)$  and  $C_{i+1}(0)$ , for  $i = 1, \dots, m - 1$ . We require that  $C_i(1) = C_{i+1}(0)$  ( $C^0$ -continuity), and typically that the derivatives of  $C_i$  at 1 and of  $C_{i+1}$  at 0 agree up to second order derivatives. This is called  $C^2$ -continuity, and it ensures that the tangents agree as well as the curvatures.

There are a number of interpolation problems, and we consider one of the most common problems which can be stated as follows:

**Problem:** Given  $N + 1$  data points  $x_0, \dots, x_N$ , find a  $C^2$  cubic spline curve  $F$ , such that  $F(i) = x_i$ , for all  $i$ ,  $0 \leq i \leq N$  ( $N \geq 2$ ).

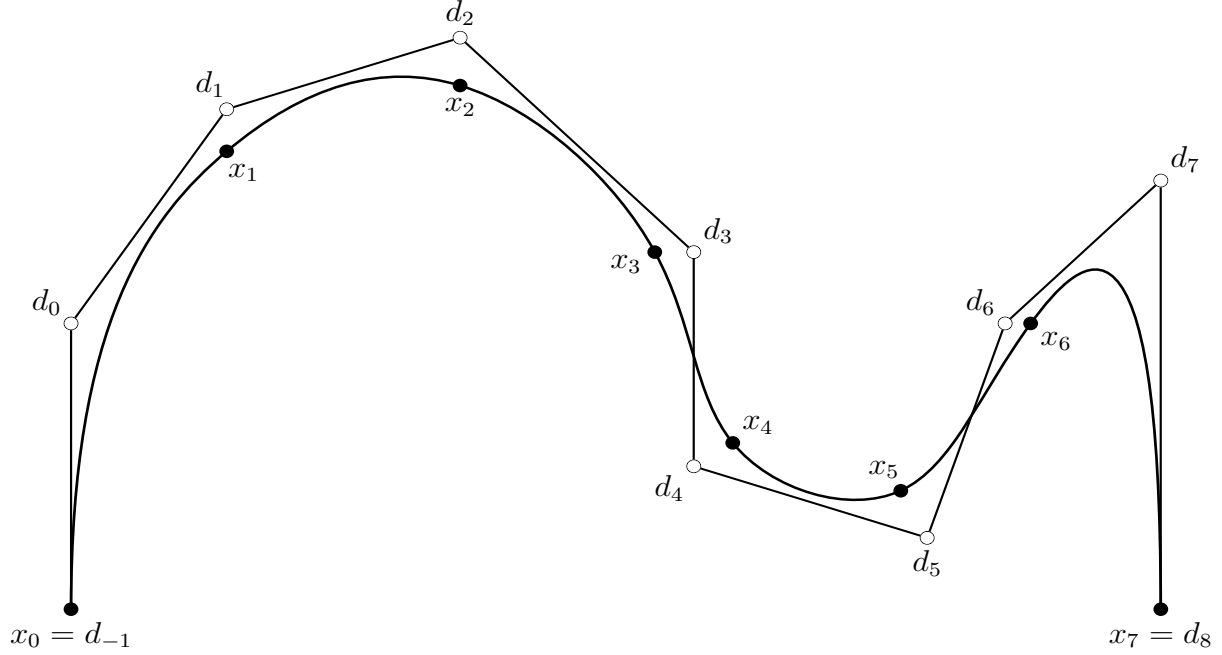


Figure 1: A  $C^2$  cubic interpolation spline curve passing through the points  $x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7$

A way to solve this problem is to find  $N + 3$  auxiliary points  $d_{-1}, \dots, d_{N+1}$  called *de Boor control points* from which  $N$  Bézier curves can be found. Actually,

$$d_{-1} = x_0 \quad \text{and} \quad d_{N+1} = x_N$$

so we only need to find  $N + 1$  points  $d_0, \dots, d_N$  (See figure 2).

It turns out that the  $C^2$ -continuity constraints on the  $N$  Bézier curves yield only  $N - 1$  equations, so  $d_0$  and  $d_N$  can be chosen arbitrarily. In practice,  $d_0$  and  $d_N$  are chosen according to various “end conditions,” such as prescribed velocities at  $x_0$  and  $x_N$ . For the time being, we will assume that  $d_0$  and  $d_N$  are given.

Figures 1 and 2 illustrates an interpolation problem involving  $N + 1 = 7 + 1 = 8$  data points. The control points  $d_0$  and  $d_7$  were chosen arbitrarily.

It can be shown that  $d_1, \dots, d_{N-1}$  are given by the linear system

$$\begin{pmatrix} \frac{7}{2} & 1 & & & \\ 1 & 4 & 1 & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & 1 & 4 & 1 \\ & & & 1 & \frac{7}{2} \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_{N-2} \\ d_{N-1} \end{pmatrix} = \begin{pmatrix} 6x_1 - \frac{3}{2}d_0 \\ 6x_2 \\ \vdots \\ 6x_{N-2} \\ 6x_{N-1} - \frac{3}{2}d_N \end{pmatrix}. \quad (1)$$

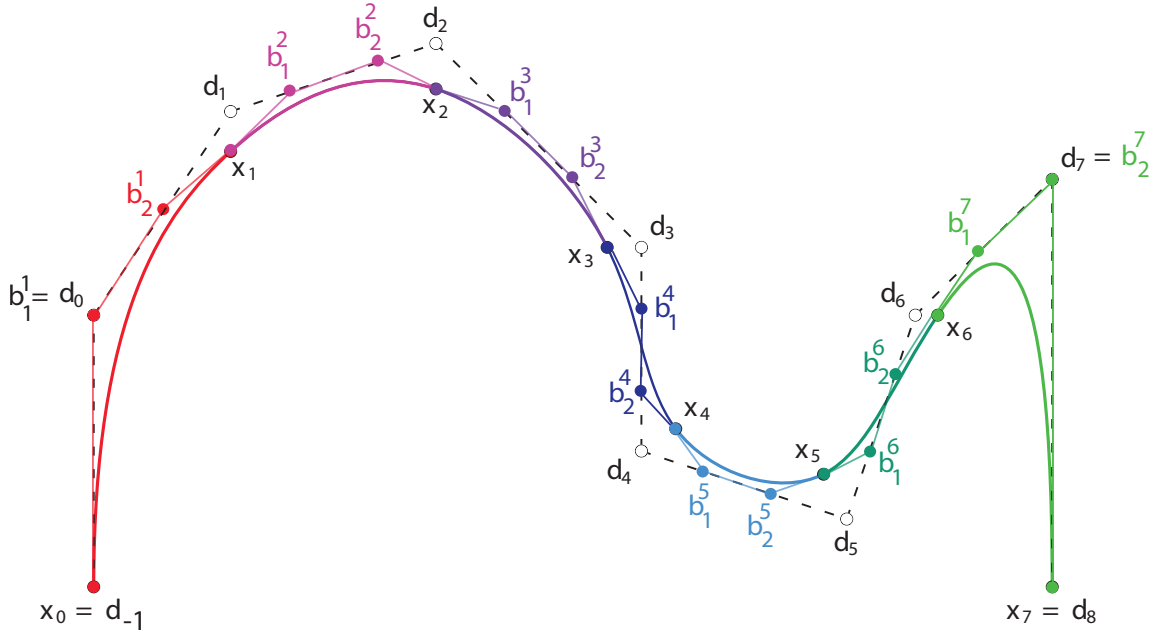


Figure 2: Figure 1 with color coding added and the Bézier control points plotted

The derivation of the above system assumes that  $N \geq 4$ . If  $N = 3$ , this system reduces to

$$\begin{pmatrix} \frac{7}{2} & 1 \\ 1 & \frac{7}{2} \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 6x_1 - \frac{3}{2}d_0 \\ 6x_2 - \frac{3}{2}d_3 \end{pmatrix} \quad (2)$$

When  $N = 2$ , it can be shown that  $d_1$  is given by

$$d_1 = 2x_1 - \frac{1}{2}d_0 - \frac{1}{2}d_2. \quad (3)$$

Observe that when  $N \geq 3$ , the above matrix is strictly diagonally dominant, so it is invertible. Actually, the above system needs to be solved for the  $x$ -coordinates and for the  $y$ -coordinates of the  $d_i$ s (and also for the  $z$ -coordinates, if the points are in  $\mathbb{R}^3$ ). Once the above system is solved, the Bézier cubics  $C_1, \dots, C_N$  are determined as follows (we assume  $N \geq 2$ ): For  $2 \leq i \leq N - 1$ , the control points  $(b_0^i, b_1^i, b_2^i, b_3^i)$  of  $C_i$  are given by

$$\begin{aligned} b_0^i &= x_{i-1} \\ b_1^i &= \frac{2}{3}d_{i-1} + \frac{1}{3}d_i \\ b_2^i &= \frac{1}{3}d_{i-1} + \frac{2}{3}d_i \\ b_3^i &= x_i. \end{aligned} \quad (4)$$

The control points  $(b_0^1, b_1^1, b_2^1, b_3^1)$  of  $C_1$  are given by

$$\begin{aligned} b_0^1 &= x_0 \\ b_1^1 &= d_0 \\ b_2^1 &= \frac{1}{2}d_0 + \frac{1}{2}d_1 \\ b_3^1 &= x_1, \end{aligned} \tag{5}$$

and the control points  $(b_0^N, b_1^N, b_2^N, b_3^N)$  of  $C_N$  are given by

$$\begin{aligned} b_0^N &= x_{N-1} \\ b_1^N &= \frac{1}{2}d_{N-1} + \frac{1}{2}d_N \\ b_2^N &= d_N \\ b_3^N &= x_N. \end{aligned} \tag{6}$$

See figure 2 for an illustration of these points.

It is easy to show that the tangent vectors  $m_0$  at  $x_0$  and  $m_N$  at  $x_N$  are given by

$$\begin{aligned} m_0 &= 3(d_0 - x_0) \\ m_N &= 3(x_N - d_N). \end{aligned} \tag{7}$$

### End Conditions.

One method to determine the points  $d_0$  and  $d_N$  is the *natural end condition*, which consists in setting the second derivatives at  $x_0$  and at  $x_N$  to be zero, that is,

$$C_1''(0) = 0, \quad C_N''(1) = 0.$$

(Part 1) (**10 points**) Prove that the second derivative at  $b_0$  of a Bézier cubic specified by the control points  $(b_0, b_1, b_2, b_3)$  is

$$6(b_0 - 2b_1 + b_2),$$

and the second derivative at  $b_3$  is

$$6(b_1 - 2b_2 + b_3).$$

Prove that our system becomes

$$\begin{pmatrix} 4 & 1 & & & 0 \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ 0 & & 1 & 4 & 1 \\ & & & 1 & 4 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_{N-2} \\ d_{N-1} \end{pmatrix} = \begin{pmatrix} 6x_1 - x_0 \\ 6x_2 \\ \vdots \\ 6x_{N-2} \\ 6x_{N-1} - x_N \end{pmatrix},$$

where  $d_0, d_N$  are given by

$$\begin{aligned} d_0 &= \frac{2}{3}x_0 + \frac{1}{3}d_1 \\ d_N &= \frac{1}{3}d_{N-1} + \frac{2}{3}x_N. \end{aligned}$$

Note that  $d_0$  is on the line segment  $(x_0, d_1)$  ( $1/3$  of the way from  $x_0$ ) and  $d_N$  is on the line segment  $(d_{N-1}, x_N)$  ( $1/3$  of the way from  $x_N$ ). For this proof, you may use Equation (1).

In the above derivation we assumed that  $N \geq 4$ . If  $N = 3$ , show that this system reduces to

$$\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 6x_1 - x_0 \\ 6x_2 - x_3 \end{pmatrix}.$$

(2) After computing  $d_1, \dots, d_{N-1}$  by solving the linear system in (1) using the **Matlab** command `\`, use your program from Project 1B to compute the control points for the Bézier curves  $C_1, \dots, C_N$  and to plot these Bézier segments (for  $t \in [0, 1]$ ) to visualize the interpolating spline.

(Part 2)(i) (**50 points**) Code a first version that takes as input two column vectors  $x$  and  $y$  of dimension  $N + 1$  containing the  $x$  and  $y$  coordinates of the  $N + 1$  data points  $x_0, x_1, \dots, x_N$  ( $N \geq 2$ ). The output should be two column vectors  $dx$  and  $dy$  of dimension  $N + 3$  consisting of the de Boor control points obtained by solving the linear system

$$\begin{pmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & 1 & 4 & 1 \\ & & & 1 & 4 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_{N-2} \\ d_{N-1} \end{pmatrix} = \begin{pmatrix} 6x_1 - x_0 \\ 6x_2 \\ \vdots \\ 6x_{N-2} \\ 6x_{N-1} - x_N \end{pmatrix},$$

with  $d_{-1} = x_0$ ,  $d_{N+1} = x_N$ , and

$$\begin{aligned} d_0 &= \frac{2}{3}x_0 + \frac{1}{3}d_1 \\ d_N &= \frac{1}{3}d_{N-1} + \frac{2}{3}x_N. \end{aligned}$$

This should be a function of the form:

```
function [dx, dy] = interpnatxy(x,y,show_plot)
```

(provided in project zip file).

(Part 2)(ii) (**20 points**) Compute the  $x$  and the  $y$  coordinates  $Bx$  and  $By$  of the  $N$  Bézier segments  $C_1, \dots, C_N$  computed by the method of Project 1B, as  $N \times 4$  matrices. For visual

correctness, also plot the curve. To achieve this, code the function `interpnatxy` modified from 2(i) to make use of the function `bspline2b` from Project 1B.

Run your program on the sequences of data points given by their  $x$  and  $y$  coordinates as the column vector  $x$  and  $y$  specified below using the output script:

```
x1 = [3.6942; 1.3690; 2.9865; 5.8509; 8.1929; 8.2098; 6.8281]
y1 = [1.2144; 3.5925; 7.3933; 7.9217; 6.9665; 4.0396; 1.5600]
x2 = [3.9806; 2.2789; 3.6942; 6.8618; 7.1820]
y2 = [2.1087; 4.2429; 7.0884; 6.9461; 4.3852]
x3 = [4.2334; 1.0826; 1.3016; 4.9579; 8.2435; 4.8062]
y3 = [1.0315; 3.6941; 5.6250; 7.9624; 5.5640; 5.8486]
x4 = [4.6040; 2.2283; 3.3741; 2.1609; 7.2494; 6.8955; 9.1702]
y4 = [1.3364; 1.6616; 3.5722; 6.8242; 8.6535; 3.7957; 2.8608]
x5 = [2.7310; 1.3599; 1.1662; 1.9709; 4.7876; 7.0827; 6.2630;
      4.2809; 3.9232; 4.9367; 8.3048; 9.0052; 7.6639]
y5 = [1.1106; 2.4716; 5.3639; 7.3677; 8.3129; 7.6134; 5.3072;
      5.0614; 2.8875; 1.0539; 0.8648; 2.5095; 3.7760]
```

Running your program on the data points specified by  $x5$  and  $y5$  you should get the interpolating curve shown in Figure 3.

(Part 2)(iii) (**20 points**) Then code a second version that takes the data points  $x_0, \dots, x_N$  as seen input using the function `getpoints` of Project 1B. This version does not need to output  $dx, dy, Bx$  and  $By$ , but should plot the interpolating curve. This should be done automatically using the output script.

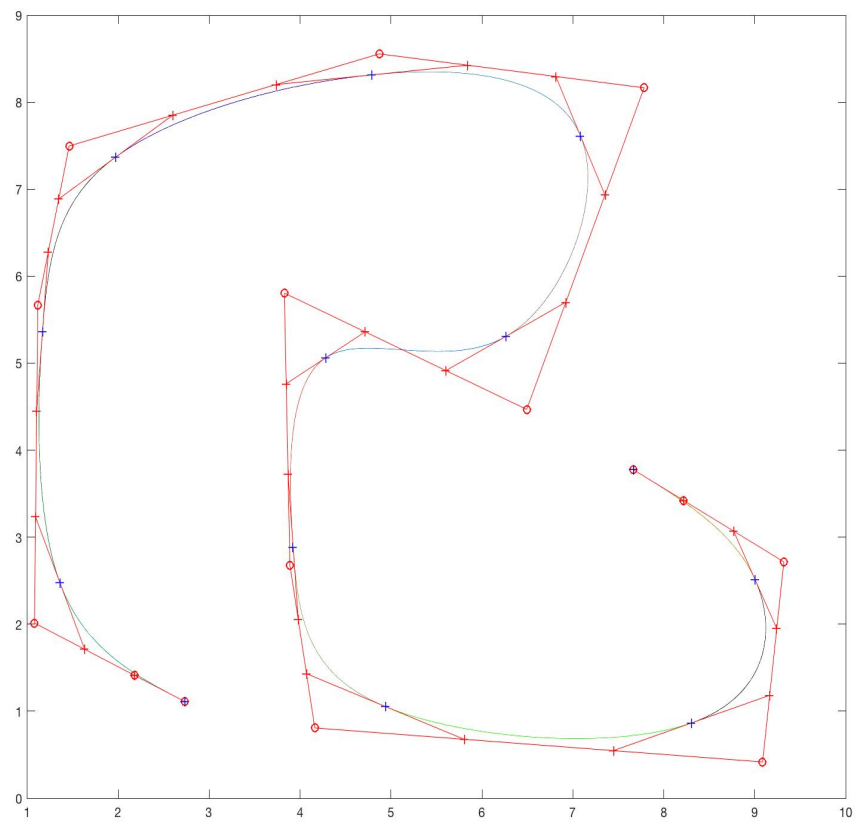


Figure 3: The interpolating curve specified by  $x5$  and  $y5$ .