Chapter 1: Probability and Counting

Introduction to Probability, 2nd Edition: Blitzstein & Hwang

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Based on the textbook

Introduction

Why Study Probability?

- Mathematics: Logic of certainty
- Probability: Logic of uncertainty
- Applications:
  - Statistics (foundation for data analysis)
  - Physics (quantum mechanics, statistical mechanics)
  - Biology (genetics, evolutionary biologiy, ecology)
  - Computer science (randomized algorithms, machine learning)
  - Finance (stock modeling, risk assessment)
  - Medicine (clinical trials)
  - Everyday life (decision making, understanding ”coincidences”)

The Probabilistic Model

The Probabilistic Model

A probabilistic model is a mathematical description of an uncertain situation. It consists of:

- **Sample space** (Ω or S): set of all possible outcomes
- **Probability law**: assigns probabilities to events (subsets of outcomes)

Definition (Sample Space)
The **sample space** S is the set of all possible outcomes of an experiment.

Requirements:

- Outcomes must be **distinct** and **mutually exclusive**
- The sample space must be **collectively exhaustive**
- When the experiment is carried out, exactly one outcome occurs

Examples:

- Coin flip: $S = \{\text{Heads}, \text{Tails}\}$
- Roll a die: $S = \{1, 2, 3, 4, 5, 6\}$
- Two dice: $S = \{(1,1), (1,2), \ldots, (6,6)\}$

[1] https://en.wikipedia.org/wiki/Sample_space

Events[2]

Definition (Event)
An **event** is a subset of the sample space S. We say A occurred if the actual outcome is in A.

Special cases:

- Single outcome: $\{\text{Heads}\}$
- Multiple outcomes: $\{2, 4, 6\}$ (rolling an even number)
- Entire sample space: S (always occurs)
- Empty set: $\emptyset$ (never occurs)

[2] https://en.wikipedia.org/wiki/Event_(probability_theory)

Set Operations for Events[4]

For events $A, B \subseteq S$:

- **Union** $A \cup B$: *A or B* occurs (or both)
- **Intersection** $A \cap B$: Both *A and B* occur
- **Complement** A^c: A does *not* occur

De Morgan's Laws[3]:

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

[3] https://en.wikipedia.org/wiki/De_Morgan's_laws
[4] https://en.wikipedia.org/wiki/Algebra_of_sets

Naive Definition of Probability

Naive Definition of Probability[5]

Definition (Naive Probability)
For an event A in a finite sample space S:

$$P_{\text{naive}}(A) = \frac{|A|}{|S|} = \frac{\text{\# outcomes favorable to } A}{\text{total \# outcomes}}$$

Assumptions:

1. Finite sample space
2. All outcomes equally likely

[5] https://en.wikipedia.org/wiki/Classical_definition_of_probability

When is the Naive Definition Applicable?

When applicable:

- Symmetry (fair coin, well-shuffled deck)
- Equally likely by design (simple random sample)
- As a null model

Pitfall: Misapplying without justification

Bad reasoning: "Either it will happen or it won't, so probability is 1/2"

Example: Life on Mars

- Wrong: $P(\text{life}) = 1/2$, $P(\text{intelligent life}) = 1/2$
- Why wrong? These events shouldn't have equal probability

Leibniz's Mistake[6]

Problem: Roll two fair dice. Which is more likely: sum of 11 or sum of 12?

Leibniz's incorrect reasoning:

- Sum 11: {5,6} (one way)
- Sum 12: {6,6} (one way)
- Conclusion: Both equally likely

Correct solution:

- Label dice A and B
- Sum 11: (5,6), (6,5) → $P = 2/36 = 1/18$
- Sum 12: (6,6) → $P = 1/36$

Lesson: Always label objects to avoid treating them as indistinguishable!

[6] https://en.wikipedia.org/wiki/Gottfried_Wilhelm_Leibniz

Counting Methods

Theorem (Multiplication Rule)
If Experiment A has a outcomes, and for each of these, Experiment B has b outcomes, then the compound experiment has $a \cdot b$ outcomes.

Example: 10 runners for 1st, 2nd, 3rd place

$$10 \times 9 \times 8 = 720 \text{ possibilities}$$

Generalization: For k experiments with $n_1, n_2, \ldots, n_k$ outcomes:

$$n_1 \times n_2 \times \cdots \times n_k \text{ total outcomes}$$

[7] https://en.wikipedia.org/wiki/Rule_of_product

Permutations[8] (ordered arrangements):

- n distinct objects taken k at a time: $P(n, k) = n(n-1)\cdots(n-k+1)$
- All n objects: $n! = n(n-1)\cdots 2 \cdot 1$

Combinations[9] (unordered selections):

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Counts number of k-element subsets of an n-element set.

Example: $\binom{4}{2} = 6$ committees from 4 people

[8] `https://en.wikipedia.org/wiki/Permutation`
[9] `https://en.wikipedia.org/wiki/Combination`

Strategy: If we count each possibility exactly c times, divide by c

Example: Choose a 2-person committee from 4 people

- **Incorrect**: $4 \times 3 = 12$ (counts each committee twice: AB and BA)
- **Correct**: $(4 \times 3)/2 = 6$

Binomial Coefficient[10]:

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$$

Counts number of k-element subsets of an n-element set.

[10] `https://en.wikipedia.org/wiki/Binomial_coefficient`

Birthday Problem[11]

What's the probability that in a group of k people, at least two share a birthday?

$$P(\text{no match}) = \frac{365 \times 364 \times \cdots \times (365 - k + 1)}{365^k}$$

Surprising results:

- With 23 people: $P(\text{match}) > 0.5$
- With 57 people: $P(\text{match}) > 0.99$

Intuition: Many possible pairs ($\binom{k}{2}$) increase collision probability.

[11] `https://en.wikipedia.org/wiki/Birthday_problem`

Full House in Poker: A Counting Example

Problem: Probability of a full house (3 of one rank, 2 of another) in 5-card poker[12]

Solution:

1. Total hands: $\binom{52}{5} = 2,598,960$
2. Choose rank for triple: 13 choices
3. Choose 3 suits for triple: $\binom{4}{3} = 4$ choices
4. Choose different rank for pair: 12 choices
5. Choose 2 suits for pair: $\binom{4}{2} = 6$ choices

$$P(\text{full house}) = \frac{13 \times 4 \times 12 \times 6}{\binom{52}{5}} = \frac{3,744}{2,598,960} \approx 0.00144$$

[12] https://en.wikipedia.org/wiki/Poker_probability

Newton-Pepys Problem[13]

Which is most likely?

- *A*: At least one 6 in 6 dice
- *B*: At least two 6's in 12 dice
- *C*: At least three 6's in 18 dice

Isaac Newton correctly solved this problem in correspondence with Samuel Pepys (1693).

[13] https://en.wikipedia.org/wiki/Newton–Pepys_problem

Newton-Pepys Solution

$$P(A) = 1 - \left(\frac{5}{6}\right)^6 \approx 0.67$$

$$P(B) = 1 - \left[\left(\frac{5}{6}\right)^{12} + \binom{12}{1}\left(\frac{5}{6}\right)^{11}\left(\frac{1}{6}\right)\right] \approx 0.62$$

$$P(C) = 1 - \left[\left(\frac{5}{6}\right)^{18} + \binom{18}{1}\left(\frac{5}{6}\right)^{17}\left(\frac{1}{6}\right) + \binom{18}{2}\left(\frac{5}{6}\right)^{16}\left(\frac{1}{6}\right)^2\right] \approx 0.60$$

A has the highest probability.

Newton's insight: While expected number of 6's increases (1, 2, 3), the probability of meeting or exceeding these thresholds decreases.

Story Proofs

What is a Story Proof?

A **story proof** proves a combinatorial identity by interpreting both sides as counting the same thing in different ways.

Advantages:

- More intuitive than algebra
- Provides insight into *why* identity holds
- Avoids messy calculations

Example 1: Symmetry of Binomial Coefficients

$$\binom{n}{k} = \binom{n}{n-k}$$

Story: Choosing k people for a committee = choosing $n - k$ people to *exclude*.

Both sides count the same thing: number of k-element subsets.

Vandermonde's Identity[14]

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j} \binom{n}{k-j}$$

Story: Choose k people from m juniors and n seniors.

Count directly: $\binom{m+n}{k}$

Count by cases: If we choose j juniors, we must choose $k-j$ seniors. Sum over all possible j:

$$\sum_{j=0}^{k} \binom{m}{j} \binom{n}{k-j}$$

Both count the same thing, so they must be equal.

[14] `https://en.wikipedia.org/wiki/Vandermonde's_identity`

Axiomatic Probability

Kolmogorov Axioms[15]

Definition (Probability Space)
A probability space consists of:

- Sample space S
- Probability function P mapping events $A \subseteq S$ to $[0, 1]$

Axioms (Kolmogorov, 1933):

1. **Nonnegativity**: $P(A) \geq 0$ for all events A
2. **Normalization**: $P(S) = 1$
3. **Countable Additivity**: If $A_1, A_2, \ldots$ are disjoint, then:

$$P\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} P(A_j)$$

[15] https://en.wikipedia.org/wiki/Probability_axioms

Interpretations of Probability

- **Frequentist**[16]: long-run frequency
- **Bayesian**[17]: degree of belief
- **Pebble World**: mass distribution (total mass = 1)
- **Axiomatic**: mathematical structure satisfying the three axioms

All interpretations must satisfy the axioms to be mathematically consistent.

[16] `https://en.wikipedia.org/wiki/Frequentist_probability`
[17] `https://en.wikipedia.org/wiki/Bayesian_probability`

Properties Derived from Axioms

Theorem: For any events A and B:

1. $P(A^c) = 1 - P(A)$
2. $P(\emptyset) = 0$
3. If $A \subseteq B$, then $P(A) \leq P(B)$ (monotonicity)
4. $P(A) \leq 1$ for all A
5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Inclusion-Exclusion Principle[18]

For events $A_1, A_2, \ldots, A_n$:

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} P(A_1 \cap \cdots \cap A_n)$$

For 3 events:

$$\begin{aligned} P(A \cup B \cup C) = {} & P(A) + P(B) + P(C) \\ & - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ & + P(A \cap B \cap C) \end{aligned}$$

[18] `https://en.wikipedia.org/wiki/Inclusion–exclusion_principle`

de Montmort's Matching Problem[19]

Problem: Shuffle n cards numbered 1 to n. Flip cards one by one, saying "1, 2, ..., n". Win if any card matches its number.

Solution: Let A_i = "card i is in position i"

Using inclusion-exclusion:

$$P(\text{win}) = P\left(\bigcup_{i=1}^{n} A_i\right) = 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n+1}\frac{1}{n!}$$

[19] https://en.wikipedia.org/wiki/Matching_problem

For large n:

$$P(\text{win}) \approx 1 - \frac{1}{e} \approx 0.632$$

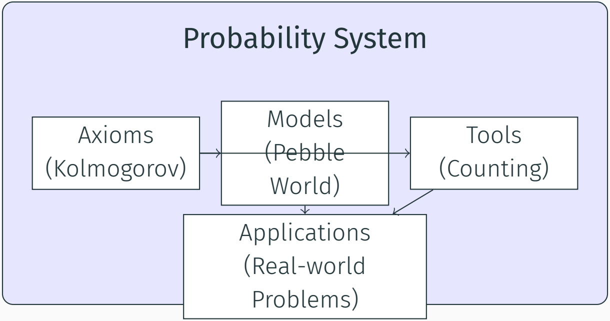
Interesting: Probability approaches $1 - 1/e$, not 0 or 1!

Historical note: Pierre Remond de Montmort (1678-1719) studied this problem in his 1708 book *Essai d'analyse sur les jeux de hazard*.

Recap and Analogy

1. **Sample space** S: Set of all possible outcomes
2. **Event**: Subset of S
3. **Naive probability**: $P(A) = |A|/|S|$ (requires equally likely outcomes)
4. **Counting tools**:
    - Multiplication rule
    - Permutations: $n!$, $n(n-1)\cdots(n-k+1)$
    - Combinations: $\binom{n}{k}$
    - Adjusting for overcounting
5. **Story proofs**: Interpret both sides of identity
6. **Axiomatic probability**: Nonnegativity, Normalization, Additivity
7. **Properties**: Complements, monotonicity, inclusion-exclusion

Probability as a Coherent System



Foundation: Axioms provide rigorous mathematical basis for all probability theory.

Step-by-Step Calculations

Birthday Problem: Step-by-Step Calculation

Problem: With $k = 23$ people, what's P(at least one birthday match)?

Step 1: Calculate P(no match)

$$P(\text{no match}) = \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \cdots \times \frac{365-22}{365}$$

Step 2: Use product notation

$$P(\text{no match}) = \prod_{i=0}^{22} \frac{365-i}{365} = \frac{365 \times 364 \times \cdots \times 343}{365^{23}}$$

Step 3: Take complement

$$P(\text{match}) = 1 - P(\text{no match}) = 1 - \frac{365 \times 364 \times \cdots \times 343}{365^{23}}$$

Step 4: Approximate using Taylor expansion

$$P(\text{match}) \approx 1 - e^{-\frac{23 \times 22}{2 \times 365}} \approx 1 - e^{-0.693} \approx 1 - 0.500 = 0.500$$

Birthday Problem: Detailed Calculation for k=23

Exact calculation step-by-step:

1. **Numerator**: $365 \times 364 \times \cdots \times 343$
   First few terms: $365 \times 364 = 132,860$
   $\times 363 = 48,228,180$
   $\times 362 = 17,458,601,160$
   $\cdots$ (continue to 343)
2. **Denominator**: $365^{23} \approx 4.85 \times 10^{58}$
3. **Ratio**: Approximately 0.492702
4. **Final probability**: $1 - 0.492702 \approx 0.507298 \approx 50.73\%$

For k=57:

- Number of pairs: $\binom{57}{2} = \frac{57 \times 56}{2} = 1,596$
- Approximate: $P(\text{no match}) \approx e^{-1596/(2 \times 365)} \approx e^{-2.187} \approx 0.112$
- $P(\text{match}) \approx 1 - 0.112 = 0.888 \approx 88.8\%$

Full House Probability: Step-by-Step Calculation

Step 1: Total number of 5-card hands

$$\binom{52}{5} = \frac{52!}{5! \cdot 47!} = \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} = 2,598,960$$

Step 2: Choose rank for triple

13 choices (A, 2, 3, ..., K)

Step 3: Choose 3 suits for the triple

$$\binom{4}{3} = \frac{4!}{3! \cdot 1!} = 4 \text{ choices}$$

(Choose which 3 of the 4 suits appear)

Full House Calculation (Continued)

Step 4: Choose different rank for the pair

12 choices (can't use the same rank as the triple)

Step 5: Choose 2 suits for the pair

$$\binom{4}{2} = \frac{4!}{2! \cdot 2!} = \frac{4 \times 3}{2} = 6 \text{ choices}$$

Step 6: Apply multiplication rule

$$13 \times 4 \times 12 \times 6 = 3,744 \text{ possible full house hands}$$

Step 7: Calculate probability

$$P(\text{full house}) = \frac{3,744}{2,598,960} = \frac{3,744 \div 48}{2,598,960 \div 48} = \frac{78}{54,145} \approx 0.0014406$$

Step 8: Interpretation

$\approx 0.144\%$ chance, or about 1 in 694 hands

Newton-Pepys Problem: Step-by-Step for Event A

Event A: At least one 6 in 6 dice

Step 1: Calculate probability of no 6's in 6 dice

$$P(\text{no } 6) = \left(\frac{5}{6}\right)^6$$

Step 2: Compute $\left(\frac{5}{6}\right)^6$:

$$\frac{5}{6} = 0.8333\ldots$$
$$(0.8333)^2 = 0.6944$$
$$(0.6944) \times (0.8333) = 0.5787$$
$$(0.5787) \times (0.8333) = 0.4823$$
$$(0.4823) \times (0.8333) = 0.4019$$
$$(0.4019) \times (0.8333) = 0.3349$$

Step 3: Complement

$$P(A) = 1 - 0.3349 = 0.6651 \text{ (exact: } 1 - (5/6)^6 = 0.6651)$$

Newton-Pepys: Step-by-Step for Event B

Event B: At least two 6's in 12 dice

Step 1: Calculate probability of 0 or 1 six

$$P(0) = \left(\frac{5}{6}\right)^{12} \quad \text{and} \quad P(1) = \binom{12}{1}\left(\frac{5}{6}\right)^{11}\left(\frac{1}{6}\right)$$

Step 2: Calculate $\left(\frac{5}{6}\right)^{12}$

$$\left(\frac{5}{6}\right)^{6} = 0.3349 \text{ (from previous)}$$

$$\left(\frac{5}{6}\right)^{12} = (0.3349)^2 = 0.1122$$

Step 3: Calculate $P(1)$

$$\binom{12}{1} = 12$$

$$\left(\frac{5}{6}\right)^{11} = \left(\frac{5}{6}\right)^{12} \times \frac{6}{5} = 0.1122 \times 1.2 = 0.1346$$

$$P(1) = 12 \times 0.1346 \times \frac{1}{6} = 12 \times 0.02243 = 0.2692$$

Step 4: Sum $P(0) + P(1)$

$$P(0) + P(1) = 0.1122 + 0.2692 = 0.3814$$

Step 5: Complement for at least two 6's

$$P(B) = 1 - 0.3814 = 0.6186$$

Exact calculation:

$$P(B) = 1 - \left[\left(\frac{5}{6}\right)^{12} + 12\left(\frac{5}{6}\right)^{11}\left(\frac{1}{6}\right)\right] \approx 0.6187$$

Note: $P(B) < P(A)$ because expected number of 6's is 2, but variance makes getting exactly 0 or 1 more likely than intuition suggests.

Newton-Pepys: Step-by-Step for Event C

Event C: At least three 6's in 18 dice

Step 1: Calculate probability of 0, 1, or 2 sixes

$$P(0) = \left(\frac{5}{6}\right)^{18}$$

$$P(1) = \binom{18}{1}\left(\frac{5}{6}\right)^{17}\left(\frac{1}{6}\right)$$

$$P(2) = \binom{18}{2}\left(\frac{5}{6}\right)^{16}\left(\frac{1}{6}\right)^{2}$$

Step 2: Calculate $\left(\frac{5}{6}\right)^{18}$

$$\left(\frac{5}{6}\right)^{6} = 0.3349$$

$$\left(\frac{5}{6}\right)^{18} = (0.3349)^{3} = 0.0376$$

Step 3: Calculate $P(1)$

$$\binom{18}{1} = 18$$

$$\left(\frac{5}{6}\right)^{17} = \left(\frac{5}{6}\right)^{18} \times \frac{6}{5} = 0.0376 \times 1.2 = 0.0451$$

$$P(1) = 18 \times 0.0451 \times \frac{1}{6} = 18 \times 0.00752 = 0.1354$$

Step 4: Calculate $P(2)$

$$\binom{18}{2} = \frac{18 \times 17}{2} = 153$$

$$\left(\frac{5}{6}\right)^{16} = \left(\frac{5}{6}\right)^{18} \times \left(\frac{6}{5}\right)^{2} = 0.0376 \times 1.44 = 0.0541$$

$$P(2) = 153 \times 0.0541 \times \left(\frac{1}{6}\right)^{2} = 153 \times 0.0541 \times 0.02778 = 153 \times 0.001503 = 0.2299$$

Step 5: Sum $P(0) + P(1) + P(2)$

$$P(0) + P(1) + P(2) = 0.0376 + 0.1354 + 0.2299 = 0.4029$$

Step 6: Complement for at least three 6's

$$P(C) = 1 - 0.4029 = 0.5971$$

Exact calculation:

$$P(C) = 1 - \left[\left(\frac{5}{6}\right)^{18} + 18\left(\frac{5}{6}\right)^{17}\left(\frac{1}{6}\right) + 153\left(\frac{5}{6}\right)^{16}\left(\frac{1}{6}\right)^{2}\right] \approx 0.5973$$

Comparison:

$$P(A) \approx 0.665 > P(B) \approx 0.619 > P(C) \approx 0.597$$

de Montmort's Problem: Step-by-Step for n=4

Problem: 4 cards numbered 1-4, shuffled. Win if any card matches its position.

Step 1: Define events Let $A_i =$ "card i is in position i" for $i = 1, 2, 3, 4$

Step 2: Use inclusion-exclusion

$$P(\text{win}) = P(A_1 \cup A_2 \cup A_3 \cup A_4)$$

$$= \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - P(A_1 \cap A_2 \cap A_3 \cap A_4)$$

Step 3: Calculate individual terms

$$P(A_i) = \frac{1}{4} \quad \text{(1 favorable position out of 4)}$$

$$P(A_i \cap A_j) = \frac{1}{4 \times 3} = \frac{1}{12} \quad \text{(2 fixed positions, 2 cards left for 2 positions)}$$

$$P(A_i \cap A_j \cap A_k) = \frac{1}{4 \times 3 \times 2} = \frac{1}{24}$$

$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = \frac{1}{4!} = \frac{1}{24}$$

Step 4: Count number of terms

- $\sum_i P(A_i)$: 4 terms, each 1/4
- $\sum_{i<j} P(A_i \cap A_j)$: $\binom{4}{2} = 6$ terms, each 1/12
- $\sum_{i<j<k} P(A_i \cap A_j \cap A_k)$: $\binom{4}{3} = 4$ terms, each 1/24
- $P(A_1 \cap A_2 \cap A_3 \cap A_4)$: 1 term, 1/24

Step 5: Substitute and calculate

$$
\begin{aligned}
P(\text{win}) &= 4 \times \frac{1}{4} - 6 \times \frac{1}{12} + 4 \times \frac{1}{24} - \frac{1}{24} \\
&= 1 - \frac{6}{12} + \frac{4}{24} - \frac{1}{24} \\
&= 1 - \frac{1}{2} + \frac{3}{24} \\
&= \frac{1}{2} + \frac{1}{8} = \frac{4}{8} + \frac{1}{8} = \frac{5}{8} = 0.625
\end{aligned}
$$

de Montmort's Problem: General Formula Derivation

General case: n cards

Step 1: Probability exactly k matches

$$P(\text{exactly k fixed points}) = \frac{\binom{n}{k} \times D_{n-k}}{n!}$$

where D_m = number of derangements of m items (permutations with no fixed points)

Step 2: Derangement formula

$$D_m = m!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^m \frac{1}{m!}\right)$$

Step 3: Probability of at least one match

$$\begin{aligned} P(\text{at least one match}) &= 1 - P(\text{no matches}) = 1 - \frac{D_n}{n!} \\ &= 1 - \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right) \\ &= \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n+1} \frac{1}{n!} \end{aligned}$$

de Montmort's Problem: Limit as n→∞

Step 1: Recognize Taylor series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Step 2: For $x = -1$

$$e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots$$

Step 3: Relate to our probability

$$P(\text{win}) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots$$

Compare with:

$$e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots$$

So:

$$P(\text{win}) = 1 - \left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots\right) = 1 - e^{-1}$$

Step 4: Final limit

$$\lim_{n\to\infty} P(\text{win}) = 1 - \frac{1}{e} \approx 1 - 0.3679 = 0.6321$$

Poker Hand Probabilities: Comparison Table

Calculate probabilities for various poker hands:

1. **Royal Flush**: $\frac{4}{\binom{52}{5}} = \frac{4}{2{,}598{,}960} \approx 0.00000154$
2. **Straight Flush**: $\frac{36}{\binom{52}{5}} \approx 0.0000139$
3. **Four of a Kind**: $\frac{13 \times 48}{\binom{52}{5}} \approx 0.000240$
4. **Full House**: $\frac{13 \times 4 \times 12 \times 6}{\binom{52}{5}} \approx 0.00144$
5. **Flush**: $\frac{\binom{13}{5} \times 4 - 40}{\binom{52}{5}} \approx 0.00197$
6. **Straight**: $\frac{10 \times 4^5 - 40}{\binom{52}{5}} \approx 0.00392$
7. **Three of a Kind**: $\frac{13 \times \binom{4}{3} \times \binom{12}{2} \times 4^2}{\binom{52}{5}} \approx 0.0211$
8. **Two Pair**: $\frac{\binom{13}{2} \times \binom{4}{2}^2 \times 44}{\binom{52}{5}} \approx 0.0475$
9. **One Pair**: $\frac{13 \times \binom{4}{2} \times \binom{12}{3} \times 4^3}{\binom{52}{5}} \approx 0.423$
10. **High Card**: $\frac{\binom{13}{5} - 10 \times (4^5 - 4)}{\binom{52}{5}} \approx 0.501$

Poker Hand Calculation: Three of a Kind

Step-by-step calculation for Three of a Kind:

Step 1: Choose rank for the triple

$$13 \text{ choices}$$

Step 2: Choose 3 suits for the triple

$$\binom{4}{3} = 4 \text{ choices}$$

Step 3: Choose 2 other ranks (different from triple rank and from each other)

$$\binom{12}{2} = \frac{12 \times 11}{2} = 66 \text{ choices}$$

Step 4: Choose suits for the 2 other cards

$$4 \times 4 = 16 \text{ choices (each can be any of 4 suits)}$$

Step 5: Apply multiplication rule

$$13 \times 4 \times 66 \times 16 = 13 \times 4 = 52,\ 52 \times 66 = 3,432,\ 3,432 \times 16 = 54,912$$

Step 6: Calculate probability

Birthday Problem: Exact Formula Derivation

Derivation of the exact formula:

Step 1: Probability first person has unique birthday

$$\frac{365}{365} = 1$$

Step 2: Probability second has different birthday than first

$$\frac{364}{365}$$

Step 3: Probability third has different birthday than first two

$$\frac{363}{365}$$

Step 4: Continue for k people

$$P(\text{all different}) = \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \cdots \times \frac{365-k+1}{365}$$

Step 5: Use factorial notation

$$P(\text{all different}) = \frac{365!}{(365-k)! \cdot 365^k}$$

Birthday Problem: Approximation Derivation

Approximation using Taylor series:

Step 1: Take natural log

$$\ln P(\text{all different}) = \sum_{i=0}^{k-1} \ln\left(1 - \frac{i}{365}\right)$$

Step 2: Use approximation $\ln(1 - x) \approx -x$ for small x

$$\ln P(\text{all different}) \approx -\sum_{i=0}^{k-1} \frac{i}{365} = -\frac{1}{365} \cdot \frac{k(k-1)}{2}$$

Step 3: Exponentiate

$$P(\text{all different}) \approx e^{-\frac{k(k-1)}{2\times 365}}$$

Step 4: Complement

$$P(\text{match}) \approx 1 - e^{-\frac{k(k-1)}{730}}$$

Step 5: Solve for when $P(\text{match}) = 0.5$

$$e^{-\frac{k(k-1)}{730}} = 0.5 \Rightarrow \frac{k(k-1)}{730} = \ln 2 \approx 0.693$$

$$k(k-1) \approx 730 \times 0.693 \approx 506 \Rightarrow k \approx 22.5$$

References and Further Reading

- Blitzstein, J. K., & Hwang, J. (2019). *Introduction to Probability* (2nd ed.). CRC Press.
- Wikipedia: Probability
- Wikipedia: Combinatorics
- Wikipedia: Andrey Kolmogorov
- Stanford Encyclopedia of Philosophy: Interpretations of Probability

Next Chapter: Conditional Probability and Bayes' Theorem