

Likelihood

Hobbs 2015

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December 29, 2023

Probability in Biology: Hobbs 4.1

- ▶ Example: Tadpole observation in a pond.

Example: Tadpole Observation

Scenario:

- ▶ Collecting data on the number of tadpoles per volume of water in a pond.
- ▶ Observed 14 tadpoles in a 1 L sample.
- ▶ **TRUE** average number of tadpoles per liter of water in the pond is 23.

First Observation:

- ▶ It is Poisson
- ▶ Probability of observing 14 tadpoles:
 $P(y_1 = 14 | \lambda = 23) = \text{Poisson}(y_1 = 14 | \lambda = 23) = 0.0136.$

Second Observation:

- ▶ Probability of observing 34 tadpoles:
 $P(y_2 = 34 | \lambda = 23) = \text{Poisson}(y_2 = 34 | \lambda = 23) = 0.0069.$

Joint Probability:

- ▶ Assuming independence: Joint probability =
 $0.0136 \times 0.0069 = 9.38 \times 10^{-5}.$

Independence of Observations

- ▶ Independence assumption: Knowledge of one observation tells us nothing about the other.
- ▶ Joint probability calculation extended to any number of independent observations.

Remarks

- ▶ Probability calculations provide insights into the likelihood of observations given a fixed average.
- ▶ Independence assumption crucial for joint probability calculations.
- ▶ The Poisson distribution to model catching probabilities.

Probability in Biology: Hobbs 4.2

- ▶ Investigating decomposition of leaf litter over time.
- ▶ Using a simple model of exponential decay: $\mu_t = e^{-kt}$.
- ▶ Data: y_t - observed proportions, **modeled with a beta distribution**.
- ▶ Parameters: k (decay rate) and σ^2 (variance).

Beta Distribution for y_t

- Model the probability density of y_t with a beta distribution:

$$y_t | \mu_t, \sigma^2 \sim \text{beta}(\alpha_t, \beta_t)$$

- Moment matching for α_t and β_t :

$$\alpha_t = \mu_t^2 - \mu_t^3 - \mu_t \sigma^2$$

$$\beta_t = \mu_t - 2\mu_t^2 + \mu_t^3 - \sigma^2 + \mu_t \sigma^2$$

Conditional on Decay Rate k and σ^2

- ▶ Conditional on known, fixed decay rate $k = 0.01 \text{ day}^{-1}$ and known, fixed $\sigma^2 = 6 \times 10^{-4}$:
- ▶ Calculate parameters for the beta distribution on day 30:

$$\alpha_{30} = 236.33$$

$$\beta_{30} = 82.68$$

Probability Density Calculation

- ▶ Given $y_{30} = 0.7$, calculate the probability density:

$$f(y_{30} = 0.7) = 4.040$$

- ▶ Interpretation: The probability that 70% of the mass remains at time $t = 30$ is 4.040.

Remarks

- ▶ The beta distribution to model decay over time.
- ▶ Moment matching provides a method for estimating distribution parameters.

Introduction to Likelihood

- ▶ Likelihood measures the support provided by the observed data for different values of the parameter in a statistical model.
- ▶ The likelihood function is the foundation of maximum likelihood estimation (MLE).

Likelihood Function

- ▶ The likelihood function, denoted as $L(\theta; \mathbf{x})$, represents the probability of observing the given data \mathbf{x} for various parameter values θ in the model.
- ▶ The likelihood function is not a probability distribution but provides a basis for estimating parameters.

Likelihood Example: Coin Toss

- ▶ Consider a simple example: coin toss.
- ▶ Let θ be the probability of getting heads ($\theta \in [0, 1]$).
- ▶ If we observe k heads in n tosses, the likelihood function is given by the binomial distribution:

$$L(\theta; k, n) = \binom{n}{k} \theta^k (1 - \theta)^{n-k}$$

Interpretation of Likelihood

- ▶ Likelihood is not a probability, but it measures the compatibility of the observed data with different parameter values.
- ▶ Larger likelihood values indicate a better fit of the model to the observed data.
- ▶ The goal is to find the parameter values that maximize the likelihood, known as maximum likelihood estimation (MLE).

Likelihood in MLE

- ▶ Maximum Likelihood Estimation (MLE) aims to find the parameter values that maximize the likelihood function.
- ▶ MLE is a common method for estimating parameters in statistical models.
- ▶ It provides point estimates that make the observed data most probable under the assumed model.

Likelihood Example: Population Growth

- ▶ Consider a simple population growth model: $N_t = N_0 \cdot e^{rt}$, where N_t is the population size at time t , N_0 is the initial population size, r is the growth rate, and e is the base of the natural logarithm.
- ▶ Likelihood function: $L(r|\mathbf{y})$, where \mathbf{y} is the observed population size over time.

Likelihood Example: Phylogenetic Trees

- ▶ In evolutionary biology, likelihood is extensively used in phylogenetic analysis.
- ▶ Given a phylogenetic tree and DNA sequence data, the likelihood of observing the given sequences under different substitution models is calculated.
- ▶ MLE finds the tree and model parameters that maximize the probability of the observed data.

Probability Distribution vs. Likelihood Function

- ▶ Key distinction: Treatment of parameters and data.
- ▶ In a probability distribution:
 - ▶ Parameter is fixed.
 - ▶ Data are random variables.
- ▶ In a likelihood function:
 - ▶ Data are fixed.
 - ▶ Parameters are variable.

Examining the Relationship Through Plots

- ▶ Probability density function with a fixed θ and varying y :
 - ▶ Area under the curve equals 1 (Figure 4.2.1A).
- ▶ Probability density function with fixed y and varying θ :
 - ▶ Likelihood profile obtained (Figure 4.2.1B).
 - ▶ Area under the curve does not equal 1.

Relationship for Discrete Data

- ▶ Probability mass function $y|\theta$ with fixed θ and varying y :
 - ▶ Sum of probabilities equals 1 (Figure 4.2.1C).
- ▶ Probability mass function with fixed y and varying θ :
 - ▶ Likelihood profile generated (Figure 4.2.1D).
 - ▶ Sum of probabilities does not equal 1.

Units of the Likelihood Profile

- ▶ Units on the y-axis are arbitrary.
- ▶ Can be scaled to any quantity.
- ▶ Likelihood profile often scaled so that the peak equals 1.
- ▶ Scaling achieved by dividing all likelihoods by the maximum likelihood.
- ▶ Does not alter the relationship between likelihood and probability.

Illustration Through Figure 4.2.1

- ▶ Figure 4.2.1 visually illustrates the subtle distinction.
- ▶ Parameter θ is not fixed in the likelihood framework.
- ▶ Likelihood functions do not define the probability or probability density of θ .

Notation Considerations

- ▶ Some authors use $L(\theta; y)$ instead of $L(\theta|y)$.
- ▶ Emphasizes that likelihood functions focus on the dependence of parameters on fixed data.

Remarks

- ▶ Probability distributions and likelihood functions highlighted through plots.
- ▶ Probability and likelihood coincide **ONLY** when parameters are treated as fixed.

Likelihood as a Relative Measure

- ▶ Likelihood emphasizes the relativity of evidence.
- ▶ The meaningfulness of the likelihood of a parameter value is realized when compared to alternative values.

Arbitrary Scaling in Likelihood Profile

- ▶ Likelihood profiles use arbitrary scaling on the y-axis.
- ▶ The arbitrary constant 'c' allows the profile values to take any magnitude, limiting insights into a single parameter value.

Importance of Comparison

- ▶ Understanding parameters relies on comparing the likelihood of one value with another.
- ▶ The constant of proportionality becomes inconsequential in this comparative analysis.

Likelihood Ratios

- ▶ Likelihood ratios are crucial for comparing evidence for alternative parameter values.
- ▶ The ratio $\frac{L(\theta_1|y)}{L(\theta_2|y)}$ encapsulates the comparative evidence.

Interpreting Likelihood Ratios

- ▶ The likelihood principle states that all information about alternative parameter values is in the likelihood ratio.
- ▶ Interpretation involves understanding how the data supports one parameter value relative to another.

Logarithmic Transformation

- ▶ The natural logarithm of the likelihood ratio is often used to quantify support.
- ▶ Formally defined as the support for one parameter value over another, conditional on the data.

Basis for Comparison

- ▶ The ratio of likelihoods or the difference between log likelihoods establishes the foundation for evaluating evidence.
- ▶ Particularly important in ecological problems for determining the parameter value that garners maximum support.

Maximum Likelihood Estimate (MLE)

- ▶ MLE involves finding the parameter value that maximizes the likelihood function.
- ▶ Analytical methods for simple models, numerical techniques for complex scenarios.