Appendix A

Probability Distributions and Conjugate Priors

TABLE A.1
Probability Distributions Used in Ecological Modeling to Represent Stochasticity in Discrete Random Variables (z)

Distribution	*Random variable*	*Parameters*	*Moments*
Poisson $[z\|\lambda] = \frac{\lambda^z e^{-\lambda}}{z}$	Counts of things that occur randomly over time or space, e.g., the number of birds in a forest stand, the number of fish in a kilometer of river, the number of prey captured per minute	λ, the mean number of occurrences per time or space $\lambda = \mu$	$\mu = \lambda$ $\sigma^2 = \lambda$
Binomial $[z \mid \eta, \phi] = \binom{\eta}{z}\phi^z (1-\phi)^{\eta-z}$ $\binom{\eta}{z} = \frac{\eta!}{z!(\eta-z)!}$	Number of "successes" on a given number of trials, e.g., number of survivors in a sample of individuals, number of plots containing an exotic species from a sample, number of terrestrial pixels that are vegetated in an image	η, the number of trials ϕ, the probability of a success $\phi = 1 - \sigma^2/\mu$ $\eta = \mu^2/\left(\mu - \sigma^2\right)$	$\mu = \eta\phi$ $\sigma^2 = \eta\phi(1-\phi)$
Bernoulli $[z\|\phi] = \phi^z (1-\phi)^{1-z}$	A special case of the binomial where the number of trials= 1 and the random variable can take on values 0 or 1; widely used in survival analysis, occupancy models	ϕ, the probability that the random variable= 1 $\phi = \mu$ $\phi = 1/2 + 1/2\sqrt{1-4\sigma^2}$	$\mu = \phi$ $\sigma^2 = \phi(1-\phi)$
Negative binomial $[z\|\lambda, \kappa] = \frac{\Gamma(z+\kappa)}{\Gamma(\kappa)z!}\left(\frac{\kappa}{\kappa+\lambda}\right)^\kappa \times \left(\frac{\lambda}{\kappa+\lambda}\right)^z$	Counts of things occurring randomly over time or space, as with the Poisson; includes dispersion parameter κ allowing the variance to exceed the mean	λ, the mean number of occurrences per time or space κ, the dispersion parameter $\lambda = \mu$ $\kappa = \mu^2/(\sigma^2 - \mu)$	$\mu = \lambda$ $\sigma^2 = \lambda + \lambda^2/\kappa$

TABLE A.1
(continued)

Distribution	*Random variable*	*Parameters*	*Moments*
Multinomial $[\mathbf{z} \mid \eta, \phi] = \eta! \prod_{i=1}^{k} \frac{\phi_i^{z_i}}{z_i}$	Counts that fall into $k > 2$ categories, e.g., number of individuals in age classes, number of pixels in different landscape categories, number of species in trophic categories in a sample from a food web	$\mathbf{z}$, a vector giving the number of counts in each category ϕ, a vector of the probabilities of occurrence in each category $\sum_{i=1}^{k} \phi_i = 1$ $\sum_{i=1}^{k} z_i = \eta$	$\mu_i = \eta\phi_i$ $\sigma_i^2 = \eta\phi_i(1-\phi_i)$

Note: We use μ to symbolize the first moment of the distribution, $\mu = E(z)$, and σ^2 to symbolize the second central moment, $\sigma^2 = E\left((z-\mu)^2\right)$.

TABLE A.2
Probability distributions Used in Ecological Modeling to Represent Stochasticity in Continuous Random Variables (z)

Continuous Distributions	*Random variable* (z)	*Parameters*	*Moments*
Normal $\left[z\vert\mu, \sigma^2\right] = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$	Continuously distributed quantities that can take on positive or negative values; sums of things.	μ, σ^2	μ, σ^2
Lognormal $[z \mid \alpha, \beta]$ $\frac{1}{z\sqrt{2\pi\beta^2}} e^{-\frac{(\log(z)-\alpha)^2}{2\beta^2}}$	Continuously distributed quantities with nonnegative values. Random variables with the property that their logs are normally distributed. Thus, if z is normally distributed, then exp(z) is lognormally distributed. Represents products of things. The variance increases with the mean squared.	α, the mean of z on the log scale β, the standard deviation of z on the log scale $\alpha = \log(\text{median}(z))$ $\alpha = \log(\mu) - 1/2 \log\left(\frac{\sigma^2+\mu^2}{\mu^2}\right)$ $\beta = \sqrt{\log\left(\frac{\sigma^2+\mu^2}{\mu^2}\right)}$	$\mu = \mathrm{e}^{\alpha+\frac{\beta^2}{2}}$ $\text{median}(z_i) = e^{\alpha}$ $\sigma^2 = (e^{\beta^2} - 1)e^{2\alpha+\beta^2}$

TABLE A.2
(*continued*)

Continuous Distributions	*Random variable* (z)	*Parameters*	*Moments*
Gamma $[z\|\alpha,\beta]=\frac{\beta^{\alpha}}{\Gamma(\alpha)}z^{\alpha-1}e^{-\beta z}$ $\Gamma(\alpha)=\int_0^{\infty}t^{\alpha-1}e^{-t}\,\mathrm{d}t$.	The time required for a specified number of events to occur in a Poisson process; any continuous quantity that is nonnegative.	$\alpha=$ shape $\beta=$ rate $\alpha=\frac{\mu^2}{\sigma^2}$ $\beta=\frac{\mu}{\sigma^2}$ Note–be very careful about rate, defined as above, and scale = $\frac{1}{\beta}$.	$\mu=\frac{\alpha}{\beta}$ $\sigma^2=\frac{\alpha}{\beta^2}$
Inverse gamma $[z\|\alpha,\beta]=\frac{\beta^{\alpha}}{\Gamma(\alpha)}z^{-\alpha-1}\exp\left(\frac{-\beta}{z}\right)$	The reciprocal of a gamma-distributed random variable.	$\alpha=$ shape $\beta=$ scale $\alpha=\frac{\mu^2}{\sigma^2}+2$ $\beta=\mu\left(\frac{\mu^2}{\sigma^2}+1\right)$	$\mu=\frac{\beta}{\alpha-1}$ for $\alpha>1$ $\sigma^2=\frac{\beta^2}{(\alpha-1)^2(\alpha-2)}$ for $\alpha>2$
Exponential $[z\|\alpha,\beta]=\lambda e^{-\lambda z}$	Intervals of time between sequential events that occur randomly over time or space. If the number of events is Poisson distributed, then the times between events are exponentially distributed.	λ, the mean number of occurrences per time or space $\lambda=\frac{1}{\mu}$	$\mu=\frac{1}{\lambda}$ $\sigma^2=\left(\frac{1}{\lambda}\right)^2$
Beta $[z\|\alpha,\beta]=B\,z^{\alpha-1}(1-z)^{\beta-1}$ $B=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$ Because B is a normalizing constant, $[z\mid\alpha,\beta]\propto z^{\alpha-1}(1-z)^{\beta-1}$	Continuous random variables that can take on values between 0; and 1, any random variable that can be expressed as a proportion; survival; proportion of landscape invaded by exotic; probabilities of transition from one state to another.	$\alpha=\frac{(\mu^2-\mu^3-\mu\sigma^2)}{\sigma^2}$ $\beta=\frac{\mu-2\mu^2+\mu^3-\sigma^2+\mu\sigma^2}{\sigma^2}$	$\mu=\frac{\alpha}{\alpha+\beta}$ $\sigma^2=\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
Dirichlet $[z\|\boldsymbol{\alpha}]=\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)\times\frac{\prod_{j=1}^{k}z_j^{\alpha_j-1}}{\Gamma\left(\alpha_j\right)}$	Vectors of more than two elements of continuous random variables that can take on values between 0 and 1 and that sum to 1.	$\alpha_i=\mu_i\alpha_0$ $\alpha_0=\sum_{i=1}^{k}\alpha_i$	$\mu_i=\frac{\alpha_i}{\sum_{i=1}^{k}\alpha_i}$ $\sigma_i^2=\frac{\alpha_i(\alpha_0-\alpha_i)}{\alpha_0^2(\alpha_0+1)}$,

TABLE A.2
(*continued*)

Continuous Distributions	*Random variable* (z)	*Parameters*	*Moments*
Uniform $[z\|\alpha, \beta] =$ $\frac{1}{\beta-\alpha}$ for $\alpha \le z \le \beta$, 0 for $z < \alpha$ or $z > \beta$	Any real number.	$\alpha =$ lower limit $\beta =$ upper limit $\alpha = \mu - \sigma\sqrt{3}$ $\beta = \mu + \sigma\sqrt{3}$	$\mu = \frac{\alpha+\beta}{2}$ $\sigma^2 = \frac{(\beta-\alpha)^2}{12}$

Note: We use μ to symbolize the first moment of the distribution, $\mu = E(z)$, and σ^2 to symbolize the second central moment, $\sigma^2 = E\left((z-\mu)^2\right)$.

TABLE A.3
Table of Conjugate Distributions

Likelihood	*Prior distribution*	*Posterior distribution*
$y_i \sim$ binomial (n, ϕ)	$\phi \sim$ beta (α, β)	$\phi \sim$ beta $\left(\sum y_i + \alpha, n - \sum y_i + \beta\right)$
$y_i \sim$ Bernoulli(ϕ)	$\phi \sim$ beta (α, β)	$\phi \sim$ beta $\left(\sum_{i=1}^n y_i + \alpha, \sum_{i=1}^n (1-y_i) + \beta\right)$
$y_i \sim$ Poisson (λ)	$\lambda \sim$ gamma (α, β)	$\lambda \sim$ gamma $\left(\alpha + \sum_{i=1}^n y_i,\ \beta + n\right)$
$y_i \sim$ normal$\left(\mu, \sigma^2\right)$; σ^2 is known	$\mu \sim$ normal(μ_0, σ_0^2)	$\mu \sim$ normal $\left(\frac{\left(\frac{\mu_0}{\sigma_0^2} + \frac{\sum_{i=1}^n y_i}{\sigma^2}\right)}{\left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right)}, \left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right)^{-1}\right)$
$y_i \sim$ normal$\left(\mu, \sigma^2\right)$; μ is known.	$\sigma^2 \sim$ inverse gamma(α, β)	$\sigma^2 \sim$ inverse gamma $\left(\alpha + \frac{n}{2}, \beta + \frac{\sum_{i=1}^n (y_i-\mu)^2}{2}\right)$
$y_i \sim$ lognormal(μ, σ^2); μ is known.	$\sigma^2 \sim$ inverse gamma(α, β),	$\sigma^2 \sim$ inverse gamma $\left(n/2 + \alpha, \frac{(\sum_{i=1}^n \log(y_i) - \mu)^2}{2} + \beta\right)$
$y_i \sim$ lognormal $\left(\mu, \sigma^2\right)$; σ^2 is known.	$\mu \sim$ normal $\left(\mu_0, \sigma_0^2\right)$	$\mu \sim$ normal $\left(\frac{\left(\frac{\mu_0}{\sigma_0^2} + \frac{\sum_{i=1}^n \log y_i}{\sigma^2}\right)}{\left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right)}, \left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right)^{-1}\right)$