In[1234]:=

```
Nvar = 1;
ε = 0.1;
c = Nvar / (Nvar + 1) ε;
N = 4;

A0 = Table[Piecewise[{{2, i == j}, {-1, (i == j - 1) || (i == j + 1)}}, 0], {i, 1, N}, {j, 1, N}];
δA = Table[Piecewise[{{c / (i + j), i ≠ j}}, 0], {i, 1, N}, {j, 1, N}];
A = A0 + δA;
Ahat = A[[All, 1 ;; -2]];

{Q, R} = QRDecomposition[Ahat];
Q = Transpose@Q; (* for some reason Mathematica returns Q as transposed *)

x0 = Table[i * i, {i, 1, N - 1}];
b = Ahat.x0;

LLSx = Inverse[R].Transpose[Q].b;
LLSx2 = LeastSquares[Ahat, b]; (* Should give the same result as formula above *)

eigenvals = Reverse@Eigenvalues[A];
{ShurQ, ShurT} = SchurDecomposition[A];
(* Should give the same eigenvalues as method above *)
(* Eigenvalues will be stored on the main diagonal of 'T' *)
{HessP, HessH} = HessenbergDecomposition[A];

Framed@"Problem"
Row@{"A₀ = ", A0 // MatrixForm}
Row@{"δA = ", δA // MatrixForm}
Row@{"A = ", A // MatrixForm}
Row@{"Â = ", Ahat // MatrixForm}
Framed@"QR decomposition"
Row@{"Q = ", Q // N // MatrixForm}
Row@{"R = ", R // N // MatrixForm}
Row@{"QᵀQ = ", Transpose[Q].Q // MatrixForm}
Row@{"QR = ", Q.R // MatrixForm}
Framed@"LLS"
Row@{"x₀ = ", x0 // MatrixForm}
Row@{"b = ", b // MatrixForm}
Row@{"x_LLS (formula) = ", LLSx // MatrixForm}
Row@{"x_LLS (built-in) = ", LLSx2 // MatrixForm}
Framed@"Eigenvalue problem"
Row@{"{λᵢ}_{i=1}^N = ", eigenvals // MatrixForm}
```

```
Row@{"P_Hessenberg = ", HessP // MatrixForm}
Row@{"H_Hessenberg = ", HessH // MatrixForm}
Row@{"P_Hess H_Hess P_Hess^T = ", HessP.HessH.Transpose[HessP] // MatrixForm}
Row@{"Q_Shur = ", ShurQ // MatrixForm}
Row@{"T_Shur = ", ShurT // MatrixForm}
Row@{"Q_Shur T_Shur Q_Shur^T = ", ShurQ.ShurT.Transpose[ShurQ] // MatrixForm}
```

Out[1251]=

Problem

Out[1252]=

$$A_0 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

Out[1253]=

$$\delta A = \begin{pmatrix} 0 & 0.0166667 & 0.0125 & 0.01 \\ 0.0166667 & 0 & 0.01 & 0.00833333 \\ 0.0125 & 0.01 & 0 & 0.00714286 \\ 0.01 & 0.00833333 & 0.00714286 & 0 \end{pmatrix}$$

Out[1254]=

$$A = \begin{pmatrix} 2 & -0.983333 & 0.0125 & 0.01 \\ -0.983333 & 2 & -0.99 & 0.00833333 \\ 0.0125 & -0.99 & 2 & -0.992857 \\ 0.01 & 0.00833333 & -0.992857 & 2 \end{pmatrix}$$

Out[1255]=

$$\hat{A} = \begin{pmatrix} 2 & -0.983333 & 0.0125 \\ -0.983333 & 2 & -0.99 \\ 0.0125 & -0.99 & 2 \\ 0.01 & 0.00833333 & -0.992857 \end{pmatrix}$$

Out[1256]=

QR decomposition

Out[1257]=

$$Q = \begin{pmatrix} -0.897375 & -0.360924 & -0.195125 \\ 0.44121 & -0.726754 & -0.397811 \\ -0.0056086 & 0.584353 & -0.604274 \\ -0.00448688 & -0.00970474 & 0.662213 \end{pmatrix}$$

Out[1258]=

$$R = \begin{pmatrix} -2.22872 & 1.77035 & -0.454777 \\ 0. & -1.67719 & 1.89332 \\ 0. & 0. & -1.47464 \end{pmatrix}$$

Out[1259]=

$$Q^T Q = \begin{pmatrix} 1. & 3.8706 \times 10^{-17} & 4.25007 \times 10^{-17} \\ 3.8706 \times 10^{-17} & 1. & 1.9082 \times 10^{-17} \\ 4.25007 \times 10^{-17} & 1.9082 \times 10^{-17} & 1. \end{pmatrix}$$

Out[1260]=

$$QR = \begin{pmatrix} 2. & -0.983333 & 0.0125 \\ -0.983333 & 2. & -0.99 \\ 0.0125 & -0.99 & 2. \\ 0.01 & 0.00833333 & -0.992857 \end{pmatrix}$$

Out[1261]=

LLS

Out[1262]=

$$x_0 = \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix}$$

Out[1263]=

$$b = \begin{pmatrix} -1.82083 \\ -1.89333 \\ 14.0525 \\ -8.89238 \end{pmatrix}$$

Out[1264]=

$$x_{LLS}\ (\text{formula}) = \begin{pmatrix} 1. \\ 4. \\ 9. \end{pmatrix}$$

Out[1265]=

$$x_{LLS}\ (\text{built-in}) = \begin{pmatrix} 1. \\ 4. \\ 9. \end{pmatrix}$$

Out[1266]=

Eigenvalue problem

Out[1267]=

$$\{\lambda_i\}_{i=1}^{N} = \begin{pmatrix} 0.411851 \\ 1.37333 \\ 2.6081 \\ 3.60672 \end{pmatrix}$$

Out[1268]=

$$P_{Hessenberg} = \begin{pmatrix} 1. & 0. & 0. & 0. \\ 0. & -0.999868 & -0.0124878 & 0.01044 \\ 0. & 0.0127102 & -0.999688 & 0.0215112 \\ 0. & 0.0101681 & 0.0216411 & 0.999714 \end{pmatrix}$$

Out[1269]=

$$H_{Hessenberg} = \begin{pmatrix} 2. & 0.983464 & 0. & 0. \\ 0.983464 & 2.02474 & -0.979765 & 3.46945 \times 10^{-18} \\ 0. & -0.979765 & 2.01824 & 1.0023 \\ 0. & 0. & 1.0023 & 1.95703 \end{pmatrix}$$

Out[1270]=

$$P_{Hess} H_{Hess} P_{Hess}^{T} = \begin{pmatrix} 2. & -0.983333 & 0.0125 & 0.01 \\ -0.983333 & 2. & -0.99 & 0.00833333 \\ 0.0125 & -0.99 & 2. & -0.992857 \\ 0.01 & 0.00833333 & -0.992857 & 2. \end{pmatrix}$$

Out[1271]=

$$Q_{Shur} = \begin{pmatrix} -0.369478 & 0.603705 & 0.365213 & -0.604686 \\ 0.599839 & -0.371863 & 0.601326 & -0.374591 \\ -0.603372 & -0.366152 & 0.605026 & 0.368535 \\ 0.37366 & 0.602653 & 0.372787 & 0.598513 \end{pmatrix}$$

Out[1272]=

$$T_{Shur} = \begin{pmatrix} 3.60672 & 1.2837 \times 10^{-16} & 7.27229 \times 10^{-17} & 1.08263 \times 10^{-16} \\ 0. & 2.6081 & 2.34037 \times 10^{-16} & 3.69448 \times 10^{-16} \\ 0. & 0. & 0.411851 & -2.20616 \times 10^{-17} \\ 0. & 0. & 0. & 1.37333 \end{pmatrix}$$

Out[1273]=

$$Q_{Shur} T_{Shur} Q_{Shur}^{T} = \begin{pmatrix} 2. & -0.983333 & 0.0125 & 0.01 \\ -0.983333 & 2. & -0.99 & 0.00833333 \\ 0.0125 & -0.99 & 2. & -0.992857 \\ 0.01 & 0.00833333 & -0.992857 & 2. \end{pmatrix}$$