Simplex Method

October 15, 2018

Problem

Given the following linear system and objective function, find the optimal solution.

$$\max 8x_1 + 6x_2 + 2x_3$$

$$\begin{cases} 2x_1 + x_2 + x_3 \leq 4 \\ x_1 + 4x_2 \leq 3 \\ 1/2x_2 + x_3 \leq 6 \end{cases}$$

Solution

Add slack variables to turn all inequalities to equalities.

$$\begin{cases} 2x_1 + x_2 + x_3 + s_1 = 4 \\ x_1 + 4x_2 + s_2 = 3 \\ 1/2x_2 + x_3 + s_3 = 6 \end{cases}$$

Create the initial tableau of the new linear system.

$$\begin{array}{c c} \left[\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & b \\ \hline 2 & 1 & 1 & 1 & 0 & 0 & 4 \\ 1 & 4 & 0 & 0 & 1 & 0 & 3 \\ 0 & 1/2 & 1 & 0 & 0 & 1 & 6 \\ \hline -8 & -6 & -2 & 0 & 0 & 0 & 0 \end{array}\right] & \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ \\ \end{array} \end{array}$$

There are negative elements in the bottom row, so the current solution is not optimal. Thus, pivot to improve the current solution. The entering variable is x_1 and the departing variable is s_1.

Perform elementary row operations until the pivot element is 1 and all other elements in the entering column are 0.

$$\begin{array}{c c} \left[\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & b \\ \hline 1 & 1/2 & 1/2 & 1/2 & 0 & 0 & 2 \\ 0 & 7/2 & -1/2 & -1/2 & 1 & 0 & 1 \\ 0 & 1/2 & 1 & 0 & 0 & 1 & 6 \\ \hline 0 & -2 & 2 & 4 & 0 & 0 & 16 \end{array}\right] & \begin{array}{c} \\ x_1 \\ s_2 \\ s_3 \\ \\ \end{array} \end{array}$$

There are negative elements in the bottom row, so the current solution is not optimal. Thus, pivot to improve the current solution. The entering variable is x_2 and the departing variable is s_2.

Perform elementary row operations until the pivot element is 1 and all other elements in the entering column are 0.

$$\begin{array}{c} \left[\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & b \\ \hline 1 & 0 & 4/7 & 4/7 & -1/7 & 0 & 13/7 \\ 0 & 1 & -1/7 & -1/7 & 2/7 & 0 & 2/7 \\ 0 & 0 & 15/14 & 1/14 & -1/7 & 1 & 41/7 \\ \hline 0 & 0 & 12/7 & 26/7 & 4/7 & 0 & 116/7 \end{array}\right] \end{array} \begin{array}{c} \\ x_1 \\ x_2 \\ s_3 \\ \\ \end{array}$$

There are no negative elements in the bottom row, so we know the solution is optimal. Thus, the solution is:

$$s_1 = 0, s_2 = 0, s_3 = \frac{41}{7}, x_1 = \frac{13}{7}, x_2 = \frac{2}{7}, x_3 = 0, z = \frac{116}{7}$$