MATHEMATICAL FINANCE CHEAT SHEET

Normal Random Variables

A random variable X is Normal $\mathbf{N}(\mu,\sigma^2)$ (aka. *Gaussian*) under a measure $\mathbf{P}$ if and only if

$$\mathbf{E_P}\left[e^{\theta X}\right]=e^{\theta\mu+\frac{1}{2}\theta^2\sigma^2},\quad \text{for all real } \theta.$$

A standard normal $Z\sim\mathbf{N}(0,1)$ under a measure $\mathbf{P}$ has density

$$\phi(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}.\qquad \mathbf{P}[Z\le x]=\Phi(x):=\int_{-\infty}^{x}\phi(z)\,dz.$$

Let $X=(X_1,X_2,\dots,X_n)'$ with $X_i\sim\mathbf{N}(\mu_i,q_{ii})$ and $\mathbf{Cov}[X_i,X_j]=q_{ij}$ for $i,j=1,\dots,n$. We call $\mu:=(\mu_1,\dots,\mu_n)'$ the *mean* and $Q:=(q_{ij})_{i,j=1}^n$ the *covariance matrix* of X. Assume $\det Q>0$, then X has a *multivariate normal distribution* if it has the density

$$\phi(x)=\frac{1}{\sqrt{(2\pi)^n\det Q}}\exp\left(-\frac{1}{2}(x-\mu)'Q^{-1}(x-\mu)\right),\quad x\in\mathbf{R}^n.$$

We write $X\sim\mathbf{N}(\mu,Q)$ if this is the case. Alternatively, $X\sim\mathbf{N}(\mu,Q)$ under $\mathbf{P}$ if and only if

$$\mathbf{E_P}[e^{\theta' X}]=\exp\left(\theta'\mu+\frac{1}{2}\theta'Q\theta\right),\quad\text{for all }\theta\in\mathbf{R}^n.$$

If $Z\sim\mathbf{N}(0,Q)$ and $c\in\mathbf{R}^n$ then $X=c'Z\sim\mathbf{N}(0,c'Qc)$. If $C\in\mathbf{R}^{m\times n}$ (i.e., $m\times n$ matrix) then $X=CZ\sim\mathbf{N}(0,CQC')$ and CQC' is a $m\times m$ covariance matrix.

Gaussian Shifts

If $Z\sim\mathbf{N}(0,1)$ under a measure $\mathbf{P}$, h is an integrable function, and c is a constant then

$$\mathbf{E_P}[e^{cZ}h(Z)]=e^{c^2/2}\mathbf{E_P}[h(Z+c)].$$

Let $X\sim\mathbf{N}(0,Q)$, h be a integrable function of $x\in\mathbf{R}^n$, and $c\in\mathbf{R}^n$. Then

$$\mathbf{E_P}[e^{c'X}h(X)]=e^{\frac{1}{2}c'Qc}\mathbf{E_P}[h(X+c)].$$

Correlating Brownian Motions

Let $(W(t))_{t\ge0}$ and $(\widetilde{W}(t))_{t\ge0}$ be independent Brownian motions. Given a correlation coefficient $\rho\in[-1,1]$, define

$$\widehat{W}(t):=\rho W(t)+\sqrt{1-\rho^2}\,\widetilde{W}(t),$$

then $(\widehat{W}(t))_{t\ge0}$ is a Brownian motion and $\mathbf{E}[W(t)\widehat{W}(t)]=\rho t$.

Identifying Martingales

If $X_t=X(t)$ is a diffusion process satisfying

$$dX(t)=\mu(t,X_t)\,dt+\sigma(t,X_t)dW(t)$$

and $\mathbf{E_P}[(\int_0^T\sigma(s,X_s)^2\,ds)^{1/2}]<\infty$ (or, $\sigma(t,x)\le c|x|$ as $|x|\to\infty$), then

X is a martingale $\iff$ X is driftless (i.e., $\mu(t)\equiv0$ with $\mathbf{P}$-prob. 1).

Novikov's Condition

In the case $dX(t)=\sigma(t)X(t)\,dW(t)$ for some $\mathscr{F}$-previsible process $(\sigma(t))_{t\ge0}$, then we have the simpler condition

$$\mathbf{E_P}\left[\exp\left(\frac{1}{2}\int_0^T\sigma(s)^2\,ds\right)\right]<\infty\Rightarrow X\text{ is a martingale.}$$

Itô's Formula

For $X_t=X(t)$ given by $dX(t)=\mu(t)\,dt+\sigma(t)\,dW(t)$ and a function $g(t,x)$ that is twice differentiable in x and once in t. Then for $Y(t)=g(t,X_t)$, we have

$$dY(t)=\frac{\partial g}{\partial t}(t,X_t)\,dt+\frac{\partial g}{\partial x}(t,X_t)\,dX_t+\frac{1}{2}\sigma(t)^2\frac{\partial^2 g}{\partial x^2}(t,X_t)dt.$$

The Product Rule

Given $X(t)$ and $Y(t)$ adapted to the same Brownian motion $(W(t))_{t\ge0}$,

$$dX(t)=\mu(t)dt+\sigma(t)dW(t),\quad dY(t)=\nu(t)\,dt+\rho(t)\,dW(t).$$

Then $d(X(t)Y(t))=X(t)\,dY(t)+Y(t)\,dX(t)+\underbrace{d\langle X,Y\rangle(t)}_{\sigma(t)\rho(t)\,dt}$.

In the other case, if $X(t)$ and $Y(t)$ are adapted to two different and independent Brownian motions $(W(t))_{t\ge0}$ and $(\widetilde{W}(t))_{t\ge0}$,

$$dX(t)=\mu(t)\,dt+\sigma(t)\,dW(t),\quad dY(t)=\nu(t)\,dt+\rho(t)\,d\widetilde{W}(t).$$

Then $d(X(t)Y(t))=X(t)\,dY(t)+Y(t)\,dX(t)$ as $d\langle X,Y\rangle(t)=0$.

Radon-Nikodým Derivative

Given $\mathbf{P}$ and $\mathbf{Q}$ equivalent measures and a time horizon T, we can define a random variable $\frac{d\mathbf{Q}}{d\mathbf{P}}$ defined on $\mathbf{P}$-possible paths, taking positive real values, such that

- $\mathbf{E_Q}[X_T]=\mathbf{E_P}\left[\frac{d\mathbf{Q}}{d\mathbf{P}}X_T\right]$, for all claims X_T knowable by time T,
- $\mathbf{E_Q}[X_t|\mathscr{F}_s]=\zeta_s^{-1}\mathbf{E_P}[\zeta_t X_t|\mathscr{F}_s]$, for $s\le t\le T$,

where ζ_t is the process $\mathbf{E_P}[\frac{d\mathbf{Q}}{d\mathbf{P}}|\mathscr{F}_t]$.

Cameron-Martin-Girsanov Theorem

If $(W(t))_{t\ge0}$ is a $\mathbf{P}$-Brownian motion and $(\gamma(t))_{t\ge0}$ is an $\mathscr{F}$-previsible process satisfying the boundedness condition $\mathbf{E_P}\left[\exp\left(\frac{1}{2}\int_0^T\gamma(t)^2\,dt\right)\right]<\infty$, then there exists a measure $\mathbf{Q}$ such that:

- $\mathbf{Q}$ is equivalent to $\mathbf{P}$,
- $\frac{d\mathbf{Q}}{d\mathbf{P}}=\exp\left(-\int_0^T\gamma(t)\,dW(t)-\frac{1}{2}\int_0^T\gamma(t)^2\,dt\right)$,
- $\widetilde{W}(t):=W(t)+\int_0^t\gamma(s)\,ds$ is a $\mathbf{Q}$-Brownian motion.

In other words, $W(t)$ is a drifting $\mathbf{Q}$-Brownian motion with drift $-\gamma(t)$ at time t.

Cameron-Martin-Girsanov Converse

If $(W(t))_{t\ge0}$ is a $\mathbf{P}$-Brownian motion, and $\mathbf{Q}$ is a measure equivalent to $\mathbf{P}$, then there exists a $\mathscr{F}$-previsible process $(\gamma(t))_{t\ge0}$ such that

$$\widetilde{W}(t):=W(t)+\int_0^t\gamma(s)\,ds$$

is a $\mathbf{Q}$-Brownian motion. That is, $W(t)$ plus drift $\gamma(t)$ is a $\mathbf{Q}$-Brownian motion. Additionally,

$$\frac{d\mathbf{Q}}{d\mathbf{P}}=\exp\left(-\int_0^t\gamma(t)\,dW(t)-\frac{1}{2}\int_0^T\gamma(t)^2\,dt\right).$$

Martingale Representation Theorem

Suppose $(M(t))_{t\ge0}$ is a $\mathbf{Q}$-martingale process whose volatility $\sqrt{\mathbf{E_Q}[M(t)^2]}=\sigma(t)$ satisfies $\sigma(t)\ne0$ for all t (with $\mathbf{Q}$-probability one). Then if $(N(t))_{t\ge0}$ is any other $\mathbf{Q}$-martingale, there exists an $\mathscr{F}$-previsible process $(\phi(t))_{t\ge0}$ such that $\int_0^T\phi(t)^2\sigma(t)^2\,dt<\infty$ (with $\mathbf{Q}$-prob. one), and N can be written as

$$N(t)=N(0)+\int_0^t\phi(s)\,dM(s),$$

or in differential form, $dN(t)=\phi(t)\,dM(s)$. Further, ϕ is (essentially) unique.

Multidimensional Diffusions, Quadratic Covariation, and Itô's Formula

If $X:=(X_1,X_2,\dots,X_n)'$ is a n-dimensional diffusion process with form

$$X(t)=X(0)+\int_0^t\mu(s)\,ds+\int_0^t\Sigma(s)\,dW(s),$$

where $\Sigma(t)\in\mathbf{R}^{n\times m}$ and W is a m-dimensional Brownian motion. The *quadration covariation* of the components X_i and X_j is

$$\langle X_i,X_j\rangle(t)=\int_0^t\Sigma_i(s)'\Sigma_j(s)\,ds,$$

or in differential form $d\langle X_i,X_j\rangle(t)=\Sigma_i(t)'\Sigma_j(t)\,dt$, where $\Sigma_i(t)$ is the i^{th} column of $\Sigma(t)$. The *quadratic variation* of $X_i(t)$ is $\langle X_i\rangle(t)=\int_0^t\Sigma_i(s)'\Sigma_i(s)\,ds$.

The *multi-dimensional Itô formula* for $Y(t)=f(t,X_1(t),\dots,X_n(t))$ is

$$dY(t)=\frac{\partial f}{\partial t}(t,X_1(t),\dots,X_n(t))dt+\sum_{i=1}^n\frac{\partial f}{\partial x_i}(t,X_1(t),\dots,X_n(t))dX_i(t)$$
$$+\frac{1}{2}\sum_{i,j=1}^n\frac{\partial^2 f}{\partial x_i\partial x_j}(t,X_1(t),\dots,X_n(t))d\langle X_i,X_j\rangle(t).$$

The *(vector-valued) multi-dimensional Itô formula* for

$$Y(t)=f(t,X(t))=(f_1(t,X(t)),\dots,f_n(t,X(t)))'$$

where $f_k(t,X)=f_k(t,X_1,\dots,X_n)$ and $Y(t)=(Y_1(t),Y_2(t),\dots,Y_n(t))'$ is given componentwise (for $k=1,\dots,n$) as

$$dY_k(t)=\frac{\partial f_k(t,X(t))}{\partial t}dt+\sum_{i=1}^n\frac{\partial f_k(t,X(t))}{\partial x_i}dX_i(t)$$
$$+\frac{1}{2}\sum_{i,j=1}^n\frac{\partial^2 f_k(t,X(t))}{\partial x_i\partial x_j}d\langle X_i,X_j\rangle(t).$$

Stochastic Exponential

The *stochastic exponential* of X is $\mathscr{E}_t(X)=\exp(X(t)-\frac{1}{2}\langle X\rangle(t))$. It satisfies

$$\mathscr{E}(0)=1,\quad \mathscr{E}(X)\mathscr{E}(Y)=\mathscr{E}(X+Y)e^{\langle X,Y\rangle},\quad \mathscr{E}(X)^{-1}=\mathscr{E}(-X)e^{\langle X,X\rangle}.$$

The process $Z=\mathscr{E}(X)$ is a positive process and solves the SDE

$$dZ=Z\,dX,\quad Z(0)=e^{X(0)}.$$

Solving Linear ODEs

The linear *ordinary differential equation*

$$\frac{dz(t)}{dt}=m(t)+\mu(t)z(t),\quad z(a)=\zeta,$$

for $a\le t\le b$ has solution given by

$$z(t)=\zeta\epsilon_t+\int_a^t\epsilon_t\epsilon_u^{-1}m(u)\,du,\qquad \epsilon_t:=\exp\left(\int_a^t\mu(u)\,du\right),$$
$$=\zeta\exp\Big(\int_a^t\mu(u)\,du\Big)+\int_a^t m(u)\exp\Big(\int_u^t\mu(r)\,dr\Big)\,du.$$

Solving Linear SDEs

The linear *stochastic differential equation*

$$dZ(t)=[m(t)+\mu(t)Z(t)]\,dt+[q(t)+\sigma(t)Z(t)]\,dW(t),\quad Z(a)=\zeta,$$

for $a\le t\le b$ has solution given by

$$Z(t)=\zeta\mathscr{E}_t+\int_a^t\mathscr{E}_t\mathscr{E}_u^{-1}[m(u)-q(u)\sigma(u)]\,du+\int_a^t\mathscr{E}_t\mathscr{E}_u^{-1}q(u)\,dW(u),$$

where $\mathscr{E}_t:=\mathscr{E}_t(X)$ and $X(t)=\int_a^t\mu(u)\,du+\int_a^t\sigma(u)\,dW(u)$. In other words,

$$\mathscr{E}_t=\exp\left(\int_a^t\mu(u)\,du+\int_a^t\sigma(u)\,dW(u)-\frac{1}{2}\int_a^t\sigma(u)^2\,du\right).$$

Fundamental Theorem of Asset Pricing

Let X be some $\mathscr{F}_T$-measurable claim, payable at time T. The arbitrage-free price $\mathscr{V}$ of X at time t is

$$\mathscr{V}(t) = \mathbf{E}_{\mathbf{Q}}\left[\exp\left(-\int_t^T r(s)\,ds\right)X\middle|\mathscr{F}_t\right],$$

where $\mathbf{Q}$ is the risk-neutral measure.

Market Price Of Risk

Let $X_t = X(t)$ be the price of a non-tradable asset with dynamics $dX(t) = \mu(t)\,dt + \sigma(t)d\,W(t)$ where $(\sigma(t))_{t\geq 0}$ and $(\mu(t))_{t\geq 0}$ are previsible processes and $(W(t))_{t\geq 0}$ is a $\mathbf{P}$-Brownian motion. Let $Y(t) := f(X_t)$ be the price of a tradable asset where $f : \mathbf{R} \to \mathbf{R}$ is a deterministic function. Then the *market price of risk* is

$$\gamma(t) := \frac{\mu_t f'(X_t) + \frac{1}{2}\sigma_t^2 f''(X_t) - r f(X_t)}{\sigma_t f'(X_t)},$$

and the behaviour of X_t under the risk-neutral measure $\mathbf{Q}$ is given by

$$dX(t) = \sigma(t)d\widetilde{W}(t) + \frac{r f(X_t) - \frac{1}{2}\sigma_t^2 f''(X_t)}{f'(X_t)}\,dt.$$

Black's Model

Consider a European option with strike price K on a asset with value V_T at maturity time T. Let F_T be the forward price of V_T, F_0 the current forward price. If $\log V_T \sim \mathbf{N}(F_0, \sigma^2 T)$ then the Call and Put prices are given by

$$\mathscr{C} = P(0,T)(F_0\Phi(d_1) - K\phi(d_2)),\ \mathscr{P} = P(0,T)(K\Phi(-d_2) - F_0\Phi(-d_1)),$$

where $d_1 = \dfrac{\log(\mathbf{E}(V_T)/K) + \sigma^2 T/2}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

Forward Rates, Short Rates, Yields, and Bond Prices

The *forward rate* at time t that applies between times T and S is defined as

$$F(t,T,S) = \frac{1}{S-T}\log\frac{P(t,T)}{P(t,S)}.$$

The *instantaneous forward rate* at time t is $f(t,T) = \lim_{S\to T} F(t,T,S)$. The *instantaneous risk-free rate* or *short rate* is $r(t) = \lim_{T\to t} f(t,T)$. The *cash account* is given by

$$B(t) = \exp\left(\int_0^t r(s)\,ds\right),$$

and satisfies $dB(t) = r(t)B(t)\,dt$ with $B(0) = 1$. The instantaneous forward rates and the yield can be written in terms of the bond prices as

$$f(t,T) = -\frac{\partial}{\partial T}\log P(t,T),\quad R(t,T) = -\frac{\log P(t,T)}{T-t}.$$

Conversely,

$$P(t,T) = \exp\left(-\int_t^T f(t,u)\,du\right)\quad\text{and}\quad P(t,T) = \exp(-(T-t)R(t,T)).$$

Short Rate and No-Arbitrage Models

The short-rate $r(t)$ follows a process of the form

$$dr(t) = a(t,r(t))\,dt + b(t,r(t))\,dW(t),$$

where $a(t,r)$ and $b(t,r)$ are chosen following:

Model	$a(t,r)$	$b(t,r)$	P?	MR?	CF?
Merton	μ	σ	N	N	Y
Dothan	$\mu r(t)$	$\sigma r(t)$	Y	N	Y
Vasicek	$\alpha(\mu - r(t))$	σ	N	Y	Y
CIR	$\alpha(\mu - r(t))$	$\sigma\sqrt{r(t)}$	Y	Y	Y
Pearson-Sun	$\alpha(\mu - r(t))$	$\sigma\sqrt{r(t)-\beta}$	Y	Y	Y
Ho & Lee	$\theta(t)$	σ	N	N	Y
Hull & White	$\alpha(\mu(t) - r(t))$	σ	N	Y	Y
Extended Vasicek	$\alpha(t)(\mu(t) - r(t))$	$\sigma(t)$	N	Y	Y
Black-Karasinski	$\alpha r(t)(\mu(t) - \ln r(t))$	$\sigma(t)r(t)$	Y	Y	N

P means the process stays positive, MR means r_t is mean-reverting, and CF means that a closed-form solution exists for bond prices and for European put and call options.

Bond Pricing for Affine Models

Given an affine short-rate model

$$dr(t) = (b(t) + \beta(t)r(t))dt + \sqrt{a(t) + \alpha(t)r(t)}\,d\widetilde{W}(t),$$

the zero-coupon T-bond prices at time t have the form

$$P(t,T) = \exp(A(t,T) - B(t,T)r(t)),$$

where the functions A and B satisfy the system of ODEs:

$$\frac{d}{dt}A(t,T) = -\frac{1}{2}a(t)B^2(t,T) + b(t)B(t,T),\qquad A(T,T) = 0,$$
$$\frac{d}{dt}B(t,T) = \frac{1}{2}\alpha(t)B^2(t,T) - \beta(t)B(t,T) - 1,\qquad B(T,T) = 0.$$

The Heath-Jarrow-Morton Framework

Given a initial forward curve $T \mapsto f(0,T)$ then, for every maturity T and under the real-world probability measure $\mathbf{P}$, the forward rate process $t \mapsto f(t,T)$ follows

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)\,ds + \int_0^t \sigma(s,T)'\,dW(s),\quad t \leq T,$$

where $\alpha(t,T) \in \mathbf{R}$ and $\sigma(t,T) := (\sigma_1(t,T),\ldots,\sigma_n(t,T))$ satisfy the technical conditions: (1) α and σ are previsible and adapted to $\mathscr{F}_t$; (2) $\int_0^T\int_0^T |\alpha(s,t)|\,ds\,dt < \infty$ for all T; (3) $\sup_{s,t\leq T}\|\sigma(s,t)\| < \infty$ for all T. The short-rate process is given by

$$r(t) = f(t,t) = f(0,t) + \int_0^t \alpha(s,t)\,ds + \int_0^t \sigma(s,t)\,dW(s),$$

so the cash account and zero coupon T-bond prices are well-defined and obtained through

$$B(t) = \exp\left(\int_0^t r(s)\,ds\right),\quad P(t,T) = \exp\left(-\int_t^T f(t,u)\,du\right).$$

The discounted asset price $Z(t,T) = P(t,T)/B(t)$ satisfies

$$dZ(t,T) = Z(t,T)\Big[\underbrace{\left(\frac{1}{2}S^2(t,T) - \int_t^T \alpha(t,u)\,du\right)}_{b(t,T)}dt + S(t,T)'\,dW(t)\Big],$$

where $S(s,T) := -\int_s^T \sigma(s,u)\,du$. The *HJM drift condition* states that

$$\mathbf{Q}\text{ is EMM (i.e., no arbitrage for bonds)}\iff b(t,T) = -S(t,T)\gamma(t)',$$

where $\widetilde{W}(t) := W(t) - \int_0^t \gamma(s)\,ds$ is a $\mathbf{Q}$-Brownian motion. If this holds, then under $\mathbf{Q}$, the forward rate process follows

$$f(t,T) = f(0,T) + \int_0^t \underbrace{\left(\sigma(s,T)\int_s^T \sigma(s,u)'\,du\right)}_{\text{HJM drift}}ds + \int_0^t \sigma(s,T)\,d\widetilde{W}(s),$$

and the discounted asset $Z(t,T)$ satisfies $dZ(t,T) = Z(0,T)\mathscr{E}_t(X)$ with

$$X(t) = \int_0^t S(s,T)'\,d\widetilde{W}(s).$$

The LIBOR Market Model

For a tenor $\delta > 0$, the *LIBOR rate* $L(T,T,T+\delta)$ is the rate such that an investment of 1 at time T will grow to $1 + \delta L(T,T,T+\delta)$ at time $T+\delta$. The *forward LIBOR rate* (i.e., a contract made at time t under which we pay 1 at time T and receive back $1 + \delta L(t,T,T+\delta)$ at time $T+\delta$) is defined as

$$L(t,T) := L(t,T,T+\delta) = \frac{1}{\delta}\left(\frac{P(t,T)}{P(t,T+\delta)} - 1\right),$$

and satisfies $L(T,T) = L(T,T,T+\delta)$.
Under the real-world probability measure $\mathbf{P}$, The LMM assumes that each LIBOR process $(L(t,T_m))_{0\leq t\leq T_m}$ satisfies

$$dL(t,T_m) = L(t,T_m)\left[\mu(t,L(t,T_m))\,dt + \lambda_m(t,L(t,T_m))'\,dW(t)\right],$$

where $W = (W^1,\ldots,W^d)$ is a d-dimensional Brownian motion with instantaneous correlations

$$d\langle W^i, W^j\rangle(t) = \rho_{i,j}(t)\,dt,\quad i,j = 1,2,\ldots,d.$$

The function $\lambda(t,L) : [0,T_j]\times\mathbf{R} \to \mathbf{R}^{N\times d}$ is the volatility, and $\mu(t) : [0,T_j] \to \mathbf{R}$ is the drift.
Let $0 \leq m, n \leq N-1$. Then the dynamics of $L(t,T_m)$ under the forward measure $\mathbf{P}_{T_{n+1}}$ is for $m < n$ given by

$$dL(t,T_m) = L(t,T_m)\left[-\lambda(t,T_m)\sum_{r=m+1}^{n}\sigma_{T_r,T_{r+1}}(t)'\,dt + \lambda(t,T_m)d\,W^m(t)\right]$$

For $m = n$,

$$dL(t,T_m) = L(t,T_m)\lambda(t,T_m)d\,W_t^m$$

and for $m > n$ we have

$$dL(t,T_m) = L(t,T_m)\left[\lambda(t,T_m)\sum_{r=n+1}^{m}\sigma_{T_r,T_{r+1}}(t)'\,dt + \lambda(t,T_m)d\,W_t^m\right]$$