

Algebra on two dimensions

Math and modeling for high school, Lecture 2

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LINEAR GRAPHS

Here are two linear graphs.
Note that both cross the origin.



How do we find a vector,
parallel to the graph?

Answer: any pair (x, y) that solve the line equation is such vector.

Example: consider line $y = -3x$. The following vectors lie on that line: $(1, -3)$, $(-1, 3)$, $(3, -9)$, etc. In a more standard vector notation they are:

$$\mathbf{r} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} 3 \\ -9 \end{bmatrix} \quad (1)$$

Note that scaling a vector does not change the fact that it lies on the line: if $\mathbf{r} = \begin{bmatrix} r_x \\ r_y \end{bmatrix}$ lies on a line \mathcal{L} , then $\lambda\mathbf{r}$ also lies on \mathcal{L} (where $\lambda \in \mathbb{R}$).

In fact, we can say the following about a line:

Theorem

If $\mathbf{r} = \begin{bmatrix} r_x \\ r_y \end{bmatrix}$ lies on a line \mathcal{L} , then all points on \mathcal{L} can be found as $\lambda \mathbf{r}$.

Let us prove it. If $\mathcal{L} = \{(x, y) : y = ax\}$ and $\mathbf{r} \in \mathcal{L}$, then $r_y = ar_x$. Then $\lambda r_y = \lambda ar_x$ and $(\lambda r_x, \lambda r_y) \in \mathcal{L}$. □

LINEAR GRAPHS

That was difficult-looking. But graphically, the proof is quite obvious. We are trying to prove that any point on the line can be reached by scaling the vector lying on that line:



How do we plot a line? If we want to plot it as a collection of points \mathbf{p}_i , we could generate them by the following formula:

$$\mathbf{p}_i = \lambda_i \mathbf{v} \tag{2}$$

where \mathbf{v} is a vector on that line and λ_i is a sequence of numbers, for example:

$$\lambda_i = -10, -9.99, -9.98, \dots 10 \tag{3}$$

Then we can plug the points \mathbf{p}_i into Python library `matplotlib.pyplot.plot`.

If we have two vectors which are not co-linear, we can describe any point on the plane as their linear combination:

Let \mathbf{r}_A be the position of point A. It can be found as
$$\mathbf{r}_A = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2$$



Let us examine the equation $\mathbf{r}_A = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2$ more carefully. First, we can re-write in in matrix form:

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \mathbf{r}_A \quad (4)$$

And we know how to solve it. Denoting the matrix $\mathbf{M} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$, we can say that as long as \mathbf{M} is full rank, the system can be solved exactly. And \mathbf{M} is full rank, as long as \mathbf{v}_1 and \mathbf{v}_2 are linearly independent.

We can call λ_1 and λ_2 *coordinates* of \mathbf{r}_A in the *basis* $\mathbf{v}_1, \mathbf{v}_2$.

If $\mathbf{M} = [\mathbf{v}_1 \quad \mathbf{v}_2]$ is full rank, we say that it *spans the plane*. That simply means - any point on the plane can be expressed by its coordinates λ_1 and λ_2 .

If you know coordinates λ_1 and λ_2 of a point B , you can find its position:

$$\mathbf{r}_B = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 \quad (5)$$

If we want to check whether or not two vectors in \mathbb{R}^2 are not parallel (are linearly independent), we can see if the following determinant Δ is not zero:

$$\Delta = \det([\mathbf{v}_1 \quad \mathbf{v}_2]) \quad (6)$$

DOT PRODUCT

Given two vectors $\mathbf{a} = [a_x \ a_y]$ and $\mathbf{b} = [b_x \ b_y]$, you can define their dot product as:

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y \quad (7)$$

We can also define dot product as follows:

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos(\varphi) \quad (8)$$

where φ is the angle between the two vectors. The two definitions can be used to compute the angle between vectors:

$$\cos(\varphi) = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} \quad (9)$$

DOT PRODUCT

Some examples. If $\mathbf{a} = [1 \ 0]$ and $\mathbf{b} = [0 \ 2]$, you can find their dot product as $\mathbf{a} \cdot \mathbf{b} = 1 \cdot 0 + 0 \cdot 2 = 0$.

Meaning the angle between the two vectors is:

$$\cos(\varphi) = \frac{0}{1 \cdot 2} = 0, \quad \varphi = \pi/2 \quad (10)$$

If $\mathbf{a} = [1 \ 1]$ and $\mathbf{b} = [-2 \ 0]$, their dot product is:

$$\mathbf{a} \cdot \mathbf{b} = 1 \cdot (-2) + 1 \cdot 0 = -2 \quad (11)$$

The angle between the two vectors is:

$$\cos(\varphi) = \frac{-2}{\sqrt{2} \cdot 2} = -\sqrt{2}/2, \quad \varphi = 3\pi/4 \quad (12)$$

Given one line, how do we find another, perpendicular (*orthogonal*) to it?

Given one vector, how do we find another, perpendicular to it? Well, by a reasonable definition, two orthogonal vectors would have angle $\varphi = \pi/2$ between them. So, if \mathbf{a} and \mathbf{b} are orthogonal, their dot product is:

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos(\pi/2) = 0 \quad (13)$$

Which in turn means:

$$a_x b_x + a_y b_y = 0 \quad (14)$$

Consider the following problem. Given vector $\mathbf{r} = [r_x \ r_y]$, find vector $\mathbf{n} = [n_x \ n_y]$, orthogonal to it.

We know that their dot product has to be zero:

$$r_x n_x + r_y n_y = 0 \quad (15)$$

We can rewrite it in matrix form:

$$\begin{bmatrix} r_x & r_y \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} = 0 \quad (16)$$

$$\mathbf{r}^\top \mathbf{n} = 0 \quad (17)$$

Meaning we can find \mathbf{n} using `lstsq`.

We could find \mathbf{n} using `lstsq`. But it looks too complicated for such a simple task.

Alternatively, we can consider $r_x n_x + r_y n_y = 0$. From this, $n_y = -\frac{r_x}{r_y} n_x$. So, for example, vector $\mathbf{n} = [1 \ -\frac{r_x}{r_y}]$ is orthogonal to \mathbf{r} .

Note that in any case, we can decide we want \mathbf{n} to have certain lengths, let us say $\|\mathbf{n}\| = 1$. To achieve that we can *normalize* it:

$$\mathbf{n}^* = \begin{bmatrix} 1 \\ -r_x/r_y \end{bmatrix} \quad (18)$$

$$\mathbf{n} = \frac{\mathbf{n}^*}{\|\mathbf{n}^*\|} \quad (19)$$

AFFINE LINES

If a line does not cross the origin, its equation would have the *affine* form $y = ax + b$:

If $x = 0$, then $y = b$.



AFFINE LINES

Consider the problem: find a line \mathcal{L} that is h -distance away from the line $y = ax$:



The line \mathcal{L} is given by its equation $y = ax + b$. All we need to do is to find b , since a is already known. Here is one way to solve it:

- Find \mathbf{v} on the line $y = ax$; $\mathbf{v} = [1, a]$.
- Find \mathbf{n} orthogonal to \mathbf{v} , $\|\mathbf{n}\| = 1$ (see previous slides).
- Point $h\mathbf{n} = [hn_x, hn_y]$ lies on \mathcal{L} , so $hn_y = ahn_x + b$.
- From this, we know that $b = h(n_y - an_x)$. Done.

Too many steps for such a simple problem? Maybe. But - it is hard to make a mistake doing it this way.

THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Extra-math-for-high-school

Check Moodle for additional links, videos, textbook suggestions.

