Derivatives

Math and modeling for high school

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Consider a curve and its *tangent*:

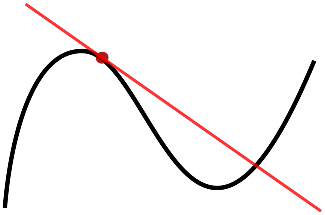


Figure 1: Curve and its tangent

Given a curve $\mathscr{L}$: $y = f(x)$ we can find a *tangent* line $y = ax + b$ to the curve at point x_0. A tangent is such a line that touches the curve $\mathscr{L}$ at the point $(x_0,\, f(x))$

The tangent is such a line that touches the curve $\mathscr{L}$ at the point $(x_0,\, f(x_0))$ and has the same *rate of change* as the function at this point.

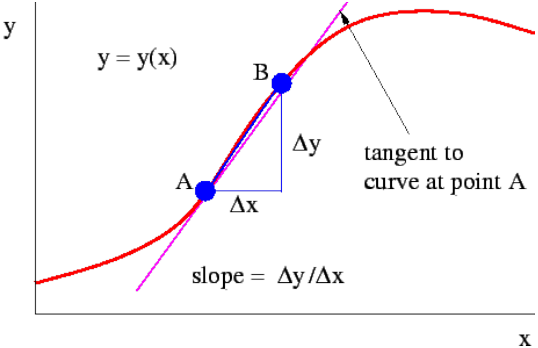
Rate of change of a function

What is a rate of change of a function? Consider function $y = f(x)$. How does y changes if you increase x a little?

We can ask about it in this way: Let $\Delta y = f(x + \Delta x) - f(x)$, where Δx is a small increase in x, and Δy captures the corresponding increase in y. What is the ration of Δy to Δx?

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{1}$$

Rate of change of a function



Making Δx smaller and smaller we find the true slope of the function:

$$a = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{2}$$

Notice that the tangent line to a function $y = f(x)$ at a point x_0 is given its equation:

$$y = ax + b \tag{3}$$

$$a = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \tag{4}$$

$$b = f(x_0) - ax_0 \tag{5}$$

We have a special name for this limit $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$. We call it *derivative of* $f(x)$.

Derivative

$\frac{d}{dx}f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ is called a derivative of $f(x)$ with respect to x.

Consider a function $y = x$. What is its derivative?

$$\frac{d}{dx}x = \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x} = 1 \tag{6}$$

Consider a function $y = 10x$. What is its derivative?

$$\frac{d}{dx}10x = \lim_{\Delta x \to 0} \frac{10x + 10\Delta x - 10x}{\Delta x} = 10 \tag{7}$$

Consider a function $y = x^2$. What is its derivative?

$$\frac{d}{dx}(x^2) = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} =$$
$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x} =$$
$$= \lim_{\Delta x \to 0} \frac{2x\Delta x}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta x^2}{\Delta x} = 2x$$

So, $\frac{d}{dx}(x^2) = 2x$.

Properties of derivatives

In general, derivatives have some interesting properties:

Distributive

$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

Multiplicative

$\frac{d}{dx}(af(x)) = a\frac{d}{dx}f(x)$

In general, derivative is a linear operation.

Consider a function $y = x^n$. Its derivative can be found as:

Derivative of x^n

$\frac{d}{dx}x^n = nx^{n-1}$

Let us see some examples:

$$\frac{d}{dx}x^7 = 7x^6 \tag{8}$$

$$\frac{d}{dx}2x^3 = 6x^2 \tag{9}$$

$$\frac{d}{dx}(x^2 - x + 6) = 2x - 1 \tag{10}$$

Coming back to our original example. We want to find a tangent line to a function $y = f(x)$.

The slope of the tangent like is given by the derivative of $y = f(x)$: $a = \frac{d}{dx} f(x)$, evaluated as the point x_0.

There are two equivalent ways to define the tangent line. First is to give it as $y = ax + b$, where $b = f(x_0) - ax_0$. Second is to give it as $y = a(x - x_0) + c$, where $c = f(x_0)$. Both are used in practice.

Now we can find an angle between the slope of the tangent line and the horizon. How? Remember the second lecture: we find a vector on the line, in this case $[1,\ a]$ would do, and find an angle it makes with the horizontal vector $[1,\ 0]$ using dot product.

But we do not need to go by such a complicated rout. There is a property of a tangent line (which you can prove on your own) that the derivative of $y = f(x)$ is equal to the tangent of the angle between its slope and the horizon:

$$\tan(\varphi) = \frac{d}{dx} f(x) \tag{11}$$

Knowing a tangent vector $[1,\ a]$ to a function $y = f(x)$ we can find a normal vector to that function. How? Remember that normal vector is orthogonal to a tangent vector, so we can find it by considering their dot product.

Why would you need to know normal vectors? In practice, they are needed for applications from 3D printing to walking robotics - in finding normals to surfaces. Consider a task: draw a curve that is one unit away from the curve $y = 10x^2$ in the normal direction. See if you can solve it by using derivatives, tangents and normals.

Thank you!

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Extra-math-for-high-school

Check Moodle for additional links, videos, textbook suggestions.