Algebra on three dimensions

Math and modeling for high school, Lecture 3

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Given two vectors $\mathbf{a} = [a_x \ a_y \ a_z]$ and $\mathbf{b} = [b_x \ b_y \ b_z]$, their cross product is:

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \times \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix} \tag{1}$$

Vector $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. Another definition of cross product is:

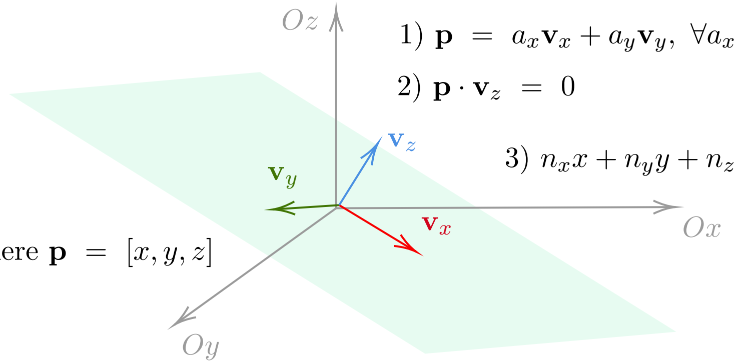
$$\mathbf{a} \times \mathbf{b} = ||\mathbf{a}|| \ ||\mathbf{b}|| \sin(\varphi) \ \mathbf{n} \tag{2}$$

where $\mathbf{n}$ is a unit vector, orthogonal to both $\mathbf{a}$ and $\mathbf{b}$, whose direction can be determined with a right-hand-rule.

Note two important properties of the cross product. First, if the angle between the two vectors is 0, meaning they are parallel, their cross product will have to be zero, since $\sin(0) = 0$. This allows cross product to be used to detect linear dependence of two vectors in 3D.

Also, cross product can be used to produce a vector, orthogonal to a pair of other vectors.

Consider a plane in 3D space, passing through the origin. We can characterize it with two vectors $\mathbf{v}_x$, $\mathbf{v}_y$ lying on it, and/or with a vector orthogonal to it $\mathbf{v}_z$.



1) $\mathbf{p} \ = \ a_x \mathbf{v}_x + a_y \mathbf{v}_y, \ \forall a_x, a_y$

2) $\mathbf{p} \cdot \mathbf{v}_z \ = \ 0$

3) $n_x x + n_y y + n_z z = 0$

where $\mathbf{p} \ = \ [x, y, z]$

Any point $\mathbf{p}$ on the plane can be found as a linear combination of $\mathbf{v}_x$, $\mathbf{v}_y$ and it is orthogonal to $\mathbf{v}_z$. Both of these define a plane.

Planes

Consider a plane, for which we know that vector $\mathbf{n} = [1 \;\; -2 \;\, 1]$ is orthogonal to it. Then, for any point $\mathbf{p} = [x \;\, y \;\, z]$ on the plane it is true that:

$$\mathbf{n} \cdot \mathbf{p} = 0 \tag{3}$$

$$x - 2y + z = 0 \tag{4}$$

This makes clear the connection between the two *representations* of a plane: in $n_x x + n_y y + n_z z = 0$ representation the coefficients n_x, n_y, n_z are the elements of the vector $\mathbf{n}$ in $\mathbf{n} \cdot \mathbf{p} = 0$ representation.

Conversely, if a plane is described as $ax + by + cx = 0$, then vector $[a \;\, b \;\, c]$ is orthogonal to it.

In the $\mathbb{R}^2$ two linear independent vectors span the whole $\mathbb{R}^2$ space.

Theorem

If two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are lying on a plane ρ and are linearly independent, they span ρ.

...which just means any point $\mathbf{p} \in \rho$ can be expressed as $\mathbf{p} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2$, where $\alpha_1, \alpha_2 \in \mathbb{R}$.

Theorem

If three vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent, they span $\mathbb{R}^3$.

Consider plane ρ and two linearly independent vectors $\mathbf{v}_1$, $\mathbf{v}_2 \in \rho$. Then, coordinates of any point $\mathbf{p} \in \rho$ can be described as:

$$\mathbf{p} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 \tag{5}$$

where α_1 and α_2 can be found as:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}^+ \mathbf{p} \tag{6}$$

which is a way to say "use least squares methods, with matrix $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$ and vector $\mathbf{p}$". The $^+$ reads as "pseudoinverse."

Consider three linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$. Then, coordinates of any point $\mathbf{p} \in \mathbb{R}^3$ can be described as:

$$\mathbf{p} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 \tag{7}$$

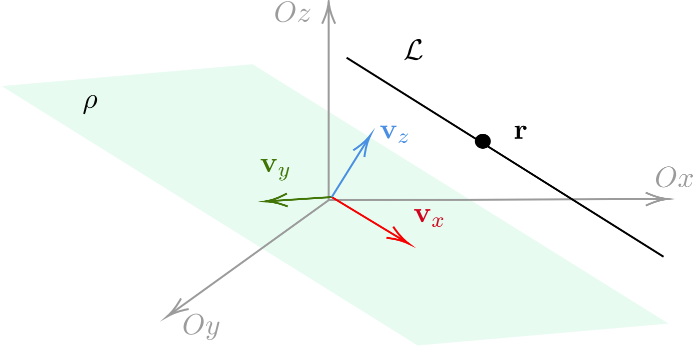
where α_1, α_2, α_3 can be found as:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}^{-1} \mathbf{p} \tag{8}$$

which is a way to say "solve system with matrix $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$ and vector $\mathbf{p}$". The $^{-1}$ reads as "inverse."

Line parallel to a plane

Let us check if a line $\mathcal{L}$ and a plane ρ are parallel. If they are, let us find distance between them.



$\mathcal{L}$ is defined by a point $\mathbf{r} \in \mathcal{L}$ and vector $\mathbf{u} \parallel \mathcal{L}$. Plane ρ passes through the origin and is defined by its normal $\mathbf{v}_z$ and tangents $\mathbf{v}_x$, $\mathbf{v}_y$.

Line parallel to a plane

First, let us observe that for the line $\mathcal{L}$ to be parallel to the plane, it has to be orthogonal to the plane's normal, i.e. $\mathbf{v}_z$. Which is the same as $\mathbf{u} \cdot \mathbf{v}_z = 0$.

Second, let us find how $\mathbf{r}$ can be expressed as a linear combination of $\mathbf{v}_x$, $\mathbf{v}_y$ and $\mathbf{v}_z$:

$$\mathbf{r} = \alpha_x \mathbf{v}_x + \alpha_y \mathbf{v}_y + \alpha_z \mathbf{v}_z \tag{9}$$

$$\begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix} = \begin{bmatrix} \mathbf{v}_x & \mathbf{v}_y & \mathbf{v}_z \end{bmatrix}^{-1} \mathbf{r} \tag{10}$$

Now consider point $\mathbf{r}_\rho = \alpha_x \mathbf{v}_x + \alpha_y \mathbf{v}_y \in \rho$. It is the closest point on ρ to the point $\mathbf{r}$. So the distance to the line is $||\mathbf{r}_\rho - \mathbf{r}|| = ||\alpha_z \mathbf{v}_z|| = \alpha_z ||\mathbf{v}_z||$ and if $||\mathbf{v}_z|| = 1$, then the distance between line and plane is α_z.

Wait, but why is $\mathbf{r}_\rho = \alpha_x \mathbf{v}_x + \alpha_y \mathbf{v}_y \in \rho$ the closest point to on ρ to the point $\mathbf{r}$?

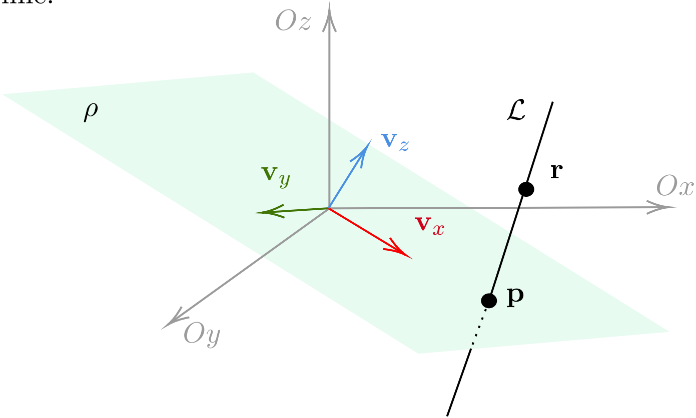
Assume the opposite. Let $\mathbf{y} = \mathbf{r}_\rho + \beta_x \mathbf{v}_x + \beta_y \mathbf{v}_y$ be the point which is closer to $\mathbf{r}$. Let us find distances $||\mathbf{r}_\rho - \mathbf{r}||$ and $||\mathbf{y} - \mathbf{r}||$:

$$||\mathbf{r}_\rho - \mathbf{r}|| = ||\alpha_z \mathbf{v}_z|| = \alpha_z \tag{11}$$

$$||\mathbf{y} - \mathbf{r}|| = ||\beta_x \mathbf{v}_x + \beta_y \mathbf{v}_y + \alpha_z \mathbf{v}_z|| \geq \alpha_z \tag{12}$$

Note that the last inequality is due to Pythagorean theorem, as both $\mathbf{v}_x$ and $\mathbf{v}_y$ are orthogonal to $\mathbf{v}_z$. $\square$

Consider a plane ρ and a line $\mathcal{L}$ passing through the point $\mathbf{r}$. Find coordinates of the point of intersection $\mathbf{p}$ of the plane and line.



Direction of the $\mathcal{L}$ is defined by vector $\mathbf{u}$. Orientation of ρ is given by vector $\mathbf{v}_z$.

We know that $\mathbf{p} \in \mathcal{L}$, so $\mathbf{p} = \lambda \mathbf{u} + \mathbf{r}$.

And $\mathbf{p} \in \rho$, so $\mathbf{p} \cdot \mathbf{v}_z = 0$, hence:

$$(\lambda \mathbf{u} + \mathbf{r}) \cdot \mathbf{v}_z = 0 \tag{13}$$

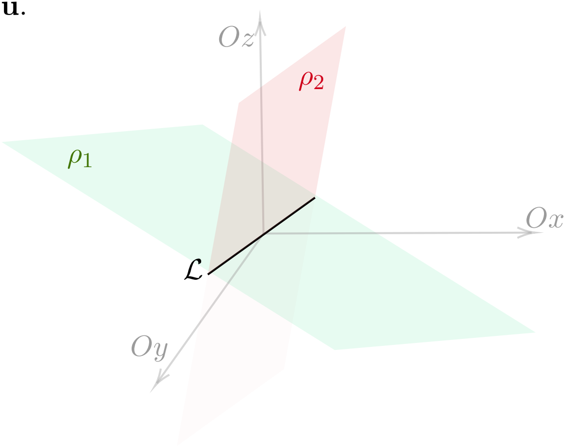
$$\lambda \mathbf{u} \cdot \mathbf{v}_z + \mathbf{r} \cdot \mathbf{v}_z = 0 \tag{14}$$

$$\lambda = -\frac{\mathbf{r} \cdot \mathbf{v}_z}{\mathbf{u} \cdot \mathbf{v}_z} \tag{15}$$

And thus we get our solution:

$$\mathbf{p} = -\frac{\mathbf{r} \cdot \mathbf{v}_z}{\mathbf{u} \cdot \mathbf{v}_z} \mathbf{u} + \mathbf{r} \tag{16}$$

Consider two planes ρ_1 and ρ_2, find a line $\mathcal{L}$ that lies on their intersection. The plane ρ_1 is given by its normal $\mathbf{n}_1$ and plane ρ_2 is given by its normal $\mathbf{n}_2$. Direction of $\mathcal{L}$ is given by a vector $\mathbf{u}$.



Since $0 \in \rho_1$ and $0 \in \rho_2$, meaning that $0 \in \mathcal{L}$.
Note that $\mathbf{u} \in \rho_1$ and $\mathbf{u} \in \rho_2$. Hence:

$$\mathbf{u} \cdot \mathbf{n}_1 = 0; \quad \mathbf{u} \cdot \mathbf{n}_2 = 0 \tag{17}$$

And we know how to find a vector in $\mathbb{R}^3$ perpendicular to two other non-parallel vectors in $\mathbb{R}^3$: we use cross-product.

$$\mathbf{u} = \mathbf{n}_1 \times \mathbf{n}_2 \tag{18}$$

If we need $\mathbf{u}$ to be unit length, we normalize it.

Thank you!

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Extra-math-for-high-school

Check Moodle for additional links, videos, textbook suggestions.