Differential equations

Math and modeling for high school

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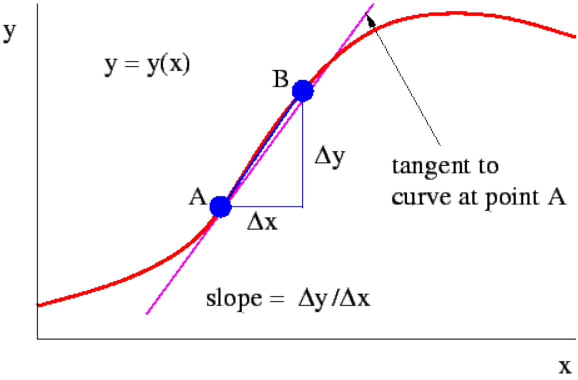
Fall 2022

Consider a function $y = y(x)$, with a derivative $\frac{d}{dx}y = \dot{y}$. Assume that $y(0) = 10$ and $\dot{y}(0) = 1$. Can we find (approximately) what $y(\Delta x)$ is, if Δx is very small?

We can reasonably say that $y(\Delta x) \approx y(0) + \dot{y}(0)\Delta x$.



More generally:

$$y(x + \Delta x) \approx y(x) + \dot{y}(x)\Delta x$$

If this approximation is accurate enough, it appears we could draw a graph of a function while knowing only its slopes (derivatives) $\dot{y} = \dot{y}(x)$.

Consider a "usual" function $y = y(x)$. For example, $y(x) = x^3 - 1$. We define it by identifying how y depends on x. In other words - we explain which y we expect for a given x.

But what if instead we identify what rate of change (slope) of the function we expect for a given x? Can we define a graph as $\dot{y} = \dot{y}(x)$?

Let us look at an example. Consider a function $\dot{y} = 2x$.

We remember that a derivative of x^2 is $2x$. So, is graph $\dot{y} = 2x$ equivalent to $y = x^2$?

Remember that $\frac{d}{dx}(x^2 - 1) = 2x$, and $\frac{d}{dx}(x^2 + 20) = 2x$, and $\frac{d}{dx}(x^2 + 3) = 2x$.

Differential equation

So, $\dot{y} = 2x$ is not equivalent to any one of those graphs, it actually describes a family of graphs $y = x^2 + c$, $c = const$. We call it *differential equation*.

Given differential equation $\dot{y} = 2x$ and knowing what the function is equal to at a single point (if $x = 0$, then $y = 7$ - as an example) is sufficient to recover the function; not a family of graphs, but one concrete graph.

Let us test it. We know that $y = x^2 + c$, and also that if $x = 0$, then $y = 7$. So, $7 = 0^2 + c$, hence $c = 7$. The graph we recovered is:

$$y = x^2 + 7 \tag{1}$$

If we know what y is when $x = 0$, this is called *initial conditions*. Knowing a differential equation and initial conditions we can *solve* it - which means we can recover a single graph it represents for those initial conditions.

This makes sense if we think back to our approximation $y(x + \Delta x) \approx y(x) + \dot{y}(x)\Delta x$. We can use it to continue a graph from a certain point, but not to draw it from scratch. It is very explicit at the point $x = 0$:

$$y(\Delta x) \approx y(0) + \dot{y}(0)\Delta x$$

If we know $y(0)$ and how to find $\dot{y}$, we can approximate the whole graph.

Some differential equations naturally make sense:

$$\dot{y} = x$$

$$\dot{y} = 4x^3 + 1$$

$$\dot{y} = -1$$

But they don't need to be simple. For example, we can write the following differential equation:

$$\dot{y} = -y$$

.

And indeed, if we know what y is when $x = 0$, then we can approximate its solution.

Thank you!

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Extra-math-for-high-school

Check Moodle for additional links, videos, textbook suggestions.