

Algebra in 3D: examples

Math and modeling for high school

by Sergei Savin

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PROBLEM STATEMENT

Given a cube $ABCD A_1 B_1 C_1 D_1$ with a side length 9. Point $K \in BB_1$, with $\|KB\| = 7$ (meaning the distance from K to B is 7). Plane α passes through K and C_1 , and is parallel to BD_1 . P is a point of intersection of α with $A_1 B_1$

- Prove that $\|A_1 P\|/\|PB_1\| = 2.5$.
- Find angle between α and $BB_1 C_1$.

Our approach here can be straight-forward: we compute everything that can be computed exactly.

Later we can discard useless steps, but it can be counter-productive to start discarding them before we know how we will arrive at the solution.

COMPUTATIONAL WAY OF LOOKING AT THE PROBLEM

For example, assuming that $C_1 = [0, 0, 0]$ (why? so α passes through the origin), and orienting the axes such that $D_1 = [9, 0, 0]$, $B_1 = [0, 0, 9]$, we get:

$$A_1 = [9, 0, 9] \tag{1}$$

$$A = [9, 9, 9] \tag{2}$$

$$B = [0, 9, 9] \tag{3}$$

$$C = [0, 9, 0] \tag{4}$$

$$D = [9, 9, 0] \tag{5}$$

Do we need all these points? No. Is it trivial to find their coordinates? Yes. Is it easier to think about the problem when you know all their coordinates? Yes.

Since $K \in BB_1$ and $\|KB\| = 7$, and $B = [0, 9, 9]$, $B_1 = [0, 0, 9]$, we can find coordinates of K :

$$K = [0, 2, 9] \quad (6)$$

There are two ways to arrive there. 1) You know that K is $7/9$ of the way between B and B_1 , so:

$$K = B + \frac{7}{9}(B_1 - B) = \frac{2}{9}B + \frac{7}{9}B_1.$$

Or, 2) The direction from B to B_1 is given as $\mathbf{b} = [0, -1, 0]$, and we know that the distance is 7, so $K = B + 7\mathbf{b}$.

COMPUTATIONAL WAY OF LOOKING AT THE PROBLEM

Plane α passes through K and C_1 , and is parallel to BD_1 . The fact that it passes through K and C_1 , it means that a vector from K to C_1 is tangent to the plane. So is a vector from B to D_1 . Let us find those two vectors:

$$\mathbf{r}_{BD_1} = B - D_1 = [0, 9, 9] - [9, 0, 0] = [-9, 9, 9] \quad (7)$$

$$\mathbf{r}_{KC_1} = K - C_1 = [0, 2, 9] - [0, 0, 0] = [0, 2, 9] \quad (8)$$

We can find the norm to the plane by taking a cross product of \mathbf{r}_{BD_1} and \mathbf{r}_{KC_1} :

$$\mathbf{n} = \mathbf{r}_{BD_1} \times \mathbf{r}_{KC_1} = \begin{bmatrix} 81 - 18 \\ 0 + 81 \\ -18 - 0 \end{bmatrix} = \begin{bmatrix} 63 \\ 81 \\ -18 \end{bmatrix} \quad (9)$$

FINDING POINT P

P is a point of intersection of α with A_1B_1 . Let us find a vector from A_1 to B_1 :

$$\mathbf{r}_{A_1B_1} = A_1 - B_1 = [9, 0, 9] - [0, 0, 9] = [9, 0, 0] \quad (10)$$

Let us denote coordinates of P as \mathbf{p} and coordinates of B_1 as \mathbf{b}_1 . We know that $\mathbf{p}^\top \mathbf{n} = 0$ and $\mathbf{p} = \lambda \mathbf{r}_{A_1B_1} + \mathbf{b}_1$. Therefore:

$$(\lambda \mathbf{r}_{A_1B_1} + \mathbf{b}_1)^\top \mathbf{n} = 0 \quad (11)$$

$$\lambda = -\frac{\mathbf{b}_1^\top \mathbf{n}}{\mathbf{r}_{A_1B_1}^\top \mathbf{n}} \quad (12)$$

$$\mathbf{b}_1^\top \mathbf{n} = 0 + 0 + 9 \cdot 2 = -162 \quad (13)$$

$$\mathbf{r}_{A_1B_1}^\top \mathbf{n} = -9 \cdot 11 + 0 + 0 = 567 \quad (14)$$

$$\lambda = -\frac{-162}{567} = 2/7 \quad (15)$$

TASK 1

We found that $\lambda = 2/7$ and $\mathbf{p} = \lambda \mathbf{r}_{A_1 B_1} + \mathbf{b}_1$. Then:

$$\mathbf{p} = 2/7[9, 0, 0] + [0, 0, 9] = [18/7, 0, 9] \quad (16)$$

We can find $\|A_1 P\|$ and $\|B_1 P\|$; These vectors are $A_1 P = [45/7, 0, 0]$ and $B_1 P = [18/7, 0, 0]$. With that we know:

$$\|A_1 P\| = 45/7 \quad (17)$$

$$\|B_1 P\| = 18/7 \quad (18)$$

$$\|A_1 P\|/\|B_1 P\| = 45/18 = 5/2 \quad (19)$$

TASK 2

Find angle between α and BB_1C_1 . This is the same as angle between the normals to these planes. First we find the norm to the plane BB_1C_1 :

$$B - B_1 = [0, 9, 0] \quad (20)$$

$$B_1 - C_1 = [0, 0, 9] \quad (21)$$

$$\mathbf{m} = (B - B_1) \times (B_1 - C_1) = [81, 0, 0] \quad (22)$$

With that we know normals to both α and BB_1C_1 , and can find the angle between them using dot product.

TASK 2

We know that:

$$\mathbf{m} \cdot \mathbf{n} = \|\mathbf{n}\| \|\mathbf{m}\| \cos(\varphi), \quad (23)$$

where φ is the angle we seek. We need to find $\mathbf{m} \cdot \mathbf{n}$, as well as $\|\mathbf{m}\|$ and $\|\mathbf{n}\|$:

$$\mathbf{m} \cdot \mathbf{n} = 63 \cdot 81 + 0 + 0 = 5103 \quad (24)$$

$$\|\mathbf{n}\| = \sqrt{10854} \quad (25)$$

$$\|\mathbf{m}\| = 81 \quad (26)$$

$$\cos(\varphi) = \frac{\mathbf{m} \cdot \mathbf{n}}{\|\mathbf{n}\| \|\mathbf{m}\|} = 5103 / (81\sqrt{10854}) \approx 0.6047 \quad (27)$$

$$\varphi = 52.79^\circ \quad (28)$$

THE CODE IN MATLAB FOR THIS PROBLEM IS:

```
0      dx = [9;0;0]; dy = [0;9;0]; dz = [0;0;9];
      C1 = sym([0;0;0]);
2      D1 = C1 + dx; A1 = C1 + dx + dz; B1 = C1 + dz;
      B = C1 + dy + dz;
4
      K = (2/9) * B + (7/9) * B1;
6      BD1 = B - D1;
      KC1 = K - C1;
8      n = cross(BD1, KC1);
      lambda = -dot(B1, n) / dot(dx, n);
10     P = lambda*dx + B1;
      A1P = A1 - P;
12     B1P = B1 - P;
      disp( norm(A1P) / norm(B1P) ) %task 1
14
      m = cross(dy, dz);
16     cos_phi = dot(n, m) / (norm(n) * norm(m));
      %task 2:
18     phi = round(double(acos(cos_phi))*180/pi, 3)
```

HOMEWORK

Consider a cube $ABCD A_1 B_1 C_1 D_1$ with a side length 3. Point $S_1 \in DB_1$, with $\|DS_1\| = 1$ (meaning the distance from D to S_1 is 1). Point $S_2 \in AB_1$, with $\|AS_2\| = 2$ (meaning the distance from A to S_2 is 2). Plane α_1 passes through S_1 , S_2 and D . Plane α_2 is orthogonal to $C_1 S_1$ through C . Point P is an intersection between the plane α_1 and the line passing through points D_1 , B .

- Find distance between P and A .
- Find angle between α_1 and α_2 .
- Find distance between α_1 and all vertices of the cube.
- Prove that P lies on $B_1 S_1$.

THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Extra-math-for-high-school

Check Moodle for additional links, videos, textbook suggestions.

