

# Algebra in 3D: examples

## Math and modeling for high school

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# PROBLEM STATEMENT

Given a cube  $ABCD A_1 B_1 C_1 D_1$  with a side length 9. Point  $K \in BB_1$ , with  $\|KB\| = 7$  (meaning the distance from  $K$  to  $B$  is 7). Plane  $\alpha$  passes through  $K$  and  $C_1$ , and is parallel to  $BD_1$ .  $P$  is a point of intersection of  $\alpha$  with  $A_1 B_1$

- Prove that  $\|A_1 P\|/\|PB_1\| = 2.5$ .
- Find angle between  $\alpha$  and  $BB_1 C_1$ .

Our approach here can be straight-forward: we compute everything that can be computed exactly.

Later we can discard useless steps, but it can be counter-productive to start discarding them before we know how we will arrive at the solution.

# COMPUTATIONAL WAY OF LOOKING AT THE PROBLEM

For example, assuming that  $C_1 = [0, 0, 0]$  (why? so  $\alpha$  passes through the origin), and orienting the axes such that  $D_1 = [9, 0, 0]$ ,  $B_1 = [0, 0, 9]$ , we get:

$$A_1 = [9, 0, 9] \tag{1}$$

$$A = [9, 9, 9] \tag{2}$$

$$B = [0, 9, 9] \tag{3}$$

$$C = [0, 9, 0] \tag{4}$$

$$D = [9, 9, 0] \tag{5}$$

Do we need all these points? No. Is it trivial to find their coordinates? Yes. Is it easier to think about the problem when you know all their coordinates? Yes.

Since  $K \in BB_1$  and  $\|KB\| = 7$ , and  $B = [0, 9, 9]$ ,  $B_1 = [0, 0, 9]$ , we can find coordinates of  $K$ :

$$K = [0, 2, 9] \quad (6)$$

There are two ways to arrive there. 1) You know that  $K$  is  $7/9$  of the way between  $B$  and  $B_1$ , so:

$$K = B + \frac{7}{9}(B_1 - B) = \frac{2}{9}B + \frac{7}{9}B_1.$$

Or, 2) The direction from  $B$  to  $B_1$  is given as  $\mathbf{b} = [0, -1, 0]$ , and we know that the distance is 7, so  $K = B + 7\mathbf{b}$ .

# COMPUTATIONAL WAY OF LOOKING AT THE PROBLEM

Plane  $\alpha$  passes through  $K$  and  $C_1$ , and is parallel to  $BD_1$ . The fact that it passes through  $K$  and  $C_1$ , it means that a vector from  $K$  to  $C_1$  is tangent to the plane. So is a vector from  $B$  to  $D_1$ . Let us find those two vectors:

$$\mathbf{r}_{BD_1} = B - D_1 = [0, 9, 9] - [9, 0, 0] = [-9, 9, 9] \quad (7)$$

$$\mathbf{r}_{KC_1} = K - C_1 = [0, 2, 9] - [0, 0, 0] = [0, 2, 9] \quad (8)$$

We can find the norm to the plane by taking a cross product of  $\mathbf{r}_{BD_1}$  and  $\mathbf{r}_{KC_1}$ :

$$\mathbf{n} = \mathbf{r}_{BD_1} \times \mathbf{r}_{KC_1} = \begin{bmatrix} 81 - 18 \\ 0 + 81 \\ -18 - 0 \end{bmatrix} = \begin{bmatrix} 63 \\ 81 \\ -18 \end{bmatrix} \quad (9)$$

# FINDING POINT P

$P$  is a point of intersection of  $\alpha$  with  $A_1B_1$ . Let us find a vector from  $A_1$  to  $B_1$ :

$$\mathbf{r}_{A_1B_1} = A_1 - B_1 = [9, 0, 9] - [0, 0, 9] = [9, 0, 0] \quad (10)$$

Let us denote coordinates of  $P$  as  $\mathbf{p}$  and coordinates of  $B_1$  as  $\mathbf{b}_1$ . We know that  $\mathbf{p}^\top \mathbf{n} = 0$  and  $\mathbf{p} = \lambda \mathbf{r}_{A_1B_1} + \mathbf{b}_1$ . Therefore:

$$(\lambda \mathbf{r}_{A_1B_1} + \mathbf{b}_1)^\top \mathbf{n} = 0 \quad (11)$$

$$\lambda = -\frac{\mathbf{b}_1^\top \mathbf{n}}{\mathbf{r}_{A_1B_1}^\top \mathbf{n}} \quad (12)$$

$$\mathbf{b}_1^\top \mathbf{n} = 0 + 0 + 9 \cdot 2 = -162 \quad (13)$$

$$\mathbf{r}_{A_1B_1}^\top \mathbf{n} = -9 \cdot 11 + 0 + 0 = 567 \quad (14)$$

$$\lambda = -\frac{-162}{567} = 2/7 \quad (15)$$

# TASK 1

We found that  $\lambda = 2/7$  and  $\mathbf{p} = \lambda \mathbf{r}_{A_1 B_1} + \mathbf{b}_1$ . Then:

$$\mathbf{p} = 2/7[9, 0, 0] + [0, 0, 9] = [18/7, 0, 9] \quad (16)$$

We can find  $\|A_1 P\|$  and  $\|B_1 P\|$ ; These vectors are  $A_1 P = [45/7, 0, 0]$  and  $B_1 P = [18/7, 0, 0]$ . With that we know:

$$\|A_1 P\| = 45/7 \quad (17)$$

$$\|B_1 P\| = 18/7 \quad (18)$$

$$\|A_1 P\|/\|B_1 P\| = 45/18 = 5/2 \quad (19)$$



## TASK 2

Find angle between  $\alpha$  and  $BB_1C_1$ . This is the same as angle between the normals to these planes. First we find the norm to the plane  $BB_1C_1$ :

$$B - B_1 = [0, 9, 0] \quad (20)$$

$$B_1 - C_1 = [0, 0, 9] \quad (21)$$

$$\mathbf{m} = (B - B_1) \times (B_1 - C_1) = [81, 0, 0] \quad (22)$$

With that we know normals to both  $\alpha$  and  $BB_1C_1$ , and can find the angle between them using dot product.

## TASK 2

We know that:

$$\mathbf{m} \cdot \mathbf{n} = \|\mathbf{n}\| \|\mathbf{m}\| \cos(\varphi), \quad (23)$$

where  $\varphi$  is the angle we seek. We need to find  $\mathbf{m} \cdot \mathbf{n}$ , as well as  $\|\mathbf{m}\|$  and  $\|\mathbf{n}\|$ :

$$\mathbf{m} \cdot \mathbf{n} = 63 \cdot 81 + 0 + 0 = 5103 \quad (24)$$

$$\|\mathbf{n}\| = \sqrt{10854} \quad (25)$$

$$\|\mathbf{m}\| = 81 \quad (26)$$

$$\cos(\varphi) = \frac{\mathbf{m} \cdot \mathbf{n}}{\|\mathbf{n}\| \|\mathbf{m}\|} = 5103 / (81\sqrt{10854}) \approx 0.6047 \quad (27)$$

$$\varphi = 52.79^\circ \quad (28)$$

# THE CODE IN MATLAB FOR THIS PROBLEM IS:

```
0      dx = [9;0;0]; dy = [0;9;0]; dz = [0;0;9];
      C1 = sym([0;0;0]);
2      D1 = C1 + dx; A1 = C1 + dx + dz; B1 = C1 + dz;
      B = C1 + dy + dz;
4
      K = (2/9) * B + (7/9) * B1;
6      BD1 = B - D1;
      KC1 = K - C1;
8      n = cross(BD1, KC1);
      lambda = -dot(B1, n) / dot(dx, n);
10     P = lambda*dx + B1;
      A1P = A1 - P;
12     B1P = B1 - P;
      disp( norm(A1P) / norm(B1P) ) %task 1
14
      m = cross(dy, dz);
16     cos_phi = dot(n, m) / (norm(n) * norm(m));
      %task 2:
18     phi = round(double(acos(cos_phi))*180/pi, 3)
```

# HOMEWORK

Consider a cube  $ABCD A_1 B_1 C_1 D_1$  with a side length 3. Point  $S_1 \in DB_1$ , with  $\|DS_1\| = 1$  (meaning the distance from  $D$  to  $S_1$  is 1). Point  $S_2 \in AB_1$ , with  $\|AS_2\| = 2$  (meaning the distance from  $A$  to  $S_2$  is 2). Plane  $\alpha_1$  passes through  $S_1$ ,  $S_2$  and  $D$ . Plane  $\alpha_2$  is orthogonal to  $C_1 S_1$  through  $C$ . Point  $P$  is an intersection between the plane  $\alpha_1$  and the line passing through points  $D_1$ ,  $B$ .

- Find distance between  $P$  and  $A$ .
- Find angle between  $\alpha_1$  and  $\alpha_2$ .
- Find distance between  $\alpha_1$  and all vertices of the cube.
- Prove that  $P$  lies on  $B_1 S_1$ .

# THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:  
[github.com/SergeiSa/Extra-math-for-high-school](https://github.com/SergeiSa/Extra-math-for-high-school)

Check Moodle for additional links, videos, textbook suggestions.

