

# Algebra on two dimensions

## Math and modeling for high school, Lecture 2

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Fall 2022

# LINEAR GRAPHS

Here are two linear graphs.  
Note that both cross the origin.



How do we find a vector,  
parallel to the graph?

Answer: any pair  $(x, y)$  that solve the line equation is such vector.

Example: consider line  $y = -3x$ . The following vectors lie on that line:  $(1, -3)$ ,  $(-1, 3)$ ,  $(3, -9)$ , etc. In a more standard vector notation they are:

$$\mathbf{r} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} 3 \\ -9 \end{bmatrix} \quad (1)$$

Note that scaling a vector does not change the fact that it lies on the line: if  $\mathbf{r} = \begin{bmatrix} r_x \\ r_y \end{bmatrix}$  lies on a line  $\mathcal{L}$ , then  $\lambda\mathbf{r}$  also lies on  $\mathcal{L}$  (where  $\lambda \in \mathbb{R}$ ).

In fact, we can say the following about a line:

## Theorem

*If  $\mathbf{r} = \begin{bmatrix} r_x \\ r_y \end{bmatrix}$  lies on a line  $\mathcal{L}$ , then all points on  $\mathcal{L}$  can be found as  $\lambda \mathbf{r}$ .*

Let us prove it. If  $\mathcal{L} = \{(x, y) : y = ax\}$  and  $\mathbf{r} \in \mathcal{L}$ , then  $r_y = ar_x$ . Then  $\lambda r_y = \lambda ar_x$  and  $(\lambda r_x, \lambda r_y) \in \mathcal{L}$ . □

# LINEAR GRAPHS

That was difficult-looking. But graphically, the proof is quite obvious. We are trying to prove that any point on the line can be reached by scaling the vector lying on that line:



How do we plot a line? If we want to plot it as a collection of points  $\mathbf{p}_i$ , we could generate them by the following formula:

$$\mathbf{p}_i = \lambda_i \mathbf{v} \tag{2}$$

where  $\mathbf{v}$  is a vector on that line and  $\lambda_i$  is a sequence of numbers, for example:

$$\lambda_i = -10, -9.99, -9.98, \dots 10 \tag{3}$$

Then we can plug the points  $\mathbf{p}_i$  into Python library `matplotlib.pyplot.plot`.

If we have two vectors which are not co-linear, we can describe any point on the plane as their linear combination:

Let  $\mathbf{r}_A$  be the position of point A. It can be found as  
$$\mathbf{r}_A = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2$$



Let us examine the equation  $\mathbf{r}_A = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2$  more carefully. First, we can re-write in in matrix form:

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \mathbf{r}_A \quad (4)$$

And we know how to solve it. Denoting the matrix  $\mathbf{M} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$ , we can say that as long as  $\mathbf{M}$  is full rank, the system can be solved exactly. And  $\mathbf{M}$  is full rank, as long as  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are linearly independent.

We can call  $\lambda_1$  and  $\lambda_2$  *coordinates* of  $\mathbf{r}_A$  in the *basis*  $\mathbf{v}_1, \mathbf{v}_2$ .



If  $\mathbf{M} = [\mathbf{v}_1 \ \mathbf{v}_2]$  is full rank, we say that it *spans the plane*. That simply means - any point on the plane can be expressed by its coordinates  $\lambda_1$  and  $\lambda_2$ .

If you know coordinates  $\lambda_1$  and  $\lambda_2$  of a point  $B$ , you can find its position:

$$\mathbf{r}_B = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 \quad (5)$$

If we want to check whether or not two vectors in  $\mathbb{R}^2$  are not parallel (are linearly independent), we can see if the following determinant  $\Delta$  is not zero:

$$\Delta = \det([\mathbf{v}_1 \ \mathbf{v}_2]) \quad (6)$$

# DOT PRODUCT

Given two vectors  $\mathbf{a} = [a_x \ a_y]$  and  $\mathbf{b} = [b_x \ b_y]$ , you can define their dot product as:

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y \quad (7)$$

We can also define dot product as follows:

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos(\varphi) \quad (8)$$

where  $\varphi$  is the angle between the two vectors. The two definitions can be used to compute the angle between vectors:

$$\cos(\varphi) = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} \quad (9)$$

# DOT PRODUCT

Some examples. If  $\mathbf{a} = [1 \ 0]$  and  $\mathbf{b} = [0 \ 2]$ , you can find their dot product as  $\mathbf{a} \cdot \mathbf{b} = 1 \cdot 0 + 0 \cdot 2 = 0$ .

Meaning the angle between the two vectors is:

$$\cos(\varphi) = \frac{0}{1 \cdot 2} = 0, \quad \varphi = \pi/2 \quad (10)$$

If  $\mathbf{a} = [1 \ 1]$  and  $\mathbf{b} = [-2 \ 0]$ , their dot product is:

$$\mathbf{a} \cdot \mathbf{b} = 1 \cdot (-2) + 1 \cdot 0 = -2 \quad (11)$$

The angle between the two vectors is:

$$\cos(\varphi) = \frac{-2}{\sqrt{2} \cdot 2} = -\sqrt{2}/2, \quad \varphi = 3\pi/4 \quad (12)$$

Given one line, how do we find another, perpendicular (*orthogonal*) to it?

Given one vector, how do we find another, perpendicular to it? Well, by a reasonable definition, two orthogonal vectors would have angle  $\varphi = \pi/2$  between them. So, if  $\mathbf{a}$  and  $\mathbf{b}$  are orthogonal, their dot product is:

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos(\pi/2) = 0 \quad (13)$$

Which in turn means:

$$a_x b_x + a_y b_y = 0 \quad (14)$$

Consider the following problem. Given vector  $\mathbf{r} = [r_x \ r_y]$ , find vector  $\mathbf{n} = [n_x \ n_y]$ , orthogonal to it.

We know that their dot product has to be zero:

$$r_x n_x + r_y n_y = 0 \quad (15)$$

We can rewrite it in matrix form:

$$\begin{bmatrix} r_x & r_y \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} = 0 \quad (16)$$

$$\mathbf{r}^\top \mathbf{n} = 0 \quad (17)$$

Meaning we can find  $\mathbf{n}$  using `lstsq`.

We could find  $\mathbf{n}$  using `lstsq`. But it looks too complicated for such a simple task.

Alternatively, we can consider  $r_x n_x + r_y n_y = 0$ . From this,  $n_y = -\frac{r_x}{r_y} n_x$ . So, for example, vector  $\mathbf{n} = [1 \ -\frac{r_x}{r_y}]$  is orthogonal to  $\mathbf{r}$ .

Note that in any case, we can decide we want  $\mathbf{n}$  to have certain lengths, let us say  $\|\mathbf{n}\| = 1$ . To achieve that we can *normalize* it:

$$\mathbf{n}^* = \begin{bmatrix} 1 \\ -r_x/r_y \end{bmatrix} \quad (18)$$

$$\mathbf{n} = \frac{\mathbf{n}^*}{\|\mathbf{n}^*\|} \quad (19)$$

# AFFINE LINES

If a line does not cross the origin, its equation would have the *affine* form  $y = ax + b$ :

If  $x = 0$ , then  $y = b$ .



# AFFINE LINES

Consider the problem: find a line  $\mathcal{L}$  that is  $h$ -distance away from the line  $y = ax$ :





The line  $\mathcal{L}$  is given by its equation  $y = ax + b$ . All we need to do is to find  $b$ , since  $a$  is already known. Here is one way to solve it:

- Find  $\mathbf{v}$  on the line  $y = ax$ ;  $\mathbf{v} = [1, a]$ .
- Find  $\mathbf{n}$  orthogonal to  $\mathbf{v}$ ,  $\|\mathbf{n}\| = 1$  (see previous slides).
- Point  $h\mathbf{n} = [hn_x, hn_y]$  lies on  $\mathcal{L}$ , so  $hn_y = ahn_x + b$ .
- From this, we know that  $b = h(n_y - an_x)$ . Done.

Too many steps for such a simple problem? Maybe. But - it is hard to make a mistake doing it this way.

# THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:  
[github.com/SergeiSa/Extra-math-for-high-school](https://github.com/SergeiSa/Extra-math-for-high-school)

Check Moodle for additional links, videos, textbook suggestions.

