Algebra in 3D: examples

Math and modeling for high school

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Problem statement

Given a cube $ABCDA_1B_1C_1D_1$ with a side length 9. Point $K \in BB_1$, with $||KB|| = 7$ (meaning the distance from K to B is 7). Plane α passes through K and C_1, and is parallel to BD_1. P is a point of intersection of α with A_1B_1

- Prove that $||A_1P||/||PB_1|| = 2.5$.
- Find angle between α and BB_1C_1.

Computational way of looking at the problem

Our approach here can be straight-forward: we compute everything that can be computed exactly.

Later we can discard useless steps, but it can be counter-productive to start discarding them before we know how we will arrive at the solution.

Computational way of looking at the problem

For example, assuming that $C_1 = [0,\ 0,\ 0]$ (why? so α passes through the origin), and orienting the axes such that $D_1 = [9,\ 0,\ 0]$, $B_1 = [0,\ 0,\ 9]$, we get:

$$A_1 = [9,\ 0,\ 9] \tag{1}$$
$$A = [9,\ 9,\ 9] \tag{2}$$
$$B = [0,\ 9,\ 9] \tag{3}$$
$$C = [0,\ 9,\ 0] \tag{4}$$
$$D = [9,\ 9,\ 0] \tag{5}$$

Do we need all these points? No. Is it trivial to find their coordinates? Yes. Is it easier to think about the problem when you know all their coordinates? Yes.

Computational way of looking at the problem

Since $K \in BB_1$ and $||KB|| = 7$, and $B = [0,\ 9,\ 9]$, $B_1 = [0,\ 0,\ 9]$, we can find coordinates of K:

$$K = [0,\ 2,\ 9] \tag{6}$$

There are two ways to arrive there. 1) You know that K is $7/9$ of the way between B and B_1, so:
$K = B + \frac{7}{9}(B_1 - B) = \frac{2}{9}B + \frac{7}{9}B_1$.

Or, 2) The direction from B to B_1 is given as $\mathbf{b} = [0,\ -1,\ 0]$, and we know that the distance is 7, so $K = B + 7\mathbf{b}$.

Computational way of looking at the problem

Plane α passes through K and C_1, and is parallel to BD_1. The fact that it passes through K and C_1, it means that a vector from K to C_1 is tangent to the plane. So is a vector from B to D_1. Let us find those two vectors:

$$\mathbf{r}_{BD_1} = B - D_1 = [0,\ 9,\ 9] - [9,\ 0,\ 0] = [-9,\ 9,\ 9] \tag{7}$$

$$\mathbf{r}_{KC_1} = K - C_1 = [0,\ 2,\ 9] - [0,\ 0,\ 0] = [0,\ 2,\ 9] \tag{8}$$

We can find the norm to the plane by taking a cross product of $\mathbf{r}_{BD_1}$ and $\mathbf{r}_{KC_1}$:

$$\mathbf{n} = \mathbf{r}_{BD_1} \times \mathbf{r}_{KC_1} = \begin{bmatrix} 81 - 18 \\ 0 + 81 \\ -18 - 0 \end{bmatrix} = \begin{bmatrix} 63 \\ 81 \\ -18 \end{bmatrix} \tag{9}$$

Finding point P

P is a point of intersection of α with A_1B_1. Let us find a vector from A_1 to B_1:

$$\mathbf{r}_{A_1B_1} = A_1 - B_1 = [9,\ 0,\ 9] - [0,\ 0,\ 9] = [9,\ 0,\ 0] \tag{10}$$

Let us denote coordinates of P as $\mathbf{p}$ and coordinates of B_1 as $\mathbf{b}_1$. We know that $\mathbf{p}^\top\mathbf{n} = 0$ and $\mathbf{p} = \lambda\mathbf{r}_{A_1B_1} + \mathbf{b}_1$. Therefore:

$$(\lambda\mathbf{r}_{A_1B_1} + \mathbf{b}_1)^\top\mathbf{n} = 0 \tag{11}$$

$$\lambda = -\frac{\mathbf{b}_1^\top\mathbf{n}}{\mathbf{r}_{A_1B_1}^\top\mathbf{n}} \tag{12}$$

$$\mathbf{b}_1^\top\mathbf{n} = 0 + 0 + 9 \cdot 2 = -162 \tag{13}$$

$$\mathbf{r}_{A_1B_1}^\top\mathbf{n} = -9 \cdot 11 + 0 + 0 = 567 \tag{14}$$

$$\lambda = -\frac{-162}{567} = 2/7 \tag{15}$$

Task 1

We found that $\lambda = 2/7$ and $\mathbf{p} = \lambda \mathbf{r}_{A_1B_1} + \mathbf{b}_1$. Then:

$$\mathbf{p} = 2/7[9,\ 0,\ 0] + [0,\ 0,\ 9] = [18/7,\ 0,\ 9] \tag{16}$$

We can find $||A_1P||$ and $||B_1P||$; These vectors are $A_1P = [45/7,\ 0,\ 0]$ and $B_1P = [18/7,\ 0,\ 0]$. With that we know:

$$||A_1P|| = 45/7 \tag{17}$$

$$||B_1P|| = 18/7 \tag{18}$$

$$||A_1P||/||B_1P|| = 45\ 18 = 5/2 \tag{19}$$

Task 2

Find angle between α and BB_1C_1. This is the same as angle between the normals to these planes. First we find the norm to the plane BB_1C_1:

$$B - B_1 = [0,\ 9,\ 0] \tag{20}$$

$$B_1 - C_1 = [0,\ 0,\ 9] \tag{21}$$

$$\mathbf{m} = (B - B_1) \times (B_1 - C_1) = [81,\ 0,\ 0] \tag{22}$$

With that we know normals to both α and BB_1C_1, and can find the angle between them using dot product.

Task 2

We know that:

$$\mathbf{m} \cdot \mathbf{n} = ||\mathbf{n}||\ ||\mathbf{m}||\cos(\varphi), \tag{23}$$

where φ is the angle we seek. We need to find $\mathbf{m} \cdot \mathbf{n}$, as well as $||\mathbf{m}||$ and $||\mathbf{n}||$:

$$\mathbf{m} \cdot \mathbf{n} = 63 \cdot 81 + 0 + 0 = 5103 \tag{24}$$

$$||\mathbf{n}|| = \sqrt{10854} \tag{25}$$

$$||\mathbf{m}|| = 81 \tag{26}$$

$$\cos(\varphi) = \frac{\mathbf{m} \cdot \mathbf{n}}{||\mathbf{n}||\ ||\mathbf{m}||} = 5103/(81\sqrt{10854}) \approx 0.6047 \tag{27}$$

$$\varphi = 52.79^\circ \tag{28}$$

The code in Matlab for this problem is:

```
dx = [9;0;0]; dy = [0;9;0]; dz = [0;0;9];
C1 = sym([0;0;0]);
D1 = C1 + dx; A1 = C1 + dx + dz; B1 = C1 + dz;
B = C1 + dy + dz;

K = (2/9) * B + (7/9) * B1;
BD1 = B - D1;
KC1 = K - C1;
n = cross(BD1, KC1);
lambda = -dot(B1, n) / dot(dx, n);
P = lambda*dx + B1;
A1P = A1 - P;
B1P = B1 - P;
disp( norm(A1P) / norm(B1P) ) %task 1

m = cross(dy, dz);
cos_phi = dot(n, m) / (norm(n) * norm(m));
%task 2:
phi = round(double(acos(cos_phi))*180/pi, 3)
```

Homework

Consider a cube $ABCDA_1B_1C_1D_1$ with a side length 3. Point $S_1 \in DB_1$, with $||DS_1|| = 1$ (meaning the distance from D to S_1 is 1). Point $S_2 \in AB_1$, with $||AS_2|| = 2$ (meaning the distance from A to S_2 is 2). Plane α_1 passes through S_1, S_2 and D. Plane α_2 is orthogonal to C_1S_1 through C. Point P is an intersection between the plane α_1 and the line passing through points D_1, B.

- Find distance between P and A.
- Find angle between α_1 and α_2.
- Find distance between α_1 and all vertices of the cube.
- Prove that P lies on B_1S_1.

Thank you!

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Extra-math-for-high-school

Check Moodle for additional links, videos, textbook suggestions.