Algebra on two dimensions

Math and modeling for high school, Lecture 2

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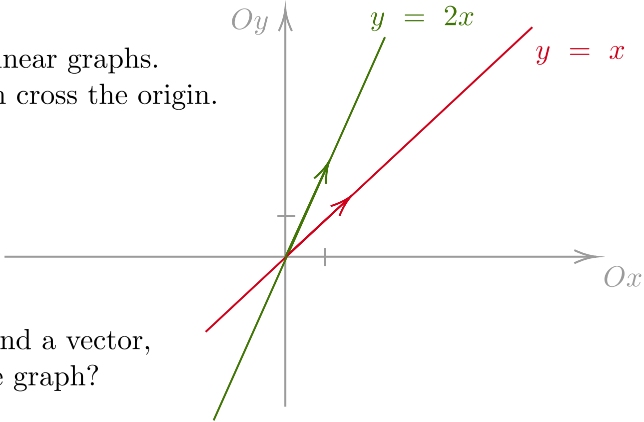
Here are two linear graphs.
Note that both cross the origin.



How do we find a vector,
parallel to the graph?

Answer: any pair (x, y) that solve the line equation is such vector.

Example: consider line $y = -3x$. The following vectors line on that line: $(1,\ -3)$, $(-1,\ 3)$, $(3,\ -9)$, etc. In more standard vector notation they are:

$$\mathbf{r} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} 3 \\ -9 \end{bmatrix} \tag{1}$$

Note that scaling a vector does not change the fact that it lines on the line: if $\mathbf{r} = \begin{bmatrix} r_x \\ r_y \end{bmatrix}$ lies on a line $\mathcal{L}$, than $\lambda \mathbf{r}$ also lies on $\mathcal{L}$ (where $\lambda \in \mathbb{R}$).

In fact, we can say the following about a line:

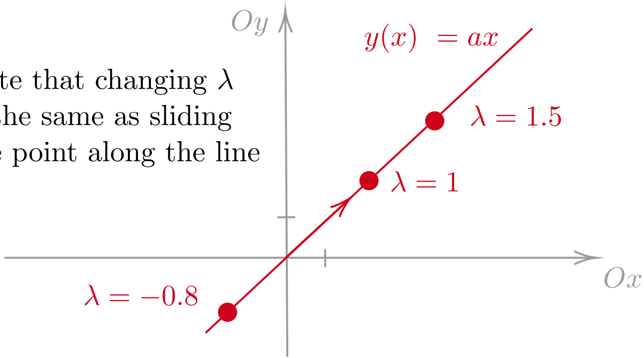
Theorem

If $\mathbf{r} = \begin{bmatrix} r_x \\ r_y \end{bmatrix}$ *lies on a line* $\mathcal{L}$*, then all points on* $\mathcal{L}$ *can be found as* $\lambda \mathbf{r}$*.*

Let us prove it. If $\mathcal{L} = \{(x, y) : \ y = ax\}$ and $\mathbf{r} \in \mathcal{L}$, then $r_y = a r_x$. Then $\lambda r_y = \lambda a r_x$ and $(\lambda r_x, \lambda r_y) \in \mathcal{L}$. $\square$

That was difficult-looking. But graphically, the proof is quite obvious. We are trying to prove that any point in the line can be reached by scaling the vector lying on that line:

Note that changing λ is the same as sliding the point along the line



How do we plot a line? If we want to plot it as a collection of points $\mathbf{p}_i$, we could generate them by the following formula:

$$\mathbf{p}_i = \lambda_i \mathbf{v} \tag{2}$$

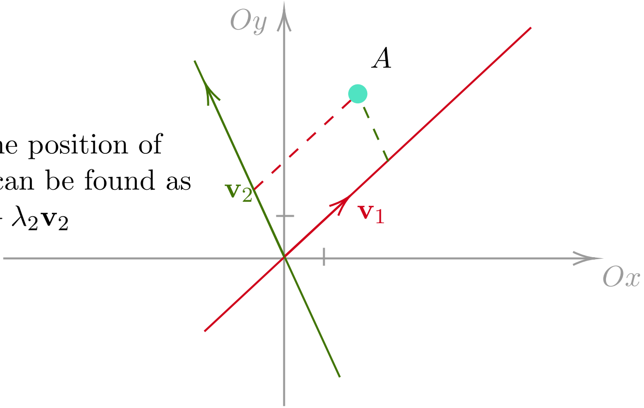
where $\mathbf{v}$ is a vector on that line and λ_i is a sequence of numbers, for example:

$$\lambda_i = -10,\ -9.99,\ -9.98,\ ...\ 10 \tag{3}$$

Then we can plug the points $\mathbf{p}_i$ into Python library `matplotlib.pyplot.plot`.

If we have two vectors which are not co-linear, we can describe any point on the plane as their linear combination:

Let $\mathbf{r}_A$ be the position of point A. It can be found as $\mathbf{r}_A = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2$



Let us examine the equation $\mathbf{r}_A = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2$ more carefully. First, we can re-write in in matrix form:

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \mathbf{r}_A \tag{4}$$

And we know how to solve it. Denoting the matrix $\mathbf{M} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$, we can say that as long as $\mathbf{M}$ is full rank, the system can be solved exactly. And $\mathbf{M}$ is full rank, as long as $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

We can call λ_1 and λ_2 *coordinates* of $\mathbf{r}_A$ in the *basis* $\mathbf{v}_1$ and $\mathbf{v}_2$.

If $\mathbf{M} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$ is full rank, we say that it *spans the plane*. That simply means - any point on the plane can be expressed by its coordinates λ_1 and λ_2.

If you know coordinates λ_1 and λ_2 of a point B, you can find its position:

$$\mathbf{r}_B = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 \tag{5}$$

If we want to check if two vectors in $\mathbb{R}^2$ are not parallel (are linearly independent), we can see if the following determinant Δ is not zero:

$$\Delta = \det\left(\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\right) \tag{6}$$

Dot product

Given two vectors $\mathbf{a} = [a_x \;\; a_y]$ and $\mathbf{b} = [b_x \;\; b_y]$, you can define their dot product as:

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y \tag{7}$$

We can also define dot product as follows:

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}|| \; ||\mathbf{b}|| \cos(\varphi) \tag{8}$$

where φ is the angle between the two vectors. The two definitions can be used to compute the angle between vectors:

$$\cos(\varphi) = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}|| \; ||\mathbf{b}||} \tag{9}$$

Some examples. If $\mathbf{a} = [1\ 0]$ and $\mathbf{b} = [0\ 2]$, you can find their dot product as $\mathbf{a} \cdot \mathbf{b} = 1 \cdot 0 + 0 \cdot 2 = 0$.
Meaning the angle between the two vectors is:

$$\cos(\varphi) = \frac{0}{1 \cdot 2} = 0, \quad \varphi = \pi/2 \tag{10}$$

If $\mathbf{a} = [1\ 1]$ and $\mathbf{b} = [-2\ 0]$, their dot product is:

$$\mathbf{a} \cdot \mathbf{b} = 1 \cdot (-2) + 1 \cdot 0 = -2 \tag{11}$$

The angle between the two vectors is:

$$\cos(\varphi) = \frac{-2}{\sqrt{2} \cdot 2} = -\sqrt{2}/2, \quad \varphi = 3\pi/4 \tag{12}$$

Given one line, how do we find another, perpendicular (*orthogonal*) to it?

Given one vector, how do we find another, perpendicular to it? Well, by a reasonable definition, two orthogonal vectors would have angle $\varphi = \pi/2$ between them. So, if If **a** and **b** are orthogonal, their dot product is:

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}||\ ||\mathbf{b}|| \cos(\pi/2) = 0 \tag{13}$$

Which in turn means:

$$a_x b_x + a_y b_y = 0 \tag{14}$$

Consider the following problem. Given vector $\mathbf{r} = [r_x \; r_y]$, find vector $\mathbf{n} = [n_x \; n_y]$, orthogonal to it.

We know that their dot product has to be zero:

$$r_x n_x + r_y n_y = 0 \tag{15}$$

We can rewrite it in matrix form:

$$\begin{bmatrix} r_x & r_y \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} = 0 \tag{16}$$

$$\mathbf{r}^\top \mathbf{n} = 0 \tag{17}$$

Meaning we can find $\mathbf{n}$ using `lstsq`.

We could find $\mathbf{n}$ using `lstsq`. But it looks too complicated for such a simple task.

Alternatively, we can consider $r_x n_x + r_y n_y = 0$. From this, $n_y = -\frac{r_x}{r_y} n_x$. So, for example, vector $\mathbf{n} = [1 \;\; -\frac{r_x}{r_y}]$ is orthogonal to $\mathbf{r}$.

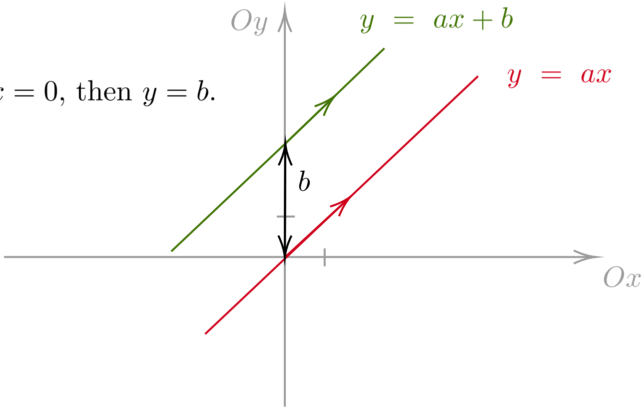
Note that in any case, we can decide we want $\mathbf{n}$ to have certain lengths, let us say $||\mathbf{n}|| = 1$. To achieve that we can *normalize* it:

$$\mathbf{n}^* = \begin{bmatrix} 1 \\ -r_x/r_y \end{bmatrix} \tag{18}$$

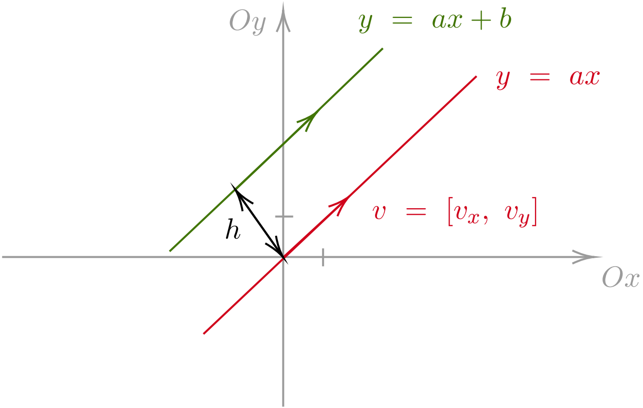
$$\mathbf{n} = \frac{\mathbf{n}^*}{||\mathbf{n}^*||} \tag{19}$$

If a line does not cross the origin, its equation would have the *affine* form $y = ax + b$:

If $x = 0$, then $y = b$.



Consider the problem: find a line $\mathcal{L}$ that is h-distance away from the line $y = ax$:



The line $\mathcal{L}$ is given by its equation $y = ax + b$. All we need to do is to find b, since a is already known. Here is one way to solve it:

- Find $\mathbf{v}$ on the line $y = ax$; $\mathbf{v} = [1,\ a]$.
- Find $\mathbf{n}$ orthogonal to $\mathbf{v}$, $||\mathbf{n}|| = 1$ (see previous slides).
- Point $h\mathbf{n} = [hn_x,\ hn_y]$ lies on $\mathcal{L}$, so $hn_y = ahn_x + b$.
- From this, we know that $b = h(n_y - an_x)$. Done.

Too many steps for such a simple problem? Maybe. But - it is hard to make a mistake doing it this way.

Thank you!

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Extra-math-for-high-school

Check Moodle for additional links, videos, textbook suggestions.