

# Differential equations

## Math and modeling for high school

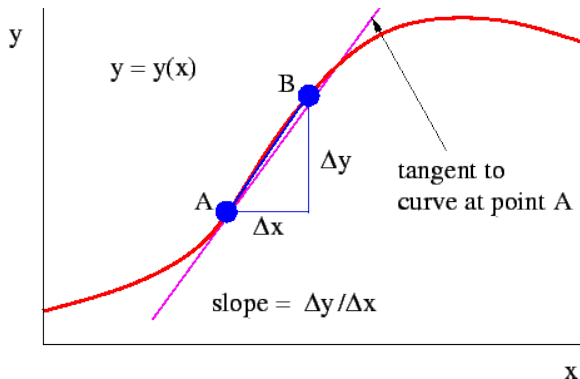
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# DERIVATIVES AND PREDICTIONS

Consider a function  $y = y(x)$ , with a derivative  $\frac{d}{dx}y = \dot{y}$ . Assume that  $y(0) = 10$  and  $\dot{y}(0) = 1$ . Can we find (approximately) what  $y(\Delta x)$  is, if  $\Delta x$  is very small?

We can reasonably say that  $y(\Delta x) \approx y(0) + \dot{y}(0)\Delta x$ .



More generally:

$$y(x + \Delta x) \approx y(x) + \dot{y}(x)\Delta x$$

If this approximation is accurate enough, it appears we could draw a graph of a function while knowing only its slopes (derivatives)  $\dot{y} = \dot{y}(x)$ .

# DERIVATIVES AND EQUATIONS

Consider a “usual” function  $y = y(x)$ . For example,  $y(x) = x^3 - 1$ . We define it by identifying how  $y$  depends on  $x$ . In other words - we explain which  $y$  we expect for a given  $x$ .

But what if instead we identify what rate of change (slope) of the function we expect for a given  $x$ ? Can we define a graph as  $\dot{y} = \dot{y}(x)$ ?

Let us look at an example. Consider a function  $\dot{y} = 2x$ .

We remember that a derivative of  $x^2$  is  $2x$ . So, is graph  $\dot{y} = 2x$  equivalent to  $y = x^2$ ?

Remember that  $\frac{d}{dx}(x^2 - 1) = 2x$ , and  $\frac{d}{dx}(x^2 + 20) = 2x$ , and  $\frac{d}{dx}(x^2 + 3) = 2x$ .

## Differential equation

So,  $\dot{y} = 2x$  is not equivalent to any one of those graphs, it actually describes a family of graphs  $y = x^2 + c$ ,  $c = \text{const}$ . We call it *differential equation*.

# INITIAL CONDITIONS

Given differential equation  $\dot{y} = 2x$  and knowing what the function is equal to at a single point (if  $x = 0$ , then  $y = 7$  - as an example) is sufficient to recover the function; not a family of graphs, but one concrete graph.

Let us test it. We know that  $y = x^2 + c$ , and also that if  $x = 0$ , then  $y = 7$ . So,  $7 = 0^2 + c$ , hence  $c = 7$ . The graph we recovered is:

$$y = x^2 + 7 \tag{1}$$

# INITIAL CONDITIONS

If we know what  $y$  is when  $x = 0$ , this is called *initial conditions*. Knowing a differential equation and initial conditions we can *solve* it - which means we can recover a single graph it represents for those initial conditions.

This makes sense if we think back to our approximation  $y(x + \Delta x) \approx y(x) + \dot{y}(x)\Delta x$ . We can use it to continue a graph from a certain point, but not to draw it from scratch. It is very explicit at the point  $x = 0$ :

$$y(\Delta x) \approx y(0) + \dot{y}(0)\Delta x$$

If we know  $y(0)$  and how to find  $\dot{y}$ , we can approximate the whole graph.

Some differential equations naturally make sense:

$$\dot{y} = x$$

$$\dot{y} = 4x^3 + 1$$

$$\dot{y} = -1$$

But they don't need to be simple. For example, we can write the following differential equation:

$$\dot{y} = -y$$

.

And indeed, if we know what  $y$  is when  $x = 0$ , then we can approximate its solution.



# THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:  
[github.com/SergeiSa/Extra-math-for-high-school](https://github.com/SergeiSa/Extra-math-for-high-school)

Check Moodle for additional links, videos, textbook suggestions.

