Задача № 5 из ТР (вариант 1)

$$P_2 = \{ p(t) \in \mathbb{R}[t] ; \deg P(t) \leq 2 \} =$$

$$= \{ p(t) = at^2 + bt + c, \quad a, b, c \in \mathbb{R} \}.$$

$$A p(t) = \frac{d}{dt} [(t+1) \cdot p(t)] = (t+1) \cdot p'(t) +$$

$$+ (t+1)' \cdot p(t) = (t+1) \cdot p'(t) + 1 \cdot p(t).$$

① $\boxed{A \in L(P_2, P_2)}$

0) $\boxed{p(t) \in P_2 \Rightarrow A(p(t)) \in P_2}$

$$A P(t) = A(at^2 + bt + c) = (t+1) \cdot (2at + b) +$$

$$+ (at^2 + bt + c) = 2at^2 + bt + 2at + b + at^2 + bt + c =$$

$$= 3at^2 + (2a + 2b) \cdot t + (b + c) \in P_2, \text{ т.е. } A : P_2 \to P_2$$

1) $\boxed{A(p_1(t) + p_2(t)) = A p_1(t) + A p_2(t)}$

$$A(p_1(t) + p_2(t)) = (t+1)(p_1'(t) + p_2'(t)) + (p_1(t) +$$

$$+ p_2(t)) = (t+1) \cdot p_1'(t) + p_1(t) + (t+1) \cdot p_2'(t) + p_2(t) =$$

$$= A p_1(t) + A p_2(t), \quad \forall p_1, p_2 \in P_2$$

2) $\boxed{A(\alpha \cdot p(t)) = \alpha \cdot A p(t), \ \forall \alpha \in \mathbb{R}}$

$$A(\alpha \cdot p(t)) = (t+1) \cdot (\alpha \cdot p'(t)) + \alpha \cdot p(t) =$$

$= \alpha \cdot (t+1) \cdot p'(t) + \alpha \cdot p(t) = \alpha \cdot ((t+1) \cdot p'(t) + p(t)) =$

$= \alpha \cdot Ap(t), \ \forall \alpha \in \mathbb{R}.$

$\left.\begin{array}{l} 0) \\ 1) \\ 2) \end{array}\right\} \Rightarrow A$ - линейный оператор $\Leftrightarrow A \in L(P_2, P_2)$.

② $e = \langle t^2, t, 1 \rangle$

$At^2 = (t+1) \cdot (t^2)' + t^2 = 2t^2 + 2t + t^2 = 3t^2 + 2t =$

$= e \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = eA^1$

$At = (t+1) \cdot (t)' + t = t + 1 + t = 2t + 1 = e \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = eA^2$

$At^0 = (t+1) \cdot (t^0)' + t^0 = 1 = e \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = eA^3$

$A_e = (A^1, A^2, A^3) = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$

③ $\boxed{\exists A^{-1} \Leftrightarrow \det A_e \neq 0}$

$\det A_e = \begin{vmatrix} 3 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 6 \Rightarrow \exists A^{-1}.$

~~det A =~~

$A^{-1} = \frac{1}{\det A} \cdot \hat{A}_e = \frac{1}{6} \cdot \begin{pmatrix} 2 & -2 & 2 \\ 0 & 3 & -3 \\ 0 & 0 & 6 \end{pmatrix}^T = \frac{1}{6} \cdot \begin{pmatrix} 2 & 0 & 0 \\ -2 & 3 & 0 \\ 2 & -3 & 6 \end{pmatrix} =$

$= \begin{pmatrix} 1/3 & 0 & 0 \\ -1/3 & 1/2 & 0 \\ 1/3 & -1/2 & 1 \end{pmatrix}$

Проверка!

$$A_e \circ A^{-1} = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1/3 & 0 & 0 \\ -1/3 & 1/2 & 0 \\ 1/3 & -1/2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4 $\exists A^{-1} \Leftrightarrow \ker A = \{\bar{0}\} \Leftrightarrow \operatorname{def} A = 0$

$\ker A = \{\bar{0}\}$

$\operatorname{def} A = 0.$

5 $\exists A^{-1} \Leftrightarrow \operatorname{Im} A = V \Leftrightarrow \operatorname{rk} A = \dim V.$

$\operatorname{Im} A = P_2$

$\operatorname{rk} A = 3.$