On asymptotic power of the new test for equality of two distributions

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Abstract. The paper introduces a new test for equality of two distributions in a class of models. We proved analytically and by stochastic simulation that the test possesses high efficiency. For the case of normal and Cauchy distributions that differ only by shift the asymptotic power of the test appears to be approximately the same as for the Wilcoxon-Mann-Whitney, the Kolmogorov-Smirnov and the Anderson-Darling tests. But if the distributions differ by scale parameters the power of the new test is considerably better.

Keywords: Test for equality of two distributions, Asymptotic power, Cauchy distribution, Normal distribution

1 Formulation of the problem

Let us consider the classical problem of testing hypothesis on the equality of two distributions

$$H_0 \, : \, F_1 = F_2 \tag{1}$$

against the alternative

$$H_1 \, : \, F_1 \neq F_2 \tag{2}$$

In the case of two independent samples $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_m)$ with the distributions functions F_1 and F_2 respectively.

It is well known (see e.g. [1]) that in the case when both distributions differ only by the means and are normal the classical Student test has a few optimal properties. If the distributions are not normal but still differs only by means a widely popular Wilcoxon-Mann-Whitney (WMW) U-statistic is often used instead. However, it can be shown that if two normal populations differ only in variances, the power of WMW test is very low. If distributions are arbitrary there are some universal techniques such as tests by Kolmogorov-Smirnov and Cramer-von Mises (see [2]) and the Anderson-Darling test (see [3]) that can be applied but in many cases these tests can be not powerful.

Recently [4] suggested the test based on U-statistics with the logarithmic kernel and provided its numerical justification for one and many dimensional cases in comparison with a few alternative techniques. However,to the best authors knowledge there are no analytical results about its asymptotic power. Here we introduce a similar but different test and provide a few analytical results on its power.

2 The new test and its statistical motivation

Assume that the distribution functions F_1 and F_2 belongs to the class of distribution functions of random variables ξ, such that

$$E[\ln(1+\xi^2)] < \infty. \tag{3}$$

Many distributions and, in particular, the Cauchy distribution have this property.

Among all distributions with given left hand side of (3) the Cauchy's one has the maximum entropy.

Consider the following test

$$\Phi_A = \frac{1}{n(n-1)} \sum_{1\le i<j\le n} g(X_i - X_j), \Phi_B = \frac{1}{m(m-1)} \sum_{1\le i<j\le m} g(Y_i - Y_j), \tag{4}$$

$$\Phi_{AB} = -\frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} g(X_i - Y_j), \Phi_{nm} = \Phi_A + \Phi_B + \Phi_{AB}, \tag{5}$$

where

$$g(u) = -\ln(1 + |u|^2)$$

is under a constant term precision the logarithm of the density of the standard Cauchy distribution. (Note that Zech and Aslan (2005) took $g(u) = \ln(|u|)$).

We would like to have a test that is appropriate for the case where the basic distribution belongs to a rather general class of distributions and the alternative distribution differ only by shift and scale transformation.

In particular, we consider the class of distributions satisfying (3), but the approach can be generalized for other classes of distributions.

Consider the class of distributions given by the property (3). Note that if the parameters are known the test based on likelihood ratio is the most powerful among tests with given parameters.

The test suggested above can be considered as an approximation of logarithm of this ratio for the Cauchy distribution. We suppose tha it will be very efficient for all distributions with property (3).

3 The analytical study of asymptotic power

Let us consider the case of two distributions having the property (3) and, in particular, the two that differ only by a shift. To simplify notations assume that $m = n$. The case $m \neq n$ is similar. Now the criterion (4) - (5) assumes the form

$$T_n = \Phi_{nn} = \frac{1}{n^2} \sum_{i,j=1}^{n} \ln(1+(X_i-Y_j)^2) - \frac{1}{n(n-1)} \sum_{1 \le i < j \le n} \ln(1+(X_i-X_j)^2) \quad (6)$$

$$-\frac{1}{n(n-1)} \sum_{1 \le i < j \le n} \ln(1+(Y_i-Y_j)^2). \quad (7)$$

Denote by $C(u,v)$ the Cauchy distribution with the density function

$$v/(\pi(v^2 + (x-u)^2)).$$

Let $f_1(x)$ denotes the density of F_1 and $f_2(x)$ denotes the density of F_2. Denote

$$J_h = \int_R g(x - y - |h|/\sqrt{n}) f_1(x) f_2(y) dx dy,$$

where $g(u) = -\ln(1 + |u|^2)$.

If there exists the limit

$$lim_{n\to\infty} n(J_h - J_0) \quad (8)$$

denote it by $J^*(h)$.

The basic analytical result of the present paper is the following

Theorem 1. *Consider the problem of testing hypothesis on the equality of two distributions (1)-(2) where both functions have the property (3).Then*
(i) under the condition $n \to \infty$ the distribution function of nT_n converges under H_0 to that of the random variable

$$(aZ + b)^2, \quad (9)$$

where Z has the normal distribution with zero expectation and variance equal to 1, $a^2 = J_0/3, b = 0$.

(ii) Let $F_1(x) = F(x), F_2 = F(x + \theta)$, where F is an arbitrary distribution function with property (3) and $\theta = h/\sqrt{n}, h$ is an arbitrary given number. Then the distribution function of nT_n converges under H_1 to that of the random variable

$$(aZ + b)^2,$$

where $a^2 = J_0/3, b = 0$ for the case of H_0 and $a^2 = J_0/3, b^2 = J^*(h)$ for H_1. In this case the power of the criterion T_n with significance α is asymptotically equal to that is given by the formula

$$Pr\{Z \ge z_{1-\alpha/2} - \sqrt{\frac{3J^*(h)}{J_0}}\} + Pr\{Z \le -z_{1-\alpha/2} - \sqrt{\frac{3J^*(h)}{J_0}}\}.$$

(iii) If $F_1 = C(\nu, 1), F_2 = C(\nu+\theta, 1)$ *then in the part (ii)* $a^2 = (2/3)\ln 3, b = h/3$ *and*

$$Pr\{Z \geq z_{1-\alpha/2} - (1/\sqrt{6\ln 3})h\} + Pr\{Z \leq -z_{1-\alpha/2} - (1/\sqrt{6\ ln3})h\}.$$

The proof of the theorem is given in the Appendix.

4 Simulation results

We found by a stochastic simulation that the formula presents an approximation of the power of the test T_n with a good accuracy (see tables 1-3 below).

At the tables 1-12 results for cases $n = 100$, 500, 1000 and different values of h with $\alpha = 0.05$ are given for normal and Cauchy distributions that differ either by shift or by scale parameters. The critical values were calculated in two ways: by simulation of the initial distribution and by random permutations (we used 800 random permutation in all cases). It worth to be noted that the results are very similar. Since the permutation technique is more universal, it can be recommended for practical applications.

Note that in all these cases when the distributions differ only in the scale parameters the power of T_n and that of the Wilcoxon-Mann-Whitney, the Kolmogorov-Smirnov and the Anderson-Darling tests were approximately equal to each other. It can be pointed out also that if the variances are not standard but are known we should simply make the corresponding normalisation. But for the cases where the distributions differ in scale parameters the Wilcoxon-Mann-Whitney is not appropriate at all and the power of the Kolmogorov-Smirnov and the Anderson-Darling tests is considerably lower.

Table 1. Cauchy distribution, $X \sim C(0, 1)$, $Y \sim C(h/\sqrt{n}, 1)$, $n = 100$

h	$T_n, perm$	T_n, sim	$formulae$	$wilcox.test$	$ks.test$	$ad.test$
1	6.4	6.3	6.8	6.6	6.3	7.1
2	10.1	10.6	12.2	11.9	11.1	11.6
3	19.6	20.3	21.5	20.5	20.2	20.7
5	50.9	50.5	49.5	48.5	53.1	52.2
7	82	82.3	77.8	77.2	83.6	80.7
9	96.7	96.8	93.9	91.5	96.5	95.2

5 Conclusion

In this paper we suggested a new test for equality of two distributions. Its asymptotic power was analytically established for the case of distributions that differ only by shift. For the case of Cauchy distribution this formula was presented

Table 2. Cauchy distribution, $X \sim C(0,1)$, $Y \sim C(h/\sqrt{n}, 1)$, $n = 500$

h	$T_n, perm$	T_n, sim	$formula$	$wilcox.test$	$ks.test$	$ad.test$
1	5.8	6.1	6.8	6.4	6.4	7.1
2	11.6	11.6	12.2	12.6	13.9	12.2
3	21	21.8	21.5	22.2	24.3	22.8
5	50.9	51	49.5	48	57.9	50.3
7	82.2	82.4	77.8	75.6	85.9	81.1
9	96.2	96.5	93.9	93.2	97.2	96.0

Table 3. Cauchy distribution, $X \sim C(0,1)$, $Y \sim C(h/\sqrt{n}, 1)$, $n = 1000$

h	$T_n, perm$	T_n, sim	$formula$	$wilcox.test$	$ks.test$	$ad.test$
1	6.3	6	6.8	6.8	8.1	6.8
2	11.4	11.9	12.2	12.9	13.4	12.9
3	21	20.9	21.5	22.8	26.2	22.2
5	53.6	53.6	49.5	50.8	59.6	54.2
7	84	84.5	77.8	79.5	87.6	84.4
9	96.6	96.6	93.9	93.2	98.3	96.3

Table 4. Cauchy distribution, $X \sim C(0,1)$, $Y \sim C(0, 1 + h/\sqrt{n})$, $n = 100$

h	$T_n, perm$	T_n, sim	$wilcox.test$	$ks.test$	$ad.test$
2	10.6	11.9	5.4	5.4	6.9
4	27.6	29.8	5.5	8.7	11.3
6	49.4	53.6	5.5	15.9	22.2
8	68.8	73.5	5.5	25	37.7
10	84.2	87.1	5.2	36.4	55.4

Table 5. Cauchy distribution, $X \sim C(0,1)$, $Y \sim C(0, 1 + h/\sqrt{n})$, $n = 500$

h	$T_n, perm$	T_n, sim	$wilcox.test$	$ks.test$	$ad.test$
2	9.4	10	4.5	6.3	6.2
4	28.5	30.6	4.8	14	12.3
6	54.5	56.5	5	26.1	29.7
8	79.5	80.5	5.2	43.3	51.0
10	93	94	5.2	62.2	74.2

Table 6. Cauchy distribution, $X \sim C(0,1)$, $Y \sim C(0, 1 + h/\sqrt{n})$, $n = 1000$

h	$T_n, perm$	T_n, sim	$wilcox.test$	$ks.test$	$ad.test$
2	10.2	10.5	5	7.6	7.3
4	32.4	33.8	5.2	13.8	14.9
6	61.1	62.8	5.2	27.9	32.8
8	84.8	85.6	5.2	47.4	59.7
10	96.1	97.1	5.4	67.9	82.8

Table 7. Normal distribution, $X \sim N(0,1)$, $Y \sim N(h/\sqrt{n},1)$, $n=100$

h	$T_n, perm$	T_n, sim	$wilcox.test$	$ks.test$	$ad.test$
1	11.1	11.3	12.5	9.5	12.2
2	29.3	29	31.1	20.5	29.6
3	52.4	53.4	55.8	42	55
4	77.5	77.5	80.6	64.9	78.9
5	91.9	92.5	93.1	84.7	93.1

Table 8. Normal distribution, $X \sim N(0,1)$, $Y \sim N(h/\sqrt{n},1)$, $n=500$

h	$T_n, perm$	T_n, sim	$wilcox.test$	$ks.test$	$ad.test$
1	9.2	8.9	9.6	8.3	9.0
2	23.9	23.9	26.3	20.6	25.4
3	47.3	48.9	51.7	41.4	49.7
4	75.3	75.1	77.8	66.9	76.9
5	91.1	91	92.8	86.1	92.6

Table 9. Normal distribution, $X \sim N(0,1)$, $Y \sim N(h/\sqrt{n},1)$, $n=1000$

h	$T_n, perm$	T_n, sim	$wilcox.test$	$ks.test$	$ad.test$
1	11	11.3	11.5	10	11.6
2	26.4	27.4	28.5	22	27.7
3	51.3	51.6	54.2	44.6	52.9
4	76.7	77	79.3	68.9	77.9
5	91.6	91.2	92.7	86.6	92.1

Table 10. Normal distribution, $X \sim N(0,1)$, $Y \sim N(0,1+h/\sqrt{n})$, $n=100$

h	$T_n, perm$	T_n, sim	$wilcox.test$	$ks.test$	$ad.test$
1	8.1	8.7	6.4	5.3	7.3
2	15	17.4	6.3	7.2	12.7
3	30.5	34.2	6.6	10.7	24.0
4	50.6	57.1	6.7	16.7	39.9
5	70.8	76.7	6.5	24.8	59.9

Table 11. Normal distribution, $X \sim N(0,1)$, $Y \sim N(0,1+h/\sqrt{n})$, $n=500$

h	$T_n, perm$	T_n, sim	$wilcox.test$	$ks.test$	$ad.test$
1	8.3	8.4	5	7.4	7.7
2	15.4	16.7	5.1	10.3	12.8
3	33.2	34.7	5.4	16.4	28.3
4	60	63.3	5.6	25.3	52.6
5	83.1	86.3	5.5	40.4	78.1

Table 12. Normal distribution, $X \sim N(0,1)$, $Y \sim N(0, 1 + h/\sqrt{n})$, $n = 1000$

h	$T_n, perm$	T_n, sim	$wilcox.test$	$ks.test$	$ad.test$
1	6.7	6.9	5.4	6	6.7
2	15.1	16.4	5.5	9.9	13.1
3	33.2	36	5.4	16.1	30.6
4	62.2	64	5.6	27.5	56.8
5	84.6	86.6	5.4	43.6	81.1

in a closed form since it proved to be possible to calculate the corresponding integrals analytically. Also the integrals in the formula can be evaluated though the sample data in a general case.

By stochastic simulation we found that for the normal and Cauchy distributions that differ only by shift the power of the new test is approximately equal to that of the Wilcoxon-Mann-Whitney, the Kolmogorov-Smirnov and the Anderson-Darling tests. However if the distributions differ by the scale parameter our simulations show that the new test is considerably better than the Kolmogorov-Smirnov and the Anderson-Darling tests. And in this case the Wilcoxon-Mann-Whitney test is not appropriate at all.

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6 Appendix

Proof of Theorem 1.

Let us begin with studying the asymptotic behaviour of the magnitude $E(nT_n)^2$.

Lemma 1. *(1) If hypothesis H_0 is satisfied and F_1 possesses property (3) then there exists a finite limit of $d = \lim E(nT_n)^2$ with $n \to \infty$.*

(2)If hypothesis H_1 is satisfied for $F_2(x) = F_1(x - \theta), \theta = h/\sqrt{n}$ and F_1 and F_2 possess property (3) then there exists a finite limit of $E(nT_n)^2$ for $n \to \infty$ and it is given by the formula

$$d + 2J^*(h)J_0 + J^*(h)^2.$$

Proof of the lemma.
Note that $(nT_n)^2$ is equal to

$$n^2[\frac{1}{n^2} \sum_{i,j=1}^{n} [g(X_i - Y_j) - J_0] - \frac{1}{n(n-1)} \sum_{i<j,i,j=1}^{n} [g(X_i - X_j) - J_0] -$$

$$\frac{1}{n(n-1)}[\sum_{i<j,i,j=1}^{n}[g(Y_i - Y_j) - J_0]]^2,$$

where $g(z) = \ln(1 + z^2)$.
The idea of the proof consists in the splitting the three squares of three sums including in this sum and three pairwise products into peculiar sums of the identical structure. Then to each peculiar sum either law of large numbers or central limit theorem is applied.

The square of the first sum,

$$n^2\{\frac{1}{n^2}\sum_{i,j=1}^{n}[g(X_i - Y_j) - J_0]\}^2,$$

can be represented by sum of the following peculiar sums.

$$1)\ n^2(\frac{1}{n^2})^2\sum_{i,j=1}^{n}[(g(X_i - Y_j) - J_0]^2,$$

$$2a)\ n^2(\frac{1}{n^2})^2\sum_{i,j=1,i\neq j}^{n}\sum_{k=1}^{n}[(g(X_i - Y_j) - J_0][(g(X_k - Y_j) - J_0],$$

$$2b)\ n^2(\frac{1}{n^2})^2\sum_{i,j=1,i\neq j}^{n}\sum_{k=1}^{n}[g(Y_k - X_i) - J_0][g(Y_k - X_j) - J_0],$$

$$3)\ n^2(\frac{1}{n^2})^2\sum_{i,j=1}^{n}\sum_{l,k=1,(l\neq k)or(i\neq j)}^{n}[g(X_i - Y_j) - J_0][g(X_l - Y_k)^2) - J_0].$$

Similar expression can be obtained for each of the two other squares and three pairwise products. Note that the limit with $n \to \infty$ for the peculiar sums of the type 1) is finite due to the law of large numbers.

The peculiar sums of type 3) consist of multiplications of indepent terms with zero expectation. Therefore for any n the expectation of these peculiar sums is zero and the limit is also equal to 0.

Consider the peculiar sum $ES^2_{xy,2a}$. It can be written as $I_1 - I_2$,

$$I_1 = \frac{1}{n}\sum_{k=1}^{n}\{[\sum_{i=1}^{n}(g(x_k - y_i) - J_0)]/\sqrt{n}][\sum_{j=1}^{n}g(x_k - y_j) - J_0)]/\sqrt{n}]\},$$

$$I_2 = \frac{1}{n^2}\sum_{i,j=1}^{n}(g(x_i - y_j) - J_0)^2.$$

Note that I_2 tends with $n \to \infty$ to a finite limit due to the law of large numbers. And due to the central limit theorem under fixed X_k the random variable

$$\sum_{i=1}^{n} [g(x_k - y_i) - J_0]/\sqrt{n}$$

tends to normal random variable with zero expectation and a finite variance.

Other sums of type 2) have a similar behaviour. Thus under H_0 there exists a finite limit

$$\lim_{n\to\infty} E(nT_n)^2.$$

Denote this limit by d.

Thus the first part of the lemma is proved.

Let now H_1 holds with $|h| > 0$. In this case we need only to study additionally the behaviour of the following sums

$$n^2\{\frac{1}{n^2}\sum_{i,j=1}^{n}[g(X_i - Y_j) - J_0]\}^2,$$

$$S_{xy,xx,2} = n^2(\frac{1}{2n(n-1)n^2})\sum_{k=1}^{n}\sum_{i,j=1,i\neq j,k,j\neq k}^{n}[(\ln(1+(X_k-Y_i)^2)-J_0][(\ln(1+(X_k-X_j)^2)-J_0],$$

$$S_{xy,xx,3} = n^2(\frac{1}{2n(n-1)n^2})\sum_{i,j=1}^{n}\sum_{k=1,l=1,l\neq k}^{n}[(\ln(1+(X_k-Y_i)^2)-J_0][(\ln(1+(X_l-X_j)^2)-J_0].$$

and $S_{xy,yy,2}, S_{xy,xy,3}$ that are determined in a similar way that $S_{xy,xx,2}, S_{xy,xx,3}$. By a direct calculation it can be verified that

$$\lim_{n\to\infty} E(nT_n)^2 = d + 2J^*(h)J_0 + J^*(h)^2.$$

Lemma is proved.

Lemma 2. *For $g(x) = x^2$ the following identity holds*

$$\Phi_{nm} = (\bar{x} - \bar{y})^2,$$

where

$$\bar{x} = (\sum_{i=1}^{n} X_i)/n, \bar{y} = (\sum_{i=1}^{m} Y_i)/m.$$

The proof follows from the known formula [see e.g. [5], p.296]

$$\frac{1}{n(n-1)}\sum_{1\leq i<j\leq n}(X_i - X_j)^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}(X_i - \bar{x})^2.$$

by direct calculations.

Assume that H_0 holds. Let C be an arbitrary positive number,

$$\tilde{X} = (\tilde{X_1}, \dots, \tilde{X_n}), \;\; \tilde{Y} = (\tilde{Y_1}, \dots, \tilde{Y_n}),$$

where $\tilde{X_i} = X_i$, if $|X_i| \leq C$ and $\tilde{X_i} = C$ if $X_i > 0$, $\tilde{X_i} = -C$ if $X_i < 0$ otherwise. And $\tilde{Y_i}$ are determined similarly. Note that $0 \leq \ln(1+x^2) \leq x^2$. Therefore there exists a value t that depends on $\tilde{X}$ and $\tilde{Y}$ such that

$$n\{\frac{1}{n^2}\sum_{i,j=1}^{n} \ln(1+(\tilde{X_i}-\tilde{Y_j})^2 - \frac{1}{n(n-1)}\sum_{i<j} \ln(1+(\tilde{X_i}-\tilde{X_j})^2) - \tag{10}$$

$$\frac{1}{n(n-1)}\sum_{i<j} \ln(1+(\tilde{Y_i}-\tilde{Y_j})^2)\} = t(\sum_{i=1}^{n} \tilde{X_i}/\sqrt{n} - \sum_{i=1}^{n} \tilde{Y_i}/\sqrt{n})^2. \tag{11}$$

For constructing the right hand side we applied Lemma 2. Note that for distributions F_1 and F_2 satisfying (3) it follows from Lemma 1 that the variance of the left hand side is finite. Therefore the variance of the right hand side is also finite for arbitrary C. Passing to the limit with $n \to \infty$ we obtain due to the central limit theorem that the right hand side has the limit distribution of the form (9) where Z has the normal distribution with zero expectation and variance equal to 1. And its variance is equal to the variance of the left hand side of (11). Since C is arbitrary we obtain that the limiting distribution has the required form for H_0.

For determining a and b in the part (ii) of the theorem we now can use the equality

$$E((aZ+b)^2)^2 = \lim_{n\to\infty} E(nT_n)^2, \tag{12}$$

that follows from (11).

Since $EZ^2 = 1, EZ^4 = 3$, we have for the left hand side of (12)

$$3a^4 + 6a^2b^2 + b^4. \tag{13}$$

The formula for the left hand side follows from Lemma 1. And the asymptotic behaviour of the power follows from the asymptotic normality of $\sqrt{nT_n}$. In order to calculate the right hand side of (12) in (iii) the following result is crucial.

Lemma 3. *If X and Y are independent random variables with the distribution $C(0,1)$, then*

$$E\ln(1+(X-Y)^2) = \ln 9, \;\;\; E\ln(1+(X-Y-\theta)^2) - \ln 9 = ln(1+\theta^2/9).$$

In order to prove this Lemma we need the following integrals

$$\int_R \frac{\ln(1+(x-y)^2)}{\pi(1+y^2)} dy = \ln(4+x^2),$$

$$\int_R \frac{\ln(4+x^2)}{\pi(1+x^2)} dx = \ln 9,$$

([6] 4.296.2 and 4.295.7.)

$$\int_R \frac{\ln(4+(x+\theta)^2)}{\pi(x^2+1)}dx = \ln(9+\theta^2),$$

[see [7], formula (2.6.14.19)]. Using these integrals we obtain

$$E\ln(1+(X-Y-\theta)^2) - \ln 9 = 2\int_R\int_R \frac{\ln(1+(x-y-\theta)^2)}{\pi^2(1+x^2)(1+y^2)}dxdy - \ln 9$$

$$= \int_R \frac{\ln(4+(y+\theta)^2)}{\pi(1+y^2)}dy - \ln 9 = \ln(9+\theta^2) - \ln 9 = \ln(1+\theta^2/9).$$

Submitting here $\theta = 0$ we obtain both formulas of the Lemma. Note that $\theta^2 = nh^2$ and

$$\lim_{n\to\infty} n\ln(1+\theta^2/9) = (1/9)h^2.$$

Therefore we obtain for the right hand side of (8) with some algebra

$$3a^4 + \frac{(2\ln 9)h^2}{9} + \frac{h^4}{81}. \tag{14}$$

From Lemma 3 and (14) we obtain

$$b = \frac{1}{3}h,\ a^2 = \frac{2}{3}\ln 3.$$

The formula for the power follows from the form of the limiting distribution (9).

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