Log-Laplace distributions

Tomasz J. Kozubowski
University of Nevada, Reno

Krzysztof Podgorski
Lund University

LOG-LAPLACE DISTRIBUTIONS

Tomasz J. Kozubowski[1]

Department of Mathematics, University of Nevada, Reno, NV 89557

Krzysztof Podgórski

Department of Mathematical Sciences, Indiana University - Purdue University Indianapolis, Indianapolis, IN 46202

Abstract: We present a comprehensive theory and review historical development of the log-Laplace distributions, which can be thought of as exponential functions of skew Laplace laws and have power tail behavior at zero and infinity. We give new results on their properties, representations, and characterizations, discuss estimation of their parameters, and briefly review their applications.

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1. Introduction

Log-Laplace models appeared sporadically in the statistical, economic as well as science literature over the past seventy years. Only the last decade brought them into the literature more frequently in the context of applied modeling. In fact, they appeared most often as models for data sets with particular properties or were derived as the most natural models based on the properties of the studied processes. Thus, the works that mention them are widely scattered through the literature and often take the point of view of their applications rather than their statistical properties and characterizations. As a result, there is no comprehensive theory of the log-Laplace laws. The purpose of this work is to provide a unified approach to bridge the many avenues via which log-Laplace distributions appeared in the literature. We present an overview of the historical development and applications of the log-Laplace models together with new results on their properties with particular attention to their stability with respect to geometric products. We also discuss some theoretical reasons behind their popularity in applied modeling.

Perhaps the most common reason for using log-Laplace models are their power tails and self-similarity. Self-similar power laws abound throughout natural phenomena (see, e.g., Schroeder (1991)). The Vilfredo Pareto's power law of incomes is one of the classical examples of power laws. It states that, for large x, the population proportion of incomes exceeding x is proportional to $x^{-\alpha}$ (see Pareto (1897)). Notably, the *Pareto distribution*, denoted by $\mathcal{P}(\alpha)$ is the simplest one with that asymptotic property at infinity. Its probability density function (p.d.f.) is

$$f(x) = \frac{\alpha}{x^{\alpha+1}}, \quad x \in (1, \infty), \quad \alpha > 0. \tag{1}$$

Pareto law has been studied extensively in the last century and found numerous applications beyond economics (see, e.g., Arnold (1983) or Johnson et al. (1994)). Interestingly, the distribution of the reciprocal of a Pareto random variable is another classical power law - the *power function distribution* given by the density

$$f(x) = \alpha x^{\alpha-1}, \quad x \in (0, 1), \quad \alpha > 0. \tag{2}$$

This distribution, denoted by $\mathcal{B}(\alpha)$, is a special case of the beta distribution and has power asymptotics at zero (see, e.g., Johnson et al. (1995)).

Log-Laplace distribution can be derived by combining the two power laws described above and has power tails at zero and at infinity. With an additional scale parameter, the resulting distribution is given by the probability density function

$$g(x) = \frac{1}{\delta}\frac{\alpha\beta}{\alpha+\beta}\begin{cases} \left(\frac{x}{\delta}\right)^{\beta-1} & \text{for } 0 < x < \delta \\ \left(\frac{\delta}{x}\right)^{\alpha+1} & \text{for } x \geq \delta. \end{cases} \tag{3}$$

Plotted on the log-log scale this density has a distinct "tent" shape. Empirical evidence of double power tails first appeared in Champernowne (1953) in connection with income modeling. However, it is only in the recent years that power laws received an increased attention following Mandelbrot's work on fractals, and the double power-tail phenomenon has been observed in a variety of fields, including archaeology (see, e.g., Fieller (1993), Fieller and Nicholson (1991), Flenley et al. (1987), Olbricht (1982)), biology (see, e.g., Sznajd-Weron and Weron (2001)), economics (see, e.g., Amaral et al. (1997), Buldyrev et al. (1997), Lee et al. (1998), Reed (2001), Stanley (2000), Stanley et al. (1996), Takayasu and Okuyama (1998)), environmental science (see, e.g., Bagnold (1937, 1954), Fieller et al. (1984a)), finance (see, e.g., Mercik and Weron (1999), Rachev et al. (1997), Weron et al. (1999ab)), and physics (see, e.g., Barndorff-Nielsen (1979)).

The log-Laplace distribution has been discovered and rediscovered over the years. Originally its symmetric form and later the asymmetric model (3) were used for modeling various phenomena by a number of researchers. A random variable with p.d.f. (3) with $\alpha = \beta$ is "symmetric" in the sense that the variable and its reciprocal have the same distribution. Perhaps for the first time symmetric log-Laplace law appeared in Fréchet (1939) as a model for income when the moral fortune, that is the logarithm of income, was assumed to have the classical Laplace distribution (see Fréchet (1958), also Chipman (1985)). This introduction of *log-Laplace* distribution parallels that of the lognormal distribution and is the most common in the literature (see Johnson (1954), Johnson et al. (1994), Kotz et al. (1985, 2001), Uppuluri (1981)). Inoue (1978) derived the symmetric log-Laplace distribution from his stochastic model for income distribution, fitted it to income data by maximum likelihood and reported a better fit than that of a lognormal model traditionally used in this area. The symmetric log-Laplace distribution also appeared in connection with stability and approximation of geometric products in Klebanov et al. (1989) as well as in Mittnik and Rachev (1993), where it was called *symmetric Pareto* distribution.

The general, not necessarily symmetric log-Laplace p.d.f. (3) can be derived in the same way as the symmetric one, as the distribution of e^X, where X is an *asymmetric Laplace* (AL) variable with density

$$f(x) = \frac{\alpha\beta}{\alpha+\beta}\begin{cases} \exp\left(-\alpha(x-\theta)\right), & \text{for } x \geq \theta, \\ \exp\left(\beta(x-\theta)\right), & \text{for } x < \theta, \end{cases} \tag{4}$$

and $\delta = e^\theta$ (see Kotz et al. (2001)). Thus, we refer to (3) as the *skew log-Laplace* law. It was called *double Pareto* in Reed (2001). The skew log-Laplace law has been obtained by Hartley and Revankar (1974) as a model for under-reported data (such as income or property values), and was followed upon in Hinkley and Revankar (1977) who presented maximum likelihood estimators of the related skew Laplace parameters. Uppuluri (1981) derived this distribution from a set of properties about the dose-response curve for radiation carcinogenesis. Along with the log-Cauchy distribution, log-Laplace provided the best fit to pharcokinetic data in Lindsey et al. (2000). Both symmetric and skew log-Laplace distributions can also be derived as ratios of two independent Pareto variables (see Pederzoli and Rathie (1980)).

Skewed log-Laplace distribution has been used extensively as a model for particle size data (see, Fieller (1993), Fieller and Flenley (1987), Fieller and Gilbertson (1985), Fieller and Nicholson (1991), Fieller et al. (1984abc, 1987, 1988, 1990, 1992ab), Flenley (1985), Flenley et al. (1987), Jones and McLachlan (1989), and Olbricht (1982)). As commented by Fieller et al. (1992a), particle size distributions play an important role in disciplines as diverse as archaeology (e.g., quartz inclusions in the fabric of pottery), fuel technology (e.g., droplets of rocket propellant), medicine (e.g., blood cells), and geology (e.g., grains of sand). Following Bagnold's (1937, 1954) observation that when plotted on the log-log scales empirical sand data p.d.f. is "tent shaped", Barndorff-Nielsen (1977) and Bagnold and Barndorff-Nielsen (1980) proposed the *log-hyperbolic* models (of which log-Laplace is a limiting case) for particle size data. Log hyperbolic density plotted on log-log scale is a hyperbola with "tent shaped" straight line asymptotes. Barndorff-Nielsen (1977) offers a theoretical justification for this model in the fact that log-Laplace distribution is a mixture of lognormal laws. These arise naturally in particle size modeling because of the random breakage process (see Epstein (1947), Halmos (1944), Kolmogorov (1941)). Sichel (1973) explains that mixing of lognormals is caused by spatial variability in the observations.

In the works mentioned above the authors report excellent fit of the log-Laplace model to the particle size distribution. They advocate skew log-Laplace law as the preferred model for particle size studies because it not only reflects the observed tent-shaped p.d.f.'s (see, e.g., Flenley et al. (1987)) but also in practical application it is much simpler computationally than the more complex log-hyperbolic model (see, e.g., Fieller et al. (1992a)).

Log-Laplace models have been recently proposed for *growth rates* of diverse processes such as annual gross domestic product (see Lee et al. (1998)), stock prices (see, e.g., Madan et al. (1998)), interest or foreign currency exchange rates (see Kozubowski and Podgórski (1999, 2000, 2001, 2002)), company sizes (see Amaral et al. (1998), Buldyrev et al. (1997), Stanley et al. (1996), Takayasu and Okuyama (1998)), and other processes (see, e.g., Stanley (2000) and Reed (2001)). Reed (2001) offers the fact that (3) is obtained when an exponential Brownian motion is stopped at a random exponential time, which follows from the fact that Laplace distribution is a mixture of normal laws (see Kotz et al. (2001)), as a possible explanation of the wide applicability of this distribution for modeling growth rates.

Let us note another distribution with an *asymptotic* double tail behavior - the *conditionally exponential dependence* (CED) model (see Jurlewicz et al. (1996)). Although this distribution shares theoretical justification for modeling tent-shaped data (see Rachev et al. (1997), Sznajd-Weron and Weron (2001), Weron et al. (1999ab)), it lacks the simplicity (no closed form expressions for the density function) and computational advantages of the log-Laplace model.

In the remainder of this work we review fundamental properties of the skew log-Laplace distributions (3) and present some new results that may explain their successful applications. The properties we focus on include:

- *Pareto-type tails* at zero and infinity, that is

$$P(Y > x) \sim C_1 x^{-\alpha} \text{ as } x \to \infty \text{ and } P(0 < Y \leq x) \sim C_2 x^{\beta} \text{ as } x \to 0^+.$$

- *Invariance with respect to scaling and exponentiation* which is natural property of variables describing multiplicative processes such as growth.
- *Mixtures of log-normal distributions* which provides theoretical justification in the areas where lognormal models are routinely applied.
- A representation as an *exponential growth-decay process over random exponential time* which extends a similar property of the Pareto distribution by allowing decay in addition to growth.
- *Simplicity* which allows for efficient practical applications (especially by non-statisticians) and thus gives an advantage over many other models for heavy power tails, such as CED, stable, or geometric stable laws.
- *The upper tail index is not bounded from above* which adds flexibility over some other models for heavy tail data such as stable or geometric stable laws where its value is limited by two.
- *Maximum entropy property* which is desirable in many applications.
- *Stability with respect to geometric multiplication* which may play a fundamental role in modeling growth rates.
- *Limiting distribution* of geometric products which leads to useful approximations.
- *Straightforward extension to the multivariate setting* which allows modeling of correlated multivariate rate data, such as joint returns on portfolios of securities.

2. Basics

If X has an asymmetric Laplace (AL) distribution given by (4), then the density of $Y = e^X$ is given by (3) with $\delta = e^{\theta}$.

Definition 2.1. *A random variable distributed according to the density (3) is said to have a log-Laplace (LL) distribution with parameters $\delta > 0$, $\alpha > 0$, and $\beta > 0$. This distribution is denoted by $\mathcal{LL}(\delta, \alpha, \beta)$.*

A $LL(\delta,\alpha,\beta)$ distribution is concentrated on $[0,\infty)$. The quantity δ is a scale parameter, and α and β are the tail parameters at $x\to\infty$ and $x\to 0^+$:

$$P(Y>x)\sim C_1x^{-\alpha} \text{ as } x\to\infty \text{ and } P(0<Y\le x)\sim C_2x^{\beta} \text{ as } x\to 0^+.$$

Figure 1 shows selected LL densities. Observe that they are not smooth at $x=\delta$.

The following simple result describes the important closure property of log-Laplace distributions with respect to multiplication and exponentiation. Here we write $W\stackrel{d}{=}V$ if the variables W and V have the same distribution.

Proposition 2.1. *Let $Y\stackrel{d}{=}\mathcal{LL}(\delta,\alpha,\beta)$, and let $c>0$, $r\ne 0$. Then*
(i) $cY\stackrel{d}{=}\mathcal{LL}(c\delta,\alpha,\beta)$,

(ii) $Y^r\stackrel{d}{=}\begin{cases}\mathcal{LL}(\delta^r,\alpha/r,\beta/r) & r>0,\\ \mathcal{LL}(\delta^r,\beta/|r|,\alpha/|r|) & r<0.\end{cases}$

In particular, if $Y\stackrel{d}{=}\mathcal{LL}(\delta,\alpha,\beta)$, then

$$\frac{1}{Y}\stackrel{d}{=}\mathcal{LL}(1/\delta,\beta,\alpha), \tag{5}$$

and if $\alpha=\beta$ we have the reciprocal property $1/Y\stackrel{d}{=}Y$.

The LL distribution function (c.d.f.) is

$$G(x)=\begin{cases}0 & \text{for } x<0\\ \frac{\alpha}{\alpha+\beta}(\frac{x}{\delta})^{\beta} & \text{for } 0\le x<\delta\\ 1-\frac{\beta}{\alpha+\beta}(\frac{\delta}{x})^{\alpha} & \text{for } x\ge\delta.\end{cases} \tag{6}$$

Consequently, the quantile function $G^{-1}(q)$, $q\in(0,1)$, has the following explicit form:

$$G^{-1}(q)=\delta\begin{cases}\left[q\frac{\alpha+\beta}{\alpha}\right]^{1/\beta} & \text{for } q\in\left(0,\frac{\alpha}{\alpha+\beta}\right]\\ \left[(1-q)\frac{\alpha+\beta}{\beta}\right]^{-1/\alpha} & \text{for } q\in\left(\frac{\alpha}{\alpha+\beta},1\right).\end{cases} \tag{7}$$

Remark 2.1. Let us note the following extreme cases of $\mathcal{LL}(1,\alpha,\beta)$ distributions, which can be derived by taking the appropriate limits of the LL c.d.f. (6). When $\alpha=0$, $\beta>0$ or when $\alpha>0$, $\beta=0$ the distribution will be degenerate distribution at 1 and at 0, respectively. When either $\alpha\to\infty$ or $\beta\to\infty$, then in the limit we obtain the beta and Pareto distributions,

$$\mathcal{LL}(1,\infty,\beta)\stackrel{d}{=}\mathcal{B}(\beta),\ \ \mathcal{LL}(1,\alpha,\infty)\stackrel{d}{=}\mathcal{P}(\alpha). \tag{8}$$

These can be thought of as "one-sided" special cases of LL distributions (just as the exponential and negative exponential are parallel special cases of the Laplace distributions).

The characteristic function of a log-Laplace distribution can be expressed in terms of the confluent hypergeometric function

$$M(a,b,z) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n}{(b)_n}\frac{z^n}{n!},$$

where $(a)_n = a(1+a)\cdot\dots\cdot(n-1+a)$ for $a>0$, $b>0$ $z \in \mathbb{C}$, and the generalized Fresnel integrals

$$C(x,a) = \int_x^\infty t^{a-1}\cos t\, dt, \;\; S(x,a) = \int_x^\infty t^{a-1}\sin t\, dt,$$

defined for $a<0$ and $x>0$.

Proposition 2.2. *The characteristic function of $\mathcal{LL}(1,\alpha,\beta)$ random variable Y has the form*

$$Ee^{itY} = \frac{\alpha}{\alpha+\beta}M(\beta,\beta+1,it) + \frac{\alpha\beta}{\alpha+\beta}t^\alpha\left[C(t,-\alpha)+iS(t,-\alpha)\right].$$

Proof. Using the representation (19) of a LL variable as a mixture of beta and Pareto distributions, we obtain

$$Ee^{itY} = \frac{\alpha}{\alpha+\beta}\phi_1(t) + \frac{\beta}{\alpha+\beta}\phi_2(t),$$

where $\phi_1(t)$ and $\phi_2(t)$ is the characteristic function of $\mathcal{B}(\beta)$ and $\mathcal{P}(\alpha)$ distributions, respectively. The first characteristic function is equal to $M(\beta,\beta+1,it)$, see Abramowitz and Stegan (1964). The other one can be obtained from the definition of generalized Fresnel integrals after applying simple change of variables. ☐

Let us note that the moment generating function of a log-Laplace $\mathcal{LL}(1,\alpha,\beta)$ r.v. Y is defined only for $s \le 0$; for $s<0$ it is of he form

$$Ee^{sY} = \frac{\alpha\beta}{\alpha+\beta}\left[\Gamma(\beta)\gamma^*(\beta,-s) + (-s)^\alpha\Gamma(-\alpha,-s)\right], \tag{9}$$

where $\gamma^*(a,s)$ and $\Gamma(a,s)$ are the incomplete gamma functions

$$\gamma^*(a,s) = \frac{s^{-a}}{\Gamma(a)}\int_0^s x^{a-1}e^{-x}dy \;\text{ and }\; \Gamma(a,s) = \int_s^\infty x^{a-1}e^{-x}dy.$$

However, since the moment generating function is not defined in an open neighborhood of zero, it is not a convenient tool for computing moments of the log-Laplace distributions.

We finish this section with the following result that provides an explicit form of the Lorenz Curve corresponding to the log-Laplace distribution. The Lorenz curve, which is commonly used in economics to derive various measures of inequality, is defined as

$$L(p) = (EX)^{-1}\int_0^p G^{-1}(t)dt, \;\; p \in (0,1), \tag{10}$$

where X is a random variable with finite mean and c.d.f. G.

Proposition 2.3. *Let $X \sim \mathcal{LL}(\delta, \alpha, \beta)$, where $\alpha > 1$. Then the Lorenz curve of X is*

$$L(p) = \begin{cases} \frac{\alpha-1}{\alpha}\left(\frac{\alpha+\beta}{\alpha}\right)^{1/\beta} p^{1/\beta+1} & \text{for } 0 < p < \frac{\alpha}{\alpha+\beta}, \\ \frac{\alpha-1}{\alpha+\beta} - \frac{\beta+1}{\beta}\left(\frac{\beta}{\alpha+\beta}\right)^{1/\alpha}\left[(1-p)^{1-1/\alpha} - \left(\frac{\beta}{\alpha+\beta}\right)^{1-1/\alpha}\right] & \text{for } \frac{\alpha}{\alpha+\beta} < p < 1. \end{cases}$$

Proof. The above formula results from a straightforward integration of the LL quantile function (7) coupled with the mean formula given in Table 1 (note that the mean is finite only for $\alpha > 1$). □

2.1. **Moments and related parameters.** We summarize moments and related parameters of LL distributions in Table 1. Note that LL distributions are heavy tailed and some moments do not exist. The mean and the variance are finite only if $\alpha > 1$ and $\alpha > 2$, respectively. Due to reciprocal properties of these laws, the harmonic mean is of the same form as the reciprocal of the mean. We also note that LL distributions are unimodal with the mode at δ when $\beta > 1$ and the mode at zero when $\beta < 1$. For $\beta = 1$ the p.d.f. is constant on $(0, \delta)$ and the mode is not unique (see Figure 1).

2.2. **Multivariate extension.** An extension to the multivariate case is straightforward. Let $\mathbf{X} = (X_1, \ldots, X_d)'$ have a multivariate asymmetric Laplace distribution $\mathcal{AL}_d(\mathbf{m}, \mathbf{\Sigma})$ on R^d, given by the characteristic function

$$\Psi(\mathbf{t}) = \left[1 + \frac{1}{2}\mathbf{t}'\mathbf{\Sigma}\mathbf{t} - i\mathbf{m}'\mathbf{t}\right]^{-1}, \tag{11}$$

where $\mathbf{t}'$ denotes a transpose of $\mathbf{t}$, $\mathbf{m} \in R^d$, and $\mathbf{\Sigma}$ is a $d \times d$ non-negative definite symmetric matrix. Then, analogous to the one-dimensional case, we can define a d-dimensional log-Laplace distribution as that of the random vector

$$\mathbf{Y} = e^{\mathbf{X}} = (e^{X_1}, \ldots, e^{X_d})'. \tag{12}$$

If $\mathbf{\Sigma}$ is positive-definite, then the distribution is truly d-dimensional and the corresponding density function g can be derived easily from that of the Laplace distribution (see Kotz et al. (2001)). Namely,

$$g(\mathbf{y}) = f(\log \mathbf{y})\left(\prod_{i=1}^{d} y_i\right)^{-1}, \quad \mathbf{y} = (y_1, \ldots, y_d)' > \mathbf{0}, \tag{13}$$

where $\log \mathbf{y} = (\log y_1, \ldots, \log y_d)$ is defined componentwise and

$$f(\mathbf{x}) = \frac{2e^{\mathbf{x}'\mathbf{\Sigma}^{-1}\mathbf{m}}}{(2\pi)^{d/2}|\Sigma|^{1/2}}\left(\frac{\mathbf{x}'\mathbf{\Sigma}^{-1}\mathbf{x}}{2 + \mathbf{m}'\mathbf{\Sigma}^{-1}\mathbf{m}}\right)^{v/2} K_v\left(\sqrt{(2 + \mathbf{m}'\mathbf{\Sigma}^{-1}\mathbf{m})(\mathbf{x}'\mathbf{\Sigma}^{-1}\mathbf{x})}\right) \tag{14}$$

is the density of the skew Laplace distribution ($\mathbf{x} \neq \mathbf{0}$). Here, $v = 1 - d/2$ and K_v is the modified Bessel function of the third kind (see, e.g., Abramowitz

Parameter	Value
r^{th} moment	$EY^r = \delta^r \frac{\alpha\beta}{(\alpha-r)(\beta+r)}, \quad -\beta < r < \alpha$
Mean	$EY = \delta \frac{\alpha\beta}{(\alpha-1)(\beta+1)}, \ \ 1 < \alpha$
Variance	$E[(Y-EY)^2] = \delta^2 \left(\frac{\alpha\beta}{(\alpha-2)(\beta+2)} - \left[\frac{\alpha\beta}{(\alpha-1)(\beta+1)} \right]^2 \right), \ \ 2 < \alpha$
Median	$G^{-1}(1/2) = \begin{cases} \delta \left[\frac{\alpha+\beta}{2\alpha}\right]^{1/\beta} & \text{for } \alpha \geq \beta, \\ \delta \left[\frac{\alpha+\beta}{2\beta}\right]^{-1/\alpha} & \text{for } \alpha \leq \beta \end{cases}$
Mode	$m = \begin{cases} \delta & \text{for } \beta > 1 \\ 0 & \text{for } 0 < \beta < 1 \end{cases}$
Geometric mean	$e^{E \log Y} = \delta e^{\frac{1}{\alpha} - \frac{1}{\beta}}$
Harmonic mean	$[E(1/Y)]^{-1} = \left(\delta \frac{\alpha\beta}{(\alpha+1)(\beta-1)} \right)^{-1}, \ \ 1 < \beta$

Table 1. Moments and other parameters of $Y \sim \mathcal{LL}(\delta, \alpha, \beta)$.

and Stegun (1964)). As is the case for Laplace distribution, the density can be written in closed form for odd values of d.

Multivariate LL distributions retain main properties of univariate LL laws, including the stability and limiting properties with respect to geometric multiplication. In addition, all marginal distributions are of the same type. In particular, each component of a multivariate LL random vector is univariate LL.

3. Representations

Log-Laplace distributions can be represented in terms of other well-known distributions, including the lognormal, exponential, uniform, Pareto, and beta distributions. We skip straightforward derivations of the results presented below, which can be obtained from the corresponding properties of Laplace laws (see, e.g., Kotz et al. (2001)). Throughout this section Y stands for a general log-Laplace variable with density (3).

3.1. **Relation to the lognormal distribution.** Recall that the lognormal distribution with parameters μ and σ is the distribution of e^X, where X is normal with mean μ and variance σ^2. We use $\mathcal{LN}(\mu, \sigma)$ to denote such a distribution or random variable with this distribution. A *standard lognormal* distribution corresponds to $\mathcal{LN}(0, 1)$ and the relation between a general log-normal random variable X and a standard one R is given by

$$X \stackrel{d}{=} e^{\mu} R^{\sigma}.$$

The log-Laplace distribution $\mathcal{LL}(\delta, \alpha, \beta)$ can be viewed as $\mathcal{LN}(\mu, \sigma)$, where the parameters μ and σ are random. More specifically, the variable $Y \sim \mathcal{LL}(\delta, \alpha, \beta)$ has the representation

$$Y \stackrel{d}{=} e^{\mu} R^{\sigma},$$

where μ and σ are the following random variables:

$$\mu = \log \delta + \left(\frac{1}{\alpha} - \frac{1}{\beta} \right) E \text{ and } \sigma = \sqrt{\frac{2E}{\alpha \beta}}.$$

Here, E is a standard exponential variable independent of R. This is a direct consequence of the fact that an asymmetric Laplace r.v. can be viewed as a normal variable with the above random mean μ and standard deviation σ (see Kotz et al. (2001)).

Remark 3.1. This representation can be expressed as

$$Y \stackrel{d}{=} e^{B(E)}, \tag{15}$$

where B is a Brownian motion with a drift, independent of the standard exponential variable E. More precisely, if $B(t)$ has normal distribution with mean $\log \delta + (\mu - \sigma^2/2)t$ and variance $\sigma^2 t$, then (15) holds, where α and $-\beta$ are the two positive roots of the quadratic equation

$$\frac{1}{2}\sigma^2 x^2 + \left(\mu - \frac{1}{2}\sigma^2 \right) x - 1 = 0,$$

cf. Reed (2001). This reflects the fact that Y is equal in distribution to $Y(1)$, where $\{Y(t) = \exp L(t), t \geq 0\}$ is a *geometric Laplace motion*. Here, $L(t) \stackrel{d}{=} B(G(t))$ is the *Laplace motion* (also known as *variance gamma process*, see Madan et al. (1998)), which is a Brownian motion subordinated to an independent gamma process $\{G(t), t \geq 0\}$, see Kotz et al. (2001) for more details. These generalized log-Laplace distributions found applications in quality control (see Rowland and Sichel (1960)). In financial application, this random time change reflects the fact that economic time is measured differently than the calendar time. Subordinated processes for modeling stock prices were first considered by Clark (1973), and are now common in financial modeling (see, e.g., Rachev and Mittnik (2000) and references therein).

3.2. **Relations to exponential distribution.** The following representation of Y is a direct consequence of the fact that a skew Laplace variable arises as a difference of two independent exponential variables. We have

$$Y \stackrel{d}{=} \delta e^{\frac{1}{\alpha}E_1 - \frac{1}{\beta}E_2}, \tag{16}$$

where E_1 and E_2 are two i.i.d. standard exponential variables. This representation can be written equivalently as

$$Y \stackrel{d}{=} \delta e^{JE},$$

where E is a standard exponential variable and J is an independent of E variable taking on values $-1/\beta$ and $1/\alpha$ with probabilities

$$p = \frac{\alpha}{\alpha+\beta} \text{ and } q = \frac{\beta}{\alpha+\beta}, \tag{17}$$

respectively. The last representation has an interesting practical interpretation: Y can be viewed as an exponential growth (or decay) for an exponentially distributed period of time. The growth and decay occur with rates $1/\beta$ and $1/\alpha$ and probabilities q and p given above. A Pareto random variable has similar interpretation of exponential growth over a random time (see Arnold (1983), p. 41).

3.3. **Relation to uniform distribution.** Let U_1 and U_2 be independent random variables distributed uniformly on $[0,1]$. Then we have

$$Y \stackrel{d}{=} \delta \frac{U_1^{1/\beta}}{U_2^{1/\alpha}}.$$

This representation can be obtained directly by computing densities or using the corresponding representation of an asymmetric Laplace random variable (see Kotz et al. (2001)). It may be particularly useful in simulations, as uniform variate generators are readily available. Note that the special case $\alpha = \beta = 1$ was obtained by Lukacs and Laha (1964) and Richmond and Oettli (1966).

3.4. **Relations to beta and Pareto distributions.** When $\alpha > 0$ and U is a standard uniform variable, then $U^{1/\alpha}$ has the beta distribution $\mathcal{B}(\alpha)$ with density (2). This leads to

$$Y \stackrel{d}{=} \delta \frac{B_1}{B_2},$$

where B_1 and B_2 are independent $\mathcal{B}(\beta)$ and $\mathcal{B}(\alpha)$ variables, respectively. Note that since $1/B_2$ has the Pareto $\mathcal{P}(\alpha)$ distribution with density (1), we also have

$$Y \stackrel{d}{=} \delta B_1 P_1 \stackrel{d}{=} \delta \frac{P_1}{P_2}, \tag{18}$$

where all variables above are mutually independent and $P_1 \stackrel{d}{=} \mathcal{P}(\alpha)$ and $P_2 \stackrel{d}{=} \mathcal{P}(\beta)$.

A related representation is obtained by noting that the conditional distributions of Y given $Y > \delta$ and Y given $Y < \delta$ are the same as those of δP and δB, respectively, where $P \stackrel{d}{=} \mathcal{P}(\alpha)$ and $B \stackrel{d}{=} \mathcal{P}(\beta)$. This can be written as

$$Y \stackrel{d}{=} \delta[IB + (1-I)P], \tag{19}$$

where B and P are independent variables specified above and I is a Bernoulli variable, independent of B and P, taking on values 1 and 0 with probabilities p and q given in (17). Thus, every log-Laplace variable can be regarded as a mixture of a Pareto and a beta variables.

Remark 3.2. The representation of a log-Laplace variable as the ratio of two independent Pareto variables (cf. Pederzoli and Rathie (1980)) is related to an application of this distribution in the modeling of underreported data (see Hartley and Revankar (1974)). Noting that in many applications (particularly for income or property values) reported values often underestimate the true value Y^*, Hartley and Revankar (1974) consider the model $Y = Y^* - U$, where Y is the observable variable and U $(0 \le U \le 1)$ is a positive underreporting error. Now, if Y^*/δ has the Pareto $\mathcal{P}(\alpha)$ distribution and the proportion of Y^* that is underreported, $W^* = U/Y^*$, has the beta distribution $\mathcal{B}(1,\beta)$ with density

$$f(x) = \beta(1-x)^{\beta-1}, \;\; 0 \le x \le 1, \;\; \beta > 0,$$

and is independent of Y^*, then Y has the log-Laplace distribution with density (3). Indeed, we have

$$Y^* - U = Y^*(1 - U/Y^*),$$

where the two factors are independent and the variable $1-U/Y^*$ is the reciprocal of the Pareto $\mathcal{P}(\beta)$ variable, i.e., has the beta $\mathcal{B}(\beta)$ distribution. Thus, we obtain the representation of the log-Laplace random variable as the product of beta and Pareto distributions as discussed above.

4. **Characterizations and related properties**

In this section we review important characterizations of log-Laplace laws. Some of the results below are scattered in the literature, while others appear to be new. Since many of them follow from analogous properties of Laplace or Pareto distributions, we just outline the proofs and refer the interested reader to appropriate references.

4.1. **Multiplicative divisibility properties.** It is well known that both normal and lognormal distributions are infinitely divisible. It is interesting that while skew Laplace distributions are infinitely divisible, the log-Laplace laws do not share this property, since they are distributions of ratios of independent Pareto variables (and such ratios are not infinitely divisible, see Rohatgi et al.

(1990)). However, log-Laplace laws do possess divisibility properties with respect to *multiplication*, which follow from additive properties of Laplace laws.

First, consider deterministic multiplication. The following result is related to infinite divisibility of skew Laplace distribution (see Kotz et al. (2001), Proposition 3.4.1) and can be deduced from representation (16) and additive properties of standard gamma distribution $\mathcal{G}(\alpha, 1)$ with density

$$f(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}, \quad x > 0. \tag{20}$$

Proposition 4.1. *Let* $Y \sim \mathcal{LL}(\delta, \alpha, \beta)$.
(i) For all natural $n \geq 1$ *we have*

$$Y \stackrel{d}{=} \prod_{i=1}^{n} Y_{ni}, \tag{21}$$

where the Y_{ni}*'s are i.i.d. variables distributed as*

$$Y_{ni} \stackrel{d}{=} \delta^{1/n} \exp\left(\frac{1}{\alpha} G_1 - \frac{1}{\beta} G_2\right), \tag{22}$$

where G_1 *and* G_2 *are i.i.d. gamma* $\mathcal{G}(1/n, 1)$ *variables.*
(ii) If $Y_1, \ldots, Y_n$ *are i.i.d. copies of* Y*, then*

$$\prod_{i=1}^{n} Y_i \stackrel{d}{=} \delta^n \exp\left(\frac{1}{\alpha} G_1 - \frac{1}{\beta} G_2\right), \tag{23}$$

where this time G_1 *and* G_2 *are i.i.d. gamma* $\mathcal{G}(n, 1)$ *variables.*

This result reflects the fact that $\mathcal{LL}(\delta, \alpha, \beta)$ is the distribution of $Y(1)$, where $\{Y(t),\ t \geq 0\}$ is a geometric Laplace motion discussed in Section 3.1. This process admits the representation

$$Y(t) \stackrel{d}{=} \delta^t \exp\left(\frac{1}{\alpha} G_1(t) - \frac{1}{\beta} G_2(t)\right), \tag{24}$$

where the quantities $\{G_i(t),\ t \geq 0\}$, $i = 1, 2$, are i.i.d. (gamma) Lévy processes (processes with independent increments starting at zero and such that $G_i(1)$ has the standard exponential distribution). The representation of the Laplace motion that appears in the exponent above, termed *variance gamma process* in Madan and Seneta (1990), has an important practical interpretation in mathematical finance: G_1 and G_2 are two independent processes that account for price increases and decreases, respectively (see Madan et al. (1998), and also Kotz et al. (2001) for further information on Laplace motion).

We now consider products with a *random* number of terms ν_p, which has a geometric distribution with mean $1/p$ and the probability mass function

$$P(\nu_p = k) = (1-p)^{k-1} p, \quad k = 1, 2, 3, \ldots. \tag{25}$$

We shall see that this distribution plays an important role is several characterizations of LL laws. The following result reflects the fact that skew Laplace variables are geometric infinitely divisible (see, e.g., Kotz et al. (2001), Proposition 3.4.3) and geometric sums of i.i.d. skew Laplace variables are skew Laplace.

Proposition 4.2. *Let $Y \sim \mathcal{LL}(1,\alpha,\beta)$ and let ν_p be a geometric variable given by (25).*
(i) For all $p \in (0,1)$ we have

$$Y \stackrel{d}{=} \prod_{i=1}^{\nu_p} Y_{pi}, \tag{26}$$

where the Y_{pi}'s are i.i.d. $\mathcal{LL}(1,\alpha(p),\beta(p))$ random variables independent of ν_p, where

$$\alpha(p) = \frac{1}{2}\left\{\sqrt{(\beta-\alpha)^2 + 4\alpha\beta\frac{1}{p}} + \alpha - \beta\right\} \tag{27}$$

and

$$\beta(p) = \frac{1}{2}\left\{\sqrt{(\beta-\alpha)^2 + 4\alpha\beta\frac{1}{p}} + \beta - \alpha\right\}. \tag{28}$$

(ii) If $Y_1, Y_2, \ldots$ are i.i.d. copies of Y, then for each $p \in (0,1)$ the geometric product $\prod_{i=1}^{\nu_p} Y_i$ has the $\mathcal{LL}(1,\alpha(1/p),\beta(1/p))$ distribution, where $\alpha(\cdot)$ and $\beta(\cdot)$ are given by (27)-(27).

Note that if $\alpha = \beta$ then $\alpha(1/p) = \beta(1/p) = \alpha\sqrt{p}$. Thus, in view of Part (ii) of the above result, we obtain the stability property (with respect to geometric multiplication) of symmetric LL distributions discussed in Section 4.3 (since $\mathcal{LL}(1,\alpha(1/p),\alpha(1/p))$ is the distribution of $Y^{1/\sqrt{p}}$ by Proposition 2.1).

4.2. **Distributional limits of geometric products.** Since Laplace distributions are limits of appropriately normalized sums $X_1 + \cdots + X_{\nu_p}$ of i.i.d. r.v.'s with a geometric number of terms, it follows that log-Laplace laws arise as limits in the geometric multiplication scheme. The following result can be deduced from the definition of geometric stable laws (which are the limiting distributions of geometric compounds) and the fact that geometric stable laws with finite variance are exponential and Laplace distributions (c.f., Kotz et al. (2001), Proposition 3.4.4).

Proposition 4.3. *The class of log-Laplace distributions with $\delta = 1$ that includes the extreme 1-sided cases (8) coincides with the class of non-degenerate distributional limits of*

$$T_p = \left\{\prod_{i=1}^{\nu_p} c_p Y_i\right\}^{a_p} \tag{29}$$

as $p \to 0$, where $Y_1, Y_2, \ldots$ are i.i.d. non-degenerate, positive r.v.'s with $E(\log Y_i)^2 < \infty$ and ν_p is a geometric r.v. with mean $1/p$, independent of

the Y_i's. Moreover, if $E[\log Y_i] = \mu$ and $Var[\log Y_i] = \sigma^2$, then the normalizing sequences in (29) may be taken as follows:
(i) If $\mu = 0$, then $a_p = \sqrt{p}$ and $c_p = 1$, in which case T_p converges to the symmetric $\mathcal{LL}(1, \sqrt{2}/\sigma, \sqrt{2}/\sigma)$ variable.
(ii) If $\mu \neq 0$, then either

$$a_p = \sqrt{p}, \quad c_p = e^{\mu(p^{1/2}-1)}, \tag{30}$$

in which case T_p converges in distribution to the $\mathcal{LL}(1, \alpha, \beta)$ random variable with

$$\alpha = \frac{2}{\sqrt{2\sigma^2 + \mu^2} + \mu}, \quad \beta = \frac{2}{\sqrt{2\sigma^2 + \mu^2} - \mu},$$

or $a_p = p$, $c_p = 1$, in which case T_p converges to $\mathcal{P}(1/\mu)$ variable if $\mu > 0$ and to $\mathcal{B}(1/\mu)$ variable if $\mu < 0$.

Remark 4.1. The convergence of T_p to a one-sided distribution when $\mu \neq 0$ and to the symmetric LL distribution when $\mu = 0$ (with the above specific values of the normalizing constants) was shown by Klebanov et al. (1989). Note that the former case follows from the law of large numbers (LLN) applied to the variables $X_i = \log Y_i$ and does not require the condition that the second moment of X_i be finite (see, e.g., Klebanov et al. (1989)). Further, in this case the variables Y_i do not have to be i.i.d. - the result holds as long as the weak LLN holds for the X_i's. In addition, if the variables X_i satisfy the strong LLN, then we can even relax the condition that the Y_i's be independent of ν_p, since in this case the geometric sum of the X_i's still converges to the exponential law as noted by Brown (1990). For more general results on the convergence of random products with other integer-valued number of terms see, e.g., Kowalski and Rychlik (1995).

Remark 4.2. The above results can be used to approximate large geometric products of i.i.d. variables. Klebanov et al. (1989) discussed how these can arise in mathematical economics. Suppose that the capital of 1 unit is invested at time $t = 0$, and denote the return on the investment over the time period $(i-1, i)$ by Y_i. Then the total return on the investment over n periods is $\prod_{i=1}^{n} Y_i$. Similarly, the geometric product $\prod_{i=1}^{\nu_p} Y_i$ has an interpretation of the total return till the occurrence of some (rare) event that changes the market conditions. The time of the event is represented by a geometric random variable ν_p (assuming that the event can occur in each period with the same probability p, independently of any other periods) with a small value of p (rare event). Then, if the geometric mean of the returns is greater than 1 (that is $E \log Y_i = \mu > 0$) then the return over the entire random period ν_p has approximately Pareto distribution $\mathcal{P}(1/\mu)$, and consequently the investment is profitable (as the return will be greater than one with probability one). On the other hand, if $\mu < 0$, then the total return will have the beta distribution and consequently the business will fail.

4.3. **Stability with respect to multiplication.** Distributions that are stable in the sense that an appropriately normalized quantity

$$S_n = X_1 \circ \cdots \circ X_n \tag{31}$$

has the same distribution as X_1, where the X_i's are i.i.d. variables, n is either deterministic or random, and "$\circ$" is an operation such as addition, multiplication, maximum, or minimum, play an important role in stochastic modeling (see, e.g., Mittnik and Rachev (1993)). It is well known that Pareto variables are stable with respect to geometric multiplication, see, e.g., Klebanov et al. (1989). More precisely, if the Y_i's are i.i.d. $\mathcal{P}(\alpha)$ variables, then

$$\left\{ \prod_{i=1}^{\nu_p} c_p Y_i \right\}^{a_p} \stackrel{d}{=} Y \ \text{ for all } \ p \in (0,1), \tag{32}$$

where $c_p = 1$, $a_p = p$, and ν_p is a geometric random variable with mean $1/p$ and probability function (25), independent of the Y_i's. Similar property is obviously shared by the reciprocal of Pareto variable (beta distribution). Notice that these distributions are supported on $(1, \infty)$ and $(0, 1)$, respectively.

The above property of Pareto distribution is a consequence of the stability property of exponential distribution with respect to geometric convolutions (see Arnold (1973)),

$$a_p \sum_{i=1}^{\nu_p} X_i \stackrel{d}{=} X \ \text{ for all } \ p \in (0,1), \tag{33}$$

where $a_p = p$ and the X_i's are i.i.d. exponential variables (since $Y = e^{X_1}$ is Pareto). Note that in this relation the variables X_i have mean different than zero (and the Pareto variables Y_i in (32) have a geometric mean different than 1). If the variables in (33) have mean zero (and finite variance), then they must have a symmetric Laplace distribution (see, e.g., Lin (1994)). In view of the above, it is not surprising that log-Laplace distributions enjoy the stability property with respect to geometric multiplication. The following characterization of the log-Laplace distribution follows from an analogous result for geometric convolutions (see Kozubowski (1994), Theorem 3.2)

Proposition 4.4. *Let $Y, Y_1, Y_2, \ldots$ be positive, non-degenerate i.i.d. random variables with $E[\log Y]^2 < \infty$ and with geometric mean equal to one. Further, let ν_p be a geometric random variable with mean $1/p$, independent of the Y_i's. Then, the geometric multiplicative stability property (32) holds if and only if Y has a $\mathcal{LL}(1, \alpha, \alpha)$ distribution with some $\alpha > 0$. Moreover, we must necessarily have $a_p = \sqrt{p}$ and $c_p = 1$.*

Remark 4.3. If the geometric mean is not equal to one, then it follows from Theorem 3.2 in Kozubowski (1994) that the variables Y_i must have either Pareto $\mathcal{P}(\alpha)$ or $\mathcal{B}(\beta)$ distribution for some $\alpha, \beta > 0$. These characterizations (in a slightly less general form where a_p is assumed to be either p or $\sqrt{p}$) of symmetric log-Laplace and Pareto distributions appeared in Klebanov et al.

(1989), see also Kowalski and Rychlik (1995) for some extensions. Mittnik and Rachev (1993) refer to the $\mathcal{LL}(\delta, \alpha, \alpha)$ distribution as two parameter *symmetric Pareto* in their systematic account of alternative stable distributions (that is stable with respect to the operation $\circ$ as described above). If we do not have any moment restrictions, then variables that satisfy (32) arise as exponents of geometric stable laws (see, e.g., Klebanov et al. (1984), Kozubowski (1994), Kozubowski and Rachev (1999)).

While non-symmetric LL laws do not possess the above stability property, they do enjoy a *conditional stability property* that may account for their applicability in modeling growth processes (see Kozubowski and Podgórski (2002)). Let (R_i) be a sequence of i.i.d. non-negative random variables that may represent growth rates $P(i+1)/P(i)$ of some process $P(i)$, and define

$$P_i = R_i \vee 1 = \max(R_i, 1), \quad B_i = R_i \wedge 1 = \min(R_i, 1).$$

Suppose that at first we have G_1 consecutive R_i's greater than 1 (the first period of growth), and D_1 consecutive R_i's less than 1 (the first period of decline). Similarly, following the first period of decline, there are G_2 consecutive R_i's greater than 1 followed by D_2 consecutive R_i's less than 1, and so on. Then, (G_k) and (D_k) are independent sequences of i.i.d. geometric random variables with parameters $p = P(R_i < 1)$ and $q = 1 - p$, respectively. Let P_k^G be the *cumulative growth* over the kth period of uninterrupted growth and let B_k^D be the *cumulative decline* over the kth period of uninterrupted decline, so that $P_1^G = P_1 \times P_2 \times \cdots \times P_{G_1+D_1}$, $P_2^G = P_{G_1+D_1+1} \times P_{G_1+D_1+2} \times \cdots \times P_{G_1+G_2+D_1+D_2}$, $B_1^D = B_1 \times B_2 \times \cdots \times B_{G_1+D_1}$ and so on. Then

$$Y_k = P_k^G \cdot B_k^D \tag{34}$$

represents the accumulated growth during the kth period of growth followed by the kth period of decline. As shown by Kozubowski and Podgórski (2002), the log-Laplace growth model has the following stability property.

Proposition 4.5. *If in the above model the R_i's have $\mathcal{LL}(1, \alpha, \beta)$ distribution, then the P_k^G's are i.i.d. Pareto $\mathcal{P}(\frac{\alpha^2}{\alpha+\beta})$ variables while the B_k^D's are i.i.d. beta $\mathcal{B}(\frac{\beta^2}{\alpha+\beta})$ variables (independent of the P_k^G's). Further, the cumulative growth rates Y_k given by (34) are i.i.d. with the $\mathcal{LL}(1, \frac{\alpha^2}{\alpha+\beta}, \frac{\beta^2}{\alpha+\beta})$ distribution.*

Remark 4.4. The geometric random variable that represents the number of consecutive R_i's greater than 1 is not independent of the R_i's, so the fact that the return over a growth period has Pareto distribution (which is the distribution of $R_i | R_i > 1$) is not a simple consequence of the stability property of this distribution with respect to geometric multiplication (cf. *Remark 4.3*).

If the stability relation (33) with $a_p = \sqrt{p}$ is written in terms of ch.f.'s, it becomes clear that it also corresponds to the following relations among random variables:

$$X \stackrel{d}{=} \sqrt{p} I X_1 + (1 - I)(X_2 + \sqrt{p} X_3) \stackrel{d}{=} \sqrt{p} X_1 + (1 - I) X_2, \tag{35}$$

where X, X_1, X_2, X_3 are i.i.d. symmetric variables and I is a Bernoulli random variable with $P(I = 1) = p$, independent of the X_i's (see, e.g., Kotz et al. (2001)). Exponentiating the relation (35) we obtain another characterization of the log-Laplace distribution (cf. Kotz et al. (2001), Proposition 2.2.8).

Proposition 4.6. *Let Y, Y_1, Y_2, Y_3 be positive, non-degenerate i.i.d. random variables with $Var[\log Y] = \sigma^2$. Let I be a Bernoulli variable with $P(I = 1) = p$, independent of Y_1, Y_2, Y_3. Then, the following statements are equivalent:*
(i) $Y \stackrel{d}{=} IY_1^{\sqrt{p}} + (1 - I)Y_2Y_3^{\sqrt{p}}$ for all $p \in [0, 1]$.
(ii) $Y \stackrel{d}{=} Y_1^{\sqrt{p}}Y_2^{1-I}$ for all $p \in [0, 1]$.
(iii) Y has the log-Laplace distribution $\mathcal{LL}(1, \alpha, \alpha)$ for some $\alpha > 0$.

Finally we provide one more characterization of "symmetric" log-Laplace distributions through products of i.i.d. variables, which is a direct consequence of similar property (with products replaced by sums) of symmetric Laplace distributions (see Kotz et al. (2001), Proposition 2.2.11).

Proposition 4.7. *Let $Y, Y_1, Y_2, ...$ be positive, non-degenerate i.i.d. random variables with $E[\log Y]^2 < \infty$, and let B_n have a beta $\mathcal{B}(n - 1)$ distribution (2), independent of the Y_i's. Then, the following statements are equivalent:*
(i) For all integers $n > 1$,

$$\left\{\prod_{i=1}^{n} Y_i\right\}^{\sqrt{1-B_n}} \stackrel{d}{=} Y. \tag{36}$$

(ii) Y possesses the $\mathcal{LL}(1, \alpha, \alpha)$ distribution with some $\alpha > 0$.

Note that Pareto distributions $\mathcal{P}(\alpha)$ admit similar characterization as the distributions with finite $E\log^2 Y$ supported of the interval $(1, \infty)$ and such that

$$\left\{\prod_{i=1}^{n} Y_i\right\}^{(1-B_n)} \stackrel{d}{=} Y. \tag{37}$$

The same relation characterizes power laws (2) within the class of distribution supported on $(0, 1)$ (with the same moment restrictions). These facts are derived by taking the logarithm of both sides of (37) leading to a relation that characterizes the exponential distribution (see, e.g., Kotz and Steutel (1988)).

4.4. **A characterization through the hazard function.** The *hazard function* (hazard rate) $h(x) = g(x)/(1 - G(x))$, where g and G are the p.d.f. and the c.d.f., respectively, plays a fundamental role in survival analysis. This function is also known as the *conditional failure rate* in reliability, the *force of mortality* in actuarial science and demography, the *intensity function* in stochastic processes, and the *inverse of the Mill's ratio* in economics. The hazard function of the LL distribution is of the Pareto type for large x and for $\delta = 1$ is given by

$$h(x) = \alpha \begin{cases} \frac{\beta x^{\beta-1}}{\alpha+\beta-\alpha x^{\beta}} & \text{for } 0 < x < 1 \\ \frac{1}{x} & \text{for } x \geq 1. \end{cases} \tag{38}$$

Note that by Proposition 2.1, if $X \sim \mathcal{LL}(1, \alpha, \beta)$, then the hazard function of the variable $1/X$ is of the form β/x for $x > 1$. In fact we have the following characterization of the log-Laplace distribution.

Proposition 4.8. *Let X be a continuous random variable with support on $(0, \infty)$ whose density is continuous at $x = 1$. Then the hazard functions of X and $1/X$ for $x > 1$ are of the form α/x and β/x, respectively (where $\alpha, \beta > 0$) if and only if X has the $\mathcal{LL}(1, \alpha, \beta)$ distribution.*

Proof. Let F and f be the c.d.f. and the p.d.f. of X, respectively. Since

$$h(x) = \frac{f(x)}{1 - F(x)} = -\frac{d}{dx}\log(1 - F(x)) = \frac{\alpha}{x}, \quad x > 1, \tag{39}$$

we immediately obtain

$$1 - F(x) = Cx^{-\alpha}, \quad x > 1, \tag{40}$$

where $C = 1 - F(1)$. Similarly, we have

$$h(x) = \frac{g(y)}{1 - G(y)} = -\frac{d}{dy}\log(1 - G(y)) = \frac{\beta}{y}, \quad y > 1, \tag{41}$$

where G and g are the c.d.f. and the p.d.f. of $Y = 1/X$. Noting that $G(y) = 1 - F(y)$ and $g(y) = f(y)/y^2$, we obtain

$$\frac{f(x)}{F(x)} = \frac{\beta}{x}, \quad x > 1, \tag{42}$$

leading to

$$F(x) = Dx^{\beta}, \quad 0 < x < 1, \tag{43}$$

where $D = F(1)$. Since obviously $C + D = 1$ and also $C\alpha = D\beta$ by the continuity of the density at $x = 1$, we obtain

$$C = \frac{\beta}{\alpha + \beta}, \quad D = \frac{\alpha}{\alpha + \beta}, \tag{44}$$

which proves the assertion. $\square$

Remark 4.5. This characterization was essentially proved by Uppuluri (1981), although it was stated differently in terms of the properties of the dose response curve for radiation carcinogenesis. Let $F(x)$ and $G(x) = 1 - F(x)$ be the cumulative proportion of deaths and survivals, respectively, at the dose level x. Then, discretizing (39) we obtain

$$\frac{G(x + \Delta x) - G(x)}{G(x)} \approx -\alpha\frac{\Delta x}{x}, \quad x > 1, \tag{45}$$

so that, as stated by Uppuluri (1981), "at large doses ($x > 1$) the percent increase in the cumulative proportion of survivors (the left-hand-side of (45)) is proportional to the percent decrease in the dose". Similarly, discretizing (42) we have

$$\frac{F(x+\Delta x) - F(x)}{F(x)} \approx \beta \frac{\Delta x}{x}, \quad 0 < x < 1, \tag{46}$$

or as stated by Uppuluri (1981), "at small doses, the percent increase in the cumulative proportion of deaths is proportional to the percent increase in the dose". This characteristic and interpretation of the LL distribution function may have applications in other areas where life-time distributions are of interest (i.e., actuarial science).

4.5. **Multiplicative memoryless property.** The survival function of the classical Pareto $\mathcal{P}(\alpha)$ variable X satisfies the equation

$$P(X > x_1 x_2) = P(X > x_1)P(X > x_2), \quad x_1, x_2 > 1. \tag{47}$$

This property, which actually characterizes the Pareto distribution (see, e.g., Arnold (1983)), is essentially a multiplicative memoryless property,

$$P(X > x_1 x_2 | X > x_1) = P(X > x_2), \quad x_1, x_2 > 1, \tag{48}$$

and follows from the relation between Pareto and exponential distributions (see, e.g., Galambos and Kotz (1978)). Since a log-Laplace variable can be expressed in terms of Pareto variables (see Section 3.4), it is not surprising that this distribution has a similar property.

Proposition 4.9. *Let Y be a continuous random variable with support on $(0, \infty)$ whose density is continuous at $y = 1$. Then, we have*

$$P(Y > y_1 y_2 | Y > y_1) = P(Y > y_2 | Y > 1), \quad y_1, y_2 > 1, \tag{49}$$

and

$$P(Y < y_1 y_2 | Y < y_1) = P(Y < y_2 | Y < 1), \quad 0 < y_1, y_2 < 1, \tag{50}$$

if and only if Y has the $\mathcal{LL}(1, \alpha, \beta)$ distribution for some $\alpha, \beta > 0$.

Proof. The fact that log-Laplace distribution $\mathcal{LL}(1, \alpha, \beta)$ satisfies the two equations can be verified directly using the explicit form of the distribution function. Conversely, assume that (49) and (50) hold. Note that since $y_1, y_2 > 1$, the first relation can be written as

$$P(Y > y_1 y_2 | Y > 1) = P(Y > y_1 | Y > 1)P(Y > y_2 | Y > 1), \quad y_1, y_2 > 1. \tag{51}$$

But in view of our discussion above, this shows that the conditional distribution of Y given $Y > 1$ is Pareto $\mathcal{P}(\alpha)$ with some $\alpha > 0$. Similarly, since the second relation can be written as

$$P(X > x_1 x_2 | X > x_1) = P(X > x_2 | X > 1), \quad x_1, x_2 > 1, \tag{52}$$

where $X = 1/Y$ and $x_i = 1/y_i$, $i = 1, 2$, we conclude that the conditional distribution of $1/Y$ given $Y < 1$ is again Pareto $\mathcal{P}(\beta)$ with some $\beta > 0$.

Consequently, Y admits the representation (19) (with $\delta = 1$), where B and P are independent $\mathcal{B}(\beta)$ and $\mathcal{P}(\alpha)$ variables, and I is a Bernoulli r.v., independent of B and P, with $P(I = 1) = p$ for some $p \in (0, 1)$. Finally, since the density of Y is continuous at 1, we conclude that p must be as in (17), so that Y has a log-Laplace distribution. $\square$

4.6. **Maximum entropy property.** An important functional of the distribution of a r.v. Y is the *entropy*, defined by

$$H(Y) = E[-\log f(Y)], \tag{53}$$

where f is a density of Y. This basic concept of information theory is a measure of uncertainty associated with the distribution of Y. According to the maximum entropy principle, of all distributions that satisfy certain constrains (dictated by a particular application), one should select one with maximum entropy. This inferential procedure has been used in a variety of fields, including statistical mechanics, statistics, queuing theory, stock market, analysis, image analysis, and reliability (see, e.g., Kapur (1993)).

Proposition 4.10. *The entropy of* $Y \stackrel{d}{=} \mathcal{LL}(\delta, \alpha, \beta)$ *is given by*

$$H(Y) = 1 + \log\delta + \frac{1}{\alpha} - \frac{1}{\beta} + \log\left(\frac{1}{\alpha} + \frac{1}{\beta}\right). \tag{54}$$

Proof. First note that for an arbitrary real random variable X with a density f, the variable $Y = e^X$ has the entropy

$$H(Y) = H(X) + EX.$$

Indeed, we have

$$\begin{aligned} H(Y) &= -\int \log\left\{\frac{1}{y} f(\log y)\right\} \frac{1}{y} f(\log y) dy \\ &= -\int \log\left\{\frac{1}{e^x} f(x)\right\} f(x) dx = \int x f(x) dx - \int f(x) \log f(x) dx. \end{aligned}$$

Recall that for the corresponding asymmetric Laplace variable X with the density (4), we have

$$H(X) = 1 + \log\left(\frac{1}{\alpha} + \frac{1}{\beta}\right),$$

and

$$EX = \log\delta + \frac{1}{\alpha} - \frac{1}{\beta},$$

see Kotz et al. (2001). This leads to (54). $\square$

It is well known that among all continuous distributions on $(1, \infty)$ having a fixed value of the geometric mean, the entropy is maximized by the Pareto distribution (see Ord et al. (1981)). As we show below, similar property is shared by the log-Laplace distribution. This is also related to the fact that the skew Laplace distribution provides the largest entropy among all continuous distributions supported on R with given the first moment and the first absolute moment (see Kotz et al. (2002)).

Proposition 4.11. *Let $\mathcal{D}$ be the class of all continuous random variables Y with non-vanishing densities on $(0,\infty)$ and with fixed geometric means of Y and $\max(Y,1/Y)$, so that*

$$e^{E\log Y} = e^{c_1} \in (0,\infty) \ \textit{and} \ e^{E\log\max(Y,1/Y)} = e^{c_2} > 1 \ \textit{for} \ Y \in \mathcal{D}, \tag{55}$$

where $|c_1| < c_2$. Then, the maximum entropy is attained by the log-Laplace r.v. Y^* with density (3), where $\delta = 1$,

$$\alpha = \frac{2}{\sqrt{c_2+c_1}(\sqrt{c_2+c_1}+\sqrt{c_2-c_1})}, \tag{56}$$

and

$$\beta = \frac{2}{\sqrt{c_2-c_1}(\sqrt{c_2+c_1}+\sqrt{c_2-c_1})}. \tag{57}$$

Moreover, the maximum entropy is

$$\max_{Y\in\mathcal{D}} H(Y) = H(Y^*) = 1 + c_1 + 2\log\frac{\sqrt{c_2+c_1}+\sqrt{c_2-c_1}}{\sqrt{2}}. \tag{58}$$

Proof. Consider the class $\mathcal{C}$ of all random variables X on $\mathbb{R}$ such that

$$EX = c_1 \ \text{and} \ E|X| = c_2.$$

Clearly, for any $X \in \mathcal{C}$ we have $Y = e^X \in \mathcal{D}$, and for $Y \in \mathcal{D}$ we have $X = \log Y \in \mathcal{C}$ (note that $\log\max(Y,1/Y) = |\log Y|$). Since the value $EX = c_1$ is fixed and $H(Y) = H(X) + E(X)$ (see the proof of Proposition 4.10), X maximizes the entropy within the class $\mathcal{C}$ if and only if $Y = e^X$ does within the class $\mathcal{D}$. By Kotz et al. (2002), X has the skew Laplace distribution with density (4) where $\theta = 0$ and α and β are given by (56) - (57). Consequently, the corresponding Y will have the $\mathcal{LL}(\delta,\alpha,\beta)$ distribution as specified in the statement of the proposition. The value of the maximum entropy (58) is obtained by applying Proposition 4.10 with the above values of the parameters. $\square$

Remark 4.6. Note that if $c_1 = 0$ then $\alpha = \beta = 1/c_2$ and the entropy is maximized by the symmetric log-Laplace distribution. In this case the maximal entropy is $1 + \log(2c_2)$. The same holds when we delete the first condition in (55) (cf. Kotz et al. (2001), Proposition 2.4.7). On the other hand, if we drop the second condition in (55), then the maximal entropy is provided by the lognormal distribution (and by an exponential distribution under the condition $EY = c$). Finally, if we only require that $E\log Y = c_1$ and consider distributions supported on (c,∞) for some $c > 0$, then the maximal entropy distribution is Pareto (see Ord et al. (1981)).

5. Maximum Likelihood Estimation

In this section we discuss maximum likelihood estimators (MLE's) of the log-Laplace parameters δ, α, and β. The results below can be derived from those for the asymmetric Laplace distributions (see Section 3.5 in Kotz et al. (2001) for details).

5.1. **Information matrices.** The Fisher information matrix for the $\mathcal{LL}(\delta, \alpha, \beta)$ random variable is straightforward to compute, and is given by (cf. Hinkley and Revankar (1977)),

$$I(\delta,\alpha,\beta) = \begin{bmatrix} \frac{\alpha\beta}{\delta^2} & -\frac{1}{\delta}\frac{\beta}{\alpha+\beta} & \frac{1}{\delta}\frac{\alpha}{\alpha+\beta} \\ -\frac{1}{\delta}\frac{\beta}{\alpha+\beta} & \frac{1}{\alpha^2} - \frac{1}{(\alpha+\beta)^2} & -\frac{1}{(\alpha+\beta)^2} \\ \frac{1}{\delta}\frac{\alpha}{\alpha+\beta} & -\frac{1}{(\alpha+\beta)^2} & \frac{1}{\beta^2} - \frac{1}{(\alpha+\beta)^2} \end{bmatrix}.$$

5.2. **The case of unknown δ.** Consider first the case when only δ is unknown. Since $\delta = e^\theta$ and the MLE of θ is known in terms of $X_i = \log Y_i$, we easily obtain the estimator

$$\hat{\delta}_n = \begin{cases} Y_{1:n} & \text{if } \frac{\alpha}{\beta} < \frac{1}{n-1}, \\ Y_{j:n} & \text{if } \frac{j-1}{n-(j-1)} \le \frac{\alpha}{\beta} < \frac{j}{n-j}, \ j = 2, 3, \ldots, n-1, \\ Y_{n:n} & \text{if } \frac{\alpha}{\beta} \ge n-1. \end{cases}$$

Moreover this estimator is asymptotically normal and efficient.

5.3. **The case of unknown α and β.** Another important practical case is when δ is known. Here, the MLE's take the following explicit form:

$$\widehat{\alpha} = \frac{n}{\log\left(\prod_{i=1}^n \left(\frac{Y_i}{\delta} \vee 1\right)\right) + \sqrt{\log\left(\prod_{i=1}^n \left(\frac{Y_i}{\delta} \vee 1\right)\right) \log\left(\prod_{i=1}^n \left(\frac{\delta}{Y_i} \vee 1\right)\right)}},$$

$$\widehat{\beta} = \frac{n}{\log\left(\prod_{i=1}^n \left(\frac{\delta}{Y_i} \vee 1\right)\right) + \sqrt{\log\left(\prod_{i=1}^n \left(\frac{Y_i}{\delta} \vee 1\right)\right) \log\left(\prod_{i=1}^n \left(\frac{\delta}{Y_i} \vee 1\right)\right)}}.$$

Since the estimators in the Laplace case are asymptotically normal and efficient, these properties are shared by the estimators in the log-Laplace case. The asymptotic covariance matrix can be easily computed as the inverse of the corresponding information matrix (see Hartley and Revankar (1974)),

$$\Sigma_{\alpha,\beta} = \frac{\alpha\beta}{2}\begin{bmatrix} \left(\frac{\alpha}{\beta}+1\right)^2 - 1 & 1 \\ 1 & \left(\frac{\beta}{\alpha}+1\right)^2 - 1 \end{bmatrix}.$$

5.4. **The case where all parameters are unknown.** When all parameters are unknown, the estimators can be obtained as follows (cf. Hartley and Revankar (1974), Hinkley and Revankar (1977)). Let $a(\delta)$ and $b(\delta)$ be the right-hand-sides of the above expressions for $\widehat{\alpha}$ and $\widehat{\beta}$, respectively. Then the MLE of δ is the order statistic $\hat{\delta} = Y_{k:n}$ that minimizes the function

$$h(\delta) = \log(a(\delta) + b(\delta)) + \frac{a(\delta)b(\delta)}{a(\delta) + b(\delta)},$$

and the other two MLE's are

$$\widehat{\alpha} = a(\hat{\delta}), \quad \widehat{\beta} = b(\hat{\delta}).$$

These three estimators are jointly asymptotically normal and efficient with the covariance matrix obtained by inverting the information matrix, see Hinkley and Revankar (1977).

Remark 5.1. The maximum likelihood estimators of the unknown parameters exist and are asymptotically normal and efficient for any situation when some of the parameters are known. We presented only the cases which are likely to be of use in practical problems.

6. **Summary**

Most of the past applications of the log-Laplace models stemmed from the need to capture the observed double power tail behavior of the data. However, we believe and argue in this paper, that it is not only the power tail, but also other important properties of the log-Laplace distributions that give them their modeling advantages and explain their reported excellent fit to the data. In conclusion, log-Laplace laws with their stability properties should have a promising future in stochastic modeling.

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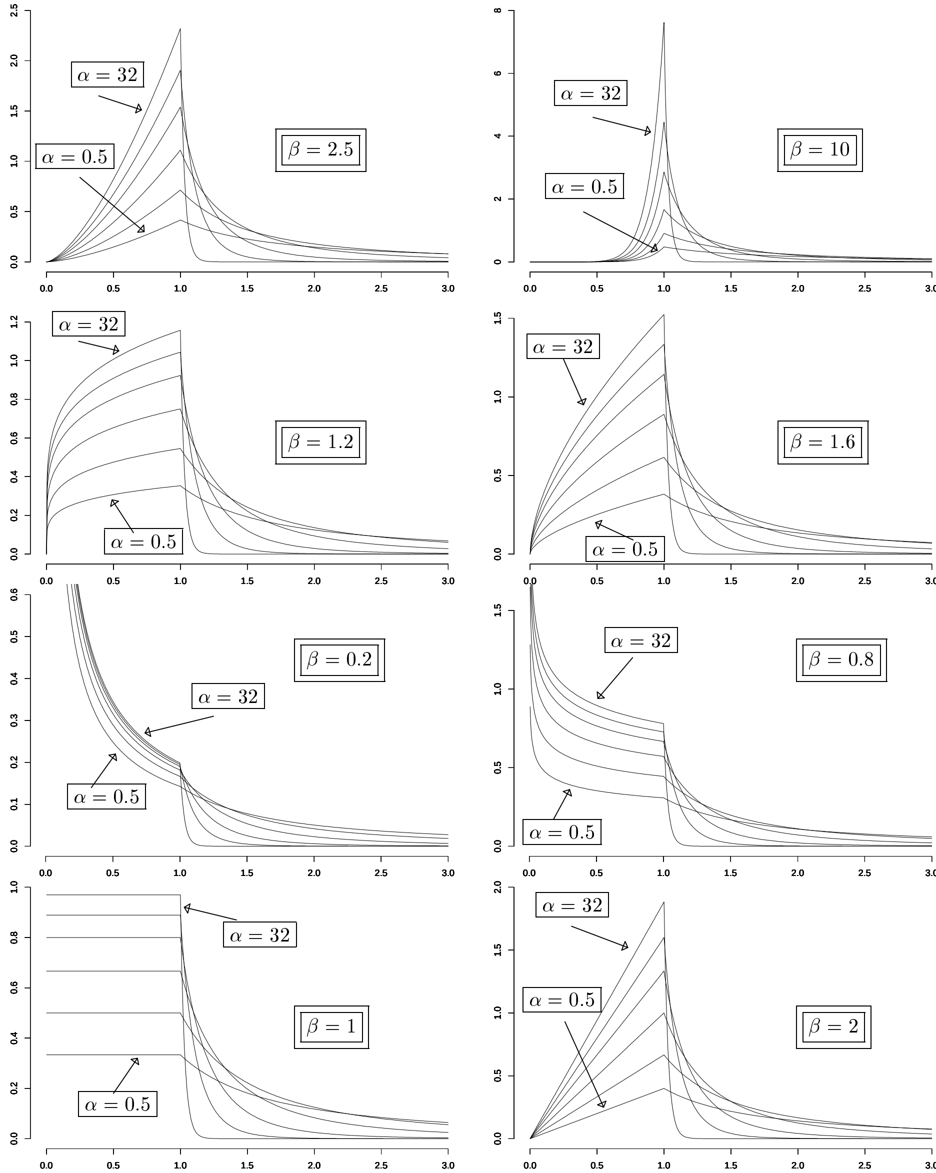

Figure 1. Densities of log-Laplace distributions with the scale parameter $\delta = 1$ and $\alpha = 0.5, 1, 2, 4, 8, 32$. *Top row:* Case $\beta > 2$ – the density is increasing and concave up on $(0, 1)$ and is equal to zero at the origin with the unique mode at $x = 1$; *Second row:* Case $1 < \beta < 2$ – the ensity is increasing and concave down on $(0, 1)$ and is equal to zero at the origin with the unique mode at $x = 1$; *Third row:* Case $0 < \beta < 1$ – the density is decreasing and concave up on $(0, 1)$ and converging to infinity at the origin; *Bottom row:* Case $\beta = 1$ – the density is constant equal to $\frac{\alpha}{\alpha+1}$ on $(0, 1)$ and case $\beta = 2$ – the density is increasing and linear on $(0, 1)$ and is equal to zero at the origin.