Special Relativity - Summary 2

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This summary is based on the book chapters 2.4 - 2.7 and 3.1 - 3.5 of Robert Resnick: Introduction to Special Relativity

1 Relativistic kinematics

1.1 A physical look at the main features of the Lorentz transformation

The main **features of the Lorentz transformation**:

1. Lengths perpendicular to the motion are the same in both frames
2. The time intervals indicated by a clock are longer if the clock is moving with respect to the observer, e.g. moving clocks run slower.This is also known as time dilation. Factor of $\sqrt{1 - v^2/c^2}$
3. Lengths parallel to the relative motion are contracted compared to the rest length. $L = L'\sqrt{1 - v^2/c^2}$ (Length contraction is a necessary consequence of time dilation)
4. Two clocks which are synchronised and separated in one inertial frame are observed to be out of sync in another inertial frame. This phase difference comes from the $\frac{v}{c^2}x'$ in the time transformation.

Everyday **examples for special relativity**: electromagnetism, GPS - the calculations need to take relativity into account for accurate location determination. Another example is high energy (particle) physics, where particles move with relativistic speeds. Particle physics is the combination of quantum mechanics and special relativity.

What is an observer in SR? - an observer is effectively a measurement at a certain point in space and time.

1.2 Relativistic addition of velocities

Relativistic or Einstein velocity addition:

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$

Here v is the velocity of the frame and u is the velocity of an object.
- If u and v are smaller than c $\rightarrow$ classical velocity addition $u = u' + v$
- If $u' = c \rightarrow u = c$ regardless of v (Light is moving)
- The velocity of light is always the same, no mater of the light source is moving. This is in agreement with experiments.
- Gives back one of the postulates $\rightarrow$ speed of light is always c in all inertial systems.
- The addition of 2 velocities $< c$ can not exceed the speed of light.
- It is possible to observe faster speed than c, however these are always geometric effects. e.g. superluminar motion in astrophysics, where a jet ejected from an AGN (active galactic nucleus, which is an accreting supermassive black hole in the centre of a galaxy) seems to be moving faster than the speed of light. This is a geometric effect of the projection. In 3D space the jet moves slower than c.

General velocity addition:

$$u'_x = \frac{u_x - v}{1 - u_x v/c^2} \qquad u_x = \frac{u'_x + v}{1 + u'_x v/c^2}$$

$$u'_y = \frac{u_y\sqrt{1 - v^2/c^2}}{1 - u_x v/c^2} \qquad u_y = \frac{u'_y\sqrt{1 - v^2/c^2}}{1 + u'_x v/c^2}$$

$$u'_z = \frac{u_z\sqrt{1 - v^2/c^2}}{1 - u_x v/c^2} \qquad u_z = \frac{u'_z\sqrt{1 - v^2/c^2}}{1 + u'_x v/c^2}$$

Transverse velocity components (u_y, u_z) of an object in the S frame are related to both the transverse component (u'_y, u'_z) and to the parallel component (u'_x) of the velocity of the object in the S' frame. $\rightarrow$ The result is not simple because neither observer is proper.

If we choose a frame where $u'_x = 0 \rightarrow u_y = u'_y\sqrt{1-v^2/c^2}$ and $u_z = u'_z\sqrt{1-v^2/c^2}$
- No length contraction is involved in the transverse direction
- Time dilation is responsible for the $\sqrt{1-v^2/c^2}$ factor

1.3 Relativistic acceleration transformation

$$a'_x = a_x\frac{(1-v^2/c^2)^{3/2}}{(1-u_x v/c^2)^3}$$

Similar formula for a_y and a_z.
- Acceleration depends on the inertial frame in which it is measured
- When u and v $<< c$ we get back the classical result
- In SR frames do not accelerate, but objects in the frame can accelerate

1.4 Relativistic aberration of light

Relativistic equation for aberration of light:

$$tan\theta = \frac{sin\theta'\sqrt{1-\beta^2}}{cos\theta'+\beta}$$

Relates the propagation angle θ, θ' as seen from two inertial frames.
The inverse transformation is:

$$tan\theta' = \frac{sin\theta\sqrt{1-\beta^2}}{cos\theta-\beta}$$

1.5 Relativistic equation for the Doppler effect

$$\nu = \frac{\nu'(1+\beta cos\theta')}{\sqrt{1-\beta^2}}$$

The inverse transformation is:

$$\nu' = \frac{\nu(1-\beta cos\theta')}{\sqrt{1-\beta^2}}$$

Classical case:
If $v << c$ we get the classical formula: $\nu = \nu'(1+\beta cos\theta)$
- If $\theta = 0° \rightarrow$ source is moving towards the observer: $\nu = \nu'(1+\beta) \rightarrow$ frequency is grater than the proper frequency (ν') ("blueshift")
- If $\theta = 180° \rightarrow$ source is moving away from the observer: $\nu = \nu'(1-\beta) \rightarrow$ frequency is less than the proper frequency ("redshift")
- If $\theta = 90° \rightarrow$ source is moving perpendicular compared to the observer: no Doppler effect.

SR:
If v is not small, we get relativistic or second order effects.
- **Longitudinal Doppler effect:**
- If $\theta = 0° \rightarrow$ source is moving towards the observer: $\nu = \nu'\sqrt{\frac{c+v}{c-v}} \rightarrow$ frequency is grater than the proper frequency (ν') ("blueshift")
- If $\theta = 180° \rightarrow$ source is moving away from the observer: $\nu = \nu'\sqrt{\frac{c-v}{c+v}} \rightarrow$ frequency is less than the proper frequency ("redshift")
- **Transverse Doppler effect:**
- If $\theta = 90° \rightarrow \nu = \nu'\sqrt{1-\beta^2} \rightarrow$ Purely relativistic effect. $\rightarrow$ We observe a lower frequency compared to the proper frequency. ("redshift")
- The transverse Doppler effect can be interpreted as a time dilation effect. $\rightarrow$ Experiments confirm relativistic time dilation.

2 Relativistic dynamics

- SR is in contrast with some of the classical laws of physics
- In the classical case we can accelerate a body to any speed $\Leftrightarrow$ SR the maximum speed is c

2.1 Relativistic momentum

Redefine momentum so that it is invariant under Lorentz transformation. The classical definition of the momentum: $\bar{p} = m\bar{u}$ is not invariant under Lorentz transformation. $\rightarrow$ We need to update the definition: using **relativistic mass**: $m = m_0/\sqrt{1 - u^2/c^2}$.

Relativistic momentum:

$$\bar{p} = m\bar{u} = \frac{m_0\bar{u}}{\sqrt{1 - u^2/c^2}}$$

- In the classical case mass is invariant. In SR mass is not invariant (note: the rest mass is invariant m_0).
- When u = 0 $\rightarrow$ the body is at rest $m = m_0$
- The mass (m) depends on the velocity, but it is independent of the direction of the motion
- m_0 rest mass is also called proper mass.

General momentum:

When there are x, y, z, components $u^2 = u_x^2 + u_y^2 + u_z^2$

$$p_x = \frac{m_0 u_x}{\sqrt{1 - u^2/c^2}}$$

$$p_y = \frac{m_0 u_y}{\sqrt{1 - u^2/c^2}}$$

$$p_z = \frac{m_0 u_z}{\sqrt{1 - u^2/c^2}}$$

Note: u^2/c^2 has the full magnitude of u and not just the components.

2.2 Alternative views on mass in relativity

The relativistic momentum and mass can be interpreted in two different ways.

classical momentum:

$$p_x = m_0 \frac{dx}{dt}$$

To make the momentum relativistic we can either treat the mass as relativistic and bringing in a factor of $\frac{1}{\sqrt{1-u^2/c^2}}$ or we treat the velocity component relativisticly and the $\frac{1}{\sqrt{1-u^2/c^2}}$ factor is attributed to time dilation. Both interpretations are correct and can be used for different problems.

As $u \rightarrow c$ $m \rightarrow \infty$ for the relativistic mass. The inertia of a body increases with velocity $\rightarrow$ c can not be reached for objects with mass.

2.3 Relativistic force and dynamics of a single body

In relativistic mechanics Newton's 2nd law generalises:

$$\bar{F} = \frac{d}{dt}(\bar{p}) = \frac{d}{dt}\left(\frac{m_0\bar{u}}{\sqrt{1 - u^2/c^2}}\right)$$

- If there is no external force $\rightarrow$ the momentum is conserved.
- If there is an external force $\rightarrow$ the momentum change is equal to the total impulse given to the system by the external force.
- Newtonian mechanics: The kinetic energy is defined to be equal to the work done by an external force in increasing the speed from 0 to some value.

Relativistic kinetic energy:

$$K = m_0c^2\left[\frac{1}{\sqrt{1-u^2/c^2}} - 1\right]$$

The **total energy** of the particle: $E = m_0c^2 + K$ is the sum of the rest energy and the kinetic energy.
Rest energy: m_0c^2 is the energy of the particle at rest ($u = 0$)

- Total energy is often used in high energy and particle physics.
- if $v << c$ we get the classical expression: $K = 1/2m_0c^2$
- if $u \to c$ $K \to \infty$ an infinite amount of work would be needed to accelerate a particle to c
- $K = (m - m_0)c^2$ a change in kinetic energy is related to change in the relativistic mass
- There is a connection between K and the momentum: $E = c\sqrt{p^2 + m_0^2c^2}$
- Another often used version: $\frac{dE}{dp} = \frac{pc^2}{E} = u$

2.3.1 Acceleration of a particle under a single force

$$\bar{F} = m\frac{d\bar{u}}{dt} + \frac{\bar{u}(\bar{F}\cdot\bar{u})}{c^2}$$

$$\bar{a} = \frac{d\bar{u}}{dt} = \frac{\bar{F}}{m} - \frac{\bar{u}}{mc^2}(\bar{F}\cdot\bar{u})$$

The acceleration by a single force is not necessarily parallel to the force, because the last term is parallel to $\bar{u}$.

There are two simple cases where $\bar{a}$ is parallel to $\bar{F}$:
1. If $\bar{F}$ is parallel to $\bar{u}$:

$$F_{\parallel} = \frac{m_0}{(1-u^2/c^2)^{3/2}}a_{\parallel}$$

longitudinal mass:

$$\frac{m_0}{(1-u^2/c^2)^{3/2}}$$

2. If $\bar{F}$ is perpendicular to $\bar{u}$:

$$F_{\perp} = \frac{m_0}{\sqrt{1-u^2/c^2}}a_{\perp}$$

transverse mass:

$$\frac{m_0}{\sqrt{1-u^2/c^2}}$$

example: force on a charged particle moving in a magnetic field $\bar{B}(\bar{F} = q\bar{u}\times\bar{B})$

Experimental support for the relativistic mass: An e^- from a β decay ($n^0 \to p^+ + e^- + \bar{\nu}_e$) enters a velocity selector, which determines the speed of the e^- that can emerge from the chamber. Then the particles with the same speed enter a magnetic field, where they are going to move on a curved path that is dependent on their momentum (mass). The radius of the path can be measured $\to$ relativistic mass can get calculated.