Special Relativity - Summary 2 - Relativistic dynamics

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This summary is based on the book Robert Resnick: Introduction to Special Relativity, chapters 3.1 - 3.7

- SR is in contrast with some of the classical laws of physics
- In the classical case we can accelerate a body to any speed $\Leftrightarrow$ SR the maximum speed is c

1 Relativistic momentum

Redefine momentum so that it is invariant under Lorentz transformation. The classical definition of the momentum: $\bar{p} = m\bar{u}$ is not invariant under Lorentz transformation. $\rightarrow$ We need to update the definition: using **relativistic mass**: $m = m_0/\sqrt{1 - u^2/c^2}$.

Relativistic momentum:

$$\bar{p} = m\bar{u} = \frac{m_0\bar{u}}{\sqrt{1 - u^2/c^2}}$$

- In the classical case mass is invariant.
- In SR mass is not invariant (note: the rest mass is invariant m_0).
- When u = 0 $\rightarrow$ the body is at rest $m = m_0$
- m_0 rest mass is also called proper mass.
- The mass (m) depends on the velocity, but it is independent of the direction of the motion

General momentum:

When there are x, y, z, components $u^2 = u_x^2 + u_y^2 + u_z^2$

$$p_x = \frac{m_0 u_x}{\sqrt{1 - u^2/c^2}}$$

$$p_y = \frac{m_0 u_y}{\sqrt{1 - u^2/c^2}}$$

$$p_z = \frac{m_0 u_z}{\sqrt{1 - u^2/c^2}}$$

Note: u^2/c^2 has the full magnitude of u and not just the components.

1.1 Alternative views on mass in relativity

The relativistic momentum and mass can be interpreted in two different ways.

classical momentum:

$$p_x = m_0 \frac{dx}{dt}$$

To make the momentum relativistic we can either treat the mass as relativistic and bringing in a factor of $\frac{1}{\sqrt{1-u^2/c^2}}$ or we treat the velocity component relativisticly and the $\frac{1}{\sqrt{1-u^2/c^2}}$ factor is attributed to time dilation. Both interpretations are correct and can be used for different problems.

As $u \rightarrow c$ $m \rightarrow \infty$ for the relativistic mass. The inertia of a body increases with velocity $\rightarrow$ c can not be reached for objects with mass.

2 Relativistic force and dynamics of a single body

In relativistic mechanics Newton's 2nd law generalises:

$$\bar{F} = \frac{d}{dt}(\bar{p}) = \frac{d}{dt}\left(\frac{m_0\bar{u}}{\sqrt{1 - u^2/c^2}}\right)$$

- If there is no external force $\rightarrow$ the momentum is conserved.

- If there is an external force → the momentum change is equal to the total impulse given to the system by the external force.
- Newtonian mechanics: The kinetic energy is defined to be equal to the work done by an external force in increasing the speed from 0 to some value.

Relativistic kinetic energy:

$$K = m_0c^2 \left[\frac{1}{\sqrt{1-u^2/c^2}} - 1 \right]$$

The **total energy** of the particle: $E = m_0c^2 + K$ is the sum of the rest energy and the kinetic energy.
Rest energy: m_0c^2 is the energy of the particle at rest ($u = 0$)

- Total energy is often used in high energy and particle physics.
- if $v << c$ we get the classical expression: $K = 1/2m_0c^2$
- if $u \to c$ $K \to \infty$ an infinite amount of work would be needed to accelerate a particle to c
- $K = (m - m_0)c^2$ a change in kinetic energy is related to change in the relativistic mass
- There is a connection between K and the momentum: $E = c\sqrt{p^2 + m_0^2c^2}$
- Another often used version: $\frac{dE}{dp} = \frac{pc^2}{E} = u$

2.1 Acceleration of a particle under a single force

$$\bar{F} = m\frac{d\bar{u}}{dt} + \frac{\bar{u}(\bar{F}\cdot\bar{u})}{c^2}$$

$$\bar{a} = \frac{d\bar{u}}{dt} = \frac{\bar{F}}{m} - \frac{\bar{u}}{mc^2}(\bar{F}\cdot\bar{u})$$

The acceleration by a single force is not necessarily parallel to the force, because the last term is parallel to $\bar{u}$.

There are two simple cases where $\bar{a}$ is parallel to $\bar{F}$:
1. If $\bar{F}$ is parallel to $\bar{u}$:

$$F_{\parallel} = \frac{m_0}{(1-u^2/c^2)^{3/2}}a_{\parallel}$$

longitudinal mass:

$$\frac{m_0}{(1-u^2/c^2)^{3/2}}$$

2. If $\bar{F}$ is perpendicular to $\bar{u}$:

$$F_{\perp} = \frac{m_0}{\sqrt{1-u^2/c^2}}a_{\perp}$$

transverse mass:

$$\frac{m_0}{\sqrt{1-u^2/c^2}}$$

example: force on a charged particle moving in a magnetic field $\bar{B}(\bar{F} = q\bar{u}\times\bar{B})$

Experimental support for the relativistic mass: An e^- from a β decay ($n^0 \to p^+ + e^- + \bar{\nu}_e$) enters a velocity selector, which determines the speed of the e^- that can emerge from the chamber. Then the particles with the same speed enter a magnetic field, where they are going to move on a curved path that is dependent on their momentum (mass). The radius of the path can be measured → relativistic mass can get calculated.

3 The equivalence of mass and energy

The rest mass of a body is equivalent to energy → **rest-mass energy**.
- This follows from the Lorentz transformation and the momentum conservation.
- The total energy is conserved in inelastic collisions
- The conservation of total energy is equivalent to conservation of rest mass

- Mass and energy are equivalent $\rightarrow$ they form **a single invariant: mass-energy**
- relation: $E = mc^2$ or $m = E/c^2$ $\rightarrow$ we can convert one into the other
- The conversion of mass into energy is often used in nuclear physics: e.g. eV or MeV units for the mass of particles
- Or a mass less particle (e.g. photon) can get a "mass" assigned to it based on its energy
- We can think of the kinetic energy as an **external energy** and of the rest-mass energy as an internal energy.
- **Internal energy** is to the largest part mad up of the total rest-mass energy of the constituent fundamental particles, but it can also include:
  - molecular motion,
  - intermolecular potential energy,
  - atomic potential energy,
  - nuclear potential energy
- The rest mass or (proper mass) of a body is generally not constant.
- We can assign a rest mass to any collection of particles in motion in a frame at which the centre of mass is at rest.
- In classical physics we have 2 conservation laws: energy and mass conservation $\rightarrow$ in SR we have 1: mass-energy conservation.

4 Lorentz transformation of momentum, energy, mass and force

4.1 Momentum and energy

$$p_x = \frac{1}{\sqrt{1 - v^2/c^2}}\left(p'_x + \frac{E'v}{c^2}\right) \qquad p'_x = \frac{1}{\sqrt{1 - v^2/c^2}}\left(p_x - \frac{Ev}{c^2}\right)$$

$$p_y = p'_y$$

$$p_z = p'_z$$

$$E = \frac{1}{\sqrt{1 - v^2/c^2}}\left(E' + vp_x\right) \qquad E' = \frac{1}{\sqrt{1 - v^2/c^2}}\left(E - vp_x\right)$$

Very similar to the Lorentz transformation of x, y, z, t:

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \qquad p'_x = \frac{p_x - v(E/c^2)}{\sqrt{1 - v^2/c^2}}$$

$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}} \qquad \frac{E'}{c^2} = \frac{(E/c^2) - (v/c^2)p_x}{\sqrt{1 - v^2/c^2}}$$

- When relativity is put into 4 dimensional space-time form a four-vector can be created out of the space and time coordinates
- Similarly a four-vector can be created out of the three momentum components and the energy
- If energy and momentum are conserved in one interaction in an inertial frame $\rightarrow$ then they are also conserved for the interaction in another inertial frame
- If momentum is conserved $\rightarrow$ energy must be conserved as well.

4.2 Mass

The transformation of mass follows directly from the transformation of energy:

$$m = \frac{m'(1 + u'_x v/c^2)}{\sqrt{1 - v^2/c^2}} \qquad m' = \frac{m(1 - u_x v/c^2)}{\sqrt{1 - v^2/c^2}}$$

4.3 Force

$$F_x = F'_x + \frac{u'_y v}{c^2 + u'_x v} F'_y + \frac{u'_z v}{c^2 + u'_x v} F'_z \qquad F'_x = F_x - \frac{u_y v}{c^2 - u_x v} F_y - \frac{u_z v}{c^2 - u_x v} F_z$$

$$F_y = \frac{\sqrt{1 - v^2/c^2}}{1 + u'_x v/c^2} F'_y \qquad F'_y = \frac{\sqrt{1 - v^2/c^2}}{1 - u_x v/c^2} F_y$$

$$F_z = \frac{\sqrt{1 - v^2/c^2}}{1 + u'_x v/c^2} F'_z \qquad F'_z = \frac{\sqrt{1 - v^2/c^2}}{1 - u_x v/c^2} F_z$$

A more **compact version**:

$$\frac{1}{\sqrt{1 - v^2/c^2}} = \gamma$$

$$F_x = \frac{F'_x + (v/c^2)\bar{u}' \cdot \bar{F}'}{1 + u'_x v/c^2} \qquad F'_x = \frac{F_x - (v/c^2)\bar{u} \cdot \bar{F}}{1 - u_x v/c^2}$$

$$F_y = \frac{F'_y}{1 + u'_x v/c^2} \qquad F'_y = \frac{F_y}{1 - u_x v/c^2}$$

$$F_z = \frac{F'_z}{1 + u'_x v/c^2} \qquad F'_z = \frac{F_z}{1 - u_x v/c^2}$$

F_x depends on $\bar{u} \cdot \bar{F}$ the power developed by the force in another frame $\rightarrow$ power and force are related in 4D space, similar to momentum and energy.

The 4th relation that goes together with the force is:

$$\bar{u} \cdot \bar{F} = \frac{\bar{u}' \cdot \bar{F}' + vF_x}{1 + u'_x v/c^2} \qquad \bar{u}' \cdot \bar{F}' = \frac{\bar{u} \cdot \bar{F} - vF_x}{1 - u'_x v/c^2}$$

Special case: if a body is at rest ($u' = 0$) in the S' frame and it is subjected to a force with components F'_x, F'_y, F'_z

$$F_x = F'_x$$

$$F_y = \frac{F'_y}{\gamma} = \frac{F'_y}{\sqrt{1 - v^2/c^2}}$$

$$F_z = \frac{F'_z}{\gamma} = \frac{F'_z}{\sqrt{1 - v^2/c^2}}$$

The force is the gratest in the rest frame of the object and smaller in all other inertial frames.