Differential Equations Solutions

(Questions 1-10 carry 1 mark each)

1) The partial differential equation $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}$ is a

(A) Linear equation of order 2
(B) Non-linear equation of order 1
(C) Linear equation of order 1
(D) Non-linear equation of order 2

Answer: (D)

2) The solution of the equation $x\frac{dy}{dx} + y = 0$ passing through the point (1, 1) is

(a) x
(b) x^2
(c) x^{-1}
(d) x^{-2}

Ans. (c)

$$x\frac{dy}{dx} + y = 0$$

$$x\frac{dy}{dx} = -y$$

$$\frac{dy}{y} = -\frac{dx}{x}$$

$$\int \frac{1}{y}dy = \int \frac{-1}{x}dx$$

$$\ln y = -\ln x + c$$

when $y = 1,\ x = 1$

$$c = 0$$

$$\Rightarrow \quad y = \frac{1}{x} = x^{-1}$$

3) Let $r = x^2 + y - z$ and $z^3 - xy + yz + y^3 = 1$. Assume that x and y are independent variables. At $(x, y, z) = (2, -1, 1)$, the value (correct to two decimal places) of $\frac{\partial r}{\partial x}$ is ______.

(a) 4.15
(b) 4.25
(c) 4.50
(d) 4.75

Ans. (4.50)

$$r = x^2 + y - z \quad \text{...(i)}$$

$$z^3 - xy + yz + y^3 = 1 \quad \text{...(ii)}$$

$$\frac{\partial r}{\partial x} = 2x - \frac{\partial z}{\partial x} \quad \text{...(iii)}$$

$$3z^2 \frac{\partial z}{\partial x} - y + y\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{y}{3z^2 + y}$$

By substituting $\frac{\partial z}{\partial x}$ in equation (iii), we get,

$$\frac{\partial r}{\partial x} = 2x - \frac{y}{3z^2 + y}$$

At (2, −1, 1),

$$\frac{\partial r}{\partial x} = 2(2) - \frac{(-1)}{3(1)^2 + (-1)} = 4 + \frac{1}{2} = 4.50$$

4) Consider a function u which depends on position x and time t. The partial differential equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ is known as the

(a) Wave equation (b) Heat equation
(c) Laplace's equation (d) Elasticity equation

Ans. **(b)**

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \text{ is known as heat equation}$$

5) Consider the following partial differential equation for u(x,y) with the constant c > 1 :

$$\frac{\partial u}{\partial y}+c\frac{\partial u}{\partial x}=0$$

Solution of this equation is

(A) $u(x,y)=f(x+cy)$ (B) $u(x,y)=f(x-cy)$

(C) $u(x,y)=f(cx+y)$ (D) $u(x,y)=f(cx-y)$

Key: **(B)**

Exp: Given $\frac{\partial u}{\partial y}+c\frac{\partial u}{\partial x}=0$

$$\frac{\partial u}{\partial x}=f'(x-cy)$$

$$\frac{\partial u}{\partial x}=-c\,f'(x-cy)$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$$

$$u(x,y)=f(x-cy)$$

6) The type of partial differential equation $\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}+3\frac{\partial^2 P}{\partial x\partial y}+2\frac{\partial P}{\partial x}-\frac{\partial P}{\partial y}=0$ is

(A) elliptic (B) parabolic (C) hyperbolic (D) none of these

Key: (C)

Exp: Comparing the given equation with the general form of second order partial differential equation, we have A=1, B=3, C=1 $\Rightarrow B^2-4AC=5>0$

$\therefore$ P.D.E is Hyperbola.

7) The solution of the partial differential equation $\frac{\partial u}{\partial t}=\alpha\frac{\partial^2 u}{\partial x^2}$ is of the form

(A) $C\cos(kt)\left[C_1e^{\left(\sqrt{k/\alpha}\right)x}+C_2e^{-\left(\sqrt{k/\alpha}\right)x}\right]$

(B) $Ce^{kt}\left[C_1e^{\left(\sqrt{k/\alpha}\right)x}+C_2e^{-\left(\sqrt{k/\alpha}\right)x}\right]$

(C) $Ce^{kt}\left[C_1\cos\left(\sqrt{k/\alpha}\right)x+C_2\sin-\left(\sqrt{k/\alpha}\right)x\right]$

(D) $C\sin(kt)\left[C_1\cos\left(\sqrt{k/\alpha}\right)x+C_2\sin-\left(\sqrt{k/\alpha}\right)x\right]$

Key: (B)

Exp: The P.D.E $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$ …. (1) is called 1-D heat equations.

Then the solution of (1) is

$$u(x,t) = (A\cos px + B\sin px)C.e^{-p^2\alpha.t}$$

Put $-p^2\alpha = k \Rightarrow p = \sqrt{\frac{-k}{\alpha}} = \sqrt{\frac{k}{\alpha}}.i$

$\therefore$ (1) becomes

$$u(x,t) = \left(A\cosh\sqrt{k/\alpha}.x + B\sinh\sqrt{k/\alpha}.x\right).C.e^{kt}$$

$$= C.e^{kt}.\left[A.\left\{\frac{e^{\sqrt{\frac{k}{\alpha}}x} + e^{-\sqrt{\frac{k}{\alpha}}x}}{2}\right\} + B.\left\{\frac{e^{\sqrt{\frac{k}{\alpha}}.x} - e^{-\sqrt{\frac{k}{\alpha}}x}}{2}\right\}\right]$$

$$= C.e^{kt}\left[e^{(\sqrt{k/\alpha}).x}.\left\{\frac{A+B}{2}\right\} + e^{-(\sqrt{k/\alpha}).x}.\left\{\frac{A-B}{2}\right\}\right]$$

$$= C.e^{kt}\left[c_1e^{(\sqrt{k/\alpha}).x} + c_2e^{-(\sqrt{k/\alpha}).x}\right]$$

8) If y is the solution of the differential equation

$$y^3\frac{dy}{dx} + x^3 = 0,$$

$$y(0) = 1$$

the value of $y(-1)$ is

(a) −2 (b) −1

(c) 0 (d) 1

Ans. (c)

$$y^3 \frac{dy}{dx} = -x^3$$

$$y^3 dy = -x^3 \ dx$$

$$\int y^3 dy = -\int x^3 dx$$

$$\frac{y^4}{4} = \frac{-x^4}{4} + C$$

$$\frac{x^4 + y^4}{4} = C$$

$$y(0) = 1,$$

$$\frac{0+1}{4} = C$$

$$C = \frac{1}{4}$$

$$x^4 + y^4 = 1$$

$$y^4 = 1 - x^4$$

$$y = \sqrt[4]{1-x^4}$$

When, $x = -1$

$$y = 0$$

9) The solution of the initial value problem $\frac{dy}{dx} = -2xy;\ y(0) = 2$ is

(A) $1 + e^{-x^2}$ (B) $2e^{-x^2}$ (C) $1 + e^{x^2}$ (D) $2e^{x^2}$

Answer : (B)

10) Given that $\ddot{x} + 3x = 0$, and $x(0) = 1, \dot{x}(0) = 0$, what is $x(1)$?

(A) -0.99 (B) -0.16 (C) 0.16 (D) 0.99

Answer : (B)

(Questions 11-20 carry 2 marks each)

11) Given the ordinary differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

with $y(0) = 0$ and $\frac{dy}{dx}(0) = 1$, the value of $y(1)$ is ________ (correct to two decimal places).

(a) 1.46 (b) 3.68
(c) 2.05 (d) 4.56

Ans. (1.4678)

$$(D^2 + D - 6)y = 0$$

$$y(0) = 0,$$

$$y'(0) = 1$$

$$(D + 3)(D - 2)y = 0$$

$$D = 2, -3$$

$$\text{C.F.} = C_1e^{2x} + C_2e^{-3x}$$

$$y = C_1e^{2x} + C_2e^{-3x}$$

$$y(0) = 0$$

So, $$0 = C_1 + C_2 \quad \ldots(i)$$

$$\frac{dy}{dx} = 2C_1e^{2x} - 3C_2e^{-3x}$$

$$y'(0) = 1,$$

$$1 = 2C_1 - 3C_2 \quad \ldots(ii)$$

From equation (i) and (ii),

$$C_1 = \frac{1}{5},$$

$$C_2 = \frac{-1}{5}$$

$$y = \frac{1}{5}e^{2x} - \frac{1}{5}e^{-3x}$$

When, $$x = 1$$

$$y(1) = \frac{e^2 - e^{-3}}{5} = 1.4678$$

12) The solution (up to three decimal places) at $x = 1$ of the differential equation $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$ subject to boundary conditions y(0) = 1 and $\frac{dy}{dx}=(0)=-1$ is ____

(a) 0.16 (b) 0.26
(c) 0.36 (d) 0.46

Ans. **(0.36)**

$$(D^2 + 2D + 1)y = 0 \quad (\therefore \text{ Roots are } -1, -1)$$

$$CF = (C_1 + C_2 x)\, e^{-x}$$

$$y = C_1 e^{-x} + C_2 x\, e^{-x} \qquad \text{...(i)}$$

$$y(0) = 1 \qquad 1 = C_1 \qquad \text{...(ii)}$$

$$y' = C_1 e^{-x} + C_2 (e^{-x} - x e^{-x})$$

$$y'(0) = -1, \qquad -1 = -C_1 + C_2 \qquad \text{...(iii)}$$

From eq. (ii) and (iii),

$$C_1 = 1,\ C_2 = 0$$

$$\therefore \qquad y = e^{-x}$$

At $x = 1$, $y = e^{-1} = \frac{1}{e} = 0.368$

13) Consider the differential equation $3y''(x)+27y(x)=0$ with initial conditions y(0) = 0 and $y'(0)=2000$. The value of y at x = 1 is ________.

(a) 78 (b) 86
(c) 94 (d) 102

Key: **93 to 95**

Exp: $3y''(x)+27y(x)=0,\ y(0)=0,\ y'(0)=2000$

Auxillary equation, $3m^2+27=0 \Rightarrow m^2+9=0 \Rightarrow m=0+3i$

$y_c = c_1\cos 3x + c_2 \sin 3x$ and $y_p = 0$

$\therefore y_c = c_1\cos 3x + c_2 \sin 3x$

$y(0)=0 \Rightarrow c_1+0=0 \Rightarrow c_1=0$

$\therefore y = c_2 \sin 3x$

$y' = 3c_2\cos 3x$

$$y'(0)=2000 \Rightarrow 2000 = 3c_2 \Rightarrow c_2 = \frac{2000}{3}$$

$$\therefore y = \frac{2000}{3}\sin 3x,\ \ y(1) = \frac{2000}{3}\sin 3 = 94.08$$

14) Consider the following differential equation:

$$x(ydx + xdy)\cos\frac{y}{x} = y(xdy - ydx)\sin\frac{y}{x}$$

Which of the following is the solution of the above equation (c is an arbitrary constant)?

(A) $\frac{x}{y}\cos\frac{y}{x} = c$ (B) $\frac{x}{y}\sin\frac{y}{x} = c$ (C) $xy\cos\frac{y}{x} = c$ (D) $xy\sin\frac{y}{x} = c$

Answer: (C)

Exp: Given D.E

$$x(ydx + xdy)\cos\frac{y}{x} = y(xdy - ydx)\sin\frac{y}{x}$$

$$\Rightarrow x(ydx + xdy)\cos\frac{y}{x} + \left(-\sin\frac{y}{x}\right)y(xdy - ydx) = 0$$

$$\Rightarrow (ydx + xdy)\cos\left(\frac{y}{x}\right) + \left(-\sin\frac{y}{x}\right)\frac{y(xdy - ydx)}{x} = 0$$

$$\Rightarrow (ydx + xdy)\cos\left(\frac{y}{x}\right) + (xy)\left(-\sin\frac{y}{x}\right)\left(\frac{xdy - ydx}{x^2}\right) = 0$$

By observing, the above equation is $d\left((xy)\cos\frac{y}{x}\right) = 0$

By integrating, $xy\cos\left(\frac{y}{x}\right) = c$

15) Consider the following second order linear differential equation

$$\frac{d^2y}{dx^2} = -12x^2 + 24x - 20$$

The boundary conditions are at x = 0, y = 5 and at x = 2, y = 21

The value of at x = 1 is ___________.

(a) -1 (b) -2
(c) -3 (d) -4

Answer: -2

Exp: Given

$$\frac{d^2y}{dx^2} = -12x^2 + 24x - 20$$

$$y(0) = 5 \quad y(2) = 21$$

$$y(1) = ?$$

Auxillary equation $m^2 = 0$

$$m = 0, 0$$

$$y_c = (c_1 + c_2x)e^{0x} = c_1 + c_2x$$

$$y_p = \frac{1}{D^2}\left(-12x^2 + 24x - 20\right)$$

$$= -12\frac{x^4}{12} + 24.\frac{x^3}{6} - 20.\frac{x^2}{2!}$$

$$= -x^4 + 4x^3 - 10x^2$$

$$y = c_1 + c_2x - 10x^2 + 4x^3 - x^4$$

$$y(0) = 5 \quad \Rightarrow c_1 = 5$$

$$y(2) = 21 \quad \Rightarrow 21 = 5 + 2c_2 + 40 + 32 - 16$$

$$21 = 2c_2 + 61$$

$$c_2 = -20$$

$$y = 5 - 20x + 10x^2 + 4x^3 - x^4$$

$$y(1) = 5 - 20 + 10 + 4 - 1$$

$$= -2$$

16) The respective expressions for complimentary function and particular integral part of the solution of the differential equation $\frac{d^4y}{dx^4} + 3\frac{d^2y}{dx^2} = 108x^2$ are

(A) $\left[c_1 + c_2x + c_3 \sin\sqrt{3x} + c_4 \cos\sqrt{3x}\right]$ and $\left[3x^4 - 12x^2 + c\right]$

(B) $\left[c_2x + c_3 \sin\sqrt{3x} + c_4 \cos\sqrt{3x}\right]$ and $\left[5x^4 - 12x^2 + c\right]$

(C) $\left[c_1 + c_3 \sin\sqrt{3x} + c_4 \cos\sqrt{3x}\right]$ and $\left[3x^4 - 12x^2 + c\right]$

(D) $\left[c_1 + c_2x + c_3 \sin\sqrt{3x} + c_4 \cos\sqrt{3x}\right]$ and $\left[5x^4 - 12x^2 + c\right]$

Key: (A)

Exp: D.E is $(D^4+3D^2).y=108x^2,\ D=\frac{d}{dx}$

A.E:- $m^4+3m^2=0 \Rightarrow m^2(m^2+3)=0 \Rightarrow m=0,0,\pm\sqrt{3}i$

$\therefore C.F=(C_1+C_2x)+C_3\sin(\sqrt{3}x)+C_4\cos(\sqrt{3}x)$

and $P.I=\frac{1}{D^4+3D^2}(108x^2)$

$$=\frac{1}{3D^2\left[1+\frac{D^2}{3}\right]}(108x^2)=\frac{36}{D^2}\left[1+\frac{D^2}{3}\right]^{-1}(x^2)$$

$$=\frac{36}{D^2}\left[1-\frac{D^2}{3}+\ldots\right](x^2)=\frac{36}{D^2}\left[x^2-\frac{1}{3}(2)+0\right]$$

$$=\int\int\left(36x^2-\frac{2}{3}\right)dxdx=36\left(\frac{x^4}{(4)(3)}-\frac{2}{3}\frac{x^2}{(2)(1)}\right)=3x^4-12x^2$$

17) The solution of the equation $\frac{dQ}{dt}+Q=1$ with $Q=0$ at $t=0$ is

(A) $Q(t)=e^{-t}-1$ (B) $Q(t)=1+e^{-t}$

(C) $Q(t)=1-e^{t}$ (D) $Q(t)=1-e^{-t}$

Key: (D)

Exp: $\frac{d\theta}{dt}+\theta=1$ and $\theta=0$ at $t=0$

Comparing with first order linear differential equations

$\frac{dQ}{dt}+pQ=q$ where $p=1;q=1$

$I.F=\int_e pdt=e^t$

$Q.(IF)=\int 1.(IF)dt+c$

$Q.e^t=\int e^t dt+c$

$Q.e^t=e^t+c$

$Q=0$ at $t=0 \Rightarrow 0.1=1+c \Rightarrow c=-1$

$\therefore Q.e^t=e^t-1 \Rightarrow Q=1-e^{-t}$

18) Consider the following second-order differential equation:

$$y'' - 4y' + 3y = 2t - 3t^2$$

The particular solution of the differential equations is

(A) $-2-2t-t^2$ (B) $-2t-t^2$ (C) $2t-3t^2$ (D) $-2-2t-3t^2$

Key: (A)

Exp: given $y^{11} - 4y^1 + 3y = 2t - 3t^2$

$$\Rightarrow \left(D^2 - 4D + 3\right) y = \left(2t - 3t^2\right)$$

By the definition of particular solution

$$y_p = \frac{1}{D^2 - 4D + 3}\left(2t - 3t^2\right)$$

$$\Rightarrow \left(D^2 - 4D + 3\right) y_p = 2t - 3t^2$$

verifying options, option (a) satisfies,

$$\left(D^2 - 4D + 3\right)\left(-2 - 2t - t^2\right)$$

$$= -2 + 8 + 8t - 6 - 6t - 3t^2 = 2t - 3t^2$$

19) The solution to the differential equation $\frac{d^2u}{dx^2} - k\frac{du}{dx} = 0$ where k is a constant, subjected to the boundary conditions u(0)=0 and u(L)=U, is

(A) $u = U\frac{x}{L}$ (B) $u = U\left(\frac{1 - e^{kx}}{1 - e^{kL}}\right)$

(C) $u = U\left(\frac{1 - e^{-kx}}{1 - e^{-kL}}\right)$ (D) $u = U\left(\frac{1 + e^{-kx}}{1 + e^{-kL}}\right)$

Answer: (B)

$$\frac{d^2u}{dx^2} - K\frac{du}{dx} = 0$$

$D^2 - kD = 0 \qquad D(D - K) = 0$

$D = 0, \;\; D = K$

$u = C_1e^0 + C_2e^{kx}$

$u = C_1 + C_2e^{kx}$

$u(0) = 0$

$\therefore C_1 + C_2 = 0 \ldots\ldots\ldots\ldots.(1)$

$u(L) = u$

$u = C_1 + C_2e^{kL} = U \ldots\ldots\ldots\ldots.(2)$

solving (1) and (2)

$$C_1 = \frac{U}{1-e^{kL}},\ C_2 e^{kx}\ \frac{-U}{1-e^{kL}}$$

$$u = U\left(\frac{1-e^{kL}}{1-e^{kx}}\right)$$

20) Fine the solution of $\frac{d^2y}{dx^2} = y$ which passes through the origin and the point $\left(\ln 2, \frac{3}{4}\right)$,

(A) $y = \frac{1}{2}e^x - e^{-x}$

(B) $y = \frac{1}{2}\left(e^x + e^{-x}\right)$

(C) $y = \frac{1}{2}\left(e^x - e^{-x}\right)$

(D) $y = \frac{1}{2}e^x + e^{-x}$

Answer: (C)

Exp: $\frac{d^2y}{dx^2} = y \Rightarrow \left(D^2 - 1\right)y = 0$

$D^2 - 1 = 0$

$\Rightarrow D = \pm 1$

$y = c_1 e^x + c_2 e^{-x}$

Passes through (0,0) and $\left(142, \frac{3}{4}\right)$

(0,0)

$\Rightarrow 0 = C_1 + C_2$ ___________(1)

$\left(142, \frac{3}{4}\right)$

$$\frac{3}{4} = C_1 e^{142} + C_2 e^{-142} = C_1 2 + \frac{C_2}{2}$$

$\Rightarrow 2C_1 + \frac{1}{2}C_2 = \frac{3}{4}$ ______(2)

solving (1) and (2)

$\Rightarrow C_1 = \frac{1}{2}$

$C_2 = -\frac{1}{2}$

$$\therefore y = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$$

$$= \frac{1}{2}\left(e^x - e^{-x}\right)$$