Limits and Calculus Solutions

(Questions 1-10 carry 1 mark each)

1) According to the Mean Value Theorem, for a continuous function $f(x)$ in the interval $[a, b]$, there exists a value ξ in this interval such that $\int_n^b f(x)dx =$

(a) $f(\xi)(b-a)$ (b) $f(b)(\xi-a)$

(c) $f(a)(b-\xi)$ (d) 0

Ans. (a)

$$\int_a^b f(x)dx = f(\xi)(b-a)$$

2) The Fourier cosine series for an even function $f(x)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$$

The value of the coefficient a_2 for the function $f(x) = \cos^2(x)$ in $[0, \pi]$ is

(a) –0.5 (b) 0.0

(c) 0.5 (d) 1.0

Ans. (c)

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$f(x) = \frac{1}{2} + \frac{\cos 2x}{2}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos nx$$

$$a_0 = 1$$

$$a_1 = 0,$$

$$a_2 = \frac{1}{2}$$

3) The value of $\lim_{x\to 0} \dfrac{1-\cos\left(x^2\right)}{2x^4}$ is

(A) 0 (B) $\frac{1}{2}$ (C) $\frac{1}{4}$ (D) undefined

Answer: (A)

Exp: $\lim\limits_{x\to 0} \dfrac{1-\cos\left(x^2\right)}{2x^4} = \dfrac{0}{0}$

Using L Hospital Rule

$$\lim_{x\to 0} \frac{\left(\sin x^2\right)2x}{-8x^3} = \frac{0}{0}$$

$$\lim_{x\to 0} \frac{\left(\cos x^2\right)2x2x + \left(\sin x^2\right)2}{24x^2}$$

$$\lim_{x\to 0} \frac{\left(\cos x^2\right)4x^2 + 2\sin x^2}{24x^2}$$

$$= \lim_{x\to 0} \frac{\left(-\sin x^2\right)4x^2 + \cos x^2\,(8x) + 2\left(\cos x^2\right)2x}{48x}$$

$$= \lim_{x\to 0} \frac{\left(-\cos x^2\right)2x\left(4x^2\right) + \left(-\sin x^2\right)(8x) + \left(-\sin x^2\right)(2x)(8x) + 12\cos x^2 + \left(-\sin x^2\right)(2x)4x}{48}$$

$$= \frac{0}{48} = 0$$

4) At $x = 0$, the function $f(x) = |x|$ has

(A) A minimum (B) A maximum

(C) A point of inflexion (D) neither a maximum nor minimum

Answer: (A)

Exp: For negative values of x, f(x) will be positive

For positive values of x, f(x) will be positive

$\therefore$ minimum value of f(x) will occur at $x = 0$

5) The values of x for which the function

$f(x)=\dfrac{x^2-3x-4}{x^2+3x-4}$ is **NOT** continuous are

(A) 4 and −1 (B) 4 and 1 (C) −4 and 1 (D) −4 and −1

Key: (C)

Exp: The function $f(x)=\dfrac{x^2-3x-4}{x^2+3x-4}$ is not continuous at $x=-4\,\&\,1$; since f(x) does not exists at x=-4 &1.

6) Laplace transform of cos(ω*t*) is

(A) $\dfrac{s}{s^2+\omega^2}$ (B) $\dfrac{\omega}{s^2+\omega^2}$ (C) $\dfrac{s}{s^2-\omega^2}$ (D) $\dfrac{\omega}{s^2-\omega^2}$

Key: (A)

Exp: By the L.T of standard functions

7) $\underset{x\to 0}{lt}\dfrac{\log_e(1+4x)}{e^{3x}-1}$ is equal to

(A) 0 (B) $\dfrac{1}{12}$ (C) $\dfrac{4}{3}$ (D) 1

Key: (C)

Exp: $\lim\limits_{x\to 0}\dfrac{\log_e(1+4x)}{e^{3x}-1}\quad\left(\dfrac{0}{0}\right)$

$$\lim_{x\to 0}\frac{\frac{1}{1+4x}.4}{e^{3x}.3}=\frac{4}{(1+4.0)e^0.3}=\frac{4}{3}$$

8) The Laplace transform of te^t is

(A) $\dfrac{s}{(s+1)^2}$ (B) $\dfrac{1}{(s-1)^2}$ (C) $\dfrac{1}{(s+1)^2}$ (D) $\dfrac{s}{s-1}$

Key: **(B)**

Exp: $L\{te^t\}=\dfrac{1}{(s-1)^2};\quad\left(\begin{array}{l}\therefore L\{e^{at}f(t)\}=F(s-a)\\ \text{where } F(s)=L\{f(t)\}\end{array}\right)$

9) The Value of $\lim_{x\to 0}\dfrac{x^3-\sin(x)}{x}$ is

(A) 0 (B) 3 (C) 1 (D) -1

Key: **(D)**

Exp: $\underset{x\to 0}{\ell t}\dfrac{x^3-\sin x}{x}=\left(\underset{x\to 0}{\ell t}\,x^2\right)-\left(\underset{x\to 0}{\ell t}\dfrac{\sin x}{x}\right)=0-1=-1$

10) $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{2x}$ is equal to

(A) e^{-2} (B) e (C) 1 (D) e^2

Answer: (D)

Exp: $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{2x}$

$$=\left(\lim_{x\to\infty}\left(x+\frac{1}{x}\right)^{x}\right)^{2}$$

$$=e^2$$

(Questions 11-20 carry 2 marks each)

11) The value of the integral $\int_0^{\pi} x\cos^2 x\,dx$ is

(a) $\frac{\pi^2}{8}$ (b) $\frac{\pi^2}{4}$

(c) $\frac{\pi^2}{2}$ (d) π^2

Ans. (b)

The value of $\int_0^{\pi} x\cos^2 x\,dx$

$$= \int_0^{\pi}\left(\frac{x}{2}+\frac{x\cos 2x}{2}\right)dx$$

$$= \left.\frac{x^2}{4}\right|_0^{\pi} + \frac{1}{2}\left(\frac{x\sin 2x}{2}+\frac{\cos 2x}{4}\right)$$

$$= \frac{\pi^2}{4}+\frac{1}{2}\left\{\left(0+\frac{1}{4}\right)-\left(0+\frac{1}{4}\right)\right\}$$

$$= \frac{\pi^2}{4}+\frac{1}{2}\left(\frac{1}{4}-\frac{1}{4}\right)$$

$$= \frac{\pi^2}{4}$$

12) The value of the definite integral $\int_1^e \sqrt{x}\ \ln(x)dx$ is

(A) $\frac{4}{9}\sqrt{e^3}+\frac{2}{9}$ (B) $\frac{2}{9}\sqrt{e^3}-\frac{4}{9}$ (C) $\frac{2}{9}\sqrt{e^3}+\frac{4}{9}$ (D) $\frac{4}{9}\sqrt{e^3}-\frac{2}{9}$

Answer: (C)

$$\int_1^e \sqrt{x}\ln(x)dx$$

$$= \left[\ln(x)\times\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^e - \int\left[\frac{1}{x}\times\frac{x^{\frac{3}{2}}}{\frac{2}{3}}\right]dx$$

$$= \left[\ln(x)\times x^{\frac{3}{2}}\times\frac{2}{3}-\frac{4}{9}\times x^{\frac{3}{2}}\right]_1^e$$

$$= \frac{2}{9}\sqrt{e^3}+\frac{4}{9}$$

13) $\lim_{x\to\infty} \sqrt{x^2+x-1} - x$ is

(A) 0 (B) ∞ (C) 1/2 (D) $-\infty$

Key: (C)

Exp: $\lim\limits_{x\to\infty}\left(\sqrt{x^2+x-1}-x\right)\times\dfrac{\sqrt{x^2+x-1}+x}{\sqrt{x^2+x-1}+x}$

$$= \lim_{x\to\infty}\frac{x^2+x-1-x^2}{\sqrt{x^2+x-1}+x}$$

$$= \lim_{x\to\infty}\frac{x\left(1-\frac{1}{x}\right)}{x\sqrt{1+\frac{1}{x}-\frac{1}{x^2}}+1} = \frac{1-0}{\sqrt{1+0-0}+1} = \frac{1}{2}$$

14) Consider the function $f(x) = 2x^3 - 3x^2$ in the domain $[-1, \ 2]$. The global minimum of $f(x)$ is ________

(a) -1 (b) -5

(c) 5 (d) -10

Key: -5

Exp: Given that, $f(x) = 2x^3 - 3x^2$

$\Rightarrow f'(x) = 0 \Rightarrow 6x^2 - 6x = 0$

$\Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0$

$\Rightarrow x = 0;\ x = 1$

are Stationary points.

$\therefore$ $f''(x) = 12x - 6$

$f''(0) = -6 > 0$

$\therefore$ f(x) has maximum at x = 0.

$f''(1) = 12(1) - 6 = 6 > 0$

$\therefore$ f(x) has minimum at x = 1.

$\therefore$ $f(1) = 2 - 3 = -1 \rightarrow$ local minimum value

But $f(-1) = -2 - 3 = -5$

$\therefore$ Global minimum of f(x) = -5

15) The value of

$$\int_0^\infty \frac{1}{1+x^2}\,dx + \int_0^\infty \frac{\sin x}{x}\,dx \text{ is}$$

(A) $\frac{\pi}{2}$ (B) π (C) $\frac{3\pi}{2}$ (D) 1

Key: (B)

Exp: $\int_0^\infty \frac{1}{1+x^2}dx = \left[\tan^{-1}x\right]_0^\infty = \tan^{-1}\infty - \tan^{-1}0 = \frac{\pi}{2}$

and $L(\sin x) = \frac{1}{s^2+1} \Rightarrow L\left(\frac{\sin x}{x}\right) = \int_s^\infty \frac{1}{s^2+1}dx$ (Using "Division by x")

$$= \left[\tan^{-1}s\right]_s^\infty = \tan^{-1}\infty - \tan^{-1}(s) = \cot^{-1}(s)$$

$\Rightarrow \int_0^\infty e^{-sx}.\frac{\sin x}{x}dx = \cot^{-1}(s)$ (Using definition of Laplace transform)

Put s=0, we get

$$\int_0^\infty \frac{\sin x}{x}dx = \cot^{-1}(0) = \frac{\pi}{2}$$

$$\propto \int_0^\infty \frac{1}{1+x^2}dx + \int_0^\infty \frac{\sin x}{x}dx = \pi$$

16) Consider the following definite integral:

$$I = \int_0^1 \frac{\left(\sin^{-1}x\right)^2}{\sqrt{1-x^2}}dx$$

The value of the integral is

(A) $\frac{\pi^3}{24}$ (B) $\frac{\pi^3}{12}$ (C) $\frac{\pi^3}{48}$ (D) $\frac{\pi^3}{64}$

Key: (A)

Exp: given $I = \int_0^1 \frac{\left(\sin^{-1}x\right)^2}{\sqrt{1-x^2}}dx$

$$= \left.\frac{\left(\sin^{-1} x\right)^3}{3}\right|_0^1 \qquad \left(\because \int f^n(x) f^1(x)dx = \frac{f^{n+1}}{n+1}\right)$$

$$= \frac{1}{3}\left[\left(\sin^{-1}\right)^3 - \sin^{-1} 0\right] = \frac{1}{3}\left[\left(\frac{\pi}{2}\right)^3 - 0\right] = \frac{\pi^3}{24}$$

17) $\lim_{x \to \infty}\left(1+\frac{1}{x}\right)^{2x}$ is equal to

(A) e^{-2} (B) e (C) 1 (D) e^2

Answer: (D)

Exp: $\lim_{x \to \infty}\left(1+\frac{1}{x}\right)^{2x}$

$$= \left(\lim_{x \to \infty}\left(x+\frac{1}{x}\right)^x\right)^2$$

$$= e^2$$

18) The optimum value of the function $f(x) = x^2 - 4x + 2$ is

(A) 2 (maximum) (B) 2 (minimum) (C) −2 (maximum) (D) −2 (minimum)

Key: (D)

Exp: $f'(x) = 0 \Rightarrow 2x - 4 = 0$

$\Rightarrow x = 2$(statinary point)

$f''(x) = 2 > 0 \Rightarrow f(x)$ is minimum at x=2

And the minimum value is f(2)

i.e., $(2)^2 - 4(2) + 2 = -2$

$\therefore$ The optimum value of f(x) is -2 (minimum)

19) What is the value of $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{xy}{x^2+y^2}$?

(A) 1 (B) −1 (C) 0 (D) Limit does not exist

Key: (D)

Exp: (i) $\lim\limits_{x\to\infty} \frac{xy}{x^2+y^2} = \lim\limits_{y\to 0}\left(\frac{0}{0^2+y^2}\right) = 0$ (i .e., put x=0 and then y=0)

(ii) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{xy}{x^2+y^2} = \lim\limits_{x\to 0}\left(\frac{0}{x^2+0}\right) = 0$ (i.e., put y=0 and then x=0)

(iii) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{xy}{x^2+y^2} = \lim\limits_{x\to 0} \frac{x(m.x)}{x^2+m^2x^2}$ (i.e., put $y = mx$)

$$= \lim_{x\to\infty}\left(\frac{m}{1+m^2}\right) = \frac{m}{1+m^2},$$ which depends on ‘m’.

Hence, the limit does not exists.

20) $\lim\limits_{x\to 0}\left(\frac{\tan x}{x^2-x}\right)$ is equal to ________

(a) -1 (b) -2
(c) -3 (d) -4

Key: (-1)

Exp: $\underset{x\to 0}{\text{lt}} \frac{\text{Tanx}}{x^2-x} = \underset{x\to 0}{\text{lt}} \frac{\text{Tanx}}{x(x-1)}$

$$= \underset{x\to 0}{\text{lt}} \frac{\text{Tanx}}{x} \cdot \underset{x\to 0}{\text{lt}} \frac{1}{(x-1)} = 1\times -1 = -1$$

EXTRA:

23. Let x be a continuous variable defined over the interval $(-\infty,\infty)$, and $f(x)=e^{-x-e^{-x}}$.

The integral $g(x)=\int f(x)\,dx$ is equal to

(A) $e^{e^{-x}}$ (B) $e^{-e^{-x}}$

(C) $e^{-e^{x}}$ (D) e^{-x}

Key: (B)

Exp: $f(x)=e^{-x-e^{-x}}$ $x\in(-\infty,\infty)$ is a continuous variable

$g(x)=\int f(x)\,dx=\int e^{-x-e^{-x}}dx=\int e^{-x}.e^{-e^{-x}}dx$

put $e^{-x}=t$

$-e^{-x}dx=dt$

$\therefore g(x)=\int e^{-t}(-dt)=-\left(\dfrac{e^{-t}}{-1}\right)=e^{-t}=e^{-e^{-x}}$