Vector Calculus Solutions

(Questions 1-10 carry 1 mark each)

1) The value of integral

$$\oiint_S \vec{r} \cdot \hat{n} ds$$

over the closed surface S bounding a volume, where $\vec{r} = x\hat{i} + y\hat{i} + z\hat{k}$ is the position vector and $\hat{n}$ is the normal to the surface S, is

(a) V (b) $2\,V$

(c) $3\,V$ (d) $4\,V$

Ans. (c)

By Gauss Divergence Theorem

$$\iint_s \vec{r} \cdot \hat{n} ds = \iiint_V \nabla \cdot \vec{r} dV = \iiint_V 3\, dV = 3V$$

2) The divergence of the vector field $\vec{u} = e^x \left(\cos y \hat{i} + \sin y \hat{j}\right)$ is

(a) 0 (b) $e^x \cos y + e^x \sin y$

(c) $2e^x \cos y$ (d) $2\, e^x \sin y$

Ans. (c)

$$\vec{u} = e^x \cos y \hat{i} + e^x \cdot \sin y \hat{j}$$

$$\nabla \cdot \vec{u} = \frac{\partial}{\partial x}(u_1) + \frac{\partial}{\partial y}(u_2)$$

$$= \frac{\partial}{\partial x}\left(e^x \cdot \cos y\right) + \frac{\partial}{\partial y}\left(e^x \cdot \sin y\right)$$

$$= e^x \cos y + e^x \cos y$$

$$\nabla \cdot \vec{u} = 2e^x \cdot \cos y$$

3) For a position vector $r = x\hat{i} + y\hat{j} + z\hat{k}$ the norm of the vector can be defined as $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$. Given a function $\phi = \ln|\vec{r}|$, its gradient $\nabla\phi$ is

(a) $\vec{r}$

(b) $\dfrac{\vec{r}}{|\vec{r}|}$

(c) $\dfrac{\vec{r}}{\vec{r}\cdot\vec{r}}$

(d) $\dfrac{\vec{r}}{|\vec{r}|^3}$

Ans. (c)

$$\phi = \ln r$$
$$\nabla\phi = \nabla(\ln r)$$
$$f(r) = \ln(r)$$
$$f'(r) = \frac{1}{r}$$
$$\nabla f(r) = \frac{f'(r)}{r}\cdot\vec{r} = \left(\frac{1}{r}\right)\times\left(\frac{1}{r}\right)\cdot\vec{r}$$
$$\nabla f(r) = \frac{\vec{r}}{r^2}$$

4) Curl of vector V(x,y,z) = $2x^2i + 3z^2j + y^3k$ at $x = y = z = 1$ is

(A) $-3i$ (B) $3i$ (C) $3i - 4j$ (D) $3i - 6k$

Answer: (A)

Exp: Curl of $V(x,y,z) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2 & 3z^2 & y^3 \end{vmatrix}$

$$= i\left[3y^2 - 6z\right] + j[0-0] + k[0-0]$$
$$= \left(3y^2 - 6z\right)i\Big|_{x=y=z=1}$$
$$= -3i$$

5) Let ϕ be an arbitrary smooth real valued scalar function and V be an arbitrary smooth vector valued function in a three-dimensional space. Which one of the following is an identity?

(A) $\mathrm{Curl}\left(\phi\overline{V}\right)=\nabla\left(\phi \mathrm{Div}\,\overline{V}\right)$ (B) $\mathrm{Div}\,\overline{V}=0$

(C) $\mathrm{Div}\,\mathrm{Curl}\,\overline{V}=0$ (D) $\mathrm{Div}\left(\phi\overline{V}\right)=\phi\mathrm{Div}\,\overline{V}$

Answer: (C)

6) The divergence of the vector $-yi+xj$ __________

(a) -1 (b) 0
(c) 1 (d) 2

Key: **0 to 0**

Exp: Let $\vec{F}=-yi+xj$

$$\text{divergence of } \vec{F}=\frac{\partial}{\partial x}(-y)+\frac{\partial}{\partial y}(x)=0$$

7) The divergence of the vector field $V = x^2i+2y^3j+z^4k$ at $x=1,\ y=2,\ z=3$ is_______

(a) 64 (b) 128
(c) 134 (d) 172

Key: **(134)**

Exp: **Given**

$$v=x^2i+2y^3J+z^4k$$

$$\text{div } v=\frac{\partial}{\partial x}\left(x^2\right)+\frac{\partial}{\partial y}\left(2y^3\right)+\frac{\partial}{\partial z}\left(z^4\right)$$

$$=2x+6y^2+4z^3$$

$$\text{div } v\big|_{(1,2,3)}=2+24+108=134$$

8) Which one of the following describes the relationship among the three vectors, $\hat{i}+\hat{j}+\hat{k}, 2\hat{i}+3\hat{j}+\hat{k}$ and $5\hat{i}+6\hat{j}+4\hat{k}$?

(A) The vectors are mutually perpendicular (B) The vectors are linearly dependent

(C) The vectors are linearly independent (D) The vectors are unit vectors

Answer : (B)

9) Curl of vector $\vec{F}=x^2z^2\hat{i}-2xy^2z\hat{j}+2y^2z^3\hat{k}$ is

(A) $\left(4yz^3+2xy^2\right)\hat{i}+2x^2z\hat{j}-2y^2z\hat{k}$ (B) $\left(4yz^3+2xy^2\right)\hat{i}-2x^2z\hat{j}-2y^2z\hat{k}$

(C) $2xz^2\hat{i}-4xyz\hat{j}+6y^2z^2\hat{k}$ (D) $2xz^2\hat{i}+4xyz\hat{j}+6y^2z^2\hat{k}$

Answer : (A)

10) Divergence of the vector field $x^2z\hat{i}+xy\hat{j}-yz^2\hat{k}$ at (1, −1, 1) is

(A) 0 (B) 3 (C) 5 (D) 6

Answer : (C)

(Questions 11-20 carry 2 marks each)

11) The value of the integral $\int_0^2 \int_0^x e^{x+y}dydx$ is

(A) $\frac{1}{2}(e-1)$ (B) $\frac{1}{2}\left(e^2-1\right)^2$ (C) $\frac{1}{2}\left(e^2-1\right)$ (D) $\frac{1}{2}\left(e-\frac{1}{2}\right)^2$

Answer : (B)

12) The value of

$\int_C \left[\left(3x-8y^2\right)dx+\left(4y-6xy\right)dy\right]$, (where C is boundary of th region bounded by x = 0, y = 0 and x + y = 1 is) is ______________

(a) 1.66 (b) 2.66
(c) 3.66 (d) 4.66

Answer: 3.66

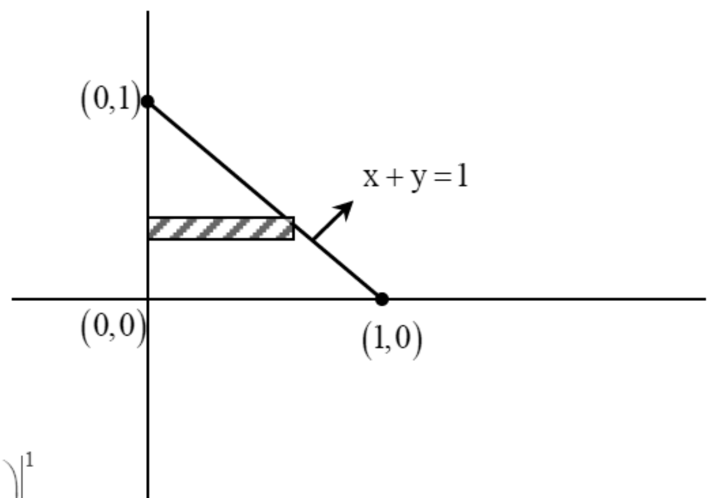
Exp: $x=0$ to $x=1-y$

&

$y=0$ to $y=1$

By Green's theorem, $\int\limits_C \underbrace{\left(3x-8y^2\right)}_{m}dx+\underbrace{\left(4y-6xy\right)}_{N}dy$

$$=\iint\left(\frac{\partial N}{\partial x}-\frac{\partial m}{\partial y}\right)dxdy$$



$$=\int\limits_{y=0}^{1}\int\limits_{x=0}^{1-y}\left[-6y-(-16y)\right]dxdy$$

$$=\int\limits_{y=0}^{1}\left[\int\limits_{x=0}^{1-y}22ydx\right]dy$$

$$=22\int\limits_{y=0}^{1}yx\Big|_0^{1-y}dy$$

$$=22\int\limits_{y=0}^{1}y\left[(1-y)-0\right]dy=22\left(\frac{y^2}{2}-\frac{y^3}{3}\right)\Bigg|_0^1$$

$$=22\left(\frac{1}{3}-\frac{1}{3}\right)=\frac{11}{3}=3.66$$

13) The value (up to two decimal places) of a line integral $\int\limits_C \vec{F}(\vec{r})\cdot d\vec{r}$, for $\vec{F}(\vec{r})=x^2\hat{i}+y^2\hat{j}$ along C which is a straight line joining (0, 0) to (1, 1) is _________.

(a) 0.66 (b) 1.66
(c) 2.66 (d) 3.66

Ans. (0.666)

$$\bar{F} = x^2\bar{i} + y^2\bar{j}$$

$$\int \bar{F}\cdot d\bar{r} = \int (x^2\bar{i} + y^2\bar{j})\cdot(dx\bar{i} + dy\bar{j})$$

$$= \int x^2dx + y^2dy$$

(0, 0) to (1, 1) line is $y = x$

$$= \int x^2dx + x^2dx = \int_0^1 2x^2dx$$

$$= 2\left(\frac{x^3}{3}\right)\Bigg|_0^1 = \frac{2}{3} = 0.666$$

14) The velocity field of an incompressible flow is given by $V = (a_1x + a_2y + a_3z)i + (b_1x + b_2y + b_3z)j + (c_1x + c_2y + c_3z)k$, where $a_1 = 2$ and $c_3 = -4$. The value of b_2 is ________.

(a) -1 (b) 0
(c) 1 (d) 2

Answer: 2

Exp: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

$$a_1 + b_2 + c_3 = 0$$

$$2 - 4 + b_2 = 0$$

$$b_2 = 2$$

15) A scalar potential ϕ has the following gradient. $\nabla\phi = yz\hat{i} + xz\hat{j} + xy\hat{k}$. Consider the integral

$\int_c \nabla\phi . d\bar{r}$ on the curve $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

The curve C is parameterized as follows: $\begin{cases} x = t \\ y = t^2 \text{ and } 1 \le t \le 3. \\ z = 3t^2 \end{cases}$

The value of the integral is______ .

(a) 324 (b) 428
(c) 622 (d) 726

Key: 726

Exp: $\int_c \nabla\phi . d\bar{r} = \int_c \left(yz\hat{i} + xz\hat{j} + xy\hat{k}\right).\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$

$$= \int_c yzdx + xzdy + xydz \qquad \ldots \quad (1)$$

$\therefore x = t;\ y = t^2;\ z = 3t^2$

$\Rightarrow dx = dt \Rightarrow dy = 2tdt \Rightarrow dz = 6tdt$

From (1): $\int_c \nabla\phi . d\bar{r} = \int_c t^2(3t^2)dt + t\left(3t^2\right)2tdt + t\left(t^2\right)6tdt$

$$= \int_{t=1}^{3} \left[3t^4 + 6t^4 + 6t^4\right]dt$$

$$= \int_{t=1}^{3} 15t^4 dt = 15\left[\frac{t^5}{5}\right]_1^3 = 3\left[3^5 - 1\right]$$

$$= 726$$

16) The value of the line integral $\oint\limits_C \bar{F}.\bar{r}'\, ds$, where C is a circle of radius $\frac{4}{\sqrt{\pi}}$ units is ____________.

Here, $\bar{F}(x,y) = y\hat{i} + 2x\hat{j}$ and $\bar{r}'$ is the **UNIT** tangent vector on the curve C at an arc length s from a reference point on the curve. $\hat{i}$ and $\hat{j}$ are the basis vectors in the *x-y* Cartesian reference. In evaluating the line integral, the curve has to be traversed in the counter-clockwise direction.

(a) 4 (b) 8
(c) 12 (d) 16

Key: 16

Exp: By Green's theorem,

$$\int_c \bar{F}.\bar{r}'\, ds = \int_c ydx + 2xdy = \iint (2-1)dxdy$$

$$= \pi\left(\frac{4}{\sqrt{\pi}}\right)^2 = 16$$

17) The surface integral $\iint_S F.ndS$ over the surface S of the sphere $x^2 + y^2 + z^2 = 9$, where F=(x+y) i+ (x+z) j+(y+z) k and n is the unit outward surface normal, yields ______.

(a) 226 (b) 324
(c) 428 (d) 522

Key: **225 to 227**

Exp: $\vec{F} = (x+y)i + (x+z)J + (y+z)k$

$$\text{div}\vec{F} = \frac{\partial}{\partial x}(x+y) + \frac{\partial}{\partial y}(x+z) + \frac{\partial}{\partial z}(y+z) = 1+0+1 = 2$$

By divergence theorem,

$$\iint_S \vec{F}.\hat{n}\,dS = \int_V \text{div}\vec{F}dV \quad \text{where V is volume of given surface of sphere } x^2+y^2+z^2=9$$

$$= \int_V 2dV = 2V = 2\times\frac{4\pi(27)}{3} = 72\pi = 226.1947$$

18) For the vector $\vec{V} = 2yz\hat{i} + 3xz\hat{j} + 4xy\hat{k}$, the value of $\nabla.\left(\nabla\times\vec{V}\right)$ is ____________

(a) -1 (b) 0
(c) 1 (d) 2

Key: **0 to 0**

Exp: $\vec{V} = 2yzi + 3xzj + 4xyk$

we know that $\nabla.(\nabla\times V) = 0$ for any vector V

19) The following surface integral is to be evaluated over a sphere for the given steady velocity vector field, $F = xi + yj + zk$ defined with respect to a Cartesian coordinate system having i, j, and k as unit base vectors.

$$\iint_S \frac{1}{4}(F.n)dA$$

Where S is the sphere, $x^2 + y^2 + z^2 = 1$ and n is the outward unit normal vector to the sphere. The value of the surface integral is

(A) π (B) 2π (C) $3\frac{\pi}{4}$ (D) 4π

Answer: (A)

Explanations:- $\frac{1}{4}\iiint_V \operatorname{div}\bar{F}\, dv$ (Using divergence theorem)

$$= \frac{1}{4}\iiint_V 3\, dv = \frac{3}{4} \times \text{volume of the sphere}$$

$$= \frac{3}{4} \times \frac{4}{3} \times (1)^3 = \pi \text{ as radius} = 1$$

20) The directional derivative of the field $u(x,y,z) = x^2 - 3yz$ in the direction of the vector $(\hat{i} + \hat{j} - 2\hat{k})$ at point $(2,-1,4)$ is ____ .

(a) -5.72 (b) -4.68

(c) 4.68 (d) 5.72

Answer: -5.72

Exp: Let $u(x,y,z) = x^2 - 3yz$

$$\vec{a} = i + j - 2k \text{ and } P(2,-1,4)$$

$$\nabla u = i\frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y} + k\frac{\partial u}{\partial z}$$

$$= i2x + j(-3z) + k(-3y)$$

$$\nabla u\big|_{(2,-1,4)} = 4i - 12j + 3k$$

$$|\vec{a}| = \sqrt{1+1+4} = \sqrt{6}$$

$$\text{directional derivative} = \nabla u.\hat{a}$$

$$= (4i - 12j + 3k).\frac{(i + j - 2k)}{\sqrt{6}}$$

$$= \frac{4 - 12 - 6}{\sqrt{6}}$$

$$= \frac{-14}{\sqrt{6}} = -5.72$$