Numerical Methods Solutions

(Questions 1-10 carry 1 mark each)

1) The quadratic equation $2x^2 - 3x + 3 = 0$ is to be solved numerically starting with an initial guess as $x_0 = 2$. The new estimate of x after the first iteration using Newton-Raphson method is __________.

(a) 1 (b) 2
(c) 3 (d) 4

Ans. (1)

Given $f(x) = 2x^2 - 3x + 3,\ x_0 = 2$

$$f'(x) = 4x - 3$$

By Newton-Rapshon

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{2(2)^2 - 3(2) + 3}{4(2) - 3}$$

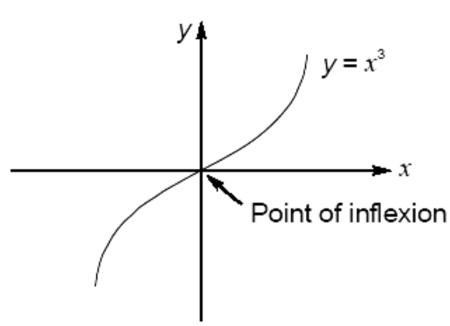
$$= 2 - \frac{5}{5} = 1$$

2) At the point $x = 0$, the function $f(x) = x^3$ has

(a) local maximum (b) local minimum
(c) both local maximum and minimum (d) neither local maximum nor local minimum

Ans. (d)

$$f(x) = x^3 \text{ at } x = 0$$



At $x = 0$, the function $y = x^3$ has neither minima nor maxima.

3) Match the CORRECT pairs:

Numerical Integration Scheme Order of Fitting Polynomial

P. Simpson’s 3/8 Rule	1. First
Q. Trapezoidal Rule	2. Second
R. Simpson’s 1/3 Rule	3. Third

(A) P-2; Q-1; R-3

(B) P-3; Q-2; R-1

(C) P-1; Q-2; R-3

(D) P-3; Q-1; R-2

Answer: (D)

4) Using a unit step size, the value of integral $\int_1^2 x \ln x \, dx$ by trapezoidal rule is ______

(a) 0.69 (b) 0.79
(c) 0.89 (d) 0.99

Answer: 0.69

Exp:

x	1	2
y = lhx	0	2lh2

By Trapezoidal Rule.

$$\int_1^2 x \ln x \, dx = \frac{1}{2}[0 + 2\ln 2] = \ln 2 = 0.69$$

5) Solve the equation $x = 10\cos(x)$ using the Newton-Raphson method. The initial guess is $x = \pi / 4$. The value of the predicted root after the first iteration, up to second decimal, is __________

(a) 1.56 (b) 2.56
(c) 3.56 (d) 4.56

Key: 1.56

Exp: By Newton-Raphson method; the iterative formula for finding approximate root at $(n+1)^{th}$ iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}; \text{ where } x = 0, 1, 2 \ldots\ldots$$

Putting n = 0; then

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \qquad \ldots (1)$$

Let $f(x) = x - 10\cos x \Rightarrow f(x_0) = f\left(\frac{\pi}{4}\right) = \frac{\pi}{4} - \frac{10}{\sqrt{2}}$.

$\Rightarrow f'(x) = 1 + 10\sin x$

From (1); $\Rightarrow f'(x_0) = f'\left(\frac{\pi}{4}\right) = 1 + \frac{10}{\sqrt{2}}$

$$\therefore \quad x_1 = \frac{\pi}{4} - \left[\frac{\frac{\pi}{4} - \frac{10}{\sqrt{2}}}{1 + \frac{10}{\sqrt{2}}}\right] \cong 1.56$$

6) Numerical integration using trapezoidal rule gives the best result for a single variable function, which is

(A) linear (B) parabolic (C) logarithmic (D) hyperbolic

Key: (A)

7) The root of the function $f(x) = x^3+x-1$ obtained after first iteration on application of Newton-Raphson scheme using an initial guess of $x_0=1$ is

(A) 0.682 (B) 0.686 (C) 0.750 (D) 1.000

Key: (C)

Exp: We have $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

For n=0, $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$f(x) = x^3 + x - 1 \Rightarrow f'(x) = 3x^2 + 1$

given $x_0 = 1$

$f(x_0) = f(1) = 1, \quad f'(x_0) = f'(1) = 4$

$\Rightarrow x_1 = 1 - \frac{1}{4} = \frac{3}{4} = 0.75$

8) In Newton-Raphson iterative method, the initial guess value (x_{ini}) is considered as zero while finding the roots of the equation: $f(x) = -2 + 6x - 4x^2 + 0.5x^3$. The correction, Δx, to be added to x_{ini} in the first iteration is ___________.

(a) 1 (b) 1/2
(c) 1/3 (d) 1/4

Answer: 0.3333

Exp: $f(x) = -2+6x-4x^2+(0.5)x^3$

$x_0 = 0$

$f'(x) = 6 - 8x + 1.5x^2$

$f(0) = -2 \qquad f'(0) = 6$

By Newton-Raphson method

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(-2)}{6}$$

$$= \frac{2}{6}$$

$$= 0.3333$$

$$\Delta x = x_1 - x_0 = 0.3333 - 0 = 0.3333$$

9) Newton-Raphson method is to be used to find root of equation $3x-e^x+\sin x=0$. If the initial trial value for the root is taken as 0.333, the next approximation for the root would be ____________.

(*note: answer up to three decimal*)

(a) 1.360 (b) 0.360
(c) 1.560 (d) 0.560

Key: (0.36)

Exp: Let $f(x) = 3x - e^x + \sin x$ and $x_0 = 0.333 \approx \frac{1}{3}$

$$\Rightarrow f'(x) = 3 - e^x + \cos x$$

$$f(x_0) = -0.069 \text{ and } f'(x_0) = 2.55$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \text{(Using Newton– Rapshon method)}$$

$$= 0.333 + \frac{0.069}{2.55} = 0.360 \text{ is the required next approximation}$$

10) The values of function f (x) at 5 discrete point are given below:

x	0	0.1	0.2	0.3	0.4
f(x)	0	10	40	90	160

Using Trapezoidal role with step size of 0.1, the value of $\int_0^{0.4} f(x)dx$ is __________

(a) 20 (b) 22
(c) 27 (d) 30

Answer: 22

Exp:

x	0	0.1	0.2	0.3	0.4
y = f(x)	0	10	40	90	160
	y_0	y_1	y_2	y_3	y_4

$$\int_0^{0.4} f(x)dx = \int_0^{0.4} ydx = \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \frac{0.1}{2}[(0+160) + 2(10+40+90)] = 22$$

(Questions 11-20 carry 2 marks each)

11) An explicit forward Euler method is used to numerically integrate the differential equation

$$\frac{dy}{dt} = y$$

using a time step of 0.1. With the initial condition $y(0) = 1$,t he value of y(1) computed by this method is _______(correct to two decimal places).

a) 2.59
b) 3.59
c) 4.59
d) 5.59

Ans. (2.5937)

$$y_1 = y_0 + hf(t_0, y_0)$$
$$= y_0 + hy_0$$
$$= 1 + 0.1\ (1)$$
$$y_1 = 1.1$$
$$y_2 = y_1 + hf(t_1,\ y_1)$$
$$= y_1 + h.y_1$$
$$= 1.1 + 0.1(1.1)$$
$$y_2 = 1.21$$
$$y_3 = y_2 + hf(t_2, y_2)$$
$$= y_2 + h.y_2$$
$$= 1.21 + 0.1 \times 1.21$$
$$y_3 = 1.331$$
$$y_4 = y_3 + h.f(t_3,\ y_3)$$
$$= y_3 + h.y_3$$
$$= 1.331 + 0.1 \times 1.331$$
$$y_4 = 1.4641$$
$$y_5 = y_4 + h.f(t_4,\ y_4)$$
$$= y_4 + h.y_4$$
$$= 1.4641 + 0.1 \times (1.4641)$$
$$= 1.61051$$
$$y_6 = y_5 + h.f(t_5,\ y_5)$$
$$= y_5 + h.y_5$$
$$= 1.61051 + 0.1 \times 1.61051$$
$$y_6 = 1.771561$$
$$y_7 = y_6 + h.f(t_6,\ y_6)$$
$$= y_6 + h \times y_6$$
$$= 1.771561 + 0.1 \times 1.771561 = 1.9487$$
$$y_8 = y_7 + h.f(t_7,\ y_7)$$
$$= y_7 + h.y_7$$
$$= 1.9487 + 0.1 \times (1.9487)$$
$$y_8 = 2.14357$$
$$y_9 = y_8 + h.f(t_8,\ y_8)$$
$$= y_8 + h.y_8 = 2.14357 + 0.1 \times 2.14357$$
$$y_9 = 2.3579$$
$$y_{10} = y_9 + h.f(t_9,\ y_9) = y_9 + h.y_9$$
$$= 2.3579 + 0.1 \times (2.3579)$$
$$y_{10} = 2.5937$$

12) Consider an ordinary differential equation $\frac{dx}{dt} = 4t + 4$ If $= x_0$ at t = 0, the increment in x calculated using Runge-Kutta fourth order multi-step method with a step size of Δt = 0.2 is

(A) 0.22 (B) 0.44 (C) 0.66 (D) 0.88

Answer : (D) 0.88

13) Newton-Raphson method is used to find the roots of the equation, $x^3 + 2x^2 + 3x - 1 = 0$. If the initial guess is $x_0 = 1$, then the value of x after 2nd iteration is ____________.

(a) 0.10 (b) 0.20
(c) 0.30 (d) 0.40

Answer: 0.30

Exp: By Newton-Raphson Method,

$$1^{st} \text{ iteration, } x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1 - \frac{f(1)}{f'(1)} = 1 - \frac{5}{10} = \frac{1}{2}$$

$$\text{Where } f(x) = x^3 + 2x^2 + 3x - 1 \Rightarrow f(1) = 5$$

$$f'(x) = 3x^2 + 4x + 3 \Rightarrow f'(1) = 10$$

$$2^{nd} \text{ iteration, } x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.5 - \frac{f(0.5)}{f'(0.5)} = 0.3043$$

14) The error in numerically computing the integral $\int_0 (\sin x + \cos x)dx$ using the trapezoidal rule with three intervals of equal length between 0 and π is_______ .

(a) 0.178 (b) 0.278
(c) 0.378 (d) 0.478

Key: 0.178

Exp: $h = \frac{b-a}{n} = \frac{\pi - 0}{3} = \frac{\pi}{3}; f(x) = \sin x + \cos x$

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π
$y = f(x)$	1	1.37	0.37	−1

By trapezoidal rule; we have the approximate value of the integral is

$$\int_0^\pi (\sin x + \cos x)dx = \frac{\pi/3}{2}\left[\left[1+(-1)\right] + 2(1.37+0.37)\right]$$
$$\cong 1.822$$

Exact value of the integral is

$$\int_0^\pi (\sin x + \cos x)dx = \left(-\cos x + \sin x\right)_0^\pi = 1-(-1) = 2$$

$$\therefore \text{Error} = \text{Exact value} - \text{Approximate value}$$
$$= 2 - 1.822 \cong 0.178$$

15) P(0,3), Q(0.5, 4), and R (1,5) are three points on the curve defined by f(x). Numerical integration is carried out using both Trapezoidal rule and Simpson's rule within limits x = 0 and x =1 for the curve. The difference between the two results will be.

(A) 0 (B) 0.25 (C) 0.5 (D) 1

Key: **(A)**

Exp: Let

x	0	0.5	1
y	3	4	5

Trapezoidial rule

$$\int_0^1 f(x)dx = \frac{0.5}{2}\left[(3+5)+2(4)\right] = \frac{0.5}{2} \times 16 = 4$$

Simpsons rule

$$\int_0^1 f(x)dx = \frac{0.5}{3}\left[(3+5)+0+4(4)\right] = \frac{0.5}{3} \times 24 = 4$$

Difference $= 0$

16) Gauss-Seidel method is used to solve the following equations (as per the given order):

$$x_1 + 2x_2 + 3x_3 = 5$$
$$2x_1 + 3x_2 + x_3 = 1$$
$$3x_1 + 2x_2 + x_3 = 3$$

Assuming initial guess as $x_1 = x_2 = x_3 = 0$, the value of x_3 after the first iteration is ____

(a) -5 (b) -6
(c) -7 (d) -8

Key: -6

Exp: $x_1^{(1)} - 0 - 0 = 5$ ...(1)

$2x_1^{(1)} + 3x_2^{(1)} + 0 = 1$ ...(2)

$3x_1^{(1)} + 2x_2^{(1)} + x_3^{(1)} = 3$ ...(3)

$\therefore$ From equation (1) $x_1^{(1)} = 5$

From equation (2), $2x_1^{(1)} + 3x_2^{(1)} = 1$

$$= 3x_2^{(1)} = 1 - 2x_1^{(1)}$$

$$= 1 - 2(5)$$

$$\Rightarrow 3x_2^{(1)} = -9$$

$$\Rightarrow x_2^{(1)} = \frac{-9}{3} = -3 \Rightarrow x_2^{(1)} = -3$$

From equation (3), $x_3^{(1)} = 3 - 3x_1^{(1)} - 2x_2^{(1)}$

$$= 3 - 3(5) - 2(-3)$$

$$= 3 - 15 + 6 = -6$$

$$\Rightarrow x_3^{(1)} = -6$$

$\therefore$ After the first iteration, the value of x_3 is -6.

17) For step-size $\Delta x = 0.4$, the value of following integral using Simpson's 1/3 rule is ________.

$$\int_0^{0.8} \left(0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5\right) dx$$

(a) -3.829 (b) -3.729
(c) -3.629 (d) -3.629

Answer: -3.8293

Exp: Given $h = \Delta x = 0.4$

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

$$x_0 = 0 \quad x_n = 0.8 \Rightarrow n = \frac{0.8 - 0}{0.4} = 0$$

x	0	0.4	0.8
$y = f(x)$	0.2	24.456	−126.744

By Simpson's $\frac{1}{3}$ Rule

$$\int_0^{0.8} f(x)\,dx = \frac{0.4}{3}\left[(0.2 - 126.744) + 4(24.456)\right] = -3.8293$$

18) Consider the equation $\frac{du}{dt} = 3t^2 + 1$ with $u = 0$ at $t = 0$. This is numerically solved by using the forward Euler method with a step size. $\Delta t = 2$. The absolute error in the solution at the end of the first time step is_________

(a) 5 (b) 6
(c) 7 (d) 8

Key: (8)

Exp: **Approximation value by Euler's Method:**

$$\frac{du}{dt} = 3t^2 + 1 \ ; \ u(0) = 0; \ h = \Delta t = 2$$

$$u(2) = u(0) + hf(0,0), \ f(u,t) = 3t^2 + 1$$

$$= 0 + 2(0 + 1) = 2$$

Exact value:

$$du = (3t^2 + 1)dt \qquad (\text{variable separable})$$

$$\Rightarrow u = t^3 + t + c \ \text{ is slоution}$$

$$u(0) = 0 \Rightarrow 0 = c$$

$$u = t^3 + t$$

$$u(2) = 8 + 2 = 10$$

$$\therefore \text{ absolute error } = |10 - 2| = 8$$

19) Simpson's $\frac{1}{3}$ rule is used to integrate the function $f(x) = \frac{3}{5}x^2 + \frac{9}{5}$ between x = 0 and x = 1 using the least number of equal sub-intervals. The value of the integral is ___________

(a) 0.0108 (b) 0.0208
(c) 0.0308 (d) 0.0408

Answer: 0.0208

Exp:

x	0	$\frac{1}{2}$	1
$y = f(x) = \frac{3}{5}x^2 + \frac{9}{5}$	$\frac{9}{5}$	$\frac{39}{20}$	$\frac{12}{5}$

$$\int_0^1 y\,dx = \frac{\left(\frac{1}{2}\right)}{2}\left[\left(\frac{9}{5} + \frac{12}{5}\right) + 4\left(\frac{39}{20}\right)\right]$$

$$= 0.0208$$

20) Function f is known at the following points:

x	0	0.3	0.6	0.9	1.2	1.5	1.8	2.1	2.4	2.7	3.0
f(x)	0	0.09	0.36	0.81	1.44	2.25	3.24	4.41	5.76	7.29	9.00

The value of $\int_0^3 f(x)\,dx$ computed using the trapezoidal rule is

(A) 8.983 (B) 9.003 (C) 9.017 (D) 9.045

Ans: (D)

Exp: $$\int_0^3 f(x)\,dx = \frac{h}{2}\left[f(x_0) + f(x_{10}) + 2\left(f(x_1) + f(x_2) + \ldots + f(x_9)\right)\right]$$

$$= \frac{0.3}{2}\left[9.00 + 2(25.65)\right] = 9.045$$