Linear Algebra Solutions

(Questions 1-10 carry 1 mark each)

1) The rank of the matrix $\begin{bmatrix} -4 & 1 & -1 \\ -1 & -1 & -1 \\ 7 & -3 & 1 \end{bmatrix}$ is

(a) 1 (b) 2
(c) 3 (d) 4

Ans. (b)

$$\begin{bmatrix} -4 & 1 & -1 \\ -1 & -1 & -1 \\ 7 & -3 & 1 \end{bmatrix}$$

$$R_1 \longleftrightarrow R_2 \qquad \begin{bmatrix} -1 & -1 & -1 \\ -4 & 1 & -1 \\ 7 & -3 & 1 \end{bmatrix}$$

$$R_2 - 4R_1, R_3 + 7R_1 \qquad \begin{bmatrix} -1 & -1 & -1 \\ 0 & 5 & 3 \\ 0 & -10 & -6 \end{bmatrix}$$

$$R_3 + 2R_2 \qquad \begin{bmatrix} -1 & -1 & -1 \\ 0 & 5 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

No. of non zero rows = 2

rank = 2

2) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ then $\det(A^{-1})$ is ________ (correct to two decimal places).

(a) 0 (b) 1/4
(c) 1/2 (d) 3/4

Ans. (0.25)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|A| = 4$$

$$|A^{-1}| = \frac{1}{|A|} = \frac{1}{4}$$

3) Consider matrix $A = \begin{bmatrix} k & 2k \\ k^2 - k & k^2 \end{bmatrix}$ and vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. The number of distinct real values of k for which the equation $AX = 0$ has infinitely many solution is _______.

(a) 1 (b) 2
(c) 3 (d) 4

Ans. (2)

$AX = 0$ has infinitely many solutions

So, $|A| = 0$

$$\begin{vmatrix} k & 2k \\ k^2 - k & k^2 \end{vmatrix} = 0$$

$$k^3 - 2k^3 + 2k^2 = 0$$

$$k^2(2 - k) = 0$$

$k = 0, 2 \Rightarrow$ "two" distinct values of k

4) Consider a matrix $A = uv^T$ where $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Note that v^T denotes the transpose of v. The largest eigenvalue of A is ________.

(a) 1 (b) 2
(c) 3 (d) 4

Ans. (3)

$$u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \; v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A = uv^T$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$(1 - \lambda)\,(2 - \lambda) - 2 = 0$$

$$\lambda^2 - 3\lambda = 0$$

$$\lambda(\lambda - 3) = 0$$

$$\lambda = 0$$

or, $\lambda = 3$

The largest eigen value is 3.

5) For the given orthogonal matrix Q,

$$Q = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix}$$

The inverse is

(a) $\begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix}$

(b) $\begin{bmatrix} -\frac{3}{7} & -\frac{2}{7} & -\frac{6}{7} \\ \frac{6}{7} & -\frac{3}{7} & -\frac{2}{7} \\ -\frac{2}{7} & -\frac{6}{7} & \frac{3}{7} \end{bmatrix}$

(c) $\begin{bmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix}$

(d) $\begin{bmatrix} -\frac{3}{7} & \frac{6}{7} & -\frac{2}{7} \\ -\frac{2}{7} & -\frac{3}{7} & -\frac{6}{7} \\ -\frac{6}{7} & -\frac{2}{7} & \frac{3}{7} \end{bmatrix}$

Ans. (c)

$$|Q| = \frac{3}{7}\left(-\frac{9}{49}-\frac{12}{49}\right)-\frac{2}{7}\left(\frac{18}{49}-\frac{4}{49}\right)+\frac{6}{7}\left(\frac{-36}{49}-\frac{6}{49}\right) = -1$$

$$\text{Adj. } Q = \begin{bmatrix} -\frac{21}{49} & \frac{42}{49} & -\frac{14}{49} \\ -\frac{14}{49} & -\frac{21}{49} & -\frac{42}{49} \\ -\frac{42}{49} & -\frac{14}{49} & \frac{21}{42} \end{bmatrix}$$

$$\therefore \qquad Q^{-1} = \frac{Adj\,Q}{|Q|} = \begin{bmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix}$$

Or $\because$ Q is orthogonal

$$\therefore \qquad Q^{-1} = Q^T$$

6) Which one of the following matrices is singular?

(a) $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 3 \\ 6 & 2 \end{bmatrix}$

Ans. (c)

Option (a): $|A| = 6 - 5 = 1$

Option (b): $|A| = 9 - 4 = 5$

Option (c): $|A| = 12 - 12 = 0$

Option (d): $|A| = 8 - 18 = -10$

Hence matrix (c) is singular.

7) The eigen values of symmetric matrix are all

(A) Complex with non-zero positive imaginary part

(B) Complex with non-zero negative imaginary part

(C) Real

(D) Pure imaginary

Answer: (C)

8) The solution to the system of equations

$$\begin{bmatrix} 2 & 5 \\ -4 & 3 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 2 \\ -30 \end{Bmatrix} \text{ is}$$

(A) 6, 2 (B) −6, 2 (C) −6, −2 (D) 6, −2

Key: (D)

Exp: By verification method; $\begin{bmatrix} 2 & 5 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ -2 \end{bmatrix} = \begin{Bmatrix} 2 \\ 30 \end{Bmatrix}$

9) The condition for which the eigen values of the matrix

$A = \begin{bmatrix} 2 & 1 \\ 1 & k \end{bmatrix}$ are positive, is

(A) $k > 1/2$ (B) $k > -2$ (C) $k > 0$ (D) $k < -1/2$

Key: (A)

Exp: By the properties of eigen values & eigen vectors, if all the principal minors of 'A' are +Ve then all the eigen values of 'A' are also +Ve.

$\therefore |A_{2\times2}| > 0 \text{ for } k > \frac{1}{2}$

So $k > \frac{1}{2}$

10) A real square matrix A is called skew-symmetric if

(A) $A^T = A$

(B) $A^T = A^{-1}$

(C) $A^T = -A$

(D) $A^T = A + A^{-1}$

Key: (C)

(Questions 11-20 carry 2 marks each)

11) The rank of the following matrix is $\begin{pmatrix} 1 & 1 & 0 & -2 \\ 2 & 0 & 2 & 2 \\ 4 & 1 & 3 & 1 \end{pmatrix}$

(a) 1
(b) 2
(c) 3
(d) 4

Ans. (b)

$$A = \begin{pmatrix} 1 & 1 & 0 & -2 \\ 2 & 0 & 2 & 2 \\ 4 & 1 & 3 & 1 \end{pmatrix}$$

$$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 4R_1$$

$$\begin{pmatrix} 1 & 1 & 0 & -2 \\ 0 & -2 & 2 & 6 \\ 0 & -3 & 3 & 9 \end{pmatrix}$$

$$R_3 \to R_3 - \frac{3}{2}R_2$$

$$\begin{pmatrix} 1 & 1 & 0 & -2 \\ 0 & -2 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

No. of non zero rows $= 2$

rank of $A = 2$

12) For a given matrix $\begin{bmatrix} 4-3i & i \\ -i & 4+3i \end{bmatrix}$, where is $i = \sqrt{-1}$, the inverse of matrix P is

(A) $\frac{1}{24}\begin{bmatrix} 4-3i & i \\ -i & 4+3i \end{bmatrix}$

(B) $\frac{1}{25}\begin{bmatrix} i & 4-3i \\ 4+3i & -i \end{bmatrix}$

(C) $\frac{1}{24}\begin{bmatrix} 4+3i & -i \\ i & 4-3i \end{bmatrix}$

(D) $\frac{1}{25}\begin{bmatrix} 4+3i & -i \\ i & 4-3i \end{bmatrix}$

Answer: (C)

Exp: $|P| = (4+3i)(4-3i)-(i)(-i) = 16+9-1 = 24$

$$\text{adj}\,P = \begin{bmatrix} 4-3i & -i \\ i & 4+3i \end{bmatrix}$$

$$\therefore P^{-1} = \frac{1}{24}\begin{bmatrix} 4-3i & -i \\ i & 4+3i \end{bmatrix}$$

13) The number of linearly independent eigenvectors of matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is ____ .

(a) 1
(b) 2
(c) 3
(d) 4

Key: 2

Exp: Here $\lambda = 2,2,3$

For $\lambda = 2$, No. of L.I eigen vectors

$$= 3 - \text{rank of}\,(A - 2I) = 3 - 2 = 1$$

For $\lambda = 3$, No. of L.I eigen vectors $=1$

$\therefore$ Total L.I eigen vectors = 2

14) Consider the matrix $A = \begin{bmatrix} 50 & 70 \\ 70 & 80 \end{bmatrix}$ whose eigenvectors corresponding to eigenvalues λ_1 and λ_2 are $x_1 = \begin{bmatrix} 70 \\ \lambda_1 - 50 \end{bmatrix}$ and $x_2 = \begin{bmatrix} \lambda_2 - 80 \\ 70 \end{bmatrix}$. respectively. The value of $x_1^T x_2$ is_________

(a) 1
(b) 2
(c) 3
(d) 4

Key: **0 to 0**

Exp: $A = \begin{bmatrix} 50 & 70 \\ 70 & 80 \end{bmatrix}$

Eigen vectors are $X_1 = \begin{pmatrix} 70 \\ \lambda_1 - 50 \end{pmatrix}$; $X_2 = \begin{pmatrix} \lambda_2 - 80 \\ 70 \end{pmatrix}$

Where λ_1, λ_2 Eigen values of A

$$X_1^T X_2 \left(70 \quad \lambda_1 - 50\right)\begin{pmatrix} \lambda_2 - 80 \\ 70 \end{pmatrix} = 70(\lambda_2 - 80) + (\lambda_1 - 50)70$$

$$= 70\lambda_2 - 5600 + 70\lambda_1 - 3500 = 70(\lambda_1 + \lambda_2) - 9100$$

$$= 70(130) - 9100 = 9100 - 9100 = 0$$

$$\begin{pmatrix} \therefore \text{sum of eigen values} = \lambda_1 + \lambda_2 \\ \text{Trace} = 50 + 80 = 130 \end{pmatrix}$$

15) Consider the matrix $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$.

Which one of the following statements about P is INCORRECT?

(A) Determinant of P is equal to 1.

(B) P is orthogonal.

(C) Inverse of P is equal to its transpose.

(D) All Eigen values of P are real numbers

Key: (D)

Exp: $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$

$$|P| = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} - 0\right) - 0 + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2} + \frac{1}{2} = 1$$

$$P.P^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ P is an orthogonal matrix

(A) Is correct $\Rightarrow$ Inverse of P is its transpose only

$\therefore$ (B) and (C) both are correct

$\therefore$ (D) is incorrect

16) The smallest and largest Eigen values of the following matrix are:

$$\begin{bmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{bmatrix}$$

(A) 1.5 and 2.5 (B) 0.5 and 2.5 (C) 1.0 and 3.0 (D) 1.0 and 2.0

Answer: (D)

Exp: Let $A = \begin{bmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{bmatrix}$

Characteristic equation is

$|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -2 & 2 \\ 4 & -4-\lambda & 6 \\ 2 & -3 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$$

$$\Rightarrow (\lambda-1)(\lambda^2 - 3\lambda + 2) = 0$$

$$(\lambda-1)(\lambda-1)(\lambda-2) = 0$$

$$\lambda = 1, 2$$

17) The two Eigen values of the matrix $\begin{bmatrix} 2 & 1 \\ 1 & p \end{bmatrix}$ have a ratio of 3 : 1 for $p = 2$. What is another value of p for which the Eigen values have the same ratio of 3 : 1?

(A) –2 (B) 1 (C) 7/3 (D) 14/3

Answer: (D)

Exp: Let $A = \begin{bmatrix} 2 & 1 \\ 1 & p \end{bmatrix}$

Given that two eigen values of A are in 3:1

Ratio for p = 2

$\Rightarrow$ Characteristic equation λ^2-4λ+3 = 0 (by substituting p=2)

$\Rightarrow \lambda=1,3$

If we take $p = \frac{14}{3}$ then $A = \begin{bmatrix} 2 & 1 \\ 1 & \frac{14}{3} \end{bmatrix}$

$$\Rightarrow \lambda^2 - \left(2 + \frac{14}{3}\right)\lambda + \left(\frac{28}{3} - 1\right) = 0$$

$$\Rightarrow \lambda^2 - \frac{20}{3}\lambda + \frac{25}{3} = 0$$

$$\Rightarrow 3\lambda^2 - 20\lambda + 25 = 0$$

$$\lambda = 5, \frac{5}{3}$$

Eigen values $5, \frac{5}{3}$ are in ratio 3:1

$\therefore p = \frac{14}{3}$

18) Consider the following linear system.

$x + 2y - 3z = a$

$2x + 3y + 3z = b$

$5x + 9y - 6z = c$

This system is consistent if a,b and c satisfy the equation

(A) $7a - b - c = 0$ (B) $3a + b - c = 0$

(C) $3a - b + c = 0$ (D) $7a - b + c = 0$

Key: (B)

19) Consider the matrix $\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}$. Which one of the following statements is TRUE for the eigenvalues and eigenvectors of this matrix?

(A) Eigenvalue 3 has a multiplicity of 2, and only one independent eigenvector exists.

(B) Eigenvalue 3 has a multiplicity of 2, and two independent eigenvectors exist.

(C) Eigenvalue 3 has a multiplicity of 2, and no independent eigenvector exists.

(D) Eigenvalues are 3 and -3, and two independent eigenvectors exist.

Key: (A)

Exp: Let $A = \begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}$

Characteristic equations is $\lambda^2 - 6\lambda + 9 = 0 \Rightarrow \lambda = 3, 3$

Eigen value 3 has multiplicity 2.

Eigen vectors corresponding to $\lambda = 3$ is $(A - 3I)X = 0$

$$\begin{pmatrix} 5-3 & -1 \\ 4 & 1-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$R_2 \to R_2 - 2R_1 \Rightarrow \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$e(A) = 1$

Number of linearly independent eigen vectors corresponding to eigen value $\lambda = 3$ is

n-r=2-1=1 where n= no. of unknowns, r= rank of $(A - \lambda I)$

$\therefore$ One linearly independent eigen vector exists corresponding to $\lambda = 3$

20) If $A=\begin{bmatrix}1 & 5\\ 6 & 2\end{bmatrix}$ and $B=\begin{bmatrix}3 & 7\\ 8 & 4\end{bmatrix}$, AB^T is equal to

(A) $\begin{bmatrix}38 & 28\\ 32 & 56\end{bmatrix}$ (B) $\begin{bmatrix}3 & 40\\ 42 & 8\end{bmatrix}$

(C) $\begin{bmatrix}43 & 27\\ 34 & 50\end{bmatrix}$ (D) $\begin{bmatrix}38 & 32\\ 28 & 56\end{bmatrix}$

Key: (A)

Exp: $A=\begin{pmatrix}1 & 5\\ 6 & 2\end{pmatrix}$ $B=\begin{pmatrix}3 & 7\\ 8 & 4\end{pmatrix}$

$$AB^T=\begin{pmatrix}1 & 5\\ 6 & 2\end{pmatrix}\begin{pmatrix}3 & 8\\ 7 & 4\end{pmatrix}$$

$$=\begin{bmatrix}38 & 28\\ 32 & 56\end{bmatrix}$$

EXTRA:

21. The matrix P is the inverse of a matrix Q. If I denotes the identity matrix, which one of the following options is correct?

(A) $PQ = I$ but $QP \neq I$ (B) $QP = I$ but $PQ \neq I$

(C) $PQ = I$ and $QP = I$ (D) $PQ - QP = I$

Key: (C)

Exp: Given P is inverse of Q

$\Rightarrow PQ = QP = I$