Probability and Statistics Solutions

(Questions 1-10 carry 1 mark each)

1) Four red balls, four green balls and four blue balls are put in a box. Three balls are pulled our of the box at random one after another without replacement. The probability that all the three balls are red is

(a) 1/72 (b) 1/55

(c) 1/36 (d) 1/27

Ans: **(b)**

Probability that all the three balls are red is

$$= R \cdot R.R.$$

$$= \frac{4}{12} \times \frac{3}{11} \times \frac{2}{10} = \frac{24}{1320} = \frac{1}{55}$$

~~4R~~	4G	4B
~~3R~~		
2R		

2) A six-faced fair dice is rolled five times. The probability (in%) of obtaining "ONE" at least four times is

(a) 33.3 (b) 3.33

(c) 0.33 (d) 0.0033

Ans: **(c)**

A dice is rolled 5 times

$$n = 5$$

$$P = \text{(Probability of getting 1)} = \frac{1}{6}$$

$$q = 1 - \frac{1}{6} = \frac{5}{6}$$

Probability of getting 1 at least 4 times is

$$P(x \geq 4) = P(x = 4) + P(x = 5)$$

$$= {}^{n}C_{4} p^{4} q^{n-4} + {}^{n}C_{5} p^{5} q^{n-5}$$

$$= {}^{5}C_{4} p^{4} q^{1} + {}^{5}C_{5} p^{5} q^{0}$$

$$= 5 \times \left(\frac{1}{6}\right)^{4} \left(\frac{5}{6}\right)^{1} + 1 \times \left(\frac{1}{6}\right)^{5} \times \left(\frac{5}{6}\right)^{0}$$

$$= \frac{25}{(6)^5} + \frac{1}{(6)^5} = \frac{26}{(6)^5} = 0.0033$$

$$\% \text{ probability} = 0.33\%$$

3) The arrival of customers over fixed time intervals in a bank follow a Poisson distribution with an average of 30 customers/hour. The probability that the time between successive customer arrival is between 1 and 3 minutes is ________ (correct to two decimal places).

(a) 0.28 (b) 0.38

(c) 0.48 (d) 0.58

Ans. **(0.383)**

Given, arrival rate, $\lambda = 30/\text{hour}$

$$\lambda = \frac{1}{2} \text{ min.}$$

$$P = \text{prob.} = 1 - e^{-\lambda t}$$

$$P(1) = 1 - e^{-\frac{1}{2}\times 1} = 0.393$$

$$P(3) = 1 - e^{-\lambda t} = 1 - e^{-\frac{1}{2}\times 3}$$

$$= 1 - e^{-1.5} = 0.7768$$

$$P(1 \leq T \leq 3 \text{ min}) = 0.7768 - 0.393 = 0.383$$

4) Let X_1 and X_2 be two independent exponentially distributed random variables with means 0.5 and 0.25 respectively. Then $Y = \min(X_1, X_2)$ is

(a) exponentially distributed with mean $\frac{1}{6}$

(b) exponentially distributed with mean 2

(c) normally distributed with mean $\frac{3}{4}$

(d) normally distributed with mean $\frac{1}{6}$

Ans. **(a)**

$$\text{Mean } (x_1) = 0.5$$

$$\frac{1}{\lambda_1} = 0.5$$

$$\lambda_1 = \frac{1}{0.5} = 2$$

$$\text{Mean } (x_2) = 0.25$$

$$\frac{1}{\lambda_2} = 0.25$$

$$\lambda_2 = \frac{1}{0.25} = 4$$

$$y = \text{mean } (x_1, x_2)$$

$$\text{Mean } (y) = \frac{1}{\lambda_1 + \lambda_2} = \frac{1}{2+4} = \frac{1}{6}$$

5) Let X_1, X_2, X_3 and X_4 be independent normal random variables with zero mean and unit variance. The probability that X_4 is the smallest among the four is _________.

(a) 0.0 (b) 0.25
(c) 0.50 (d) 1.0

Ans. (0.25)

$$P(X_4 \text{ is smallest}) = \frac{3!}{4!} = \frac{1}{4} = 0.25$$

6) A random variable X takes values –0.5 and 0.5 with probabilities $\frac{1}{4}$ and $\frac{3}{4}$, respectively. The noisy observation of X is $Y = X + Z$, where Z has uniform probability density over the interval (–1, 1). X and Z are independent. If the MAP rule based detector outputs $\hat{X}$ as

$$\hat{X} = \begin{cases} -0.5, & Y < \alpha \\ 0.5, & Y \geq \alpha, \end{cases}$$

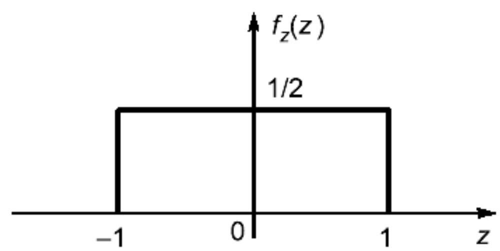
then the value of α (accurate to two decimal places) is ______.

(a) 0.0 (b) -0.25
(c) -0.50 (d) -1.0

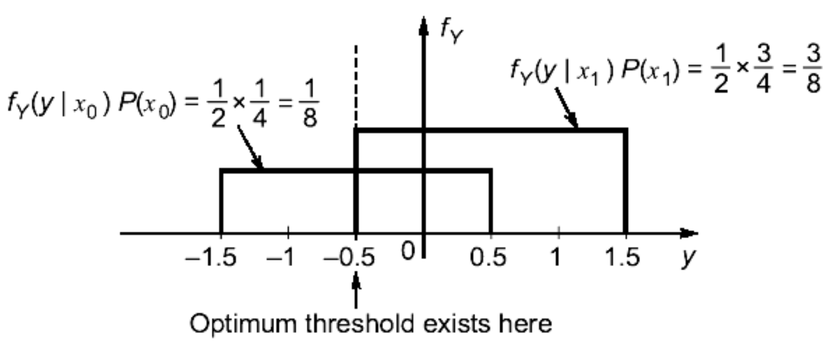
Ans. (–0.50)

$$P(x_0) = \frac{1}{4}$$

$$P(x_1) = \frac{3}{4}$$



MAP criteria, $f_Y(y \mid x_0)\, P(x_0) \underset{x_1}{\overset{x_0}{\gtrless}} f_Y(y \mid x_1) P(x_1)$



So, $\alpha = -0.50$

7) Probability (up to one decimal place) of consecutively picking 3 red balls without replacement from a box containing 5 red balls and 1 white ball is ________.

(a) 0.2 (b) 0.3

(c) 0.4 (d) 0.5

Ans. (0.5)

Probability, $\bar{P} = \frac{5}{6} \times \frac{4}{5} \times \frac{3}{4} = \frac{1}{2} = 0.5$

8) Let X be a nominal variable with mean 1 and variance 4. The probability P (X < 0) is

(A) 0.5

(B) Greater than zero and less than 0.5

(C) Greater than 0.5 and less than 1

(D) 1.0

Answer: (B)

Explanations:- $P(x < 0) = P\left(\frac{x-\mu}{\sigma} < \frac{0-\mu}{\sigma}\right) - P(Z < -0.5)$

$= P(Z > 0.5) = 0.5 - P(0 < Z < 0.5)$,

which is greater than zero and less than 0.5

9) A sample of 15 data is a follows: 17, 18, 17, 17, 13, 18, 5, 5, 6, 7, 8, 9, 20, 17, 3. The mode of the data is

(A) 4 (B) 13 (C) 17 (D) 20

Key: (C)

Exp: We know that mode is the value of the data which occurred most of

$\therefore 17$ is mode

10) There vendors were asked to supply a very high precision component. The respective probabilities of their meeting the strict design specifications are 0.8, 0.7 and 0.5. Each vendor supplies one component. The probability that out of total three components supplied by the vendors, at least one will meet the design specification is __________

(a) 0.87 (b) 0.97

(c) 0.77 (d) 0.17

Answer: 0.97

Exp: Probability (at least one will meet specification) = 1-probability (none will meet specification)

$= 1-(1-0.8)\times(1-0.7)\times(1-0.5)$

$= 1-0.2\times0.3\times0.5$

$= 1-0.03$

$= 0.97$

(Questions 11-20 carry 2 marks each)

11) The probability that a screw manufactured by a company is defective is 0.1. The company sells screws in packets containing 5 screws and gives a guarantee of replacement if one or more screws in the packet are found to be defective. The probability that a packet would have to be replaced is ______________.

(a) 0.409 (b) 0.509
(c) 0.609 (d) 0.709

Key: 0.40951

Exp: Probability that a packet would have to be replaced i.e., $P[X \ge 1] = ?$ [Let 'x' denote the number of defective screws]

$$\Rightarrow P[X \ge 1] = 1 - P[X < 1]$$
$$= 1 - P[X = 0]$$
$$= 1 - 5C_0 (0.1)^0 (0.9)^5$$
$$= 1 - (0.9)^5 \approx 0.40951$$

Since by the Binomial distribution when P=probability of defective screw.

12) Three cards were drawn from a pack of 52 cards. The probability that they are a king, a queen, and a jack is

(A) $\frac{16}{5525}$ (B) $\frac{64}{2197}$ (C) $\frac{3}{13}$ (D) $\frac{8}{16575}$

Key: (A)

Exp: Required probability $= \frac{{}^4C_1 \times {}^4C_1 \times {}^4C_1}{{}^{52}C_3} = \frac{64}{22100} = \frac{16}{5525}$

13) The probability of obtaining at least two "SIX" in throwing a fair dice 4 time is

(A) 425/432 (B) 19/144 (C) 13/144 (D) 125/432

Answer: (B)

Exp: $n = 4; \quad p = \frac{1}{6}$

$$\Rightarrow q = 1 - \frac{1}{6} = \frac{5}{6}$$
$$p(x \ge 2) = 1 - p(x < 2)$$
$$= 1 - [p(x = 0) + p(x = 1)]$$
$$= 1 - \left[4C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^4 + 4C_1 \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^3\right] = \frac{19}{144}$$

14) For probability density function of a random variable, x is

$$f(x) = \frac{x}{4}\left(4 - x^2\right) \text{for } 0 \le x \le 2$$

$$= 0 \text{ otherwise}$$

The mean μ_z of the random varialbe is _____

(a) 16/15 (b) 15/16

(c) 8/15 (d) 15/8

Answer: 1.0667

Exp: $f(x) = \frac{x}{4}\left(4 - x^2\right) \quad 0 \le x \le 2$

$$\text{mean} = \mu_x = E(x)$$

$$= \int_0^2 x f(x)\, dx$$

$$= \int_0^2 x\left(\frac{x}{4}\right)\left(4 - x^2\right) dx$$

$$= \frac{1}{4}\int_0^0 \left(4x^2 - x^4\right) dx$$

$$= \frac{1}{4}\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2$$

$$= \frac{1}{4}\left[4.\frac{8}{3} - \frac{32}{5}\right]$$

$$= \frac{32}{4}\left[\frac{1}{3} - \frac{1}{5}\right]$$

$$= 8\left[\frac{2}{15}\right] = \frac{16}{15} = 1.0667$$

15) Probability density function of a random variable X is given below

$$f(x)=\begin{cases}0.25 & \text{if } 1\le x\le 5\\ 0 & \text{otherwise}\end{cases}$$

$P(X \le 4)$ is

(A) $\frac{3}{4}$ (B) $\frac{1}{2}$ (C) $\frac{1}{4}$ (D) $\frac{1}{8}$

Key: (A)

Exp: $$P(x\le 4)=\int_{-\infty}^{4} f(x)dx=\int_{-\infty}^{1}(0)dx+\int_{1}^{4}(0.25)\,dx+\int_{4}^{\infty}(0)dx$$

$$=\frac{1}{4}(x)_1^4=\frac{1}{4}(4-1)=\frac{3}{4}$$

16) If f(x) and g(x) are two probability density functions.

$$f(x) = \begin{cases} \frac{x}{a} + 1 : -a \le x < 0 \\ -\frac{x}{a} + 1 : 0 \le x \le a \\ 0 : \text{otherwise} \end{cases}$$

$$g(x) = \begin{cases} -\frac{x}{a} : a \le x < 0 \\ \frac{x}{a} : 0 \le x \le a \\ 0 : \text{otherwise} \end{cases}$$

Which one of the following statements is true?

(A) Mean of $f(x)$ and $g(x)$ are same; Variance of $f(x)$ and $g(x)$ are same

(B) Mean of $f(x)$ and $g(x)$ are same; Variance of $f(x)$ and $g(x)$ are different

(C) Mean of $f(x)$ and $g(x)$ are different; Variance of $f(x)$ and $g(x)$ are same

(D) Mean of $f(x)$ and $g(x)$ are different; Variance of $f(x)$ and $g(x)$ are different

Key: (B)

Exp: Mean of f(x) is $E(x) = \int_{-a}^{0} x.\left(\frac{x}{a}+1\right)dx + \int_{0}^{a} x.\left(\frac{-x}{a}+1\right)dx$

$$= \left(\frac{x^3}{3a} + \frac{x^2}{2}\right)_{-a}^{0} + \left(\frac{-x^3}{3a} + \frac{x^2}{2}\right)_{0}^{a} = 0$$

Variance of f(x) is $E\left(x^2\right) - \{E(x)\}^2$ where

$$E(x^2) = \int_{-a}^{0} x^2.\left(\frac{x}{a}+1\right)dx + \int_{0}^{a} x^2.\left(\frac{-x}{a}+1\right)dx$$

$$= \left(\frac{x^4}{4a} + \frac{x^3}{3}\right)_{-a}^{0} + \left(\frac{-x^4}{4a} + \frac{x^3}{3}\right)_{0}^{a} = \frac{a^3}{6}$$

$\Rightarrow$ Variance is $\frac{a^3}{6}$

Next, mean of g(x) is $E(x)=\int_{a}^{0} x.\left(\frac{-x}{a}\right)dx+\int_{0}^{a} x.\left(\frac{x}{a}\right)dx=0$

Variance of g(x) is $E(x^2)-\{E(x)\}^2$, where

$E(x^2)=\int_{-a}^{0} x^2.\left(\frac{-x}{a}\right)dx+\int_{0}^{a} x^2.\left(\frac{x}{a}\right)dx=\frac{a^3}{2}$

$\Rightarrow$ Variance is $\frac{a^3}{2}$

$\therefore$ Mean of f(x) and g(x) are same but variance of f(x) and g(x) are different.

17) For the function $f(x)=a+bx,\ 0\le x\le 1,$ to be a valid probability density function, which one of the following statements is correct?

(A) a = 1, b = 4 (B) a = 0.5, b = 1 (C) a = 0, b = 1 (D) a = 1, b = -1

Key: (B)

Exp: $f(x)=a+bx \quad 0\le x\le 1$ is a valid probability density function

i.e., $\int_0^1 f(x)dx=1$

$\int_0^1 (a+bx)\,dx=1$

$\left[ax+\frac{bx^2}{2}\right]_0^1=1\Rightarrow a+\frac{b}{2}=1$

$\Rightarrow 2a+b=2$

a=0.5, b=1 satisfies the above relation

18) X and Y are two random independent events. It is known that P(X) = 0.40 and $P\left(X\cup Y^C\right)=0.7.$ Which one of the following is the value of $P(X\cup Y)$?

(A) 0.7 (B) 0.5 (C) 0.4 (D) 0.3

Key: (A)

Exp: $P\left(X \cup Y^{c}\right)=0.7 \Rightarrow P(x)+P(y^{c})-P(x).P(y^{c})=0.7$

(Since, x,y are independent events)

$\Rightarrow P(x)+1-P(y)-P(x)\{1-P(y)\}=0.7$

$\Rightarrow P(y)-P(x \cap y)=0.3---(1)$

$P(x \cup y)=P(x)+P(y)-P(x \cap y)=0.4+0.3=0.7 \ 3$

Second Method:

We know that $\Rightarrow P(x \cup y^{'})=P(x)+P\left[(x \cup y)^{'}\right]$

$\Rightarrow 0.7=0.4+1-P(x \cup y)$

$\Rightarrow P(x \cup y)=0.7$

19) A two-faced fair coin has its faces designated as head (H) and tail (T). This coin is tossed three times in succession to record the following outcomes: H, H, H. If the coin is tossed one more time, the probability (up to one decimal place) of obtaining H again, given the previous realizations of H, H and H, would be________

(a) 0 (b) 0.5

(c) 0.25 (d) 0.75

Key: **(0.5)**

Exp: Given first three are already heads. If the coin is tossed again, the outcome does not depends on previous outcomes.

Probability getting head $==\frac{1}{2}=0.5$

(or)

probability of first three are heads

$=P(H \times H \times H)=\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}=\frac{1}{8}$

Probability of fourth time head is

$=P(H \times H \times H \times H)=\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}=\frac{1}{16}$

Given condition is that $(H.H.H)$ is already realized

The required probability $=\frac{1/16}{1/8}=0.5$

20) Two people. P and Q, decide to independently roll two identical dice, each with 6 faces. Numbered 1 to 6. The person with the lower number wins. In case of a tie, they roll the dice repeatedly until there is no tie. Define a trial as a throw of the dice by P and Q. Assume that all 6 numbers on each dice are equi-probable and that all trials are independent. The probability (rounded to 3 decimal places) that one of them wins on the third trial is ________.

(a) 0.123 (b) 0.223

(c) 0.023 (d) 0.333

Ans. (0.023)

P(one of them wins in 3rd trial)

= P(Ist trial is Tie) × P(IInd trial is Tie) × P(one of them wins 3rd trial)

P(Tie in any trial) = P(P=1 and Q=1) + P(P=2 and Q=2) + ... + P(P=6 and Q=6)

$$= \frac{1}{36}+\frac{1}{36}+\frac{1}{36}+\frac{1}{36}+\frac{1}{36}+\frac{1}{36}$$

$$= \frac{6}{36}=\frac{1}{6}$$

P(one of them wins) = 1 − P(Tie)

$$= 1-\frac{1}{6}=\frac{5}{6}$$

So required probability $= \frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}=\frac{5}{216}=0.023$ (rounded to 3 decimal places)

Extra

1. The spot speeds (expressed in km/hr) observed at a road section are 66, 62, 45, 79, 32, 51, 56, 60, 53, and 49. The median speed (expressed in km/hr) is _______.

(Note: answer with one decimal accuracy)

Key: (54.5)

Exp. Median speed is the speed at the middle value in series of spot speeds that are arranged in ascending order. 50% of speed values will be greater than the median 50% will be less than the median.

Ascending order of spot speed studies are

32,39,45,51,53,56,60,62,66,79

Median speed $=\frac{53+56}{2}=54.5\text{ km / hr}$