State Sum Invariants of Links and Manifolds

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May 24, 2009

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In this section we will study the theory behind the decomposition of tensor products of irreducible representations of an algebra $U\mathfrak{sl}_2$ into direct sums of irreducibles. This has physical meaning in the theory of angular momentum, but we do not discuss this aspect. There is a diagrammatic aspect to this theory, which will be useful for calculations and for when we discuss the quantum analogue of these ideas. The diagrammatical approach was first developed by Penrose in his *spin networks*.

2.1 The Algebra $U\mathfrak{sl}_2$ and its Representations

Let us review some basic facts about the Lie algebra $\mathfrak{sl}_2$ and the representation theory of its universal enveloping algebra $U\mathfrak{sl}_2$. It is the $\mathbb{C}$-vector space generated by elements E, F, H and subject to the relations

$$[H, E] = 2E \qquad [H, F] = -2F \qquad [E, F] = H$$

Since the representation theory of $\mathfrak{sl}_2$ is identical to that of its universal enveloping algebra we will actually concern ourselves with this algebra most of the time. Recall that this is the associative $\mathbb{C}$-algebra generated by E, F, H and subject to the relations

$$HE - EH = 2E \qquad HF - FH = -2F \qquad EF - FE = H$$

It is useful to first know that $U\mathfrak{sl}_2$'s representation theory is semi-simple; that is, every representation decomposes as a direct sum of irreducibles.

Theorem 2.1. *Every representation of $U\mathfrak{sl}_2$ is completely reducible, i.e. is a direct sum of irreducibles.*

Proof. The easiest way to see this is to observe that the representation theory of $U\mathfrak{sl}_2$ is identical to the representation theory of $SU(2)$ (as a Lie group). Since $SU(2)$ is a compact Lie group we can equip it with a Haar measure so that every representation of $SU(2)$ is actually unitary, and every unitary representation is completely reducible. $\square$

The most obvious representation of $U\mathfrak{sl}_2$ is the trivial 1-dimensional representation $V_0 = \mathbb{C}$, where E, F, H all act as multiplication by 0, and it is clearly irreducible. There is a more interesting representation of $U\mathfrak{sl}_2$ on a 2-dimensional vector space $V_1 = \mathbb{C}e_0 \oplus \mathbb{C}e_1$ by realizing that we can identify $\mathfrak{sl}_2$ with $T_I\, SL(2)$, the traceless complex matrices. So we let E, F, H act as multiplication by the following matrices

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We can easily compute the relations

$$
\begin{array}{llll}
Ee_0 & = & 0 & \quad Ee_1 & = & e_0 \\
Fe_0 & = & e_1 & \quad Fe_1 & = & 0 \\
He_0 & = & e_0 & \quad He_1 & = & -e_1
\end{array}
$$

This representation is clearly irreducible, for any non-zero sub-representation must contain one of e_0 or e_1 either of which generates all of V_1. From this representation, often called the fundamental

representation, we can construct many representations by taking tensor powers. We extend the action of E, F, H on V_1 to the tensor power $V_1^{\otimes k}$ by forcing E, F, H to act like a derivation

$$Ev_{\epsilon_1} \otimes \cdots \otimes v_{\epsilon_k} = \sum_{i=1}^{k} v_{\epsilon_1} \otimes \cdots \otimes Ev_{\epsilon_i} \otimes \cdots \otimes v_{\epsilon_k} \quad , \epsilon_i = 0 \text{ or } 1$$

and so on. For $k \geq 2$ these representations will *never* be irreducible. For example, in $V_1 \otimes V_1$ there is a sub-representation isomorphic to the trivial representation by taking the subspace generated by $e_0 \otimes e_1 - e_1 \otimes e_0$. One can easily check that E, F, H act trivially on this vector, hence it generates a sub-representation isomorphic to V_0.

Let us define a new family of representations $V_k := S^k(V_1)$ by taking the k-th symmetric power of V_1. The action of E, F, H on the k-th tensor power of V_1 descends to the quotient by symmetric tensors, so we get a well-defined action of E, F, H on V_k. Clearly V_k is of dimension $(k+1)$ over $\mathbb{C}$. We can think of elements of V_k as homogeneous, symmetric polynomials in e_0, e_1 of degree k, hence it is linearly spanned by the monomials $e_i := e_0^i e_1^{k-i}$, $i = 0, \ldots, k$. We can easily write down the actions of E, F, H on this basis:

$$\begin{array}{rcl} Ee_i & = & (k-i)e_{i+1} \\ Fe_i & = & ie_{i-1} \\ He_i & = & (2i-k)e_i \end{array} \qquad (2.1)$$

All of the representations V_k are irreducible of dimension $(k+1)$. Any non-zero sub-representation of V_k must contain one of the vectors e_i, and by applying E and F repeatedly we can obtain the other e_j's (see equation 2.1), hence V_k is irreducible.

For any representation V of $U\mathfrak{sl}_2$, a **weight vector** of V is an eigenvector v of the action of H, and v's eigenvalue is called its **weight**. For example, the representation V_k has each e_i as a weight vector of weight $(2i - k)$ (see equation 2.1). For a weight $\lambda \in \mathbb{C}$ of V, let $V[\lambda] \subseteq V$ denote the subspace of weight vectors of weight λ. A weight λ for V such that $\operatorname{Re}\lambda \geq \operatorname{Re}\lambda'$ for every other weight λ' of V is called a **highest weight** of V, and the vectors in $V[\lambda]$ are called the **highest weight vectors**.

Lemma 2.1. *For any representation V of $U\mathfrak{sl}_2$ and weight λ we have*

$$EV[\lambda] \subset V[\lambda + 2]$$
$$FV[\lambda] \subset V[\lambda - 2]$$

Proof. Let $v \in V[\lambda]$, then

$$HEv = EHv + 2Ev = \lambda Ev + 2Ev = (\lambda + 2)Ev$$

hence $Ev \in V[\lambda + 2]$. A similar argument works for F. $\qquad\square$

Lemma 2.2. *Let W be any finite dimensional representation of $U\mathfrak{sl}_2$ with a weight vector v of weight λ such that $Ev = 0$. Then λ is a non-negative integer, and W contains a sub-representation isomorphic to the irreducible representation V_λ such that the highest weight vector e_0^λ corresponds to v.*

Proof. $\qquad\square$

An easy corollary of this lemma is the following classification of $U\mathfrak{sl}_2$-representations.

Corollary 2.1. *Every irreducible representation of $U\mathfrak{sl}_2$ is isomorphic to some V_k $(k \geq 0)$.*

Proof. $\qquad\square$

2.2 A Diagrammatic Calculus for the Representation Theory of $U\mathfrak{sl}_2$

There is a diagrammatic way of describing intertwining maps between tensor powers of the fundamental representation V_1 of $U\mathfrak{sl}_2$. This was originally invented by Roger Penrose in his *spin networks*, and later generalized to a q-deformed recoupling theory by Louis Kauffman. We begin with two simple maps:

$$\cup : \mathbb{C} \to V_1 \otimes V_1$$

$$\cap : V_1 \otimes V_1 \to \mathbb{C}$$

defined by

$$\cup(1) = \sqrt{-1}(e_0 \otimes e_1 - e_1 \otimes e_0)$$

$$\cap(e_0 \otimes e_0) = \cap(e_1 \otimes e_1) = 0$$

$$\cap(e_0 \otimes e_1) = \sqrt{-1} = -\cap(e_1 \otimes e_0)$$

We also define $| : V_1 \to V_1$ to be the identity, and $\times : V_1 \otimes V_1 \to V_1 \otimes V_1$ to be the flip map $\times (a \otimes b) = b \otimes a$.

Lemma 2.3. *The maps* $\cup, \cap, |, \times$ *are all intertwiners of the $U\mathfrak{sl}_2$ action on V_1, and they satisfy the following properties:*

1. *(Fundamental Binor Identity)* $\times = | \otimes | + \cup \circ \cap$

2. *(Zig-Zag Identity)* $(| \otimes \cap) \circ (\cup \otimes |) = | = (\cap \otimes |) \circ (| \otimes \cup)$

3. $(\cap \otimes |) \circ (| \otimes \times) = (| \otimes \cap) \circ (\times \otimes |)$

4. $(\times \otimes |) \circ (| \otimes \cup) = (| \otimes \times) \circ (\cup \otimes |)$

5. *(Circle Evaluation)* $\cap \circ \cup$ *is the linear map* $\mathbb{C} \to \mathbb{C}$ *given by* $z \mapsto -2z$.

Proof. The proof of these statements is just routine computations. $\square$

Consider an immersion of a 1-manifold C (with or without boundary) into the manifold $\mathbb{R} \times [0,1]$, such that the image of ∂C lies in $\mathbb{R} \times \{0, 1\}$ (see figure 2.1). If we take a generic immersion, then we can divide the interval $[0,1]$ into pieces $c_1 = 0 < c_1 < \cdots < c_{n-1} < c_n = 1$ such that the piece of C lying in $\mathbb{R} \times (c_{i-1}, c_i)$ looks like all vertical strands $|| \cdots ||$, or many vertical strands with a cup $|| \cdots | \cup | \cdots ||$, a cap $|| \cdots | \cap | \cdots ||$ or a crossing $|| \cdots | \times | \cdots ||$. We will consider such immersions only up to *regular homotopy*. That is, we consider two immersions $\varphi_0, \varphi_1 : C \to \mathbb{R} \times [0,1]$ equivalent if there is a homotopy φ_t between φ_0 and φ_1 such that φ_t is an immersion for all t. Such homotopies are generated by an immersed version of the Reidemeister II move $\times \leftrightarrow)($ and Reidemeister III move $\times \leftrightarrow \times$. Note that we do not have a Reidemeister I move $) \leftrightarrow \wp$, but a consequence of the immersed Reidemeister II move is the following: $) \leftrightarrow \wp$.

To each immersion of a 1-manifold into $\mathbb{R} \times [0,1]$ we will associate an intertwiner of tensor powers of V_1. In particular, suppose a generic immersion as n boundary points on $\mathbb{R} \times 0$ and n boundary points on $\mathbb{R} \times 1$. Then we associate an intertwiner $V_1^{\otimes n} \to V_1^{\otimes m}$ in the following way. Divide the interval $[0,1]$ into pieces as before so that each slice of the immersion consists of a bunch
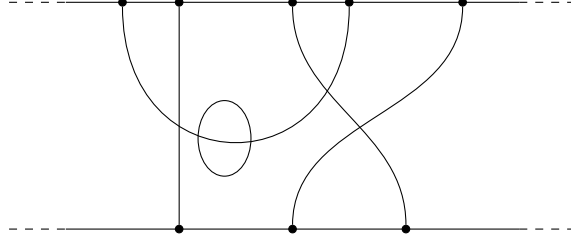
Figure 2.1: An immersed 1-manifold in $\mathbb{R} \times [0,1]$

of vertical lines and at most one cup, cap or crossing. To each such slice we associate the obvious intertwiners:

$$
\begin{array}{rcl}
|\cdots| & \mapsto & \mathrm{id} \otimes \cdots \otimes \mathrm{id} \\
|\cdots|\cup|\cdots| & \mapsto & \mathrm{id} \otimes \cdots \otimes \cup \otimes \cdots \otimes \mathrm{id} \\
|\cdots|\cap|\cdots| & \mapsto & \mathrm{id} \otimes \cdots \otimes \cap \otimes \cdots \otimes \mathrm{id} \\
|\cdots|\times|\cdots| & \mapsto & \mathrm{id} \otimes \cdots \otimes \times \otimes \cdots \otimes \mathrm{id}
\end{array}
$$

By composing the maps from each slice we get an intertwiner $V_1^{\otimes n} \to V_1^{\otimes m}$, and this is how diagrams of immersions of 1-manifolds induce maps on tensor powers of V_1. We can also consider formal linear combinations of diagrams of immersions, and they will induce form linear combinations of intertwiners. This gives us a diagrammatic interpretation of the identities in lemma 2.3. For example, the fundamental binor identity is simply

and the zig-zag identity can be drawn as

Further, it is easy to see that the maps satisfy the immersed versions of the Reidemeister II and III moves

This shows that the induced map $V_1^{\otimes n} \to V_1^{\otimes m}$ depends only on the regular homotopy type of the immersion.

3 Quantum Recoupling Theory

3.1 The Hopf Algebra $U_q\mathfrak{sl}_2$ and its Representations

3.2 A Diagrammatic Calculus for the Representation Theory of $U_q\mathfrak{sl}_2$