

# **DISCRETE MATHEMATICS AND ITS APPLICATIONS**

**Book: Discrete Mathematics and Its Applications**

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**Sixth Edition**

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# Chapter 1

## The Foundations: Logic and Proofs

# Objectives

- Explain what makes up a correct mathematical argument
- Introduce tools to construct arguments

# Contents

- 1.1-Propositional Logic
- 1.2-Propositonal Equivalences
- 1.3-Predicates and Quantifiers
- 1.4-Nested Quantifiers
- 1.5-Rules of Inference

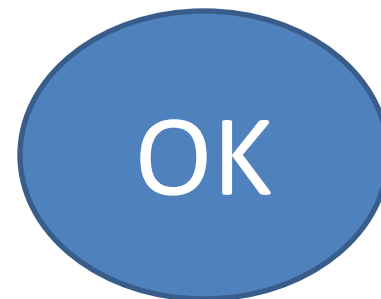
# 1.1- Propositional Logic

1.1.1- Definitions and Truth Table

1.1.2- Precedence of Logical Operators

# 1.1.1- Definitions and Truth Table

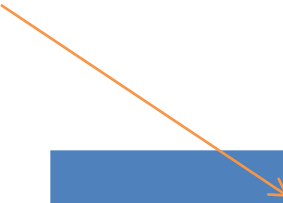
- **Proposition** is a declarative sentence that is either *true* or *false* but *not both*.
- Proposition is a sentence that declares a *fact*.
- Examples:
  - \* Bà Trưng is one of descendants of Bà Trưng \*
  - Ha Noi is not the capital of Vietnam
  - \*  $1+5 < 4$
  - \* What time is it?
  - \*  $X+Y=Z$



# 1.1.1- Definitions...

- **Truth table**

- I am a girl



p
True/ T / 1
False / F / 0

## 1.1.1- Definitions...

- **Negation** of proposition  $p$  is the statement "It is not case that  $p$ ".
- Notation:  $\neg p$  (or  $\bar{p}$ )

$p$	$\bar{p}$
1	0
0	1



# 1.1.1- Definitions...

- **Conjunction** of propositions  $p$  and  $q$  is the proposition “ $p$  and  $q$ ” and denoted by  $p \wedge q$

$p$	$q$	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

# 1.1.1- Definitions...

- **Disjunction** of propositions  $p$  and  $q$  is the proposition “ $p$  or  $q$ ” and denoted by  $p \vee q$

$p$	$q$	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

## 1.1.1- Definitions...

- Exclusive-or (xor) of propositions  $p$  and  $q$ , denoted by  $p \oplus q$

$p$	$q$	$p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

# 1.1.1- Definitions...

- Implication:  $p \rightarrow q$  (p implies q)
- $p$ : *hypothesis / antecedent / premise*
- $q$ : *conclusion / consequence*
- $p \rightarrow q$  can be expressed as:
  - $q$  if  $p$
  - If  $p$ , then  $q$
  - $p$  is sufficient condition for  $q$
  - $q$  is necessary condition for  $p$

p	q	$p \rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

“If  $1 + 1 = 3$ , then dogs can fly”  
 $\rightarrow$  TRUE  
 $(p \rightarrow q)$   
 $p=0, q=0$ ,  
 so  $(p \rightarrow q)$  is true.

# 1.1.1- Definitions...

- Biconditional statement  $p \leftrightarrow q$  is the proposition “p if and only if q”
- $p \rightarrow q$  (p **only if** q) and  $p \leftarrow q$  (p **if** q)

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \leftrightarrow q$
0	0	1	1	1	1
0	1	1	0	0	0
1	0	0	1	0	0
1	1	1	1	1	1

## 1.1.2- Precedence of Logical Operators

(1) Parentheses from inner to outer

(2)  $\neg$

(3)  $\wedge$

(4)  $\vee$

(5)  $\rightarrow$

(6)  $\leftrightarrow$

# 1.2- Propositional Equivalences

1.2.1- Tautology and Contradiction

1.2.2- Logical Equivalences

1.2.3- De Morgan's Laws

## 1.2.1- Tautology and Contradiction

- Tautology is a proposition that is *always true*
- Contradiction is a proposition that is *always false*
- When  $p \leftrightarrow q$  is tautology, we say “p and q are called logically equivalence”. Notation:  $p \equiv q$



## Example 3 p.23

- Show that  $p \rightarrow q$  and  $\neg p \vee q$  are logically equivalent.

<b>TABLE 4 Truth Tables for <math>\neg p \vee q</math> and <math>p \rightarrow q</math>.</b>				
$p$	$q$	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

## 1.2.2- Logical Equivalences...

Equivalence	Name
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination Laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent Laws
$\neg(\neg p) \equiv p$	Double Negation Laws
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative Laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative Laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive Laws

## 1.2.2- Logical Equivalences...

Equivalence		Name
$\neg (p \wedge q) \equiv \neg p \vee \neg q$ $\neg p \wedge \neg q$	$\neg (p \vee q) \equiv$	De Morgan Laws
$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$	Absorption Laws
$p \vee \neg p \equiv T$	$p \wedge \neg p \equiv F$	Negation Laws

# 1.2.2- Logical Equivalences...

Equivalences	Equivalences
$p \rightarrow q \equiv \neg p \vee q$	$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
$p \rightarrow q \equiv \neg q \rightarrow \neg p$	$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \vee q \equiv \neg p \rightarrow q$	$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$p \wedge q \equiv \neg (p \rightarrow \neg q)$	$\neg (p \leftrightarrow q) \equiv p \leftrightarrow \neg q$
$\neg (p \rightarrow q) \equiv p \wedge \neg q$	
$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$	
$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$	
$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$	
$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$	

# 1.3- Predicates and Quantifiers

- Introduction
- Predicates
- Quantifiers

## 1.3.1- Introduction

- A type of logic used to express the meaning of a wide range of statements in mathematics and computer science in ways that permit us to reason and explore relationships between objects.

## 1.3.2- Predicates – vị từ

- $X > 0$
- $P(X)$  = “X is a prime number” , called propositional function at X.
- $P(2)$  = “2 is a prime number”  $\equiv \text{True}$
- $P(4)$  = “4 is a prime number”  $\equiv \text{False}$

## 1.3.2- Predicates – vị từ

- $Q(X_1, X_2, \dots, X_n)$  , n-place/ n-ary predicate
- Example: “ $x=y+3$ ”  $\rightarrow Q(x,y)$   
 $Q(1,2) \equiv “1=2+3” \equiv \text{false}$   
 $Q(5,2) \equiv “5=2+3” \equiv \text{true}$



## 1.3.2- Predicates...

- Predicates are pre-conditions and post-conditions of a program.
- If  $x > 0$  then  $x := x + 1$ 
  - Predicate: “ $x > 0$ ”  $\rightarrow P(x)$
  - Pre-condition:  $P(x)$
  - Post-condition:  $P(x)$
- $T := X;$   
 $X := Y;$   
 $Y := T;$ 
  - Pre-condition: “ $x = a$  and  $y = b$ ”  $\rightarrow P(x, y)$
  - Post-condition: “ $x = b$  and  $y = a$ ”  $\rightarrow Q(x, y)$

Pre-condition ( $P(\dots)$ ) : condition describes valid input.

Post-condition ( $Q(\dots)$ ) : condition describes valid output of the codes.

**Show the verification that a program always produces the desired output:**

$P(\dots)$  is true

Executing Step 1.

Executing Step 2.

.....

$Q(\dots)$  is true

## 1.3.3- Quantifiers – Lượng từ

- The words in natural language: all, some, many, none, few, ....are used in quantifications.
- Predicate Calculus : area of logic that deals with predicates and quantifiers.
- The ***universal quantification*** of  $P(x)$  is the statement “ $P(x)$  for all values of  $x$  in the domain”. Notation :  $\forall xP(x)$
- The ***existential quantification*** of  $P(x)$  is the statement “There exists an element  $x$  in the domain such that  $P(x)$ ”. Notation :  $\exists xP(x)$
- Uniqueness quantifier:  $\exists!x P(x)$  or  $\exists_1xP(x)$
- $\forall xP(x) \vee Q(y)$  :
  - $x$  is a bound variable
  - $y$  is a free variable

## 1.3.4- Quantifiers and Restricted Domains

$$\forall x < 0 (x^2 > 0), \forall y \neq 0 (y^3 \neq 0), \exists z > 0 (z^2 = 2)$$



$$\forall x (x < 0 \rightarrow x^2 > 0), \forall y (y \neq 0 \rightarrow y^3 \neq 0), \exists z (z > 0 \wedge z^2 = 2)$$

Restricted domains

## 1.3.5- Precedence of Quantifiers

- Quantifiers have higher precedence than all logical operators from propositional calculus.
- $\forall x P(x) \vee Q(x) \rightarrow (\forall x P(x)) \vee Q(x)$
- $\forall$  has higher precedence. So,  $\forall$  affects on  $P(x)$  only.

## 1.3.6- Logical Equivalences Involving Quantifiers

Statements involving predicates and quantifiers are **logically equivalent if and only if they have the same truth value** no matter which predicates are substituted into the statements and which domain of discourse is used for the variables in these propositional functions.

- $\forall x (P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$   
 – Proof: page 39

Expression	Equivalence	Expression	Negation
$\neg \exists x P(x)$	$\forall x \neg P(x)$	$\exists x P(x)$	$\forall x \neg P(x)$
$\neg \forall x P(x)$	$\exists x \neg P(x)$	$\forall x P(x)$	$\exists x \neg P(x)$

## 1.3.7- Translating

- For every student in the class has studied calculus
- For every student in the class, that student has studied calculus
- For every student  $x$  in the class,  $x$  has studied calculus
- $\forall x (S(x) \rightarrow C(x))$

# Negating nested quantifiers

$$\begin{aligned}
 \neg \forall x \exists y (xy=1) &\equiv \exists x \neg \exists y (xy=1) \quad // \text{ De Morgan laws} \\
 &\equiv (\exists x) (\forall y) \neg (xy=1) \\
 &\equiv (\exists x) (\forall y) (xy \neq 1)
 \end{aligned}$$

**TABLE 2** De Morgan's Laws for Quantifiers.

<i>Negation</i>	<i>Equivalent Statement</i>	<i>When Is Negation True?</i>	<i>When False?</i>
$\neg \exists x P(x)$	$\forall x \neg P(x)$	For every $x$ , $P(x)$ is false.	There is an $x$ for which $P(x)$ is true.
$\neg \forall x P(x)$	$\exists x \neg P(x)$	There is an $x$ for which $P(x)$ is false.	$P(x)$ is true for every $x$ .

# 1.5- Rules of Inference

- Definitions
- Rules of Inferences

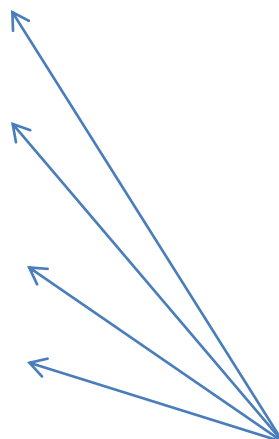


## 1.5.1- Definitions

- **Proposition 1** // Hypothesis
- Proposition 2
- Proposition 3
- Proposition 4
- Proposition 5
- .....
- **Conclusion**

Arguments 2,3,4  
are premises of  
argument 5

Arguments  
Propositional  
Equivalences



## 1.5.2- Rules Inferences

Rule	Tautology	Name
$\frac{p \quad p \rightarrow q}{\therefore q}$	$[p \wedge (p \rightarrow q)] \rightarrow q$ <b>You work hard</b> <b>If you work hard then you will pass the examination</b> <b><math>\therefore</math> you will pass the examination</b>	Modus ponens
$\frac{\neg q \quad p \rightarrow q}{\therefore \neg p}$	$[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$ <b>She did not get a prize</b> <b>If she is good at learning she will get a prize</b> <b><math>\therefore</math> She is not good at learning</b>	Modus tollens

## 1.5.2- Rules Inferences

Rule	Tautology	Name
$  \begin{array}{l}  p \\  \rightarrow q \\  \hline q \rightarrow r \\  \therefore p \rightarrow r  \end{array}  $	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ <p>If the prime interest rate goes up then the stock prices go down.</p> <p>If the stock prices go down then most people are unhappy.</p> <p>If the prime interest rate goes up then most people are unhappy.</p>	Hypothetical syllogism

# Rules Inferences...

Rule	Tautology	Name
$p \vee q$ $\neg p$ $\therefore q$	$[(p \vee q) \wedge \neg p] \rightarrow q$ Power puts off or <b>the lamp is malfunctional</b> Power doesn't put off <b>the lamp is malfunctional</b>	Disjunctive syllogism
$\underline{p}$ $\therefore p \vee q$	$p \rightarrow (p \vee q)$ It is below freezing now <b>It is below freezing now</b> or <b>raining now</b>	Addition
$\underline{p \wedge q}$ $\therefore p$	$(p \wedge q) \rightarrow p$ <b>It is below freezing now</b> and <b>raining now</b> <b>It is below freezing now</b>	Simplification

# Rules Inferences...

Rule	Tautology	Name
$\frac{p \quad q}{\therefore p \wedge q}$	$[(p) \wedge (q)] \rightarrow (p \wedge q)$	Conjunction
$\frac{p \vee q \quad \neg p \vee r}{\therefore q \vee r}$	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$ Jasmin is skiing OR it is not snowing It is snowing OR Bart is playing hockey Jasmin is skiing OR Bart is playing hockey	Resolution

## 1.5.3- Fallacies

- If **you do every problem in this book** then **you will learn discrete mathematic**

You learned mathematic

$$(p \rightarrow q) \wedge q$$

$$= (\neg p \vee q) \wedge q$$

(absorption law)

$$= q$$

$\Rightarrow$  No information for p

p can be true or false  $\Rightarrow$  You may learn discrete mathematic but you might do some problems only.

# Fallacies...

- $(p \rightarrow q) \wedge q \rightarrow p$  is not a tautology  
( it is false when  $p = 0, q = 1$ )
- $(p \rightarrow q) \wedge \neg p \rightarrow \neg q$  is not a tautology  
(it is false when  $p = 0, q = 1$ )

$$\begin{array}{c} p \\ \hline p \rightarrow q \\ \hline \therefore q \end{array}$$

$$\begin{array}{c} \neg q \\ \hline p \rightarrow q \\ \hline \therefore \neg p \end{array}$$

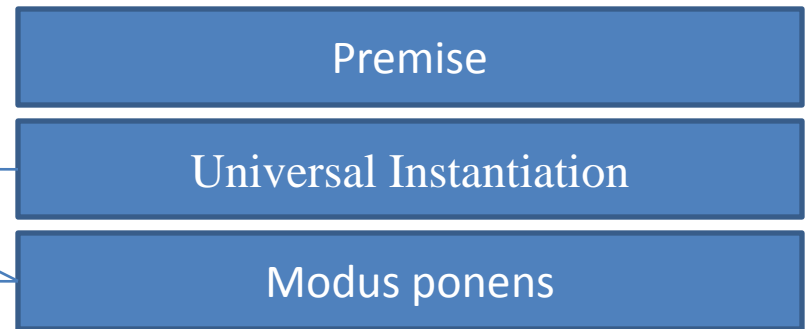
# 1.5.4- Rules of Inference for Quantified Statements

Rule	Name
$\frac{\forall xP(x)}{\therefore P(c)}$	Universal Instantiation
$\frac{P(c) \text{ for arbitrary } c}{\therefore \forall xP(x)}$	Universal generalization
$\frac{\exists xP(x)}{\therefore P(c) \text{ for some element } c}$	Existential instantiation
$\frac{P(c) \text{ for some element } c}{\therefore \exists xP(x)}$	Existential generalization



# Rules of Inference for Quantified Statements...

- “All student are in this class had taken the course PFC”
- “HB is in this class”
- “Had HB taken PFC?”
- $\forall x(P(x) \rightarrow Q(x))$
- $P(HB) \rightarrow Q(HB)$
- $P(HB)$
- $Q(HB)$  // conclusion



# Summary

- Propositional Logic
- Propositional Equivalences
- Predicates and Quantifiers
- Nested Quantifiers
- Rules and Inference

**THANK YOU**