DISCRETE MATHEMATICS AND ITS APPLICATIONS

Book: Discrete Mathematics and Its Applications

Author: Kenneth H. Rosen

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Chapter 1
The Foundations: Logic and Proofs

Objectives

- Explain what makes up a correct mathematical argument
- Introduce tools to construct arguments

Contents

1.1- Propositional Logic

1.1.1- Definitions and Truth Table

1.1.2- Precedence of Logical Operators

1.1.1- Definitions and Truth Table

- **Proposition** is a declarative sentence that is either ***true*** or ***false*** but ***not both***.
- Proposition is a sentence that declares a ***fact***.
- Examples:

  * Bà Tưng is one of descendants of Bà Trưng *
  Ha Noi is not the capital of Vietnam

  * 1+5 < 4

  * What time is it?

  * X+Y=Z

OK

No OK

1.1.1- Definitions…

- **Truth table**
  - I am a girl

p
True/ T / 1
False / F / 0

1.1.1- Definitions…

- Negation of proposition p is the statement " It is not case that p".
- Notation: ¬p (or $\overline{p}$)

p	$\overline{p}$
1	0
0	1

1.1.1- Definitions…

- Conjunction of propositions p and q is the proposition “ p and q” and denoted by p^q

p	q	p^q
0	0	0
0	1	0
1	0	0
1	1	1

1.1.1- Definitions…

- Disjunction of propositions p and q is the proposition "p or q" and denoted by p ∨ q

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

1.1.1- Definitions…

- Exclusive-or (xor) of propositions p and q, denoted by $p \oplus q$

p	q	$p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

1.1.1- Definitions…

- Implication: $p \longrightarrow q$ (p implies q)
- p: *hypothesis / antecedent / premise*
- q: *conclusion / consequence*
- $p \longrightarrow q$ *can be expressed as:*

- *q if p*
- *If p, then q*
- *p is sufficient condition for q*
- *q is necessary condition for p*

p	q	$p \longrightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

"If 1 + 1 = 3, then dogs can fly"
→ TRUE
($p \longrightarrow q$)
p=0, q=0 ,
so ($p \longrightarrow q$) is true.

1.1.1- Definitions…

- Biconditional statement $p \leftrightarrow q$ is the proposition " p if and only if q"
- $p \rightarrow q$ (p ***only if*** q) and $p \leftarrow q$ (p ***if*** q)

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \land (q \rightarrow p)$	$p \leftrightarrow q$
0	0	1	1	1	1
0	1	1	0	0	0
1	0	0	1	0	0
1	1	1	1	1	1

1.1.2- Precedence of Logical Operators

(1) Parentheses from inner to outer

(2) ¬

(3) ∧

(4) ∨

(5) →

(6) ↔

1.2- Propositional Equivalences

1.2.1- Tautology and Contradiction

1.2.2- Logical Equivalences

1.2.3- De Morgan’s Laws

1.2.1- Tautology and Contradiction

- Tautology is a proposition that is ***always true***
- Contradiction is a proposition that is ***always false***
- When p $\leftrightarrow$ q is tautology, we say “p and q are called logically equivalence”. Notation: p $\equiv$ q

Example 3 p.23

- *Show that* $p \rightarrow q$ *and* $\neg p \vee q$ *are logically equivalent.*

TABLE 4 **Truth Tables for $\neg p \vee q$ and $p \rightarrow q$.**

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

1.2.2- Logical Equivalences…

Equivalence	Name
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination Laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent Laws
$\neg(\neg p) \equiv p$	Double Negation Laws
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative Laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative Laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive Laws

1.2.2- Logical Equivalences…

Equivalence	Name
$\neg (p \wedge q) \equiv \neg p \vee \neg q$ $\neg (p \vee q) \equiv \neg p \wedge \neg q$	De Morgan Laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption Laws
$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$	Negation Laws

1.2.2- Logical Equivalences…

Equivalences	Equivalences
$p \to q \equiv \neg p \vee q$	$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \to q \equiv \neg q \to \neg p$	$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \vee q \equiv \neg p \to q$	$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$p \wedge q \equiv \neg (p \to \neg q)$	$\neg (p \leftrightarrow q) \equiv p \leftrightarrow \neg q$
$\neg(p \to q) \equiv p \wedge \neg q$	
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$	
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$	
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$	
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$	

1.3- Predicates and Quantifiers

- Introduction
- Predicates
- Quantifiers

1.3.1- Introduction

- A type of logic used to express the meaning of a wide range of statements in mathematics and computer science in ways that permit us to reason and explore relationships between objects.

1.3.2- Predicates – vị từ

- X > 0
- P(X)="X is a prime number" , called propositional function at X.
- P(2)=""2 is a prime number" ≡True
- P(4)="4 is a prime number" ≡False

1.3.2- Predicates – vị từ

- $Q(X_1, X_2, \ldots, X_n)$, n-place/ n-ary predicate
- Example: "x=y+3" ➔ Q(x,y)

  $Q(1,2) \equiv$ "1=2+3" $\equiv$ false

  $Q(5,2) \equiv$ "5=2+3" $\equiv$ true

1.3.2- Predicates…

- Predicates are pre-conditions and post-conditions of a program.
- If x>0 then x:=x+1
  - Predicate: “x>0” ➔ P(x)
  - Pre-condition: P(x)
  - Post-condition: P(x)
- T:=X;
  X:=Y;
  Y:=T;
  - Pre-condition: “x=a and y=b” ➔ P(x, y)
  - Post-condition: “x=b and y=a” ➔ Q(x, y)

Pre-condition (P(…)) : condition describes valid input.
Post-condition (Q(…)) : condition describes valid output of the codes.
Show the verification that a program always produces the desired output:
P(…) is true
Executing Step 1.
Executing Step 2.
…..
Q(…) is true

1.3.3- Quantifiers – Lượng từ

- The words in natural language: all, some, many, none, few, ….are used in quantifications.
- Predicate Calculus : area of logic that deals with predicates and quantifiers.
- The ***universal quantification*** *of P(x) is the statement "P(x) for all values of x in the domain".* Notation : $\forall x P(x)$
- The ***existential quantification*** *of P(x) is the statement "There exists an element x in the domain such that P(x)".* Notation : $\exists x P(x)$
- **Uniqueness quantifier**: $\exists! x\ P(x)$ or $\exists_1 x P(x)$
- $\forall x P(x) \vee Q(y)$:
    - x is a bound variable
    - y is a free variable

1.3.4- Quantifiers and Restricted Domains

$\forall x{<}0(x^2 > 0)$, $\forall y \neq 0(y^3 \neq 0)$, $\exists z{>}0(z^2 = 2)$

$\Leftrightarrow$

$\forall x(x<0 \rightarrow x^2 > 0)$, $\forall y(y \neq 0 \rightarrow y^3 \neq 0)$, $\exists z(z>0 \wedge z^2 = 2)$

Restricted domains

1.3.5- Precedence of Quantifiers

- Quantifiers have higher precedence than all logical operators from propositional calculus.
- $\forall x P(x) \lor Q(x) \rightarrow (\forall x P(x)) \lor Q(x)$
- $\forall$ has higher precedence. So, $\forall$ affects on $P(x)$ only.

1.3.6- Logical Equivalences Involving Quantifiers

Statements involving predicates and quantifiers are ***logically equivalent if and only if they have the same truth value*** no matter which predicates are substituted into the statements and which domain of discourse is used for the variables in these propositional functions.

- $\forall x\,(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$
  - Proof: page 39

Expression	Equivalence	Expression	Negation
$\neg\exists x P(x)$	$\forall x\, \neg P(x)$	$\exists x P(x)$	$\forall x\, \neg P(x)$
$\neg\, \forall x P(x)$	$\exists x\, \neg P(x)$	$\forall x P(x)$	$\exists x\, \neg P(x)$

1.3.7- Translating

- For every student in the class has studied calculus
- For every student in the class, that student has studied calculus
- For every student x in the class, x has studied calculus
- $\forall x \, (S(x) \rightarrow C(x))$

Negating nested quantifiers

$$\neg \forall x \exists y (xy=1) \equiv \exists x \neg \exists y (xy=1) \quad \text{// De Morgan laws}$$

$$\equiv (\exists x)(\forall y) \neg(xy=1)$$

$$\equiv (\exists x)(\forall y)(xy \neq 1)$$

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

1.5- Rules of Inference

- Definitions
- Rules of Inferences

1.5.1- Definitions

- Proposition 1 // Hypothesis
- Proposition 2
- Proposition 3
- Proposition 4
- Proposition 5
- ………
- Conclusion

Arguments 2,3,4 are premises of argument 5

Arguments Propositional Equivalences

1.5.2- Rules Inferences

Rule	Tautology	Name
p $p \rightarrow q$ $\therefore q$	$[p \wedge (p \rightarrow q)] \rightarrow q$ **You work hard** If **you work hard** then **you will pass the examination** $\therefore$ **you will pass the examination**	Modus ponen
$\neg q$ $p \rightarrow q$ $\therefore \neg p$	$[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$ **She did not get a prize** If **she is good at learning she will get a prize** $\therefore$ **She is not good at learning**	Modus tollen

1.5.2- Rules Inferences

Rule	Tautology	Name
$p \to q$ $q \to r$ $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$ If **the prime interest rate goes up** then **the stock prices go down.** If **the stock prices go down** then **most people are unhappy**. If **the prime interest rate goes up** then **most people are unhappy**.	Hypothetical syllogism

Rules Inferences…

Rule	Tautology	Name
$p \vee q$ $\underline{\neg p}$ $\therefore q$	$[(p \vee q) \wedge \neg p] \rightarrow q$ **Power puts off** or **the lamp is malfunctional** **Power doesn't put off** **the lamp is malfunctional**	Disjunctive syllogism
$\underline{\quad p}$ $\therefore p \vee q$	$p \rightarrow (p \vee q)$ **It is below freezing now** **It is below freezing now or raining now**	Addition
$\underline{\quad p \wedge q}$ $\therefore p$	$(p \wedge q) \rightarrow p$ **It is below freezing now and raining now** **It is below freezing now**	Simplification

Rules Inferences…

Rule	Tautology	Name
p q $\therefore p \wedge q$	$[(p) \wedge (q)) \rightarrow (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$ **Jasmin is skiing OR it is not snowing** **It is snowing OR Bart is playing hockey** **Jasmin is skiing OR Bart is playing hockey**	Resolution

1.5.3- Fallacies

- If **you do every problem in this book** then you will learn discrete mathematic

  You learned mathematic

  $(p \to q) \wedge q$

  $=(\neg p \vee q) \wedge q$

  (absorption law)

  $= q$

  $\Rightarrow$ No information for p

  p can be true or false ➔ You may learn discrete mathematic but you might do some problems only.

Fallacies…

- $(p \rightarrow q) \wedge q \rightarrow p$ is not a tautology

  (it is false when p = 0, q = 1)
- $(p \rightarrow q) \wedge \neg p \rightarrow \neg q$ is not a tautology

  (it is false when p = 0, q = 1)

$$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}$$

$$\begin{array}{l} \neg q \\ \underline{p \rightarrow q} \\ \therefore \neg p \end{array}$$

1.5.4- Rules of Inference for Quantified Statements

Rule	Name
$\underline{\quad \forall x P(x)}$ $\therefore P(c)$	Universal Instantiation
$\underline{\quad P(c) \text{ for arbitrary } c}$ $\therefore \forall x P(x)$	Universal generalization
$\underline{\exists x P(x)}$ $\therefore P(c)$ for some element c	Existential instantiation
$\underline{P(c) \text{ for some element } c}$ $\therefore \exists x P(x)$	Existential generalization

Rules of Inference for Quantified Statements…

- "All student are in this class had taken the course PFC"
- "HB is in this class"
- "Had HB taken PFC?"
- $\forall x(P(x) \rightarrow Q(x))$
- $P(HB) \rightarrow Q(HB)$
- $P(HB)$
- $Q(HB)$ // conclusion

Premise

Universal Instantiation

Modus ponens

Summary

- Propositional Logic
- Propositional Equivalences
- Predicates and Quantifiers
- Nested Quantifiers
- Rules and Inference

Fpt University

THANK YOU