Chapter 2
Basic Structures
Sets, Functions
Sequences, and Sums

Objectives

- Sets
- Set operations
- Functions
- Sequences
- Summations

2.1- Sets

- An unordered collection of objects
- The objects in a set are called the elements, or members. A set is said to contain its elements.
- Some important sets in discrete mathematics

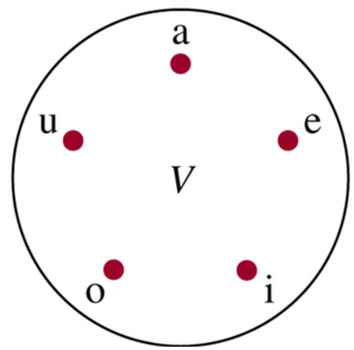
N = { 0,1,2,3,4,… }

Z = { … , -2,-1,0,1,2,…}　　　Z^+ = {0,1,2,…}

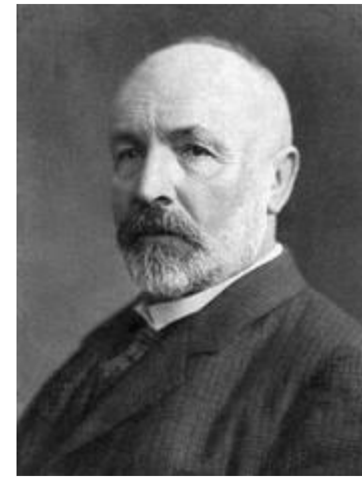
R: the set of real numbers

$$Q = \left\{ r = \frac{p}{q} \middle| p \in Z, 0 \neq q \in Z \right\}$$

$$V = \{a, u, o, i, e\}$$



Venn Diagram for V


G. Cantor

a∈A : a is an element of the set A // a belongs to A

a∉A: a is not an element of A

Sets…

Definitions:

- ***Finite set***: Set has n elements, n is a nonnegative integer
- A set is an ***infinite*** set if it is not finite
- ***Cardinality*** of a set |S|: Number of elements of S
- $\varnothing$: ***empty*** set (null set), the set with no element

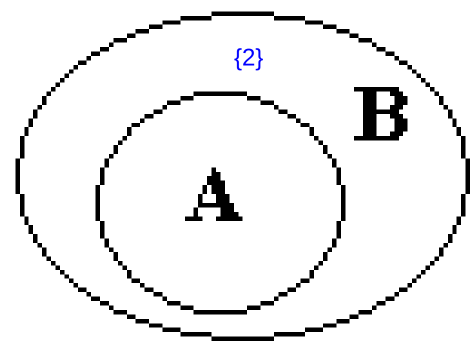
{1,1,1,}={1}

- Two sets are ***equal*** ←→ they have the same elements

  A = B if and only if $\forall x\ (x \in A \leftrightarrow x \in B)$
- $A \subseteq B$: the set A is a ***subset*** of the set B

  $A \subseteq B$ if and only if $\forall x\ (x \in A \rightarrow x \in B)$
- $A \subset B$: A is a ***proper subset*** of B

  $A \subset B$ if and only if $(A \subseteq B) \wedge (A \neq B)$



Venn diagram shows that A is a subset of B

Theorem 1

For every set S ,

i) $\varnothing \subseteq S$ ii) $\mathrm{S} \subseteq \mathrm{S}$

Pr*oof*

$i)\ (\mathrm{x} \in \varnothing) \equiv False$

$So\ \forall \mathrm{x}\left(x \in \varnothing \rightarrow x \in S\right) \equiv \mathrm{True}$

$\mathrm{ii)}\ \forall \mathrm{x}\left(x \in S \rightarrow x \in S\right) \equiv True$

a) $\emptyset \in \{\emptyset\}$

b) $\emptyset \in \{\emptyset, \{\emptyset\}\}$

c) $\{\emptyset\} \in \{\emptyset\}$

d) $\{\emptyset\} \in \{\{\emptyset\}\}$

e) $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$

f) $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$

$x \in \{x\}$ b) $\{x\} \subseteq \{x\}$ c) $\{x\} \in \{x\}$

$\{x\} \in \{\{x\}\}$ e) $\emptyset \subseteq \{x\}$ f) $\emptyset \in \{x\}$

Power Sets

Given a set S, **power set** P(S) of S is a set of all subsets of the set S.

S= { 1,2,3}

P(S)= {Ø,

{a,b}
{{};{a},{b},{a,b}}=P({a,b})

{1}, {2},{3},

{1,2}, {1,3},{2,3},

{1,2,3}}

Cartesian Products

- The **ordered** n-tuple $(a_1,a_2,\ldots,a_n)$ is the **ordered collection** that has a_1 as its first element, a_2 as its second element, …, and a_n as its n^{th} element.
- Let A and B be sets. The Cartesian product of A and B, denoted by AxB,

$$A\times B=\{(a,b)\,|\,a\in A, b\in B\}$$

For example

$A=\{a,b\}$ $B=\{1,2,3\}$ Tìm AxA?

$$A\times B=\{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\}$$

Cartesian Products…

- The Cartesian product of $A_1, A_2, \ldots, A_n$, denoted $A_1 x A_2 x \ldots x A_n$, is the set of ordered n- tuples $(a_1, a_2, \ldots, a_n)$,

$$A_1 \times A_2 \times \ldots \times A_n = \left\{ (a_1, a_2, \ldots, a_n) \middle| a_i \in A_i, \forall i = \overline{1, n} \right\}$$

For example

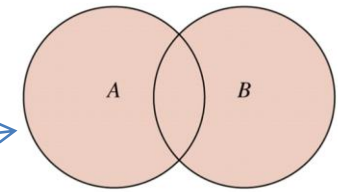
$A = \{a, b\}$ $B = \{1, 2, 3\}, C = \{0, 1\}$

- AxBxC= {(a,1,0),(a,1,1),(a,2,0),(a,2,1),(a,3,0),(a,3,1),
(b,1,0),(b,1,1),(b,2,0),(b,2,1),(b,3,0),(b,3,1) }

2.2- Set Operations

The *Union* of sets A and B, denoted by $A \cup B$

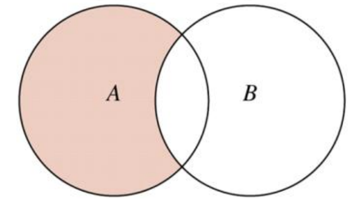
$$A \cup B = \{x \mid x \in A \vee x \in B\}$$



{1,2,3} u {2,4,5} = {1,2,3,4,5}
{1,2,3} and {2,4,5}={2}
{1,2,3} - {2,4,5}={1,3}
{2,4,5}-{1,2,3}={4,5}
{1,2,3} Xor {2,4,5} ={1,3,4,5}

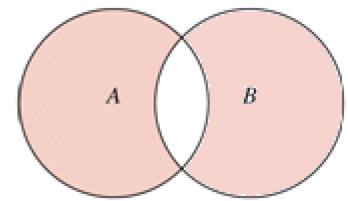
The difference of A and B, denoted by A-B

$$A\text{-}B = \{x \mid x \in A \wedge x \notin B\}$$

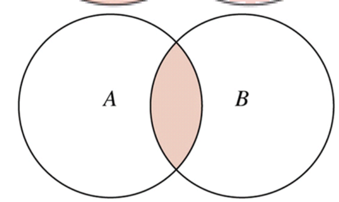


The symmetric difference of A and B, denoted by $A \oplus B$

$$A \oplus B = A \cup B - A \cap B = \{x \mid (x \in A \vee x \in B) \wedge (x \notin A \cap B)\}$$

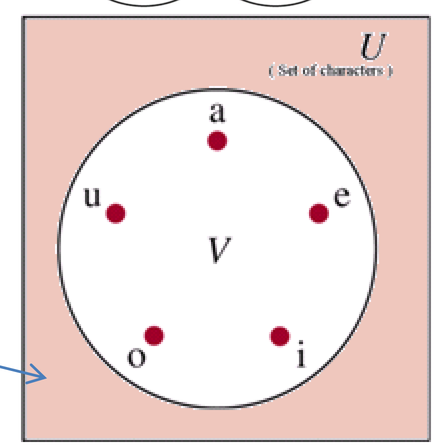


$$Intersection: A \cap B = \{x \mid x \in A \wedge x \in B\}$$



U is the universal set, complement of A is denoted by $\overline{A}$

$$\overline{A} = U\text{-}A = \{x \mid x \notin A\}$$



U={1,2,3,4,5}
A={1,3}
-A={2,4,5}

Set Identities

Identity – See proofs : pages 125, 126	Name
$A \cup \varnothing = A$ $\quad$ $A \cap \mathbf{U} = \mathbf{A}$	Identity laws
$A \cup \mathbf{U} = \mathbf{U}$ $\quad$ $A \cap \varnothing = \varnothing$	Domination laws
$A \cup A = A$ $\quad$ $A \cap A = A$	Idempotent laws
$\overline{\overline{A}} = A$	Complementation law
$A \cup B = B \cup A$ $\quad$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $\quad$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\quad$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan laws
$A \cup (A \cap B) = A$ $\quad$ $A \cap (A \cup B) = A$	Absorption
$A \cup \overline{A} = U$ $\quad$ $A \cap \overline{A} = \varnothing$	Complement laws

Generalized Unions and Intersections

$$A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i = \left\{ x \middle| x \in A_i, \forall i = 1, 2, \ldots, n \right\}$$

$$A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$$

$$= \left\{ x \middle| x \in A_1 \vee x \in A_2 \vee x \in A_3 \vee \ldots \vee x \in A_n \right\}$$

Computer Representation of Sets

- Use bit string U={1,2,3,4,5,6,7,8,9,10}
- A= {1,3,5,7,9 } ➔ A = "1010101010"
- B= { 1,8,9} ➔ B = "1000000110"

1010101110

1+1=1
1+0=0+1=1
0+0=0

Computer Representation of Sets

- A = “1010101010”
- B = “1000000110”

1-1=0
1-0=1
0-1=0
0-0=0

$$A \cup B = 10\ 1010\ 1010 \vee 10\ 0000\ 0110 = 10\ 1010\ 1110$$

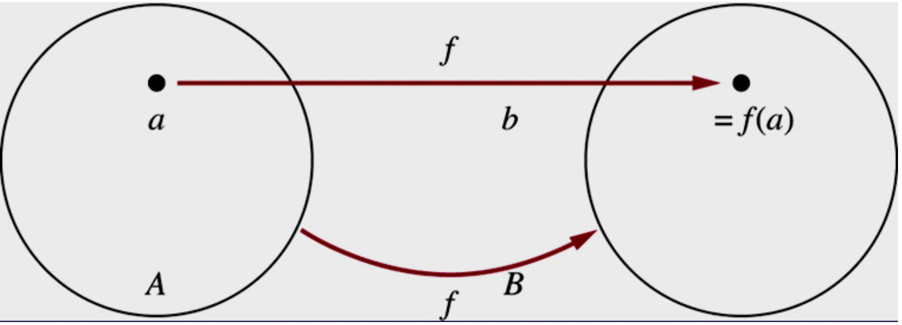
$$A \cup B = \{1, 3, 5, 7, 8, 9\}$$

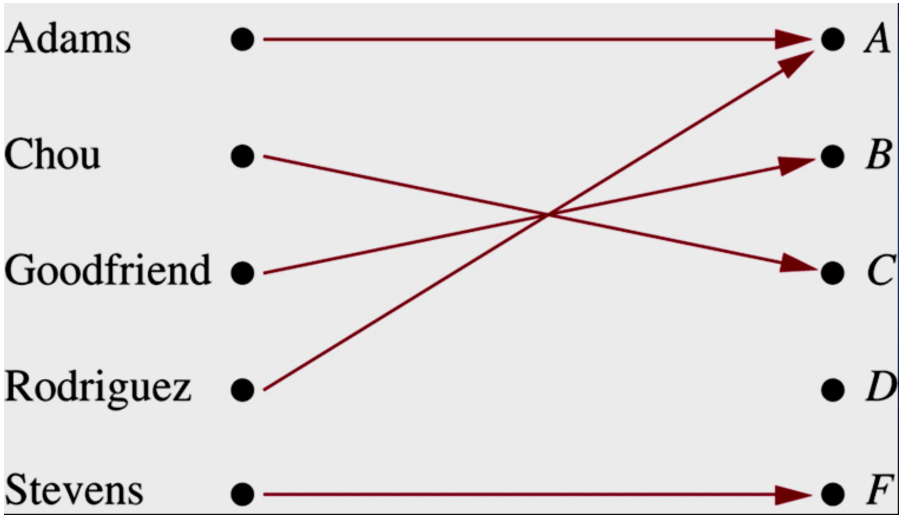
$$\mathrm{A} \cap \mathrm{B} = 10\ 1010\ 1010 \wedge 10\ 0000\ 0110 = 10\ 0000\ 0010$$

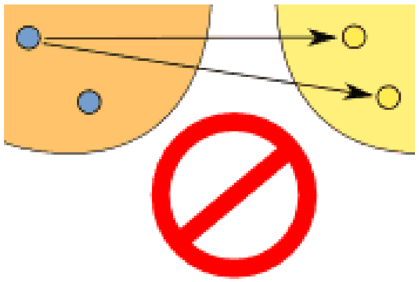
$$\mathrm{A} \cap \mathrm{B} = \{1, 9\}$$

2.3. Functions / Mappings / Transformations…

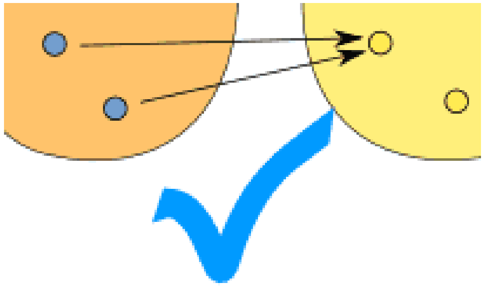
- **f: A $\rightarrow$ B :** function f from A to B (or function f maps A to B)
- A: **domain** of f
- B: **codomain** of f







(one-to-many)
This is **NOT** OK in a function



(many-to-one)
But this **is** OK in a function

Functions as sets of ordered pairs

Set of Ordered Pairs

A function can then be defined as a **set** of ordered pairs:

Example: **{(2,4), (3,5), (7,3)}** is a function that says

"2 is related to 4", "3 is related to 5" and "7 is related 3".

Also, notice that:

- the domain is **{2,3,7}** (the input values)
- and the range is **{4,5,3}** (the output values)

Functions / Mappings / Transformations…

What are functions?

- $f: \mathbb{Z} \rightarrow \mathbb{R} : f(x) = x^2 + 2$
- $f: \mathbb{Z} \rightarrow \mathbb{R} : f(x) = 1/(x-1)^2 + 5x$
- $f: \mathbb{R} \rightarrow \mathbb{R} : f(x) = (2x+5)/7$
- $f: \mathbb{Z} \rightarrow \mathbb{R} : f(x) = (2x+5)^2/(7-2x)$

Some Important Functions

f(2.1)=2
f(2)=2
f(-2.1)=-3
c(2.1)=3
c(2)=2
c(-2.1)=-2

See Figure 10 – Page 143

Floor function

f: $\mathbb{R} \rightarrow \mathbb{Z}$ such that f(x)= $\lfloor x \rfloor$ = largest integer that less than or equal to x, $\lfloor x \rfloor \le x$

Ceiling function

f: $\mathbb{R} \rightarrow \mathbb{Z}$ such that f(x)= $\lceil x \rceil$ = smallest integer that greater than or equal to x, $x \le \lceil x \rceil$

One-to-One/ Injective functions

Function f is one-to-one (or injective) if and only if

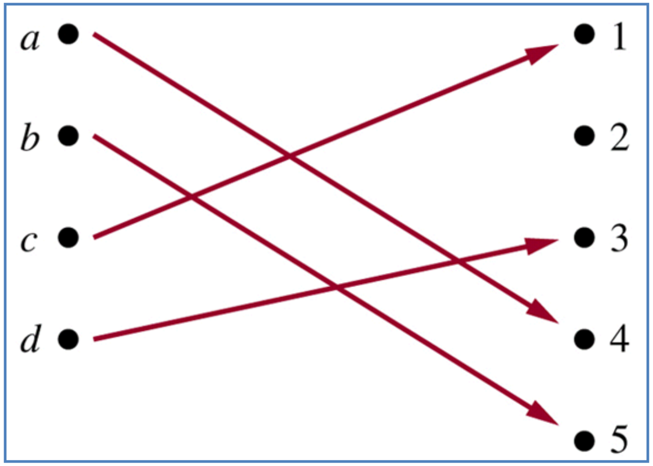
$a \neq b \rightarrow f(a) \neq f(b)$

for all a and b in the domain of f.

- $f : \mathbb{Z} \rightarrow \mathbb{Z}$, $f(x) = x^2$

  f is not one-to-one

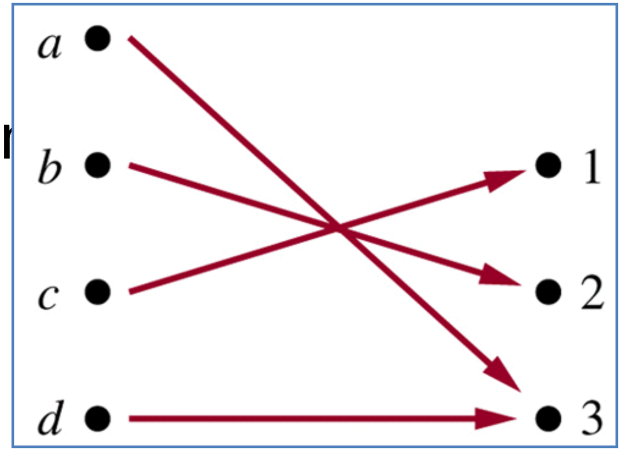
  (we have $f(-1) = f(1)$)



Onto Functions

A function f from A to B is called **onto, or surjective**, iff

for every element b in B there is an element a in A with f(a)=b.

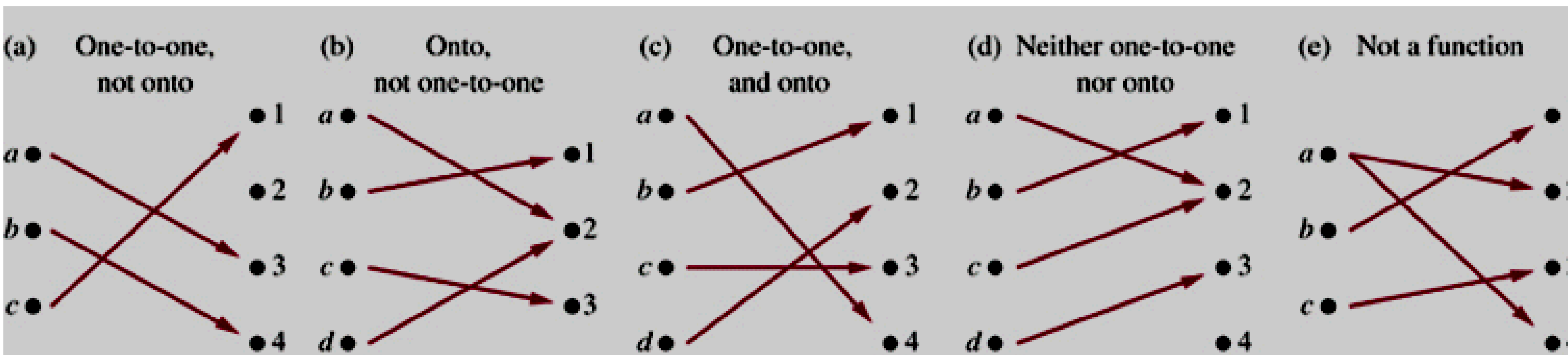


- f: $\mathbb{Z} \rightarrow \mathbb{Z}$, f(m) =m-1

  f is **onto** because $\forall y \in \mathbb{Z}$, y=f(m)=m-1, where m=y+1

One-to-one Correspodent / Bijective Functions

Function f is a **one-to-one corespondence** or a **bijection** if it is both one-to-one and onto.

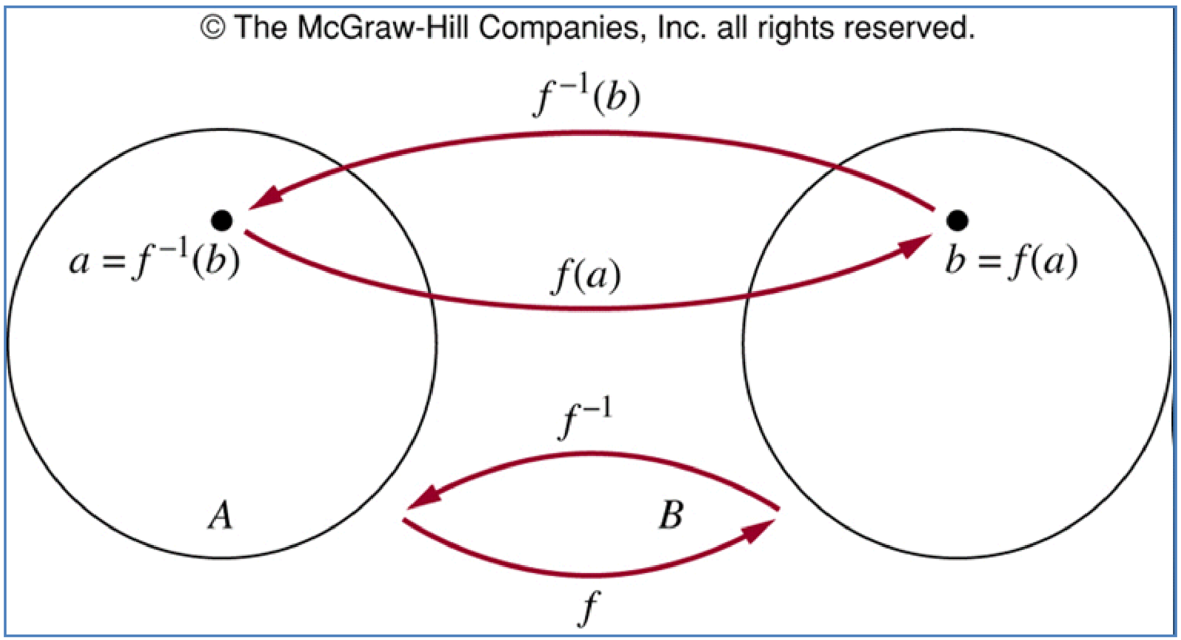


f: {A,B,…,Z} $\rightarrow${65,66,…,90} is a bijection

Inverse Functions

Let f is a bijection from A to B. The *inverse function*, denoted by f^{-1}, of f is the function that assigns to an element b belonging to B the unique element a in A such that f(a)=b. Hence $f^{-1}(b)=a$ when $f(a)=b$.

giải pt: y=f(x)
x =g(y)
g dgl f^-1
f(x) = 3x-4 =y
x = (y+4)/3
f^-1(x) = (x+4)/3
f^-1(y) = (y+4)/3
f^-1??



Inverse Functions…

$f:\mathbb{Z}\rightarrow\mathbb{Z}$ such that $f(x)=x+1$

Is f invertible? And if it is, what is its inverse?

Step 1: Show that f is onto

$f(y-1)=y$ for all y

➔ f is **onto**

Step 2: Show that f is one-to-one

$f(a)=a+1= f(b)=b+1$ ➔ $a=b$ ➔ f is **one-to-one**

➔ f is **bijection** ➔ f is **invertible**

Step 3: Find **inverse function**

$f(x)= y=x+1$ ➔ $x=f^{-1}(y)$

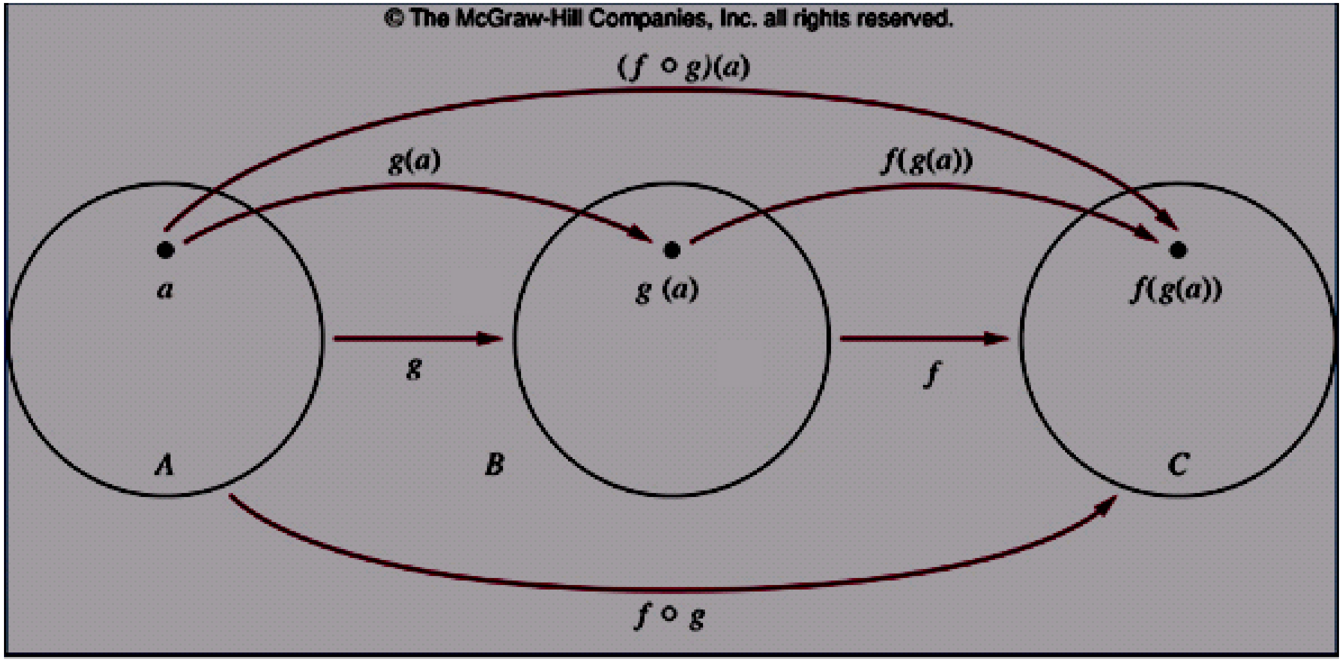
$x=y-1$ ➔ $f^{-1}(y)=y-1$

Composition of Functions

Let $g: A \rightarrow B$, $f: B \rightarrow C$

The *composition* of f and g, denoted by $f \circ g$, is defined by:

$(f \circ g)(x) = f(g(x))$



Example:

$f: \mathbb{Z} \rightarrow \mathbb{Z}$, $f(x)=x+1$

$g: \mathbb{Z} \rightarrow \mathbb{Z}$, $g(x)= x^2$

$(f \circ g)(x)= f(g(x))= f(x^2) = x^2+1$

$(g \circ f)(x) = g(f(x)) = g(x+1)= (x+1)^2$

2.4- Sequences

- Sequence : $a_1, a_2, a_3, \ldots, a_n, \ldots$

  Ex: 1,3,5,8 : Finite sequence

  Ex: 1, 1, 2, 3, 5, 8, 13,… : Infinite sequence
- A sequence is a function from a subset of integers to a set S.
- a_n : image of the integer n
- a_i : a term of the sequence
- $\{a_n = 1/n\}: \mathbb{Z}_+ \rightarrow \mathbb{R}$ ➔ 1, 1/2, 1/3, 1/4, …

Sequences…

Geometric progression

$f(n) = ar^n \rightarrow a, ar, ar^2, ar^3, \ldots, ar^n$

Arithmetic progression

$f(n) = a + nd \rightarrow a, a+d, a+2d, \ldots, a+nd$

a: initial term,

r: common ratio, a real number

d: common difference, real number

Do yourself

$b_n = (-1)^n, n>=0$

$c_n = 2(5)^n, n>=0$

$t_n = 7-3n, n>=0$

$a_n = -1 + 4n, n>=0$

Some Useful Sequences

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, ...
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, ...
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, ...

Summations

$$a_m + a_{m+1} + a_{m+2} + \ldots + a_n = \sum_{j=m}^{n} a_j = \sum\nolimits_{j=m}^{n} a_j = \sum\nolimits_{m \le j \le n} a_j$$

a : Sequence

j : Index of summation

m: Lower limit

n : Upper limit

```
// 1 + 2 +3+4+...+n
long  sum1 ( int n)   // n additions
{  long S=0;
   for (int i=1; i<=n; i++) S+= i;
   return S;
}
// 1 addition, 1 multiplication, 1 division
long sum2 (int n)
{  return  ((long)n) * (n+1)/2;
}
```

See examples 10, 11. Page 154

Summations….

Theorem 1- (Summation of geometric series)

$$a + ar + ar^2 + \ldots + ar^n = \sum_{i=0}^{n} ar^i = \begin{cases} \dfrac{ar^{n+1} - a}{r - 1} & \textit{If } r \neq 1 \\ (n+1)a & \textit{If } r = 1 \end{cases}$$

See the proofs in page 155

Some Useful Summation Formulae

Sum	Closed Form
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

See example 15, page 157

```
#include <stdio.h>
#include <conio.h>
// computing sigma from 1 to n of k^2
double sigma(int n)
{   double m=(double)n;
    return m*(m+1)*(m+m+1)/6;
}
void main()
{   int n1, n2;
    clrscr();
    scanf("%d%d",&n1,&n2);
    printf("%lf",sigma(n2)-sigma(n1-1));
    getch();
}
```

Cardinality

- Cardinality = number of elements in a set.
- The sets A and B have the same cardinality if and only if there is a one-to-one correspondence from A to B
- A set that is either finite or has the same cardinality as the set of positive integers is called ***countable***.
- A set that is not countable is called ***uncountable.***
- When a infinite set S is countable, we denote the cardinality of S is $|S| = \aleph_0$ (aleph null)
- For example, $|\mathbb{N}| = \aleph_0$ because $\mathbb{N}$ is countable and infinite but $\mathbb{R}$ is uncountable and infinite, and we say $|\mathbb{R}| = 2^{\aleph_0}$

Examples p.159, 160

sets	countable	uncountable	cardinality
$\{a, b, \ldots, z\}$, $\{x \mid x^5 - 3x^2 - 11 = 0\}$, …	✓	✗	$< \infty$
$\{0, 2, 4, \ldots, \}$	✓	✗	$\aleph_0$
N, Z^+, Z, Q, $Z \times Z$, …	✓	✗	$\aleph_0$
$\{x \mid 0 < x < 1\}$, R,…	✗	✓	$2^{\aleph_0}$

Summary

- Sets
- Set operations
- Functions
- Sequences
- Summations

Thanks