Chapter 3
The Fundamentals: Algorithms
The Integers

Objectives

- Algorithms
- The Growth of Functions
- Complexity of Algorithms
- The Integers and Division
- Primes and Greatest Common Divisors
- Integers and Algorithms

3.1- Algorithms

An algorithm is a finite set of precise instructions for performing a computation or for solving a problem.

Specifying an algorithm: natural language/ pseudocode

Properties of an algorithm

- Input
- Output
- Definiteness
- Correctness
- Effectiveness
- Generality

Finding the Maximum Element in a Finite Sequence

Procedure max ($a_1, a_2, a_3, \ldots, a_n$: integers)

max:=a_1

for i:=2 **to** n

if max < a_i **then** max:= a_i

{max is the largest element}

The Linear Search

Procedure linear search (x: integer, $a_1, a_2, \ldots, a_n$:distinct integers)

$i := 1$

while $i \leq n$ **and** $x \neq a_i$

$i := i + 1$

if $i \leq n$ **then** location:= i

else location:=0

{location is the subscript of the term that equals x, or is 0 if x is not found}

The Binary Search

procedure **binary search** (x:integer, $a_1, a_2, \ldots, a_n$: increasing integers)

$i := 1$ { i is left endpoint of search interval}

$j := n$ { j is right endpoint of search interval}

while $i < j$

begin

$m := \lfloor (i+j)/2 \rfloor$

if $x > a_m$ **then** $i := m+1$

else $j := m$

end

if $x = a_i$ **then** $location := i$

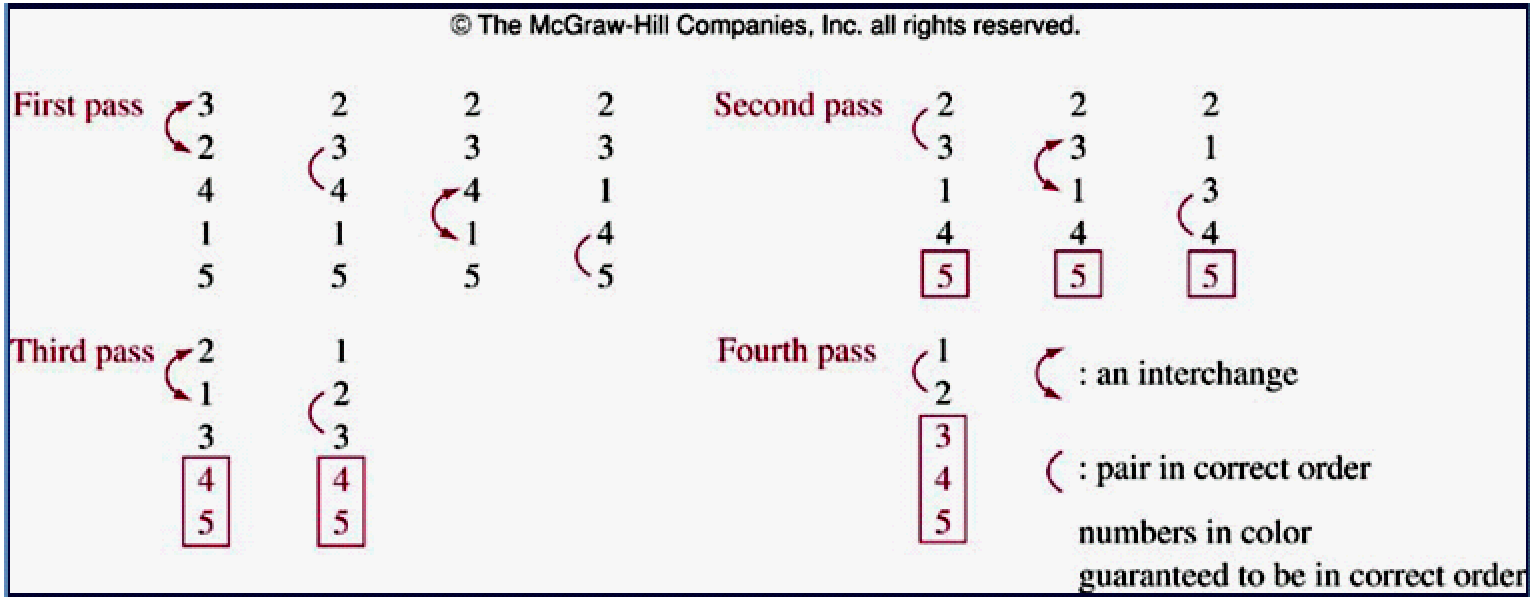
else $location := 0$

{location is the subscript of the term that equals x, or is 0 if x is not found}

Sorting

- Putting elements into a list in which the elements are in increasing order.
- There are some sorting algorithms
- Bubble sort
- Insertion sort
- Selection sort (exercise p. 178)
- Binary insertion sort (exercise p. 179)
- Shaker sort (exercise p.259)
- Merge sort and quick sort (section 4.4)
- Tournament sort (10.2)

Bubble Sort



procedure bubble sort ($a_1, a_2, \ldots, a_n$:real numbers with $n \geq 2$)

for $i := 1$ **to** $n-1$

    for $j := 1$ **to** $n - i$

        if $a_j > a_{j+1}$ **then interchange** a_j **and** a_{j+1}

{$a_1, a_2, \ldots, a_n$ are sorted}

Insertion Sort

procedure insertion sort ($a_1, a_2, \ldots, a_n$:real numbers with $n \geq 2$)

for $j := 2$ **to** n { j: position of the examined element }

begin

{ finding out the right position of a_j }

$i := 1$

while $a_j > a_i$ $i := i+1$

$m := a_j$ { save a_j }

{ moving j-i elements backward }

for $k := 0$ to $j-i-1$ $a_{j-k} := a_{j-k-1}$

{move a_j to the position i}

$a_i := m$

end

{$a_1, a_2, \ldots, a_n$ are sorted} It is usually not the most efficient

a: 1 2 3 6 7 8 **5** 9 12 11

j= 7

i=4

m=**5**

a: 1 2 3 $\boxed{6\ 7\ 8}$ **5** 9 12 11

a: 1 2 3 6 $\boxed{6\ 7\ 8}$ 9 12 11

a: 1 2 3 **5** 6 7 8 9 12 11

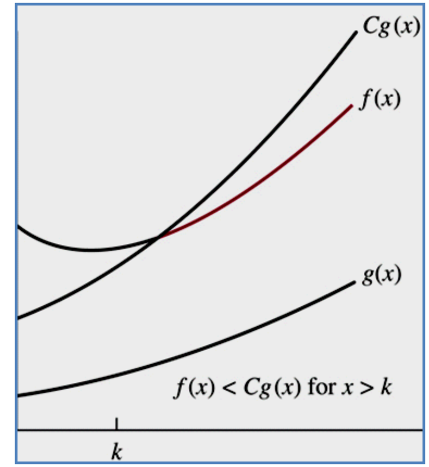
Greedy Algorithm

- They are usually used to solve optimization problems: Finding out a solution to the given problem that either minimizes or maximizes the value of some parameter.
- Selecting the best choice at each step, instead of considering all sequences of steps that may lead to an optimal solution.
- Some problems:
  - Finding a route between two cities with smallest total mileage (number of miles that a person passed).
  - Determining a way to encode messages using the fewest bits possible.
  - Finding a set of fiber links between network nodes using the least amount of fiber.

3.2- The Growth of Functions

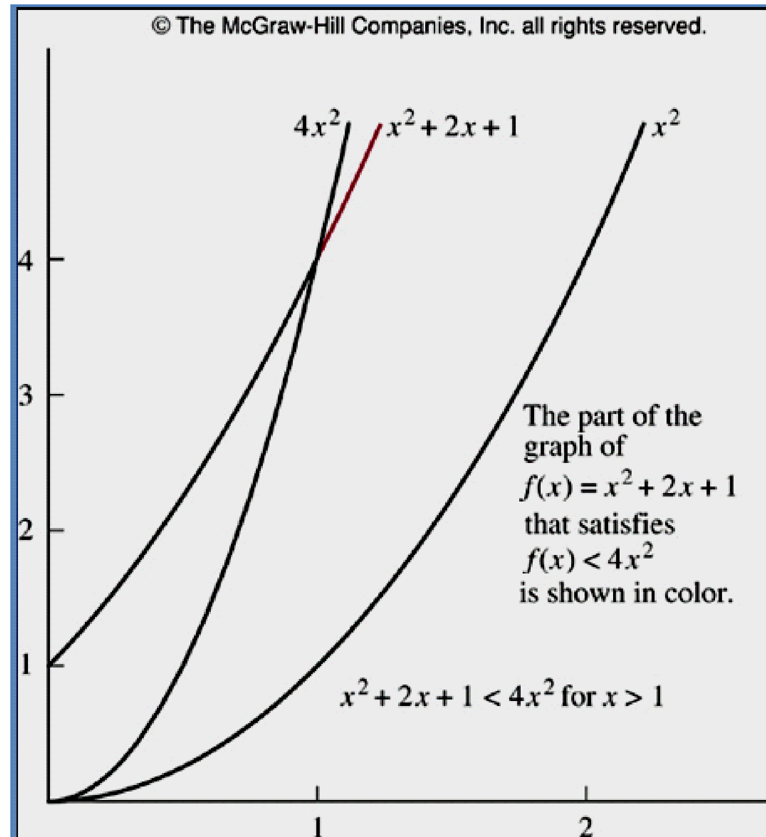
- The complexity of an algorithm that acts on a sequence depends on the number of elements of sequence.
- The growth of a function is an approach that help selecting the right algorithm to solve a problem among some of them.
- Big-O notation is a mathematical representation of the growth of a function.

3.2.1-Big-O Notation



Definition:

Let *f* and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that f(x) is O(g(x)) if there are constants C and k such that $|f(x)| \le C|g(x)|$ whenever $x>k$



Example***:*** **Show that $f(x)=x^2 + 2x +1$ is $O(x^2)$**

- Examine with $x>1$ ➔ $x^2 > x$
- ➔ $f(x)=x^2 + 2x +1 < x^2 + 2x^2 + x^2$
- ➔ $f(x) < 4x^2$
- ➔ Let $g(x)= x^2$
- ➔ $C=4$, $k=1$, $|f(x)| \le C|g(x)|$
- ➔ $f(x)$ is $O(x^2)$

Big-O: Theorem 1

Let $f(x)=a_n x^n + a_{n-1}x^{n-1}+\ldots+a_1x+a_0$, where $a_0,a_1,\ldots,a_n$ are real number, then $f(x)$ is $O(x^n)$

If $x>1$

$|f(x)| = | a_n x^n + a_{n-1}x^{n-1}+\ldots+a_1x+a_0 |$

$\leq | a_n x^n | + |a_{n-1}x^{n-1}| + \ldots + |a_1 x| + |a_0|$ { triangle inequality }

$\leq x^n (| a_n | + |a_{n-1}x^{n-1}/x^n| + \ldots + |a_1 x/x^n| + |a_0/x^n|)$

$\leq x^n (| a_n | + |a_{n-1}/x| + \ldots + |a_1/x^{n-1}| + |a_0/x^n|)$

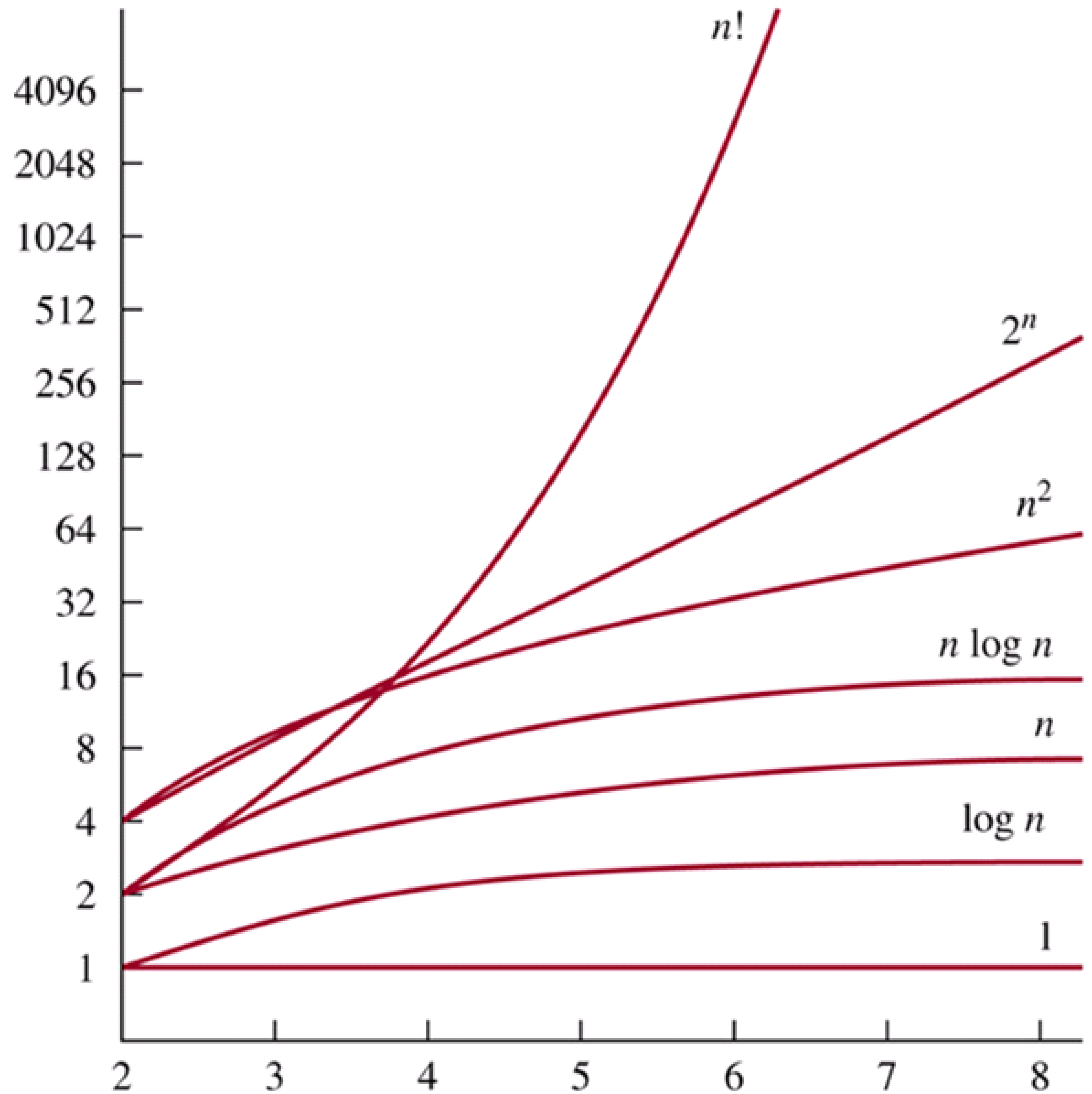
$\leq x^n (| a_n | + |a_{n-1}| + \ldots + |a_1| + |a_0|)$

Let $C = | a_n | + |a_{n-1}| + \ldots + |a_1| + |a_0|$

$|f(x)| \leq Cx^n$

➔ **$f(x) = O(x^n)$**

The Growth of Combinations of Functions



Big-O : Theorems

Theorem 2:

$f_1(x)=O(g_1(x)) \wedge f_2(x)=O(g_2(x))$

$\rightarrow (f_1+f_2)(x) = O(\max(|g_1(x)|,|g_2(x)|))$

Theorem 3:

$f_1(x)=O(g_1(x)) \wedge f_2(x)=O(g_2(x))$

$\rightarrow (f_1 f_2)(x) = O(g_1 g_2(x)))$

Corollary 1:

$f_1(x)=O(g\,(x)) \wedge f_2(x)=O(g\,(x)) \rightarrow (f_1+f_2)(x) = O(g(x))$

3.2.2- Big-Omega and Big-Theta Notation

- Big-O does not provide the lower bound for the size of f(x)
- Big-Ω, Big- θ were introduced by Donald Knuth in the 1970s
- Big-Ω provides the lower bound for the size of f(x)
- Big- θ provides the upper bound and lower bound on the size of f(x)

Big-Omega and Big-Theta Notation

- Definitions

  $\exists c>0, k\ x \geq k \wedge |f(x)| \geq C|(g(x)| \rightarrow |f(x)| = \Omega(g(x))$

  $f(x)= O(g(x)) \wedge f(x)= \Omega(g(x)) \rightarrow f(x) = \theta(g(x))$

  If $f(x) = \theta(g(x))$ then $f(x)$ is of order $g(x)$

Show that $f(x)=1+2+\ldots+n$ is $\theta(n^2)$

Examining $x>0$

$f(x)=1+2+\ldots+n = n(n+1)/2 = (n^2 +n) /2$

$f(x) \leq (2n^2)/2$

$f(x) \leq n^2$

➔Let $c_1=1/2$, $c_2=1$, $g(x)= n^2$

➔ $c_1g(x) \leq f(x) \leq c_2g(x)$

➔ $f(x) = \theta (n^2)$ with $x>0$

Big-Omega and Big-Theta Notation

- Theorem 4

  Let $f(x)=a_n x^n + a_{n-1}x^{n-1}+\ldots+a_1x+a_0$, where $a_0,a_1,\ldots,a_n$ are real number, then $f(x)$ is of order x^n

3.3- Complexity of Algorithms

- Computational complexity = Time complexity + space complexity.
- Time complexity can be expressed in terms of the number of operations used by the algorithm.
  - Average-case complexity
  - Worst-case complexity
- Space complexity will not be considered.

Demo 1

Describe the time complexity of the algorithm for finding the largest element in a set:

Procedure max ($a_1, a_2, \ldots, a_n$: integers)

$max := a_1$

for $i := 2$ to n

If $max < a_i$ **then** $max := a_i$

Time Complexity is θ(n)

i	Number of comparisons
2	2
3	2
…	2
n	2
n+1	1, $max < a_i$ is omitted

2(n-1) +1 = 2n-1 comparisions

Demo 2

Describe the average-case time complexity of the linear-search algorithm :

Procedure **linear search** (x: integer, $a_1, a_2, \ldots, a_n$:distinct integers)

i:=1

while ($i \le n$ **and** $x \ne a_i$) i:=i+1

if $i \le n$ **then** location:= i

else location:=0

i	Number of comparisons done
1	2
2	4
…	
n	2n
n+1	1, $x \ne a_i$ is omitted

Avg-Complexity= [(2+4+6+…+2n)]/n +1 +1
= [2(1+2+3+…+n) /n+2
= [2n(n+1)/2]/n + 2
=[n(n+1)]/n + 2
= n+1 + 2 = n+3
= θ(n)

See demonstrations about the worst-case complexity: Examples 5,6 pages 195, 196

Understanding the Complexity of Algorithms

Complexity	Terminology	Problem class
$\Theta(1)$	Constant complexity	Tractable (dễ), class P
$\Theta(\log n)$	Logarithmic complexity	Class P
$\Theta(n)$	Linear complexity	Class P
$\Theta(n \log n)$	n log n complexity	Class P
$\Theta(n^b)$, $b \geq 1$, integer	Polynominal complexity	Intractable, class NP
$\Theta(b^n)$, $b>1$	Exponential complexity	
$\Theta(n!)$	Factorial complexity	

NP : Non-deterministic Polynomial time

3.4- The Integers and Division

Definition: If a and b are integers with $a \neq 0$, we say that a divides b if there is an integer c such that $b=ac$.

When a divides b, we say that:

a is a factor of b

b is a multiple of a

Notation: $a|b$: a divides b $a \nmid b$: a does not divide b

Example:

$3 \nmid 7$ because $7/3$ is not an integer

$3|12$ because $12/3=4$, remainder=0

Corollary 1:

$a|b \wedge a|c \rightarrow a|(mb+nc)$, m,n are integers

The Division Algorithm

Theorem 2: Division Algorithm: Let a be an integer and d a positive integer. Then there are unique integers q and r, with $0 \le r < d$, such that $a=dq+r$

d: divisor , r : remainder, q: quotient (thương số)

Note: r can not be negative

Definition: $a=dq+r$

a: dividend d: divisor

q: quotient r : remainder,

q= a div d r = a mod d

Example:

101 is divided by 11 :101 = 11.9 + 2 ➔ q=9, r=2

-11 is divided by 3 : 3(-4)+ 1 ➔ q=-4, r=1

No OK : -11 is divided by 3 : 3(-3)-2 ➔ q=-3, **r = -2**

Modular Arithmetic

Definition: a, b: integers, m: positive integer.

a is called ***congruent*** to b modulo m if $m | a-b$

Notations:

$a \equiv b \pmod{m}$, a is congruent to b modulo m

$a \not\equiv b \pmod{m}$, a is not congruent to b mod m

Examples:

15 is congruent to 6 modulo 3 because $3 | 15-6$

15 is **not** congruent to 7 modulo 3 because $3 \nmid 15-7$

Modular Arithmetic

Theorem 3

a,b: integers, m: positive integer

$a \equiv b \pmod{m} \leftrightarrow a \bmod m = b \bmod m$

Proof

(1) $a \equiv b \pmod{m} \rightarrow a \bmod m = b \bmod m$

$a \equiv b \pmod{m} \rightarrow m \mid a-b \rightarrow a-b = km \rightarrow a = b + km$

$\rightarrow a \bmod m = (b + km) \bmod m$

$\rightarrow a \bmod m = b \bmod m$ { $km \bmod m = 0$ }

(2) $a \bmod m = b \bmod m \rightarrow a \equiv b \pmod{m}$

$a = k1m + c \wedge b = k2m + c \rightarrow a-b = (k1-k2)\, m$ { suppose $a>b$)

$= km$ { $k = k1-k2$}

$\rightarrow m \mid a-b \rightarrow a = b \pmod{m}$

Modular Arithmetic…

Theorem 4

a,b: integers, m: positive integer

a and b are congruent modulo m if and only if there is an integer k such that a = b + km

Proof

(1) $a \equiv b \pmod{m} \rightarrow a = b + km$

$a \equiv b \pmod{m} \rightarrow m \mid a-b \rightarrow a-b = km$ { from definition of division}

(2) $a = b + km \rightarrow a \equiv b \pmod{m}$

$a = b + km \rightarrow a-b=km \rightarrow m \mid a-b \rightarrow a \equiv b \pmod{m}$

Modular Arithmetic…

Theorem 5

m: positive integer

$a \equiv b \pmod{m} \wedge c \equiv d \pmod{m} \rightarrow$

$a+c \equiv b+d \pmod{m} \wedge ac \equiv bd \pmod{m}$

Proof : See page 204

Corollary 2:

m : positive integer, a,b : integers

$(a+b) \bmod m = ((a \bmod m) + (b \bmod m)) \bmod m$

$ab \bmod m = ((a \bmod m)(b \bmod m)) \bmod m$

Proof: page 205

Applications of Congruences

Hashing Function: $H(k) = k \bmod m$

Using in searching data tin memory.

k: data searched, m : memory block

Examples:

H(064212848) mod 111= 14

H(037149212) mod 111= 65

Collision: $H(k_1) = H(k_2)$. For example, H(107405723) = 14

Applications of Congruences

Pseudo-random Numbers $x_{n+1}=(ax_n+c) \bmod m$

a: multiplier, c: increment, x_0: seed

with $2 \leq a<m$, $0 \leq c<m$, $0 \leq x_0<m$

Examples:

m=9 ➔ random numbers: 0..8

a= 7, c=4, x_0=3

Result: Page 207

Applications of Congruences

Cryptography: letter 1 $\rightarrow$ letter 2

Examples: shift cipher with k $f(p) = (p+k) \bmod 26$

→ $f^{-1}(p) = (p-k) \bmod 26$

Sender: (encoding)

Message: "ABC" , k=3

Using $f(p) = (p+3) \bmod 26$ // 26 characters

ABC ➔ 0 1 2 → 3 4 5 ➔ DEF

Receiver: (decoding)

DEF ➔ 3 4 5

Using $f^{-1}(p) = (p-3) \bmod 26$

3 4 5 ➔ 0 1 2 ➔ ABC

3.5- Primes and Greatest Common Divisors

Definition 1:

A positive integer p greater than 1 is called ***prime*** if the only positive factors are 1 and p

A positive integer that is greater than 1 and is *not prime* is called *composite*

Examples:

Primes: 2 3 5 7 11

Composites: 4 6 8 9

Finding very large primes: tests for supercomputers

Primes…

Theorem 1- The fundamental theorem of arithmetic:

Every positive integer greater than 1 can be written uniquely as a prime or as the product of two or more primes where the prime factors are written in order of nondecreasing size

Examples:

Primes: 37

Composite: $100 = 2.2.5.5 = 2^2 5^2$

$999 = 3.3.3.37 = 3^3 37$

Primes…

Converting a number to prime factors

Examples: 7007

Try it to 2,3,5 : 7007 can not divided by 2,3,5

7007 : 7

1001 : 7

143: 11

13: 13

0

➔ $7007 = 7^2.11.13$

Primes…

Theorem 2: If n is a composite, then n has a prime **divisor** less than or equal to $\sqrt{n}$

Proof:

n is a composite ➔ n = ab in which a or b is a prime

If $(a > \sqrt{n} \wedge b > \sqrt{n}) \rightarrow ab > n$ ➔ false

➔ Prime divisor of n less than or equal to $\sqrt{n}$

Primes…

Theorem 3: There are infinite many primes

Proof: page 212

Theorem 4: The prime number theorem:

The ratio of the number of primes not exceeding x and x/ln x approaches 1 and grows with bound (ln x: natural logarithm of x)

See page 213

Example:

$x = 10^{1000}$ → The odds that an integer near 10^{1000} is prime are approximately $1/\ln 10^{1000} \sim 1/2300$

Conjectures and Open Problems About Primes

See pages: 214, 215

- 3x + 1 conjecture
- Twin prime conjecture: there are infinitely many twin primes

Greatest Common Divisors and Least Common Multiples

Definition 2:

Let a, b be integers, not both zero. The largest integer d such that d|a and d|b is called the *greatest common divisor* of a and b.

Notation: gcd(a,b)

Example: gcd(24,36)=?

Divisors of 24: 2 3 4 6 8 12 = $2^3 3^1$

Divisors of 36: 2 3 4 6 9 12 18 = $2^2 3^2$

gcd(24,36)=12 = $2^2 3^1$ // Get factors having minimum power

Greatest Common Divisors and Least Common Multiples

Definition 3:

The integers a, b are *relatively prime* if their greatest common divisor is 1

Example:

gcd(3,7)=1 → 3,7 are relatively prime

gcd (17,22)=1 → 17,22 are relatively prime

gcd(17,34) = 17 → 17, 34 are **not** relatively prime

Greatest Common Divisors and Least Common Multiples

Definition 4:

The integers $a_1,a_2,a_3,\ldots,a_n$ are *pairwise relatively prime* if gcd(a_i,a_j)=1 whenever $1 \le i<j \le n$

Example:

7 10 11 17 23 are pairwise relatively prime

7 10 11 16 24 are **not** pairwise relatively prime

➔ Adjacent number of every composite in sequence must be a prime.

Greatest Common Divisors and Least Common Multiples

Definition 5:

The Least common multiple of the positive integer a and b is the smallest integer that is divisible by both a and b

Notation: lcm(a,b)

Example:

lcm(12,36) = 36   lcm(7,11) = 77

lcm $(2^3 3^5 7^2, 2^4 3^3) = 2^4 3^5 7^2$

$2^3 3^5 7^2, 2^4 3^3 7^0 \rightarrow 2^4 3^5 7^2$ // get maximum power

Greatest Common Divisors and Least Common Multiples

Theorem 5:

Let a, b be positive integers then

ab= gcd(a,b). lcm(a,b)

Example: gcd(8, 12) = 4 lcm(8,12)=24 ➔ 8.12 = 4.24

Proof: Based on analyzing a, b to prime factors to get gcd(a,b) and lcm(a,b)

➔ ab=gcd(a,b). lcm(a,b)

3.6- Integers and Algorithms

- Representations of Integers
- Algorithms for Integer Operations
- Modular Exponentiation
- Euclid Algorithm

Representations of Integers

Theorem 1:

Let b be a positive integer greater than 1. Then if n is a positive integer, it can be expressed uniquely in the form

$n= a_k b^k + a_{k-1} b^{k-1} + \ldots + a_1 b + a_0$

Where k is a nonnegative integer, $a_0, a_1, a_2, \ldots, a_k$ are nonnegative integers less than b and $a_k \neq 0$

Proof: page 219

Example: $(245)_8 = 2.8^2 + 4.8 + 5 = 165$

Common Bases Expansions: Binary, Octal, Decimal, Hexadecimal

Finding expansion of an integer: Pages 219, 220, 221

Algorithm 1: Constructing Base b Expansions

Procedure base b expansion (n: positive integer)

q:=n

k:=0

while $q \neq 0$

begin

a_k :=q mod b

q := $\lfloor q/b \rfloor$

k := k +1

end { The base b expansion of n is $(a_{k-1}a_{k-2}\ldots a_1a_0)$}

Algorithms for Integer Operations

Algorithm 2: Addition integers in binary format

Algorithm 3: Multiplying integers in binary format

Algorithm 4: Computing div and mod integers

Algorithm 5: Modular Exponentiation

Algorithm 2: Adding of Integers

Complexity: O (n)

$$
\begin{array}{r}
1\ 1\ 1\ 0\ 0 \\
1\ 1\ 1\ 0 \text{ (a)} \\
+\ 1\ 0\ 1\ 1 \text{ (b)} \\
\hline
1\ 1\ 0\ 0\ 1 \text{ (s)}
\end{array}
$$

procedure add (a,b: integers)

{ a: $(a_{n-1}a_{n-2}\ldots a_1a_0)_2$ b: $(b_{n-1}b_{n-2}\ldots b_1b_0)_2$ }

c:=0

for j:=0 to n-1

Begin

d:= $\lfloor (a_j+b_j+c)/2 \rfloor$ // next carry of next step

s_j= a_j+b_j+c - 2d // result bit

c:=d // updating new carry to next step

End

s_n= c // rightmost bit of result

{ The binary of expansion of the sum is $(s_ns_{n-1}\ldots s_1s_0)$}

Algorithm 3: Multiplying Integers

Complexity: O (n^2)

$$
\begin{array}{rrrrrl}
 & & 1 & 1 & 0 & (a) \\
\text{X} & & 1 & 0 & 1 & (b) \\
\hline
 & & 1 & 1 & 0 & \\
+ & 0 & 0 & 0 & 0 & \\
1 & 1 & 0 & 0 & 0 & \\
\hline
1 & 1 & 1 & 1 & 0 & (p)
\end{array}
$$

procedure multiply (a,b: integer)

{ a: $(a_{n-1}a_{n-2}\ldots a_1a_0)_2$ b: $(b_{n-1}b_{n-2}\ldots b_1b_0)_2$ }

for j:= 0 to n-1

begin

if b_j =1 then c_j := a shifted j places

end

Complexity: O (n)

{ $c_0, c_1, \ldots, c_{n-1}$ are the partial products }

p := 0

for j:= 0 to n-1

p:=p+c_j

{p is the value of ab}

Algorithm 4: Computing div and mod

```
procedure division ( a: integer; d: positive integer)
 q:=0
 r:= |a|
while r ≥ d  {quotient= number of times  of successive subtractions}
   begin
      r:= r-d
      q := q+1
   end
If a<0 and r>0  then  {updating remainder when a<0}
   begin
      r:= d-r
     q := -(q+1)
   end
{ q = a div d is the quotient, r=a mod d is the remainder}
```

Algorithm 5: Modular Exponentiation

{ b^n mod m = ? . Ex: 3^{644} mod 645 = 36 }

procedure mod_ex (b: integer, n=$(a_{k-1}a_{k-2}\ldots a_1a_0)_2$, m: positive integer)

x:=1

power := b mod m

for i:=0 to k-1

begin

if a_i=1 then x:= (x.power) mod m

power := (power.power) mod m

end

{ x equals b^n mod m }

Corollary 2: ab mod m = ((a mod m)(b mod m)) mod m

b^n mod m = result of successive factor_i mod m

The Euclidean Algorithm

Lemma: Proof: page 228

Let a= bq+r, where a, b, q, r are integers. Then gcd(a,b) = gcd(b,r)

Example: 287 = 91.3 + 14 ➔gcd(287,91) =gcd(91,14)= 7

```
procedure  gcd(a,b: positive integer)
 x:=a
 y:=b
 while y ≠ 0
  begin
    r := x mod y
    x:=y
    y:= r
   end  {gcd(a,b) is x}
```

Summary

- Algorithms
- The Growth of Functions
- Complexity of Algorithms
- The Integers and Division
- Primes and Greatest Common Divisors
- Integers and Algorithms

Thanks