Chapter 5 & 7
Counting

Objectives

- Some questions we can face:
  - “How can we determine the complexity of an algorithm?”
  - “How many cases possible are there to arrange n objects?”
  - “How can we generate input data from given initial data?”
  - A password consists of six characters. How many such password with some criteria?
  - How many bit strings of length n that do not have two consecutive 0s?

Contents

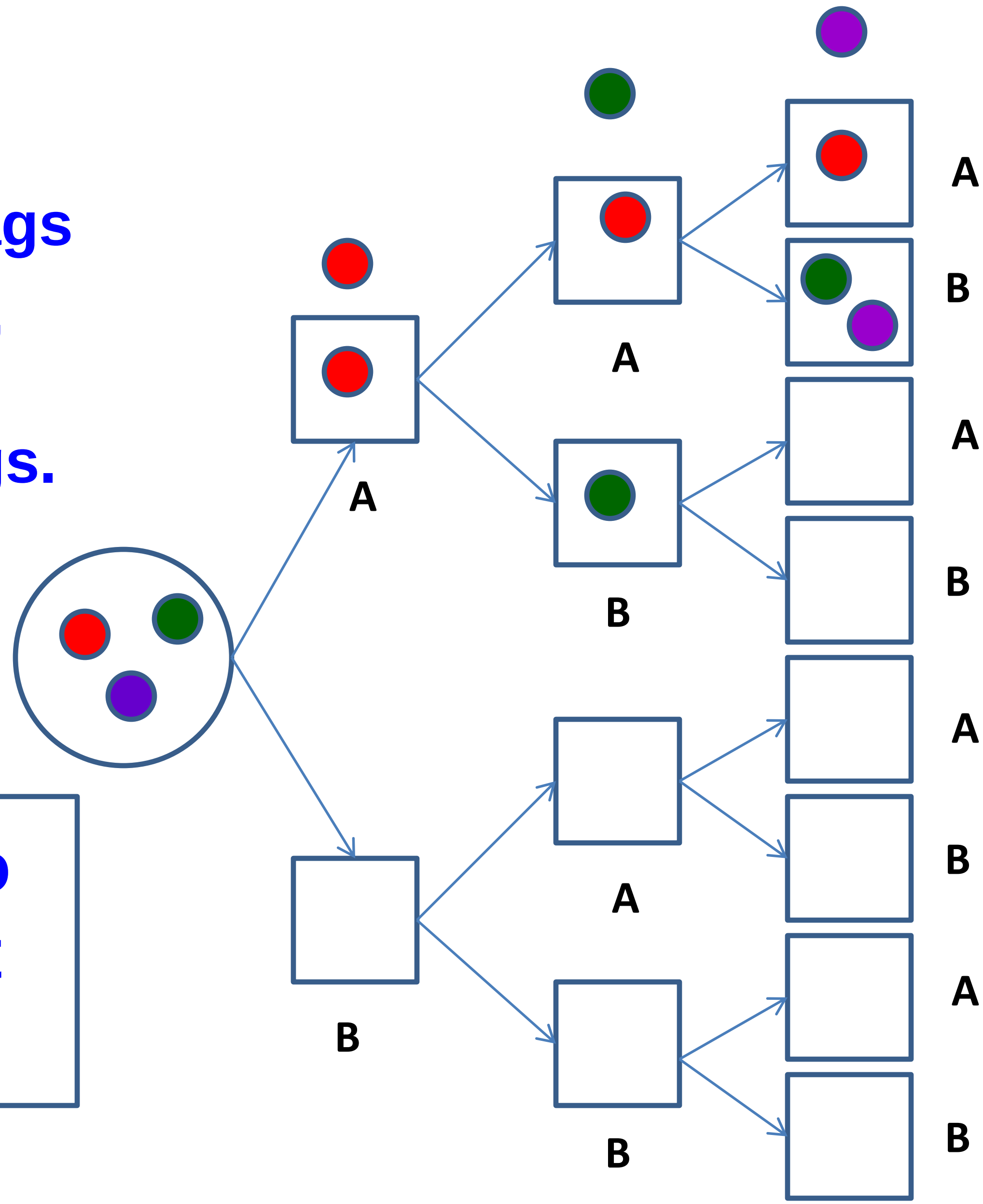
- The basics of Counting
- Recurrence relations
- Divide – and – Conquer Algorithms and recurrence relations

5.1- The Basics of Counting

- Basic Counting Principles : Product rule, sum rule, The Inclusion-Exclusion Principle
- Tree Diagrams

Basic Counting Principle

- Ex: Distribute 3 distinct balls to 2 bags
- We must do 3 times, each time we have 2 ways to select a bags.
- 2.2.2 = 2^3 ways



There are r^n ways to distribute n distinct objects to r bags

Basic Counting Principle

- **12 teams. How many ways to select 3 teams for the first, second and third teams**

I

II

III

Select 1/ 12
→ 12 ways

After select 1 team for the first position, one of 11 teams can be select into the second team
→ 11 ways

After select 1 team for the second position, one of 10 remaining teams can be select into the third team
→ 10ways

12.11.10= 1320 ways

Basic Counting Principle: Product rule

- **Suppose that a procedure can be broken down into a sequence of two tasks. If there is n_1 ways** to do the first task, there is n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.
- **Example 1:** **2 persons, 12 rooms. How many ways are there to assign different room to these two person?**

  12 ways can be applied to assign the first

  11 ways can be applied to assign the second after the first assigned.

➔ 12.11= 132 ways.

Examples 3, 4, 5,6,7,8,9: Refer to pages 336, 337

Basic Counting Principle: Sum rule

- If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do this task.
- Example 11: There are 37 faculties, 83 students. A person either a faculty or student can be choose to attend a committee. How many ways are there to select such person?

  37 + 83= 120 ways

Examples 12, 13: Refer to page 339

Basic Counting Principles…

- Example 14: Name of a variable in BASIC language, an case insensitive language, is a string of 1 or 2 characters in which the first must be a character, the second may be an alphanumeric characters, and must be different from 5 two-character keywords . How many different variable names are there?

26 characters (a..z), 10 digits (0..9)

Name with 1 character: 26

Name with 2 character: 26.36 -5 =931

➔ There are 957 different names

Basic Counting Principles…

- **Example 16 – page 341**:Counting Internet Address
- Class A address: largest networks
- Class B address: medium-sized networks
- Class C address: smallest networks.

IPv4

Bit Number	0	1	2	3	4	8	16	24	31
Class A	0	netid				hostid			
Class B	1	0	netid				hostid		
Class C	1	1	0	netid				hostid	
Class D	1	1	1	0	Multicast Address				
Class E	1	1	1	1	0	Address			

Basic Counting Principles: The Inclusion-Exclusion Principle

- Suppose that a task can be done in n_1 or n_2 ways, but that some ways in the set of n_1 ways are the same as some ways in the set of n_2 ways.
- n_s : number of ways that are the same in two sets of ways.
- Total ways $n = n_1 + n_2 - n_S$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example 17: How many 8-bit strings either start with 1 or end with 00?

$2^7 + 2^6 - 2^5 = 160$ – Refer to page 342

Basic Counting Principles: The Inclusion-Exclusion Principle

Example 18:

350 applications, in which

A1: 220 people majored in IT

A2: 147 people majored in business

A3: 51 people majored both in IT and business

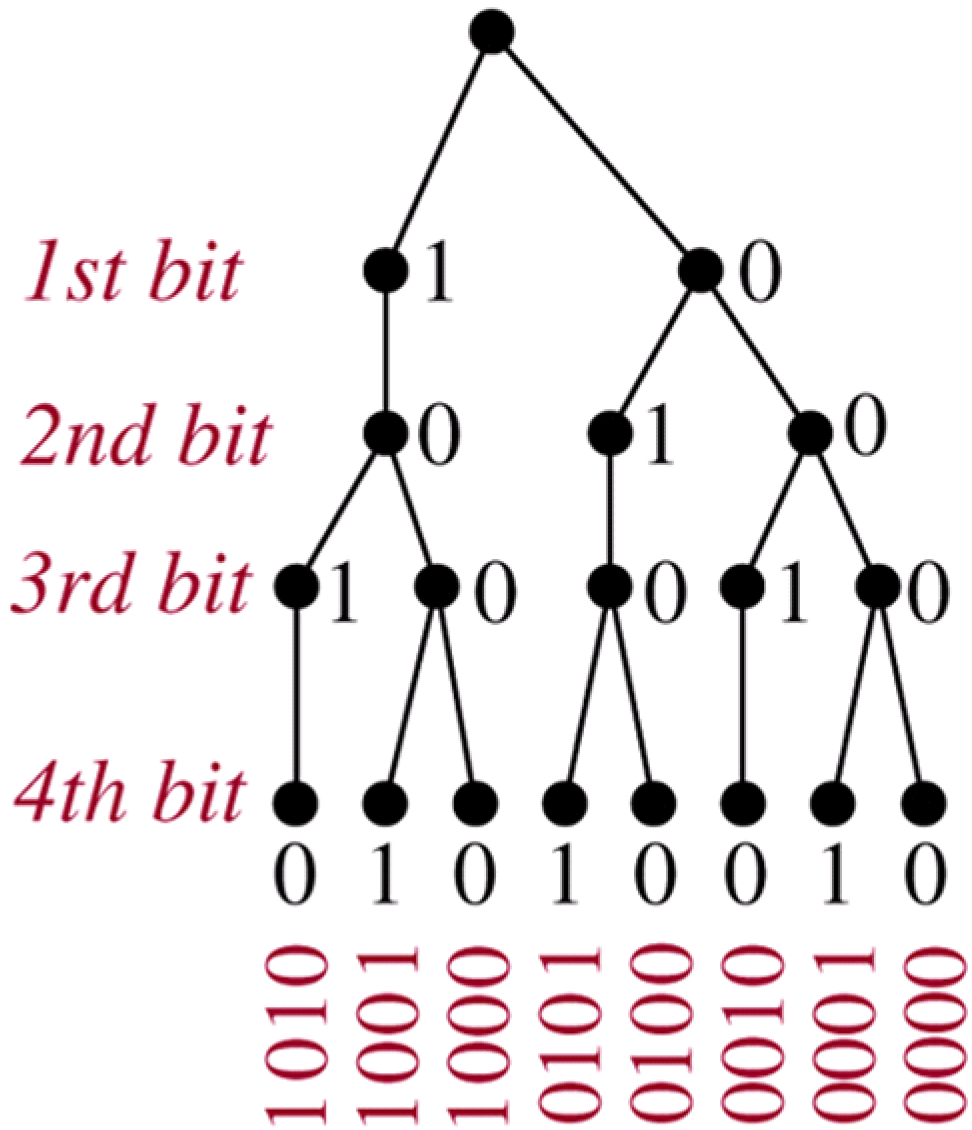
How many applicants majored **neither** in IT **nor** business?

Number of applicants who majored either IT or business = (|A1| + |A2| - |A3|) = 220+147-51=316

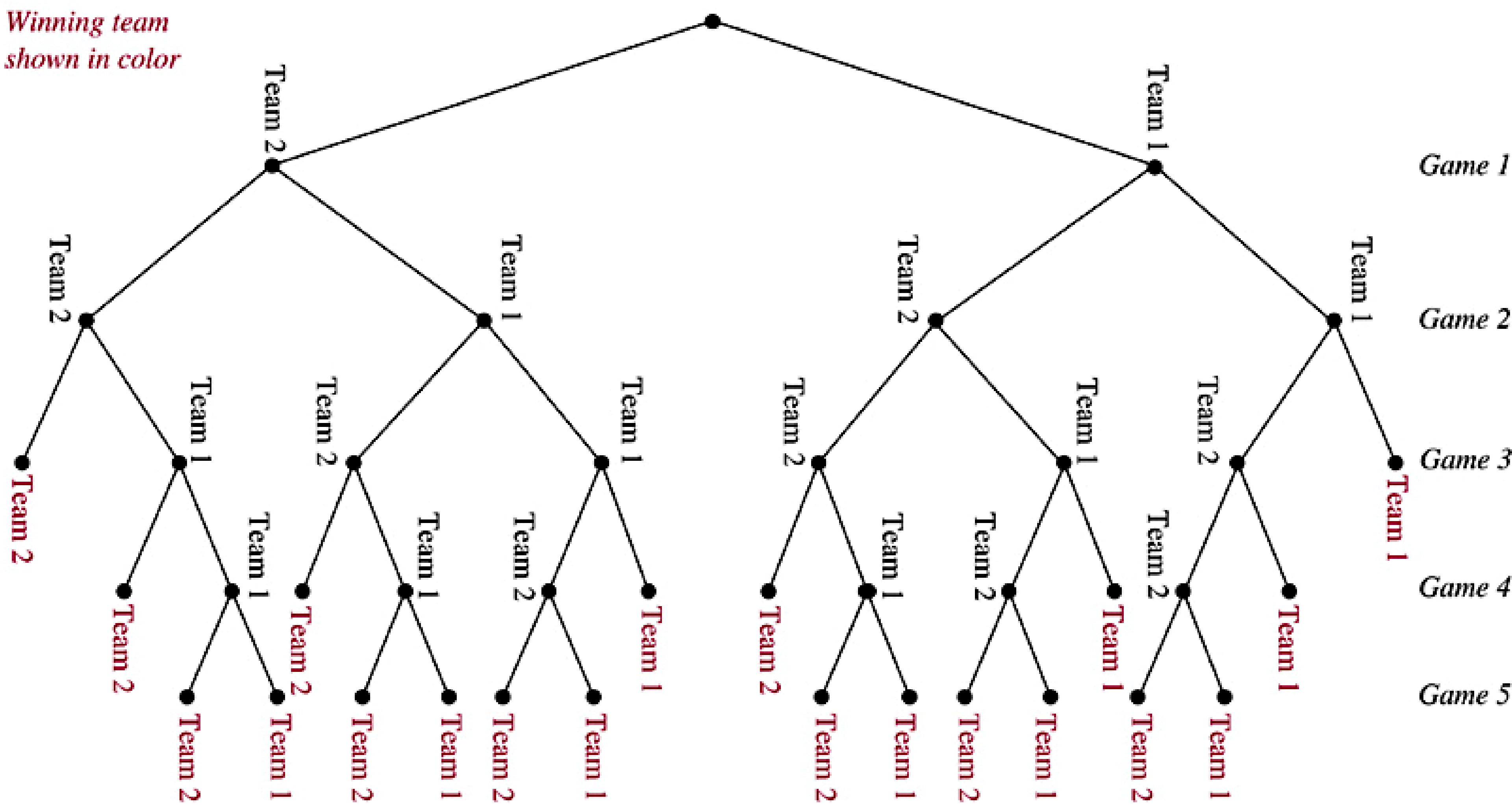
Result: 350 – 316 = 34

Tree Diagrams

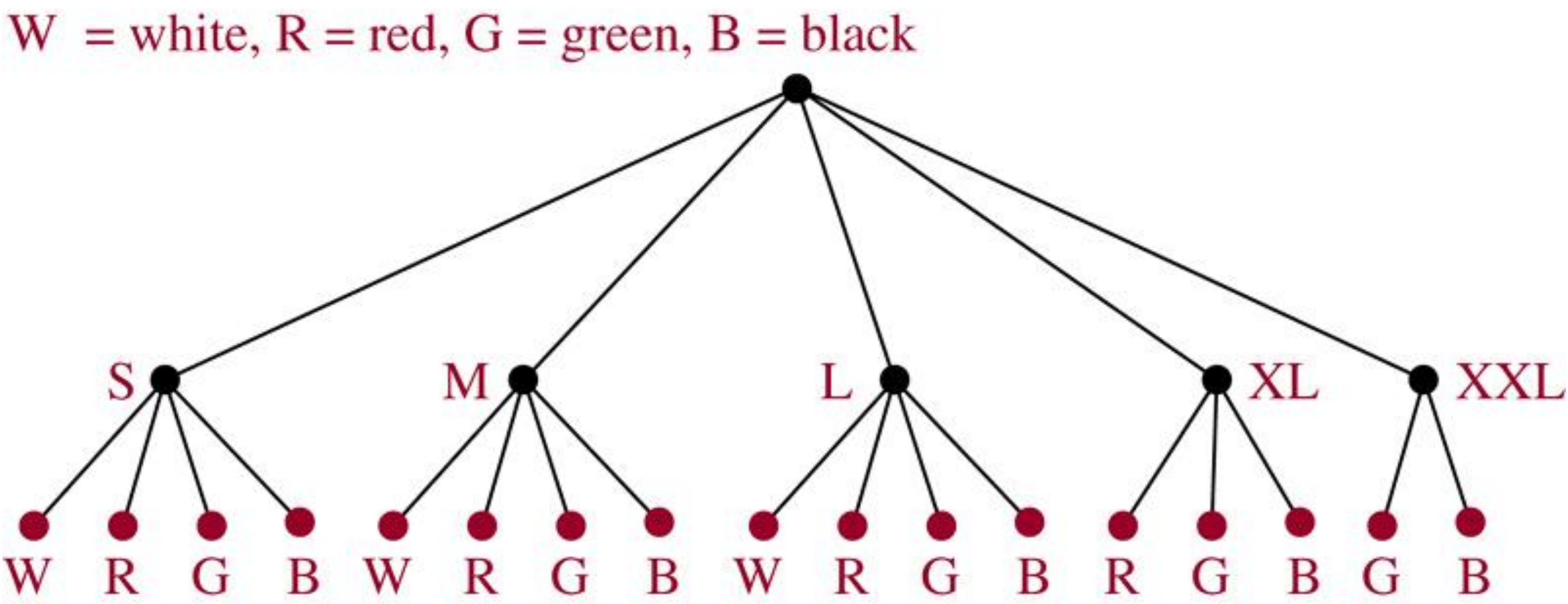
- Counting problem can be solved using **Tree Diagram.**
- **Example: How many 4-bit strings do not have two consecutive 1s?**



Example 20: A playoff between two teams consists of at lost 5 games. The first team that wins 3 games wins the playoff. In how many different ways can the playoff occur?



- Example 22: T-Shirts come in 5 sizes: S, M, L, XL, XXL. Each size comes in 4 colors, W, R, G, B excepts for XL, which comes only in R, G, B, and XXL, which comes only in G and B.
- How many different shirts a souvenir shop have to stock to have at least one of each available size and color of the T_shirt?



7.1. Recurrence Relations

- Definition 1: Recurrence relation on the sequence $\{ a_n \}$:

  a_n is expressed in terms of one or more of previous items for all n with $n \geq n_0$
- $\{a_n\}$ is called a **solution** of recurrence relation.
- Example 1: $a_0=3$, $a_1= 5$ $a_n=a_{n-1} – a_{n-2}$, $n>1$
- Example 2: Determine whether $\{a_n\} = 3n$, $n \geq 0$ is a solution of the recurrence relation $a_n= 2a_{n-1} – a_{n-2}$, $n \geq 2$?
- **Initial conditions**: Terms that **precede the first item where the recurrence takes effect.**

Modeling with Recurrence Relations

- Problem ← Modeling with recurrence relation
- Example 3 (page 451): Compound interest Problem **– Lãi gộp:**

P_0: Initial deposite

r: interest rate per year

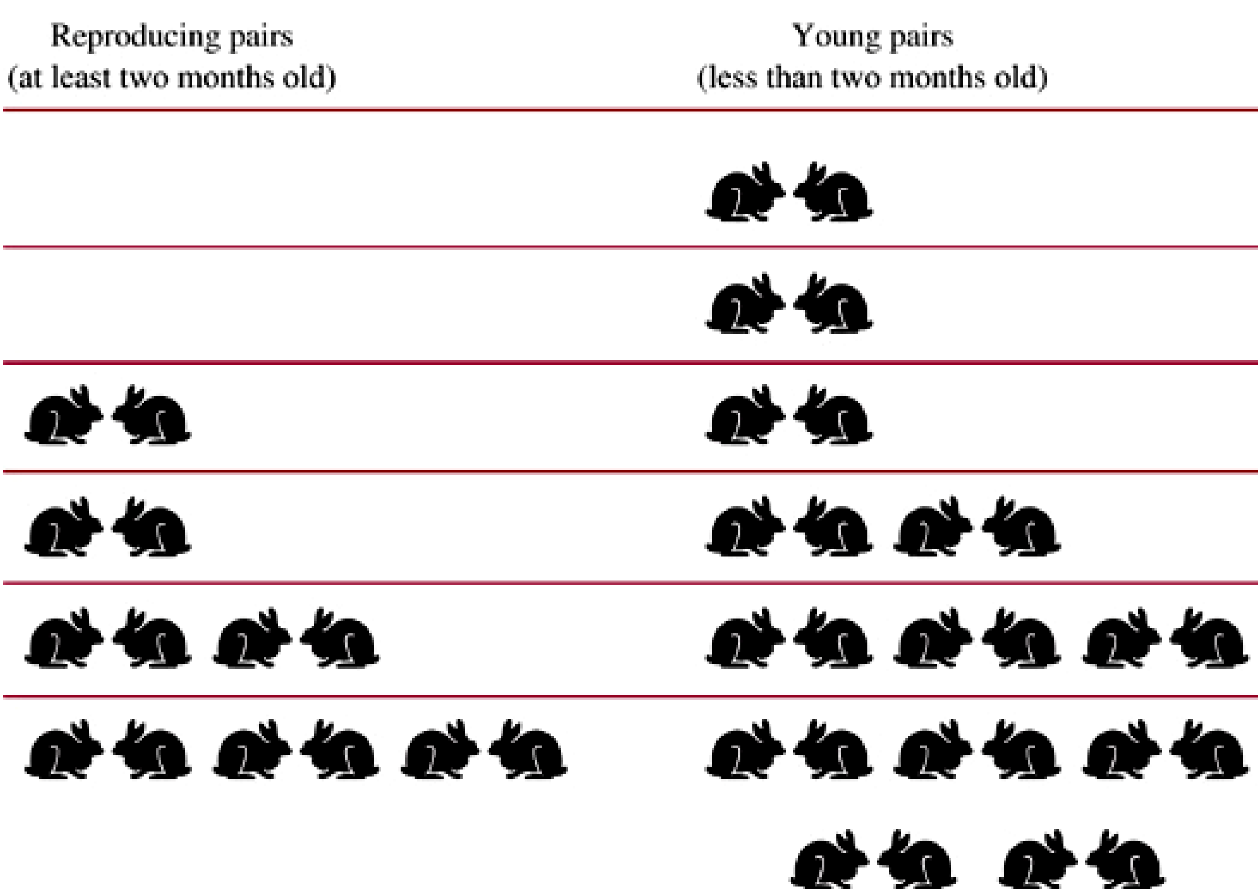
P_n: amount at the n^{th} year

$P_n = P_{n-1} + rP_{n-1}$

➔ $P_n = (1+r)^n P_0$

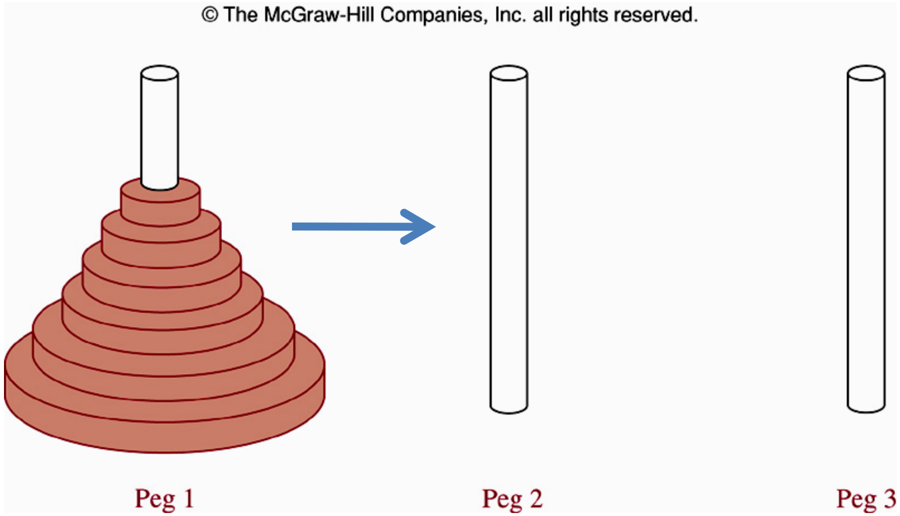
Fibonacci numbers of Leonardo Pisano

- Example 4: : $f_1=f_2=1;$ $f_n= f_{n-2} + f_{n-1}$



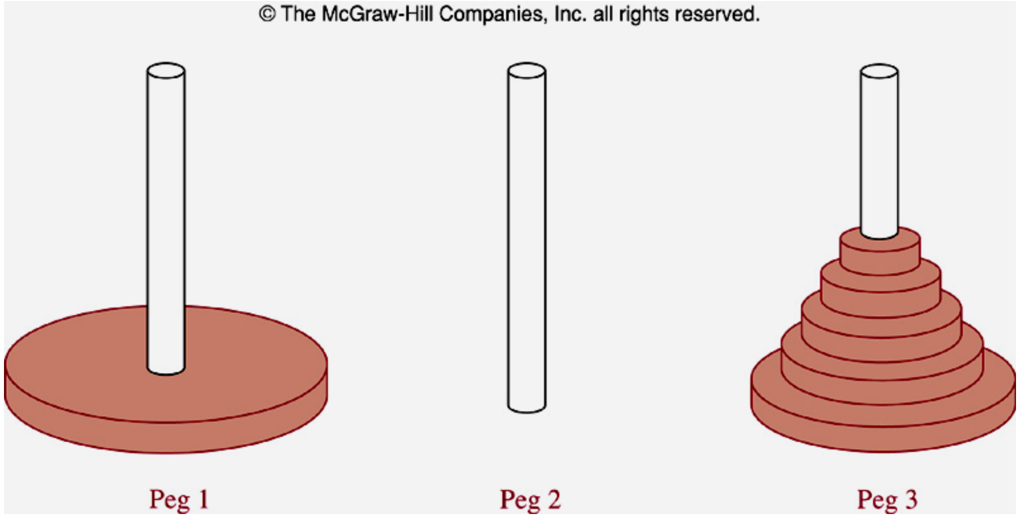
Month	Reproducing pairs	Young pairs	Total pairs
1	0	1	1
2	0	1	1
3	1	1	2
4	1	2	3
5	2	3	5
6	3	5	8

The Tower of Hanoi Problem- Ex5, page 452



How many steps this problem is solved if there is n disks on the peg 1?

The Tower of Hanoi Problem- Ex5, page 452



Let H_n is the number of moves needed to solve the Tower of Hanoi problem. We will set up a recurrence relation for the sequence H_n.

-Begin with n disk on peg 1.

-We transfer the top n-1 disk from peg1 to peg3 → H_{n-1} moves .

-We transfer the largest disk from peg1 to peg 2 → 1 move

-We transfer n-1 disk from peg3 to peg 2

→ $H_n = 2H_{n-1} + 1$

$H_1=1$ → $H_n = 2^n - 1$ (See page 453)

n=64 → $2^{64}-1=18\,446\,744\,073\,709\,551\,615$. With 1 move/sec → 500 billion years.

Find recurrence relation and give initial conditions for the number of bit strings of length n that do not have two consecutive 0s. How many such bit strings are there of length five.

- **a_n**: number of bit string of length n that do not have two consecutive 0s.
- One bit string can terminate with bit 1 or 0
- Format of a bit string that **do not have two consecutive 0s**:

End with 1	Any bit string of length n-1 with no two consecutive 0s	1
End with 0	Any bit string of length n-2 with no two consecutive 0s	1 0

→ $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$ { Fibonacci sequence }

→ $a_1=2$ { two string "0" and "1" }

→ $a_2= 3$ { "10" , "01" , "11" }

With n=5

→ $a_3 = a_2+ a_1 =5$

→ $a_4= a_3 + a_2 = 5+3=8$

→ $a_5=a_4 + a_3 = 8+5=13$

Counting valid codewords

valid codeword: String of decimal digits that contains an even number of 0 digits.

- a_n: number of valid n-digit codewords
- Find a recurrence relation for a_n
- $a_1=9$ // "1", "2", …, "9" ➔ "0" is not used
- Study a_n:

Valid codeword with n-1 digits	1..9	$9a_{n-1}$ ways
Invalid codeword with n-1 digits (number of 0s is odd)	0	$10^{n-1} - a_{n-1}$ ways

All possible codewords minus valid codewords

$$a_n = 9a_{n-1} + 10^{n-1} - a_{n-1} = 8a_{n-1} + 10^{n-1}$$

7.3. Divide-and-Conquer Algorithms and recurrence Relations

- **Divide:** Dividing a problem into one or more instances of the same problem of smaller size
- **Conquer**: Using the solutions of the smaller problems to find a solution of the original problem, perhaps with some additional work.

Divide-and-Conquer Recurrence Relations

- n : size of the original problem
- n/b : size of the sub-problem
- f(n) : number of operation required to solve the original problem.
- → f(n/b) : number of operation required to solve a sub-problem.
- g(n): overhead for additional work of the step **conquer.**
- Divide-and-conquer recurrence relation:

$$f(n) = af(n/b) + g(n)$$

Recurrence Relations for Binary Search

procedure binary-search(x, i, j)
if i>j **then** location=0
m= $\lfloor (i+j)/2 \rfloor$
if x= a_m **then** location =m
else if x< a_m **then** location= binary-search(x, i, m-1)
else location= binary-search(x, m+1, j)

$$f(n) = f(n/2) \quad +2$$

Recurrence Relations for Finding Maximum of a sequence

procedure max(i,j: integer ,a_i,a_{2+1},…,a_j: integers)
if i=j **then**
begin
max:= a_i
end
else
begin
m= $\lfloor (i+j)/2 \rfloor$
max1= max (i,m,a_i,a_{i+1},…,a_m)
max2= max (m+1,j,a_{m+1},a_{m+2},…,a_j)
if max1>max2 **then** max:= max1
else max:=max2

$$f(n) = 2f(n/2) + 1$$

Theorem 1

- Let f be an **increasing function** that satisfies the recurrence relation $f(n)= af(n/b) + c$

whenever n is divisible by b, where $a \geq 1$, b is an integer and greater than 1, and c is a positive real number. **Then**

$$f(n) \text{ is } \begin{cases} O(n^{\log_b a}) & \text{if } a>1 \\ O(\log n) & \text{if } a=1 \end{cases}$$

Furthermore, when $n=b^k$, where k is a positive integer,

$$f(n) = C_1 n^{\log_b a} + C_2$$

Where $C_1=f(1) + c/(a-1)$ and $C_2= -c/(a-1)$

Proof: page 477

Using Theorem 1

- Example 6: $f(n)= 5f(n/2) + 3$, $f(1)=7$. Find $f(2^k)$, k is a positive integer. Estimate f(n) if f is increasing function.

  Using theorem 1: $a= 5, b=2, c=3, n=2^k$

  $C_1=f(1) + c/(a-1) = 7+3/(5-1)= 7+3/4=31/4$

  $C_2= -c/(a-1) = -3/(5-1)= -3/4$

  $n^{\log a} = 2^{k \log a} = a^k$ $\{ \log a \equiv \log_2 a \}$

  $f(n)= f(2^k)\ C_1 a^k + C_2 = a^k.31/4 -3/4$

  f is increasing function , $a>1 \rightarrow f(n) = O(n^{\log a}) = O(n^{\log 5})$

- Estimate the number of comparisons used by a binary search

  $f(n) = f(n/2) +2$, $a=1 \rightarrow f(n) = O(\log n)$ // theorem 1

- Estimate the number of comparisons to locate the maximum element in a sequence

  $f(n) = 2f(n/2) +1$, $a=2 \rightarrow f(n) = O(n^{\log a}) = O(n)$ // theorem 1

Theorem 2: Master Theorem

Let f be an **increasing function that satisfies the recurrence relation $f(n) = af(n/b) + cn^d$**

Whenever $n = b^k$, where k is a positive integer, a $\geq$ 1, b is an integer greater than 1, and c and d are real numbers with c positive and d nonnegative. Then

$$f(n) \quad is \quad \begin{cases} O(n^d)\ if\ a < b^d \\ O(n^d \log n)\ if\ a = b^d \\ O(n^{\log_b a})\ if\ a > b^d \end{cases}$$

An Demonstration: Closest-Pair Problem

- n points in the plane. How to determine the closest-pair of points?
- (1) Determine the distance of every pair of points.
- (2) Determine the pair of points that have minimum distance.
- → $C(n,2)= n(n-1)/2=$ **$O(n^2)$**
- Michal Samos proposed an approach that is **O(nlogn)** only.
- Michal Samos's approach

(1) Sorting points in order of increasing x coordinates → O(nlog(n))

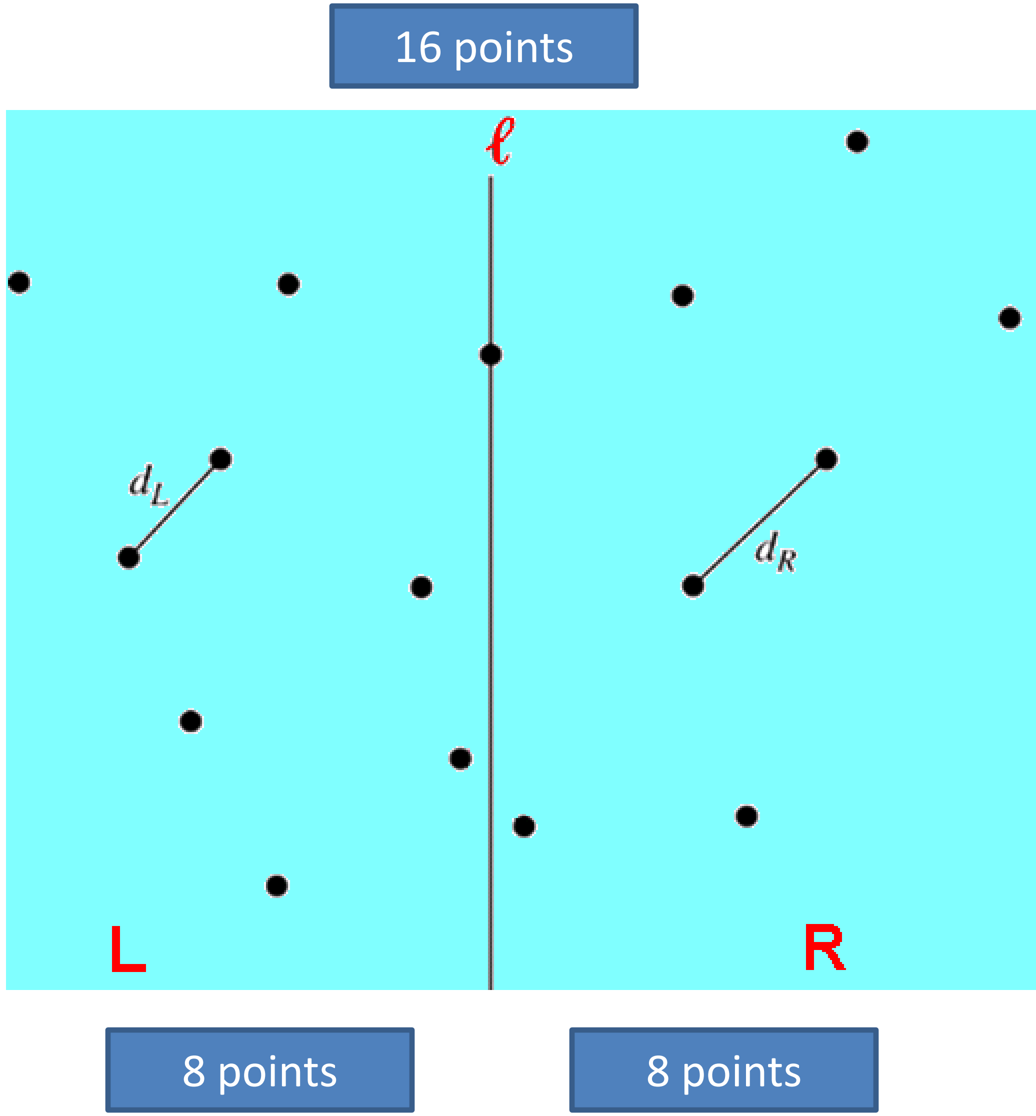
(2) Sorting points in order of increasing y coordinates → O(nlog(n))

An Demonstration: Closest-Pair Problem

(3) Using recursive approach to divide the problem into 2 subproblem with n/2 points (left and right points based on x coordinates). Let ℓ is the line that partitions two subproblems. If there is any point on this dividing line, we decide these points among the two parts if necessary)

(4) Finding out closest-pair of points in two side (d_L, d_R)

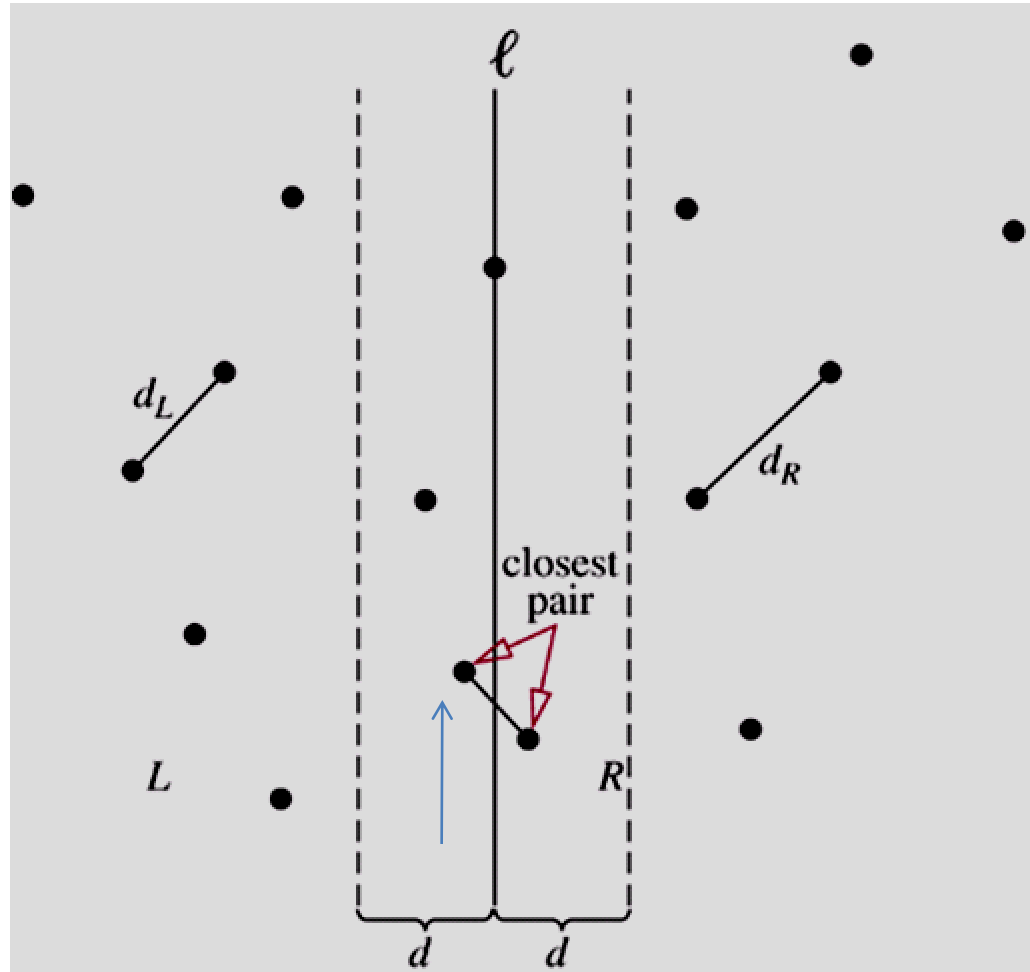
(5) Let $d=\min(d_L, d_R)$



An Demonstration: Closest-Pair Problem

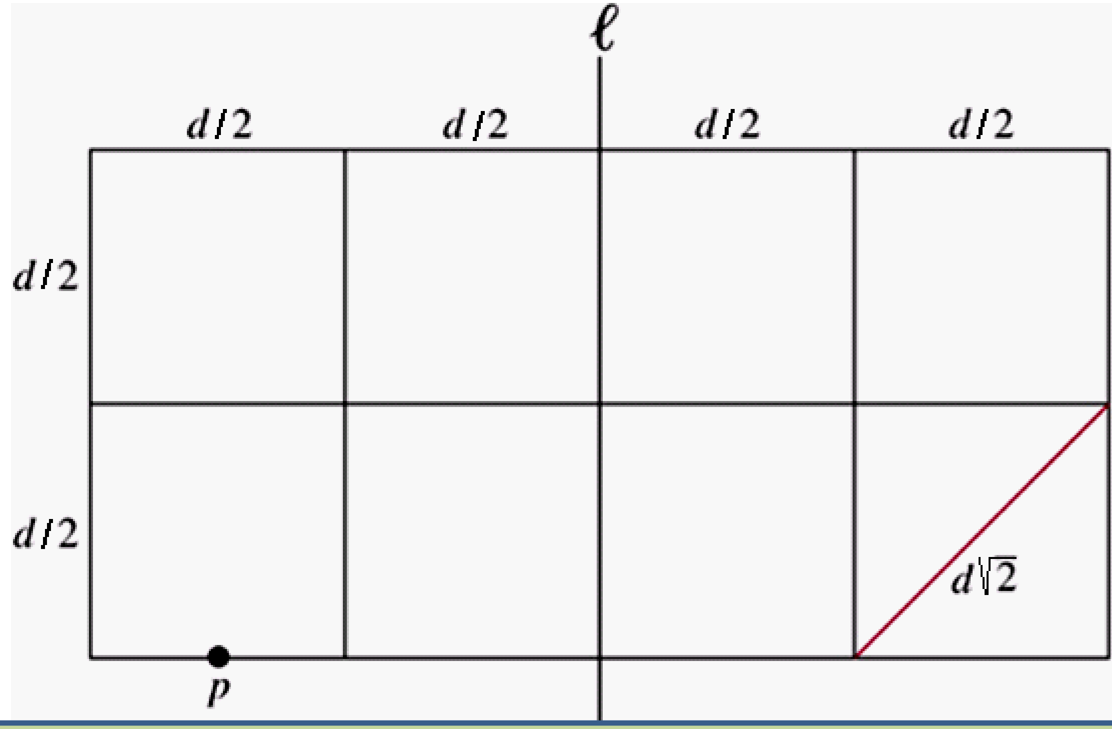
(6) Studying area [l-d,l+d]. This area may be contains the result.

(7) Because we sorted points by their y coordinate. We examine for points p in the strip of width 2d that has the line l as its center with upward direction.



An Demonstration: Closest-Pair Problem

Total number of points in the strip does not exceed n and there are at most 8 points, including p, can lie in or on the 2dxd rectangle.



➔A point will be computed with **7** others.

➔At most **7n** distances need to be compare with d to find the minimum distance between points.

➔ The increasing function f(n) satisfies the recurrence relation :

f(n) = 2f(n/2) + 7n

➔ By the Master Theorem, it follows that f(n) is **O(nlogn)**

Summary

- **Product rule**: Suppose that a procedure can be broken down into a sequence of two tasks. If there is n1 ways to do the first task, there is n2 ways to do the second task, then there are n1n2 ways to do the procedure.
- **Sum rule**: If a task can be done either in one of n1 ways or in one of n2 ways, where none of the set of n1 ways is the same as any of the set of n2 ways, then there are n1 + n2 ways to do this task.

Summary

- **Inclusion-Exclusion Principle**: Suppose that a task can be done in n1 or n2 ways, but that some ways in the set of n1 ways are the same as some ways in the set of n2 ways.

Summary

- **Recurrence relation** on the sequence $\{ a_n \}$ is an expression in which a_n is expressed in terms of one or more of previous items for all n with $n \geq n_0$. {an} is called a **solution** of recurrence relation.
- **Initial conditions**: Terms that **precede the first item where the recurrence takes effect.**
- **Modeling with recurrence relation:** Finding out an recurrence relation for input of a problem.

Summary

- **Divide:** Dividing a problem into one or more instances of the same problem of smaller size
- **Conquer**: Using the solutions of the smaller problems to find a solution of the original problem, perhaps with some additional work
- Recurrence relation: $\mathbf{f(n) = af(n/b) + g(n)}$

Summary – Theorem 1

- Let f be an **increasing function** that satisfies the recurrence relation f(n)= af(n/b) + c

  whenever n is divisible by b, where a ≥ 1 , b is an integer and greater than 1, and c is a positive real number. **Then**

$$f(n)\ is \begin{cases} O(n^{\log_b a}) & \textbf{if } a>1 \\ O(\log n) & \textbf{if } a=1 \end{cases}$$

Furthermore, when $n=b^k$, where k is a positive integer,

$$f(n) = C_1 n^{\log_b a} + C_2$$

Where C_1=f(1) + c/(a-1) and C_2= -c/(a-1)

Proof: page 477

Summary – Theorem 2- Master Theorem

Let f be an **increasing function that satisfies the recurrence relation $f(n) = af(n/b) + cn^d$**

Whenever $n = b^k$, where k is a positive integer, $a \geq 1$, b is an integer greater than 1, and c and d are real numbers with c positive and d nonnegative. Then

$$f(n) \quad is \quad \begin{cases} O(n^d)\, if\ a < b^d \\ O(n^d \log n)\, if\ a = b^d \\ O(n^{\log_b a})\, if\ a > b^d \end{cases}$$

- **THANKS**