ECE 863
Analysis of Stochastic Systems

Part III.2: Multiple Random Processes & Examples of Discrete- & Continuous-Time Processes

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ECE 863

- **Reading assignment**
  - Sections 6.1 – 6.5
  - Section 6.7

Multiple Random Processes

- **Two random processes $X(t,\zeta)$ and $Y(t,\zeta)$ can be generated by the random outcomes ζ.**
- **These two random processes, $X(t)$ and $Y(t)$, have joint statistics:**

  cross-correlation

  cross-covariance

  orthogonality

Multiple Random Processes

- **The cross-correlation function is defined as:**

$$\boxed{R_{X,Y}(t_1,t_2) = E\left[X(t_1)Y(t_2)\right].}$$

- **When Y=X, the cross-correlation function is the same as the autocorrelation function:**

$$R_{X,Y}(t_1,t_2) = R_{X,X}(t_1,t_2) = R_X(t_1,t_2).$$

Multiple Random Processes

- The <u>cross-covariance function</u> is defined as:

$$C_{X,Y}(t_1,t_2) \;=\; E\left[\{X(t_1) - m_x(t_1)\}\{Y(t_2) - m_y(t_2)\}\right]$$

$$\boxed{C_{X,Y}(t_1,t_2) = R_{X,Y}(t_1,t_2) - m_X(t_1)\,m_Y(t_2)}$$

Multiple Random Processes

- The two random processes X(t) and Y(t) are <u>orthogonal</u> if:

$$\boxed{R_{X,Y}(t_1,t_2) \;=\; 0} \qquad \boxed{\forall\; t_1 \text{ and } t_2}$$

- When X(t) and Y(t) are orthogonal:

$$C_{X,Y}(t_1,t_2) = -\,m_X(t_1)\,m_Y(t_2)$$

Multiple Random Processes

- The two random processes X(t) and Y(t) are <u>uncorrelated</u> when:

$$\boxed{C_{X,Y}(t_1,t_2) \;=\; 0} \qquad \boxed{\forall\; t_1 \text{ and } t_2}$$

- When X(t) and Y(t) are uncorrelated:

$$\boxed{R_{X,Y}(t_1,t_2) = m_X(t_1)\,m_Y(t_2)}$$

Example: Sinusoidal Amplitude

- Let ζ be a random variable, and let $X(t,\zeta)$ and $Y(t,\zeta)$ be two random processes:

$$X(t,\zeta) = \zeta\cos(t) \qquad Y(t,\zeta) = \zeta\sin(t)$$

Find the cross-covariance of $X(t,\zeta)$ and $Y(t,\zeta)$

$$C_{xy}(t_1,t_2) \;=\; R_{xy}(t_1,t_2) - m_x(t_1)\,m_y(t_2)$$

Example: Sinusoidal Amplitude

- First lets find the means:

$$m_x(t) = E[\zeta \cos(t)]$$
$$m_y(t) = E[\zeta \sin(t)]$$

$$\boxed{m_x(t) = E[\zeta]\cos(t)}$$

$$\boxed{m_y(t) = E[\zeta]\sin(t)}$$

Example: Sinusoidal Amplitude

- The cross-correlation function

$$R_{xy}(t_1,t_2) = E[\zeta\cos(t_1)\zeta\sin(t_2)]$$

$$\boxed{R_{xy}(t_1,t_2) = E\left[\zeta^2\right]\cos(t_1)\sin(t_2)}$$

Example: Sinusoidal Amplitude

- The cross-covariance function:

$$C_{xy}(t_1,t_2) = R_{xy}(t_1,t_2) - m_x(t_1)m_y(t_2)$$

$$= \left\{E\left[\zeta^2\right] - (E[\zeta])^2\right\}\cos(t_1)\sin(t_2)$$

$$\boxed{C_{xy}(t_1,t_2) = VAR(\zeta)\cos(t_1)\sin(t_2)}$$

Example: Sinusoidal Phase

- Let Θ be a uniform random variable over the interval $(-\pi, \pi)$, and let the two random processes:

$$X(t) = \cos(t+\Theta) \quad \& \quad Y(t) = \sin(t+\Theta),$$

Find the cross-covariance function of $X(t,\Theta)$ and $Y(t,\Theta)$

Example: Sinusoidal Phase

- **The mean values $m_X(t)$ and $m_Y(t)$:**

$$m_x(t) = E[\cos(t + \Theta)]$$

$$m_x(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(t + \theta)\,d\theta$$

$$\boxed{m_x(t) = 0} \quad \text{similarly} \quad \boxed{m_y(t) = 0}$$

Example: Sinusoidal Phase

- **The cross-covariance function:**

$$C_{xy}(t_1, t_2) = R_{xy}(t_1, t_2) - m_x(t_1)\,m_y(t_2)$$

$$C_{xy}(t_1, t_2) = R_{xy}(t_1, t_2)$$

$$C_{xy}(t_1, t_2) = E[\cos(t_1 + \Theta)\sin(t_2 + \Theta)]$$

Example: Sinusoidal Phase

$$R_{X,Y}(t_1, t_2) = E[\cos(t_1 + \Theta)\,\mathbf{sin}(t_2 + \Theta)]$$

$$= E\left[-\frac{1}{2}\sin(t_1 - t_2) + \frac{1}{2}\sin((t_1 + t_2) + 2\Theta)\right]$$

Since Θ is uniformly distributed $[-\pi,\pi]$

$$\Rightarrow \quad E[\sin((t_1 + t_2) + 2\Theta)] = 0.$$

$$\boxed{C_{xy}(t_1, t_2) = R_{xy}(t_1, t_2) = -\frac{1}{2}\sin(t_1 - t_2)}$$

Independent Random Processes

- **Let the two random processes X(t) and Y(t) generates the two random-variable vectors:**

$$(X_1, \ldots, X_k) = (X(t_1), \ldots, X(t_k))$$

$$(Y_1, \ldots, Y_j) = (Y(t_1'), \ldots, Y(t_j'))$$

- **The two random processes X(t) and Y(t) are independent when the two vectors are independent for all values of k, j, and time-indexes:** $t_1, \ldots, t_k \;\; \& \;\; t_1', \ldots, t_j'$.

Independent Random Processes

- Therefore, the two random processes X(t) and Y(t) are independent when:

$$F_{X_1,\ldots,X_k,Y_1,\ldots,Y_j}\left(x_1,\ldots,x_k,y_1,\ldots,y_j\right) = F_{X_1}(x_1)\ldots F_{X_k}(x_k)F_{Y_1}(y_1)\ldots F_{Y_j}\left(y_j\right)$$

$$\forall \;\; k, \;\; j, \;\; t_1,\ldots,t_k \;\; \& \;\; t_1',\ldots,t_j'$$

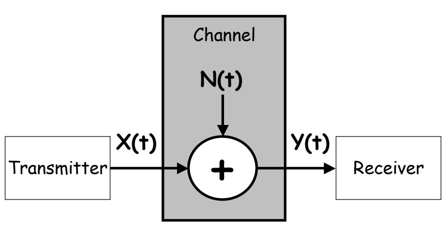
Example: Independent Noise & Signal

- Let X(t) be a random signal generated at the transmitter of a communication system with mean $m_X(t)$ and autocorrleation function $R_X(t_1,t_2)$. Let N(t) be a zero-mean additive noise (random) process, that corrupts the signal X(t). (The two processes N(t) and X(t) are independent.)

Find the cross-correlation function between the received and transmitted signals.

Example: Independent Noise & Signal

The channel has an input X(t) and output Y(t)
The cross-correlation function is a measure for how the input and output are related



Example: Independent Noise & Signal

- The additive noise N(t) leads to the process

$$Y(t) = X(t) + N(t)$$

at the receiver.

$$R_{X,Y}(t_1,t_2) = E\left[X(t_1)Y(t_2)\right]$$

$$= E\left[X(t_1)\left\{X(t_2) + N(t_2)\right\}\right]$$

Example: Independent Noise & Signal

$$R_{X,Y}(t_1,t_2) = E\left[X(t_1)X(t_2)\right] + E\left[X(t_1)N(t_2)\right]$$

$$= R_X(t_1,t_2) + E\left[X(t_1)\right]E\left[N(t_2)\right]$$

$$\boxed{R_{X,Y}(t_1,t_2) = R_X(t_1,t_2) + m_X(t_1)m_N(t_2)}$$

Example: Independent Noise & Signal

- **Since N(t) is a zero-mean random process:**

$$m_N(t) = 0 \quad \forall\, t$$

Then

$$R_{X,Y}(t_1,t_2) = R_X(t_1,t_2) + m_X(t_1)m_N(t_2)$$

$$\Rightarrow \quad \boxed{R_{X,Y}(t_1,t_2) = R_X(t_1,t_2)}$$

Example: Independent Noise & Signal

Y(t) = X(t) + N(t) $\quad m_N(t) = 0 \quad \forall\, t$

$$\boxed{R_{X,Y}(t_1,t_2) = R_X(t_1,t_2)}$$

- **Therefore, for a "zero-mean, independent, additive noise process", the cross-correlation function between the input and output processes is the same as the auto-correlation function of the input process.**
- **What's about the cross-covariance $C_{X,Y}(t_1,t_2)$?**

Discrete-Time Random Processes

- **A discrete-time random process X_{n_i} is a RP with countable time index-set:**

$$\{n_1 , n_2 , \ldots\}$$

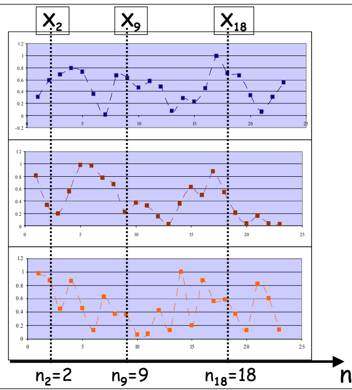
- **To simplify the notation, let the countable time indexes be integers:**

$$\{n_1 , n_2 , \ldots\} = \{1, 2, \ldots \}$$

- **Therefore, a discrete-time process X_n generates a sequence of random variables:**

$$X_1 , X_2 , \ldots$$

Discrete-Time Random Processes



IID Random processes

- A discrete-time random process X_n is an iid process when the sequence:

  $$X_1 \ , \ X_2 \ , \ \ldots.$$

  is a set of independent-identically-distributed random variables

- Therefore, the sequence have the same cdf:

$$\boxed{F_{X_1}(x) = F_{X_2}(x) \ldots = F_{X_k}(x) \triangleq F_X(x)}$$

IID Random processes

- Therefore, the joint cdf of an iid process:

$$F_{X_1,\ldots,X_k}(x_1, x_2, \ldots, x_k) = P\left[X_1 \le x_1, X_2 \le x_2, \ldots, X_k \le x_k\right]$$

$$F_{X_1,\ldots,X_k}(x_1, x_2, \ldots, x_k) = F_X(x_1) F_X(x_2) \ldots F_X(x_k)$$

By evaluating the joint cdf at the same value x:

$$\boxed{F_{X_1,\ldots,X_k}(x, x, \ldots, x) = \left(F_X(x)\right)^k}$$

IID Random processes

- For an iid process X_n:

$$\boxed{E\left[X_n\right] = m_X(n) \quad = m \qquad \forall n}$$

$$C_X(n_1, n_2) = E\left[\left(X_{n_1} - m\right)\left(X_{n_2} - m\right)\right]$$

$$\forall \ n_1 \ne n_2$$

$$C_X(n_1, n_2) = E\left[\left(X_{n_1} - m\right)\right] E\left[\left(X_{n_2} - m\right)\right]$$

IID Random processes

- **Therefore, for an iid process X_n:**

$$\forall\, n_1 \neq n_2$$
$$C_X(n_1,n_2) = E\left[\left(X_{n_1} - m\right)\right] E\left[\left(X_{n_2} - m\right)\right]$$

$$\boxed{C_X(n_1,n_2) = 0 \qquad \forall\, n_1 \neq n_2}$$

IID Random processes

- **However, for an iid process X_n:**

$$C_X(n_1,n_2) = E\left[\left(X_{n_1} - m\right)\left(X_{n_2} - m\right)\right]$$

$$\boxed{\begin{gathered}\forall\, n_1 = n_2 = n \\ C_X(n,n) = E\left[\left(X_n - m\right)^2\right] = \sigma^2\end{gathered}}$$

IID Random processes

- **Therefore, for an iid process X_n:**

$$\boxed{C_X(n_1,n_2) = 0 \qquad \forall\, n_1 \neq n_2}$$
$$\boxed{C_X(n_1,n_2) = C_X(n,n) = \sigma^2 \qquad \forall\, n_1 = n_2 = n}$$

$$\boxed{C_X(n_1,n_2) = \sigma^2\, \delta(n_1,n_2)}$$

$$\boxed{\begin{aligned}\text{where } \delta(n_1,n_2) &= 0 \quad \forall\, n_1 \neq n_2 \\ &= 1 \quad \forall\, n_1 = n_2\end{aligned}}$$

IID Random processes

- **Moreover, for an iid process X_n:**

$$R_X(n_1,n_2) = C_X(n_1,n_2) + m^2$$

$$\boxed{R_X(n_1,n_2) = \sigma^2\, \delta(n_1,n_2) + m^2}$$

IID Sum Random Processes

- Special cases of iid sum processes are the "random walk" and binomial "counting" processes
- Similar to the sum of sequences of iid RVs, the sum process:

$$S_n = X_1 + X_2 + \cdots + X_n$$

$$n = 1, 2, \ldots$$

IID Sum Random Processes

- Also similar to the sum of iid RVs, if

$E[X_1] = E[X_2] = \ldots = E[X_n] = E[X] = m$, then:

$$m_S(n) = E[S_n] \qquad \boxed{m_S(n) = nm}$$

Similarly for the variance of S_n:

$$VAR(S_n) = nVAR(X) \qquad \boxed{VAR(S_n) = n\sigma^2}$$

IID Sum Random Processes

- Note that:

$$S_{n-1} = X_1 + X_2 + \cdots + X_{n-1}$$

$$\boxed{S_n = S_{n-1} + X_n}$$

$$n = 1, 2, \ldots \text{ and where } S_0 = 0$$

Therefore, at any "discrete-time instance" n, the sum process S_n depends on the past only through the the previous instance (n-1): S_{n-1}

Example: Binomial Process

- If $X_n = I_n$ is a Bernoulli process with probability $p = P[I_n=1]$ ($P[I_n=0] = 1-p$), then the sum:

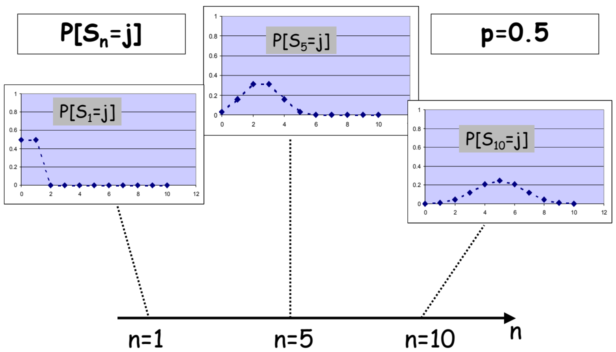
$$S_n = I_1 + I_2 \cdots I_n$$

is a "Binomial Process":

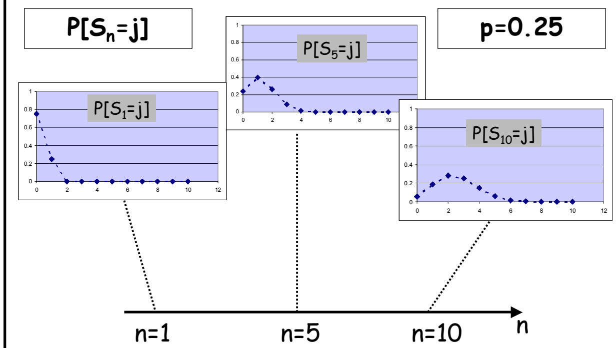
$$P[S_n = j] = \binom{n}{j} p^j (1-p)^{n-j}$$

$$\forall\, 0 \le j \le n$$

Example: Binomial Process



Example: Binomial Process



IID Sum Proc. - Covariance

- **The autocovariance of the sum process S_n can be found using:**

$$C_S(n_1,n_2) = E\left[\left(S_{n_1} - E\left[S_{n_1}\right]\right)\left(S_{n_2} - E\left[S_{n_2}\right]\right)\right]$$

$$C_S(n_1,n_2) = E\left[\left(S_{n_1} - n_1 m\right)\left(S_{n_2} - n_2 m\right)\right]$$

IID Sum Proc. - Covariance

$$C_S(n_1,n_2) = E\left[\left(S_{n_1} - n_1 m\right)\left(S_{n_2} - n_2 m\right)\right]$$

$$C_S(n_1,n_2) = E\left[\left\{\sum_{i=1}^{n_1}\left(X_i - m\right)\right\}\left\{\sum_{j=1}^{n_2}\left(X_j - m\right)\right\}\right]$$

$$C_S(n_1,n_2) = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} E\left[\left(X_i - m\right)\left(X_j - m\right)\right]$$

IID Sum Proc. - Covariance

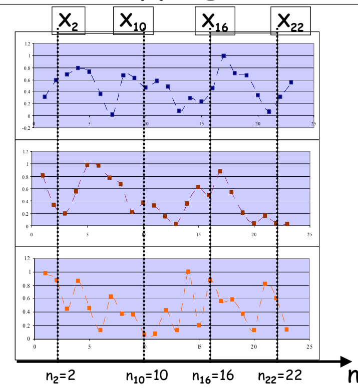
$$C_S(n_1, n_2) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} E\left[(X_i - m)(X_j - m)\right]$$

When $i \neq j$, then we have zero terms since X_i and X_j are uncorrelated (because they are independent)

$$C_S(n_1, n_2) = \sum_{i=1}^{\min(n_1, n_2)} C_X(i, i)$$

$$\boxed{C_S(n_1, n_2) = \min(n_1, n_2)\,\sigma^2}$$

Non-overlapping Intervals



IID Sum Proc. - Non-overlapping Intervals

- **Over non-overlapping intervals, the sum process S_n has non-overlapping iid RVs**
- **Let $S(n_1,n_2) = S_{n2}-S_{n1}$ and $S(n_3,n_4) = S_{n4}-S_{n3}$ where $n_1 < n_2 < n_3 < n_4$**
- **Therefore,**

$$S(n_1,n_2) = X_{n1+1} + X_{n1+2} + \ldots + X_{n2}$$

$$S(n_3,n_4) = X_{n3+1} + X_{n3+2} + \ldots + X_{n4}$$

IID Sum Proc. - Non-overlapping Intervals

- **Since $S(n_1,n_2)$ and $S(n_3,n_4)$ are functions of "non-overlapping" independent random variables, then $S(n_1,n_2)$ and $S(n_3,n_4)$ are independent:**

$$\boxed{\begin{aligned} &P[S(n_1,n_2) = k_1\ ,\ S(n_3,n_4) = k_2] \\ &= P[S(n_1,n_2) = k_1]\ P[S(n_3,n_4) = k_2] \end{aligned}}$$

The iid sum process S_n has <u>independent increments</u>

IID Sum Proc. - Non-overlapping Intervals

- **Since**

$$S(n_1,n_2) = S_{n_2} - S_{n_1} = X_{n_1+1} + X_{n_1+2} + .. + X_{n_2}$$

is the sum of (n_2-n_1) iid random variables

$$\boxed{P[\ S(n_1,n_2) = k\ ] = P[\ S_{n_2-n_1} = k\ ]}$$

The iid sum process S_n has <u>stationary increments</u>

IID Sum Proc. - Non-overlapping Intervals

- **The "<u>independent increments</u>" and "<u>stationary increments</u>" properties of the iid sum process S_n can be used to compute joint probability functions (e.g., pmf and pdf):**

$$\boxed{\begin{aligned} &P[\ S_{n_1} = k_1, S_{n_2} = k_2\ ] = \\ &P[\ S_{n_1} = k_1]\,P[\ S_{n_2} - S_{n_1} = k_2 - k_1\ ] \end{aligned}}$$

$$\boxed{\begin{aligned} &P[\ S_{n_1} = k_1, S_{n_2} = k_2\ ] = \\ &P[\ S_{n_1} = k_1]\,P[\ S_{n_2-n_1} = k_2 - k_1\ ] \end{aligned}}$$

IID Sum Proc. - Non-overlapping Intervals

- **For example,** $P[\ S_{n_1} = k_1, S_{n_2} = k_2\ ]$

$$= P[\ S_{n_1} = k_1\ ,\ \ S_{n_2} - S_{n_1} = k_2 - k_1\ ]$$

independent increments $\Rightarrow$

$$\boxed{\begin{aligned} &P[\ S_{n_1} = k_1, S_{n_2} = k_2\ ] = \\ &P[\ S_{n_1} = k_1]\,P[\ S_{n_2} - S_{n_1} = k_2 - k_1\ ] \end{aligned}}$$

$$\Downarrow$$

stationary increments $\Rightarrow$

$$\boxed{\begin{aligned} &P[\ S_{n_1} = k_1, S_{n_2} = k_2\ ] = \\ &P[\ S_{n_1} = k_1]\,P[\ S_{n_2-n_1} = k_2 - k_1\ ] \end{aligned}}$$

IID Sum Proc. - Non-overlapping Intervals

- **In general, for "discrete-valued" X_i of the iid sum process S_n=X_1+X_2+ ....+X_n ,**

$$\boxed{\begin{aligned} &P[\ S_{n_1} = k_1, S_{n_2} = k_2, ...\ S_{n_m} = k_m\ ] = \\ &P[\ S_{n_1} = k_1]\prod_{i=2}^{m} P[\ S_{n_i - n_{i-1}} = k_i - k_{i-1}\ ] \end{aligned}}$$

(See example 6.16)

IID Sum Proc. - Non-overlapping Intervals

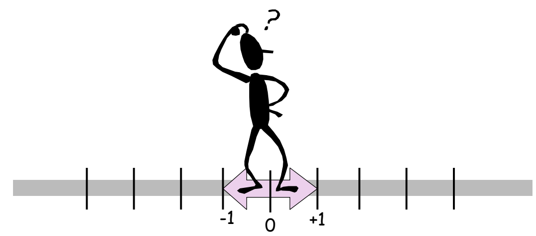
- For "continuous-valued" X_i of the iid sum process $S_n = X_1 + X_2 + \cdots + X_n$,

$$\boxed{f_{S_{n_1}, S_{n_2}, \ldots, S_{n_m}}(z_1, z_2 \ldots z_m) = f_{S_{n_1}}(z_1) \prod_{i=2}^{m} f_{S_{n_i - n_{i-1}}}(z_i - z_{i-1})}$$

(See example 6.17)

The "Random Walk" Process

- A person or an object taking a step with a random direction (e.g. +1 , -1) generates what is known as the "random walk" process.



The "Random Walk" Process

- The random outcome D_n (-1 or +1) of the random walk process at any given time-index n, n=1,2,…. can be expressed as a function of the Bernoulli process I_n (0 or +1):

$$D_n = 2I_n - 1$$

The "Random Walk" Process

- Therefore, when

$$D_n = 2I_n - 1$$

we have:

$$\boxed{D_n = \begin{cases} 1 & \text{if } I_n = 1 \\ -1 & \text{if } I_n = 0. \end{cases}}$$

$$\boxed{P[D_n = +1] = P[I_n = 1] = p}$$

$$\boxed{P[D_n = -1] = P[I_n = 0] = 1 - p}$$

The "Random Walk" Process

- The mean D_n:

$$m_D(n) = E[D_n] = E[2I_n - 1]$$

$$m_D(n) = 2E[I_n] - 1 = 2p - 1.$$

$$\boxed{m_D(n) = 2p - 1.}$$

The "Random Walk" Process

- The variance for D_n:

$$VAR[D_n] = VAR[2I_n - 1]$$

$$VAR[D_n] = 4VAR[I_n]$$

$$\boxed{VAR[D_n] = 4p(1-p)}$$

The "Random Walk" Process

- The random walk process can be expressed as the sum of the iid process D_n:

$$S_{Dn} = D_1 + D_2 + \cdots + D_n$$

Therefore, S_{Dn} represents the position of the object after taking n random steps over a straight line (one-dimensional random walk).

The "Random Walk" Process

- Now, the probability that "the object is d steps away from the origin after taking n random steps":

$$P[S_{Dn}=d]$$

is the same as the probability that the object takes k positive steps (+1) and n-k negative steps (-1): $d = k-(n-k) = 2k-n$

$$P[S_{Dn}=d] = P[S_{Dn}=2k-n]$$

The "Random Walk" Process

- **The event "S_{Dn}=2k-n" is equivalent to the event that there are k "successes" after n "trials" of a binomial process:**

$$P\left[S_{Dn} = d\right] = P\left[S_{Dn} = 2k - n\right]$$

$$\boxed{P\left[S_{Dn} = 2k - n\right] = P\left[S_n = k\right]} \quad \forall k \in \{0,1,\ldots,n\}$$

where S_n is the binomial process (i.e. sum of Bernoulli iid RVs)

The "Random Walk" Process

- **Therefore,:**

$$\boxed{\begin{aligned} P\left[S_{Dn} = d\right] &= P\left[S_{Dn} = 2k - n\right] \\ &= \binom{n}{k} p^k (1-p)^{n-k} \end{aligned}}$$

$$\forall k \in \{0,1,\ldots,n\}$$

The "Random Walk" Process

- **Since d=2k-n where k=0,1,2,…n then:**

when n is even ⇒
d is even ⇒ d ∈ {0, ± 2, ± 4,… ± n}

when n is odd ⇒
d is odd ⇒ d ∈ {± 1, ± 3,… ± n}

The "Random Walk" Process

- **Therefore:**

$$\boxed{\begin{aligned} P\left[S_{Dn} = d\right] &= P\left[S_{Dn} = 2k - n\right] \\ &= \binom{n}{\left(\frac{n+d}{2}\right)} p^{(n+d)/2} (1-p)^{(n-d)/2} \end{aligned}}$$

The "Random Walk" Process

$$P\left[S_{Dn} = d\right] = \left(\begin{array}{c} n \\ \left(\frac{n+d}{2}\right) \end{array} \right) p^{(n+d)/2} (1-p)^{(n-d)/2}$$

It is important to note that:

$\left(\frac{n+d}{2}\right)$ and $\left(\frac{n-d}{2}\right)$ are always integer

The "Random Walk" Process

- Since the random walk process is an iid sum process, then its covariance:

$$C_{S_D}(n_1, n_2) = VAR\left[D_n\right] \min(n_1, n_2)$$

$$\boxed{C_{S_D}(n_1, n_2) = 4p(1-p)\min(n_1, n_2)}$$

Continuous-Time Processes

- Continuous-time processes arises in many applications and systems (e.g. queueing systems)
- Important continuous-time processes include:

  The Wiener Process (related to Random Walk)

  The Poisson Process (related to Binomial proc.)

  Other processes are derived from the above (e.g., Random Telegraph Signal from Poisson)

The Wiener Processes

- The "random walk" process can be extended to a continuous-time process under certain conditions
- First, let the step size be of size h
- Second, let the probability of a "+h" step is the same as a "-h" step (i.e., p=1/2)
- Third, let the object takes a step (+h or -h) every δ time-units (e.g. seconds)

The Wiener Processes

- Now, we can define the sum process, $X_\delta(t)$ which represents the position of the object at time t
- Therefore, by time t, the object takes n steps, where:

$$n = \left\lfloor \frac{t}{\delta} \right\rfloor$$

$\lfloor x \rfloor$ is the largest integer smaller than or equal to x

The Wiener Processes

- Therefore, $X_\delta(t)$ is also a sum process:

$$X_\delta(t) = h\left(D_1 + D_2 + \cdots + D_{\lfloor t/\delta \rfloor}\right)$$

$$X_\delta(t) = h(D_1 + D_2 + \cdots + D_n)$$

$$\boxed{X_\delta(t) = hS_{Dn}}$$

The Wiener Processes

- Now, we can evaluate the mean and variance of $X_\delta(t)$ using results from the "random walk" process: $\boxed{X_\delta(t) = hS_{Dn}}$

$$\boxed{E[X_\delta(t)] = hE[S_{Dn}] = 0}$$

$$\boxed{VAR[X_\delta(t)] = h^2 n\ VAR[D_n] = h^2 n}$$

remember $VAR[D_n] = 4p(1-p)$

The Wiener Processes

- Now, we would like to express the variance (hn^2) as a function of time (t) rather than as a function of the integer (n): $n = \lfloor t/\delta \rfloor$
- We are also going to make the object takes very small steps (h gets very small) and much more often (δ gets very small)
- Therefore, both h and δ get small simultaneously

The Wiener Processes

- **Hence, h is "proportional to" δ :**

$$h = \sqrt{\alpha\,\delta}$$

you can think of $\sqrt{\alpha}$ as a "proportionality" constant

- **Also, as δ gets very small,**

$$n = \lfloor t/\delta \rfloor \cong t/\delta$$

$$\boxed{\lim_{\substack{h\to 0\\ \delta\to 0}} VAR\left[X_\delta(t)\right] = \lim_{\substack{h\to 0\\ \delta\to 0}} h^2 n = \alpha\, t}$$

The Wiener Processes

- **Therefore, the resulting process:**

$$X(t) = \lim_{\substack{h\to 0\\ \delta\to 0}} X_\delta(t)$$

has a variance (αt) **and zero-mean**

- **Also, X(t) results from a sum of a very large number** $(n\to\infty)$ **of iid RVs**
- **Therefore, X(t) is a Gaussian process**

The Wiener Processes

- **Therefore, the Wiener Process has a pdf:**

$$\boxed{f_{X(t)}(x) = \frac{1}{\sqrt{2\pi\alpha t}}\, e^{-x^2/2\alpha t}}$$

The Wiener Processes

- **The Wiener Process meets the "independent increments" and "stationary increments" properties:**

$$f_{X(t_1),\ldots,X(t_k)}(x_1, x_2 \ldots x_{k-1}, x_k) =$$
$$f_{X(t_1)}(x_1)\, f_{X(t_2-t_1)}(x_2 - x_1)\cdots f_{X(t_k-t_{k-1})}(x_k - x_{k-1})$$

The Wiener Processes

- Therefore,

$$f_{X(t_1),\ldots,X(t_k)}\left(x_1,x_2\ldots x_{k-1},x_k\right)=\frac{\exp\left\{-\frac{1}{2}\left[\frac{x_1^2}{\alpha t_1}+\frac{\left(x_2-x_1\right)^2}{\alpha\left(t_2-t_1\right)}+\cdots+\frac{\left(x_k-x_{k-1}\right)^2}{\alpha\left(t_k-t_{k-1}\right)}\right]\right\}}{\sqrt{(2\pi\alpha)^k t_1\left(t_2-t_1\right)\cdots\left(t_k-t_{k-1}\right)}}$$

The Wiener Processes

- Is the Wiener process a <u>Gaussian process</u> ?
- A random process X(t) is a Gaussian process when the RVs $X(t_1)$, $X(t_2)$, …. $X(t_k)$ are jointly Gaussian
- For the Wiener process X(t), the RVs: $X(t_1)$, ($X(t_2)$- $X(t_1)$), …. ($X(t_k)$- $X(t_{k-1})$) are independent Gaussian RVs

The Wiener Processes

- Since the RVs:
  $X(t_1)$, ($X(t_2)$- $X(t_1)$), …. ($X(t_k)$- $X(t_{k-1})$)
  are independent Gaussian RVs
  it can be shown that:
  $X(t_1)$, $X(t_2)$, …. $X(t_k)$ are jointly Gaussian

  Therefore:
  <u>the Wiener process is a Gaussian process</u>

The Wiener Processes

- The "independent and stationary increment" property enables us to compute the covariance:

$$C_X\left(t_1,t_2\right) = E\left[X\left(t_1\right)X\left(t_2\right)\right]$$

First let's assume that $t_1 \le t_2$

$$C_X\left(t_1,t_2\right) = E\left[X\left(t_1\right)\left\{\left(X\left(t_2\right)-X\left(t_1\right)\right)+X\left(t_1\right)\right\}\right]$$

The Wiener Processes

$$C_X(t_1,t_2) = E\left[X(t_1)\left\{\left(X(t_2)-X(t_1)\right)+X(t_1)\right\}\right]$$

$$C_X(t_1,t_2) = E\left[X(t_1)\left(X(t_2)-X(t_1)\right)\right] + E\left[\left(X(t_1)\right)^2\right]$$

$$C_X(t_1,t_2) = E\left[X(t_1)\right]E\left[X(t_2)-X(t_1)\right] + E\left[\left(X(t_1)\right)^2\right]$$

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The Wiener Processes

$$C_X(t_1,t_2) = E\left[\left(X(t_1)\right)^2\right] = VAR\left[X(t_1)\right]$$

$$C_X(t_1,t_2) = \alpha\, t_1 \quad \text{when } t_1 \le t_2$$

$$C_X(t_1,t_2) = \alpha\, t_2 \quad \text{when } t_2 \le t_1$$

$$\boxed{C_X(t_1,t_2) = \alpha \min(t_1,t_2)}$$

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The Poisson Process

- **Let assume that a random event occurs at an average rate of λ per unit-of-time. Therefore, the average number of occurrences N(t) of the event during a time interval [0,t] is (λt)**
- **If we divide the interval [0,t] into a large number (n) of subintervals with duration (δ)**
- **The subinterval (δ) is small enough such that the probability that the event occurs more than once during (δ) is negligible**

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The Poisson Process

- **Therefore, during each subinterval (δ) the event either occurs once or does not occur.**
- **Hence, the average number of occurrences of the event in these n subintervals is (np), where p is the probability that the event occurs in a given (Bernoulli) trial**
- **Now, since n=t/δ , then the average number of occurrences is np=(λt)**

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The Poisson Process

- This process can be modeled as a binomial process with parameters (n) and (p)
- We have shown that a binomial random variable converges to a Poisson random variable as (n) gets very large and (p) gets very small while $np=\lambda t$
- Therefore, N(t) is a Poisson process:

$$\boxed{P\left[N(t) = k\right] = \frac{(\lambda t)^k}{k!} e^{-\lambda t} \qquad k = 0, 1, \ldots\ldots}$$

The Poisson Process

- Since the Poisson process is derived from the binomial process, then the Poisson process also has the "independent and stationary increments" properties:
- Therefore, for $t_1 < t_2$

$$\boxed{\begin{aligned} &P\left[N(t_1) = k_1, N(t_2) = k_2\right] = \\ &P\left[N(t_1) = k_1\right] P\left[N(t_2) - N(t_1) = k_2 - k_1\right] \end{aligned}}$$

The Poisson Process

$$\begin{aligned} &P\left[N(t_1) = k_1, N(t_2) = k_2\right] = \\ &P\left[N(t_1) = k_1\right] P\left[N(t_2) - N(t_1) = k_2 - k_1\right] \end{aligned}$$

$$\boxed{\begin{aligned} &P\left[N(t_1) = k_1, N(t_2) = k_2\right] = \\ &P\left[N(t_1) = k_1\right] P\left[N(t_2 - t_1) = k_2 - k_1\right] \end{aligned}}$$

The Poisson Process

- The independent and stationary increment property of the Poisson process N(t) can be used to compute the Covariance of N(t) (with parameter λ):

$$\boxed{C_N(t_1, t_2) = \lambda \min(t_1, t_2)}$$

The Poisson Process

- The independent and stationary increment property of the Poisson process N(t) can also be used to show that the number of arrivals (k) in a time interval [0,t] are distributed independently and uniformly over that interval
- In particular, given that an event has occurred during [0,t], then the probability that the event has occurred during a smaller interval [0,x], where $x \le t$:

$$\boxed{P[X \le x] = P\left[N(x) = 1 \middle| N(t) = 1\right] = \frac{x}{t}}$$

The Poisson Process

- It is important to recall that the inter-arrival time T of a Poisson process is an exponential random variable:

$$f_T(t) = \lambda e^{-\lambda t}$$

with mean $(1/\lambda)$

- Also recall that the sum of exponential iid RVs: $S_n=T_1+T_2+\ldots.T_n$ is an "Erlang" random variable.

The Poisson Process

- Therefore, the sum of the Poisson process's inter-arrival times: $S_n=T_1+T_2+\ldots.T_n$ has an Erlang density function:

$$\boxed{f_{S_n}(y) = \frac{(\lambda y)^{n-1}}{(n-1)!}\lambda e^{-\lambda y} \qquad y \ge 0}$$