ECE 863
Analysis of Stochastic Systems
Part III.3: Stationary and Ergodic Random Processes

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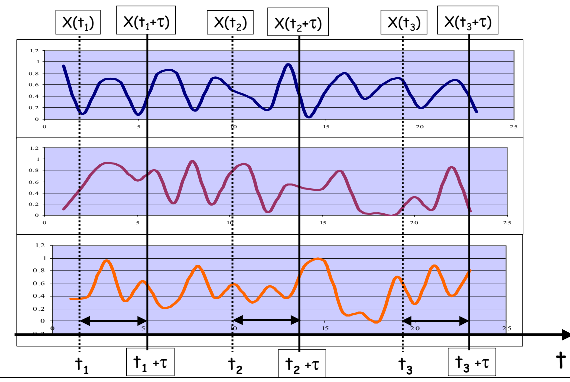
Stationary Processes



Stationary Processes

- A process X(t) is stationary when:

$$F_{X(t_1),\ldots,X(t_k)}\left(x_1,\ldots,x_k\right)$$

$$= F_{X(t_1+\tau),\ldots,X(t_k+\tau)}\left(x_1,\ldots,x_k\right)$$

$$\forall \quad \tau,\ k,\ t_1,\ldots,t_k$$

Stationary Processes

- The "first-order cdf" of a stationary process is independent of time:

$$F_{X(t)}(x) = F_{X(t+\tau)}(x) \qquad \forall\ t,\tau$$

$$\boxed{F_{X(t)}(x) = F_X(x) \qquad \forall\ t,\tau}$$

Stationary Processes

- All k^{th} moments $E[(X(t))^k]$ of a stationary process are independent of time.
- In particular, the mean and variance are independent of time:

$$m_X(t) = E[X(t)] = m \qquad \forall t$$

$$VAR[X(t)] = E\left[(X(t)-m)^2\right] = \sigma^2 \qquad \forall t$$

Wide Sense Stationary Proc.

- A process is <u>Wide Sense Stationary (WSS)</u> if:
  (a) the mean is constant;
  (b) the autocorrelation function depend only on the time difference (t_2-t_1):

$$\boxed{m_X(t) = E[X(t)] = m \qquad \forall t}$$

$$\boxed{R_X(t_1,t_2) = R_X(t_2-t_1) \qquad \forall t_1,t_2}$$

$$\Rightarrow \quad \boxed{C_X(t_1,t_2) = C_X(t_2-t_1) \qquad \forall t_1,t_2}$$

Autocorrelation of WSS Proc.

- $R_X(t_2-t_1) = R_X(\tau)$ where $\tau = t_2-t_1$
- The autocorrelation is a symmetric function of the time difference τ :

$$R_X(\tau) = R_X(-\tau)$$

- The autocorrelation function has a maximum magnitude at the origin $\tau = 0$:

$$|R_X(\tau)| \leq R_X(0)$$

Example: Sinusoidal Phase

- Let Θ be a uniform random variable over the interval $(-\pi, \pi)$, and let the random process $X(t,\Theta)$:

$$X(t) = \cos(t + \Theta)$$

Is X(t) a WSS process?

Example: Sinusoidal Phase

- **Remember that the mean:**

$$m_x(t) = E\left[\cos(t + \Theta)\right]$$

$$m_x(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(t + \theta)\, d\theta$$

$$\boxed{m_x(t) = 0}$$

Example: Sinusoidal Phase

- **And the autocorrelation function:**

$$R_x(t_1, t_2) = E\left[\cos(t_1 + \Theta)\cos(t_2 + \Theta)\right]$$

$$\boxed{R_x(t_1, t_2) = \frac{1}{2}\cos(t_1 - t_2) = \frac{1}{2}\cos(\tau)}$$

Therefore, X(t) is a WSS process

Example: Sinusoidal Phase

- **Note that $R_X(\tau)$ is symmetric and has a maximum at $\tau = 0$**
- **In this example, $R_X(\tau)$ is also periodic with a period of 2π**
- **This leads to the following general attribute:**

If $R_X(0) = R_X(d)$ $\Rightarrow$ $R_X(\tau)$ **is periodic**
In other words
$R_X(\tau) = R_X(\tau + d) \quad \forall\ \tau$

Stationary & WSS Processes

- **A stationary process is a WSS process**
- **However, the inverse is not always true**
- **For a Gaussian process X(t):**

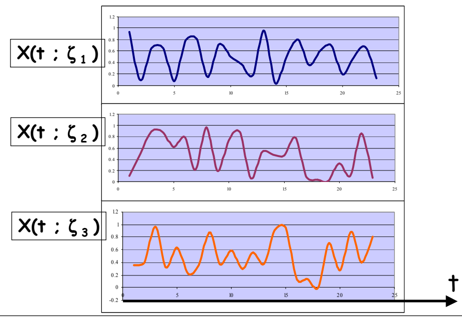
If X(t) is a Gaussian WSS process then X(t) is a stationary process

Ergodicity

- **Remember that a random process $X(t)=X(t,\zeta)$ is a collection of time functions, where each function $X(t,\zeta_1)$ is generated from one random outcome ζ_1 of a random experiment**
- **Under certain conditions, we may be able to compute some statistics (e.g. the mean) of the random process $X(t)$ by focusing on a <u>single</u> time function**

Ergidicity

$$X(t,\zeta) \qquad \zeta \in S$$



Ergidicity

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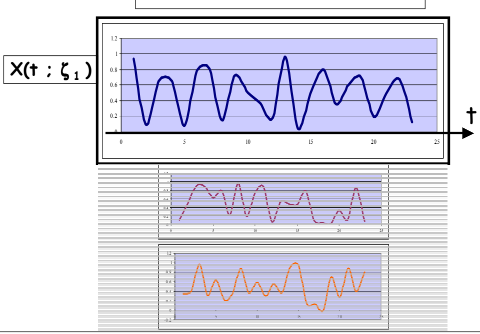


Ergodicity

- **For example, for a given outcome ζ of a random experiment, the <u>time average</u> over some time interval [-T, T] :**

$$\langle X(t) \rangle_T = \frac{1}{2T} \int_{-T}^{T} X(t,\zeta)\,dt$$

Ergodicity

- As T gets large, and if the "time average" $\langle X(t)\rangle_T$ converges to the true mean $E[X(t)]$, then we say that $X(t)$ is:

  ergodic in the mean or "mean ergodic"

Ergodicity

- It can be shown that, for WSS processes, the ergodic mean does converge to the mean $E[X(t)] = m$ under a certain condition.
- This is formulated in a theorem
  - (The complete proof of the theorem is included as an Appendix.)

Ergodicity

- Before stating the theorem, it is important to highlight three key expressions that result from the proof of the theorem:

$$\boxed{E\left[\langle X(t)\rangle_T\right] = m}$$

$$\boxed{VAR\left[\langle X(t)\rangle_T\right] = E\left[\left(\langle X(t)\rangle_T - m\right)^2\right]}$$

$$\boxed{VAR\left[X\langle(t)\rangle_T\right] = \frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)C_X(\tau)\,d\tau}$$

"Mean Ergodic" Theorem

- Let $\langle X(t)\rangle_T$ be the time-average of a WSS process $X(t)$ with mean $E[X(t)] = m$ and covariance $C_X(\tau)$, then:

$\langle X(t)\rangle_T$ converges to the mean $E[X(t)] = m$

if and only if $VAR(\langle X(t)\rangle_T)$ converges to zero

$\Rightarrow$ if & only if $\boxed{\lim_{T\to\infty}\left\{\frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)C_X(\tau)\,d\tau\right\} = 0}$

Example: Random Telegraph Process

- **Let X(t) be a <u>random telegraph</u> process with parameter α**
  - α is the average number of occurrences per unit-of-time of a Poisson process

Evaluate the variance of the time-average of X(t). Does the time-average converges to the mean E[X(t)] in a mean-square-sense?

Example: Random Telegraph Process

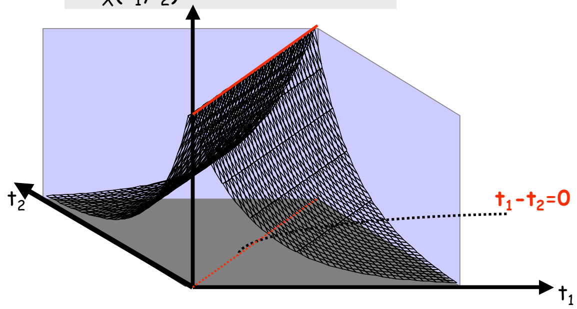
- **X(t) has a zero mean** $E[X(t)] = m = 0$, **and a covariance function:**

$$\boxed{C_X(\tau) = R_X(\tau) = e^{-2\alpha|\tau|}}$$

(See example 6.22 in the book)

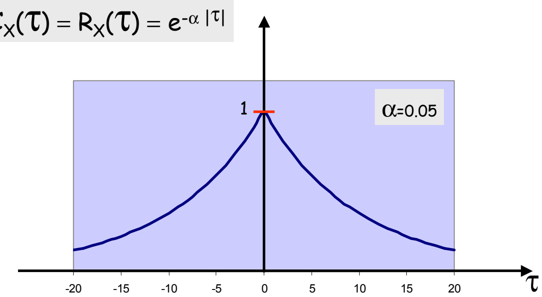
Example: Random Telegraph Process

$R_X(t_1,t_2) = e^{-\alpha|t_1-t_2|} = e^{-\alpha|\tau|}$



Example: Random Telegraph Process

$C_X(\tau) = R_X(\tau) = e^{-\alpha|\tau|}$



Example: Random Telegraph Process

- **Therefore, X(t) is a WSS process, and we can evaluate the variance of its time-average using the following:**

$$VAR\left[X\langle(t)\rangle_T\right]=\frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)C_X(\tau)d\tau$$

$$VAR\left[X\langle(t)\rangle_T\right]=\frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)e^{-2\alpha|\tau|}d\tau$$

Example: Random Telegraph Process

- **Since the integrand is symmetric then:**

$$VAR\left[X\langle(t)\rangle_T\right]=\frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)e^{-2\alpha|\tau|}d\tau$$

$$VAR\left[X\langle(t)\rangle_T\right]=\frac{2}{2T}\int_{0}^{2T}\left(1-\frac{\tau}{2T}\right)e^{-2\alpha\tau}d\tau$$

Example: Random Telegraph Process

- **Therefore,**

$$VAR\left[X\langle(t)\rangle_T\right]=\frac{2}{2T}\int_{0}^{2T}\left(1-\frac{\tau}{2T}\right)e^{-2\alpha\tau}d\tau$$

$$=\frac{1}{T}\left\{\int_0^{2T}e^{-2\alpha\tau}d\tau-\frac{1}{2T}\int_0^{2T}\tau\, e^{-2\alpha\tau}d\tau\right\}$$

Example: Random Telegraph Process

- **This leads to the following:**

$$VAR\left[X\langle(t)\rangle_T\right]=$$

$$\frac{1-e^{-4\alpha T}}{2\alpha T}-\frac{1}{8\alpha^2T^2}\left[1-(1+4\alpha T)e^{-4\alpha T}\right]$$

$$\boxed{VAR\left[X\langle(t)\rangle_T\right]=\frac{4\alpha T+e^{-4\alpha T}-1}{8\alpha^2T^2}}$$

Example: Random Telegraph Process

- As T goes to infinity, the variance of the time-average goes to zero:

$$\boxed{\lim_{T\to\infty} VAR\left[X\langle(t)\rangle_T\right] = 0}$$

- Therefore, for the "random telegraph process" the time-average converges to the mean (E[X(t)]=m=0) in a mean-square-sense:

$$\boxed{\lim_{T\to\infty} E\left[\left(\langle X(t)\rangle_T - m\right)^2\right] = 0}$$

Example: Random Telegraph Process

- Since the following condition is satisfied:

$$\boxed{\lim_{T\to\infty} E\left[\left(\langle X(t)\rangle_T - m\right)^2\right] = 0}$$

then, we say that the random telegraph process X(t) is "mean ergodic" (or "ergodic in the mean").

APPENDIX A

Proof of the

"Mean Ergodic" Theorem

"Mean Ergodic" Theorem

- Let $\langle X(t)\rangle_T$ be the time-average of a WSS process X(t) with mean E[X(t)]=m and covariance $C_X(\tau)$, then:

$$\lim_{T\to\infty} E\left[\left(\langle X(t)\rangle_T - m\right)^2\right] = 0$$

if and only if:

$$\boxed{\lim_{T\to\infty} \frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)C_X(\tau)\,d\tau = 0}$$

Ergodicity

- We need to show that as T gets large, then the time average <u>converges</u> to the mean E[X(t)] ; i.e. we need to show that X(t) is:

  <u>ergodic in the mean</u> or <u>"mean ergodic"</u>

- A key point here is the type of convergence (section 5.5)

- Here, we focus on "mean-square convergence"

Ergodicity

- The time-average converges in the "mean-square" sense to some number (a) when the following is satisfied:

$$\lim_{T\to\infty} E\left[\left(\langle X(t)\rangle_T - a\right)^2\right] = 0$$

Ergodicity

- We are interested in the convergence of the time-average to the mean E[X(t)] (i.e. a=E[X(t)])

- Although the time-average $\langle X(t)\rangle_T$ is a function of T, $\langle X(t)\rangle_T$ is NOT a function of time t

- Consequently, we are interested in processes with E[X(t)] that is NOT a function of time: E[X(t)]=m

Ergodicity

- Therefore, we will consider Wide-Sense-Stationary (WSS) processes.

- We need to derive the condition that leads to the following "mean-square-convergence" of the time-average to the mean (m) of a WSS process:

$$\lim_{T\to\infty} E\left[\left(\langle X(t)\rangle_T - m\right)^2\right] = 0$$

Ergodicity

- **First, lets look at the expected value of the time average of a WSS process**

$$E\left[\langle X(t)\rangle_T\right]=E\left[\frac{1}{2T}\int_{-T}^{T}X(t)\,dt\right]$$

$$E\left[\langle X(t)\rangle_T\right]=\frac{1}{2T}\int_{-T}^{T}E\left[X(t)\right]dt$$

$$\boxed{E\left[\langle X(t)\rangle_T\right]=m}$$

Ergodicity

- **Since the expected value of the time-average is the mean (m), then**

$$E\left[\left(\langle X(t)\rangle_T-m\right)^2\right]=$$

$$E\left[\left(\langle X(t)\rangle_T-E\left[\langle X(t)\rangle_T\right]\right)^2\right]$$

$$\boxed{E\left[\left(\langle X(t)\rangle_T-m\right)^2\right]=VAR\left[\langle X(t)\rangle_T\right]}$$

Ergodicity

- **Therefore, convergence of the time-average to the mean m (in a MS sense) is equivalent to the convergence of the "time-average-variance" to zero:**

$$\boxed{\begin{gathered}\lim_{T\to\infty} E\left[\left(\langle X(t)\rangle_T-m\right)^2\right]=0\\ \Leftrightarrow\\ \lim_{T\to\infty} VAR\left[\langle X(t)\rangle_T\right]=0\end{gathered}}$$

Ergodicity

- **Now, let's look at the condition that leads to the desired convergence:**

$$VAR\left[\langle X(t)\rangle_T\right]=E\left[\left(\langle X(t)\rangle_T-m\right)^2\right]$$

$$=E\left[\left\{\frac{1}{2T}\int_{-T}^{T}(X(t)-m)\,dt\right\}\left\{\frac{1}{2T}\int_{-T}^{T}(X(t')-m)\,dt'\right\}\right]$$

Ergodicity

$$VAR\left[\langle X(t)\rangle_T\right]=E\left[\left\{\frac{1}{2T}\int_{-T}^{T}(X(t)-m)\,dt\right\}\left\{\frac{1}{2T}\int_{-T}^{T}(X(t')-m)\,dt'\right\}\right]$$

$$=\frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T}E\left[(X(t)-m)(X(t')-m)\right]dt\,dt'$$

$$\boxed{VAR\left[\langle X(t)\rangle_T\right]=\frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T}C_X(t,t')\,dt\,dt'}$$

This is a general expression which is applicable not only for WSS processes

Ergodicity

- **If X(t) is a WSS process:**

$$C_X(t,t')=C_X(t-t')$$

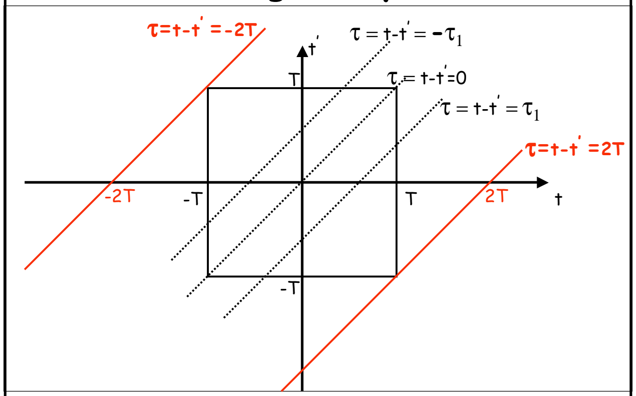
then:

$$VAR\left[\langle X(t)\rangle_T\right]=\frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T}C_X(t,t')\,dt\,dt'$$

$$\Downarrow$$

$$VAR\left[\langle X(t)\rangle_T\right]=\frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T}C_X(t-t')\,dt\,dt'$$

Ergodicity



Ergodicity

- **Since the covariance C_X is a function of the time difference $\tau=t-t'$, we can express the 2-dimensional integral as a one-dimensional integral of τ**
- **The function C_X is constant across the line $\tau=t-t'$ in the (t,t') space**
- **It can be seen that τ ranges from (-2T) to (+2T)**

Ergodicity

- It can be shown that the 2-dimensional integral:

$$VAR\left[\langle X(t)\rangle_T\right]=\frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T}C_X(t-t')\,dt\,dt'$$

is equivalent to the following integral:

$$VAR\left[X\langle(t)\rangle_T\right]=\frac{1}{4T^2}\int_{-2T}^{2T}(2T-|\tau|)C_X(\tau)\,d\tau$$

Ergodicity

- Therefore, the variance of the time average $\langle X(t)\rangle_T$ of a WSS process X(t) can be expressed as:

$$\boxed{VAR\left[X\langle(t)\rangle_T\right]=\frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)C_X(\tau)\,d\tau}$$

Ergodicity

- Consequently, if the following is satisfied:

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)C_X(\tau)\,d\tau=0$$

then the variance of the time-average converges to zero which leads to the convergence of the time-average to the mean (m) in a "mean-square-sense".

<u>END of the Proof</u>

APPENDIX B

A stationary process is a Wide-Sense-Stationary (WSS) Process

Stationary Processes

- The "second-order cdf": $F_{X(t_1),X(t_2)}(x_1,x_2)$

of a stationary process is only dependent on the time difference (t_2-t_1):

$$F_{X(t_1),X(t_2)}(x_1,x_2) = F_{X(t_1+\tau),X(t_2+\tau)}(x_1,x_2) \qquad \forall t_1,t_2,\tau$$

By setting $\tau=-t_1$:

$$\boxed{F_{X(t_1),X(t_2)}(x_1,x_2) = F_{X(0),X(t_2-t_1)}(x_1,x_2) \qquad \forall t_1,t_2}$$

Stationary Processes

- Therefore, for a stationary process, the autocorrelation and autocovariance functions depend only on the time difference (t_2-t_1):

$$R_X(t_1,t_2) = R_X(t_2-t_1) \qquad \forall t_1,t_2$$

$$C_X(t_1,t_2) = C_X(t_2-t_1) \qquad \forall t_1,t_2$$

- Therefore, a stationary process is a WSS process