Digital Signal Processing

Chapter 3: Discrete-Time Signals and Systems

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Continuous – discrete connection in time domain

- We obtained a sampled signal from sampling a continuous(-time) signal $x(t)$ by a train of Dirac delta functions

$$\boxed{x_{\Delta}(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)}$$

  T is the sampling period.

- Sampled signal $x_{\Delta}(t)$ is a continuous-time signal (independent variable is t).
- $x_{\Delta}(t)$ completely depends on $x(nT)$.

- Consider a discrete(-time) signal $x_{\mathrm{d}}(n)$: $x_{\mathrm{d}}(n) = x(nT)$.
- We have

$$\boxed{x_{\Delta}(t) = \sum_{n=-\infty}^{\infty} x_{\mathrm{d}}(n)\delta(t - nT)}$$

  relationship between the continuous signal $x_{\Delta}(t)$ and the discrete signal $x_{\mathrm{d}}(n)$.
- When we have $x_{\Delta}(t)$, it is easy to infer $x_{\mathrm{d}}(n)$ and vice versa.
- Processing the discrete signal $x_{\mathrm{d}}(n)$ is equivalent to processing the continuous signal $x_{\Delta}(t)$.

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Continuous–discrete connection via Laplace Transform

- Take the Laplace transform of $x_\Delta(t)$

$$
\begin{aligned}
X_\Delta(s) &= \int_{-\infty}^{\infty} x_\Delta(t) e^{-st} dt \\
&= \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} x_\mathrm{d}(n)\delta(t - nT) \right] e^{-st} dt \\
&= \sum_{n=-\infty}^{\infty} x_\mathrm{d}(n) \left[\int_{-\infty}^{\infty} \delta(t - nT) e^{-st} dt \right]
\end{aligned}
$$

$$
\boxed{X_\Delta(s) = \sum_{n=-\infty}^{\infty} x_\mathrm{d}(n) e^{-nsT}} \qquad (*)
$$

Equation (*) plays an important role when transforming from the continuous domain to the discrete domain and vice versa.

- Let $z = e^{sT}$, hence

$$\boxed{X_\Delta(z) = \sum_{n=-\infty}^{\infty} x_\mathrm{d}(n) z^{-n}}$$

When we ignore the sampling period T (or sampling rate), z is an independent variable. Only when we discuss about the sampling system than the meaning of z is related to $z = e^{sT}$.

- From now on, when considering discrete signals, we will not consider the sampling period/rate and just use the Z transform for any discrete signals and systems.

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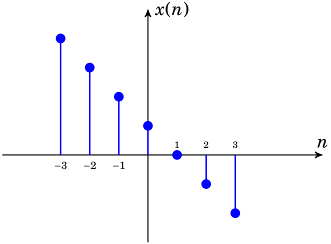
Discrete signal representation

- Discrete signals can be represented by mathematical functions, graphs or number sequences.

$$x(n) = \begin{cases} -n+1, & \text{if } -3 \leq n \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

$$x(n) = \{4, 3, 2, \underset{\uparrow}{1}, -1, -2\}$$



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- When we describe it as number sequence, the sample at $n = 0$ must be clearly determined.
- Consider the following discrete signals:

$$x_1(n) = \{\ldots, 0.25, 0.5, \underset{\uparrow}{1}, 0.5, 0.25, \ldots\}$$

$$x_2(n) = \{1.2, -3, \ldots\}$$

$$x_3(n) = \{1, -1, 3, \underset{\uparrow}{5}, 0, 4, 1\}$$

$$x_4(n) = \{1.5, 0, 7\}$$

- $x_1(n)$, $x_2(n)$, $x_3(n)$ and $x_4(n)$ have the origin point at 1, 1, 5, and 1, respectively.
- $x_1(n)$ and $x_2(n)$ have infinite number of samples.
- $x_3(n)$ and $x_4(n)$ have finite number of sample.
- $x_2(n)$ and $x_4(n)$ are equal 0 for negative n.

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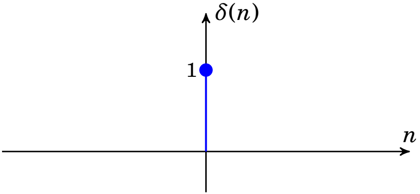
Kronecker delta

We have met the Dirac delta, $\delta(t)$, in the theory of continuous signals and systems. In the discrete domain, the Kronecker delta $\delta(n)$ has a similar role:

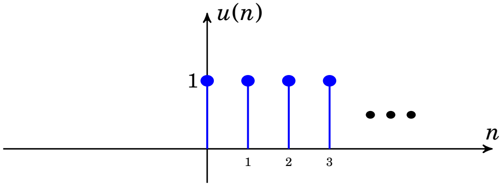
$$\boxed{\delta(n) = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{otherwise} \end{cases}}$$

Unlike the Dirac delta function, the Kronecker delta function has a unit value at $n = 0$.



The unit-step signal

$$u(n) = \begin{cases} 1, & \text{if } n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$



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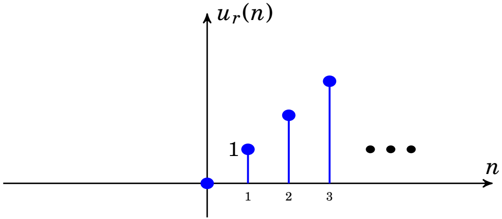
The unit-ramp signal

$$\boxed{u_r(n) = \begin{cases} n, & \text{if } n \geq 0 \\ 0, & \text{otherwise} \end{cases}}$$



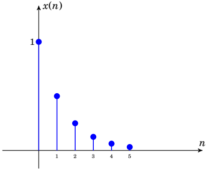
The exponential signal

- Definition:

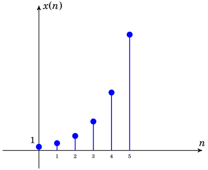
$$\boxed{x(n) = a^n}$$

where a is a constant.

- If a is real, then $x(n)$ is also real.



$0 < a < 1$



$a > 1$

- If a is complex ($a = re^{j\theta}$), then $x(n)$ is complex. Real and imaginary components are described separately: $x_R(n) = r^n \cos(n\theta)$, $x_I(n) = r^n \sin(n\theta)$.

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Energy signals and Power signals

- Energy:

$$\boxed{E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2}$$

- When E_x is finite, $x(n)$ is called an energy signal.
- Average power:

$$\boxed{P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2}$$

- When E_x is infinite, the average power of the signal may be finite of infinite. When P_x is finite, $x(n)$ is called a power signal.

Periodic signals

- $x(n)$ is periodic with period N (positive integer), if and only if

$$\boxed{x(n+N) = x(n), \qquad \text{for all } n}$$

  The smallest period of the signal is called basic period.

- A periodic continuous signal may not be periodic in the discrete domain.
- $\cos(t)$ is periodic with period 2π, that means $\cos(t+2\pi) = \cos(t)$. Its discrete version is: $x(n) = \cos(2\pi f_0 n)$, where f_0 is a positive constant. Is $x(n)$ periodic?
- Assume there exists a positive integer N such that

$$\cos(2\pi f_0 n + 2\pi f_0 N) = \cos(2\pi f_0 n)$$

  This is satisfied if $f_0 N$ is integer, hence f_0 must be a rational number p/q. Thus, $X(n)$ is periodic when $N = kq$.

- Generally, when we sample a periodic analog signal, if the sampling rate does not have a rational relation to the period of the continuous signal, the discrete signal is non-periodic.
- Average power of a periodic discrete signal with period N

$$\boxed{P = \frac{1}{N}\sum_{n=0}^{N-1} |x(n)|^2}$$

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Even and odd signals

- A real signal $x(n)$ is even if

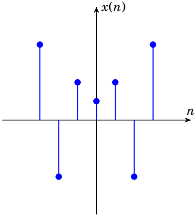
$$x(-n) = x(n), \quad \text{for all } n$$

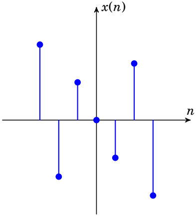
Even signal is symmetric

- A real signal $x(n)$ is odd if

$$x(-n) = -x(n), \qquad \text{for all } n$$

Odd signal is anti-symmetric. Also $x(0) = 0$.





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For an arbitrary signal $x(n)$, we can decompose it into a sum of an even signal and an odd signal:

$$x_e(n) = \frac{1}{2}\left[x(n) + x(-n)\right]$$
$$x_o(n) = \frac{1}{2}\left[x(n) - x(-n)\right]$$

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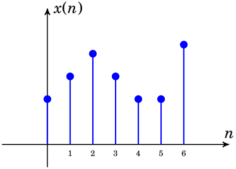
Shifting

- Shift operator, $\mathcal{D}_{n_0}\{\cdot\}$: replace n with $n - n_0$

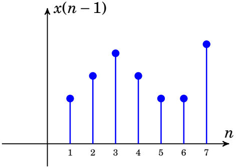
$$\boxed{\mathcal{D}_{n_0}\{x(n)\} = x(n - n_0)}$$

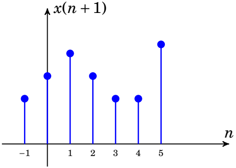
n_0 is an integer constant (negative or positive).

- $n_0 > 0$: the operator shifts the signal to the right (delay) by n_0 samples.
- $n_0 < 0$: the operator shifts the signal to the left (advance), by $|n_0|$ samples.



Original signal





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- For continuous signals, the delay operator is very complicated and the advance operator is impossible.
- However, for discrete signal, $x(n)$ is written in memory so the both delaying or advancing $x(n)$ is simple.

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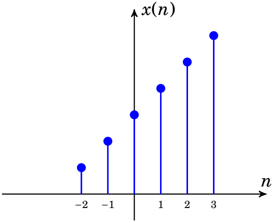
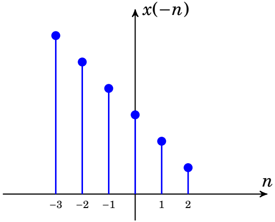
Time reversal

- Time-reversal operator $\mathcal{I}\{\cdot\}$: replace n with $-n$

$$\boxed{\mathcal{I}\{x(n)\} = x(-n)}$$





- Shifting and time-reversal are not commutative

$$\boxed{\mathcal{D}_{n_0}\{\mathcal{I}\{x(n)\}\} \neq \mathcal{I}\{\mathcal{D}_{n_0}\{x(n)\}\}}$$

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Time scaling

- Time-scaling operator: replace n with αn, α is a positive integer constant.

$$\boxed{\downarrow_\alpha \{x(n)\} = x(\alpha n)}$$

- Time-scaling operator is also called the decimator.
- Example: $x_a(t)$ is a continuous signal. Now, sample $x_a(t)$ using 2 different periods, T_1 and T_2, and obtain 2 different discrete signals $x_1(n)$ and $x_2(n)$. Let $T_1 = T$, $T_2 = 2T$, then

$$x_1(n) = x_a(nT), \qquad x_2(n) = x_a(n2T)$$

$$\Longrightarrow x_2(n) = x_1(2n)$$

$$\Longrightarrow x_2(n) \ = \ \downarrow_2 \{x_1(n)\}$$

That is, downsample $x_1(n)$ by a time-scale of $\alpha = 2$ to get $x_2(n)$.

Addition, multiplication, scaling

- Addition:

$$y(n) = x_1(n) + x_2(n), \forall n$$

- Multiplication:

$$y(n) = x_1(n) \cdot x_2(n), \forall n$$

- Amplitude-scaling:

$$y(n) = ax(n), \forall n$$

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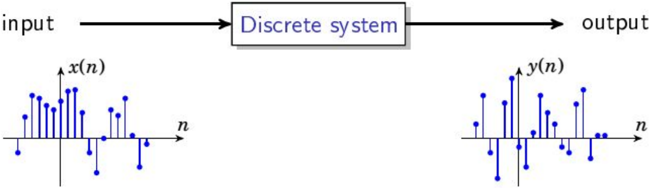
Definition

- Discrete-time system: a device or an algorithm to perform operations on discrete-time signals.
- Mathematically, a system is an operator, $\mathcal{T}\{\cdot\}$, which transforms a discrete-time signal (input signal, excitation) to another discrete-time signal (output signal, response)
- Let $x(n)$ be the input signal to $\mathcal{T}\{\cdot\}$, then the output $y(n)$ is

$$y(n) = \mathcal{T}\{x(n)\}$$



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System modeling

- Consider a class of systems modeled by linear constant coefficient difference equations (LCCDE).
- For input $x(n)$ and output $y(n)$, the system I/O relationship is given by

$$\boxed{\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)}$$

- Coefficients a_k and b_k may depend on n but are completely independent of $x(n)$ and $y(n)$.
- N and M are positive integer constants. N and M are finite, so systems represented by the LCCDE also called finite order systems.
- We can specify the LCCDE by a system diagram with three basic operators: adder, amplitude amplifier, time delay.

Example

- Consider the following LCCDE:

$$y(n) + 2y(n-1) = 3x(n) + 0,5x(n-1) + 0,6x(n-2)$$

- Question: Draw the system diagram with the three basic operators: adder, amplitude amplifier, time delay.

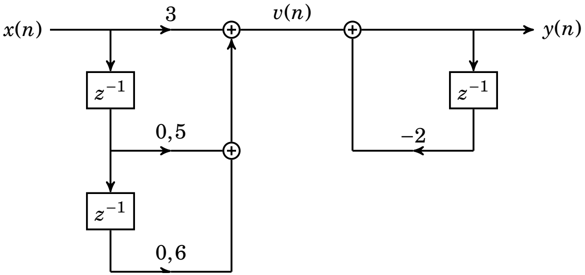
- LCCDE:

$$y(n) + 2y(n-1) = 3x(n) + 0.5x(n-1) + 0.6x(n-2)$$

- Rewrite

$$\begin{aligned} y(n) &= -2y(n-1) + 3x(n) + 0.5x(n-1) + 0.6x(n-2) \\ &= -2y(n-1) + v(n) \end{aligned}$$

- Equivalent system diagram



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Static and dynamic systems

- A discrete-time system is static (memoryless), if the value of the output $y(n)$ at time n (present time) depends only on the present sample of the input $x(n)$ at time n.
- A discrete-time system is dynamic (with memory), if the value of $y(n)$ at time n depends on several samples of $x(n)$ at different times.

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Example

What type of system are these?

$$y(n) = 10nx(n)$$

$$y(n) = -7x(n) + 0.2x^3(n)$$

$$y(n) = 2x(n) - 0.5x(n-1)$$

$$y(n) = \sum_{k=0}^{N} x(n-k)$$

$$y(n) = \sum_{k=0}^{\infty} x(n-k)$$

Example

What type of system are these?

$$y(n) = 10nx(n) \qquad \text{static}$$

$$y(n) = -7x(n) + 0.2x^3(n) \qquad \text{static}$$

$$y(n) = 2x(n) - 0.5x(n-1) \qquad \text{dynamic}$$

$$y(n) = \sum_{k=0}^{n} x(n-k) \qquad \text{dynamic with finite memory}$$

$$y(n) = \sum_{k=0}^{\infty} x(n-k) \qquad \text{dynamic with infinite memory}$$

Time-invariant systems

- Physically, a discrete-time system is time-invariant if, the response of the system remains the same to the same excitation but applied at the input of the system at different times.
- Let a system $\mathcal{T}$ be excited by $x(n)$:

$$y(n) = \mathcal{T}\{x(n)\}$$

- Let the same system $\mathcal{T}$ be also excited by $x(n - n_0)$. Its new response is

$$z(n) = \mathcal{T}\{x(n - n_0)\}$$

- This system is said to be time-invariant if $z(n) = y(n - n_0)$, that is

$$\boxed{y(n - n_0) = \mathcal{T}\{x(n - n_0)\}}$$

Example

Example 2.2.4 (Time-invariance vs Time-variance) [Proakis 4th]

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$X_\Delta(s) \longleftrightarrow x_d(n)$

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Linear system

- A system is linear if it satisfies the following two important physical properties:
  1. Homogeneity: If the system input is amplified a times, the system output is also amplified a times;
  2. Additivity: If the system input is the sum of two signals, then system output is the sum of the system responses corresponding to the two signals.
- Mathematically

$$\boxed{\mathcal{T}\left\{\sum_k a_k x_k(n)\right\} = \sum_k a_k \mathcal{T}\left\{x_k(n)\right\}}$$

- Superposition principle: The response of the system to a weighted sum of signals be equal to the corresponding weighted sum of the responses (outputs) of the system to each of the individual input signals.

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Causal systems

- A system is causal if the output occurs after the input has occured.
- Assume that the input signal $x(n)$ satisfies

$$x(n) = 0 \quad \text{with all } n < n_0.$$

If the system is causal, the output signal $y(n)$ satisfies

$$y(n) = 0 \quad \text{with all } n < n_0.$$

- This condition indicates that at the observed time n, the output $y(n)$ depends only the present and past values of the input, that is, on $x(n)$, $x(n-1)$, $x(n-2)$, ...
- If $y(n)$ depends on future values of $x(n)$ ($x(n+1)$, $x(n+2)$, ...), the system is noncausal.

Example: Accumulator

Example 2.2.2 [Proakis 4th]

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Stable systems

- A system is stable if we pull it out of its normal operational orbit after a while it will return to its normal operational orbit.
- This is the most important characteristic of a system.
- Bounded input - bounded output statiblity: For a stable system, if we excite it with an input signal with a finite amplitude, the amplitude of the output signal will also be finite.
- If there exists $M_x > 0$ such that the input $x(n)$ of a stable system satisfies

$$|x(n)| < M_x < \infty, \text{ for all } n$$

then there exists $M_y > 0$ such that the system output $y(n)$ satisfies

$$|y(n)| < M_y < \infty, \text{ for all } n$$

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- System interconnection

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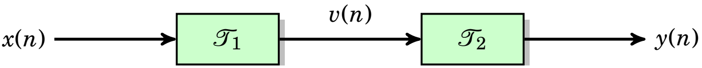
System interconnection

- Discrete-time systems are usually connected to each others to make a larger system.
- Two basic types of interconnections: serial/cascade and parallel.
- Cascade interconnection:



$$\boxed{y(n) = \mathcal{T}_2\{v(n)\} = \mathcal{T}_2\{\mathcal{T}_1\{x(n)\}\}}$$

The positions of the sub-systems are very important, because generally

$$\mathcal{T}_2\left\{\mathcal{T}_1\left\{x(n)\right\}\right\} \neq \mathcal{T}_1\left\{\mathcal{T}_2\left\{x(n)\right\}\right\}$$

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are linear and time-invariant, we can swap their positions of them without changing the output of the overall system.

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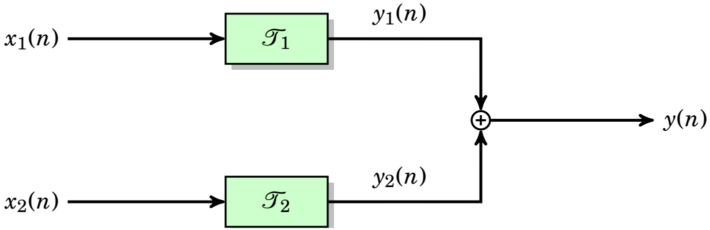
- Parallel interconnection:



$$\boxed{y(n) = y_1(n) + y_2(n) = \mathcal{T}_1\{x_1(n)\} + \mathcal{T}_2\{x_2(n)\}}$$

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Impulse response and convolution

- System: $\mathcal{T}$.
- Excitation: Kronecker delta $\delta(n)$.
- Response: $h(n) = \mathcal{T}\{\delta(n)\}$. Here, $h(n)$ is also called the impulse response of the system.
- If the length of $h(n)$ is finite, then the system is finite impulse response (FIR). Otherwise, it is infinite impulse response (IIR).

$$h_1(n) = \{\ldots, 0.25, 0.5, \underset{\uparrow}{1}, 0.5, 0.25, \ \ldots\}$$

$$h_2(n) = \{1.2, -3, \ldots\}$$

$$h_3(n) = \{1, -1, 3, \underset{\uparrow}{5}, 0, 4, 1\}$$

$$h_4(n) = \{1.5, 0, 7\}$$

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- Consider an arbitrary input $x(n)$, instead of the Kronecker delta.
- $x(n)$ can be represented as a linear combination of Kronecker signal:

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

- If the system is linear, then its response $y(n)$ to $x(n)$ is a linear combination of responses which were obtained by exciting the system with Kronecker deltas $\delta(n-k)$ for all k:

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\mathcal{T}\{\delta(n-k)\}$$

- If the system is further time-invariant, then the system response obtained by exciting the system with $\delta(n-k)$ (Kronecker delta at time k) is $h(n-k)$:

$$\boxed{y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)}$$

- The above input/output relationship is completely determined by the impulse response $h(n)$ (i.e., the knowledge of $h(n)$ is sufficient to determine the response of the system to an arbitrary input). Thus, an LTI system is completely characterized by its impulse response.

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- The I/O relationship is shortened as

$$\boxed{y(n) = x(n) \star h(n)}$$

  the operator $\star$ is called convolution.
- Thus, the output $y(n)$ of an LTI system equals the convolution of the impulse response $h(n)$ and the input $x(n)$.
- By a change of variable, $m = n - k$, we have

$$y(n) = \sum_{m=-\infty}^{\infty} h(m)x(n-m)$$

  Hence convolution is commutative:

$$\boxed{x(n) \star h(n) = h(n) \star x(n)}$$

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Interpretation of impulse response and convolution

- An LTI can be causal or non-causal.
- An LTI system is causal if and only if the impulse response

$$\boxed{h(n) = 0, \quad n < 0}$$

  Otherwise, it is non-causal.
- If $h(n) = 0$ for all $n \geq 0$, then the system is anti-causal.
- When an LTI system is causal, the convolution formula becomes

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

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- For any LTI system, the convolution formula gives the output at a particular time instant n_0

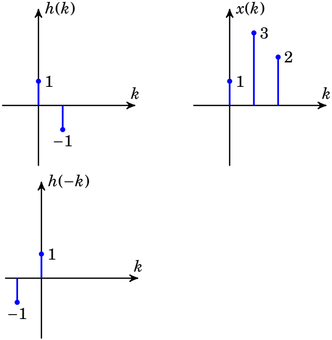
$$y(n_0) = \sum_{k=-\infty}^{\infty} x(k)h(n_0 - k)$$

This suggests the way to compute convolution.

Example: Computing convolution

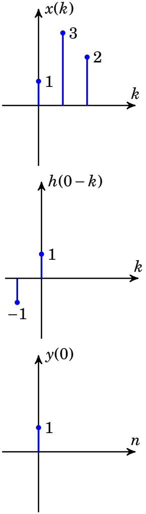
- Impulse response: $h(n) = \{1, -1\}$
- Input signal: $x(n) = \{1, 3, 2\}$
- Question: $y(n) = x(n) \star h(n) = ?$

- Impulse response: $h(n) = \{1, -1\}$
- Input signal: $x(n) = \{1, 3, 2\}$



$$y(n_0) = \sum_{k=-\infty}^{\infty} x(k)h(n_0 - k)$$

- at $n_0 = 0$:



$$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(0-k) = (1)(1) = 1$$

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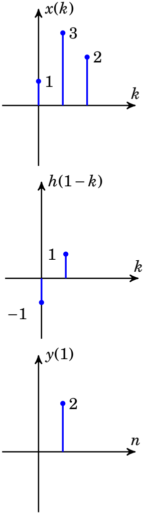
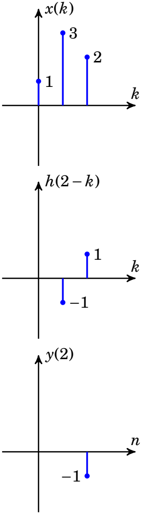
- at $n_0 = 1$:



$$y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k) = (1)(-1) + (3)(1) = 2$$

- at $n_0 = 2$:



$$y(2) = \sum_{k=-\infty}^{\infty} x(k)h(2-k) = (3)(-1) + (2)(1) = -1$$

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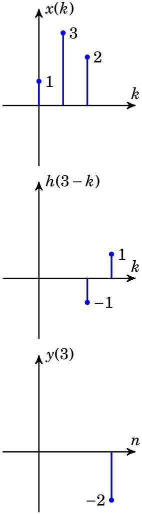
- at $n_0 = 3$:



$$y(3) = \sum_{k=-\infty}^{\infty} x(k)h(3-k) = (2)(-1) = -2$$

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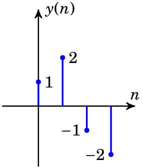
- at other n_0: $y(n_0) = 0$.
- Thus,

$$y(n) = x(n) \star h(n) = \{1, 2, -1, -2\}$$



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Impulse response of cascaded systems

- Connect two LTI systems in cascade: $\mathcal{T}_1$ $\mathcal{T}_2$, with impulse responses $h_1(n)$ and $h_2(n)$.
- Because both $\mathcal{T}_1$ and $\mathcal{T}_2$ are linear, the overall cascaded system is also linear.
- Impulse response of the cascaded system:

$$h(n) = \mathcal{T}_2\left\{\mathcal{T}_1\left\{\delta(n)\right\}\right\} = \mathcal{T}_2\left\{h(n)\right\} = h_2(n) \star h_1(n)$$

  convolution of individual impulse responses.
- Since convolution is commutative, we can exchange individual systems with cascade interconnection while still maintain the system output.

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System BIBO stability

- We have

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$
$$\leq \sum_{k=-\infty}^{\infty} |h(k)|\,|x(n-k)|$$

- An input is bounded if there exists $M_x \in \mathbb{R}, M_x > 0$ such that

$$|x(n)| \leq M_x, \qquad -\infty < n < \infty.$$

$$y(n) \leq \sum_{k=-\infty}^{\infty} |h(k)|\,|x(n-k)| \leq \sum_{k=-\infty}^{\infty} |h(k)|\, M_x$$

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- The output $y(n)$ is also bounded if there exists $M \in \mathbb{R}, M > 0$ such that

$$y(n) \leq \sum_{k=-\infty}^{\infty} |h(k)|\,|x(n-k)| \leq \sum_{k=-\infty}^{\infty} |h(k)|\, M_x < M$$

- Hence,

$$\sum_{k=-\infty}^{\infty} |h(k)| < \frac{M}{M_x} < \infty$$

- Thus, the LTI system $h(n)$ is BIBO[1] stable if

$$\boxed{\sum_{k=-\infty}^{\infty} |h(k)| < \infty}$$

[1] BIBO: bounded-input-bounded-output.

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Z transform

- Z transform (ZT) is used to analyze and represent discrete-time signals and systems.
- The ZT of $x(n)$, denoted by $X(z)$ or $\mathcal{Z}\{x(n)\}$ is defined as:

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}} \qquad (*)$$

- ZT is a sequence with the independent complex variable x. The coefficient of each z^{-n} at time index n is the sample of $x(n)$ at index n.
- Conditions for convergence of infinite sum $(*)$ is needed for the sum to be finite.
- The region which includes points z such that $X(z)$ converges is called region of convergence, (ROC).

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Example

Find the ZT of the following signals:

- $x(n) = \{1, -1, 0, \underset{\uparrow}{3}, 5, 7\}$
- The Kronecker delta sequence: $\delta(n)$

Example

- $x(n) = \{1, -1, 0, \underset{\uparrow}{3}, 5, 7\}$

  ZT of $x(n)$:

$$
\begin{aligned}
X(z) &= z^3 - z^2 + 0.z + 3z^0 + 5z^{-1} + 7z^{-2} \\
&= z^3 - z^2 + 3 + 5z^{-1} + 7z^{-2}
\end{aligned}
$$

  Thus, ZT of a finite-duration signal always converges.

- ZT of Kronecker delta $\delta(n)$:

$$X(z) = 1$$

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ZT of causal signals

- Consider an exponential signal defined as

$$x(n) = \begin{cases} a^n, & \text{if } n \geq 0 \\ 0, & \text{if } n < 0 \end{cases}$$

Thus, $x(n)$ is causal.

- Its ZT:

$$X(z) = 1 + az^{-1} + a^2 z^{-2} + \cdots = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n .$$

- We know that

$$\sum_{n=0}^{\infty} d^n = \frac{1}{1-d}, \qquad \text{with } |d| < 1.$$

Then,

$$X(z) = \frac{1}{1-az^{-1}}, \qquad \text{with } |z| > |a|,$$

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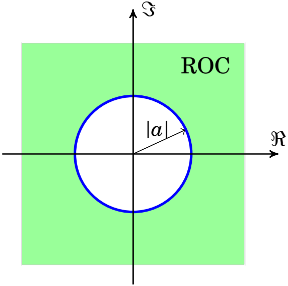
- $X(z)$ converges in the region determined by $|z| > |a|$, that is outside the circle with radius of $|a|$:



- General result: causal signals have ROC outside some circle.
- When $a = 1$, $x(n)$ becomes the unit-step signal $u(n)$, then its ZT is:

$$X(z) = \frac{1}{1 - z^{-1}}, \qquad \text{with } |z| > 1,$$

The ROC is outside the unit circle.

ZT of anti-causal signals

- Consider an exponential signal defined as

$$x(n) = \begin{cases} b^n, & \text{if } n < 0 \\ 0, & \text{if } n \geq 0 \end{cases}$$

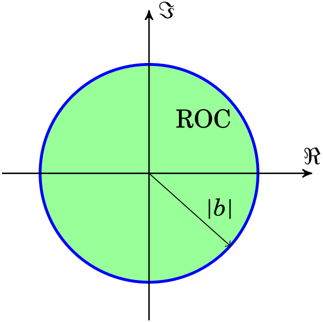
This signal vanishes at non-negative times and is thus sadi to be anti-causal.

- Its ZT:

$$X(z) = \sum_{n=-\infty}^{-1} b^n z^{-n} = \sum_{n=1}^{\infty} \left(\frac{z}{b}\right)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{z}{b}\right)^n - 1 = -\frac{1}{1 - bz^{-1}}, \qquad \text{với } |z| < |b| .$$

- $X(z)$ converges inside the circle with radius of $|b|$.



- General result: Anti-causal signals have ROC inside some circle.

ZT of non-causal signals

- A non-causal exists at both past and future times.
- Consider the following non-causal signal:

$$x(n) = \begin{cases} a^n, & \text{with } n \geq 0 \\ b^n, & \text{with } n < 0 \end{cases}$$

- Its ZT:

$$\begin{aligned} X(z) &= \cdots + b^2 z^2 + bz + 1 + az^{-1} + a^2 z^{-2} + \cdots \\ &= \sum_{n=-\infty}^{-1} b^n z^{-n} + \sum_{n=0}^{\infty} a^n z^{-n}. \end{aligned}$$

- $X(z)$ can be split into two sequences, one is causal and the other is anti-causal:

$$X(z) = -\frac{1}{1 - bz^{-1}} + \frac{1}{1 - az^{-1}}, \qquad \text{with } |z| < |b|, |z| > |a|.$$

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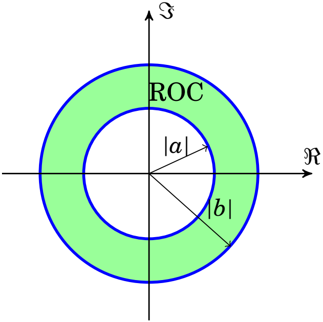
- $X(z)$ converges when $|a| < |b|$, the ROC becomes a ring:



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Frequently used ZT results

$x(n)$	$S(z)$
$\delta(n)$	1
$u(n)$	$\dfrac{1}{1-z^{-1}}$, ROC: $\lvert z\rvert > 1$
$anu(n)$	$\dfrac{az^{-1}}{\left(1-z^{-1}\right)^2}$, ROC: $\lvert z\rvert > 1$
$e^{-na}u(n)$	$\dfrac{1}{1-e^{-1}z^{-1}}$, ROC: $\lvert z\rvert > \left\lvert e^{-a}\right\rvert$
$a^n u(n)$	$\dfrac{1}{1-az^{-1}}$, ROC: $\lvert z\rvert > \lvert a\rvert$
$a^n[1-u(n)]$	$-\dfrac{1}{1-az^{-1}}$, ROC: $\lvert z\rvert < \lvert a\rvert$
$\sin(n\omega_0)u(n)$	$\dfrac{\sin(\omega_0)z^{-1}}{1-2z^{-1}\cos(\omega_0)+z^2}$
$\cos(n\omega_0)u(n)$	$\dfrac{1-\cos(\omega_0)z^{-1}}{1-2z^{-1}\cos(\omega_0)+z^2}$

Properties of ZT

$x(n)$	$S(z)$
$a_1x_1(n)+a_2x_2(n)$	$a_1x_1(z)+a_2x_2(z)$
$s(n-n_0)$	$z^{-n_0}X(z)$
$e^{-na}x(n)$	$S(e^a z)$
$\alpha^{-n}x(n)$	$S(\alpha z)$
$h(n)\star x(n)$	$H(z)X(z)$

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Inverse ZT

- The operation to transform a signal from the Z domain to the time domain (i.e., from $X(z)$ to $x(n)$) is called inverse ZT, denoted by $\mathcal{Z}^{-1}$:

$$x(n) = Z^{-1}\left\{X(z)\right\}.$$

- Cauchy's integral theorem to compute the inverse ZT:

$$x(n) = Z^{-1}\left\{X(z)\right\} = \frac{1}{2\pi j}\oint X(z) z^{-n} dz.$$

- When ZT has a rational form of z^{-1}, instead of using the Cauchy's integral theorem, we can use the table of frequently-used ZT results. The information about ROC is also important.

Example

- Given $X(z) = \dfrac{3z}{z - 0.5}, \quad |z| > 0.5$, find $x(n)$.

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Example

- Given $X(z) = \dfrac{3z}{z - 0.5}, \quad |z| > 0.5$, find $x(n)$.
  The ROC of $X(z)$ is outside the circle with radius of 0.5, so $x(n)$ is causal.
- Also
$$X(z) = \frac{3z}{z - 0.5} = 3 \times \frac{1}{1 - 0.5z^{-1}}.$$
- From table of frequently used ZT results, the following pair is found:
$$a^n u(n) \xrightarrow{\mathcal{Z}} \frac{1}{1 - az^{-1}}$$
$$\Rightarrow \frac{1}{1 - 0.5z^{-1}} \xrightarrow{\mathcal{Z}^{-1}} (0.5)^n u(n).$$
- Finally, since ZT is linear (table of properties), we have:
$$x(n) = 3(0.5)^n u(n).$$

$\mathcal{Z}^{-1}$ by partial fraction expansion

- We recognize

$$\frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

So, when expanding $X(z)$ into sum of partial fractions, instead of do it directly on $X(z)$, we do it on $X(z)/z$ and to have partial fractions of the form $1/(z - a)$.

$\mathcal{Z}^{-1}$ by partial fraction expansion

- Example: Find $x(n)$ if

$$X(z) = \frac{z(z-4)}{(z-1)(z-2)}, \qquad 1 < |z| < 2$$

$\mathcal{Z}^{-1}$ by partial fraction expansion

- Example: Find $x(n)$ if

$$X(z) = \frac{z(z-4)}{(z-1)(z-2)}, \qquad 1 < |z| < 2$$

- The ROC is a ring, so $x(n)$ is non-causal, then we should decompose it into two components: one is causal and the other is non-causal.
- Partial fraction expansion of $X(z)/z$:

$$\frac{X(z)}{z} = \frac{z-4}{(z-1)(z-2)} = \frac{3}{z-1} - \frac{2}{z-2}$$

- Then

$$X(z) = 3\frac{z}{z-1} - 2\frac{z}{z-2} = 3\frac{1}{1-z^{-1}} - 2\frac{1}{1-2z^{-1}}$$

- With $1 < |z|$, inverse ZT of $\dfrac{1}{1-z^{-1}}$ is causal.

- With $|z| < 2$, inverse ZT of $\dfrac{1}{1-2z^{-1}}$ is anti-causal.
- From table, we then have:

$$x(n) = 3u(n) + 2^{n+1}(1 - u(n))$$

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Application of ZT to LTI systems

- ZT is very useful for studying LTI systems of finite order.
- Such systems are characterized by the Linear Constant Coefficient Difference Equation (LCCDE):

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k). \qquad \text{(LCCDE)}$$

- The solution of (LCCDE), which is $y(n)$, can be represented by two methods.

Method I

- Method I: homogeneous / particular

$$\boxed{y(n) = y_\text{h}(n) + y_\text{p}(n)} \qquad \text{(Method I)}$$

- $y_\text{p}(n)$: particular solution of the LCCDE, as response to a particular input and after the initial transients have died out.
- $y_\text{h}(n)$: homogeneous solution of the following homogeneous equation (i.e. when no input is excited: zero-input):

$$\sum_{k=0}^{N} a_k y_h(n-k) = 0$$

- $y(n)$: complete solution

- $y_\mathrm{h}(n)$ has the general form

$$y_h(n) = A_1 r_1^n + A_2 r_2^n + \cdots + A_N r_N^n \qquad (*)$$

- $A_1, A_2, \ldots, A_N$ are constant.
- $r_1, r_2, \ldots, r_N$ are N solutions of

$$\boxed{a_0 r^N + a_1 r^{N-1} + \cdots + a_{N-1} r + a_N = 0} \qquad (**)$$

called characteristic equation.

- Notes: The general form of (*) is only correct if the N solutions of the characteristic equation, (**), are distinct.
- If we have a double solution, then r_k^n still show up in the system solution.
- Parameters $A_1, A_2, \ldots, A_N$ are determined by using N initial conditions of the LCCDE: $y(-1), y(-2), \ldots, y(-N)$.

Method II

- Method II: zero-input / zero-state

$$\boxed{y(n) = y_{zs}(n) + y_{zi}(n)} \qquad \text{(Method II)}$$

- $y_{zs}(n)$: zero-state solution of the LCCDE, as response to an input and with N initial conditions set to zero: the system starts from zero or is initially relaxed.
- $y_{zi}(n)$: zero-input solution of the homogeneous equation

$$\sum_{k=0}^{N} a_k y_h(n-k) = 0$$

and determined by the N initial conditions.

- Method II is equivalent to Method I, but it has precise physical interpretations. The total response of a system, $y(n)$, can be found by splitting a problem into two parts:
  - $y_{zi}(n)$: The zero-input part of the response is the response due to initial conditions alone (with the input set to zero).
  - $y_{zs}(n)$: The zero-state part of the response is the response due to the system input alone (with initial conditions set to zero).
- $y_{zs}(n)$ obeys the superposition principle (linearity): If the system is excited by a weighted sum (linear combination) of input signals, then $y_{zs}(n)$ is a weighted sum of the responses of the individual input signals.

- A system is linear if satisfies all the following:
  - $y(n) = y_{zi}(n) + y_{zs}(n)$,
  - Superposition principle applies to $y_{zs}(n)$ (zero-state linear),
  - Superposition principle applies to $y_{zi}(n)$ (zero-input linear).
- Special case: A system with zero initial conditions is linear (three requirements are satisfied since $y_{zi}(n) = 0$).
- A system described by an LCCDE is linear and time-invariant.
- From now on, we only study LTI systems defined by the LCCDE.

System transfer function

- Perform ZT on both sides of (LCCDE) (with initial conditions set to zero) and use the property of linearity of ZT:

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z),$$

$X(z)$ and $Y(z)$ are ZTs of input $x(n)$ and output $y(n)$.

- Denote

$$\boxed{H(z) = \frac{Y(z)}{X(z)}}$$

$H(z)$ is called the system transfer function.

- Then

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

- Relationship between $Y(z)$ and $X(z)$:

$$\boxed{Y(z) = H(z)X(z)}$$

- Relationship between $Y(z)$ and $X(z)$ using block diagram:

$$x(n) \longrightarrow \boxed{H(z)} \longrightarrow y(n)$$

- Excite the system with a Kronecker delta $x(n) = \delta(n)$, $(X(z) = 1)$. The ZT of the output is $Y(z) = H(z)$. Thus, the system transfer function is the ZT of the impulse response:

$$\boxed{H(z) = \mathcal{Z}\{h(n)\}}$$

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- Be careful: Two different systems may have the same ZT.

$$H(z) = \frac{z}{1 - 0.5z^{-1}}.$$

If the system is causal, the ROC is outside the circle with radius of 0.5:

$$h(n) = 3(0.5)^n u(n)$$

If the system is anti-causal, the ROC is inside the circle with radius of 0.5.

$$h(n) = -3(0.5)^n[1 - u(n)]$$

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- Equations

$$y_h(n) = A_1 r_1^n + A_2 r_2^n + \cdots + A_N r_N^n,$$

$$y(n) = y_{zs}(n) + y_{zi}(n)$$

  together imply that a causal system is only stable if $|r_k| < 1$ for all k, and an anti-causal system is only stable if $|r_k| > 1$ for all k.

- In practice, even we deal with non-causal sytems at times, the system response often starts at some finite time $-n_0$. We can delay the system by n_0 samples to make the system causal. Therefore, in this course, we only deal with systems that are causal and stable.

- Transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

  Solutions of the polynomial in the numerator ($Y(z) = 0$) are called zeros of the transfer function.
  Solutions of the polynomial in the denominator ($X(z) = 0$) are called poles.
- Poles are actually solutions of the characteristic function. Thus, the stability depends on poles. A system is stable if its poles lie inside the unit circle.

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Discrete-time Fourier transform

- The frequency spectrum of a continuous signal $x(t)$ is the Fourier transform (FT) of $x(t)$, defined as

$$\boxed{X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt}$$

- Sampling $x(t)$ by the infinite train of Dirac delta functions, with sampling period T, to obtain the sampled signal:

$$x_\Delta(t) = x(t)\Delta(t) = \sum_{k=-\infty}^{\infty} x(kT)\delta(t-kT).$$

- FT of $x_\Delta(t)$

$$X_\Delta(\Omega) = \sum_{k=-\infty}^{\infty} x(kT)e^{-jk\Omega T}$$

- Let $\omega = \Omega T$, and $x_d(n) = x(nT)$.

- Let

$$\boxed{X_\text{d}(\omega) = \sum_{n=-\infty}^{\infty} x_\text{d}(n)e^{-jn\omega}} \qquad (*)$$

- $X_\text{d}(\omega)$ in (*) is called the discrete-time Fourier transform (DTFT) of the discrete signal $x_\text{d}(n)$.
- If we do not consider T, then ω is an independent variable and thus the definition of DTFT can be applied to any discrete signals.
- $X_\text{d}(\omega)$ is a function with period of 2π, because $e^{-jn\omega} = e^{-jn(\omega+2\pi)}$. Thus, when analyzing $X_\text{d}(\omega)$, we only need to consider 1 period, $[0, 2\pi]$ or $[-\pi, \pi]$.
- ω is called the digital frequency with unit of radian/sample.

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- If $x_d(n)$ is real, then

$$X_d(-\omega) = X_d^*(\omega)$$

  Accordingly, the amplitude of $X_d(\omega)$ is an even function, its phase is an odd function. The non-negative part of the spectrum contains all spectral information. Hence, for a real signal $x_d(n)$, we only need to do frequency analysis in the range $[0, \pi]$.

- Given $X_d(\omega)$, we can obtain $X_\Delta(\Omega)$, and hence the spectrum of $x(t)$, provided that the Nyquist conditions were satisfied.

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DTFT for LTI systems

- FT plays a very important role in the theory of LTI systems
- Consider an LTI system $h(n)$ and excite it with a complex exponential signal $x(n) = e^{jn\omega_0}$
- The system output is

$$
\begin{aligned}
y(n) &= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\
&= \sum_{k=-\infty}^{\infty} h(k)e^{j(n-k)\omega_0} \\
&= \left[\sum_{k=-\infty}^{\infty} h(k)e^{-jk\omega_0}\right] e^{jn\omega_0}
\end{aligned}
$$

- Thus

$$\boxed{y(n) = H(\omega_0)e^{jn\omega_0}}$$

$H(\omega_0)$ is the FT of $h(n)$ evaluated at $\omega = \omega_0$.

- When exciting an LTI system with a complex exponential $e^{jn\omega_0}$, the output is also a complex exponential at the same frequency ω_0 but is amplified by a coefficient of $H(\omega_0)$. Thus, $H(\omega_0)$ is called the complex amplification of the system.
- Generally, $H(\omega)$ is the complex amplification of the system, as a function of ω. So, it is called the frequency response of the system.

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Connection between ZT and DTFT

- ZT

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- DTFT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-jn\omega}$$

- So, if we replace z in ZT by $e^{j\omega}$, then

$$\boxed{X(\omega) = X(z)\,\big|_{z=e^{j\omega}}} \qquad (*)$$

- Expression (*) shows a close connection between ZT and DTFT.
- DTFT only exists if the ROC of $X(z)$ contains the unit circle (this is satisfied if the system is causal).