ECE 863
Analysis of Stochastic Systems
Part III: Discrete- & Continuous-Time Random Processes

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ECE 863

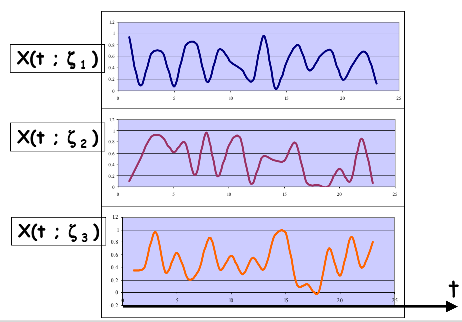
- **Reading assignments**
  - Sections 6.1 – 6.5

Random Processes

- **Remember that a random variable X(ζ) is a mapping from a "random outcome" ζ to a "real number".**
- **A Random Process X(t;ζ) is a mapping from a "random outcome" ζ to a "function"**
- **A random process can be a function of time, space, etc.**

Random Processes

$$X(t,\zeta) \qquad \zeta \in S$$



Random Processes

- For a given outcome $\zeta = \zeta_k$, the function $X(t, \zeta_k)$ is referred to as a "sample function"
- Therefore, a random process is a "collection of functions" (or a "family of functions") generated by the random outcome ζ

$$X(t,\zeta) \qquad \zeta \in S$$

Example: Sinusoidal Amplitude

- Let ζ be a uniformly distributed random variable in the interval [-1,+1].

  One can define the random process $X(t,\zeta)$:

$$X(t,\zeta) = \zeta \cos(t) \qquad -\infty < t < \infty$$

Example: Sinusoidal Amplitude

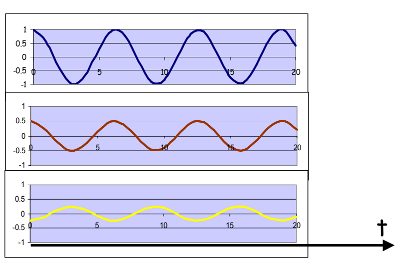
$$X(t,\zeta) = \zeta \cos(t) \qquad -\infty < t < \infty$$

$\boxed{\zeta \in [-1,+1]}$

$\zeta = 1$

$\zeta = 0.5$

$\zeta = -0.25$



Example: Sinusoidal Phase

- Let ζ be a uniformly distributed random variable in the interval $[0,2\pi]$.

  One can define the random process $Y(t,\zeta)$:

$$Y(t,\zeta) = \cos(t + \zeta)$$

Example: Sinusoidal Phase

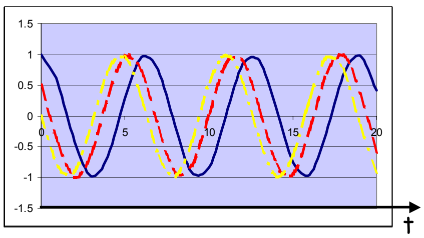
$$Y(t,\zeta) = \cos(t+\zeta)$$

$\zeta \in [0,2\pi]$

$\zeta = 0$

$\zeta = \frac{\pi}{3}$

$\zeta = \frac{\pi}{2}$



Random Processes

- **The parameter (t) of the random process $X(t,\zeta)$ can be discrete or continuous**
  - When t is continuous, we have a <u>"continuous-time" random process</u>
  - When t is discrete, we have a <u>"discrete-time" random process</u>
- **Either case t is a member of an "index set" I:**

$$X(t,\zeta) \quad t \in I \quad \zeta \in S$$

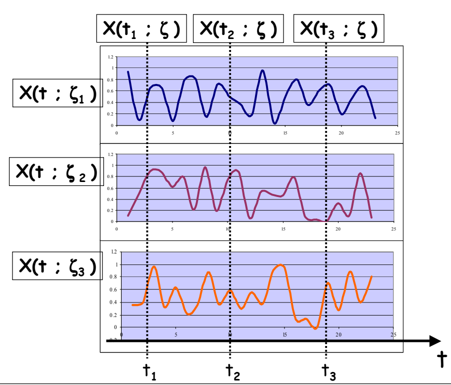
Source of randomness

Random Processes

- **At a given "time index" t, the random process $X(t,\zeta)$ represents a <u>random variable</u>**
- **As time changes, the random process generates "different" random variables:**

$X(t_1\,;\,\zeta)$ $X(t_2\,;\,\zeta)$ $X(t_3\,;\,\zeta)$ ....

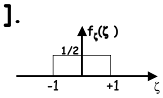
Random Processes



Example: Sinusoidal Amplitude

- Let ζ be a uniformly distributed random variable in the interval [-1,+1].

Therefore, $f_\zeta(\zeta)=1/2$.



One can define the random process $X(t,\zeta)$:

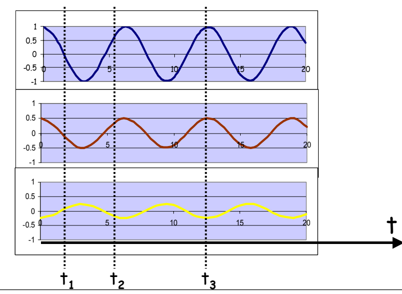
$$X(t,\zeta) = \zeta \cos(t) \qquad -\infty < t < \infty$$

Find the probability density functions (pdf) of the random variable $X(t,\zeta)$

Example: Sinusoidal Amplitude

$$X(t,\zeta) = \zeta \cos(t) \qquad -\infty < t < \infty$$

$\boxed{\zeta \in [-1,+1]}$

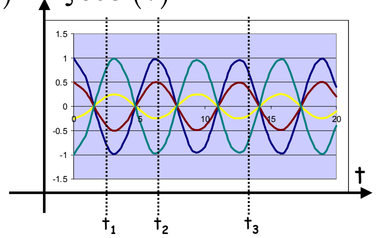


Example: Sinusoidal Amplitude

$$X(t,\zeta) = \zeta \cos(t)$$

$\boxed{\zeta \in [-1,+1]}$

For a given t_k, the RV $X(t_k,\zeta)$ varies within the interval:



$$\boxed{X(t_k,\zeta) \in [-|\cos(t_k)|, |\cos(t_k)|]}$$

Example: Sinusoidal Amplitude

- $X(t,\zeta)$ is a function of the random variable ζ.
- How would we find the pdf of $X(t,\zeta)$, given that we know the pdf of ζ?
- Do you remember the Fundamental Theorem?

Example: Sinusoidal Amplitude

- **Remember that we need to find the solution(s) to $X(t,\zeta)$. I.e. express in ζ terms x**
- **These solutions: ζ_1 , ζ_2 , . . . are functions of x**
- **Then, the Fundamental Theorem can be used as follows:**

$$f_X(x) = \frac{f_\zeta(\zeta_1)}{|\, dx(t,\zeta_1)/d\zeta \,|} + \frac{f_\zeta(\zeta_2)}{|\, dx(t,\zeta_2)/d\zeta \,|} + \cdots$$

Example: Sinusoidal Amplitude

$$X(t,\zeta) = \zeta \cos(t) \qquad \boxed{\zeta \in [-1,+1]}$$

Solving $x = \zeta \cos(t)$ leads to a single solution:

$$\boxed{\zeta = \frac{x}{\cos(t)}}$$

Taking the derivative of $x = \zeta \cos(t)$:

$$\boxed{\frac{dx}{d\zeta} = \cos(t)}$$

Example: Sinusoidal Amplitude

$$X(t,\zeta) = \zeta \cos(t) \qquad \boxed{\zeta \in [-1,+1]}$$

For a given t, the RV $X(t,\zeta)$ have a uniform distribution:

$$\boxed{f_{X(t,\zeta)}(x(t,\zeta)) = \frac{f_\zeta(\zeta)}{|\cos(t)|} = \frac{1}{2\,|\cos(t)|}}$$

$$\boxed{X(t,\zeta) \in [-|\cos(t)|,\ |\cos(t)|]}$$

Example: Sinusoidal Phase

- **Let ζ be a uniformly distributed random variable in the interval $[0, 2\pi]$.**

  One can define the random process $Y(t,\zeta)$:

$$Y(t,\zeta) = \cos(t + \zeta)$$

  Find the probability density functions (pdf) of the random variable $Y(t,\zeta)$

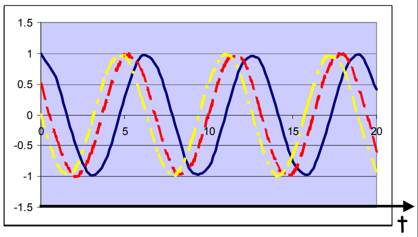
Example: Sinusoidal Phase

$$Y(t,\zeta) = \cos(t + \zeta)$$

$\boxed{\zeta \in [0,2\pi]}$

For a given t_k, the RV $Y(t_k,\zeta)$ varies within the interval:

$\boxed{Y(t,\zeta) \in [-1,+1]}$



Example: Sinusoidal Phase

- **By using the fundamental theorem:**

$$f_Y(y(t,\zeta)) = \frac{f_\zeta(\zeta_1)}{|\,dy(t,\zeta_1)/d\zeta\,|} + \frac{f_\zeta(\zeta_2)}{|\,dy(t,\zeta_2)/d\zeta\,|} + \cdots$$

we can show:

$$\boxed{f_Y(y(t,\zeta)) = \frac{1}{\pi\sqrt{1 - y^2}} \qquad |y| < 1}$$

(See examples 6.4 and 3.28 in the book)

Probability Functions for RP

- **Since a random process $X(t,\zeta)$ is a random variable at a given "time index" t, $X(t,\zeta)$ generates a <u>sequence of random variables</u> at a series of discrete time instances:**

  t_1, t_2 …t_k

$$\boxed{X_1 = X(t_1,\zeta),\quad X_2 = X(t_2,\zeta),\ldots,\quad X_k = X(t_k,\zeta)}$$

Probability Functions for RP

- **Therefore, we can define joint probability functions (cdf, pdf, pmf) of these "time samples" random variables:**

$$\boxed{X_1 = X(t_1,\zeta),\quad X_2 = X(t_2,\zeta),\ldots,\quad X_k = X(t_k,\zeta)}$$

- **For example, the joint cdf:**

$$\boxed{F_{X_1,\ldots,X_k}(x_1,x_2,\ldots,x_k) = P[X_1 \le x_1, X_2 \le x_2,\ldots,X_k \le x_k]}$$

Probability Functions for RP

- For discrete X_1 , X_2 ,… X_k , we can define the probability mass functions (pmf):

$$p_{x_1,\dots,x_k}(x_1,x_2,\dots,x_k) = P[X_1 = x_1, X_2,\dots,X_k = x_k]$$

- Similarly, for continuous X_1 , X_2 ,… X_k , we can define the joint probability density functions:

$$f_{X_1X_2\dots X_k}(x_1,x_2,\dots,x_k)$$

Moments of Random Processes

- Let $f_{X(t)}(x)$ be the pdf of the random variable X(t). We can define the kth moment:

$$E\left[(X(t))^k\right] = \int_{-\infty}^{+\infty} x^k f_{x(t)}(x)\,dx$$

Moments of Random Processes

- A special case of the kth moment is the mean:

$$m_x(t) = E[X(t)] = \int_{-\infty}^{+\infty} x f_{x(t)}(x)\,dx$$

- Therefore, in general, the mean and other moments could change with "time"

Moments of Random Processes

- If $f_{X(t)}(x)$ is the pdf of the random variable X(t), then the kth central moment:

$$E\left[(X(t)-m_X(t))^k\right] = \int_{-\infty}^{+\infty} (x-m_X(t))^k f_{x(t)}(x)\,dx$$

Moments of Random Processes

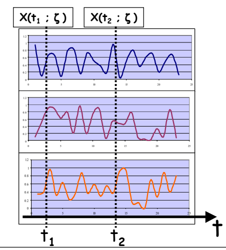
- A special case of the kth central moment is the variance:

$$\boxed{\begin{aligned} VAR(X(t)) &= E\left[(X(t)-m_X(t))^2\right] \\ &= \int_{-\infty}^{+\infty} (x-m_X(t))^2 f_{x(t)}(x)\,dx \end{aligned}}$$

Joint Moments of Random Proc.

- Two "time samples" $X(t_1)$ and $X(t_2)$ of the random process $X(t,\zeta)$ represent two random variables with a joint density function:

$$\boxed{f_{X(t_1),X(t_2)}(x,y)}$$



Joint Moments of Random Proc.

- The <u>autocorrelation</u> of the random process $X(t,\zeta)$ is defined as:

$$\boxed{\begin{aligned} R_x(t_1,t_2) &= E\left[X(t_1)X(t_2)\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{x(t_1),x(t_2)}(x,y)\,dx\,dy \end{aligned}}$$

Joint Moments of Random Proc.

- The <u>autocovariance</u> of the random process $X(t,\zeta)$ is defined as:

$$\boxed{\begin{aligned} &C_x(t_1,t_2) = \\ &E\left[\{X(t_1) - m_x(t_1)\}\{X(t_2) - m_x(t_2)\}\right] \end{aligned}}$$

What is $C_X(t_1,t_1)$?

Joint Moments of Random Proc.

- Evaluating the autocovariance at a single time sample t gives the variance at time t:

$$\begin{aligned} VAR(X(t)) &= E\left[(X(t) - m_x(t))^2\right] \\ &= C_x(t,t) \end{aligned}$$

Joint Moments of Random Proc.

- The autocorrelation and the autocovariance functions are related as follows:

$$C_x(t_1,t_2) = R_x(t_1,t_2) - m_x(t_1)m_x(t_2)$$

$$\begin{aligned} C_x(t,t) &= R_x(t,t) - m_x(t)m_x(t) \\ &= E[(X(t))^2] - (m_x(t))^2 \\ &= VAR(X(t)) \end{aligned}$$

Joint Moments of Random Proc.

- The autocorrelation coefficient (or simply correlation coefficient) is defined as:

$$\rho_x(t_1,t_2) = \frac{C_x(t_1,t_2)}{\sqrt{C_x(t_1,t_1)}\sqrt{C_x(t_2,t_2)}}$$

$$\rho_x(t_1,t_2) = \frac{C_x(t_1,t_2)}{\sqrt{VAR(t_1)}\sqrt{VAR(t_2)}} \qquad \left|\rho_x(t_1,t_2)\right| \leq 1$$

Example: Sinusoidal Amplitude

- Let ζ be a random variable, and let the random process $X(t,\zeta)$:

$$X(t,\zeta) = \zeta\cos(t)$$

Find the mean, variance, autocorrelation, and autocovariance of $X(t,\zeta)$

Example: Sinusoidal Amplitude

- **The mean:**

$$m_x(t) = E[\zeta \cos(t)]$$

$$\boxed{m_x(t) = E[\zeta]\cos(t)}$$

Example: Sinusoidal Amplitude

- **The variance:**

$$VAR(X(t)) = E\left[(\zeta\cos(t))^2\right] - (E[\zeta]\cos(t))^2$$

$$= E\left[\zeta^2\right]\cos^2(t) - (E[\zeta])^2\cos^2(t)$$

$$= \left(E\left[\zeta^2\right] - (E[\zeta])^2\right)\cos^2(t)$$

$$\boxed{VAR(X(t)) = (VAR(\zeta))\cos^2(t)}$$

Example: Sinusoidal Amplitude

- **The autocorrelation function**

$$R_x(t_1, t_2) = E[\zeta\cos(t_1)\zeta\cos(t_2)]$$

$$\boxed{R_x(t_1, t_2) = E\left[\zeta^2\right]\cos(t_1)\cos(t_2)}$$

Example: Sinusoidal Amplitude

- **The autocovariance function:**

$$C_x(t_1, t_2) = R_x(t_1, t_2) - m_x(t_1)m_x(t_2)$$

$$= \left\{E\left[\zeta^2\right] - (E[\zeta])^2\right\}\cos(t_1)\cos(t_2)$$

$$\boxed{C_x(t_1, t_2) = VAR(\zeta)\cos(t_1)\cos(t_2)}$$

Example: Sinusoidal Phase

- Let Θ be a uniform random variable over the interval $(-\pi, \pi)$, and let the random process $X(t,\Theta)$:

$$X(t) = \cos(t + \Theta)$$

Find the mean, variance, autocorrelation, and autocovariance of $X(t,\Theta)$

Example: Sinusoidal Phase

- The mean:

$$m_x(t) = E\left[\cos(t + \Theta)\right]$$

$$m_x(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(t + \theta)\,d\theta$$

$$\boxed{m_x(t) = 0}$$

Example: Sinusoidal Phase

- The autocorrelation and autocovariance functions

$$C_x(t_1,t_2) = R_x(t_1,t_2) - m_x(t_1)m_x(t_2)$$

$$C_x(t_1,t_2) = R_x(t_1,t_2)$$

$$C_x(t_1,t_2) = E\left[\cos(t_1 + \Theta)\cos(t_2 + \Theta)\right]$$

Example: Sinusoidal Phase

- The autocorrelation and autocovariance functions

$$C_x(t_1,t_2) = E\left[\cos(t_1 + \Theta)\cos(t_2 + \Theta)\right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}\left\{\cos(t_1 - t_2) + \cos\left((t_1 + t_2) + 2\theta\right)\right\}d\theta$$

$$\boxed{C_x(t_1,t_2) = R_x(t_1,t_2) = \frac{1}{2}\cos(t_1 - t_2)}$$

Example: Sinusoidal Phase

- **The variance:**

$$C_x(t_1, t_2) = R_x(t_1, t_2) = \frac{1}{2}\cos(t_1 - t_2)$$

$$C_x(t, t) = VAR(X(t))$$

$$\boxed{VAR(X(t)) = \frac{1}{2}}$$