EE 574 Detection and Estimation Theory

Lecture Presentation 4

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☞ Estimation Theory

A model of a general estimation problem is shown in Fig. 1. The model has the following components.

➢ **Parameter Space.**
The output of the source is a parameter in a parameter space. For the single parameter space, this corresponds to the line $-\infty < A < \infty$.
- The parameter is a random variable.
- The parameter is an unknown quantity but not a random variable.

➢ **Probabilistic Mapping from Parameter Space to Observation Space.**
This is the probability law that governs the effect of a on the observation.

➢ **Observation Space.**
In the classical problem, this is a finite-dimensional space. It is denoted by the vector **R**.

➢ **Estimation Rule.**
After observing **R** an estimate of the value of a is made and this estimate is denoted by $\hat{a}(\mathbf{R})$. This mapping of the observation space into an estimate is called the estimation rule.

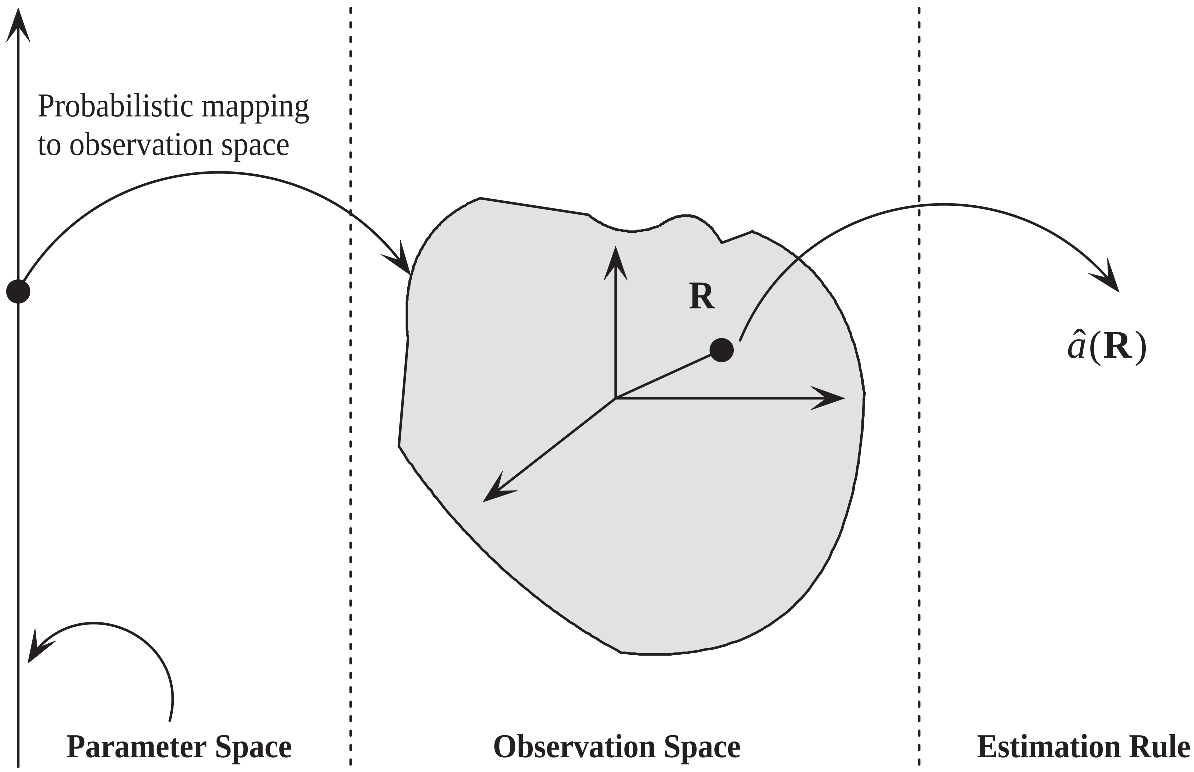

Figure 1: Estimation Model.

Random Parameters: Bayes Estimation

All pairs $[a, \hat{a}(\mathbf{R})]$ are assigned costs over the range of interest. This function of two variables is denoted as $C(a, \hat{a})$. Usually, the cost depends only on the error of the estimate which is given by

$$a_\epsilon(\mathbf{R}) = \hat{a}(\mathbf{R}) - a$$

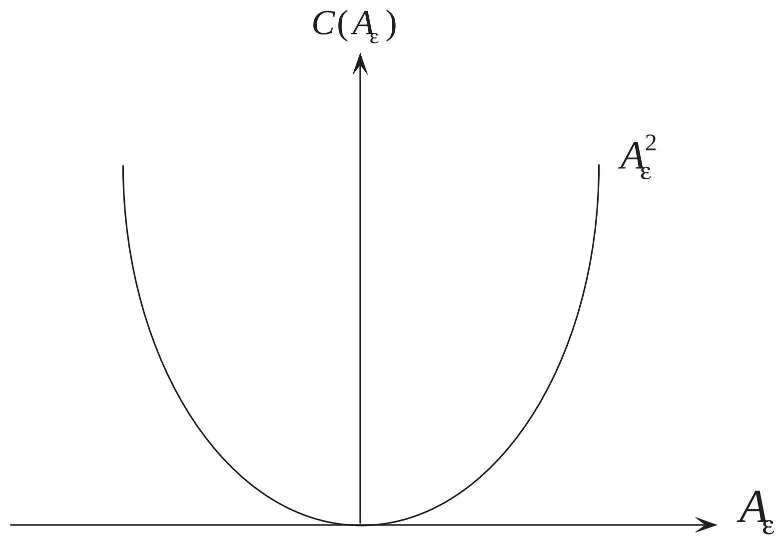

Figure 2: *Mean-square error* cost function.

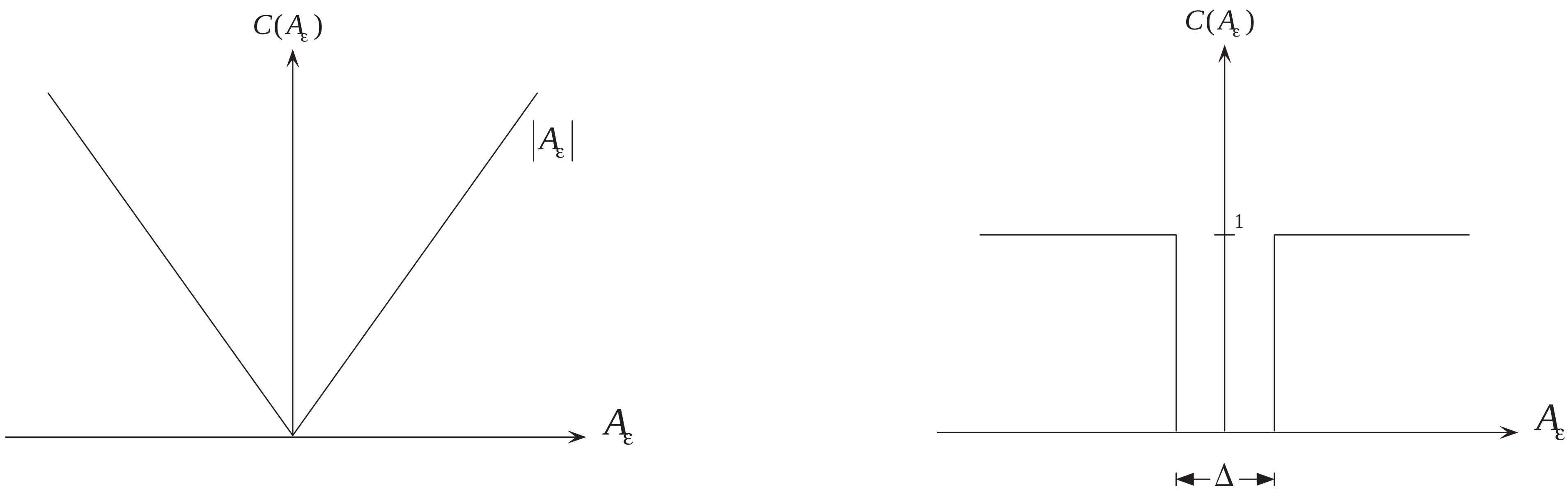

Figure 3: *Absolute error* and *uniform error* cost functions

The cost function $C(a_\epsilon)$ is a function of the error hence it is a function of a single variable. Some typical cost functions are shown Figs. 2 and 3.

➢ Squared error function.

$$C(a_\epsilon) = a_\epsilon^2$$

➢ Absolute value of the error function

$$C(a_\epsilon) = |a_\epsilon|$$

➢ Uniform error cost function

$$C(a_\epsilon) = \begin{cases} 0, & |a_\epsilon| \leq \frac{\Delta}{2} \\ 1, & |a_\epsilon| > \frac{\Delta}{2} \end{cases}$$

Just like the detection problem, it is assumed that the a priori probability density $p_a(A)$ in the random parameter estimation problem is known. If it is unknown, a procedure similar to the minimax test may be used.

The expression for the risk becomes:

$$\mathcal{R} = E\{C[A, \hat{a}(\mathbf{R})]\} = \int_{\infty}^{\infty} C[a, \hat{a}(\mathbf{R})] p_{a,\mathbf{r}}(A, \mathbf{R}) \, dA \, d\mathbf{R}. \tag{1}$$

The expectation is over the random variable a and the observed variables $\mathbf{r}$. For costs which are defined on the error only Eq.(1) becomes

$$\mathcal{R} = \int_{\infty}^{\infty} C[A - \hat{a}(\mathbf{R})] p_{a,\mathbf{r}}(A, \mathbf{R}) \, dA \, d\mathbf{R}. \tag{2}$$

Bayes estimate is the estimate that minimizes the risk. We write the joint density as:

$$p_{a,\mathbf{r}}(A, \mathbf{R}) = p_{\mathbf{r}}(\mathbf{R}) p_{a|\mathbf{r}}(A|\mathbf{R})$$

We then take the derivatives with respect to $\hat{a}$ to obtain the estimates. The Bayes estimates for the cost functions defined above are as follows:

- **Bayes estimate for the square error cost function: (Mean Square Estimate)**

$$\hat{a}_{\mathrm{ms}} = \int_{-\infty}^{\infty} A p_{a|\mathbf{r}}(A|\mathbf{R})\, dA \tag{3}$$

The term on the right hand side of Eq. (3) is the mean of the 'a posteriori density' (conditional mean).

- **Bayes estimate for the absolute value criterion**

$$\int_{-\infty}^{\hat{a}_{\mathrm{abs}}(\mathbf{R})} p_{a|\mathbf{r}}(A|\mathbf{R})\, dA = \int_{\hat{a}_{\mathrm{abs}}(\mathbf{R})}^{\infty} p_{a|\mathbf{r}}(A|\mathbf{R})\, dA \tag{4}$$

This is given by the median of the 'a posteriori' density.

- **Bayes estimate for the uniform cost function:(Maximum a Posteriori (MAP) Estimate)**

This is obtained when $\lim \Delta \to 0$ for the uniform cost function. In order to find $\hat{a}_{\mathrm{map}}(\mathbf{R})$ we must have the location of the maximum of the 'a posteriori' density.

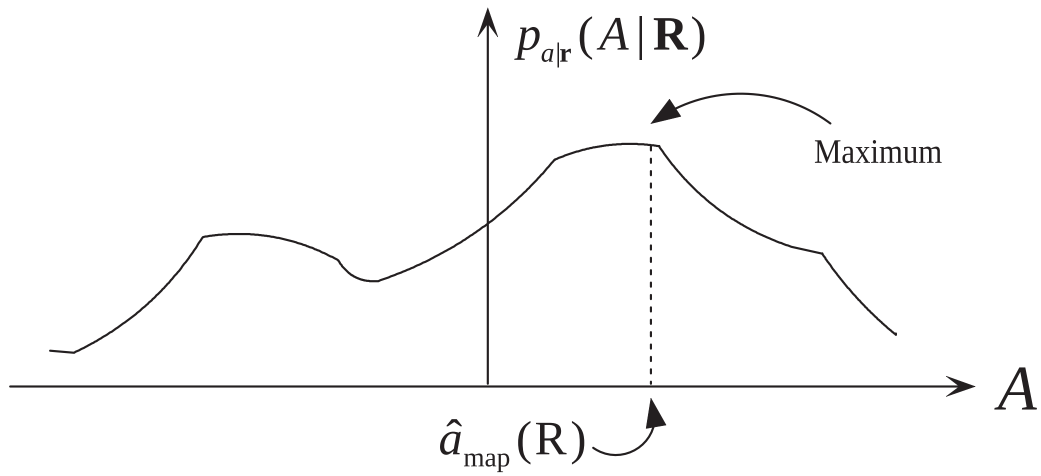

Figure 4: Maximum of the 'a posteriori' density.

Properties for the invariance of the cost functions for the estimators

1. We assume that the cost function $C(a_\epsilon)$ is a symmetric, convex upward function and that the a posteriori density $p_{a|\mathbf{r}}(A|\mathbf{R})$ is symmetric about its conditional mean: that is,

$$C(a_\epsilon) = C(-a_\epsilon) \ \ \text{Symmetry}$$

$$C(bx_1 + (1-b)x_2)) \leq bC(x_1) + (1-b)C(x_2) \;\; \text{Convexity}$$

For *non-convex functions*, we have property 2,

2. We assume that the cost function is a symmetric, nondecreasing function and that the a posteriori density $p_{a|\mathbf{r}}(A|\mathbf{R})$ is a symmetric function about its conditional mean. It is a unimodal function that satisfies the condition

$$\lim_{x \to \infty} C(x) p_{a|\mathbf{r}}(A|\mathbf{R}) = 0$$

The estimate $\hat{a}$ that minimizes any cost function in this class (satisfying the properties) is identical to $\hat{a}_{\text{ms}}$.