ECE 863
Analysis of Stochastic Systems
Part IV.3: Optimum Linear Estimation of Random Processes

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ECE 863

- **Exam 3 is on:**

  Wednesday, December 5

  during the regular time of the class

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- **Reading assignment included in Exam 3**
  - Sections 6.1-6.5 and 6.7
  - Sections 7.1, 7.2, and 7.4
- **Related lectures:**
  - Part-III (three sets)
  - Part-IV (three sets)

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- **Two Homework sets are included in Exam 3**
  - HW 8 and HW 9
- **You can:**
  - make copies of the tables that are in the inside cover of the book
  - make a copy of the Fourier transform table in the book
  - bring one sheet of paper with formulas, equations, etc. on both sides

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Exam 3 will NOT include questions from the Appendices at the end of the lecture notes

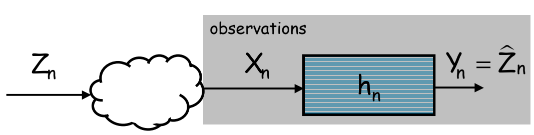
Optimum Linear Estimation

- **In many applications, we observe a random process X_n , and we would like to estimate another process Z_n based on our observations.**
- **In general, X_n and Z_n are correlated with some (known) cross-correlation function $R_{ZX}(\tau)$.**
- **We focus here on <u>jointly WSS processes</u>: X_n and Z_n**

Optimum Linear Estimation

- **In order to "estimate" Z_n from our observations of X_n , we process X_n using a linear system with a "unit-sample response" h_n:**

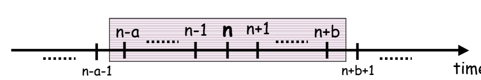
$$\hat{Z}_n = Y_n = \sum_{k=-b}^{a} h_k X_{n-k}$$



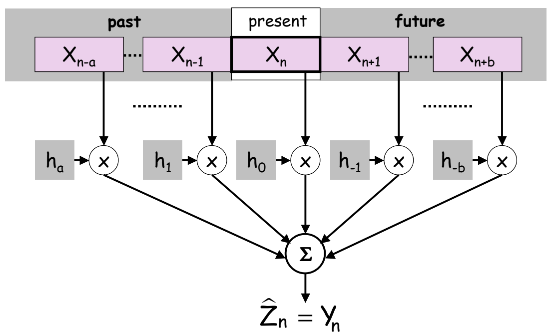
Optimum Linear Estimation

- **Therefore, we consider (a+b+1) samples of the random process X_n when "estimating" Z_n:**

$$\hat{Z}_n = Y_n = \sum_{k=-b}^{a} h_k X_{n-k}$$



Optimum Linear Estimation



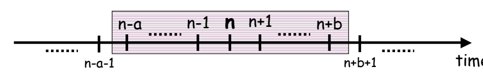
Optimum Linear Estimation

- At a given time instance (n) , the objective is to find the optimum (a+b+1) "unit-sample response" (or impulse response) coefficients:

$$h_{-b}\ ,\ \ldots\ ,\ h_{-1}\ ,\ h_0\ ,\ h_1\ ,\ \ldots,\ h_a$$

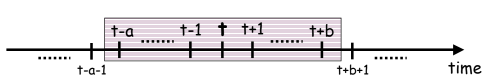
that minimize the mean-square-error:

$$E\left[e_n^2\right] = E\left[\left(Z_n - \hat{Z}_n\right)^2\right]$$



Optimum Linear Estimation

- All of the fundamental results of optimum linear estimation of random processes are applicable to both discrete and continuous-time processes
- To emphasize this notion of consistency between discrete and continuous-time optimum linear estimation, we will use the notation X_t to represent a discrete-time process



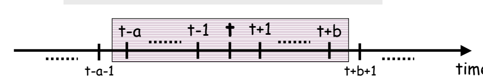
Optimum Linear Estimation

- Therefore, our objective is to find the optimum "unit-sample" response h_t:

$$\hat{Z}_t = Y_t = \sum_{\beta=-b}^{a} h_\beta X_{t-\beta}$$

which minimizes the mean-square-error:

$$E\left[e_t^2\right] = E\left[\left(Z_t - Y_t\right)^2\right]$$

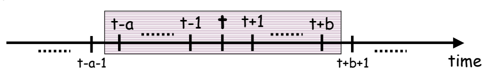


Optimum Linear Estimation

- We define the time interval over which we are observing the data:

$$I = \{t-a, \ldots, t, \ldots t+b\}$$

- The optimum solution depends on the size of the interval I.
- Let's start with the general case for any (a) and any (b).



Optimum Linear Estimation

- First, recall that for linear Minimum-Mean-Square-Error (MMSE) estimation of a random variable Z, the coefficient (h) in:

$$\hat{Z} = h\,X$$

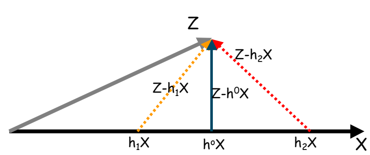
lead to the MMSE when the orthogonality principle is satisfied

$$E\left[\left(Z-\hat{Z}\right)X\right] = 0$$

Orthogonality Principle

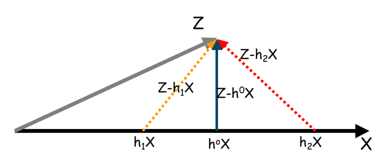
- Remember when: $\hat{Z} = h\,X$

  we are trying to minimize $E[(Z-hX)^2]$



Orthogonality Principle

- Consequently, the optimum (MMSE) estimate (i.e. h^0X) occurs when the error $(Z-hX)$ is orthogonal to the observation X



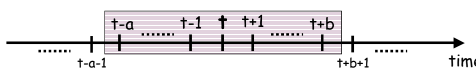
Optimum Linear Estimation

- **The orthogonality principle has to be satisfied by all (a+b+1) observations in the estimate:**

$$\hat{Z}_t = Y_t = \sum_{\beta=-b}^{a} h_\beta X_{t-\beta}$$

$$\boxed{E[e_t X_\alpha] = E\left[(Z_t - Y_t)X_\alpha\right] = 0 \qquad \forall\ \alpha \in I}$$

$$I = \{t-a, \ldots, t, \ldots, t+b\}$$



Optimum Linear Estimation

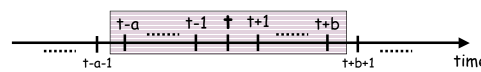
- **The orthogonality principle:**

$$\boxed{E[e_t X_\alpha] = E\left[(Z_t - Y_t)X_\alpha\right] = 0 \qquad \forall\ \alpha \in I}$$

leads to:

$$\boxed{E[Z_t X_\alpha] = E[Y_t X_\alpha] \qquad \forall\ \alpha \in I}$$

$$I = \{t-a, \ldots, t, \ldots, t+b\}$$

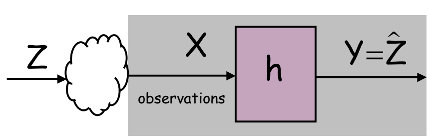


Optimum Linear Estimation

- **Therefore,**

$$\boxed{E[Z_t X_\alpha] = E[Y_t X_\alpha] \qquad \forall\ \alpha \in I}$$

$$\hat{Z}_t = Y_t = \sum_{\beta=-b}^{a} h_\beta X_{t-k} \qquad Y_t = h_t * X_t$$

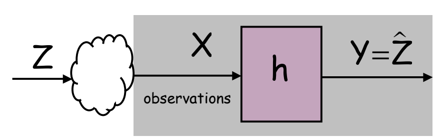


Optimum Linear Estimation

- **This leads to:**

$$\boxed{E[Z_t X_\alpha] = E[Y_t X_\alpha] \qquad \forall\ \alpha \in I} \qquad Y_t = h_t * X_t$$

$$\boxed{E[Z_t X_\alpha] = E\left[(h_t * X_t)X_\alpha\right] \qquad \forall\ \alpha \in I}$$

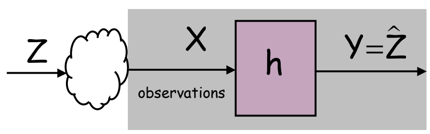


Optimum Linear Estimation

- **This leads to:**

$$\boxed{E[Z_t X_\alpha] = E\left[(h_t * X_t) X_\alpha\right] \qquad \forall\ \alpha \in I}$$

$$\boxed{E[Z_t X_\alpha] = h_t * E[X_t X_\alpha] \qquad \forall\ \alpha \in I}$$



Theorem: Optimum Linear Estimation

- **Let X_t and Z_t be jointly WSS, zero-mean processes. For a given cross-correlation function $R_{ZX}(m)$, and autocorrelation functions $R_X(m)$ and $R_Z(m)$, the filter h_t that minimizes the mean-square-error:**

$$E\left[e_t^2\right] = E\left[\left(Z_t - \hat{Z}_t\right)^2\right] \quad \text{where} \quad \hat{Z}_t = \sum_{\beta=-b}^{a} h_\beta X_{t-\beta}$$

satisfies the following:

$$\boxed{R_{ZX}(m) = \sum_{\beta=-b}^{a} h_\beta R_X(m-\beta) \qquad -b \le m \le a}$$

Theorem: Optimum Linear Estimation

- **Moreover, the Minimum-Mean-Square-Error (i.e. the mean square error associated with the optimum filter) can be expressed as follows:**

$$\boxed{E\left[\left(Z_t - \hat{Z}_t\right)^2\right] = R_Z(0) - \sum_{\beta=-b}^{a} h_\beta R_{ZX}(\beta)}$$

What does happen when Z_n and X_n are uncorrelated or independent?

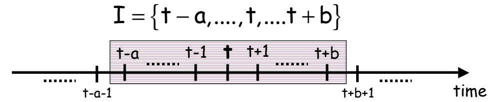
Optimum Linear Estimation

- **The solution for the optimum filter:**

$$\boxed{R_{ZX}(m) = \sum_{\beta=-b}^{a} h_\beta R_X(m-\beta) \qquad -b \le m \le a}$$

can be simplified by considering special cases of the interval I:

$$I = \{t-a, \ldots, t, \ldots t+b\}$$



Optimum Linear Estimation

- We will consider two important cases of "Infinite Impulse Response" (IIR) filters:
- **First**, we consider a non-causal IIR filter:

  $b = \infty$ and $a = \infty$

  $I = \{t-a, \ldots, t, \ldots t+b\} \Rightarrow \boxed{I = \{-\infty, \ldots t, \ldots \infty\}}$
- **Second**, we consider a causal IIR filter:

  $b = 0$ and $a = \infty$

  $I = \{t-a, \ldots, t, \ldots t+b\} \Rightarrow \boxed{I = \{-\infty, \ldots, t\}}$

Optimum Linear Estimation

$$R_{ZX}(m) = \sum_{\beta=-b}^{a} h_\beta R_X(m-\beta) \qquad -b \le m \le a$$

$b = 0$, $a = \infty$

Wiener-Hopf Equation for an optimum causal IIR filter

$$R_{ZX}(m) = \sum_{\beta=0}^{\infty} h_\beta R_X(m-\beta) \quad \forall\, m \ge 0$$

$b = \infty$, $a = \infty$

"Infinite-Smoothing" optimum (non-causal) IIR filter

$$R_{ZX}(m) = \sum_{\beta=-\infty}^{\infty} h_\beta R_X(m-\beta) \quad \forall\, m$$

Optimum Linear Estimation

$$R_{ZX}(\tau) = \int_{-b}^{a} h(\beta) R_X(\tau-\beta)\, d\beta \qquad -b \le \tau \le a$$

$b = 0$, $a = \infty$

Wiener-Hopf Equation for an optimum causal IIR filter

$$R_{ZX}(\tau) = \int_{0}^{\infty} h_\beta R_X(\tau-\beta)\, d\beta \quad \forall\, \tau \ge 0$$

$b = \infty$, $a = \infty$

"Infinite-Smoothing" optimum (non-causal) IIR filter

$$R_{ZX}(\tau) = \int_{-\infty}^{\infty} h_\beta R_X(\tau-\beta)\, d\beta \quad \forall\, \tau$$

Infinite Smoothing

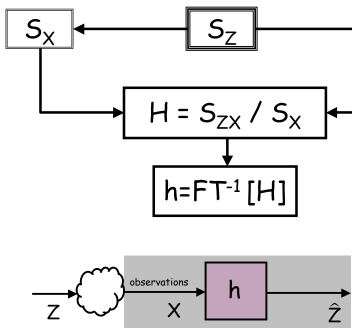
- By taking the FT of the optimum filter equation:

$$R_{ZX}(m) = \sum_{\beta=-\infty}^{\infty} h_\beta R_X(m-\beta) \quad \forall\, m \qquad R_{ZX}(\tau) = \int_{-\infty}^{\infty} h_\beta R_X(\tau-\beta)\, d\beta \quad \forall\, \tau$$

it can be easily shown that the optimum filter transfer function $H(f)$ satisfies the following:

$$S_{ZX}(f) = H(f) S_X(f) \qquad \boxed{H(f) = \frac{S_{ZX}(f)}{S_X(f)}}$$

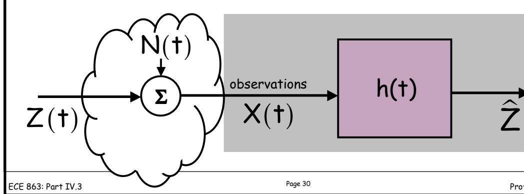
Infinite Smoothing



Example: Additive Noise

- **Find the optimum-linear, non-casual, IIR filter for the case of a <u>zero-mean</u> <u>independent</u> <u>additive</u> noise** $N(t)$:

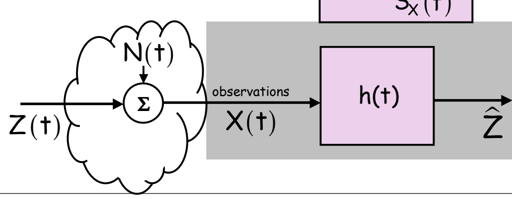
$$X(t) = Z(t) + N(t)$$



Example: Additive Noise

- **Now, we can find the optimum-linear, non-causal-IIR filter** $H(f)$ **that needs to be implemented at the receiver to estimate the original signal** $Z(t)$

$$H(f) = \frac{S_{ZX}(f)}{S_X(f)}$$



Example: Additive Noise

- **We are given:**
  - **the PSD function of Z:** $S_Z(f)$
  - **the PSD function of N:** $S_N(f)$

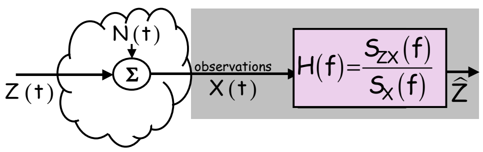
To find the optimum filter $H(f)$**, we need to evaluate the PSD functions** $S_{ZX}(f)$ **and** $S_X(f)$

$$H(f) = \frac{S_{ZX}(f)}{S_X(f)}$$

Example: Additive Noise

- Recall that for a __zero-mean__ __independent__ __additive__ noise, we can use the following:

$$R_{ZX}(\tau) = R_Z(\tau) \quad \Rightarrow \quad S_{ZX}(f) = S_Z(f)$$

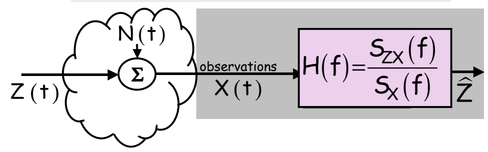


Example: Additive Noise

- Also, for a __zero-mean__ __independent__ __additive__ noise, we can use:

$$R_X(\tau) = R_Z(\tau) + R_N(\tau)$$

$$\Rightarrow \quad S_X(\tau) = S_Z(\tau) + S_N(\tau)$$

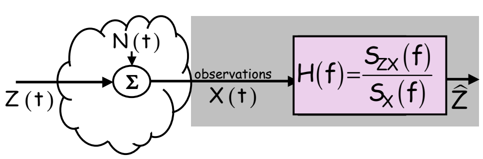


Example: Additive Noise

- Therefore:

$$S_{ZX}(f) = S_Z(f)$$

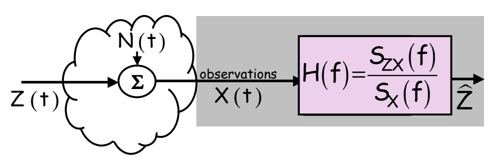
$$S_X(f) = S_Z(f) + S_N(f)$$



Example: Additive Noise

- Therefore:

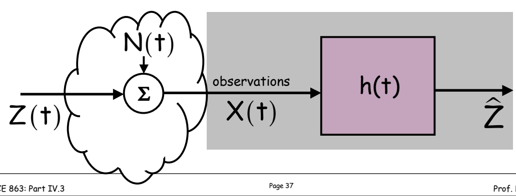
$$H(f) = \frac{S_{ZX}(f)}{S_X(f)} = \frac{S_Z(f)}{S_Z(f) + S_N(f)}$$



Example: Additive Noise

- **Now, let** $Z(t)$ **be a zero-mean WSS signal with an autocorrelation function:**

$$R_Z(\tau) = e^{-|\tau|}$$



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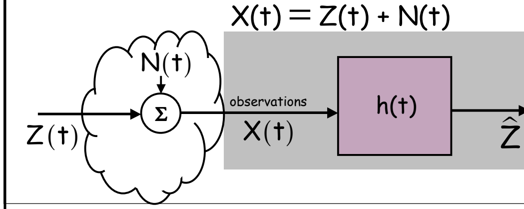
Example: Additive Noise

- **Also, let** $N(t)$ **be a zero-mean white noise process with a unit PSD:** $S_N(f) = 1$
- **Remember that** $N(t)$ **is independent of** $Z(t)$

$$X(t) = Z(t) + N(t)$$



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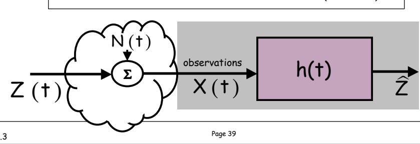
Example: Additive Noise

- **Since** $R_Z(\tau) = e^{-|\tau|}$

$$\Rightarrow \quad \boxed{S_Z(f) = \frac{2}{1+(2\pi f)^2}}$$

$$\Rightarrow \quad \boxed{S_{ZX}(f) = S_Z(f) = \frac{2}{1+(2\pi f)^2}}$$



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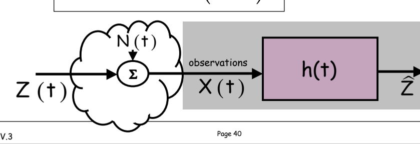
Example: Additive Noise

- **Using:** $S_X(f) = S_Z(f) + S_N(f)$

$$\Rightarrow \quad S_X(f) = \frac{2}{1+(2\pi f)^2} + 1$$

$$\Rightarrow \quad \boxed{S_X(f) = \frac{3+(2\pi f)^2}{1+(2\pi f)^2}}$$



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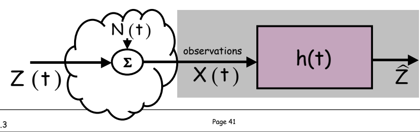
Example: Additive Noise

- **Therefore:**

$$\Rightarrow\ H(f) = \frac{S_{ZX}(f)}{S_X(f)} = \frac{S_Z(f)}{S_Z(f) + S_N(f)}$$

$$\Rightarrow\ H(f) = \frac{1+(2\pi f)^2}{3+(2\pi f)^2}\,\frac{2}{1+(2\pi f)^2}$$

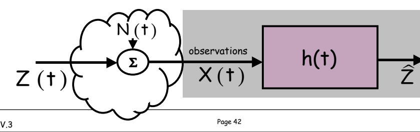


Example: Additive Noise

- **Therefore:**

$$\Rightarrow\ \boxed{H(f) = \frac{2}{3+(2\pi f)^2}}$$

$$\Rightarrow\ H(f) = \left(\frac{1}{\sqrt{3}}\right)\frac{2\sqrt{3}}{\left(\sqrt{3}\right)^2 + (2\pi f)^2}$$



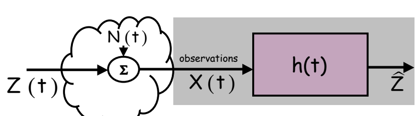
Example: Additive Noise

- **Taking the inverse FT of the following:**

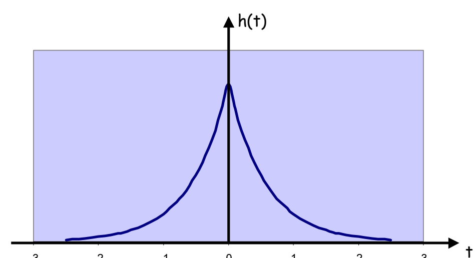
$$\Rightarrow\ H(f) = c_1\frac{2\sqrt{3}}{\left(\sqrt{3}\right)^2 + (2\pi f)^2}$$

where $c_1 = \frac{1}{\sqrt{3}}$

$$\Rightarrow\ \boxed{h(t) = c_1 e^{-\sqrt{3}|t|} \quad \forall t}$$



Example: Additive Noise



Optimum Linear Estimation

$$R_{ZX}(\tau) = \int_{-b}^{a} h(\beta) R_X(\tau - \beta)\, d\beta \qquad -b \le \tau \le a$$

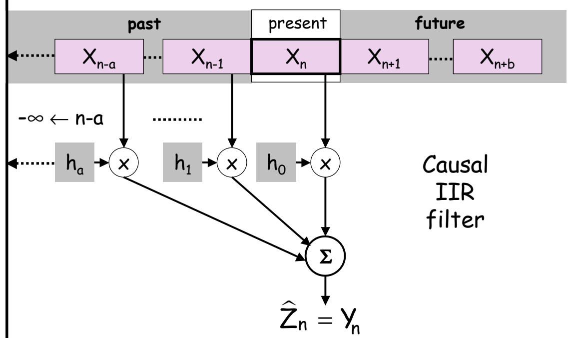
b = 0, a = ∞ → Wiener-Hopf Equation for an optimum causal IIR filter

$$R_{ZX}(\tau) = \int_{0}^{\infty} h_\beta R_X(\tau - \beta)\, d\beta \quad \forall\, \tau \ge 0$$

b = ∞, a = ∞ → "Infinite-Smoothing" optimum (non-causal) IIR filter

$$R_{ZX}(\tau) = \int_{-\infty}^{\infty} h_\beta R_X(\tau - \beta)\, d\beta \quad \forall\, \tau$$

Optimum Linear Estimation



Wiener-Hopf Equation

- **By setting** b = 0 **and** a = ∞ **, the solution to the optimum filter becomes:**

$$\boxed{R_{ZX}(m) = \sum_{\beta=0}^{\infty} h_\beta R_X(m - \beta)} \qquad \forall\, m \ge 0$$

- **This is known as the <u>Wiener-Hopf</u> Equation**

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End of Material that is included in Exam 3

Good Luck

☺

Appendix A

The Solution

for the

Wiener-Hopf Equation

Wiener-Hopf Equation

- **Although the Wiener-Hopf equation is for an IIR optimum filter, it is still difficult to solve.**
- **However, for the special case when the observed process (say X'_n) is a white-noise process, then a simplified solution can be obtained.**

Wiener-Hopf Equation

- **Therefore, if $R_{X'}(m) = \delta_m$, then the Wiener-Hopf equation becomes:**

$$R_{ZX'}(m) = \sum_{\beta=0}^{\infty} h'_\beta R_{X'}(m-\beta) \qquad \forall\, m \geq 0$$

$$R_{ZX'}(m) = \sum_{\beta=0}^{\infty} h'_\beta \delta_{m-\beta} \qquad \forall\, m \geq 0$$

$$\boxed{R_{ZX'}(m) = h'_m \qquad \forall\, m \geq 0}$$

Wiener-Hopf Equation

- **Therefore, if we have an IIR causal filter operating on a white noise process X'_n , then the optimum coefficients for that filter satisfy the following:**

$$\boxed{h'_m = \begin{cases} R_{ZX'}(m) & m \geq 0 \\ 0 & m < 0 \end{cases}}$$

$$\boxed{H'(f) = \sum_{m=0}^{\infty} R_{ZX'}(m) e^{-j2\pi fm}}$$

Wiener-Hopf Equation

- **Consequently, if we have a knowledge of the cross-power spectral density $S'_{ZX'}(f)$ (of the white noise process X'_n and the process Z_n), then the optimum filter can be expressed:**

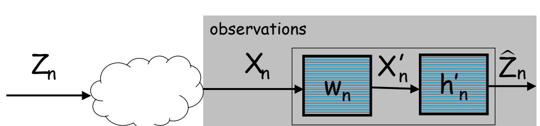
$$\boxed{h'_m = \begin{cases} FT^{-1}\left[S'_{ZX'}(f)\right] & m \geq 0 \\ 0 & m < 0 \end{cases}}$$

Wiener-Hopf Equation

- **In general, we do not observe a white noise process**
- **However, if we convert our observed process X_n to a white noise process X'_n , then we can find the optimum filter more easily**
- **Therefore, for any process X_n we can find the optimum IIR causal filter by cascading two filters**

Wiener-Hopf Equation

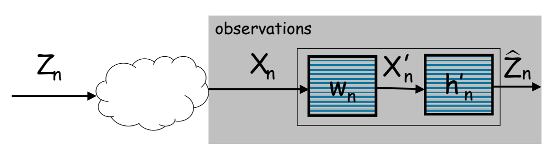
- **The first filter, w_n , converts our observed process X_n to the white noise process X'_n**
- **The second filter, h'_n , operates on the white noise process**



Wiener-Hopf Equation

- **Recall that for any filter, w_n , with transfer function $W(f)$:**

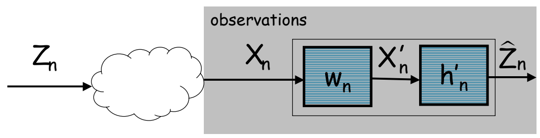
$$\boxed{S_{X'}(f) = |W(f)|^2 S_X(f)}$$



Wiener-Hopf Equation

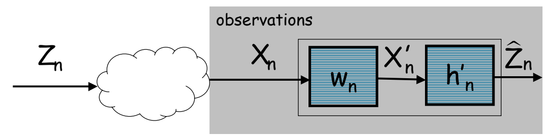
- **Here, we would like to have** $S_{X'}(f)=1$. **Therefore:**

$$S_{X'}(f) = \underbrace{|W(f)|^2 S_X(f)}_{=1} \qquad \boxed{|W(f)|^2 = \frac{1}{S_X(f)}}$$



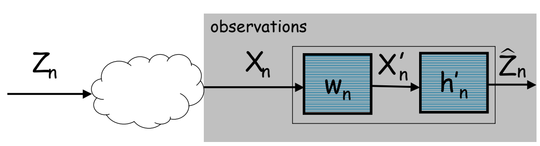
Wiener-Hopf Equation

- **Now, if we can factor:** $S_X(f) = G(f)G^*(f)$
- **Then:** $\boxed{|W(f)|^2 = \frac{1}{S_X(f)} = \frac{1}{G(f)G^*(f)}}$



Wiener-Hopf Equation

- **Therefore, if** $G(f)$ and $1/G(f)$ **are causal filters:**
- **Then:** $\boxed{W(f) = \frac{1}{G(f)}}$

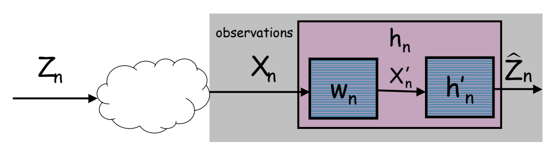


Wiener-Hopf Equation

- **Therefore, the optimum (cascaded) causal IIR filter** h_n **has a transfer function:**

$$\boxed{H(f) = W(f)H'(f) = \frac{H'(f)}{G(f)}}$$

where $S_X(f) = G(f)G^*(f)$

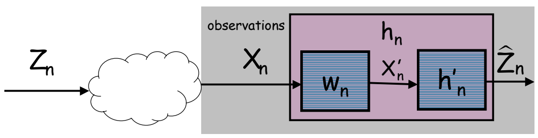


Wiener-Hopf Equation

- **We still need to express H'(f) in terms of the observed process X_n and the process Z_n**

$$\boxed{H(f) = W(f)H'(f) = \frac{H'(f)}{G(f)}}$$

where $S_X(f) = G(f)G^*(f)$

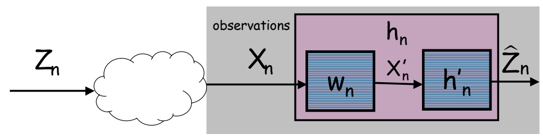


Wiener-Hopf Equation

- **Since the input to** H'(f) **is the "white noise" process** X'_n **, then:**

$$\boxed{H'(f) = \sum_{m=0}^{\infty} R_{ZX'}(m)e^{-j2\pi fm}}$$

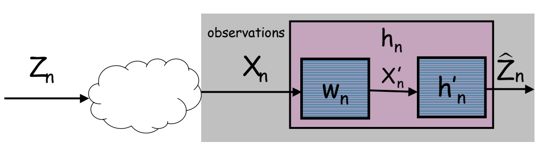
We need to express $R_{ZX'}$ in terms of R_{ZX}



Wiener-Hopf Equation

$$R_{ZX'}(k) = E\left[Z_{n+k}X'_n\right] = E\left[Z_{n+k}\left(\sum_{i=0}^{\infty} w_i X_{n-i}\right)\right]$$

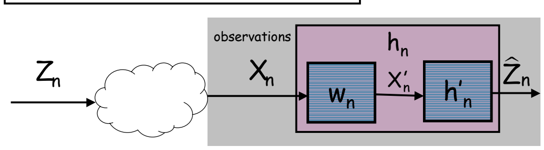
$$R_{ZX'}(k) = \sum_{i=0}^{\infty} w_i E\left[Z_{n+k}X_{n-i}\right]$$



Wiener-Hopf Equation

$$R_{ZX'}(k) = \sum_{i=0}^{\infty} w_i E\left[Z_{n+k}X_{n-i}\right]$$

$$\boxed{R_{ZX'}(k) = \sum_{i=0}^{\infty} w_i R_{ZX}(k+i)}$$

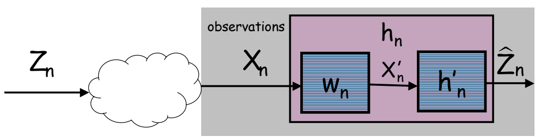


Wiener-Hopf Equation

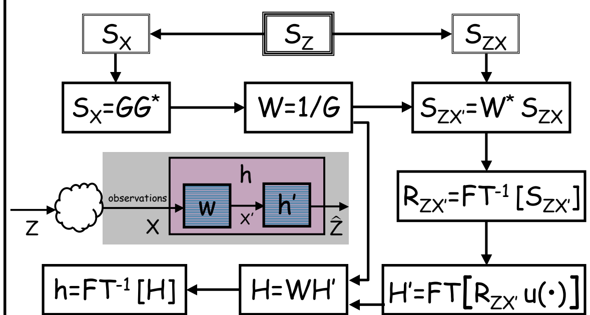
By taking the FT of:

$$R_{ZX'}(k) = \sum_{i=0}^{\infty} w_i R_{ZX}(k+i)$$

$$\boxed{S_{ZX'}(f) = W^*(f)\, S_{ZX}(f)}$$



Wiener-Hopf Equation: "Solution Roadmap"



Example: Additive Noise

- **Let $Z(t)$ be a zero-mean WSS signal generated at the output of a transmitter. $Z(t)$ has an autocorrleation function:**

$$R_Z(\tau) = e^{-|\tau|}$$

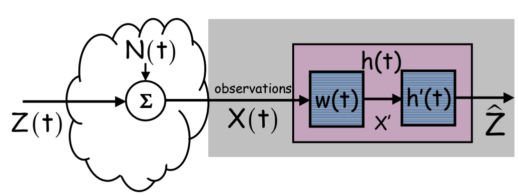
Let $N(t)$ be a zero-mean white noise process with a unit PSD. $N(t)$ is independent of $Z(t)$. The sum process $X(t)$ is observed at the input of a receiver:

$$X(t) = Z(t) + N(t)$$

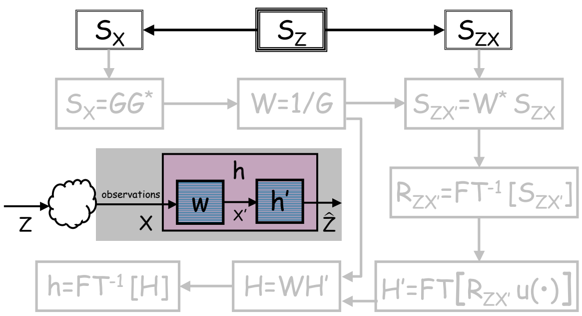
Find the optimum-linear, causal-IIR filter that needs to be implemented at the receiver to estimate the original signal $Z(t)$

Example: Additive Noise

- **First we need to find S_Z , S_X , and S_{ZX}**



Example: Additive Noise



Example: Additive Noise

- **For a zero-mean independent additive noise, we can use the following:**

$$R_{ZX}(\tau) = R_Z(\tau) \quad \Rightarrow \quad S_{ZX}(f) = S_Z(f)$$

$$R_X(\tau) = R_Z(\tau) + R_N(\tau)$$

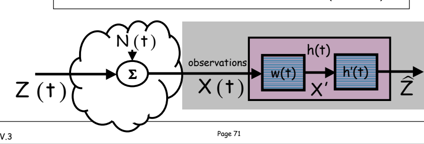
$$\Rightarrow \quad S_X(\tau) = S_Z(\tau) + S_N(\tau)$$

Example: Additive Noise

- **Since** $R_Z(\tau) = e^{-|\tau|}$

$$\Rightarrow \quad \boxed{S_Z(f) = \frac{2}{1+(2\pi f)^2}}$$

$$\Rightarrow \quad \boxed{S_{ZX}(f) = S_Z(f) = \frac{2}{1+(2\pi f)^2}}$$

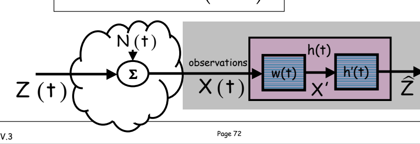


Example: Additive Noise

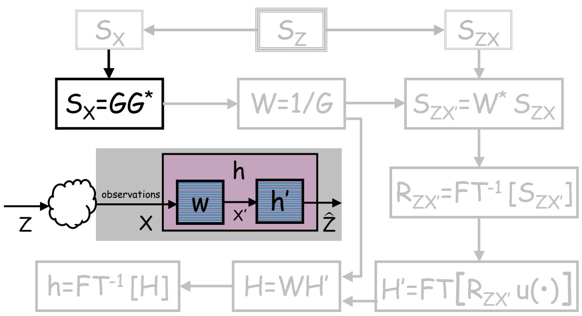
- **Using:** $S_X(f) = S_Z(f) + S_N(f)$

$$\Rightarrow \quad S_X(f) = \frac{2}{1+(2\pi f)^2} + 1$$

$$\Rightarrow \quad \boxed{S_X(f) = \frac{3+(2\pi f)^2}{1+(2\pi f)^2}}$$



Example: Additive Noise

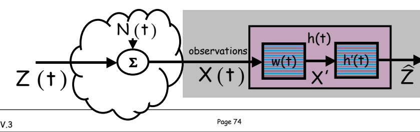


Example: Additive Noise

- **Since:**

$$\boxed{S_X(f) = \frac{3+(2\pi f)^2}{1+(2\pi f)^2}}$$

$$\Rightarrow \boxed{S_X(f) = \left(\frac{\sqrt{3}+j2\pi f}{1+j2\pi f}\right)\left(\frac{\sqrt{3}-j2\pi f}{1-j2\pi f}\right)}$$

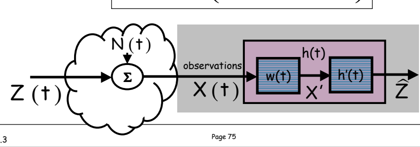


Example: Additive Noise

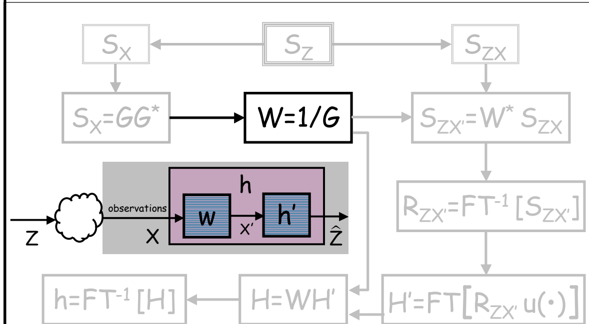
- **Therefore, we can set:**

$$S_X(f) = \left(\frac{\sqrt{3}+j2\pi f}{1+j2\pi f}\right)\left(\frac{\sqrt{3}-j2\pi f}{1-j2\pi f}\right) = G(f)G^*(f)$$

$$\Rightarrow \boxed{G(f) = \left(\frac{\sqrt{3}+j2\pi f}{1+j2\pi f}\right)}$$



Example: Additive Noise

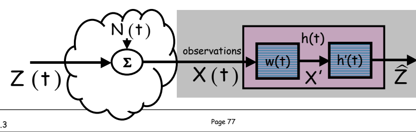


Example: Additive Noise

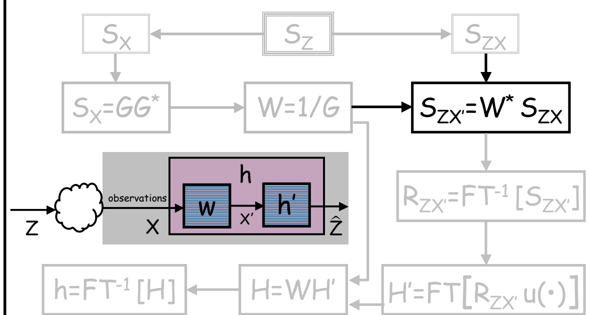
- **Since:**

$$G(f) = \left(\frac{\sqrt{3} + j2\pi f}{1 + j2\pi f} \right)$$

$$\Rightarrow \boxed{W(f) = \frac{1}{G(f)} = \left(\frac{1 + j2\pi f}{\sqrt{3} + j2\pi f} \right)}$$



Example: Additive Noise

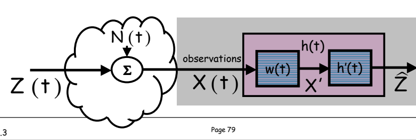


Example: Additive Noise

- **We already have:**

$$\boxed{S_{ZX}(f) = S_Z(f) = \frac{2}{1 + (2\pi f)^2}}$$

$$\& \quad \boxed{W^*(f) = \frac{1}{G^*(f)} = \left(\frac{1 - j2\pi f}{\sqrt{3} - j2\pi f} \right)}$$

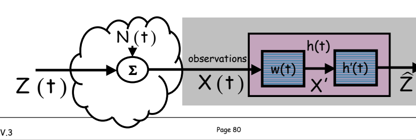


Example: Additive Noise

- **Therefore,**

$$S_{ZX'}(f) = W^*(f) S_{ZX}(f)$$

$$S_{ZX'}(f) = \left(\frac{2}{1 + (2\pi f)^2} \right) \left(\frac{1 - j2\pi f}{\sqrt{3} - j2\pi f} \right)$$

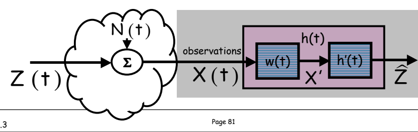


Example: Additive Noise

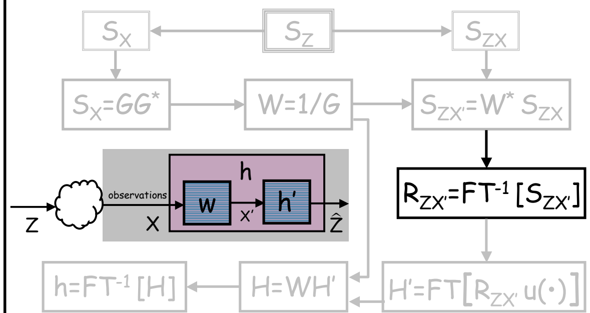
- **The following can be simplified:**

$$S_{ZX'}(f) = \left(\frac{2}{1+(2\pi f)^2}\right)\left(\frac{1-j2\pi f}{\sqrt{3}-j2\pi f}\right)$$

$$\boxed{S_{ZX'}(f) = \frac{2}{(1+j2\pi f)\left(\sqrt{3}-j2\pi f\right)}}$$



Example: Additive Noise



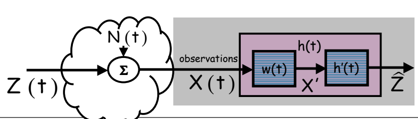
Example: Additive Noise

- **Now:** $R_{ZX'}(\tau) = FT^{-1}\left[S_{ZX'}(f)\right]$

$$S_{ZX'}(f) = \frac{2}{(1+j2\pi f)\left(\sqrt{3}-j2\pi f\right)}$$

$$S_{ZX'}(f) = \frac{c}{1+j2\pi f} + \frac{c}{\sqrt{3}-j2\pi f}$$

where $c = \frac{2}{1+\sqrt{3}}$



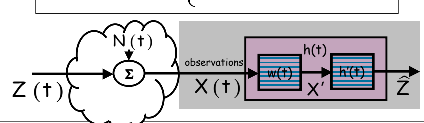
Example: Additive Noise

- **Therefore:** $R_{ZX'}(\tau) = FT^{-1}\left[S_{ZX'}(f)\right]$

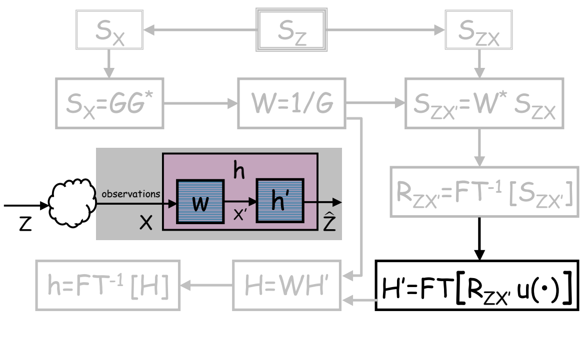
$$S_{ZX'}(f) = \frac{c}{1+j2\pi f} + \frac{c}{\sqrt{3}-j2\pi f}$$

where $c = \frac{2}{1+\sqrt{3}}$

$$\boxed{R_{ZX'}(\tau) = \begin{cases} ce^{-\tau} & \tau \geq 0 \\ ce^{\sqrt{3}\tau} & \tau < 0 \end{cases}}$$



Example: Additive Noise



Example: Additive Noise

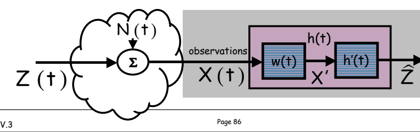
- **Since:**

$$R_{ZX'}(\tau) = \begin{cases} ce^{-\tau} & \tau \geq 0 \\ ce^{\sqrt{3}\tau} & \tau < 0 \end{cases}$$

where $c = \dfrac{2}{1+\sqrt{3}}$

$$h'(\tau) = \begin{cases} R_{ZX'}(\tau) & \tau \geq 0 \\ 0 & \tau < 0 \end{cases}$$

Remember we need a causal filter



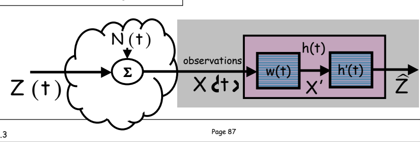
Example: Additive Noise

- **Since:**

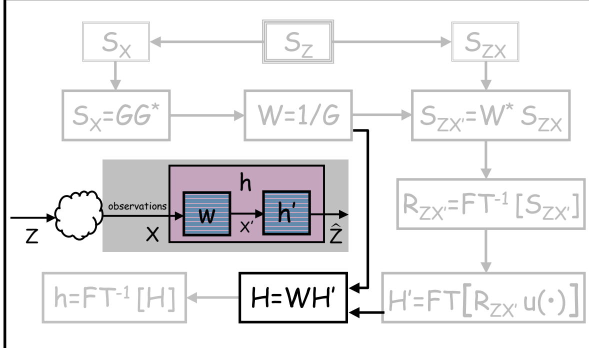
$$h'(t) = \begin{cases} ce^{-t} & t \geq 0 \\ 0 & t < 0 \end{cases} \qquad h'(t) = ce^{-t}u(t)$$

$$H'(f) = FT\left[h'(t)\right] = FT\left[ce^{-t}u(t)\right]$$

$$\boxed{H'(f) = \frac{c}{1 + j2\pi f}}$$



Example: Additive Noise

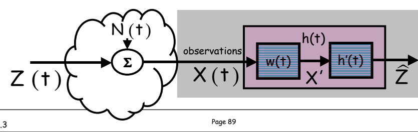


Example: Additive Noise

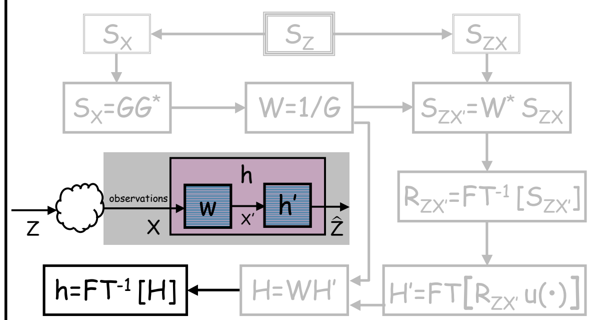
- **We already have:**

$$W(f)=\left(\frac{1+j2\pi f}{\sqrt{3}+j2\pi f}\right) \quad \& \quad H'(f)=\frac{c}{1+j2\pi f}$$

$$\boxed{H(f)=\left(\frac{c}{\sqrt{3}+j2\pi f}\right)}$$



Example: Additive Noise



Example: Additive Noise

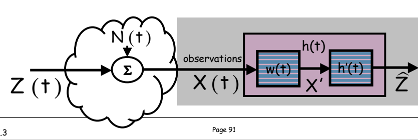
- **Now taking the inverse FT:**

$$h(t)=FT^{-1}\left[H(f)\right]=FT^{-1}\left[\frac{c}{\sqrt{3}+j2\pi f}\right]$$

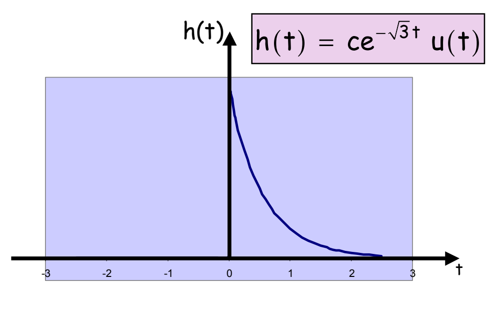
$$\boxed{h(t)\ =\ ce^{-\sqrt{3}t}\ u(t)}$$

where

$$c=\frac{2}{1+\sqrt{3}}$$



Example: Additive Noise



Optimum Linear Estimation

$$R_{ZX}(m) = \sum_{\beta=-b}^{a} h_\beta R_X(m-\beta) \qquad -b \le m \le a$$

b = 0
a = ∞

Wiener-Hopf Equation
for an optimum
causal IIR filter

$$R_{ZX}(m) = \sum_{\beta=0}^{\infty} h_\beta R_X(m-\beta)$$
$$\forall\, m \ge 0$$

b = ∞
a = ∞

"Infinite-Smoothing"
optimum
(non-causal) IIR filter

$$R_{ZX}(m) = \sum_{\beta=-\infty}^{\infty} h_\beta R_X(m-\beta)$$
$$\forall\, m$$

Optimum Linear Estimation

$$R_{ZX}(\tau) = \int_{-b}^{a} h(\beta) R_X(\tau-\beta)\, d\beta \qquad -b \le \tau \le a$$

b = 0
a = ∞

Wiener-Hopf Equation
for an optimum
causal IIR filter

$$R_{ZX}(\tau) = \int_{0}^{\infty} h_\beta R_X(\tau-\beta)\, d\beta$$
$$\forall\, \tau \ge 0$$

b = ∞
a = ∞

"Infinite-Smoothing"
optimum
(non-causal) IIR filter

$$R_{ZX}(\tau) = \int_{-\infty}^{\infty} h_\beta R_X(\tau-\beta)\, d\beta$$
$$\forall\, \tau$$

Appendix B

Derivation of the

Optimum Linear Estimator

Optimum Linear Estimation

- **Using:**

$$\boxed{E[Z_t X_\alpha] = E[Y_t X_\alpha] \qquad \forall\, \alpha \in I}$$

$$R_{ZX}(t,\alpha) = E\left[\left(\sum_{\beta=-b}^{a} h_\beta X_{t-\beta}\right) X_\alpha\right] \qquad \forall\, \alpha \in I$$

$$R_{ZX}(t,\alpha) = \sum_{\beta=-b}^{a} h_\beta E\left[X_{t-\beta} X_\alpha\right] \qquad \forall\, \alpha \in I$$

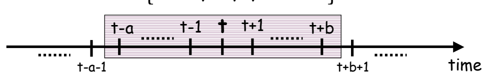
Optimum Linear Estimation

- **Therefore:**

$$R_{ZX}(t,\alpha) = \sum_{\beta=-b}^{a} h_\beta E\left[X_{t-\beta}X_\alpha\right] \qquad \forall\ \alpha \in I$$

$$R_{ZX}(t,\alpha) = \sum_{\beta=-b}^{a} h_\beta R_X(t-\alpha-\beta) \qquad \forall\ \alpha \in I$$

$$I = \{t-a,\ldots,t,\ldots t+b\}$$



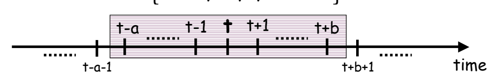
Optimum Linear Estimation

- **This equation:**

$$R_{ZX}(t,\alpha) = \sum_{\beta=-b}^{a} h_\beta R_X(t-\alpha-\beta) \qquad \forall\ \alpha \in I$$

shows that the cross-correlation function R_{ZX} **is a function of the time difference** (t-α)

$$I = \{t-a,\ldots,t,\ldots t+b\}$$



Optimum Linear Estimation

- **If we define:** $m = t-\alpha$, **then:**

$$R_{ZX}(t-\alpha) = \sum_{\beta=-b}^{a} h_\beta R_X(t-\alpha-\beta) \qquad \forall\ \alpha \in I$$

becomes:

$$\boxed{R_{ZX}(m) = \sum_{\beta=-b}^{a} h_\beta R_X(m-\beta) \qquad -b \le m \le a}$$

$$I = \{t-a,\ldots,t,\ldots t+b\}$$