ECE 863
Analysis of Stochastic Systems
Part IV.2: Linear Systems for Random Processes

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ECE 863

- Reading assignment

  Sections 7.1, 7.2, and 7.4

- Homework #8 solutions have been downloaded

- Last Homework Assignment (#9) have been downloaded

Linear Systems for Random Processes

- Linear systems play a major role in random processes
- In general, a linear system takes an input random process X(t) and generates another random process Y(t) as an output
- Linear systems can be used in many estimation, filtering, and prediction problems
- An important class of linear systems is "Time-Invariant"

Linear Time-Invariant Systems

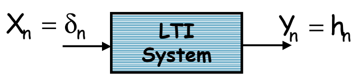
- A Linear Time-Invariant (LTI) system can be characterized by its "impulse response" h(t)
- For a discrete-time LTI system, the "unit-sample response" h_n is used:

$$X_n \longrightarrow \boxed{h_n} \longrightarrow Y_n$$

Linear Time-Invariant Systems

- The "unit-sample response" h_n of a discrete-time LTI system is measured by applying the "unit-sample" function as an input:

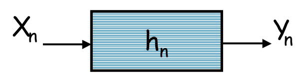
$$\delta_n = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



Linear Time-Invariant Systems

- For a LTI system with a unit-sample response h_n , the relationship between the input process X_n and the output process Y_n can be expressed using the "convolution summation":

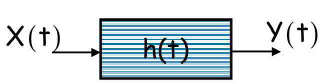
$$Y_n = \sum_{k=-\infty}^{\infty} h_k X_{n-k} = \sum_{k=-\infty}^{\infty} h_{n-k} X_k$$



Linear Time-Invariant Systems

- For a continuous-time LTI system with an impulse response h(t) , the relationship between the input process X(t) and the output process Y(t) can be expressed:

$$y(t)=\int_{-\infty}^{\infty} h(s)x(t-s)ds = \int_{-\infty}^{\infty} h(t-s)x(s)ds$$



Linear Time-Invariant Systems

- Many of the fundamental concepts of LTI systems are the same for discrete-time and continuous-time random processes

- Moving forward, we focus on developing the relationship between the input and output of LTI systems for the discrete-time case. All extensions to the continuous-time case are rather straightforward (please see the book for more details).

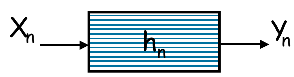
Linear Time-Invariant Systems

- **If the system response h_n has only (a+b+1) non-zero coefficients:**

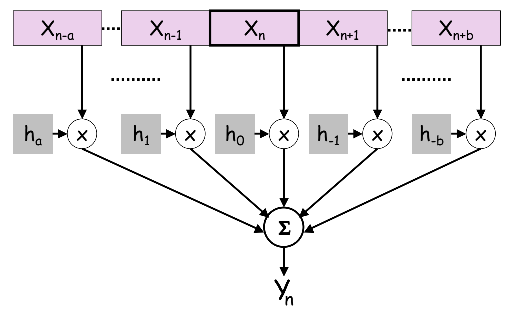
$$h_{-b}\ ,\ h_{-(b-1)}\ ,\ \ldots, h_{-1}\ ,\ h_0\ ,\ h_1\ ,\ \ldots,\ h_{a-1}\ ,\ h_a$$

then $$Y_n = \sum_{k=-\infty}^{\infty} h_k X_{n-k} = \sum_{k=-b}^{a} h_k X_{n-k}$$

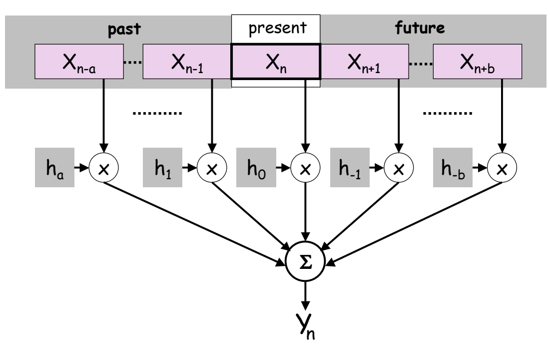
$$= h_a X_{n-a} + \cdots + h_1 X_{n-1} + h_0 X_n + h_{-1} X_{n+1} + \cdots + h_{-b} X_{n+b}$$



Linear Time-Invariant Systems



Linear Time-Invariant Systems



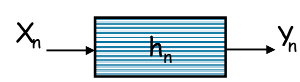
Linear Time-Invariant Systems

- **If h_n is zero for all samples $n < 0$:**

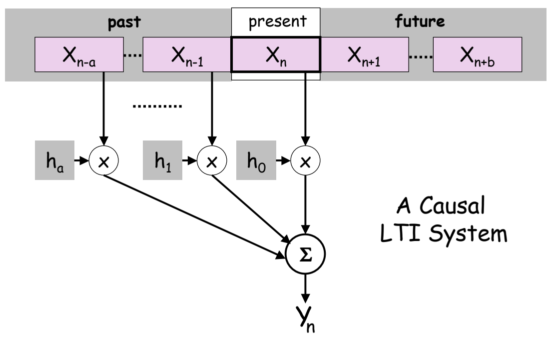
$$h_{-b}=0\ ,\ h_{-(b-1)}=0\ ,\ \ldots, h_{-1}=0$$

then the system is a "causal" LTI system

- **In this case, the system depends only on the "past" and "present" samples of the input process X_n**



Linear Time-Invariant Systems



Linear Time-Invariant Systems

- Therefore, for a causal LTI system, the output process Y_n does not depend on samples from the future:

$$Y_n = \sum_{k=-\infty}^{\infty} h_k X_{n-k} = \sum_{k=0}^{a} h_k X_{n-k}$$
$$= h_a X_{n-a} + \cdots + h_1 X_{n-1} + h_0 X_n$$

Linear Time-Invariant Systems

- Since convolution in the "time domain" is equivalent to multiplication in the "frequency domain" , it is useful to look at the frequency domain representation of LTI systems
- The "unit-sample response" h_n has a frequency-domain "Transfer Function" H(f):

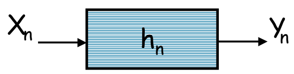
$$H(f) = \sum_{k=-\infty}^{\infty} h_k \, e^{-j2\pi fk}$$

Linear Time-Invariant Systems

- We recall that the "power spectral density" (psd) can be used to represent WSS processes in the frequency domain.
- Therefore, we focus here on LTI systems that take as an input a WSS process X_n
- In order to find the relationship between the psd $S_X(f)$ of the input X_n and the frequency-domain representation of the output Y_n , we need to see if Y_n is WSS or not.

Linear Time-Invariant Systems

- It can be shown that, if the input process X_n is WSS, then the output process Y_n is also WSS
- We assume here that the input X_n has been applied a "long time ago" (at $t = -\infty$)
- If the WSS input X_n is applied at some "recent" time (e.g., t=0), then we have to wait for a "long time" (at $t = \infty$) before the output Y_n becomes WSS

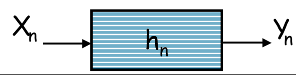


Linear Time-Invariant Systems

- Now let's look at the mean m_Y and autocorrelation function R_Y of the output process Y_n:

$$E[Y_n] = \sum_{k=-\infty}^{\infty} h_k E\left[X_{n-k}\right]$$

$$\boxed{m_Y = m_X \sum_{k=-\infty}^{\infty} h_k} \Rightarrow \boxed{m_Y = m_X H(0)}$$



Linear Time-Invariant Systems

- For a LTI system with a transfer function $H(f)$, if the input is a WSS process with a psd $S_X(f)$, then the output process is also a WSS process with a psd $S_Y(f)$:

$$\boxed{S_Y(f) = |H(f)|^2 S_X(f)}$$

This equation is applicable to both discrete-time and continuous-time WSS random processes

Example: Discrete-Time White Noise

- Let the input to a LTI system be a discrete-time white noise process X_n with a power-spectral-density:

$$S_X(f) = \sigma_X^2$$

Then the output process psd $S_Y(f)$:

$$\boxed{S_Y(f) = |H(f)|^2 S_X(f) = |H(f)|^2 \sigma_X^2}$$

where H(f) is the transfer function of the LTI system

Example: Discrete-Time White Noise

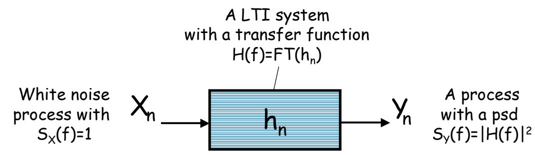
- Therefore, the white noise process X_n can be used to generate an output Y_n with any desired power spectral density $S_Y(f)$:

  In particular, if we let $S_X(f)=1$:

$$\boxed{S_Y(f) = |H(f)|^2 S_X(f) = |H(f)|^2}$$

Linear Time-Invariant Systems

- Therefore, by designing the appropriate filter with transfer function H(f) we can control the power spectral density of the output process:



Example: Moving Average Process

- For a causal LTI system with a white noise input X_n, the output process:

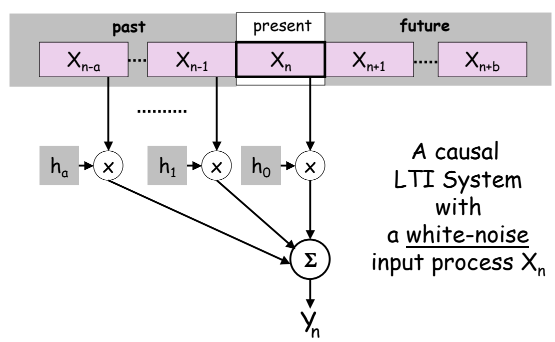
$$Y_n = \sum_{k=0}^{a} h_k X_{n-k}$$

  is knows as a <u>"Moving Average" (MA) process</u>

  In this case, the transfer function is:

$$H(f) = \sum_{k=0}^{a} h_k e^{-j2\pi fk}$$

Example: Moving Average Process



A causal LTI System with a <u>white-noise</u> input process X_n

Example: Moving Average Process

- **Let's consider a simple case of a MA process:**

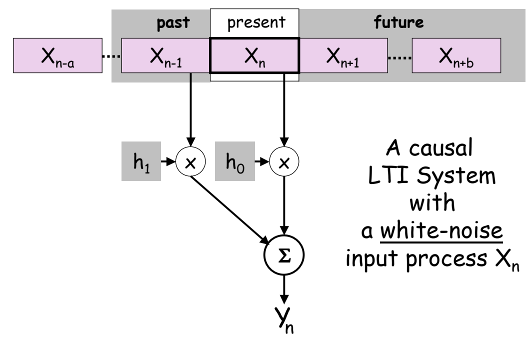
$$Y_n = h_0 X_n + h_1 X_{n-1}$$

where X_n is a white noise process with a psd:

$$S_X(f) = \sigma_X^2$$

Find the psd of the output process: $S_Y(f)$

Example: Moving Average Process



A causal LTI System with a white-noise input process X_n

Example: Moving Average Process

We can find $S_Y(f)$ using:

$$\boxed{S_Y(f) = |H(f)|^2 S_X(f)}$$

First, we need to find the transfer function $H(f)$.

Since: $Y_n = h_0 X_n + h_1 X_{n-1}$ **and** $H(f) = \sum_{k=0}^{1} h_k e^{-j2\pi fk}$

Then: $\boxed{H(f) = h_0 + h_1 e^{-j2\pi f}}$

Example: Moving Average Process

- **Therefore,**

$$|H(f)|^2 = H(f)\, H^*(f)$$
$$= \left(h_0 + h_1 e^{-j2\pi f}\right)\left(h_0 + h_1 e^{j2\pi f}\right)$$

$$|H(f)|^2 = h_0^2 + h_0 h_1 \left(e^{-j2\pi f} + e^{j2\pi f}\right) + h_1^2$$

$$\boxed{|H(f)|^2 = h_0^2 + h_1^2 + 2h_0 h_1 \cos(2\pi f)}$$

Example: Moving Average Process

- **Therefore,**

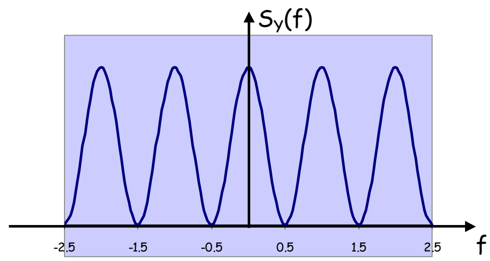
$$S_Y(f) = |H(f)|^2 S_X(f)$$

$$\boxed{S_Y(f) = \{h_0^2 + h_1^2 + 2h_0h_1\cos(2\pi f)\}\sigma_X^2}$$

For $h_0 = h_1 = 1/2$

$$S_Y(f) = \{1 + \cos(2\pi f)\}\frac{\sigma_X^2}{2}$$

Example: Moving Average Process



Example: Moving Average Process

- **Note that $S_Y(f)$ has all of the expected characteristics of a power-spectral-density functions:**

  It is symmetric (even): $S_Y(f) = S_Y(-f)$

  It is always non-negative: $S_Y(f) \geq 0$

  It is periodic with a period of 1: $S_Y(f+1) = S_Y(f)$
  (since the input process X_n is discrete-time)

Example: Moving Average Process

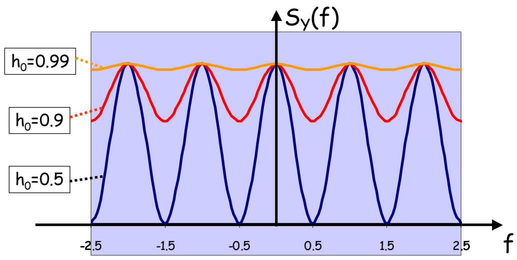
- **It is important to note that as we emphasize one of the samples (X_n or X_{n-1}), the output process Y_n will have a psd function that resembles the input psd $S_X(f)$:**

If $h_0 >> h_1$ (and let $h_0+h_1=1$), then:

$$\boxed{Y_n = h_0X_n + h_1X_{n-1} \simeq X_n} \quad \Rightarrow$$

$$S_Y(f) = \{h_0^2 + h_1^2 + 2h_0h_1\cos(2\pi f)\}\sigma_X^2 \simeq \sigma_X^2$$

Example: Moving Average Process



Example: Moving Average Process

- Finally, we can compute the autocorrelation function $R_Y(\tau)$ using:

$$R_Y(d) = \int_{-1/2}^{1/2} S_Y(f)\, e^{j2\pi fd}\, df$$

$$= \int_{-1/2}^{1/2} \left\{ \left(h_0^2 + h_1^2\right) + 2h_0h_1 \cos(2\pi f) \right\} \sigma_X^2 \, e^{j2\pi fd}\, df$$

$$\boxed{R_Y(d) = \left\{ \left(h_0^2 + h_1^2\right)\delta_d + h_0h_1\left(\delta_{d-1} + \delta_{d+1}\right) \right\} \sigma_X^2}$$

Gaussian LTI Systems

- In general, knowledge of the power spectral density $S_Y(f)$ or autocorrelation $R_Y(\tau)$ functions does not provide enough information regarding the probability density function of the output process $Y(t)$ of a LTI system.
- However, if the input process $X(t)$ is a Gaussian WSS process, then the output process $Y(t)$ is also a Gaussian WSS process.

Gaussian LTI Systems

- If the input process $X(t)$ is a Gaussian WSS process, then the mean m_Y and autocorrelation function $R_Y(\tau)$ can be used to determine all joint probability-density-functions of the output (Gaussian) process $Y(t)$
- For example, the "first order" probability density function $f_{Y(t)}(y)$ can be expressed in terms of m_y and $R_Y(\tau)$

Gaussian LTI Systems

- **Therefore, for a Gaussian WSS process** $Y(t)$**:**

$$\boxed{f_{Y(t)}(y) = \frac{1}{\sqrt{2\pi VAR[Y(t)]}} e^{-(y-m_y)^2/2VAR[Y(t)]}}$$

where: $\boxed{m_y = m_x H(0)}$

$$VAR\left[Y(t)\right] = E\left[(Y(t))^2\right] - (m_y)^2$$

$$\boxed{VAR\left[Y(t)\right] = R_Y(0) - (m_y)^2}$$

Appendix A

Derivation of the

Power-Spectral-Density (PSD) function

of the output of

a Linear-Time-Invariant (LTI) System

Linear Time-Invariant Systems

- **Now, let's evaluate the autocorrelation function** $R_Y(n+d,d)$**. We need to show that** R_Y **is only a function of (d) and not a function of the time index (n):**

$$R_Y(n+d,n) = E\left[Y_{n+d}Y_n\right]$$

$$= E\left[\left(\sum_{k_1=-\infty}^{\infty} h_{k_1} X_{n+d-k_1}\right)\left(\sum_{k_2=-\infty}^{\infty} h_{k_2} X_{n-k_2}\right)\right]$$

$$X_n \longrightarrow \boxed{h_n} \longrightarrow Y_n$$

Linear Time-Invariant Systems

- **Therefore:**

$$R_Y(n+d,n) = E\left[\left(\sum_{k_1=-\infty}^{\infty} h_{k_1} X_{n+d-k_1}\right)\left(\sum_{k_2=-\infty}^{\infty} h_{k_2} X_{n-k_2}\right)\right]$$

$$R_Y(n+d,n) = \sum_{k_2=-\infty}^{\infty} h_{k_2} \sum_{k_1=-\infty}^{\infty} h_{k_1} E\left[X_{n+d-k_1} X_{n-k_2}\right]$$

$$X_n \longrightarrow \boxed{h_n} \longrightarrow Y_n$$

Linear Time-Invariant Systems

- **Therefore:**

$$R_Y(n+d,n) = \sum_{k_2=-\infty}^{\infty} h_{k_2} \sum_{k_1=-\infty}^{\infty} h_{k_1} E\left[X_{n+d-k_1} X_{n-k_2}\right]$$

$$\boxed{R_Y(d) = \sum_{k_2=-\infty}^{\infty} h_{k_2} \left(\sum_{k_1=-\infty}^{\infty} h_{k_1} R_X(d+k_2-k_1) \right)}$$

$$X_n \longrightarrow \boxed{h_n} \longrightarrow Y_n$$

Linear Time-Invariant Systems

- **Consequently, the output process Y_n has a constant mean and an autocorrelation function $R_Y(d)$ which is only a function of the time difference (d). Therefore, Y_n is WSS.**
- **Since Y_n is WSS, then we are ready to evaluate its power-spectral-density $S_Y(f)$**

$$X_n \longrightarrow \boxed{h_n} \longrightarrow Y_n$$

Linear Time-Invariant Systems

- **By taking the Fourier Transform of:**

$$R_Y(d) = \sum_{k_2=-\infty}^{\infty} h_{k_2} \left(\sum_{k_1=-\infty}^{\infty} h_{k_1} R_X(d+k_2-k_1) \right)$$

$$S_Y(f) =$$

$$\sum_{d=-\infty}^{\infty} \left\{ \sum_{k_2=-\infty}^{\infty} h_{k_2} \left(\sum_{k_1=-\infty}^{\infty} h_{k_1} R_X(d+k_2-k_1) \right) \right\} e^{-j2\pi f d}$$

Linear Time-Invariant Systems

- **By using the following change of variables:**

$$k = (d + k_2 - k_1)$$

$$S_Y(f) =$$

$$\sum_{k=-\infty}^{\infty} \left\{ \sum_{k_2=-\infty}^{\infty} h_{k_2} \left(\sum_{k_1=-\infty}^{\infty} h_{k_1} R_X(k) \right) \right\} e^{-j2\pi f(k-k_2+k_1)}$$

$$= \left(\sum_{k=-\infty}^{\infty} R_X(k) e^{-j2\pi f k} \right) \left(\sum_{k_2=-\infty}^{\infty} h_{k_2} e^{j2\pi f k_2} \right) \left(\sum_{k_1=-\infty}^{\infty} h_{k_1} e^{-j2\pi f k_1} \right)$$

Linear Time-Invariant Systems

- **This leads to:**

$$S_Y(f) =$$

$$\left(\sum_{k=-\infty}^{\infty} R_X(k)e^{-j2\pi fk}\right)\left(\sum_{k_2=-\infty}^{\infty} h_{k_2} e^{j2\pi fk_2}\right)\left(\sum_{k_1=-\infty}^{\infty} h_{k_1} e^{-j2\pi fk_1}\right)$$

$$S_Y(f) = S_X(f)\, H^*(f)\, H(f)$$

$$S_Y(f) = |H(f)|^2 S_X(f)$$

Linear Time-Invariant Systems

- **Therefore, for a LTI system with a transfer function H(f), if the input is a WSS process with a psd $S_X(f)$, then the output process is also a WSS process with a psd $S_Y(f)$:**

$$\boxed{S_Y(f) = |H(f)|^2 S_X(f)}$$

This equation is applicable to both discrete-time and continuous-time WSS random processes

Appendix B

Autoregressive Processes

and

Moving Average Autoregressive Processes

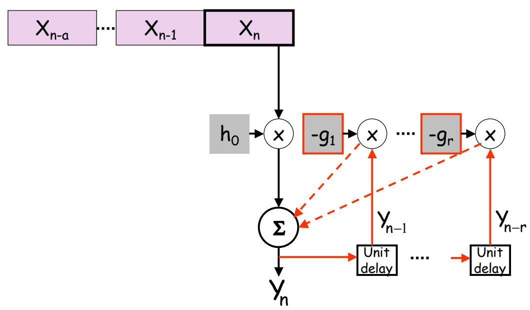
Example: Autoregressive Process

- **If the output of the LTI system at time (n) depends on: the past (r) output samples <u>and</u> the current white-noise input sample (X_n), then the process Y_n:**

$$Y_n = \sum_{m=1}^{r} (-g_m) Y_{n-m} + h_0 X_n$$

is knows as <u>"Autoregressive" (AR) process</u>

Example: Autoregressive Process



Example: Autoregressive Process

- Therefore,

$$Y_n + \sum_{m=1}^{r} g_m Y_{n-m} = h_0 X_n$$

- This leads to the following transfer function for the AR process:

$$\boxed{H(f) = \frac{h_0}{1 + \sum_{m=1}^{r} g_m e^{-j2\pi fm}}}$$

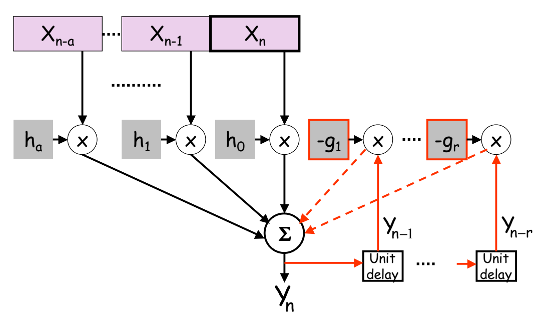
Autoregressive Moving Average Process

- If the output Y_n of the LTI system at time (n) depends on: the past (r) output samples <u>and</u> (a+1) white-noise input samples, then the process Y_n:

$$Y_n = \sum_{m=1}^{r} (-g_m) Y_{n-m} + \sum_{k=0}^{a} h_k X_{n-k}$$

is knows as <u>**"Autoregressive Moving Average" (ARMA) process**</u>

Autoregressive Moving Average Process



Autoregressive Moving Average Process

- **Therefore,**

$$Y_n + \sum_{m=1}^{r} g_m Y_{n-m} = \sum_{k=0}^{a} h_k X_{n-k}$$

- **This leads to the following transfer function for the ARMA process:**

$$\boxed{H(f) = \frac{\sum_{k=0}^{a} h_k e^{-j2\pi fk}}{1 + \sum_{m=1}^{r} g_m e^{-j2\pi fm}}}$$