ECE 863
Analysis of Stochastic Systems
Part IV.1: Power Spectral Density of Random Processes

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ECE 863

- **Reading Assignment**

  Section 7.1 – Power Spectral Density

  Section 7.2 – Linear Systems

  Section 7.4 – Optimum Linear Systems

ECE 863

- **Exam 3 is on:**

  Wednesday, December 5

  Chapter 7 reading assignment and related lecture notes are the last material included in Exam 3

Power Spectral Density

- **The "power spectral density" $S_X(f)$ measures the average energy (or power) of a random process X(t) at the frequency (f)**
- **Therefore, if the random process X(t) "changes slowly", then X(t) must have most of its power concentrated at low frequencies, (and vice versa).**

Power Spectral Density

- A slowly (or rapidly) changing random process implies a highly-correlated (or highly-uncorrelated) process across the time axis.
- For a random process we need to measure the "average change"
- The autocorrelation R_X provides a measure of the "average change" for a random process

Power Spectral Density

- Therefore, we expect the power spectral density $S_X(f)$ to be related to the autocorrelation function $R_X(\tau)$, where X(t) is a WSS process.
- We can derive the power spectral density S_X from R_X by using the Fourier Transform

Power Spectral Density

- The power spectral density $S_X(f)$ of the WSS process X_n is the discrete-time Fourier Transform of the autocorrelation $R_X(d)$:

$$\boxed{S_X(f) = \sum_{d=-\infty}^{\infty} R_X(d) e^{-j2\pi fd}}$$

Power Spectral Density

- For any discrete-time WSS random process X_n, the power spectral density $S_X(f)$ is a periodic function of (f) with a period of one:

$$S_X(f+1) = \sum_{d=-\infty}^{\infty} R_X(d) e^{-j2\pi(f+1)d}$$

$$S_X(f+1) = \sum_{d=-\infty}^{\infty} R_X(d) e^{-j2\pi fd} e^{-j2\pi d}$$

$$\boxed{S_X(f+1) = \sum_{d=-\infty}^{\infty} R_X(d) e^{-j2\pi fd} = S_X(f)}$$

Power Spectral Density

- The power spectral density $S_X(f)$ is a "non-negative" function (it is a measure of power):

$$\boxed{S_X(f) \geq 0 \qquad \forall\ f}$$

Power Spectral Density

- The autocorrelation $R_X(d)$ can be computed as the inverse Fourier Transform of the power spectral density function $S_X(f)$
- Since, for discrete-time processes, $S_X(f)$ is a periodic function of (f). Therefore,

$$\boxed{R_X(d) = \int_{-1/2}^{1/2} S_X(f)\, e^{j2\pi f d}\, df}$$

Power Spectral Density

- For a continuous-time WSS process X(t), the power spectral density $S_X(f)$ is the Fourier Transform of the autocorrelation function $R_X(\tau)$:

$$\boxed{S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau}$$

This is known as the Wiener-Khinchin Theorem

Power Spectral Density

- The autocorrelation function $R_X(\tau)$ for a continuous-time WSS process X(t) is the inverse Fourier Transform of the power spectral density $S_X(f)$:

$$\boxed{R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f\tau} df}$$

Power Spectral Density

- **Since $S_X(f)$ is the power spectral density, then integrating $S_X(f)$ over all possible frequencies gives the total "average power":**

$$\text{Total "Average Power"} = \int_{-\infty}^{\infty} S_X(f)\,df$$

$$E\left[(X(t))^2\right] = \int_{-\infty}^{\infty} S_X(f)\,df$$

$$R(\tau = 0) = \int_{-\infty}^{\infty} S_X(f)\,df$$

Power Spectral Density

- **If X(t) is a real (i.e. not a complex) process, then $R_X(\tau)$ is a real and symmetric function:**

$$R_X(\tau) = R_X(-\tau)$$

- **Therefore, the power spectral density $S_X(f)$ is also real and symmetric:**

$$S_X(f) = S_X(-f)$$

Power Spectral Density

- **For a real process X(t), the expression for $S_X(f)$ can be simplified by taking advantage of the symmetry of the autocorrelation function $R_X(\tau)$:**

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\cos(2\pi f\tau)\,d\tau$$
$$-j\int_{-\infty}^{\infty} R_X(\tau)\sin(2\pi f\tau)\,d\tau$$

$$\boxed{S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\cos(2\pi f\tau)\,d\tau}$$

Example: White Noise

- **The "white noise" process X'(t) is generated by taking the derivative of a Wiener process** X(t):

$$X'(t) = \frac{dX(t)}{dt}$$

We need to find the power spectral density $S_{X'}(f)$ of X'(t)

Example: White Noise

- Finding the power spectral density of the "white noise" process X'(t) requires computing its autocorrelation function $R_{X'}(t_1,t_2)$

- Since X'(t) is the derivative of X(t), we need to know how to take derivatives of random processes

Example: White Noise

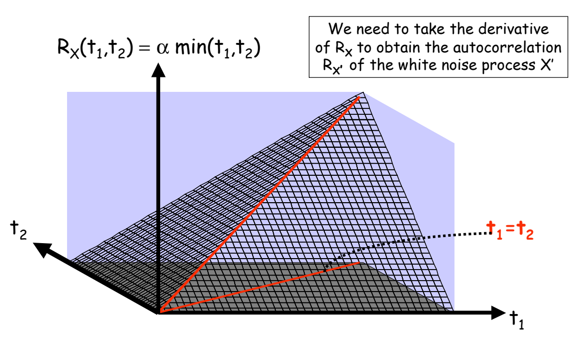
- Under certain conditions, the autocorrelation function $R_{X'}(\tau)$ of $X'(t)=dX(t)/dt$ can be obtained by taking the partial derivatives of the autocorrleation $R_X(\tau)$ of X(t):

If $$\boxed{X'(t) = \frac{dX(t)}{dt}}$$

$$\Rightarrow \boxed{R_{X'}(t_1,t_2) = \frac{\partial^2}{\partial t_1 \partial t_2} R_X(t_1,t_2)}$$

(See Appendix B for more details.)

Example: White Noise



Example: White Noise

- Recall that the autocorrelation function for the Wiener process:

$$R_X(t_1,t_2) = \alpha \min(t_1,t_2)$$

For the Wiener process, the partial derivative of $R_X(t_1,t_2)$:
$$R_{X'}(t_1,t_2) = \frac{\partial^2}{\partial t_1 \partial t_2} R_X(t_1,t_2)$$

is zero everywhere except when $t_1=t_2$

Example: White Noise

- This leads to the following expression for the autocorrelation $R_{X'}(t_1,t_2)$ of the "white noise" process:

$$R_{X'}(t_1,t_2) = \alpha\ \delta(t_1 - t_2)$$

$$\boxed{R_{X'}(\tau) = \alpha\ \delta(\tau)}$$

Example: White Noise

- Therefore, the white noise process is a WSS process
- Hence, now, we can compute the power spectral density $S_{X'}(f)$:

$$S_{X'}(f) = \int_{-\infty}^{\infty} R_{X'}(\tau)\cos(2\pi f\tau)\,d\tau$$

$$S_{X'}(f) = \int_{-\infty}^{\infty} \alpha\ \delta(\tau)\cos(2\pi f\tau)\,d\tau$$

$$\boxed{S_{X'}(f) = \alpha \qquad \forall\ f}$$

Example: White Noise

- Therefore, the "white noise" process X'(t) has a "flat" power spectral density that covers all frequencies.
- Consequently, the "white noise" process has <u>infinite power</u>!

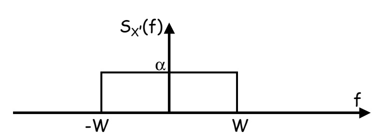
$$E\left[(X'(t))^2\right] = \int_{-\infty}^{\infty} S_{X'}(f)\,df$$

$$\boxed{E\left[(X'(t))^2\right] = R(\tau = 0) = \alpha\ \delta(0) = \alpha \int_{-\infty}^{\infty} df = \infty}$$

Example: Bandlimited White Noise

- A "bandlimited white noise" process X'(t) has a "flat" power spectral density over a certain range of frequencies: [-W,W]

$$\boxed{S_{X'}(f) = \alpha \qquad \forall\ f \in [-W,W]}$$

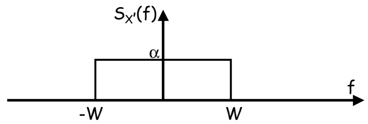


Example: Bandlimited White Noise

- The "bandlimited white noise" process X'(t) has finite power:

$$E\left[(X'(t))^2\right] = \int_{-W}^{W} S_{X'}(f)\,df$$

$$E\left[(X'(t))^2\right] = \alpha \int_{-W}^{W} df \qquad \boxed{E\left[(X'(t))^2\right] = 2W\alpha}$$



Example: Bandlimited White Noise

- The autocorrelation function $R_{X'}(\tau)$ of the bandlimited white noise process can be computed using the inverse Fourier Transform:

$$R_{X'}(\tau) = \alpha \int_{-W}^{W} e^{j2\pi f\tau} df = \alpha \int_{-W}^{W} \cos(2\pi f\tau)\, df$$

$$\boxed{R_{X'}(\tau) = \frac{(2\alpha)\sin(2\pi W\tau)}{2\pi\tau}}$$

Example: Bandlimited White Noise

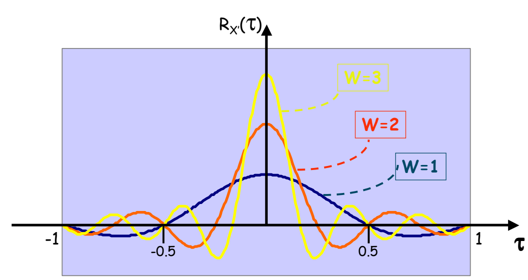
- The autocorrelation function $R_{X'}(\tau)$:

$$R_{X'}(\tau) = \frac{(2\alpha)\sin(2\pi W\tau)}{2\pi\tau}$$

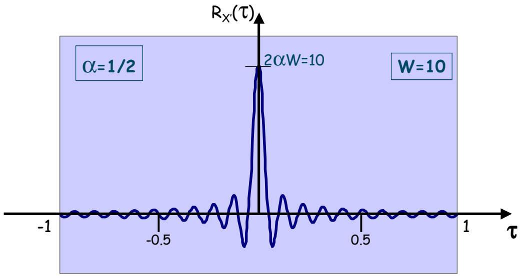
is zero at periodic values of τ:

$$R_{X'}(\tau) = 0 \qquad \forall\ \tau = \pm\frac{k}{2W},\quad k = 1,2,3,....$$

Example: Bandlimited White Noise



Example: Bandlimited White Noise



Example: Discrete-time White Noise

- Let X_n be a zero-mean discrete-time process with the following autocorrelation function:

$$\boxed{R_X(k) = \sigma_X^2 \ \delta_k} \qquad \delta_k = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

This is a "discrete time" white-noise process

Find the power spectral density $S_X(f)$

Example: Discrete-time White Noise

- First, we note that X_n is a WSS process. Since:

$$\boxed{m_X(k) = \text{constant} = 0}$$

and $\boxed{R_X(k) = \sigma_X^2 \ \delta_k}$

The auto-correlation function $R_X(k)$ is a function of the time difference (k) only

Example: Discrete-time White Noise

- Therefore, by using:

$$S_X(f) = \sum_{k=-\infty}^{\infty} R_X(k) e^{-j2\pi fk}$$

$$S_X(f) = \sum_{k=-\infty}^{\infty} \left(\sigma_X^2 \ \delta_k\right) e^{-j2\pi fk}$$

$$\boxed{S_X(f) = \sigma_X^2} \qquad \text{For all} \qquad -\frac{1}{2} < f < \frac{1}{2}$$

Cross-Power Spectral Density

- **Two processes** X(t) **and** Y(t) **have a**

  "cross-power spectral density" $S_{XY}(f)$

  if X(t) **and** Y(t) **are**

  jointly WSS processes

Cross-Power Spectral Density

- **X(t) and Y(t) are jointly WSS processes when the following conditions are satisfied:**

1) $R_X(t_1,t_2) = R_X(t_1-t_2) = R_X(\tau)$ i.e. X(t) is WSS

2) $R_Y(t_1,t_2) = R_Y(t_1-t_2) = R_Y(\tau)$ i.e. Y(t) is WSS

3) $R_{XY}(t_1,t_2) = R_{XY}(t_1-t_2) = R_{XY}(\tau)$

Cross-Power Spectral Density

- **If X(t) and Y(t) are jointly WSS processes, then:**

$$R_{XY}(\tau) = E[X(t+\tau)Y(t)]$$

$$R_{YX}(\tau) = E[Y(t+\tau)X(t)] = E[X(t)Y(t+\tau)] = R_{XY}(-\tau)$$

$$\boxed{R_{YX}(\tau) = R_{XY}(-\tau)}$$

When computing cross-functions for wide sense stationary processes, we have to use a consistent definition for τ:

$\tau = t_1 - t_2$ OR $\tau = t_2 - t_1$

Here, and in the book, we use the convention: $\tau = t_1 - t_2$

Cross-Power Spectral Density

- **If** X(t) **and** Y(t) **are continuous-time jointly WSS processes, then the cross-power spectral densities** $S_{XY}(f)$ **and** $S_{YX}(f)$ **are:**

$$\boxed{S_{XY}(f) = \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-j2\pi f\tau}d\tau}$$

$$\boxed{S_{YX}(f) = \int_{-\infty}^{\infty} R_{YX}(\tau)e^{-j2\pi f\tau}d\tau}$$

Cross-Power Spectral Density

- Since for jointly WSS processes (discrete- or continuous-time):

$$\boxed{R_{YX}(\tau) = R_{XY}(-\tau)}$$

then

$$\boxed{S_{YX}(f) = S^{*}_{XY}(f)}$$

where $S^{*}_{XY}(f)$ is the complex conjugate of $S_{XY}(f)$

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Cross-Power Spectral Density

- Similarly, if X_n and Y_n are discrete-time jointly WSS processes, then the cross-power spectral densities $S_{XY}(f)$ and $S_{YX}(f)$ are:

$$\boxed{S_{XY}(f) = \sum_{d=-\infty}^{\infty} R_{XY}(d)e^{-j2\pi fd}} \qquad R_{XY}(d) = E\left[X_{n+d}Y_n\right]$$

$$\boxed{S_{YX}(f) = \sum_{d=-\infty}^{\infty} R_{YX}(d)e^{-j2\pi fd}} \qquad R_{YX}(d) = E\left[Y_{n+d}X_n\right]$$

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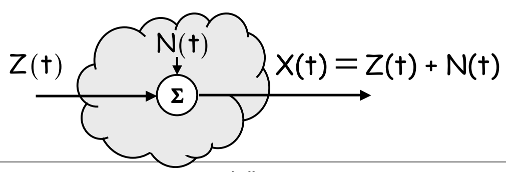
Example: Additive Noise

- A signal $Z(t)$ is corrupted by an additive noise process $N(t)$.
- $Z(t)$ and $N(t)$ are jointly WSS processes with PSD functions $S_Z(f)$, $S_N(f)$, and $S_{ZN}(f)$
- Derive an expression for the PSD $S_X(f)$ of the received process $X(t)$ and the cross-PSD function $S_{ZX}(f)$



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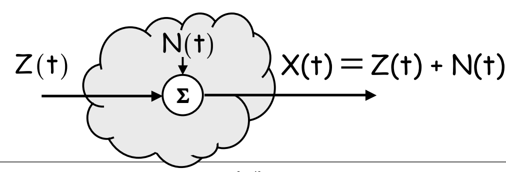
Example: Additive Noise

- First, we evaluate the PSD function $S_X(f)$
- We start by using the definition of the autocorrelation function $R_X(\tau)$
- Then, we take the Fourier Transform of $R_X(\tau)$ to express $S_X(f)$



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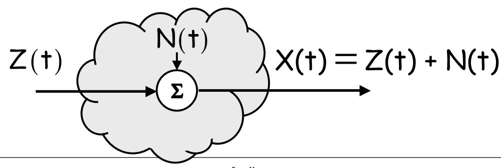
Example: Additive Noise

- **The autocorrelation function** $R_X(\tau)$

$$\boxed{R_X(\tau) = E\left[X(t+\tau)X(t)\right]}$$

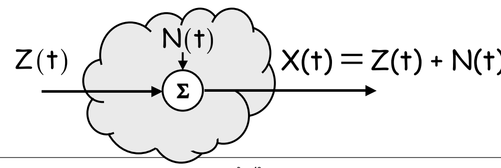
$$\boxed{R_X(\tau) = E\left[\left(Z(t+\tau)+N(t+\tau)\right)\left(Z(t)+N(t)\right)\right]}$$



Example: Additive Noise

$$\boxed{R_X(\tau) = E\left[\left(Z(t+\tau)+N(t+\tau)\right)\left(Z(t)+N(t)\right)\right]}$$

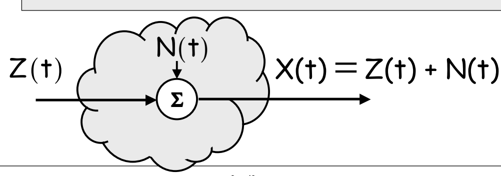
$$\boxed{\begin{aligned} R_X(\tau) = E\big[\ & Z(t+\tau)Z(t)+N(t+\tau)Z(t) \\ & +Z(t+\tau)N(t)+N(t+\tau)Z(t)\ \big] \end{aligned}}$$



Example: Additive Noise

$$\boxed{\begin{aligned} R_X(\tau) = E\big[\ & Z(t+\tau)Z(t)+N(t+\tau)Z(t) \\ & +Z(t+\tau)N(t)+N(t+\tau)N(t)\ \big] \end{aligned}}$$

$$\Rightarrow \boxed{R_X(\tau) = R_Z(\tau)+R_{NZ}(\tau)+R_{ZN}(\tau)+R_N(\tau)}$$

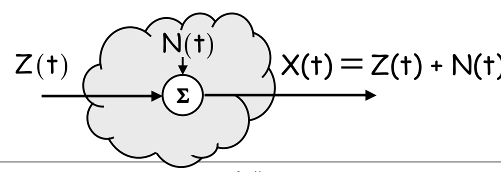


Example: Additive Noise

- **Now, by taking the Fourier transform of** $R_X(\tau)$:

$$\boxed{R_X(\tau) = R_Z(\tau)+R_{NZ}(\tau)+R_{ZN}(\tau)+R_N(\tau)}$$

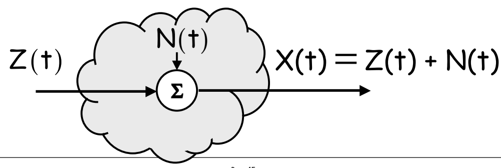
$$\Rightarrow \boxed{S_X(f) = S_Z(f)+S_{NZ}(f)+S_{ZN}(f)+S_N(f)}$$



Example: Independent Additive Noise

- **<u>IF Z(t) and N(t) are independent</u>, then:**

$$R_{NZ}(\tau) = E\left[N(t+\tau)Z(t)\right]$$
$$= E\left[N(t+\tau)\right]E\left[Z(t)\right]$$

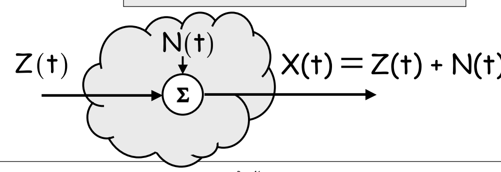


Example: Independent Additive Noise

- **Since** Z(t) **and** N(t) **are WSS processes, then they have constant means** m_Z **and** m_N **:**

$$R_{NZ}(\tau) = m_N\, m_Z$$ **Similarly,** $$R_{ZN}(\tau) = m_Z\, m_N$$

Therefore, $$R_{NZ}(\tau) + R_{ZN}(\tau) = 2\, m_Z\, m_N$$

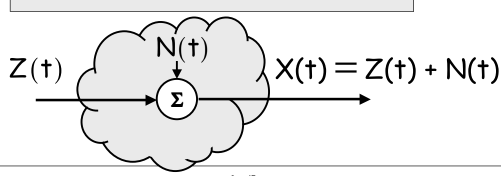


Example: Independent Additive Noise

- **Therefore, when** Z(t) **and** N(t) **are independent, then:**

$$R_X(\tau) = R_Z(\tau) + R_{NZ}(\tau) + R_{ZN}(\tau) + R_N(\tau)$$

$$\Rightarrow R_X(\tau) = R_Z(\tau) + 2m_Z\, m_N + R_N(\tau)$$

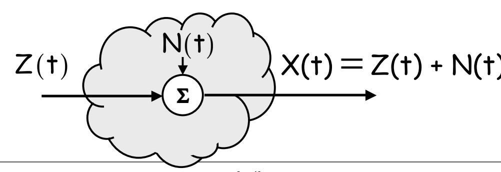


Example: Independent Additive Noise

- **Moreover, IF** N(t) **is a <u>zero-mean process</u>:**

$$R_X(\tau) = R_Z(\tau) + 2m_Z\, m_N + R_N(\tau)$$

$$\Rightarrow \boxed{R_X(\tau) = R_Z(\tau) + R_N(\tau)}$$



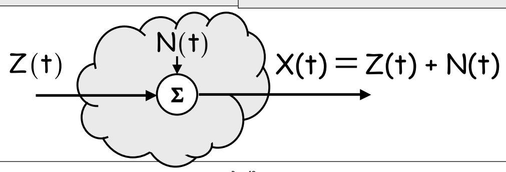
Example: Additive Noise

- **In summary:**

$$R_X(\tau) = R_Z(\tau) + R_{NZ}(\tau) + R_{ZN}(\tau) + R_N(\tau)$$

$$R_X(\tau) = R_Z(\tau) + 2m_Z m_N + R_N(\tau)$$ For independent noise

$$R_X(\tau) = R_Z(\tau) + R_N(\tau)$$ For independent & zero-mean noise



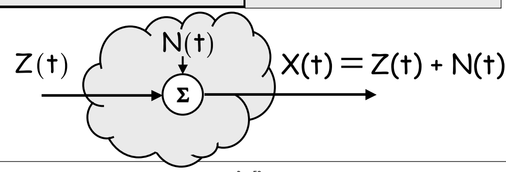
Example: Additive Noise

- **Consequently:**

$$S_X(f) = S_Z(f) + S_{NZ}(f) + S_{ZN}(f) + S_N(f)$$

$$S_X(f) = S_Z(f) + 2m_Z m_N \delta(f) + S_N(f)$$ independent noise

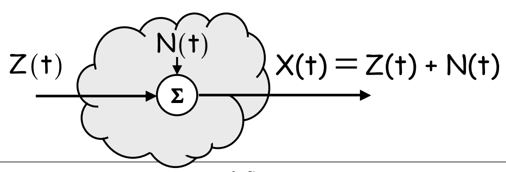
$$\boxed{S_X(f) = S_Z(f) + S_N(f)}$$ independent & zero-mean noise



Example: Additive Noise

- **Similarly, we can use the definition of the cross-correlation function $R_{ZX}(\tau)$ to derive the cross-PSD function $S_{ZX}(f)$**

$$R_{ZX}(\tau) = E\left[Z(t+\tau)X(t)\right]$$



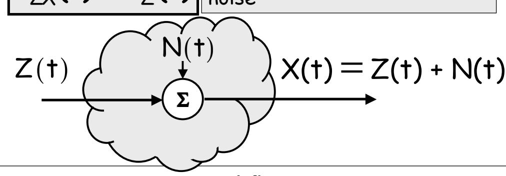
Example: Additive Noise

- **As an exercise, show the following :**

$$R_{ZX}(\tau) = R_Z(\tau) + R_{ZN}(\tau)$$

$$R_{ZX}(\tau) = R_Z(\tau) + m_Z m_N$$ For independent noise

$$\boxed{R_{ZX}(\tau) = R_Z(\tau)}$$ For independent & zero-mean noise



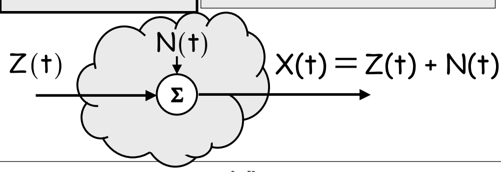
Example: Additive Noise

- **Therefore:**

$$\boxed{S_{ZX}(f) = S_Z(f) + S_{ZN}(f)}$$

$$\boxed{S_{ZX}(f) = S_Z(f) + m_Z\, m_N\, \delta(f)}$$ For independent noise

$$\boxed{S_{ZX}(f) = S_Z(f)}$$ For independent & zero-mean noise



Appendix A

Derivation of the

Power Spectral Density

Power Spectral Density

- **We start by looking at the "periodogram" of a discrete-time WSS random process X_n**
- **If X_0 , X_1 , ….X_{k-1} are k samples from the WSS process X_n , then we can take the discrete-time Fourier Transform of these samples:**

$$\boxed{\tilde{x}_k(f) = \sum_{m=0}^{k-1} X_m e^{-j2\pi f m}}$$

Power Spectral Density

- **To estimate the average power that the random process X_n has at frequency (f), we use the "periodogram estimate" :**

$$\boxed{\tilde{p}_k(f) = \frac{1}{k}\left|\tilde{x}_k(f)\right|^2}$$

where

$$\tilde{x}_k(f) = \sum_{m=0}^{k-1} X_m e^{-j2\pi f m}$$

Power Spectral Density

- **Taking the expected value of the periodogram:**

$$E\left[\tilde{p}_k(f)\right] = E\left[\frac{1}{k}\left|\tilde{x}_k(f)\right|^2\right]$$

$$E\left[\tilde{p}_k(f)\right] = \frac{1}{k}E\left[\tilde{x}_k(f)\tilde{x}_k^*(f)\right]$$

Power Spectral Density

$$E\left[\tilde{p}_k(f)\right] = \frac{1}{k}E\left[\tilde{x}_k(f)\tilde{x}_k^*(f)\right]$$

$$= \frac{1}{k}E\left[\sum_{m=0}^{k-1} X_m e^{-j2\pi fm} \sum_{m'=0}^{k-1} X_{m'} e^{j2\pi fm'}\right]$$

$$= \frac{1}{k}\sum_{m=0}^{k-1}\sum_{m'=0}^{k-1} E\left[X_m X_{m'}\right] e^{-j2\pi f(m-m')}$$

Power Spectral Density

- **Using $E[X_m X_{m'}]=R_X[m,m']$**

$$E\left[\tilde{p}_k(f)\right] = \frac{1}{k}\sum_{m=0}^{k-1}\sum_{m'=0}^{k-1} E\left[X_m X_{m'}\right] e^{-j2\pi f(m-m')}$$

$$= \frac{1}{k}\sum_{m=0}^{k-1}\sum_{m'=0}^{k-1} R_X(m,m') e^{-j2\pi f(m-m')}$$

Power Spectral Density

- **Since X_n is a WSS process, then: $R_X[m,m']=R_X[m-m']$:**

$$E\left[\tilde{p}_k(f)\right] = \frac{1}{k}\sum_{m=0}^{k-1}\sum_{m'=0}^{k-1} R_X(m-m') e^{-j2\pi f(m-m')}$$

Power Spectral Density

- If we let d=m-m′, then the double summation:

$$E\left[\tilde{p}_k(f)\right] = \frac{1}{k}\sum_{m=0}^{k-1}\sum_{m'=0}^{k-1} R_X(m-m')e^{-j2\pi f(m-m')}$$

reduces to the following single summation

$$E\left[\tilde{p}_k(f)\right] = \frac{1}{k}\sum_{d=-(k-1)}^{k-1}\left\{k-|d|\right\}R_X(d)e^{-j2\pi fd}$$

Power Spectral Density

- Therefore:

$$E\left[\tilde{p}_k(f)\right] = \frac{1}{k}\sum_{d=-(k-1)}^{k-1}\left\{k-|d|\right\}R_X(d)e^{-j2\pi fd}$$

$$\boxed{E\left[\tilde{p}_k(f)\right] = \sum_{d=-(k-1)}^{k-1}\left\{1-\frac{|d|}{k}\right\}R_X(d)e^{-j2\pi fd}}$$

Power Spectral Density

- Note that we are only considering k samples of the random process X_n
- Considering all samples of the process X_n leads to the power spectral density $S_X(f)$:

$$\boxed{\begin{aligned} S_X(f) &= \lim_{k\to\infty} E\left[\tilde{p}_k(f)\right] \\ &= \lim_{k\to\infty}\sum_{d=-(k-1)}^{k-1}\left\{1-\frac{|d|}{k}\right\}R_X(d)e^{-j2\pi fd}\end{aligned}}$$

Power Spectral Density

- Therefore, the power spectral density $S_X(f)$ of the WSS process X_n is the discrete-time Fourier Transform of the autocorrelation $R_X(d)$:

$$\boxed{S_X(f) = \sum_{d=-\infty}^{\infty} R_X(d)e^{-j2\pi fd}}$$

Power Spectral Density

- Moreover, the power spectral density $S_X(f)$ is a "non-negative" function since it results from the "periodogram estimate":

$$S_X(f) = \lim_{k\to\infty} E\left[\tilde{p}_k(f)\right] = \lim_{k\to\infty} E\left[\frac{1}{k}\left|\tilde{x}_k(f)\right|^2\right]$$

$$\boxed{S_X(f) \geq 0 \qquad \forall\ f}$$

Appendix B

Derivatives of

Random Processes

Derivative of Random Processes

- The derivative of any random process X(t):

$$X'(t) = \frac{dX(t)}{dt}$$

$$X'(t) = \lim_{\varepsilon\to 0} \frac{X(t+\varepsilon) - X(t)}{\varepsilon}$$

may exist for some sample functions and may not exist for other sample functions.

Derivative of Random Processes

- Therefore, it is customary to define the "Mean Square Derivative":

X'(t) is the "mean square derivative" of X(t) if the following is satisfied:

$$\lim_{\varepsilon\to 0} E\left[\left(\frac{X(t+\varepsilon) - X(t)}{\varepsilon} - X'(t)\right)^2\right] = 0$$

Derivative of Random Processes

- **It can be shown that if the "mean square derivative" exists, then the autocorrelation function of X'(t) can be obtained by taking the partial derivative of the autocorrleation of X(t):**

$$\boxed{R_{X'}(t_1, t_2) = \frac{\partial^2}{\partial t_1 \partial t_2} R_X(t_1, t_2)}$$

(This is a general result that is true
for any process which has a mean-square-derivative;
See section 6.6 for more details.)