Programming Homework of Probability Spring 2024

Due 1pm, May 21, 2024.

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Let $X_t \sim \mathcal{N}(0,1)$ and $W_t \sim \mathcal{N}(0,\sigma_w^2)$ be two Gaussian random variables, $\forall t \in \mathbb{N} = \{1,2,3,...\}$. In addition, X_t and W_t are statistically independent, $\forall t \in \mathbb{N}$. Furthermore, $X_1, X_2, X_3, ...$ are independent and identically distributed (IID) random variables and $W_1, W_2, W_3, ...$ are IID random variables. Let $Z_t = X_t + W_t$, $\forall t \in \mathbb{N}$.

Write a computer simulation program that works as follows. At time/iteration 0, it obtains the value of a, where

$$a = \frac{\mathbb{E}[X_t \cdot Z_t]}{\mathbb{E}[(Z_t)^2]}. \tag{1}$$

In time/iteration $t \in \mathbb{N}$, it generates a sample of (X_t, W_t, Z_t). Given Z_t, it use $\hat{X}_t$ to estimate X_t. Specifically,

$$\hat{X}_t = a \cdot Z_t. \tag{2}$$

For each $n \in \mathbb{N}$, define E_n as follows.

$$E_n = \frac{1}{n}\sum_{t=1}^{n}(\hat{X}_t - X_t)^2. \tag{3}$$

Note that E_n is the mean squared error of estimation based on n samples.

Let $S = \{1, 5, 1\times 10^1, 5\times 10^1, 1\times 10^2, 5\times 10^2, 1\times 10^3, 5\times 10^3, 1\times 10^4, 5\times 10^4, 1\times 10^5, 5\times 10^5, 1\times 10^6\}$.

(1) In Figure 1, show E_n, $\forall n \in S$, when $\sigma_w^2 \in \{10^{-3}, 10^{-2}, 10^{-1}\}$. Use logarithmic scale in the x-axis. For example, use semilogx() in Matlab.

(2) Let λ be a positive real number. Consider the case in which W_t has the following probability density function.

$$f_{W_t}(w) = \begin{cases} \frac{\lambda}{2}e^{-\lambda x}, \forall x \geq 0, \\ \frac{\lambda}{2}e^{\lambda x}, \forall x < 0. \end{cases} \tag{4}$$

In Figure 2, show E_n, $\forall n \in S$, when $\sigma_w^2 \in \{10^{-3}, 10^{-2}, 10^{-1}\}$.

(3) In Figure 3, show $E_n^{'} = \frac{1}{n}\sum_{t=1}^{n}(\hat{X}_t - X_t)^4$ for the two probability density functions of W_t, when $\sigma_w^2 = 0.16$.

(4) Write a short report that includes your figures and codes. In addition, you should explain how you obtain the answers in English. Your report should contain at most five A4 pages. Name your Matlab main program NYCU-YourStudentId.m (or your Python main program NYCU-YourStudentId.py). Name your report NYCU-YourStudentId.pdf. Name your submitted file NYCU-YourStudentId.zip. Submit your files to the new E3 website.