Lebesgue Integration Cheatsheet

Based on *Intégrale de Lebesgue sur* $\mathbb{R}$ and *sur* $\mathbb{R}^d$

Integration on $\mathbb{R}$

1. Simple Functions (Fonctions Étagées)

A function $f : X \to [0, +\infty]$ is **simple positive** ($\mathcal{E}^+(X)$) if it takes a finite number of values $\{c_1, \dots, c_n\}$ on measurable sets A_j:

$$f = \sum_{j=1}^{n} c_j \mathbf{1}_{A_j}$$

Integral Definition:

$$\int_X f \, d\lambda = \sum_{c \in f(X)} c\lambda(f^{-1}(\{c\}))$$

With convention $0 \times \infty = 0$.

2. Measurable Positive Functions

For a measurable $f : X \to [0, +\infty]$:

$$\int_X f \, d\lambda = \sup \left\{ \int_X \varphi \, d\lambda : \varphi \in \mathcal{E}^+(X), \varphi \leq f \right\}$$

Approximation: Every measurable $f \geq 0$ is the limit of an increasing sequence of simple functions $\varphi_n \nearrow f$.

3. Integrable Functions ($\mathcal{L}^1$)

A measurable function $f : X \to \mathbb{R}$ is **integrable** (denoted $f \in \mathcal{L}^1(X)$) if:

$$\int_X |f| \, d\lambda < +\infty$$

Definition of Integral:

$$\int_X f \, d\lambda = \int_X f^+ \, d\lambda - \int_X f^- \, d\lambda$$

where $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$.

Key Properties

- **Linearity:** $\int (\alpha f + \beta g) = \alpha \int f + \beta \int g$.
- **Monotonicity:** $f \leq g \implies \int f \leq \int g$.
- **Triangle Inequality:** $|\int f| \leq \int |f|$.
- **Null Sets:** $\int_N f = 0$ if $\lambda(N) = 0$.
- **Vanishing Integral:** For $f \geq 0$, $\int f = 0 \iff f = 0$ a.e.

Markov Inequality

For measurable $f \geq 0$ and $\alpha > 0$:

$$\lambda(\{x \in X : f(x) \geq \alpha\}) \leq \frac{1}{\alpha} \int_X f \, d\lambda$$

Relation to Riemann

If $f : [a, b] \to \mathbb{R}$ is continuous (or Riemann-integrable), it is Lebesgue integrable and the integrals coincide.

Limit Theorems

Monotone Convergence (Beppo-Levi)

If (f_n) is an **increasing** sequence of measurable **positive** functions ($0 \leq f_n \leq f_{n+1}$):

$$\lim_{n \to \infty} \int_X f_n \, d\lambda = \int_X \left(\lim_{n \to \infty} f_n \right) d\lambda$$

Note: Allows limit/integral exchange even if integral is infinite.

Fatou's Lemma

For any sequence of measurable **positive** functions ($f_n \geq 0$):

$$\int_X \liminf_{n \to \infty} f_n \, d\lambda \leq \liminf_{n \to \infty} \int_X f_n \, d\lambda$$

Dominated Convergence Theorem (DCT)

Let (f_n) be a sequence of measurable functions such that:

1. $f_n(x) \to f(x)$ for almost every x.
2. **Domination:** There exists $g \in \mathcal{L}^1(X)$ such that $|f_n(x)| \leq g(x)$ for all n, a.e. x.

Then $f \in \mathcal{L}^1(X)$ and:

$$\lim_{n \to \infty} \int_X f_n \, d\lambda = \int_X f \, d\lambda$$

Integrals with Parameters

Let $F(u) = \int_X f(u, x) \, d\lambda(x)$ for $u \in I$.

Continuity

F is continuous on I if:

- $x \mapsto f(u, x)$ is measurable.
- $u \mapsto f(u, x)$ is continuous a.e.
- **Domination:** $|f(u, x)| \leq g(x)$ with $g \in \mathcal{L}^1$.

Differentiability (Leibniz Rule)

F is differentiable and $F'(u) = \int \frac{\partial f}{\partial u} d\lambda$ if:

- $x \mapsto f(u, x)$ is integrable.
- $u \mapsto f(u, x)$ is differentiable a.e.
- **Domination:** $|\frac{\partial f}{\partial u}(u, x)| \leq g(x)$ with $g \in \mathcal{L}^1$.

Integration on $\mathbb{R}^d$

Measure on $\mathbb{R}^d$

Built from products of intervals (pavés).

- $\lambda_d(A \times B) = \lambda_p(A)\lambda_{d-p}(B)$.
- Sets of dimension $k < d$ (e.g., lines in $\mathbb{R}^2$) have measure 0.
- Invariant by translation and rotation (isometries).

Tonelli's Theorem (Positive Functions)

If $f : \mathbb{R}^{p+q} \to [0, +\infty]$ is measurable **positive**:

$$\int_{\mathbb{R}^{p+q}} f\, d\lambda = \int_{\mathbb{R}^q} \left(\int_{\mathbb{R}^p} f(x,y) dx \right) dy$$
$$= \int_{\mathbb{R}^p} \left(\int_{\mathbb{R}^q} f(x,y) dy \right) dx$$

Use this to check integrability (i.e., if integral is finite).

Fubini's Theorem (Integrable Functions)

If $f : \mathbb{R}^{p+q} \to \mathbb{R}$ is **integrable** ($f \in \mathcal{L}^1$), then:

- The slices $x \mapsto f(x,y)$ are in $\mathcal{L}^1$ for a.e. y.
- The integral order can be swapped (same formula as Tonelli).

Standard Strategy: Use Tonelli on $|f|$ to prove $f \in \mathcal{L}^1$, then Fubini to compute.

Change of Variables

Linear Transformation

If $T : \mathbb{R}^d \to \mathbb{R}^d$ is an isomorphism and $f \in \mathcal{L}^1$:

$$\int_{\mathbb{R}^d} f(y)\, dy = \int_{\mathbb{R}^d} f(Tx)\, |\det T|\, dx$$

General Diffeomorphism (C^1)

Let $\Phi : U \to V$ be a C^1-diffeomorphism between open sets in $\mathbb{R}^d$. $f \in \mathcal{L}^1(V)$ iff $(f \circ \Phi)|\det J_\Phi| \in \mathcal{L}^1(U)$.

$$\int_V f(y)\, dy = \int_U f(\Phi(x))\, |\det J_\Phi(x)|\, dx$$

where $J_\Phi(x)$ is the Jacobian matrix.

Example: Polar Coordinates ($\mathbb{R}^2$)

$\Phi(r, \theta) = (r\cos\theta, r\sin\theta)$. Determinant: $|\det J_\Phi| = r$.

$$\int_{\mathbb{R}^2} f\, dxdy = \int_0^\infty \int_{-\pi}^{\pi} f(r\cos\theta, r\sin\theta)\, r\, d\theta dr$$

Application: $\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$.