Identities of the Graph Variety

axiomatized by the term equation $x^2y \approx xy^2$.

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Section 1

Introduction

Graph Algebras

Introduced by McNulty and Shallon in 1987, Graph Algebras provide a method to ascribe algebraic structures to graphs. [1]

[1] Andrei V. Kelarev., Graph Algebras and Automata., Marcel Dekker, New York, 2003. [1]

Problem Statement

The objective of the following presentation is to characterise,

1. The graph variety $\mathcal{V}_G(\mathcal{G})$ generated by $\mathcal{G}$ where $\mathcal{G}$ satisfies the term equation $x^2y \approx xy^2$.[2] In other words, $Mod_G\{x^2y \approx xy^2\}$.
2. The identities of $Mod_G\{x^2y \approx xy^2\}$. That is, $Id(Mod_G\{x^2y \approx xy^2\})$.

In the end, we can describe the graph variety.

[2] By $x^2y \approx xy^2$ we mean $(xx)y \approx x(yy)$

Graph Algebras

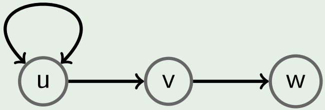
Let $G = (V, E)$ be a directed graph with vertex set V and edge set $E \subseteq V \times V$. We define the graph algebra of G, $A(G)$ with an underlying set $V \cup \{\infty\}$ where ∞ is any variable outside of V with the following operations: a nullary operation pointing to ∞ and a binary operation where given $u, v \in V \cup \{\infty\}$,

$$uv = \begin{cases} u & \text{if } (u, v) \in E \\ \infty & \text{if otherwise.} \end{cases} \tag{1}$$

Graph Algebras

Examples

Consider the following graph,



The Graph Algebra binary operation table would be,

$\cdot$	u	v	w
u	u	u	∞
v	∞	∞	w
w	∞	∞	∞

Section 2

Definitions: Terms, Graph Varieties, and Identities

The Rooted Graph Corresponding to a Term

Given a term t we can use graph algebras to create a corresponding rooted graph.

Theorem 1.1

For every *non-trivial term* t, we can associate with it a directed graph $G(t) = (V(t), E(t))$ where $V(t)$ is the set of all variables in t and $E(t)$ is defined inductively by,

- 1. $E(t) = \phi$ if t is a variable.
- 2. If $t = t_1 t_2$ then $E(t_1 t_2) = E(t_1) \cup E(t_2) \cup \{(L(t_1)), (L(t_2))\}$

The left most variable of t is denoted $L(t)$ and is the root of the graph. The empty graph ϕ is associated with any trivial term.

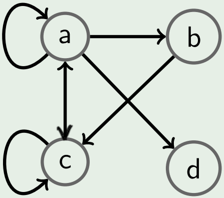
The Rooted Graph Corresponding to a Term

Examples

Consider the term

$$((a((c(ab))c))d)((bc)a)$$

we will have the rooted graph,



Evaluation Function

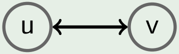
For any graph $G = (V, E)$, let the map $h : X \to V \cup \{\infty\}$ be called an *assignment*. We can extend h to another map $\bar{h} : W_\tau(X) \to V \cup \{\infty\}$ by assigning $\bar{h}(t) = h(t)$ if $t = x \in X$ and $\bar{h}(t) = \bar{h}(t_1)\bar{h}(t_2)$, when $t = t_1 t_2$ and the product is taken in $A(G)$.

$\bar{h}$ is called the evaluation function of the term t in graph G corresponding to assignment h.

Evaluation Function

Examples

Consider the term $t = xy$ and the following graph,



The assignment $h : V(t) \to \{\infty, u, v\}$ such that $h(x) = u$ and $h(y) = v$ will yield $\overline{h}(t) = \overline{h}(x)\overline{h}(y) = uv = u$.

Evaluation Function

Let s and t be terms. A graph $G = (V, E)$ is said to satisfy a term equation $s \approx t$ if and only if the graph algebra $A(G)$ satisfies $s \approx t$. That is, for every assignment $h : V(s) \cup V(t) \to V \cup \{\infty\}$ yields $\bar{h}(s) = \bar{h}(t)$.

Definition 1

If a graph G satisfies a term equation $s \approx t$, we write $G \models s \approx t$.

IdG, Mod_G, and $\mathcal{V}_G$

We can extend Definition 1 to an arbitary class of graphs $\mathcal{G}$ and any set of term equations Σ,

- $G \models \Sigma$ if $G \models s \approx t$ for all $s \approx t \in \Sigma$
- $\mathcal{G} \models s \approx t$ if $G \models s \approx t$ for all $G \in \mathcal{G}$.
- $\mathcal{G} \models \Sigma$ if $G \models \Sigma$ for all $G \in \mathcal{G}$.
- $Id\mathcal{G} = \{s \approx t |\ s, t \in W_\tau(X)$ and $G \models s \approx t\}$.
- $Mod_G\Sigma = \{G|\ G$ is a graph and $G \models \Sigma\}$.
- $\mathcal{V}_G(\mathcal{G}) = Mod_G(Id\mathcal{G})$

Remark

$\mathcal{V}_G(\mathcal{G}) = Mod_G(Id\mathcal{G})$ is called the graph variety generated by $\mathcal{G}$. If $\mathcal{V}_G(\mathcal{G}) = \mathcal{G}$ then $\mathcal{G}$ is a called a graph variety.

Problem Statement (Revisited)

The objective of the following presentation is to characterise,

1. The graph variety $\mathcal{V}_G(\mathcal{G})$ generated by $\mathcal{G}$ where $\mathcal{G}$ satisfies the term equation $x^2y \approx xy^2$. In other words, $Mod_G\{x^2y \approx xy^2\}$.
2. The identities of $Mod_G\{x^2y \approx xy^2\}$. That is, $Id(Mod_G\{x^2y \approx xy^2\})$.

In the end, we can describe the graph variety.

Proposition 1

Let $G = (V, E)$ be a graph and s and t be nontrivial terms such that $V(s) = V(t)$ and $L(s) = L(t)$. Then, $G \models s \approx t$ if and only if G exhibits the following property:

any mapping $h : V(s) \to V$ is a homomorphism from $G(s)$ into G if and only if it is a homomorphism from $G(t)$ into G.

Proven by Poschël and Wessel[3], this proposition is very useful since $x^2y \approx xy^2$ has $V(s) = V(t)$ and $L(s) = L(t)$.

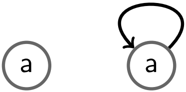
[3]Wessel, Walter., Pöschel, Reinhard. Classes of graphs definable by graph algebra identities or quasi-identities. Commentationes Mathematicae Universitatis Carolinae, 028(3):581–592, 1987.[2]

Section 3

The Graph Variety generated by $Mod_G\{x^2y \approx xy^2\}$

Graph Variety

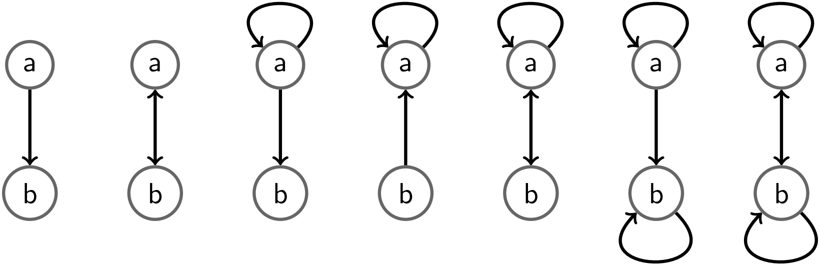
Consider the following permutations of directed graphs with 1 node.



By inspection we find that they both satisfy $x^2y \approx xy^2$.

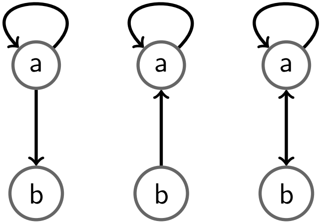
Graph Variety

Consider the following permutations of directed graphs with 2 nodes.



Graph Variety

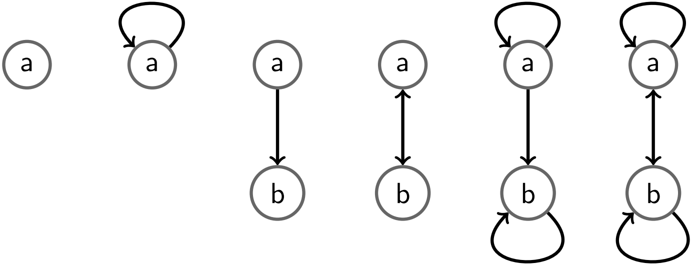
By inspection we find that these graphs do not satisfy $x^2y \approx xy^2$.



Any graph with the above three graphs as a component will not satisfy $x^2y \approx xy^2$. Note, that all of them have vertex without loops sharing an edge with a vertex with a loop.

Graph Variety

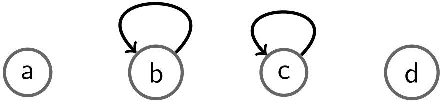
On the other hand, the graphs below satisfy $x^2y \approx xy^2$,



Any graph with the above graphs as a component will satisfy $x^2y \approx xy^2$. Note, for the graphs above, if two vertices share an edge, they either both have loops or no loops.

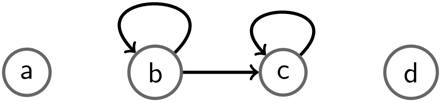
Graph Variety

In fact, this graph satisfies the term equation.



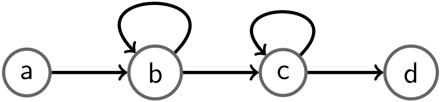
Graph Variety

So, does this graph.



Graph Variety

But not this graph,



Theorem 1

Theorem 1

Let $G = (V, E)$ be a graph. Then G belongs to the graph variety generated by $x^2y \approx xy^2$ if and only if for each component graph $K = (V_k, E_k)$ of G, $(a, a) \in E_k$ for all $a \in V_k$ or $(a, a) \notin E_k$ for all $a \in V_k$.

Sketch of Proof: $(\rightarrow)$ We shall prove using contradiction.

1. We assume that there exists a graph contrary to our theorem that satisfies the term equation.
2. Consider any one component graph H that has a node without a loop and node with loop. Recall that all nodes in a component graph is connected.
3. Consider a subgraph H' of the component graph. This will yield 3 possible cases, (in fact these are the three graphs that we found do not satisfy the term equation).
4. Find a map h such that the the graph will not satisfy the term equation.

Theorem 1

Sketch of Proof: $(\leftarrow)$

1. Let G be a graph that satisfies our property.
2. From Proposition 1, we have that $h : G(x^2y) \to G$ is a homomorphism if and only $h : G(xy^2) \to G$ is a homomorphism.
3. Assume that $h : G(x^2y) \to G$ is a homomorphism. Since, the properties of homomorphism means that a non-trivial map has x and y map to two nodes on the same component graph, the loop on x^2y will imply that all the nodes in the component graph has loops. That is, $h : G(xy^2) \to G$ is a homomorphism. Similarly assuming $h : G(xy^2) \to G$ is a homomorphism implies $h : G(x^2y) \to G$ is a homomorphism.

Section 4

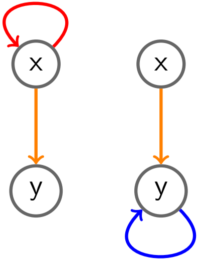
Identities

Identities

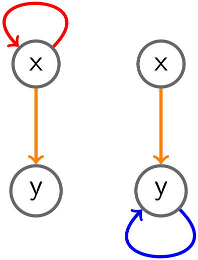
Next, we will consider the identities that are satisfied by $G \in Mod_G\{x^2y \approx xy^2\}$. Clearly, since $G \models x^2y \approx xy^2$, we have that $x^2y \approx xy^2$ is an identity of $Mod_G\{x^2y \approx xy^2\}$.

Identities

First, we note that $x^2y \approx xy^2$ has $L(s) = L(t)$ and $V(s) = V(t)$. Next, consider the rooted graph corresponding to s and t.



Identities



Recall Proposition 1, for a mapping $h : V(s) \to V \cup \{\infty\}$ to have property $h : G(s) \to G$ is a homomorphism if and only if $h : G(t) \to G$ is a homomorphism,

- $\{h(x), h(x)\} \in E$ if and only if $\{h(y), h(y)\} \in E$.
- $\{h(x), h(y)\} \in E$.

Theorem 2

Generalizing, we have the following conjecture.

Theorem 2

Let $s \approx t$ be a non-trivial term equation. Then $s \approx t$ is an identity of $Mod_G\{x^2y \approx xy^2\}$ if and only if $s \approx t$ have the following properties.

- 1. $V(s) = V(t)$
- 2. $L(s) = L(t)$
- 3. $G(s)$ has a looped vertex if and only if $G(t)$ has a looped vertex.
- 4. For any $x, y \in V(s), x \neq y, (x, y) \in E(s)$ if and only if $(x, y) \in E(t)$.

Theorem 2

Sketch of Proof: $(\rightarrow)$ We suppose that there exists a term equation that is contrary to each case and show by some arbitary evaluation map that they do not satisfy one member of our graph variety.

Remark

We use an interesting approach to prove property (3-4). That is, we show that $G(t) \in Mod_G\{x^2y \approx xy^2\}$ and that some evaluation identity map does not satisfy $G(s)$.

Theorem 2

Sketch of Proof: ($\leftarrow$) We use Proposition 1.

1. Assume that $s \approx t$ has properties (1-4). Suppose that $h : G(s) \to G$ is a homomorphism.
2. Consider any $(x', y') \in E(t)$ in the cases $x' \neq y'$ or $x' = y'$. We get that our properties implies $h : G(t) \to G$ is a homomorphism.
3. Similarly, assuming $h : G(t) \to G$ is a homomorphism implies $h : G(s) \to G$ is a homomorphism.

Section 5

Conclusion

Conclusion

Hence, we have that,

Theorem 1

Let $G = (V, E)$ be a graph. Then G belongs to the graph variety $Mod_G\{x^2y \approx xy^2\}$ if and only if for each component graph $K = (V_k, E_k)$ of G, $(a, a) \in E_k$ for all $a \in V_k$ or $(a, a) \notin E_k$ for all $a \in V_k$.

Theorem 2

Let $s \approx t$ be a non-trivial term equation. Then $s \approx t$ is an identity of $Mod_G\{x^2y \approx xy^2\}$ if and only if $s \approx t$ have the following properties.

- 1. $V(s) = V(t)$
- 2. $L(s) = L(t)$
- 3. $G(s)$ has a looped vertex if and only if $G(t)$ has a looped vertex.
- 4. For any $x, y \in V(s), x \neq y, (x, y) \in E(s)$ if and only if $(x, y) \in E(t)$.

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Marcel Dekker, New York, 2003.

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Classes of graphs definable by graph algebra identities or quasi-identities.
Commentationes Mathematicae Universitatis Carolinae, 028(3):581–592, 1987.

Section 6

Appendix

Proof of Theorem 1

($\rightarrow$) Suppose that there exists a component graph $H = (V_h, E_h)$ of G such that there are verticies $a, b \in V_h$ which $(a, a) \in E_h$ but $(b, b) \notin E_h$. Since H is connected, we have a subgraph $H' = (V_{h'}, E_{h'})$ of H where

- $V_{h'} = \{a', b'\}$, $E_{h'} = \{(a', a'), (a', b')\}$ or
- $V_{h'} = \{a', b'\}$, $E_{h'} = \{(a', b'), (b', b')\}$,
- $V_{h'} = \{a', b'\}$, $E_{h'} = \{(a', a'), (a', b'), (b', a')\}$

Let $h : \{x, y\} \rightarrow V \cup \{\infty\}$ be an evaluation function and $\bar{h}$ be an extension of h. Let $h(x) = a'$ and $h(y) = b'$. Then, we have for each case

- $\bar{h}(x^2y) = a' \neq \infty = \bar{h}(xy^2)$
- $\bar{h}(x^2y) = \infty \neq a' = \bar{h}(xy^2)$
- $\bar{h}(xy^2) = a' \neq \infty = \bar{h}(xy^2)$

That is for all cases $G \notin Mod_G x^2y \approx xy^2$ which is a contradiction.

Proof of Theorem 1

$(\leftarrow)$ Let $G = (V, E)$ be a graph which has the property that for each component graph $K = (V_k, E_k)$ of G either every vertex $a \in V_k$, $(a, a) \in E_k$ or for every vertex $a \in V_k, (a, a) \notin E_k$ but not both. Let $h : \{x, y\} \to V \cup \{\infty\}$. Suppose that $h(G(x^2y)) \to h$ is a homomorphism. Since $(x, x), (x, y) \in E(x^2y)$, we have $(h(x), h(x)), (h(x), h(y)) \in E$. Since $(h(x), h(x)) \in E_k$ and by the property of G, we have $(a, a) \in E_k$ for all $a \in V_k$. Hence $(h(y), h(y)) \in E_k$. Since $(h(x), h(y)), (h(y), h(y)) \in E_k$ we have $h(G(xy^2) \to G)$ is a homomorphism. Similarly, if $h(G(xy^2)) \to G$ is a homomorphism, then $h(G(x^2y)) \to G$ is a homomorphism. By proposition X, we get $G \models x^2y \approx xy^2$. ■

Characteristic of $Mod_G\{x^2y \approx xy^2\}$

Hence, we have the theorem

Theorem 1

Let $G = (V, E)$ be a graph. Then G belongs to the graph variety $Mod_G\{x^2y \approx xy^2\}$ if and only if for each component graph $K = (V_k, E_k)$ of G, $(a, a) \in E_k$ for all $a \in V_k$ or $(a, a) \notin E_k$ for all $a \in V_k$.

Proof of Theorem 2

($\rightarrow$) We shall assume for the sake of contradiction that s and t are non-trivial terms that satisfy $Mod_G\{x^2y \approx xy^2\}$.

1. Suppose that $V(s) \neq V(t)$. Hence, there is some $y \in V(s)$ but $y \notin V(t)$. Let $G = (V, E)$ be a graph with $V = \{a, b\}$ and $E = \{(a, a), (b, b)\}$. From Theorem 2.1, $G \in Mod_G\{x^2y \approx xy^2\}$. Let $h : V(s) \cup V(t) \rightarrow V$ be an evaluation map such that $h(x) = a$ for all $x \in V(s) \cup V(t)/\{y\}$ and $h(y) = b$. We get $\bar{h}(s) = \infty$ or $\bar{h}(s) = b$ but $\bar{h}(t) = a$. Hence, $\bar{h}(s) \neq \bar{h}(t)$ which is a contradiction.
2. Suppose that $L(s) \neq L(t)$. Let $G = (V, E)$ be a graph with $V = \{a, b\}$ and $E = \{(a, a), (b, b), (a, b), (b, a)\}$. From theorem 2.1, $G \in Mod_G\{x^2y \approx xy^2\}$. Let $h : V(s) \rightarrow V$ be an evaluation map where $h(x) = a$ for all $x \in V(s)/\{L(t)\}$ and $h(L(t)) = b$. We see that $\bar{h}(s) = \bar{h}(L(s)) = a \neq b = \bar{h}(L(t)) = \bar{h}(t)$ which is a contradiction.

Proof of Theorem 2

3. Suppose that $G(s)$ has loops but $G(t)$ has no loops. Let $G = G(t) = (V(t), E(t))$.
   From Theorem 2.1, $G \in Mod_G\{x^2y \approx xy^2\}$. Let $h : V(t) \to V$ be an evaluation identity map which yields $\overline{h}(s) = \infty \neq L(t) = \overline{h}(t)$ which is a contradiction. Similarly if we suppose that $G(t)$ has loops but $G(s)$ has no loop, we will reach a similar contradiction.
4. Suppose that there exist $x, y \in V(s), x \neq y$ such that $(x, y) \in E(s)$ but $(x, y) \notin E(t)$. Suppose we add loops into all vertices of $G(t)$ and denote this graph $G_l(t)$. From theorem 2.1, $G_l(t) \in Mod_G\{x^2y \approx xy^2\}$. Select $G = G_l(t)$. Let $h : V(s) \to V$ be an evaluation identity map. We get $\overline{h}(s) = \infty \neq L(t) = \overline{h}(t)$ which is a contradiction.

Proof of Theorem 2

($\leftarrow$) Suppose that $s \approx t$ is a term equation with properties (1 - 4). Let $G = (V, E)$, and $G \in Mod_G\{x^2y \approx xy^2\}$. Suppose that $h : G(s) \to G$ is a homomorphism. That is if $(x, y) \in E(s)$ then $(h(x), h(y)) \in E$. Let $(x', y') \in E(t)$.

1. If $x' = y'$ then $(x', x') \in E(t)$. By (3) we have there exists $z' \in V(s)$ such that $(z', z') \in E(s)$. Hence $(h(z'), h(z')) \in E$. Since $G(s)$ is a connected graph, we have that the image of h in G is a connected subgraph of G. By the properties of G we have $(h(x'), h(x')) \in E$.
2. If $x' \neq y'$, then since $(x', y') \in E(t)$ by (4) we have $(x', y') \in E(s)$. Hence, $(h(x'), h(y')) \in E$. Therefore, $h : G(t) \to G$ is a homomorphism.
3. Similarly, if $h : G(t) \to G$ is a homomorphism, then we can prove that $h : G(s) \to G$ is a homomorphism. By proposition 1, we have $G \models s \approx t$. ■

Identities of $Mod_G\{x^2y \approx xy^2\}$

Hence, we have the theorem

Theorem 2

Let $s \approx t$ be a non-trivial term equation. Then $s \approx t$ is an identity of $Mod_G\{x^2y \approx xy^2\}$ if and only if $s \approx t$ have the following properties.

1. $V(s) = V(t)$
2. $L(s) = L(t)$
3. $G(s)$ has a looped vertex if and only if $G(t)$ has a looped vertex.
4. For any $x, y \in V(s), x \neq y, (x, y) \in E(s)$ if and only if $(x, y) \in E(t)$.