Computational Math Camp

Problem Sets

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Overview

Contains problem sets for the 2019 Computational Math Camp.

Chapter 1

Linear equations, notation, sets, and functions

1.1 Simplify expressions

Simplify the following expressions as much as possible:

a. $(-x^4y^2)^2$

1. Distribute exponents over products.

$$(-1)^2x^{(2\times4)}y^{(2\times2)}$$

2. Multiply 2 and 2 together.

$$(-1)^2x^{(2\times4)}y^4$$

3. Multiply 2 and 4 together.

$$(-1)^2x^8y^4$$

4. Evaluate $(-1)^2$.

$$x^8y^4$$

b. $9(3^0)$

1. Any nonzero number to the zero power is 1.

$$9(1)$$

2. Anything times 1 is the same value.

$$9$$

c. $(2a^2)(4a^4)$

1. Combine products of like terms.

$$2a^2 \times 4a^4 = 2 \times 4a^{(2+4)}$$

2. Evaluate $2 + 4$.

$$2 \times 4a^6$$

3. Multiply 2 and 4 together.

$$8a^6$$

d. $\dfrac{x^4}{x^3}$

1. For all exponents, $\dfrac{a^n}{a^m} = a^{(n-m)}$.

$$x^{(4-3)}$$

2. Evaluate $4 - 3$.

$$x$$

e. $(-2)^{7-4}$

1. Subtract 4 from 7.

$$(-2)^3$$

2. In order to evaluate 2^3 express 2^3 as 2×2^2.

$$-2 \times 2^2$$

3. Evaluate 2^2.

$$-2 \times 4$$

4. Multiply -2 and 4 together.

$$-8$$

f. $\left(\dfrac{1}{27b^3}\right)^{1/3}$

1. Separate component terms.

$$\frac{1}{27}^{1/3} \times \frac{1}{b^3}^{1/3}$$

2. Evaluate cube roots.

$$\frac{1}{3} \times \frac{1}{b}$$

3. Combine terms.

$$\frac{1}{3b}$$

g. $y^7y^6y^5y^4$

1. Combine products of like terms.

$$y^{(7+6+5+4)}$$

2. Evaluate $7 + 6 + 5 + 4$.

$$y^{22}$$

h. $\dfrac{2a/7b}{11b/5a}$

1. Write as a single fraction by multiplying the numerator by the reciprocal of the denominator.

$$\frac{2a}{7b} \times \frac{5a}{11b}$$

2. Product property of exponents: $x^a \times x^b = x^{(a+b)}$

$$\frac{5a \times 2a}{7b \times 11b} = \frac{5 \times 2a^{1+1}}{7 \times 11b^{1+1}}$$

3. Evaluate $1 + 1$.

$$\frac{5 \times 2a^2}{7 \times 11b^2}$$

4. Multiple scalars together.

$$\frac{10a^2}{77b^2}$$

i. $(z^2)^4$

1. Nested exponents rule: $(x^a)^b = x^{ab}$

$$z^{2\times 4}$$

2. Evaluate 2×4

$$z^8$$

1.2 Simplify a (more complex) expression

Simplify the following expression:

$$(a+b)^2 + (a-b)^2 + 2(a+b)(a-b) - 3a^2$$

1. Expand $(a+b)^2$ with FOIL.

$$a^2 + 2ab + b^2 + (a-b)^2 + 2(a+b)(a-b) - 3a^2$$

2. Expand $(a-b)^2$ with FOIL.

$$a^2 + 2ab + b^2 + a^2 - 2ab + b^2 + 2(a+b)(a-b) - 3a^2$$

3. Multiply $a+b$ and $a-b$ together using FOIL.

$$a^2 + 2ab + b^2 + a^2 - 2ab + b^2 + 2(a^2 - b^2) - 3a^2$$

4. Distribute 2 over $a^2 - b^2$.

$$a^2 + 2ab + b^2 + a^2 - 2ab + b^2 + 2a^2 - 2b^2 - 3a^2$$

5. Group like terms.

$$(a^2 + a^2 + 2a^2 - 3a^2) + (b^2 + b^2 - 2b^2) + (2ab - 2ab)$$

6. Combine like terms.

$$a^2 + (b^2 + b^2 - 2b^2) + (2ab - 2ab)$$

7. Look for the difference of two identical terms.

$$a^2$$

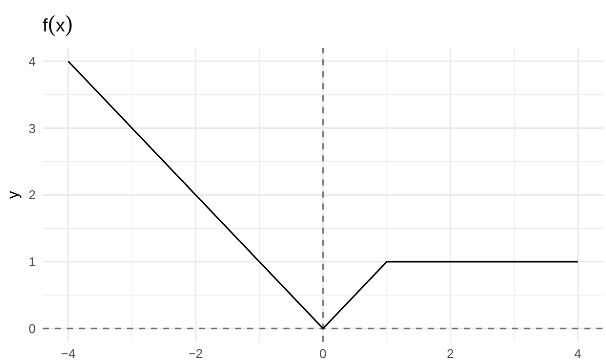
1.3 Graph sketching

Let the functions $f(x)$ and $g(x)$ be defined for all $x \in \Re$ by

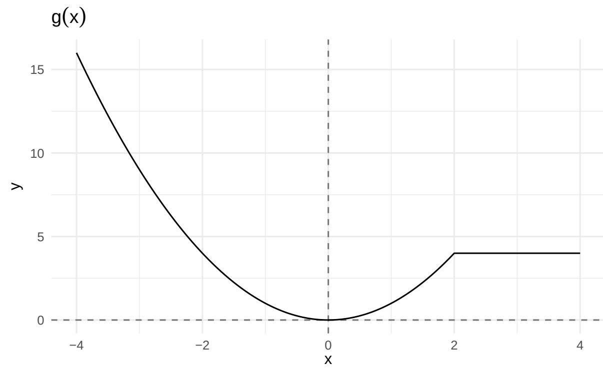
$$f(x) = \begin{cases} |x| & \text{if } x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}, \quad g(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 4 & \text{if } x \geq 2 \end{cases}$$

Sketch the graphs of:

1. $y = f(x)$



2. $y = g(x)$



3. $y = f(g(x))$

To sketch the composite function, we first evaluate $g(x)$ for different values of x, and then evaluate $f(g(x))$ for different outputs of $g(x)$.

- For $x \geq 2$, $g(x)$ is a constant value:

$$\begin{aligned} x &\geq 2 \\ g(x) &= 4 \\ f(g(x)) &= f(4) = 1 \end{aligned}$$

- For $x < 2$, $g(x)$ is not constant: $g(x) = x^2$. $f(x)$ evaluates differently depending on its input, so we have two cases based on the output of $g(x)$:
  - if $g(x) < 1$, $f(g(x)) = |g(x)| = |x^2| = x^2$. This is the case when:

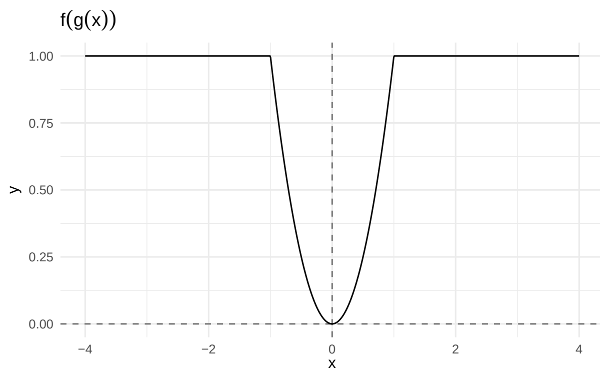
$$\begin{aligned} g(x) &< 1 \\ x^2 < 1 \text{ and } x &< 2 \\ -1 < x &< 1 \end{aligned}$$

  - if $g(x) \geq 1$, $f(g(x)) = 1$. This is the case when:

$$\begin{aligned} g(x) &\geq 1 \\ x^2 \geq 1 \text{ and } x &< 2 \\ x \leq -1 \text{ or } 1 \leq x &< 2 \end{aligned}$$

- Therefore, $f(g(x))$ has the following values:

$$f(g(x)) = \begin{cases} 1 & \text{if } x \leq -1 \\ x^2 & \text{if } -1 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$



4. $y = g(f(x))$

   To sketch the composite function, we first evaluate $f(x)$ for different values of x, and then evaluate $g(f(x))$ for different outputs of $f(x)$.

   - For $x \geq 1$, $f(x)$ is a constant value:

   $$\begin{aligned} x &\geq 1 \\ f(x) &= 1 \\ g(f(x)) &= f(1) = 1^2 = 1 \end{aligned}$$

   - For $x < 1$, $f(x)$ is not constant: $f(x) = |x|$. $g(x)$ evaluates differently depending on its input, so we have two cases based on the output of $f(x)$:
     - if $f(x) < 2$, $g(f(x)) = f(x)^2 = |x|^2 = x^2$. This is the case when:

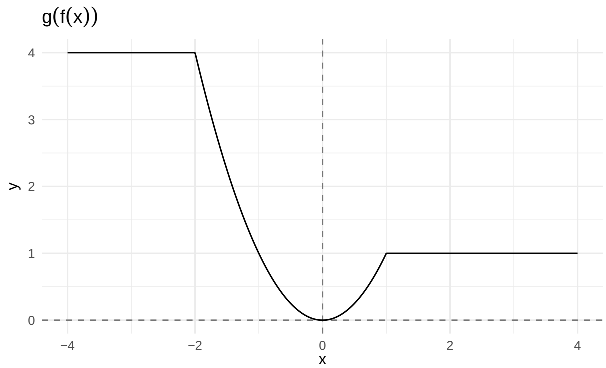
     $$\begin{aligned} f(x) &< 2 \\ |x| < 2 \text{ and } x &< 1 \\ -2 < x &< 1 \end{aligned}$$

     - if $f(x) \geq 2$, $g(f(x)) = 4$. This is the case when:

     $$\begin{aligned} f(x) &\geq 2 \\ |x| \geq 2 \text{ and } x &< 1 \\ x &\leq -2 \end{aligned}$$

   - Therefore, $g(f(x))$ has the following values:

   $$g(f(x)) = \begin{cases} 4 & \text{if } x \leq -2 \\ x^2 & \text{if } -2 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$



1.4 Root finding

Find the roots (solutions) to the following quadratic equations.

Definition 1.1 (The quadratic formula)**.**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

a. $4x^2 - 1 = 17$

- Move terms so that x is alone on the left side of the equation.

$$\begin{aligned} 4x^2 - 1 &= 17 \\ 4x^2 &= 18 \\ x^2 &= \frac{18}{4} \\ x^2 &= \frac{9}{2} \\ x &= \pm\sqrt{\frac{9}{2}} \end{aligned}$$

b. $9x^2 - 3x - 12 = 0$

- Factor the left-hand side.

$$3(x+1)(3x-4) = 0$$

- Divide both sides by 3 to simplify the equation.

$$(x+1)(3x-4) = 0$$

- Find the roots of each term in the product separately by solving for x.

$$\begin{aligned} x + 1 &= 0 \\ x &= -1 \end{aligned} \qquad \begin{aligned} 3x &= 4 \\ x &= \frac{4}{3} \end{aligned}$$

c. $x^2 - 2x - 16 = 0$

1. Complete the square

$$\begin{aligned} x^2 - 2x - 16 &= 0 \\ x^2 - 2x &= 16 \\ x^2 - 2x + 1 &= 17 \\ (x-1)^2 &= 17 \\ x - 1 &= \pm\sqrt{17} \\ x &= 1 \pm \sqrt{17} \end{aligned}$$

2. Quadratic formula

   - Using the quadratic formula, solve for x

$$\begin{aligned} x &= \frac{-(-2) \pm \sqrt{(-2)^2 - (4 \times 1 \times 16)}}{2 \times 1} \\ x &= \frac{2 \pm \sqrt{4 + 64}}{2} \\ x &= \frac{2 \pm \sqrt{68}}{2} \end{aligned}$$

   - Simplify the radical

$$\begin{aligned} x &= \frac{2 \pm \sqrt{2^2 \times 17}}{2} \\ x &= \frac{2 \pm 2\sqrt{17}}{2} \end{aligned}$$

   - Factor the greatest common divisor

$$x = 1 \pm \sqrt{17}$$

d. $6x^2 - 6x - 6 = 0$

- Divide both sides by 6 to simplify the equation.

$$x^2 - x - 1 = 0$$

- Using the quadratic formula, solve for x

$$\begin{aligned} x &= \frac{-(-1) \pm \sqrt{(-1)^2 - (4 \times 1 \times -1)}}{2 \times 1} \\ x &= \frac{1 \pm \sqrt{1 - 4(-1)}}{2} \\ x &= \frac{1 \pm \sqrt{1 + 4}}{2} \\ x &= \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

e. $5 + 11x = -3x^2$

- Move everything to the left hand side.

$$3x^2 + 11x + 5 = 0$$

- Using the quadratic formula, solve for x

$$\begin{aligned} x &= \frac{-11 \pm \sqrt{(11)^2 - (4 \times 3 \times 5)}}{2 \times 3} \\ x &= \frac{-11 \pm \sqrt{121 - 60}}{6} \\ x &= \frac{-11 \pm \sqrt{61}}{6} \end{aligned}$$

1.5 Work with sets

Using the sets

$$\begin{aligned} A &= \{2, 3, 7, 9, 13\} \\ B &= \{x : 4 \leq x \leq 8 \text{ and } x \text{ is an integer}\} \\ C &= \{x : 2 < x < 25 \text{ and } x \text{ is prime}\} \\ D &= \{1, 4, 9, 16, 25, \ldots\} \end{aligned}$$

identify the following:

1. $A \cup B$

   $E = \{2, 3, 4, 5, 6, 7, 8, 9, 13\}$, combine all integers between 4 and 8 inclusive with the numbers in set A.

2. $(A \cup B) \cap C$

   $F = \{3, 5, 7, 13\}$, Since C is only prime numbers greater than 2 and less than 25, we take all the prime numbers that are also included in E, but remember to drop out 2 since it is not included in C.

3. $C \cap D$

   $G = \emptyset$, there are no prime numbers in D, so nothing is shared between C and D.

Chapter 2

Logarithms, sequences, and limits

2.1 Simplify logarithms

Express each of the following as a single logarithm:

a. $\log(x) + \log(y) - \log(z)$

- Multiplication rule of logarithms: $\log(x \times y) = \log(x) + \log(y)$
- Division rule of logarithms: $\log(\frac{x}{y}) = \log(x) - \log(y)$
- Applying the log rules, we combine logs that are added through multiplication and then combine logs that are subtracted with division.

$$\log(x) + \log(y) - \log(z)$$

$$\log(xy) - \log(z)$$

$$\log(\frac{xy}{z})$$

b. $2\log(x) + 1$

- Exponentiation rule of logarithms: $\log(x^y) = y\log(x)$
- $\log(e) = 1$

$$2\log(x) + 1$$

$$2\log(x) + \log(e)$$

$$\log(x^2) + \log(e)$$

$$\log(ex^2)$$

c. $\log(x) - 2$

- $\log(e) = 1$

$$\log(x) - 2$$

$$\log(x) - 2\log(e)$$

$$\log(x) - \log(e^2)$$

$$\log(\frac{x}{e^2})$$

2.2 Sequences

Write down the first three terms of each of the following sequences. In each case, state whether the sequence is an arithmetic progression, a geometric progression, or neither.

a. $u_n = 4 + 3n$

$$7, 10, 13$$

Arithmetic progression.

b. $u_n = 5 - 6n$

$$-1, -7, -13$$

Arithmetic progression.

c. $u_n = 4^n$

$$4, 16, 64$$

Geometric progression.

d. $u_n = 5 \times (-2)^n$

$$-10, 20, -40$$

Geometric progression.

e. $u_n = n \times 3^n$

$$3, 18, 81$$

Neither.

2.3 Find the limit

In each of the following cases, state whether the sequence $\{u_n\}$ tends to a limit, and find the limit if it exists:

a. $u_n = 1 + \frac{1}{2}n$

No limit ($u_n \to \infty$)

b. $u_n = 1 - \frac{1}{2}n$

No limit ($u_n \to \infty$)

c. $u_n = \left(\frac{1}{2}\right)^n$

Yes. $\lim_{x\to\infty} u_n = 0$

d. $u_n = \left(-\frac{1}{2}\right)^n$

Yes. $\lim_{x\to\infty} u_n = 0$

2.4 Determine convergence or divergence

Determine whether each of the following sequences converges or diverges. If it converges, find the limit.

a. $a_n = \frac{3+5n^2}{n+n^2}$

The sequence converges to 5. We can see this by factoring n^2 from both the numerator and denominator and then cancelling it out.

$$\lim_{n\to\infty} a_n = \frac{3+5n^2}{n+n^2} = \lim_{n\to\infty} \frac{n^2\left(\frac{3}{n^2}+5\right)}{n^2\left(\frac{1}{n}+1\right)} = \lim_{n\to\infty} \frac{\left(\frac{3}{n^2}+5\right)}{\left(\frac{1}{n}+1\right)} = \frac{\lim_{n\to\infty}\frac{3}{n^2}+5}{\lim_{n\to\infty}\frac{1}{n}+1} = \frac{0+5}{0+1} = 5$$

(This is slightly curt: Make sure you know how to show that the limit of $\frac{3}{n^2}$ approaches 0.) As $n \to \infty$, $\frac{3}{n} \to 0$ and $\frac{1}{n} \to 0$. Therefore, $a_n \to 5$.

Alternatively, you could split the fraction into two terms: one with a numerator of 3, and the other with a numerator of $5n^2$. The first fraction converges to 0. (Can you show that?) Factoring out an n from both sides of the second fraction, you're left with $\frac{5n}{n+1}$; $\frac{n}{n+1}$ converges to 1, giving you $5 \times 1 = 5$.

b. $a_n = \frac{(-1)^{n-1}n}{n^2+1}$

The sequence converges to 0. To see why, take the absolute value of the sequence, then factor out and cancel n from both sides of the fraction.

$$\lim_{n\to\infty}\left|\frac{(-1)^{n-1}n}{n^2+1}\right| = \lim_{n\to\infty}\frac{1^{n-1}n}{n^2+1} = \lim_{n\to\infty}\frac{1}{n+\frac{1}{n}} = \frac{\lim_{n\to\infty}1}{\lim_{n\to\infty}(n+\frac{1}{n})} = \frac{1}{\lim_{n\to\infty}n+0} = 0$$

2.5 Find more limits

Given that

$$\lim_{x\to a} f(x) = -3, \quad \lim_{x\to a} g(x) = 0, \quad \lim_{x\to a} h(x) = 8$$

find the limits that exist. If the limit doesn't exist, explain why.

a. $\lim_{x\to a}[f(x)+h(x)] = -3+5 = 8$

b. $\lim_{x\to a}[f(x)]^2 = (-3)^2 = 9$

c. $\lim_{x\to a}\sqrt[3]{h(x)} = \sqrt[3]{8} = 2$

d. $\lim_{x\to a}\frac{1}{f(x)} = -\frac{1}{3}$

e. $\lim_{x\to a}\frac{f(x)}{h(x)} = -\frac{3}{8}$

f. $\lim_{x\to a}\frac{g(x)}{f(x)} = \frac{0}{-3} = 0$

g. $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{-3}{0}$ = Undefined – cannot divide by 0, no limit

h. $\lim\limits_{x\to a} \frac{2f(x)}{h(x)-f(x)} = \frac{2\times -3}{8-(-3)} = -\frac{6}{11}$

2.6 Find even more limits

Find the limits of the following:

a. $\lim\limits_{x\to -4} \frac{x^2+5x+4}{x^2+3x-4}$

$$\lim_{n\to -4} \frac{x^2+5x+4}{x^2+3x-4} = \lim_{n\to -4} \frac{(x+4)(x+1)}{(x+4)(x-1)} = \lim_{n\to -4} \frac{x+1}{x-1} = \frac{\lim_{n\to -4}(x+1)}{\lim_{n\to -4}(x-1)} = \frac{-3}{-5} = \frac{3}{5}$$

b. $\lim\limits_{x\to 4^-} \sqrt{16-x^2}$

$$\begin{aligned}\lim_{n\to 4^-} \sqrt{16-x^2} &= \lim_{n\to 4^-} \sqrt{(4+x)(4-x)} \\ &= \lim_{n\to 4^-} \sqrt{4+x}\sqrt{4-x} \\ &= \lim_{n\to 4^-} \sqrt{4+x} \cdot \lim_{n\to 4^-} \sqrt{4-x} \\ &= \sqrt{8}\cdot\sqrt{0} \\ &= 0\end{aligned}$$

c. $\lim\limits_{x\to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x}$

$$\begin{aligned}\lim_{n\to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x} &= \lim_{n\to -4} \frac{\frac{x+4}{4x}}{4+x} \\ &= \lim_{n\to -4} \frac{4+x}{4x} \cdot \frac{1}{4+x} \\ &= \lim_{n\to -4} \frac{1}{4x} \\ &= \frac{1}{4(-4)} \\ &= -\frac{1}{16}\end{aligned}$$

2.7 Check for discontinuities

Which of the following functions are continuous? If not, where are the discontinuities?

a. $f(x) = \dfrac{9x^3 - x}{(x-1)(x+1)}$

- Discontinuous at $x = -1, +1$ (denominator would be 0, leaving the fraction undefined)

b. $f(x) = e^{-x^2}$

- Continuous for all real numbers.

c. $f(y) = y^3 - y^2 + 1$

- All polynomials are continuous.

d. $f(x) = \begin{cases} x^3 + 1, & x > 0 \\ \frac{1}{2} \; x = 0 \\ -x^2, & x < 0 \end{cases}$

- Discontinuous at $x = 0$. This is a piecewise function. To be continuous $\lim_{x\to 0^+} f(x) = 0$. However in this function, $\lim_{x\to 0^+} f(x) = 1 \neq 0$.

Chapter 3

Differentiation

3.1 Find finite limits

Find the following finite limits:

a. $\lim_{x\to 4} x^2 - 6x + 4$

$$\begin{aligned}\lim_{x\to 4} x^2 - 6x + 4 &= 4^2 - 6(4) + 4 \\ &= 16 - 24 + 4 \\ &= -4\end{aligned}$$

b. $\lim_{x\to 0} \left[\frac{x-25}{x+5}\right]$

$$\begin{aligned}\lim_{x\to 0} \left[\frac{x-25}{x+5}\right] &= \frac{0-25}{0+5} \\ &= \frac{-25}{5} \\ &= -5\end{aligned}$$

c. $\lim_{x\to 4} \left[\frac{x^2}{3x-2}\right]$

$$\begin{aligned}\lim_{x\to 4}\left[\frac{x^2}{3x-2}\right] &= \frac{4^2}{3(4)-2}\\ &= \frac{16}{12-2}\\ &= \frac{16}{10}\\ &= \frac{8}{5}\end{aligned}$$

d. $\lim_{x\to 1}\left[\frac{x^4-1}{x-1}\right]$

The key here is to factor the initial expression in the numerator, then cancel terms out with the denominator:

$$\begin{aligned}\lim_{x\to 1}\left[\frac{x^4-1}{x-1}\right] &= \lim_{x\to 1}\left[\frac{(x-1)(x+1)(x^2+1)}{x-1}\right]\\ &= \lim_{x\to 1}[(x+1)(x^2+1)]\\ &= (1+1)(1^2+1)\\ &= (2)(2)\\ &= 4\end{aligned}$$

Alternatively, we can use L'Hôpital's Rule:

$$\begin{aligned}\lim_{x\to 1}\left[\frac{x^4-1}{x-1}\right] &= \lim_{x\to 1}\left[\frac{4x^3}{1}\right]\\ &= \frac{4(1)^3}{1}\\ &= 4\end{aligned}$$

e. $\lim_{x\to -4}\left[\frac{x^2+5x+4}{x^2+3x-4}\right]$

The key here is to factor the initial expression:

$$\begin{aligned}\lim_{x\to -4}\frac{(x+4)(x+1)}{(x+4)(x-1)} &= \lim_{x\to -4}\frac{x+1}{x-1}\\ &= \frac{\lim\limits_{x\to -4}(x+1)}{\lim\limits_{x\to -4}(x-1)}\\ &= \frac{-3}{-5}\\ &= \frac{3}{5}\end{aligned}$$

f. $\lim\limits_{x\to 4^-}\sqrt{16-x^2}$

$$\begin{aligned}\lim_{x\to 4^-}\sqrt{16-x^2} &= \lim_{x\to 4^-}\sqrt{(4+x)(4-x)}\\ &= \lim_{x\to 4^-}\sqrt{4+x}\sqrt{4-x}\\ &= \lim_{x\to 4^-}\sqrt{4+x}\cdot\lim_{x\to 4^-}\sqrt{4-x}\\ &= \sqrt{8}*\sqrt{0}\\ &= 0\end{aligned}$$

A critical aspect of this limit, which allows for it to exist, is that it is a left-hand limit.

g. $\lim\limits_{x\to -1}\left[\frac{x-2}{x^2+4x-3}\right]$

$$\begin{aligned}\lim_{x\to -1}\frac{x-2}{x^2+4x-3} &= \frac{\lim\limits_{x\to -1}(x-2)}{\lim\limits_{x\to -1}(x^2+4x-3)}\\ &= \frac{-1-2}{(-1)^2+4(-1)-3}\\ &= \frac{-3}{-6}\\ &= \frac{1}{2}\end{aligned}$$

h. $\lim\limits_{x\to -4}\left[\frac{\frac{1}{4}+\frac{1}{x}}{4+x}\right]$

$$\begin{aligned}
\lim_{x\to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x} &= \lim_{x\to -4} \frac{\frac{x+4}{4x}}{4+x} \\
&= \lim_{x\to -4} \frac{4+x}{4x}\frac{1}{4+x} \\
&= \lim_{x\to -4} \frac{1}{4x} \\
&= \frac{1}{4(-4)} \\
&= -\frac{1}{16}
\end{aligned}$$

Alternatively, we can use L'Hôpital's Rule:

$$\begin{aligned}
\lim_{x\to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x} &= \lim_{x\to -4} \frac{-\frac{1}{x^2}}{1} \\
&= \lim_{x\to -4} (-\frac{1}{x^2}) \\
&= -\frac{1}{16}
\end{aligned}$$

3.2 Find infinite limits

Find the following infinite limits:

Hint: use **L'Hôpital's Rule** to switch from $\lim_{x\to\infty}\left(\frac{f(x)}{g(x)}\right)$ to $\lim_{x\to\infty}\left(\frac{f'(x)}{g'(x)}\right)$.

a. $\lim_{x\to\infty}\left[\frac{9x^2}{x^2+3}\right]$

$$\begin{aligned}
\lim_{x\to\infty}\left[\frac{9x^2}{x^2+3}\right] &= \lim_{x\to\infty}\left[\frac{18x}{2x}\right] \\
&= 9
\end{aligned}$$

b. $\lim_{x\to\infty}\left[\frac{3x-4}{x+3}\right]$

$$\begin{aligned}
\lim_{x\to\infty}\left[\frac{3x-4}{x+3}\right] &= \lim_{x\to\infty}\left[\frac{3}{1}\right] \\
&= 3
\end{aligned}$$

c. $\lim_{x\to\infty}\left[\frac{2^x-3}{2^x+1}\right]$

Remember that $\frac{d}{dx}n^x = \log(n)n^x$:

$$\begin{aligned}\lim_{x\to\infty}\left[\frac{2^x-3}{2^x+1}\right] &= \lim_{x\to\infty}\left[\frac{\log(2)2^x}{\log(2)2^x}\right]\\ &= 1\end{aligned}$$

d. $\lim_{x\to\infty}\left[\frac{\log(x)}{x}\right]$

$$\begin{aligned}\lim_{x\to\infty}\left[\frac{\log(x)}{x}\right] &= \lim_{x\to\infty}\left[\frac{\frac{1}{x}}{1}\right]\\ &= \lim_{x\to\infty}\left[\frac{1}{x}\right]\\ &= \frac{1}{\infty}\\ &= 0\end{aligned}$$

e. $\lim_{x\to\infty}\left[\frac{3^x}{x^3}\right]$

The trick here is to repeatedly calculate the derivative of the numerator and denominators until there is no x term on the denominator. You end up calculating the third derivative, but L'Hôpital's Rule still applies.

$$\begin{aligned}\lim_{x\to\infty}\left[\frac{3^x}{x^3}\right] &= \lim_{x\to\infty}\left[\frac{\log(3)3^x}{3x^2}\right]\\ &= \lim_{x\to\infty}\left[\frac{\log^2(3)3^x}{6x}\right]\\ &= \lim_{x\to\infty}\left[\frac{\log^3(3)3^x}{6}\right]\\ &= \frac{\log^3(3)3^\infty}{6}\\ &= \infty\end{aligned}$$

f. $\lim_{y\to\infty}\left[\frac{3e^y}{y^3}\right]$

Same as above: repeatedly calculate the derivatives until the y term disappears in the denominator.

$$\begin{aligned}\lim_{x\to\infty}\left[\frac{3e^y}{y^3}\right] &= \lim_{x\to\infty}\left[\frac{3e^y}{3y^2}\right]\\ &= \lim_{x\to\infty}\left[\frac{3e^y}{6y}\right]\\ &= \lim_{x\to\infty}\left[\frac{3e^y}{6}\right]\\ &= \frac{3e^\infty}{6}\\ &= \infty\end{aligned}$$

3.3 Assessing continuity and differentiability

For each of the following functions, describe whether it is continuous and/or differentiable at the point of transition of its two formulas.

a.

$$f(x) = \begin{cases} +x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$

Solution:

$$f'(x) = \begin{cases} 2x, & x \geq 0 \\ -2x, & x < 0 \end{cases}$$

As x converges to 0 from both above and below, $f'(0)$ converges to 0, so the function is continuous and differentiable.

b.

$$f(x) = \begin{cases} +x^2+1, & x \geq 0 \\ -x^2-1, & x < 0 \end{cases}$$

Solution: This function is not continuous (and thus not differentiable). As x converges to 0 from above, $f(x)$ tends to 1, whereas x tends to 0 from below, $f(x)$ converges to -1.

c.

$$f(x) = \begin{cases} x^3, & x \leq 1 \\ x, & x > 1 \end{cases}$$

Solution: This function is continuous, since $\lim_{x\to 1} f(x) = 1$ no matter how the limit is taken. However it is not differentiable since

$$f'(x) = \begin{cases} 3x^2, & x \leq 1 \\ 1, & x > 1 \end{cases}$$

$\lim_{x \to 1^+} f'(x) = 1$, whereas $\lim_{x \to 1^-} f'(x) = 3$. The function is not smooth and continuous at $f(1)$.

d.

$$f(x) = \begin{cases} x^3, & x < 1 \\ 3x - 2, & x \geq 1 \end{cases}$$

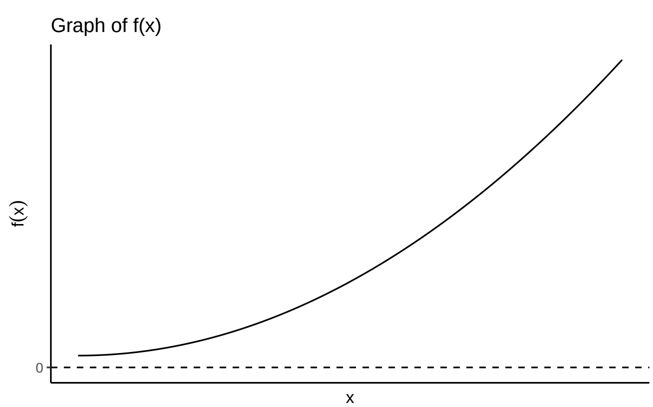
Solution: This function is continuous since $f(1)$ tends to 1 from either direction. Likewise, this function is continuous because

$$f'(x) = \begin{cases} 3x^2, & x \leq 1 \\ 3, & x > 1 \end{cases}$$

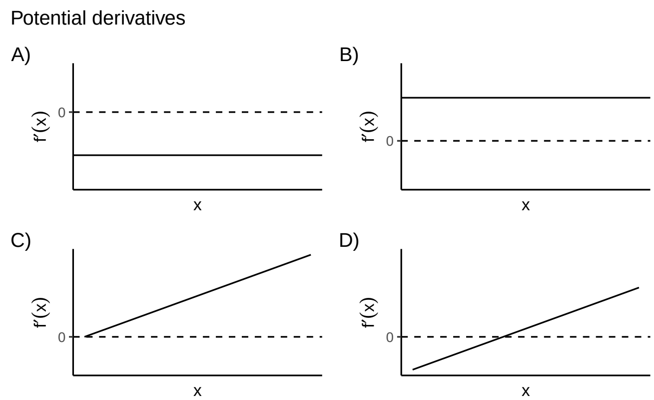
and $\lim_{x \to 1} f'(x) = 3$ from either direction.

3.4 Possible derivative

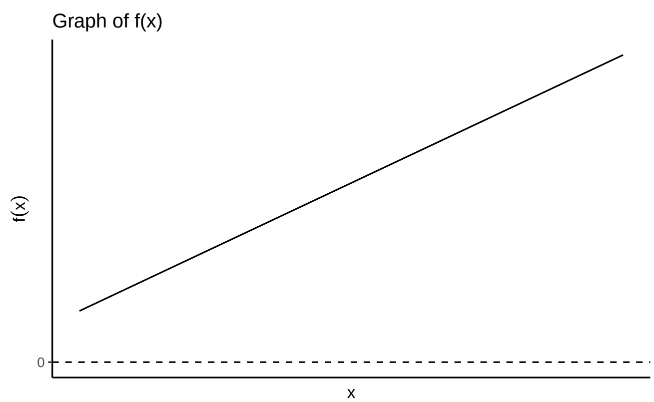
A friend shows you this graph of a function $f(x)$:



Which of the following could be a graph of $f'(x)$? For each graph, explain why or why not it might be the derivative of $f(x)$.



What if the figure below was the graph of $f(x)$? Which of the graphs might potentially be the derivative of $f(x)$ then?



Solution:

a. A doesn't work because it is negative and the function we observe is increasing in x. B is constant so this also won't work, the function we observe gets larger at an increasing, not constant rate. C seems to be a plausible candidate because an upward sloping derivative would map to the behavior of the function we observe, that $g(x)$ gets large at an increasing rate. D does not work because it suggests the function would need to be decreasing over some interval and because, when we refer back to $g(x)$, there doesn't seem to be any local minimum, maximum or a saddle point despite the graph in D crossing 0.

b. Again, A doesn't work because it is negative and the function we observe is increasing in x. B seems to be plausible as the derivative, since $g(x)$ appears to increase at a constant rate, its derivative should be flat and greater than 0. C won't work because the slope of $g(x)$ is constant and does not increase in x. D doesn't work, again because it suggests the function would need to be decreasing at some point over the interval we observe.

3.5 Calculate derivatives

Differentiate the following functions:

a. $f(x) = 4x^3 + 2x^2 + 5x + 11$

Solution: Power rule.

$$\begin{aligned} f(x) &= 4x^3 + 2x^2 + 5x + 11 \\ f'(x) &= 12x^2 + 4x + 5 \end{aligned}$$

b. $y = \sqrt{30}$

Solution: Derivative of a constant is 0.

$$\begin{aligned} y &= \sqrt{30} \\ y' &= 0 \end{aligned}$$

c. $h(t) = \log(9t + 1)$

Solution: Derivative of $\log(u)$ is $\frac{1}{u}$. Since u is a function in this problem, need to apply the chain rule to calculate the derivative of $9t + 1$ and multiply that by $\frac{1}{9t+1}$

$$\begin{aligned} h(t) &= \log(9t + 1) \\ h'(t) &= \frac{1}{9t+1} * 9 \end{aligned}$$

d. $f(x) = \log(x^2 e^x)$

Solution: Derivative of a logarithm plus the chain rule.

$$\begin{aligned} f(x) &= \log(x^2 e^x) \\ f'(x) &= \frac{1}{x^2 e^x} * (2xe^x + e^x x^2) \\ &= \frac{2xe^x + e^x x^2}{x^2 e^x} \\ &= \frac{2}{x} + 1 \end{aligned}$$

e. $h(y) = \left(\frac{1}{y^2} - \frac{3}{y^4}\right)(y + 5y^3)$

Solution: Simplify the expression first, then basic application of power rule.

$$\begin{aligned}
h(y) &= \left(\frac{1}{y^2} - \frac{3}{y^4}\right)\left(y + 5y^3\right) \\
&= \frac{y}{y^2} + \frac{5y^3}{y^2} - \frac{3y}{y^4} - \frac{15y^3}{y^4} \\
&= \frac{1}{y} + 5y - \frac{3}{y^3} - \frac{15}{y} \\
&= 5y - \frac{14}{y} - \frac{3}{y^3} \\
h'(y) &= 5 + \frac{14}{y^2} + \frac{9}{y^4}
\end{aligned}$$

f. $g(t) = \frac{3t-1}{2t+1}$

Solution: Quotient rule.

$$\begin{aligned}
g(t) &= \frac{3t-1}{2t+1} \\
g'(t) &= \frac{(3)(2t+1) - (3t-1)(2)}{(2t+1)^2} \\
&= \frac{5}{(2t+1)^2}
\end{aligned}$$

3.6 Use the product and quotient rules

Differentiate the following using both the product and quotient rules:

$$f(x) = \frac{x^2 - 2x}{x^4 + 6}$$

Solution:

a. First let's use the quotient rule:

$$\begin{aligned}
h(x) &= \frac{f(x)}{g(x)} \\
f(x) &= x^2 - 2x \\
g(x) &= x^4 + 6 \\
f'(x) &= 2x - 2 \\
g'(x) &= 4x^3 \\
h'(x) &= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \\
&= \frac{(2x-2)(x^4+6) - (x^2-2x)(4x^3)}{(x^4+6)^2} \\
&= \frac{2x^5 + 12x - 2x^4 - 12 - 4x^5 + 8x^4}{(x^4+6)^2} \\
&= \frac{-2x^5 + 6x^4 + 12x - 12}{(x^4+6)^2}
\end{aligned}$$

b. Now we can do the same thing with the product rule:

$$\begin{aligned}
j(x) &= k(x)m(x) \\
k(x) &= x^2 - 2x \\
m(x) &= (x^4+6)^{-1} \\
k'(x) &= 2x - 2 \\
m'(x) &= -(x^4+6)^{-2}(4x^3) = -\frac{4x^3}{(x^4+6)^2} \\
j'(x) &= k(x)m'(x) + k'(x)m(x) \\
&= (x^2-2x)(-\frac{4x^3}{(x^4+6)^2}) + (2x-2)(x^4+6)^{-1} \\
&= -\frac{(x^2-2x)(4x^3)}{(x^4+6)^2} + \frac{2x-2}{x^4+6} \\
&= -\frac{4x^5 - 8x^4}{x^4+6)^2} + \frac{2x-2}{x^4+6} \\
&= -\frac{4x^5 - 8x^4}{x^4+6)^2} + \frac{2x-2}{x^4+6}\frac{x^4+6}{x^4+6} \\
&= -\frac{4x^5 - 8x^4}{(x^4+6)^2} + \frac{2x^5 + 12x - 2x^4 - 12}{(x^4+6)^2} \\
&= \frac{2x^5 + 12x - 2x^4 - 12 - 4x^5 - 8x^4}{(x^4+6)^2} \\
&= \frac{-2x^5 + 6x^4 + 12x - 12}{(x^4+6)^2}
\end{aligned}$$

The quotient rule is simply a derivation of the product rule combined with the chain rule:

$$\begin{aligned} h(x) &= \frac{f(x)}{g(x)} \\ &= f(x)g(x)^{-1} \end{aligned}$$

Apply product and chain rules:

$$\begin{aligned} h'(x) &= f'(x)g(x)^{-1} + f(x)(-1)g(x)^{-2}g'(x) \\ &= f'(x)g(x)g(x)^{-2} - f(x)g(x)^{-2}g'(x) \\ &= [f'(x)g(x) - f(x)g'(x)]g(x)^{-2} \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \end{aligned}$$

which is the quotient rule.

3.7 Logarithms and exponential functions

Compute the derivative of each of the following functions:

a. $f(x) = xe^{3x}$

Solution: Use the product rule to split the function into component functions.

$$g(x) = x, \quad h(x)e^{3x}$$

Use the chain rule to solve $h'(x)$.

$$\begin{aligned} g(x) &= x \qquad & h(x) &= e^{3x} \\ g'(x) &= 1 \qquad & h'(x) &= 3e^{3x} \end{aligned}$$

$$\begin{aligned} f(x) &= g'(x)h(x) + g(x)h'(x) \\ &= 1(e^{3x}) + x(3e^{3x}) \\ &= e^{3x} + 3xe^{3x} \\ &= e^{3x}(3x + 1) \end{aligned}$$

b. $f(x) = \frac{x}{e^x}$

Solution: Use the product rule.

$$g(x) = x, \quad h(x) = \frac{1}{e^x}$$

Use the chain rule to solve $h'(x)$.

$$\begin{aligned} g(x) &= x & h(x) &= \frac{1}{e^x} \\ g'(x) &= 1 & h'(x) &= -e^{-x} \end{aligned}$$

$$\begin{aligned} f(x) &= g'(x)h(x) + g(x)h'(x) \\ &= 1(\frac{1}{e^x}) + x(-e^{-x}) \\ &= \frac{1}{e^x} - xe^{-x} \\ &= \frac{1}{e^x} - \frac{x}{e^x} \\ &= \frac{1-x}{e^x} \end{aligned}$$

c. $h(x) = \frac{x}{\log(x)}$

Solution: Use the quotient rule.

$$g(x) = x, \quad h(x) = \log(x)$$

$$\begin{aligned} f(x) &= x & g(x) &= \log(x) \\ f'(x) &= 1 & g'(x) &= \frac{1}{x} \end{aligned}$$

$$\begin{aligned} \left[\frac{f(x)}{g(x)}\right]' &= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \\ &= \frac{1(\log(x)) - x(\frac{1}{x})}{[\log(x)]^2} \\ &= \frac{\log(x) - 1}{[\log(x)]^2} \end{aligned}$$

3.8 Composite functions

For each of the following pairs of functions $g(x)$ and $h(z)$, write out the composite function $g(h[z])$ and $h(g[x])$. In each case, describe the domain of the composite function.

a. $g(x) = x^2 + 4, \quad h(z) = 5z - 1$

Solution:

$$\begin{aligned} g(h[z]) &= (5z-1)^2 + 4 \\ h(g[x]) &= 5(x^2+4) - 1 \\ &= 5x^2 + 20 - 1 \\ &= 5x^2 + 19 \end{aligned}$$

- Domain of $g(h[z])$ $x \in \Re$
- Domain of $h(g[x])$ $x \in \Re$

b. $g(x) = x^3, \quad h(z) = (z-1)(z+1)$

Solution:

$$\begin{aligned} g(h[z]) &= [(z-1)(z+1)]^3 \\ &= (z-1)^3(z+1)^3 \\ h(g[x]) &= (x^3-1)(x^3+1) \end{aligned}$$

- Domain of $g(h[z])$ $x \in \Re$
- Domain of $h(g[x])$ $x \in \Re$

c. $g(x) = 4x + 2, \quad h(z) = \frac{1}{4}(z-2)$

Solution:

$$\begin{aligned} g(h[z]) &= 4\left[\frac{1}{4}(z-2)\right] + 2 \\ &= (z-2) + 2 \\ &= z \\ h(g[x]) &= \frac{1}{4}([4x+2] - 2) \\ &= \frac{1}{4}(4x) \\ &= x \end{aligned}$$

- Domain of $g(h[z])$ $x \in \Re$
- Domain of $h(g[x])$ $x \in \Re$

d. $g(x) = \frac{1}{x}, \quad h(z) = z^2 + 1$

Solution:

$$\begin{aligned} g(h[z]) &= \frac{1}{z^2+1} \\ h(g[x]) &= \left(\frac{1}{x}\right)^2 + 1 \\ &= \frac{1}{x^2} + 1 \end{aligned}$$

- Domain of $g(h[z])$ $x \in \Re$
- Domain of $h(g[x])$ $x \in \Re : x \neq 0$

3.9 Chain rule

Use the chain rule to compute the derivative of the first three composite functions in the previous section from the derivatives of the two component functions. Then, compute each derivative directly using your expression for the composite function. Simplify and compare your answers.

a. $g(x) = x^2 + 4, \quad h(z) = 5z - 1$

Solution:

- Using component functions and the chain rule

$$g'(x) = 2x \quad h'(z) = 5$$

$$\begin{aligned} \frac{d}{dz}\{g(h[z])\} &= g'(h[z])h'(z) \\ &= 2(5z-1)(5) \\ &= 2(25z-5) \\ &= 50z - 10 \\ \frac{d}{dx}\{h(g[x])\} &= h'(g[x])g'(x) \\ &= 5(2x) \\ &= 10x \end{aligned}$$

- Using the composite function

$$\begin{aligned}
g(h[z]) &= (5z-1)^2+4 \\
&= 25z^2-10z+1+4 \\
&= 25z^2-10z+5 \\
\frac{d}{dz}g(h[z]) &= 50z-10 \\
h(g[x]) &= 5x^2+19 \\
\frac{d}{dx}h(g[x]) &= 10x
\end{aligned}$$

b. $g(x) = x^3, \quad h(z) = (z-1)(z+1)$

Solution:

- Using component functions and the chain rule

$$g'(x) = 3x^2 \quad h'(z) = 2z$$

$$\begin{aligned}
\frac{d}{dz}\{g(h[z])\} &= g'(h[z])h'(z) \\
&= 3[(z-1)(z+1)]^2(2z) \\
&= 3(z^2-1)^2(2z) \\
&= 6z(z^2-1)^2 \\
\frac{d}{dx}\{h(g[x])\} &= h'(g[x])g'(x) \\
&= 2(x^3)(3x^2) \\
&= 6x^5
\end{aligned}$$

- Using the composite function

$$\begin{aligned}
g(h[z]) &= (z-1)^3(z+1)^3 \\
&= (z-1)(z-1)(z-1)(z+1)(z+1)(z+1) \\
&= z^6-3z^4+3z^2-1 \\
\frac{d}{dz}g(h[z]) &= 6z^5-12z^3+6z \\
&= 6z(z^4-2z^2+1) \\
&= 6z(z^2-1)^2 \\
h(g[x]) &= (x^3-1)(x^3+1) \\
&= x^6-1 \\
\frac{d}{dx}h(g[x]) &= 6x^5
\end{aligned}$$

c. $g(x) = 4x + 2, \quad h(z) = \frac{1}{4}(z - 2)$

Solution:

- Using component functions and the chain rule

$$g'(x) = 4 \quad h'(z) = \frac{1}{4}$$

$$\begin{aligned}
\frac{d}{dz}\{g(h[z])\} &= g'(h[z])h'(z) \\
&= 4(\frac{1}{4}) \\
&= 1 \\
\frac{d}{dx}\{h(g[x])\} &= h'(g[x])g'(x) \\
&= \frac{1}{4}(4) \\
&= 1
\end{aligned}$$

- Using the composite function

$$\begin{aligned}
g(h[z]) &= z \\
\frac{d}{dz}g(h[z]) &= 1 \\
h(g[x]) &= x \\
\frac{d}{dx}h(g[x]) &= 1
\end{aligned}$$

Chapter 4

Critical points and approximation

4.1 Sketch a function

Sketch the graph of a function (any function you like, no need to specify a functional form) that is:

a. Continuous on $[0, 3]$ and has the following properties: an absolute minimum at 0, an absolute maximum at 3, a local maximum at 1 and a local minimum at 2.
b. Do the same for another function with the following properties: 2 is a **critical number** (i.e. $f'(x) = 0$ or $f'(x)$ is undefined), but there is no local minimum and no local maximum.

4.2 Find critical values

Find the critical values of these functions:

a. $f(x) = 5x^{3/2} - 4x$
b. $s(t) = 3t^4 + 4t^3 - 6t^2$
c. $f(r) = \dfrac{r}{r^2 + 1}$
d. $h(x) = x \log(x)$

4.3 Find absolute minimum/maximum values

Find the absolute minimum and absolute maximum values of the functions on the given interval:

a. $f(x) = 3x^2 - 12x + 5, [0, 3]$
b. $f(t) = t\sqrt{4 - t^2}, [-1, 4]$
c. $s(x) = x - \log(x), [1/2, 2]$
d. $h(p) = 1 - e^{-p}, [0, 1000]$

4.4 A function with no local minima/maxima

Demonstrate that the function $f(x) = x^5 + x^3 + x + 1$ has no local maximum and no local minimum.

4.5 Approximate root-finding

Show that the equation

$$x^7 - 6x + 4 = 0$$

has a root between 0 and 1.

a. Find an initial approximation by ignoring the term x^7.
b. Use Newton's method to find the root correct to 3 decimal places.

4.6 Apply the mean value theorem

Does a continuous, differentiable function exist on $[0, 2]$ such that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leq 2 \; \forall x$? Use the mean value theorem to explain your answer.