Assignment 2: Sequences, Limits, Derivatives, and Critical Points

Sequences

Write down the first three terms of each of the following sequences. In each case, state whether the sequence is an arithmetic progression, a geometric progression, or neither.[1]

a. $u_n = 4 + 3n$

b. $u_n = 4^n$

c. $u_n = n \times 3^n$

Find the limit

In each of the following cases, state whether the sequence $\{u_n\}$ tends to a limit, and find the limit if it exists:[2]

a. $u_n = 1 + \frac{1}{2}n$

b. $u_n = \left(\frac{1}{2}\right)^n$

Determine convergence or divergence

Determine whether each of the following sequences converges or diverges. If it converges, find the limit.[3]

a. $a_n = \frac{3+5n^2}{n+n^2}$

[1] Pemberton and Rau 5.1.1
[2] Pemberton and Rau 5.1.3
[3] Grimmer 2012 HW2.2

b. $a_n = \frac{(-1)^{n-1} n}{n^2+1}$

Find more limits

Given that

$$\lim_{x \to a} f(x) = -3, \quad \lim_{x \to a} g(x) = 0, \quad \lim_{x \to a} h(x) = 8$$

find the limits that exist. If the limit doesn't exist, explain why.[4]

a. $\lim_{x \to a} [f(x) + h(x)]$

b. $\lim_{x \to a} \sqrt[3]{h(x)}$

c. $\lim_{x \to a} \frac{g(x)}{f(x)}$

d. $\lim_{x \to a} \frac{f(x)}{g(x)}$

e. $\lim_{x \to a} \frac{2f(x)}{h(x)-f(x)}$

Find even more limits

Find the limits of the following:[5]

a. $\lim_{x \to -4} \frac{x^2+5x+4}{x^2+3x-4}$

b. $\lim_{x \to 4^-} \sqrt{16 - x^2}$

[4]Grimmer 2012 HW 2.4
[5]Grimmer 2012 HW 2.4(b)

Check for discontinuities

Which of the following functions are continuous? If not, where are the discontinuities?[6]

a. $f(x) = \dfrac{9x^3 - x}{(x-1)(x+1)}$

b. $f(x) = e^{-x^2}$

c. $f(x) = \begin{cases} x^3 + 1, & x > 0 \\ -x^2, & x \leq 0 \end{cases}$

Find finite limits

Find the following finite limits:[7]

a. $\lim\limits_{x \to 4} x^2 - 6x + 4$

b. $\lim\limits_{x \to 1} \left[\dfrac{x^4 - 1}{x - 1} \right]$

c. $\lim\limits_{x \to -4} \left[\dfrac{x^2 + 5x + 4}{x^2 + 3x - 4} \right]$

Find infinite limits

Find the following infinite limits:[8]

> Hint: use **L'Hôpital's Rule** to switch from $\lim\limits_{x \to \infty} \left(\dfrac{f(x)}{g(x)} \right)$ to $\lim\limits_{x \to \infty} \left(\dfrac{f'(x)}{g'(x)} \right)$.

a. $\lim\limits_{x \to \infty} \left[\dfrac{9x^2}{x^2 + 3} \right]$

b. $\lim\limits_{x \to \infty} \left[\dfrac{2^x - 3}{2^x + 1} \right]$

c. $\lim\limits_{x \to \infty} \left[\dfrac{3^x}{x^3} \right]$

[6] Gill 1.9

[7] a-d from Gill 5.1. e-h from Grimmer HW2.2

[8] Gill 5.3 and 5.8

Assessing continuity and differentiability

For each of the following functions, describe whether it is continuous and/or differentiable at the point of transition of its two formulas.[9]

a.

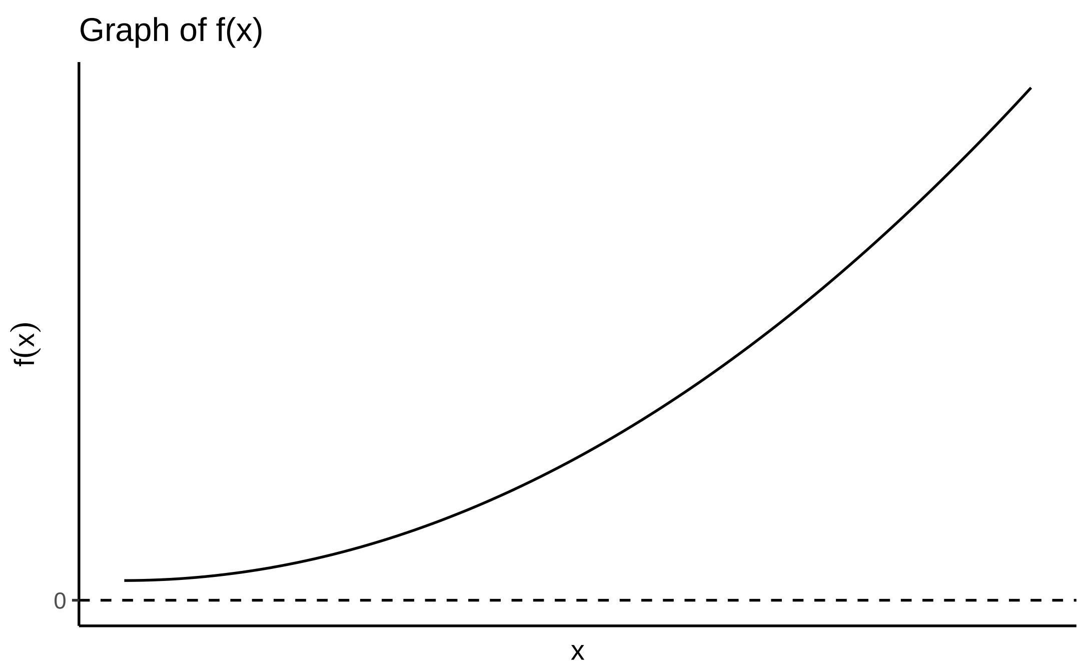
$$f(x) = \begin{cases} +x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$

b.

$$f(x) = \begin{cases} x^3, & x \leq 1 \\ x, & x > 1 \end{cases}$$

Possible derivative

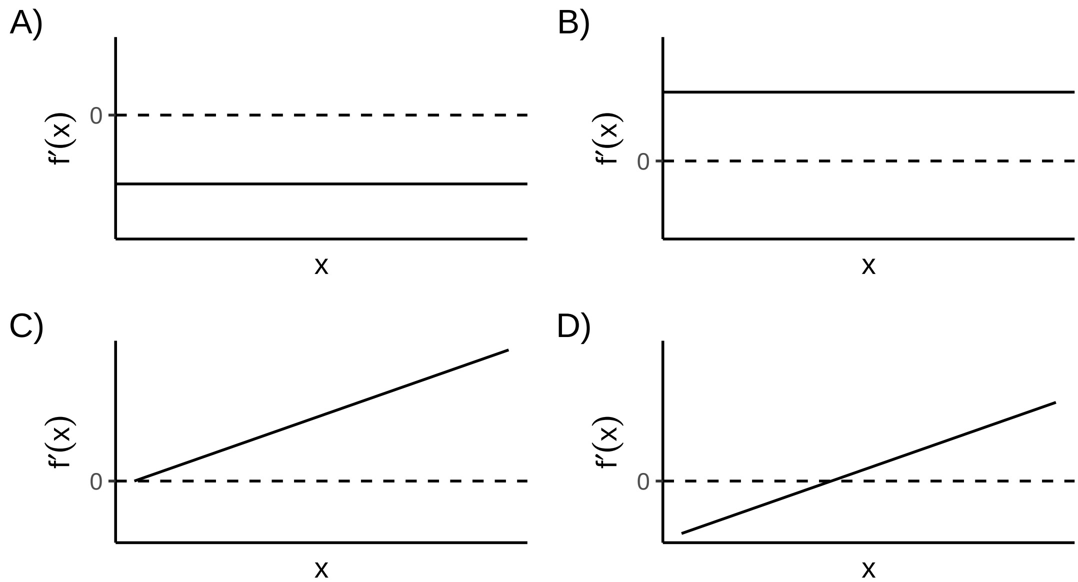
A friend shows you this graph of a function $f(x)$:[10]



Which of the following could be a graph of $f'(x)$? For each graph, explain why or why not it might be the derivative of $f(x)$.

[9]Simon and Blume 2.16
[10]Grimmer HW3.6

Potential derivatives



What if the figure below was the graph of $f(x)$? Which of the graphs might potentially be the derivative of $f(x)$ then?

Calculate derivatives

Differentiate the following functions:[11]

a. $f(x) = 4x^3 + 2x^2 + 5x + 11$

b. $y = \sqrt{30}$

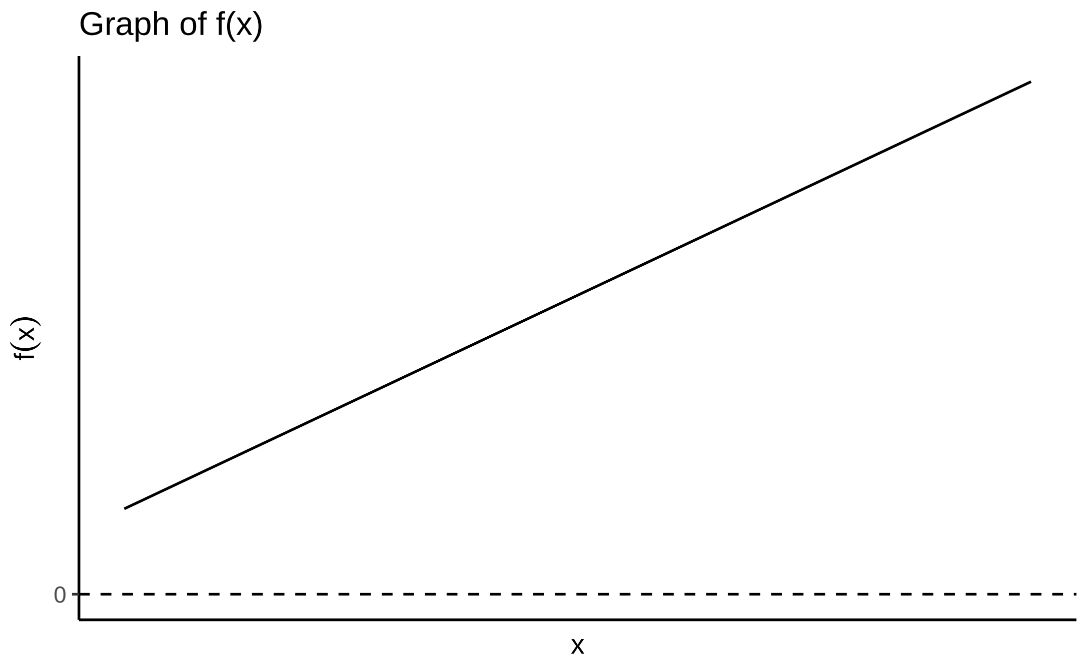
c. $h(t) = \log(9t + 1)$

d. $f(x) = \log(x^2 e^x)$

e. $h(y) = \left(\frac{1}{y^2} - \frac{3}{y^4}\right)(y + 5y^3)$

f. $g(t) = \frac{3t - 1}{2t + 1}$

[11] Grimmer HW2.3



Use the product and quotient rules

Differentiate the following function twice – once using the product rule, and once using the quotient rule:[12]

$$f(x) = \frac{x^2 - 2x}{x^4 + 6}$$

Logarithms and exponential functions

Compute the derivative of each of the following functions:[13]

a. $f(x) = xe^{3x}$
b. $f(x) = \frac{x}{e^x}$
c. $h(x) = \frac{x}{\log(x)}$

[12] Grimmer HW2.4
[13] Simon and Blume 5.8

Composite functions

For each of the following pairs of functions $g(x)$ and $h(z)$, write out the composite function $g(h[z])$ and $h(g[x])$. In each case, describe the domain of the composite function.[14]

a. $g(x) = x^2 + 4, \quad h(z) = 5z - 1$

b. $g(x) = x^3, \quad h(z) = (z - 1)(z + 1)$

Chain rule

Use the chain rule to compute the derivative of the composite functions in the previous section from the derivatives of the two component functions. Then, compute each derivative directly using your expression for the composite function. Simplify and compare your answers.[15]

a. $g(x) = x^2 + 4, \quad h(z) = 5z - 1$

b. $g(x) = x^3, \quad h(z) = (z - 1)(z + 1)$

c. $g(x) = 4x + 2, \quad h(z) = \frac{1}{4}(z - 2)$

[14]Simon and Blume 4.1

[15]Simon and Blume 4.3