Pset2 Solutions

August 30, 2024

Simplify logarithms

Express each of the following as a single logarithm.[1]

a. $\log(x) + \log(y) - \log(z)$

    1. Multiplication rule of logarithms: $\log(x \times y) = \log(x) + \log(y)$

    2. Division rule of logarithms: $\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$

    3. Applying the log rules, we combine logs that are added through multiplication and then combine logs that are subtracted with division.

$$\log(x) + \log(y) - \log(z) = \log(xy) - \log(z) = \log\left(\frac{xy}{z}\right)$$

b. $2\log(x) + 1$

    1. Exponentiation rule of logarithms: $\log(x^y) = y\log(x)$

    2. $\log(e) = 1$

$$2\log(x) + 1 = 2\log(x) + \log(e) = \log(x^2) + \log(e) = \log(ex^2)$$

c. $\log(x) - 2$

    1. $\log(e) = 1$

$$\log(x) - 2 = \log(x) - 2\log(e) = \log(x) - \log(e^2) = \log\left(\frac{x}{e^2}\right)$$

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[1] Grimmer HW1.3

Sequences

Write down the first three terms of each of the following sequences. In each case, state whether the sequence is an arithmetic progression, a geometric progression, or neither.[2]

a. $u_n = 5 + 3n$

$$8, \ 11, \ 14 \quad \text{(Arithmetic progression)}$$

b. $u_n = 3^n$

$$3, \ 9, \ 27 \quad \text{(Geometric progression)}$$

c. $u_n = n \times 3^n$

$$3, \ 18, \ 81 \quad \text{(Neither)}$$

Find the limit

In each of the following cases, state whether the sequence $\{u_n\}$ tends to a limit, and find the limit if it exists.[3]

a. $u_n = 1 + \frac{1}{2}n$

No limit $(u_n \to \infty)$

b. $u_n = \left(\frac{1}{2}\right)^n$

Yes. $\lim_{n \to \infty} u_n = 0$

c. $\lim_{x \to -4} \frac{x^2 + 5x + 4}{x^3 + 3x - 4}$

$$\lim_{x \to -4} \frac{x^2 + 5x + 4}{x^2 + 3x - 4} = \lim_{x \to -4} \frac{(x+4)(x+1)}{(x+4)(x-1)} = \lim_{x \to -4} \frac{x+1}{x-1} = \frac{\lim_{x \to -4}(x+1)}{\lim_{x \to -4}(x-1)} = \frac{-3}{-5} = \frac{3}{5}$$

Determine convergence or divergence

Determine whether each of the following sequences converges or diverges. If it converges, find the limit.[4]

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[2] Pemberton and Rau 5.1.1
[3] Pemberton and Rau 5.1.3
[4] Grimmer 2012 HW2.2

a. $a_n = \frac{3+5n^2}{n+n^2}$

    1. The sequence converges to 5. We can see this by factoring n^2 from both the numerator and denominator and then canceling it out.

$$\lim_{n\to\infty} a_n = \frac{3+5n^2}{n+n^2} = \lim_{n\to\infty} \frac{n^2\left(\frac{3}{n^2}+5\right)}{n^2\left(\frac{1}{n}+1\right)} = \lim_{n\to\infty} \frac{\left(\frac{3}{n^2}+5\right)}{\left(\frac{1}{n}+1\right)} = \frac{\lim_{n\to\infty}\frac{3}{n^2}+5}{\lim_{n\to\infty}\frac{1}{n}+1} = \frac{0+5}{0+1} = 5$$

As $n \to \infty$, $\frac{3}{n^2} \to 0$ and $\frac{1}{n} \to 0$. Therefore, $a_n \to 5$.

Alternatively, you could split the fraction into two terms: one with a numerator of 3, and the other with a numerator of $5n^2$. The first fraction converges to 0. Factoring out an n from both sides of the second fraction, you're left with $\frac{5n}{n+1}$; $\frac{n}{n+1}$ converges to 1, giving you $5 \times 1 = 5$.

b. $a_n = \frac{(-1)^{n-1}n}{n^2+1}$

    1. The sequence converges to 0. To see why, take the absolute value of the sequence, then factor out and cancel n from both sides of the fraction.

$$\lim_{n\to\infty} \left|\frac{(-1)^{n-1}n}{n^2+1}\right| = \lim_{n\to\infty} \frac{1^{n-1}n}{n^2+1} = \lim_{n\to\infty} \frac{1}{n+\frac{1}{n}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty}\left(n+\frac{1}{n}\right)} = \frac{1}{\lim_{n\to\infty} n+0} = 0$$

Find more limits

Given that

$$\lim_{x\to a} f(x) = -3, \quad \lim_{x\to a} g(x) = 0, \quad \lim_{x\to a} h(x) = 8$$

find the limits that exist. If the limit doesn't exist, explain why.[5]

a. $\lim_{x\to a}[f(x) + h(x)]$

$$\lim_{x\to a}[f(x) + h(x)] = -3 + 8 = 5$$

b. $\lim_{x\to a} \frac{f(x)}{g(x)}$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{-3}{0} = \text{Undefined}$$

Cannot divide by 0, no limit.

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[5]Grimmer 2012 HW 2.4

c. $\lim_{x \to a} \frac{2f(x)}{h(x)-f(x)}$

$$\lim_{x \to a} \frac{2f(x)}{h(x) - f(x)} = \frac{2 \times -3}{8 - (-3)} = -\frac{6}{11}$$

Check for discontinuities

Which of the following functions are continuous? If not, where are the discontinuities?[6]

a. $f(x) = \dfrac{9x^3 - x}{(x-1)(x+1)}$

   1. Discontinuous at $x = -1, +1$ (denominator would be 0, leaving the fraction undefined).

b. $f(x) = e^{-x^2}$

   1. Continuous for all real numbers.

Find finite limits

Find the following finite limits:[7]

a. $\lim_{x \to 1} \left[\dfrac{x^4 - 1}{x - 1} \right]$

   1. The key here is to factor the initial expression in the numerator, then cancel terms out with the denominator:

$$\lim_{x \to 1} \left[\frac{x^4 - 1}{x - 1} \right] = \lim_{x \to 1} \left[\frac{(x-1)(x+1)(x^2+1)}{x-1} \right]$$
$$= \lim_{x \to 1} [(x+1)(x^2+1)]$$
$$= (1+1)(1^2+1)$$
$$= (2)(2)$$
$$= 4$$

Alternatively, we can use L'Hôpital's Rule:

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[6] Gill 1.9

[7] a-d from Gill 5.1. e-h from Grimmer HW2.2

$$\lim_{x \to 1} \left[\frac{x^4 - 1}{x - 1} \right] = \lim_{x \to 1} \left[\frac{4x^3}{1} \right]$$
$$= \frac{4(1)^3}{1}$$
$$= 4$$

b. $\lim_{x \to -4} \left[\dfrac{x^2 + 5x + 4}{x^2 + 3x - 4} \right]$

1. The key here is to factor the initial expression:

$$\lim_{x \to -4} \frac{(x+4)(x+1)}{(x+4)(x-1)} = \lim_{x \to -4} \frac{x+1}{x-1}$$
$$= \frac{\lim_{x \to -4}(x+1)}{\lim_{x \to -4}(x-1)}$$
$$= \frac{-3}{-5}$$
$$= \frac{3}{5}$$

Find infinite limits

Find the following infinite limits.[8]

Hint: Use **L'Hôpital's Rule** to switch from $\lim_{x \to \infty} \left(\frac{f(x)}{g(x)} \right)$ to $\lim_{x \to \infty} \left(\frac{f'(x)}{g'(x)} \right)$.

a. $\lim_{x \to \infty} \left[\frac{9x^2}{x^2 + 3} \right]$

$$\lim_{x \to \infty} \left[\frac{9x^2}{x^2 + 3} \right] = \lim_{x \to \infty} \left[\frac{18x}{2x} \right]$$
$$= \lim_{x \to \infty} \left[\frac{18}{2} \right]$$
$$= 9$$

b. $\lim_{x \to \infty} \left[\frac{3^x}{x^3} \right]$

1. The trick here is to repeatedly calculate the derivative of the numerator and denominator until there is no x term in the denominator. You end up calculating the third derivative, but L'Hôpital's Rule still applies.

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[8]Gill 5.3 and 5.8

$$\lim_{x \to \infty} \left[\frac{3^x}{x^3} \right] = \lim_{x \to \infty} \left[\frac{\log(3)3^x}{3x^2} \right]$$

$$= \lim_{x \to \infty} \left[\frac{\log^2(3)3^x}{6x} \right]$$

$$= \lim_{x \to \infty} \left[\frac{\log^3(3)3^x}{6} \right]$$

$$= \frac{\log^3(3)3^\infty}{6}$$

$$= \infty$$

Assessing continuity and differentiability

For each of the following functions, describe whether it is continuous and/or differentiable at the point of transition of its two formulas.[9]

a. $f(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$

Solution:

$$f'(x) = \begin{cases} 2x, & x \geq 0 \\ -2x, & x < 0 \end{cases}$$

As x converges to 0 from both above and below, $f'(0)$ converges to 0, so the function is continuous and differentiable.

b. $f(x) = \begin{cases} x^3, & x \leq 1 \\ x, & x > 1 \end{cases}$

Solution: This function is continuous, since $\lim_{x \to 1} f(x) = 1$ no matter how the limit is taken. However, it is not differentiable since

$$f'(x) = \begin{cases} 3x^2, & x \leq 1 \\ 1, & x > 1 \end{cases}$$

$\lim_{x \to 1^+} f'(x) = 1$, whereas $\lim_{x \to 1^-} f'(x) = 3$. The function is not smooth and continuous at $f(1)$.

Potential Derivatives

Solution:

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[9]Simon and Blume 2.16

a.

A doesn't work because it is negative and the function we observe is increasing in x. B is constant, so this also won't work; the function we observe gets larger at an increasing, not constant rate. C seems to be a plausible candidate because an upward-sloping derivative would map to the behavior of the function we observe, that $g(x)$ gets large at an increasing rate. D does not work because it suggests the function would need to be decreasing over some interval, and when we refer back to $g(x)$, there doesn't seem to be any local minimum, maximum, or a saddle point despite the graph in D crossing 0.

b.

Again, A doesn't work because it is negative and the function we observe is increasing in x. B seems to be plausible as the derivative, since $g(x)$ appears to increase at a constant rate; its derivative should be flat and greater than 0. C won't work because the slope of $g(x)$ is constant and does not increase in x. D doesn't work, again because it suggests the function would need to be decreasing at some point over the interval we observe.

Calculate derivatives

Differentiate the following functions.[10]

a. $f(x) = 4x^3 + 2x^2 + 5x + 11$

Solution: Power rule.

$$f(x) = 4x^3 + 2x^2 + 5x + 11$$
$$f'(x) = 12x^2 + 4x + 5$$

b. $y = \sqrt{30}$

Solution: Derivative of a constant is 0.

$$y = \sqrt{30}$$
$$y' = 0$$

c. $h(t) = \log(9t + 1)$

Solution: Derivative of $\log(u)$ is $\frac{1}{u}$. Since u is a function in this problem, we need to apply the chain rule to calculate the derivative of $9t + 1$ and multiply that by $\frac{1}{9t + 1}$.

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[10]Grimmer HW2.3

$$h(t) = \log(9t + 1)$$
$$h'(t) = \frac{1}{9t + 1} \times 9$$

d. $f(x) = \log(x^2 e^x)$

Solution: Derivative of a logarithm plus the chain rule.

$$f(x) = \log(x^2 e^x)$$
$$f'(x) = \frac{1}{x^2 e^x} \times (2xe^x + e^x x^2)$$
$$= \frac{2xe^x + e^x x^2}{x^2 e^x}$$
$$= \frac{2}{x} + 1$$

e. $h(y) = \left(\frac{1}{y^2} - \frac{3}{y^4}\right)(y + 5y^3)$

Solution: Simplify the expression first, then basic application of the power rule.

$$h(y) = \left(\frac{1}{y^2} - \frac{3}{y^4}\right)(y + 5y^3)$$
$$= \frac{y}{y^2} + \frac{5y^3}{y^2} - \frac{3y}{y^4} - \frac{15y^3}{y^4}$$
$$= \frac{1}{y} + 5y - \frac{3}{y^3} - \frac{15}{y}$$
$$= 5y - \frac{14}{y} - \frac{3}{y^3}$$
$$h'(y) = 5 + \frac{14}{y^2} + \frac{9}{y^4}$$

f. $h(x) = \frac{x}{\log(x)}$

Solution: Use the quotient rule.

$$g(x) = x, \quad h(x) = \log(x)$$
$$f'(x) = 1, \quad g'(x) = \frac{1}{x}$$

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$= \frac{1(\log(x)) - x\left(\frac{1}{x}\right)}{[\log(x)]^2}$$

$$= \frac{\log(x) - 1}{[\log(x)]^2}$$

Use the product and quotient rules

Differentiate the following function twice – once using the product rule, and once using the quotient rule.[11]

$$f(x) = \frac{x^2 - 2x}{x^4 + 6}$$

Solution:

a. First let's use the quotient rule:

$$h(x) = \frac{f(x)}{g(x)}$$

$$f(x) = x^2 - 2x$$

$$g(x) = x^4 + 6$$

$$f'(x) = 2x - 2$$

$$g'(x) = 4x^3$$

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$= \frac{(2x - 2)(x^4 + 6) - (x^2 - 2x)(4x^3)}{(x^4 + 6)^2}$$

$$= \frac{2x^5 + 12x - 2x^4 - 12 - 4x^5 + 8x^4}{(x^4 + 6)^2}$$

$$= \frac{-2x^5 + 6x^4 + 12x - 12}{(x^4 + 6)^2}$$

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[11] Grimmer HW2.4

b. Now we can do the same thing with the product rule:

$$j(x) = k(x)m(x)$$
$$k(x) = x^2 - 2x$$
$$m(x) = (x^4 + 6)^{-1}$$
$$k'(x) = 2x - 2$$
$$m'(x) = -(x^4 + 6)^{-2}(4x^3) = -\frac{4x^3}{(x^4 + 6)^2}$$
$$j'(x) = k(x)m'(x) + k'(x)m(x)$$
$$= (x^2 - 2x)\left(-\frac{4x^3}{(x^4 + 6)^2}\right) + (2x - 2)(x^4 + 6)^{-1}$$
$$= -\frac{(x^2 - 2x)(4x^3)}{(x^4 + 6)^2} + \frac{2x - 2}{x^4 + 6}$$
$$= -\frac{4x^5 - 8x^4}{(x^4 + 6)^2} + \frac{2x - 2}{x^4 + 6}$$
$$= -\frac{4x^5 - 8x^4}{(x^4 + 6)^2} + \frac{2x - 2}{x^4 + 6} \times \frac{x^4 + 6}{x^4 + 6}$$
$$= -\frac{4x^5 - 8x^4}{(x^4 + 6)^2} + \frac{2x^5 + 12x - 2x^4 - 12}{(x^4 + 6)^2}$$
$$= \frac{2x^5 + 12x - 2x^4 - 12 - 4x^5 - 8x^4}{(x^4 + 6)^2}$$
$$= \frac{-2x^5 + 6x^4 + 12x - 12}{(x^4 + 6)^2}$$

The quotient rule is simply a derivation of the product rule combined with the chain rule:

$$h(x) = \frac{f(x)}{g(x)}$$
$$= f(x)g(x)^{-1}$$

Apply product and chain rules:

$$h'(x) = f'(x)g(x)^{-1} + f(x)(-1)g(x)^{-2}g'(x)$$
$$= f'(x)g(x)g(x)^{-2} - f(x)g(x)^{-2}g'(x)$$
$$= [f'(x)g(x) - f(x)g'(x)]g(x)^{-2}$$
$$= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

which is the quotient rule.

Composite functions

For each of the following pairs of functions $g(x)$ and $h(z)$, write out the composite function $g(h[z])$ and $h(g[x])$. In each case, describe the domain of the composite function.[12]

a. $g(x) = x^2 + 4, \quad h(z) = 5z - 1$

Solution:

$$g(h[z]) = (5z - 1)^2 + 4$$
$$h(g[x]) = 5(x^2 + 4) - 1$$
$$= 5x^2 + 20 - 1$$
$$= 5x^2 + 19$$

- Domain of $g(h[z])$: $z \in \mathbb{R}$
- Domain of $h(g[x])$: $x \in \mathbb{R}$

b. $g(x) = x^3, \quad h(z) = (z - 1)(z + 1)$

Solution:

$$g(h[z]) = [(z - 1)(z + 1)]^3$$
$$= (z - 1)^3 (z + 1)^3$$
$$h(g[x]) = (x^3 - 1)(x^3 + 1)$$

- Domain of $g(h[z])$: $z \in \mathbb{R}$
- Domain of $h(g[x])$: $x \in \mathbb{R}$

Chain rule

Use the chain rule to compute the derivative of the composite functions in the previous section from the derivatives of the two component functions. Then, compute each derivative directly using your expression for the composite function. Simplify and compare your answers.[13]

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[12]Simon and Blume 4.1
[13]Simon and Blume 4.3

a. $g(x) = x^2 + 4, \quad h(z) = 5z - 1$

Solution:

    Using component functions and the chain rule

$$g'(x) = 2x, \quad h'(z) = 5$$

$$\frac{d}{dz}\{g(h[z])\} = g'(h[z])h'(z)$$
$$= 2(5z - 1)(5)$$
$$= 2(25z - 5)$$
$$= 50z - 10$$

$$\frac{d}{dx}\{h(g[x])\} = h'(g[x])g'(x)$$
$$= 5(2x)$$
$$= 10x$$

    Using the composite function

$$g(h[z]) = (5z - 1)^2 + 4$$
$$= 25z^2 - 10z + 1 + 4$$
$$= 25z^2 - 10z + 5$$

$$\frac{d}{dz}g(h[z]) = 50z - 10$$
$$h(g[x]) = 5x^2 + 19$$
$$\frac{d}{dx}h(g[x]) = 10x$$

b. $g(x) = x^3, \quad h(z) = (z - 1)(z + 1)$

Solution:

    Using component functions and the chain rule

$$g'(x) = 3x^2, \quad h'(z) = 2z$$

$$\frac{d}{dz}\{g(h[z])\} = g'(h[z])h'(z)$$
$$= 3[(z-1)(z+1)]^2(2z)$$
$$= 3(z^2-1)^2(2z)$$
$$= 6z(z^2-1)^2$$

$$\frac{d}{dx}\{h(g[x])\} = h'(g[x])g'(x)$$
$$= 2(x^3)(3x^2)$$
$$= 6x^5$$

Using the composite function

$$g(h[z]) = (z-1)^3(z+1)^3$$
$$= (z-1)(z-1)(z-1)(z+1)(z+1)(z+1)$$
$$= z^6 - 3z^4 + 3z^2 - 1$$

$$\frac{d}{dz}g(h[z]) = 6z^5 - 12z^3 + 6z$$
$$= 6z(z^4 - 2z^2 + 1)$$
$$= 6z(z^2-1)^2$$

$$h(g[x]) = (x^3-1)(x^3+1)$$
$$= x^6 - 1$$

$$\frac{d}{dx}h(g[x]) = 6x^5$$

c. $g(x) = 4x + 2, \quad h(z) = \frac{1}{4}(z-2)$

Solution:

Using component functions and the chain rule

$$g'(x) = 4, \quad h'(z) = \frac{1}{4}$$

$$\frac{d}{dz}\{g(h[z])\} = g'(h[z])h'(z)$$
$$= 4 \times \frac{1}{4}$$
$$= 1$$

$$\frac{d}{dx}\{h(g[x])\} = h'(g[x])g'(x)$$
$$= \frac{1}{4} \times 4$$
$$= 1$$

Using the composite function

$$g(h[z]) = z$$
$$\frac{d}{dz}g(h[z]) = 1$$
$$h(g[x]) = x$$
$$\frac{d}{dx}h(g[x]) = 1$$