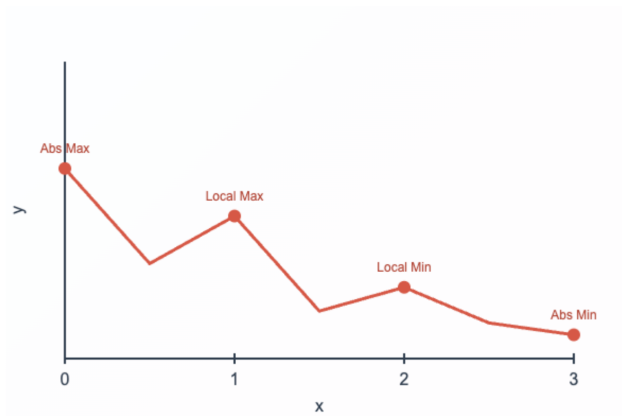
PSET 3: Critical points and approximation

1 Assignment Qs

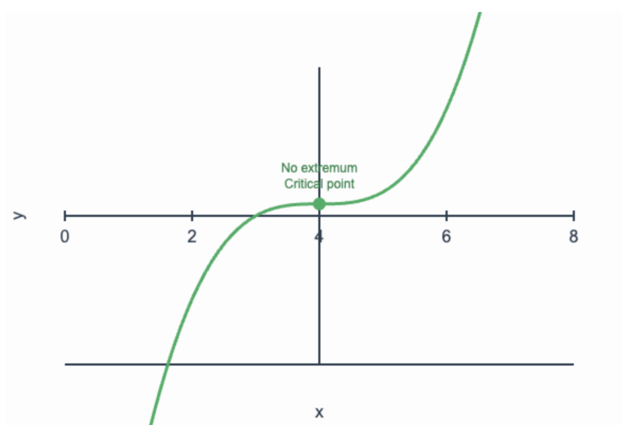
2 Sketch a function

Sketch the graph of a function (any function you like, no need to specify a functional form) that is:[1]

a. Continuous on $[0, 3]$ and has the following properties: an absolute maximum at 0, an absolute minimum at 3, a local maximum at 1 and a local minimum at 2.



b. Do the same for another function with the following properties: 4 is a **critical number** (i.e. $f'(x) = 0$ or $f'(x)$ is undefined), but there is no local minimum and no local maximum.



3 Find critical values

Find the critical values of these functions:[2]

[1] inspired by Grimmer HW3.1
[2] inspired by Grimmer HW3.2

a. $f(x) = 5x^{2/3} - 4x$
The derivative is $f'(x) = 5 \cdot \frac{2}{3}x^{-1/3} - 4 = \frac{10}{3x^{1/3}} - 4$. Critical values occur where $f'(x) = 0$ or $f'(x)$ is undefined.

- $f'(x)$ is undefined at $x = 0$.
- Set $f'(x) = 0$: $\frac{10}{3x^{1/3}} = 4 \implies 10 = 12x^{1/3} \implies x^{1/3} = \frac{10}{12} = \frac{5}{6} \implies x = \left(\frac{5}{6}\right)^3 = \frac{125}{216}$.

The critical values are $x = 0$ and $x = \frac{125}{216}$.

b. $s(t) = 3t^4 - 4t^3 + 6t^2$
The derivative is $s'(t) = 12t^3 - 12t^2 + 12t$. Since this is a polynomial, it is defined everywhere. Set $s'(t) = 0$: $12t^3 - 12t^2 + 12t = 0 \implies 12t(t^2 - t + 1) = 0$. One solution is $t = 0$. For the quadratic factor $t^2 - t + 1$, the discriminant is $\Delta = b^2 - 4ac = (-1)^2 - 4(1)(1) = -3 < 0$, so there are no real roots. The only critical value is $t = 0$.

c. $f(r) = \dfrac{r}{r^2 + r + 1}$
The denominator $r^2 + r + 1$ has a discriminant of $\Delta = 1^2 - 4(1)(1) = -3 < 0$, so it is never zero and the function is defined for all real r. Using the quotient rule: $f'(r) = \frac{(1)(r^2+r+1) - r(2r+1)}{(r^2+r+1)^2} = \frac{r^2+r+1-2r^2-r}{(r^2+r+1)^2} = \frac{-r^2+1}{(r^2+r+1)^2}$. The derivative is defined everywhere. Set $f'(r) = 0$: $-r^2 + 1 = 0 \implies r^2 = 1 \implies r = \pm 1$. The critical values are $r = 1$ and $r = -1$.

d. $h(x) = x \ln(x)$
The domain is $x > 0$. The derivative is $h'(x) = (1)\ln(x) + x(\frac{1}{x}) = \ln(x) + 1$. The derivative is defined for all x in the domain. Set $h'(x) = 0$: $\ln(x) + 1 = 0 \implies \ln(x) = -1 \implies x = e^{-1} = \frac{1}{e}$. The only critical value is $x = \frac{1}{e}$.

4 Find absolute minimum/maximum values

Find the absolute minimum and absolute maximum values of the functions on the given interval:[3]

a. $f(x) = 3x^2 - 12x + 5, [0, 1]$
Find critical points: $f'(x) = 6x - 12$. Set $f'(x) = 0 \implies 6x = 12 \implies x = 2$. This is outside the interval $[0, 1]$. We only need to check the endpoints:

- $f(0) = 3(0)^2 - 12(0) + 5 = 5$
- $f(1) = 3(1)^2 - 12(1) + 5 = 3 - 12 + 5 = -4$

Absolute maximum is 5 at $x = 0$. Absolute minimum is -4 at $x = 1$.

b. $f(t) = t^2\sqrt{9 - t^2}, [-1, 4]$
The domain of $f(t)$ is $9 - t^2 \geq 0 \implies t^2 \leq 9 \implies -3 \leq t \leq 3$. The given interval is $[-1, 4]$, so we must consider the intersection, which is $[-1, 3]$. Find the derivative: $f'(t) = 2t\sqrt{9 - t^2} + t^2 \cdot \frac{-2t}{2\sqrt{9-t^2}} = \frac{2t(9-t^2)-t^3}{\sqrt{9-t^2}} = \frac{18t-3t^3}{\sqrt{9-t^2}}$. Critical points:

- $f'(t)$ is undefined at $t = \pm 3$. $t = 3$ is an endpoint.
- $f'(t) = 0 \implies 18t - 3t^3 = 0 \implies 3t(6 - t^2) = 0$. This gives $t = 0$ or $t^2 = 6 \implies t = \pm\sqrt{6}$.

The critical points in the interval $(-1, 3)$ are $t = 0$ and $t = \sqrt{6}$ (since $\sqrt{6} \approx 2.45$). Evaluate the function at critical points and endpoints of $[-1, 3]$:

- $f(-1) = (-1)^2\sqrt{9 - 1} = \sqrt{8} = 2\sqrt{2} \approx 2.828$
- $f(0) = 0^2\sqrt{9 - 0} = 0$
- $f(\sqrt{6}) = (\sqrt{6})^2\sqrt{9 - 6} = 6\sqrt{3} \approx 10.392$

[3] inspired by Grimmer HW3.3

- $f(3) = 3^2\sqrt{9-9} = 0$

Absolute maximum is $6\sqrt{3}$ at $t = \sqrt{6}$. Absolute minimum is 0 at $t = 0$ and $t = 3$.

c. $s(x) = x - \ln(x), [1/2, 2]$
The domain is $x > 0$, which includes the interval $[1/2, 2]$. Find critical points: $s'(x) = 1 - \frac{1}{x}$. Set $s'(x) = 0 \implies 1 - \frac{1}{x} = 0 \implies x = 1$. This is in our interval. Evaluate the function at the critical point and endpoints:

- $s(1/2) = \frac{1}{2} - \ln(1/2) = \frac{1}{2} + \ln(2) \approx 0.5 + 0.693 = 1.193$
- $s(1) = 1 - \ln(1) = 1 - 0 = 1$
- $s(2) = 2 - \ln(2) \approx 2 - 0.693 = 1.307$

Absolute minimum is 1 at $x = 1$. Absolute maximum is $2 - \ln(2)$ at $x = 2$.

5 Approximate root-finding

Show that the equation

$$x^7 + 6x - 4 = 0$$

has a root between 0 and 1.[4]

Let $f(x) = x^7 + 6x - 4$. As a polynomial, $f(x)$ is continuous everywhere. We evaluate the function at the endpoints of the interval $[0, 1]$:

- $f(0) = 0^7 + 6(0) - 4 = -4$
- $f(1) = 1^7 + 6(1) - 4 = 3$

Since $f(0) < 0$ and $f(1) > 0$, by the Intermediate Value Theorem, there must exist a number $c \in (0, 1)$ such that $f(c) = 0$.

a. Find an initial approximation by ignoring the term x^7.
If we ignore the x^7 term (which is small for $x \in (0, 1)$), the equation becomes $6x - 4 = 0$. Solving for x gives $6x = 4$, so $x = \frac{4}{6} = \frac{2}{3}$. Our initial approximation is $x_0 = \frac{2}{3}$.

b. Use Newton's method to find the root correct to 3 decimal places.
Newton's method formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. We have $f(x) = x^7 + 6x - 4$ and $f'(x) = 7x^6 + 6$. Starting with $x_0 = 2/3 \approx 0.66667$:

- **Iteration 1:**
$x_1 = x_0 - \frac{x_0^7 + 6x_0 - 4}{7x_0^6 + 6} = 0.66667 - \frac{(0.66667)^7 + 6(0.66667) - 4}{7(0.66667)^6 + 6}$
$x_1 = 0.66667 - \frac{0.05854 + 4.00002 - 4}{7(0.08781) + 6} = 0.66667 - \frac{0.05856}{6.61467} \approx 0.66667 - 0.00885 \approx 0.65782$
- **Iteration 2:**
$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.65782 - \frac{(0.65782)^7 + 6(0.65782) - 4}{7(0.65782)^6 + 6}$
$x_2 = 0.65782 - \frac{0.05207 + 3.94692 - 4}{7(0.07915) + 6} = 0.65782 - \frac{-0.00101}{6.55405} \approx 0.65782 + 0.000154 \approx 0.65797$
- **Iteration 3:**
$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.65797 - \frac{(0.65797)^7 + 6(0.65797) - 4}{7(0.65797)^6 + 6}$
$x_3 = 0.65797 - \frac{0.05217 + 3.94782 - 4}{7(0.07930) + 6} = 0.65797 - \frac{-0.00001}{6.5551} \approx 0.65797 + 0.0000015 \approx 0.65797$

Since x_2 and x_3 agree to 5 decimal places, the root correct to 3 decimal places is 0.658.

[4] inspired by Pemberton and Rau 10.1.3

6 Apply the mean value theorem

Does a continuous, differentiable function exist on $[0, 4]$ such that $f(0) = -1$, $f(4) = 4$, and $f'(x) \leq 2 \forall x$? Use the mean value theorem to explain your answer.[5]

Yes, such a function exists.

The Mean Value Theorem (MVT) states that if a function f is continuous on $[a, b]$ and differentiable on (a, b), then there exists some $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

In our case, $a = 0$, $b = 4$, $f(0) = -1$, and $f(4) = 4$. The function is stated to be continuous and differentiable on the required intervals. According to the MVT, if such a function exists, there must be a point $c \in (0, 4)$ where the instantaneous rate of change equals the average rate of change.

The average rate of change is:

$$\frac{f(4) - f(0)}{4 - 0} = \frac{4 - (-1)}{4} = \frac{5}{4} = 1.25$$

So, the MVT guarantees that for such a function to exist, there must be a point c in $(0, 4)$ where $f'(c) = 1.25$.

The given condition is that $f'(x) \leq 2$ for all x. Since the required value for the derivative at point c is 1.25, and $1.25 \leq 2$, there is no contradiction. The condition can be satisfied.

For example, the linear function $f(x) = \frac{5}{4}x - 1$ satisfies all conditions:

- It is continuous and differentiable everywhere.
- $f(0) = \frac{5}{4}(0) - 1 = -1$.
- $f(4) = \frac{5}{4}(4) - 1 = 5 - 1 = 4$.
- $f'(x) = \frac{5}{4} = 1.25$, which is less than or equal to 2 for all x.

Therefore, such a function exists.

6.1 Optional!: Finding Max/Min

a. **OPTIONAL** $h(p) = 1 - e^{-p}, [0, 1000]$
The derivative is $h'(p) = -e^{-p}(-1) = e^{-p}$. The exponential function e^x is never zero, so $h'(p)$ is never zero. Thus, there are no critical points where the derivative is zero. The derivative is defined everywhere. We only need to check the endpoints of the interval $[0, 1000]$:

- $h(0) = 1 - e^0 = 1 - 1 = 0$
- $h(1000) = 1 - e^{-1000} = 1 - \frac{1}{e^{1000}}$

Since e^{-1000} is a very small positive number, $1 - e^{-1000}$ is slightly less than 1. The absolute minimum is 0 at $p = 0$. The absolute maximum is $1 - e^{-1000}$ at $p = 1000$.

b. **OPTIONAL** Demonstrate that the function $f(x) = x^5 + x^3 + x + 1$ has no local maximum and no local minimum.[6]
Local extrema can only occur at critical numbers. We find the derivative to locate any critical numbers:

$$f'(x) = 5x^4 + 3x^2 + 1$$

To find critical numbers, we set $f'(x) = 0$. However, notice that $x^4 \geq 0$ and $x^2 \geq 0$ for all real x. Therefore, $5x^4 \geq 0$ and $3x^2 \geq 0$. This means $f'(x) = 5x^4 + 3x^2 + 1 \geq 5(0) + 3(0) + 1 = 1$. Since $f'(x) \geq 1$ for all x, the derivative is never zero. The function is strictly increasing for all real numbers. A function that is strictly increasing on its entire domain has no local maximums or minimums.

[5]inspired by Grimmer HW3.5
[6]inspired by Grimmer HW3.4

6.2 AI and Resources statement

- As an AI model (Gemini), I generated these solutions based on my training data. I did not consult any external websites, academic papers, or individuals while preparing this problem set. The solutions are entirely my own work.