# Vectors, Matricies, and Operations

Computational Mathematics and Statistics Camp
University of Chicago
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1. Perform the following vector multiplication operations:

a. $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b & c \end{bmatrix}'$

$$\begin{aligned}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b & c \end{bmatrix}' &= (1)a + (1)b + 1(c) \\ &= a + b + c\end{aligned}$$

b. $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} a & b & c \end{bmatrix}$

$$\begin{aligned}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} a & b & c \end{bmatrix} &= [(1)(c) - (1)(b), (1)(a) - (1)(c), (1)(b) - (1)(a)] \\ &= [c - b, c - a, b - a]\end{aligned}$$

c. $\begin{bmatrix} -1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 3 & 12 \end{bmatrix}'$

$$\begin{aligned}\begin{bmatrix} -1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 3 & 12 \end{bmatrix}' &= (-1)(4) + (1)(3) + (-1)(12) \\ &= -13\end{aligned}$$

d. $\begin{bmatrix} -1 & 1 & -1 \end{bmatrix} \times \begin{bmatrix} 4 & 3 & 12 \end{bmatrix}$

$$\begin{aligned}\begin{bmatrix} -1 & 1 & -1 \end{bmatrix} \times \begin{bmatrix} 4 & 3 & 12 \end{bmatrix} &= [(1)(12) - (-1)(3), (-1)(4) - (-1)(12), (-1)(3) - (1)(4)] \\ &= [12 + 3, -4 + 12, -3 - 4] \\ &= [15, 8, -7]\end{aligned}$$

e. $\begin{bmatrix} 0 & 9 & 0 & 11 \end{bmatrix} \cdot \begin{bmatrix} 123.98211 & 6 & -6392.38743 & -5 \end{bmatrix}'$

$$\begin{aligned}\begin{bmatrix} 0 & 9 & 0 & 11 \end{bmatrix} \cdot \begin{bmatrix} 123.98211 & 6 & -6392.38743 & -5 \end{bmatrix}' &= (0)(123.98211) + (9)(6) + (0)(-6392.38743) + (11)(- \\ &= 0 + 54 + 0 - 55 \\ &= -1\end{aligned}$$

f. $\begin{bmatrix} 123.98211 & 6 & -6392.38743 & -5 \end{bmatrix} \cdot \begin{bmatrix} 0 & 9 & 0 & 11 \end{bmatrix}'$

$$\begin{aligned}\begin{bmatrix} 123.98211 & 6 & -6392.38743 & -5 \end{bmatrix} \cdot \begin{bmatrix} 0 & 9 & 0 & 11 \end{bmatrix}' &= (123.98211)(0) + (6)(9) + (-6392.38743)(0) + (-5)(1 \\ &= 0 + 54 + 0 - 55 \\ &= -1\end{aligned}$$

2. Find the length of the following vectors:

a. $(3, 4)$

$$\begin{aligned}\sqrt{3^2+4^2} &= \sqrt{9+16}\\ &= \sqrt{25}\\ &= 5\end{aligned}$$

b. $(0, -3)$

$$\begin{aligned}\sqrt{0^2+(-3)^2} &= \sqrt{0+9}\\ &= \sqrt{9}\\ &= 3\end{aligned}$$

c. $(1, 1, 1)$

$$\begin{aligned}\sqrt{1^2+1^2+1^2} &= \sqrt{1+1+1}\\ &= \sqrt{3}\end{aligned}$$

d. $(3, 3)$

$$\begin{aligned}\sqrt{3^2+3^2} &= \sqrt{9+9}\\ &= \sqrt{18}\\ &= \sqrt{3^2\times 2}\\ &= 3\sqrt{2}\end{aligned}$$

e. $(-1, -1)$

$$\begin{aligned}\sqrt{(-1)^2+(-1)^2} &= \sqrt{1+1}\\ &= \sqrt{2}\end{aligned}$$

f. $(1, 2, 3)$

$$\begin{aligned}\sqrt{1^2+2^2+3^2} &= \sqrt{1+4+9}\\ &= \sqrt{14}\end{aligned}$$

g. $(2, 0)$

$$\begin{aligned}\sqrt{2^2+0^2} &= \sqrt{4+0}\\ &= \sqrt{4}\\ &= 2\end{aligned}$$

h. $(1, 2, 3, 4)$

$$\begin{aligned}\sqrt{1^2+2^2+3^2+4^2} &= \sqrt{1+4+9+16}\\ &= \sqrt{30}\\ &= 2\end{aligned}$$

i. $(3, 0, 0, 0, 0)$

$$\begin{aligned}\sqrt{3^2+0^2+0^2+0^2+0^2} &= \sqrt{9+0+0+0+0}\\ &= \sqrt{3}\\ &= 3\end{aligned}$$

3. Recall a property of the **law of cosines**:

$$\cos(\theta) = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|}$$

where $\theta$ is the angle from $\mathbf{w}$ to $\mathbf{v}$ measured in radians. Of importance, arccos() is the inverse of cos():

$$\theta = \arccos\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|}\right)$$

For each of the following pairs of vectors, calculate the angle between them. Report your answers in both radians and degrees. To convert between radians and degrees:[1]

$$\text{Degrees} = \text{Radians} \times \frac{180^o}{\pi}$$

a. $\mathbf{v} = (1, 0), \quad \mathbf{w} = (2, 2)$

$$\begin{aligned}\mathbf{v}\cdot\mathbf{w} &= (1)(2)+(0)(2)\\ &= 2+0\\ &= 2\\ \|\mathbf{v}\| &= \sqrt{1^2+0^2}\\ &= \sqrt{1+0}\\ &= \sqrt{1}\\ &= 1\\ \|\mathbf{w}\| &= \sqrt{2^2+2^2}\\ &= \sqrt{4+4}\\ &= \sqrt{8}\\ &= \sqrt{2^2\times 2}\\ &= 2\sqrt{2}\\ \theta &= \arccos\left(\frac{2}{1(2\sqrt{2})}\right)\\ &= \frac{\pi}{4}\\ &= 45^\circ\end{aligned}$$

b. $\mathbf{v} = (4, 1), \quad \mathbf{w} = (2, -8)$

[1] Simon and Blume 10.12

$$
\begin{aligned}
\mathbf{v} \cdot \mathbf{w} &= (4)(2) + (1)(-8) \\
&= 8 + (-8) \\
&= 0 \\
\|\mathbf{v}\| &= \sqrt{4^2 + 1^2} \\
&= \sqrt{16 + 1} \\
&= \sqrt{17} \\
&= 1 \\
\|\mathbf{w}\| &= \sqrt{2^2 + (-8)^2} \\
&= \sqrt{4 + 64} \\
&= \sqrt{68} \\
&= \sqrt{2^2 \times 17} \\
&= 2\sqrt{17} \\
\theta &= \arccos\left(\frac{0}{1(2\sqrt{17})}\right) \\
&= \frac{\pi}{2} \\
&= 90^\circ
\end{aligned}
$$

Note: you could stop after solving $\mathbf{v} \cdot \mathbf{w}$, because the denominator will be irrelevant.

c. $\mathbf{v} = (1, 1, 0), \quad \mathbf{w} = (1, 2, 1)$

$$
\begin{aligned}
\mathbf{v} \cdot \mathbf{w} &= (1)(1) + (1)(2) + (0)(1) \\
&= 1 + 2 + 0 \\
&= 3 \\
\|\mathbf{v}\| &= \sqrt{1^2 + 1^2 + 0^2} \\
&= \sqrt{1 + 1 + 0} \\
&= \sqrt{2} \\
\|\mathbf{w}\| &= \sqrt{1^2 + 2^2 + 1^2} \\
&= \sqrt{1 + 4 + 1} \\
&= \sqrt{6} \\
\theta &= \arccos\left(\frac{3}{\sqrt{2}(\sqrt{6})}\right) \\
&= \arccos\left(\frac{3}{\sqrt{2 \times 6}}\right) \\
&= \arccos\left(\frac{3}{\sqrt{12}}\right) \\
&= \arccos\left(\frac{3}{\sqrt{2^2 \times 3}}\right) \\
&= \arccos\left(\frac{3}{2\sqrt{3}}\right) \\
&= \arccos\left(\frac{3\sqrt{3}}{2\sqrt{3}\sqrt{3}}\right) \\
&= \arccos\left(\frac{3\sqrt{3}}{2 \times 3}\right) \\
&= \arccos\left(\frac{\sqrt{3}}{2}\right) \\
&= \frac{\pi}{6} \\
&= 30^\circ
\end{aligned}
$$

d. $\mathbf{v} = (1, -1, 0), \quad \mathbf{w} = (1, 2, 1)$

$$
\begin{aligned}
\mathbf{v} \cdot \mathbf{w} &= (1)(1) + (-1)(2) + (0)(1) \\
&= 1 - 2 + 0 \\
&= -1 \\
\|\mathbf{v}\| &= \sqrt{1^2 + (-1)^2 + 0^2} \\
&= \sqrt{1 + 1 + 0} \\
&= \sqrt{2} \\
\|\mathbf{w}\| &= \sqrt{1^2 + 2^2 + 1^2} \\
&= \sqrt{1 + 4 + 1} \\
&= \sqrt{6} \\
\theta &= \arccos\left(\frac{-1}{\sqrt{2}(\sqrt{6})}\right) \\
&= \arccos\left(\frac{-1}{\sqrt{2 \times 6}}\right) \\
&= \arccos\left(\frac{-1}{\sqrt{12}}\right) \\
&= \arccos\left(\frac{-1}{\sqrt{2^2 \times 3}}\right) \\
&= \arccos\left(\frac{-1}{2\sqrt{3}}\right) \\
&\approx 106.8^\circ
\end{aligned}
$$

e. $\mathbf{v} = (1, 0, 0, 0, 0), \quad \mathbf{w} = (1, 1, 1, 1, 1)$

$$
\begin{aligned}
\mathbf{v} \cdot \mathbf{w} &= (1)(0) + (0)(1) + (0)(1) + (0)(1) + (0)(1) + (0)(1) \\
&= 1 + 0 + 0 + 0 + 0 \\
&= 1 \\
\|\mathbf{v}\| &= \sqrt{1^2 + 0^2 + 0^2 + 0^2 + 0^2} \\
&= \sqrt{1 + 0 + 0 + 0 + 0} \\
&= \sqrt{1} \\
&= 1 \\
\|\mathbf{w}\| &= \sqrt{1^2 + 1^2 + 1^2 + 1^2 + 1^2} \\
&= \sqrt{1 + 1 + 1 + 1 + 1} \\
&= \sqrt{5} \\
\theta &= \arccos\left(\frac{1}{1(\sqrt{5})}\right) \\
&= \arccos\left(\frac{1}{\sqrt{5}}\right) \\
&\approx 63.4^\circ
\end{aligned}
$$

4. Using the matrices below, calculate the following. Some may not be defined; if that is the case, say so.

$$\mathbf{A} = \begin{bmatrix} 3 \\ -2 \\ 9 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 7 & -1 & 5 \\ 0 & 2 & -4 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 3 & 1 \\ 3 & 4 \\ 3 & -7 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 0 & -4 \\ -2 & 1 & -6 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 4 & 1 & -5 \\ 0 & 7 & 7 \\ 2 & -3 & 0 \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} 2 & -8 & -5 \\ -3 & 7 & -4 \\ 1 & 0 & 3 \\ 1 & 2 & 6 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} 9 \\ -2 \\ -1 \\ 0 \end{bmatrix} \quad \mathbf{L} = \begin{bmatrix} 5 & 0 & 3 & 1 \end{bmatrix}$$

a. $\mathbf{A} + \mathbf{B}$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 3+8 \\ -2+0 \\ 9+(-1) \end{bmatrix} = \begin{bmatrix} 11 \\ -2 \\ 8 \end{bmatrix}$$

b. $-\mathbf{G}$

$$-\mathbf{G} = (-1)\begin{bmatrix} 2 & -8 & -5 \\ -3 & 7 & -4 \\ 1 & 0 & 3 \\ 1 & 2 & 6 \end{bmatrix} = \begin{bmatrix} -2 & 8 & 5 \\ 3 & -7 & 4 \\ -1 & 0 & -3 \\ -1 & -2 & -6 \end{bmatrix}$$

c. $\mathbf{D}'$

$$\mathbf{D}' = \begin{bmatrix} 3 & 3 & 3 \\ 1 & 4 & -7 \end{bmatrix}$$

d. $\mathbf{C} + \mathbf{D}$

$\mathbf{C} + \mathbf{D}$ does not exist. The matricies are not the same dimensions.

e. $3\mathbf{C} - 2\mathbf{D}'$

$$\begin{aligned} 3\mathbf{C} - 2\mathbf{D}' &= (3)\begin{bmatrix} 7 & -1 & 5 \\ 0 & 2 & -4 \end{bmatrix} - (2)\begin{bmatrix} 3 & 3 & 3 \\ 1 & 4 & -7 \end{bmatrix} \\ &= \begin{bmatrix} 21 & -3 & 15 \\ 0 & 6 & -12 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ 2 & 8 & -14 \end{bmatrix} \\ &= \begin{bmatrix} 15 & -9 & 9 \\ -2 & -2 & 2 \end{bmatrix} \end{aligned}$$

f. $\mathbf{A}' \cdot \mathbf{B}$

$$\mathbf{A} \cdot \mathbf{B} = 3(8) + (-2)(0) + 9(-1) = 24 + 0 - 9 = 15$$

g. $\mathbf{CB}$

$$
\begin{aligned}
\mathbf{CB} &= \begin{bmatrix} 7 & -1 & 5 \\ 0 & 2 & -4 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 7(8) & + & (-1)(0) & + & 5(-1) \\ 0(8) & + & 2(0) & + & (-4)(-1) \end{bmatrix} \\
&= \begin{bmatrix} 56+0-5 \\ 0+0+4 \end{bmatrix} \\
&= \begin{bmatrix} 51 \\ 4 \end{bmatrix}
\end{aligned}
$$

h. **BC**

**BC** does not exist. The matricies are non-conformable.

i. **FB**

$$
\begin{aligned}
\mathbf{FB} &= \begin{bmatrix} 4 & 1 & -5 \\ 0 & 7 & 7 \\ 2 & -3 & 0 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 4(8) & + & 1(0) & + & (-5)(-1) \\ 0(8) & + & 7(0) & + & 7(-1) \\ 2(8) & + & (-3)(0) & + & 0(-1) \end{bmatrix} \\
&= \begin{bmatrix} 32+0+5 \\ 0+0-7 \\ 16+0+0 \end{bmatrix} \\
&= \begin{bmatrix} 37 \\ -7 \\ 16 \end{bmatrix}
\end{aligned}
$$

j. **EF**

(Vertical and horizontal lines are added in the work here to clearly delineate each term.)

$$
\begin{aligned}
\mathbf{EF} &= \left[\begin{array}{ccccc|ccccc|ccccc}
5(4) & + & 2(0) & + & 3(2) & 5(1) & + & 2(7) & + & 3(-3) & 5(-5) & + & 2(7) & + & 3(0) \\ \hline
1(4) & + & 0(0) & + & (-4)(2) & 1(1) & + & 0(7) & + & (-4)(-3) & 1(-5) & + & 0(7) & + & (-4)(0) \\ \hline
-2(4) & + & 1(0) & + & (-6)(2) & -2(1) & + & 1(7) & + & (-6)(-3) & -2(-5) & + & 1(7) & + & (-6)(0)
\end{array}\right] \\
&= \begin{bmatrix} 20+0+6 & 5+14-9 & -25+14+0 \\ 4+0-8 & 1+0+12 & -5+0+0 \\ -8+0-12 & -2+7+18 & 10+7+0 \end{bmatrix} \\
&= \begin{bmatrix} 26 & 10 & -11 \\ -4 & 13 & -5 \\ -20 & 23 & 17 \end{bmatrix}
\end{aligned}
$$

k. $\mathbf{K} \cdot \mathbf{L}'$

$$
\mathbf{K} \cdot \mathbf{L}' = \begin{bmatrix} 9 \\ -2 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 0 \\ 3 \\ 1 \end{bmatrix} = 9(5) + (-2)(0) + (-1)(3) + 0(1) = 45 + 0 - 3 + 0 = 42
$$

l. $\|\mathbf{K}\|$

$$\|\mathbf{K}\| = \sqrt{\mathbf{K} \cdot \mathbf{K}} = \sqrt{9^2 + (-2)^2 + (-1)^2 + 0^2} = \sqrt{81 + 4 + 1 + 0} = \sqrt{86}$$

m. $\mathbf{G}'$

$$\mathbf{G}' = \begin{bmatrix} 2 & -8 & -5 \\ -3 & 7 & -4 \\ 1 & 0 & 3 \\ 1 & 2 & 6 \end{bmatrix}' = \begin{bmatrix} 2 & -3 & 1 & 1 \\ -8 & 7 & 0 & 2 \\ -5 & -4 & 3 & 6 \end{bmatrix}$$

n. $\mathbf{E} - 5\mathbf{I_3}$

$$\begin{aligned} \mathbf{E} - 5\mathbf{I}_3 &= \begin{bmatrix} 5 & 2 & 3 \\ 1 & 0 & -4 \\ -2 & 1 & -6 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 2 & 3 \\ 1 & 0 & -4 \\ -2 & 1 & -6 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2 & 3 \\ 1 & -5 & -4 \\ -2 & 1 & -11 \end{bmatrix} \end{aligned}$$

5. Prove the additive property of matrix transposition:

$$(\mathbf{X} + \mathbf{Y})' = \mathbf{X}' + \mathbf{Y}'$$

This is a relatively simple proof, and there are a lot of different ways to do it. The most straightforward way is simply to write out each of the two matrices and observe that they are identical. Without loss of generality, suppose $\mathbf{X}$ and $\mathbf{Y}$ are $m \times n$ matrices. We know that

$$\mathbf{X} + \mathbf{Y} = \begin{bmatrix} x_{11} + y_{11} & \dots & x_{1n} + y_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} + y_{m1} & \dots & x_{mn} + y_{mn} \end{bmatrix}$$

Then it must be that

$$(\mathbf{X} + \mathbf{Y})' = \begin{bmatrix} x_{11} + y_{11} & \dots & x_{m1} + y_{m1} \\ \vdots & \ddots & \vdots \\ x_{1n} + y_{1n} & \dots & x_{mn} + y_{mn} \end{bmatrix}$$

Now let's consider the right hand side of the equation. Since

$$\mathbf{X}' = \begin{bmatrix} x_{11} & \dots & x_{m1} \\ \vdots & \ddots & \vdots \\ x_{1n} & \dots & x_{mn} \end{bmatrix} \qquad \mathbf{Y} = \begin{bmatrix} y_{11} & \dots & y_{m1} \\ \vdots & \ddots & \vdots \\ y_{1n} & \dots & y_{mn} \end{bmatrix}$$

we know that

$$\mathbf{X}' + \mathbf{Y}' = \begin{bmatrix} x_{11} + y_{11} & \dots & x_{m1} + y_{m1} \\ \vdots & \ddots & \vdots \\ x_{1n} + y_{1n} & \dots & x_{mn} + y_{mn} \end{bmatrix}$$

which is the same as $(\mathbf{X} + \mathbf{Y})'$.

6. For two vectors in $\mathfrak{R}^3$ using

$$1 = \cos^2\theta + \sin^2\theta$$

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$$

show that the norm of the cross product between two vectors, $\mathbf{u}$ and $\mathbf{v}$, is:

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \sin(\theta)$$

- Recall from the **cosine rule**[2] the property that:

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}$$

- Square both sides.

$$\cos^2(\theta) = \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{u}\|^2 \, \|\mathbf{v}\|^2}$$

- Substitute $1 - \sin^2(\theta)$ for $cos^2(\theta)$.

$$1 - \sin^2(\theta) = \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{u}\|^2 \, \|\mathbf{v}\|^2}$$

- Multiply both sides by $\|\mathbf{u}\|^2 \, \|\mathbf{v}\|^2$.

$$\|\mathbf{u}\|^2 \, \|\mathbf{v}\|^2 - \|\mathbf{u}\|^2 \, \|\mathbf{v}\|^2 \sin^2(\theta) = (\mathbf{u} \cdot \mathbf{v})^2$$

- Isolate $\|\mathbf{u}\|^2 \, \|\mathbf{v}\|^2 \sin^2(\theta)$ on the right hand side.

$$\|\mathbf{u}\|^2 \, \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2 = \|\mathbf{u}\|^2 \, \|\mathbf{v}\|^2 \sin^2(\theta)$$

- Apply the **multiplication norm** of the standard vector norm: $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \, \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \, \|\mathbf{v}\|^2 \sin^2(\theta)$$

- Square root both sides of the equation.

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \sin(\theta)$$

[2] Gill 3.2.1