PS and Sociology Math Prefresher

Math Camp

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Agenda

- Exponents + logs (review)
- Derivatives: intro
- Derivatives FUN EXCITING RULES (chain rule! quotient rule!)

Exponents, Exponentials, Exponential functions

Recall from earlier this week:

Exponents

Exponents are where you take a variable to some power – e.g. x^a where x is a variable and a is a constant. Typically, we focus on the numerical portion of the exponent–calling it 'the exponent'.

Exponential

An exponential is the reverse of the exponent – here it is a number to the power of a variable, e.g. a^x. To get the x 'down', we need to use logarithms (aka logs).

Exponential Function

The exponential function has a particular base, e, (where e is Euler's e and is approx 2.72.)

Functions: Quadratic Functions and Polynomials

Quadratic functions : highest order (largest exponent) is equal to 2.
$y = \alpha + \beta_1 x + \beta_2 x^2$.

Higher order polynomials: more of the same, but now the highest order can be anything. Some examples include:
$y = \alpha + \beta_1 x + \beta_2 x^3$. and $y = \alpha + \beta_1 x + \beta_2 x^2 + \beta_3 x^3$.

We care because we might see these as utility functions or use higher order values as a way to represent the relationship between two variables in a linear regression.

Logs and other functions

Logs are the inverses of exponents: the power to which you raise the base, e.g. 10, to produce a given value, e.g.z **You will see logs!**

- Logarithms (typically base 10 ($log(x)$) or base e ($ln(e)$), but any base is possible, e.g. $log_{8675309}x$ (Bases aside from e and 10 will be specified).
  - $y = log(z) \leftrightarrow 10^y = z$
  - $y = ln(z) \leftrightarrow e^y = z$
- $log(1) = 0$

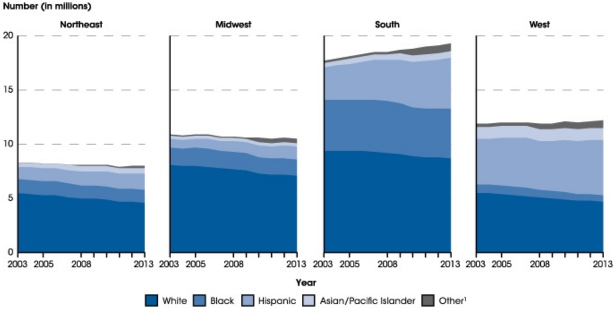
Exponents in log are different from what you might expect:

- $log(x^2) = 2(log(x))$
- $log(x/y) = log(x) - log(y)$ provided $(x, y > 0)$

Logs help weigh smaller values more heavily; adding units not linear–less meaningful for larger values ($log(100) = 2, log(1000) = 3$).

Education Enrollment in the US

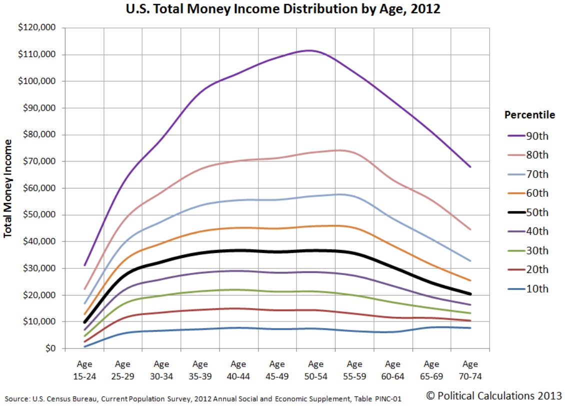
What trends in education can we surmise from these graphs?



Source: **NCES** `http://nces.ed.gov/programs/coe/indicator_cge.asp`

Income by age

At what point is your income increasing the fastest? When do earnings slow down? When do they peak?



Source: Political Calculations

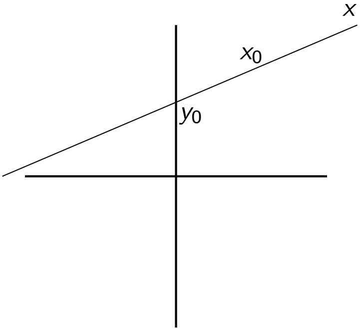
In these instances, and in many, many, *many* others, we will care about **rates of change**.

These rates may be between two points (AKA discrete change) or the rate may be at a particular point (AKA instantaneous change). We get at this by calculating the **derivative**, which we denote by $\frac{dy}{dx}$ or $f'(x)$. Both work and both mean the same.

Derivatives: Discrete Change

Slope: rate of change between two points.

$y = a + bx = y_0 + b(x - x_0)$, intercept y_0 or a.

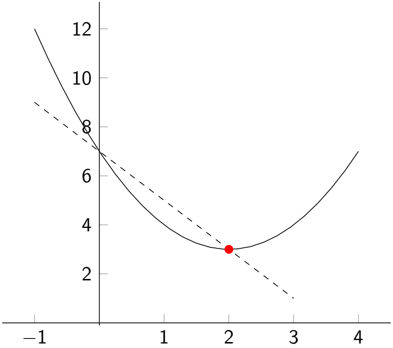


Change: Review

Derivatives allow us to focus upon rate of change.

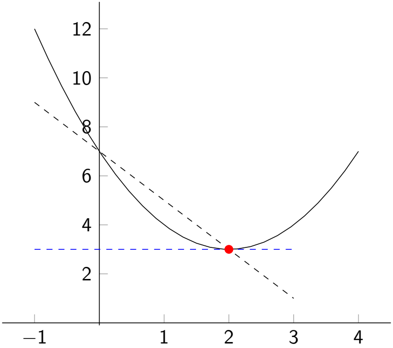
- Notation: $f'(x)$ or $\frac{dy}{dx}$
- Discrete change: time between two points
- First difference: difference between the value at time=$t-1$ to time=t
- Instantaneous change: rate of change at a particular *moment*

As the interval of change gets smaller, we approach a measure of instantaneous change



Formally, use limits to calculate this (there they are again!).

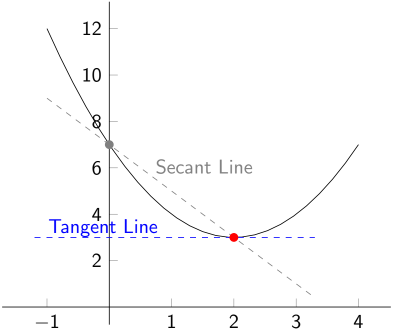
As the interval of change gets smaller, we approach a measure of instantaneous change



Formally, use limits to calculate this (there they are again!).

Secants and Tangents

Secant: slope between two points (intersects two points on a curve)
Tangent: touches the curve at any given point
As the interval of change gets smaller, we approach a measure of instantaneous change



Derivatives

To calculate the derivative, begin with the secant formula (discrete change between points), reducing the difference to some arbitrarily small value, h. As h goes to zero, we go from discrete (secant) to instantaneous (tangent).

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Example: $3x$

$$\lim_{h \to 0} \frac{3(x+h) - 3x}{h} = \lim_{h \to 0} \frac{3x + 3h - 3x}{h} = \lim_{h \to 0} \frac{3h}{h} = \lim_{h \to 0} 3 = 3$$

Try: $2x$, x^2

Note: we're using composition here! Hello, day 1!

Derivatives: $2x$

To calculate the derivative, begin with the secant formula (discrete change between points), reducing the difference to some arbitrarily small value, h. As h goes to zero, we go from discrete (secant) to instantaneous (tangent).

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Example: $2x$

$$\lim_{h \to 0} \frac{2(x+h) - 2x}{h} = \lim_{h \to 0} \frac{2x + 2h - 2x}{h} = \lim_{h \to 0} \frac{2h}{h} = \lim_{h \to 0} 2 = 2$$

Derivatives: x^2

To calculate the derivative, begin with the secant formula (discrete change between points), reducing the difference to some arbitrarily small value, h. As h goes to zero, we go from discrete (secant) to instantaneous (tangent).

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Example: x^2

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + 2h^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + 2h^2}{h}$$

$$= \lim_{h \to 0} 2x + 2h = 2x + 0 = 2x$$

Derivative as information: Rate of change

OK great – we know derivatives tell us about rates of change. So what? Why does this matter?

They give us the information about the function's *rate of change* which again matters because we can know more about the relationship between x and y – e.g. more x is always, sometimes or never associated with more y.

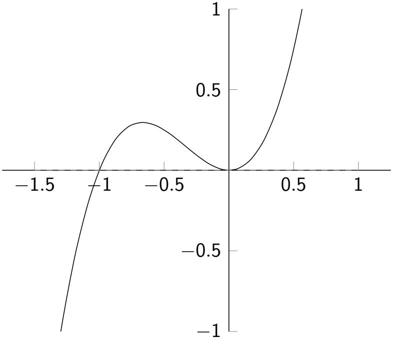
Derivative as information: Rate of change

The rate of change can tell us whether the function is increasing, decreasing or at a max/min. .

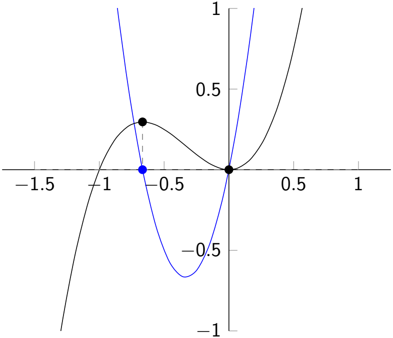
- What if derivative is positive? function is increasing
- Negative? function is decreasing
- Zero? max or min

Derivative of a function: max and min

Where are the maxima and/or minima of the function?



Derivative of a function: max and min



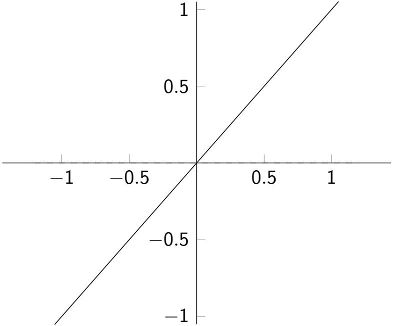
Behaving Badly

As we saw before: the function must be *continuous* on the *interval* to be differentiable

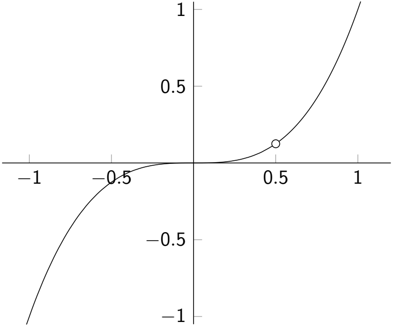
Some functions not differentiable or not differentiable at a certain point–not “well behaved” functions

Continuous function: draw without picking up a pencil

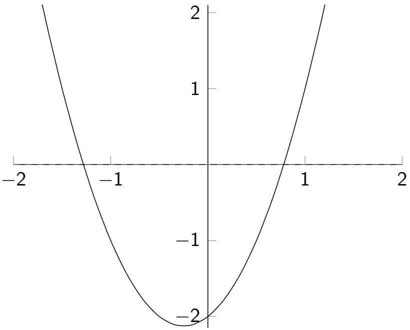
YES



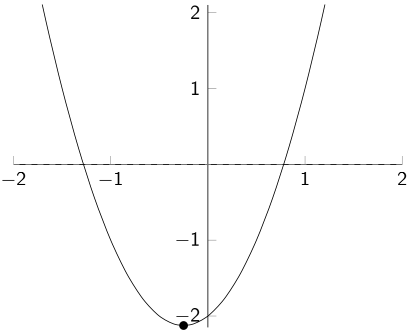
NO



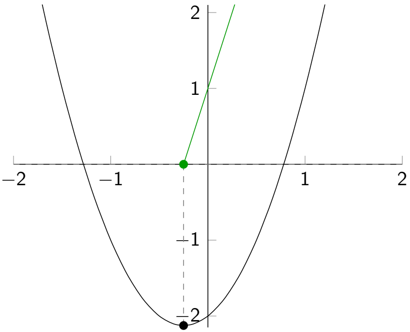
Derivative of this function



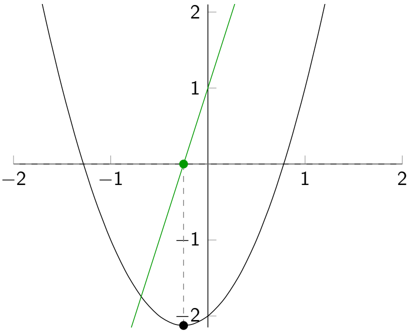
Derivative of this function



Derivative of this function



Derivative of this function



Derivatives: Calculation

While we can calculate the derivative using the formula from before, it's a bit tedious. Can't there be another way?

Think about it: do you want to do $4x^3 + 3x - 2$ using $\frac{f(x_0+h)-f(x_0)}{h}$?

Derivative Rules

We refer to derivative of $f(x)$ as $f'(x)$ below with constant k:

1. $f(k * x) = k * f(x), f'(k * x) = k * f'(x)$
   - $f(x) = 3x, f'(x) = 3$
2. $f(x) = k$ has derivative $f'(x) = 0$
   - $f(x) = 4$, $f'(x) = 0$
3. $f(x) = x^n, f'(x) = n * x^{n-1}$
   - $f(x) = x^3$, $f'(x) = 3x^2$
4. $[f(x) + g(x)]' = f'(x) + g'(x)$
   - $f(x) = 3x, g(x) = 7$, $3 + 0 = 3$
5. $[f(x) - g(x)]' = f'(x) - g'(x)$
   - $[3x - 7]'$, $3 - 0 = 3$

NOTE: $[f(x) * g(x)]! = f'(x) * g'(x)$ Ex: $(3x * 10x)'! = 30$

Derivatives Two Ways

We can check these handy formulas work as they should. Let's try. Find the derivative of $f(x) = \frac{1}{x}$

Formal Definition:

$$f'(x) = lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$
$$f'(x) = lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$
$$f'(x) = lim_{h\to 0}\frac{\frac{x}{x(x+h)}-\frac{(x+h)}{x(x+h)}}{h}$$
$$f'(x) = lim_{h\to 0}\frac{\frac{-h}{x(x+h)}}{h}$$
$$f'(x) = lim_{h\to 0}\frac{-1}{x(x+h)}$$
$$f'(x) = \frac{-1}{x(x+0)}$$
$$f'(x) = \frac{-1}{x^2}$$
$$f'(x) = -x^{-2}$$

Rules:

$$f(x) = x^n, f'(x) = n * x^{n-1}$$
$$f(x) = x^{-1}$$
$$f'(x) = -1 * x^{-2}$$
$$f'(x) = -x^{-2}$$

Practice Problems

Find where functions are continuous and find derivatives

- $f(x) = 5$,$f'(x) = 0$
- $f(x) = 3x - 7$,$f'(x) = 3$
- $f(x) = 3x^2$,$f'(x) = 6x$
- $f(x) = \frac{x^2}{x}$,$f'(x) = 1$
- $f(s) = s^{-2}$,$f'(s) = -2s^{-3}$ (not continuous at $s = 0$
- $f(y) = y(y + 7)(y - 3)$,$f'(y) = 3y^2 + 8y - 21$
- $f(z) = \frac{z^2-5z-6}{z+1}$,$f'(z) = 1$ (not continuous at $z = -1$)

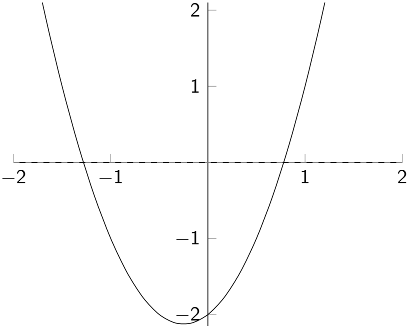
Higher Order Derivatives

Second derivatives (n^{th} derivatives): take a derivative a second (n^{th}) time
Rate of change of rate of change (velocity vs acceleration)

- $f(x) = x^5 + 3x^3 + 2x + 8$
- $f'(x) = 5x^4 + 9x^2 + 2$
- $f''(x) = 20x^3 + 18x$

Critical Points

Critical points occur where the derivative is zero. We can find them by graphing (ocular method) or plugging in values after calculating the derivative.



Critical Points

When the derivative is zero: can be local max or min
Try:

- $x^2 + 4x$: max or min? Where is the critical point? Min, critical point at $x = -2$
- -$x^2 + 4x$: max or min? Where is the critical point? Max, critical point at $x = 2$

Critical Points, part 2

Sometimes there are multiple critical points

- How many? (max) number of critical points is the highest degree of the derivative (same as finding zeroes)
- How to find? take the derivative and set to zero

Try: $f(x) = x^4 - 16x^2$

(up to) Three critical points.
$f'(x) = 4x^3 - 16x = 4x(x^2 - 4)$.
Zeroes at -2, 0, 2

Partial Derivatives

Similar to ‘regular’ derivative; treat additional variable(s) as constants. Written as ∂_x or $\frac{\partial f}{\partial x}(x, ...)$

THIS IS IMPORTANT FOR INTERACTION TERMS!

Ex: $y = 3xz$, $\partial_x = 3z$

Find ∂_x
$f(x, z) = 7xz + 4x^2 + z$ $\partial_x = 7z + 8x$
$f(x, y) = x + 4y$ $\partial_x = 1$

Brambor, Clark, and Golder (05)

Partial derivatives show how rate of change moves with another variable. What is expected change of Y in relation to X?

$Y = \beta_0 + \beta_1 X + \beta_2 Z + \beta_3 XZ + \epsilon$

$\partial_X = \beta_1 + \beta_3 Z$

How is that different from just β_1 or just β_3?

Brambor, Clark, and Golder (05)

The effect is $\partial_X = \beta_1 + \beta_3 Z$. Now, suppose Z can be 0 or 1.

$$Y = \beta_0 + \beta_1 X + \beta_2 Z + \beta_3 XZ + \varepsilon$$

Hypothesis H_1: An increase in X is associated with an increase in Y when condition Z is met, but not when condition Z is absent.

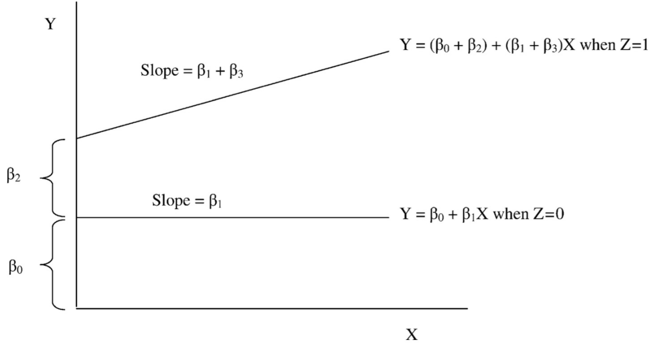


Fig. 1 A graphical illustration of an interaction model consistent with hypothesis H_1.

(Additional) Rules for derivatives

So far, we've just had simple functions but what if we are multiplying, dividing, or have an otherwise-more-advanced function?

- Product Rule: $f(x) * g(x)$
- Quotient Rule: $\frac{f(x)}{g(x)}$
- Chain Rule: $f(g(x))$
- Other: eg, exponentials: e^x, $ln(x)$

(Additional) Rules for derivatives

So far, we've just had simple functions but what if we are multiplying, dividing, or have an otherwise-more-advanced function?

- Product Rule: $f(x) * g(x)$, $x^3 * x^2$
- Quotient Rule: $\frac{f(x)}{g(x)}$, $\frac{x^4+3x}{x^2}$
- Chain Rule: $f(g(x))$, $(x^2+1)^3$ (composition!)
- Other: eg, exponentials: e^x, $ln(x)$

Product Rule

If we have two things multiplied together and need the derivative, we have two options: multiply everything and then take the derivative OR use the product rule.

Example: (3x+4)(x+2)
To take the derivative using the product rule, we do the following: $f'(x) * g(x) + g'(x)f(x)$. For us, our two functions are $f(x) = 3x + 4$ and $g(x) = x + 2$. The derivatives are $f'(x) = 3$ and $g'(x) = 1$. Then, we substitute these in:
$f'(x) * g(x) + g'(x)f(x) = 3(x + 2) + 1(3x + 4)$ Simplify to get: $3x + 6 + 3x + 4 = 6x + 10$.

To take the derivative using our previous approach, we first multiply: $3x^2 + 4x + 6x + 8 = 3x^2 + 10x + 8$. Then, just take the derivative of each term: $f'(x) = 6x + 10$.

WHY DO WITH THE PRODUCT RULE??

Product Rule

Suppose that instead you had $(3x^2 + 3x + 4)(x^3 + 2x^2 + x + 2)$. Now the product rule is looking a little nicer!

$f(x) = (3x^2 + 3x + 4)$ and $g(x) = (x^3 + 2x^2 + x + 2)$
$f'(x) = 6x + 3$ and $g'(x) = 3x^2 + 4x + 1$.

We can substitute this into the formula: $f'(x) * g(x) + g'(x)f(x)$ $(6x + 3)(x^3 + 2x^2 + x + 2) + (3x^2 + 4x + 1)(3x^2 + 3x + 4)$. This is a mess – but you have your answer at least (and much easier than doing it the long way)!

Quotient Rule

There are two ways to think about the quotient rule: a) you have something divided by something else or b) you have something multiplied by something to a negative power (chain rule, next up!) Example: $\frac{3x^2}{x+2}$. Forumla is $\frac{f'(x)*g(x)-g'(x)f(x)}{(g(x))^2}$ So, we identify the following: $f(x) = 3x^2$ and $g(x) = x + 2$ $f'(x) = 6x$ and $g'(x) = 1$. Plug in to get $\frac{6x(x+2)-1(3x^2)}{(x+2)^2} = \frac{6x^2+12x-3x^2}{(x+2)^2} = \frac{3x^2+12x}{(x+2)^2}$

Practice: $\frac{x-4}{x+5}$ $\frac{3x^3}{x+2}$
Ans: $\frac{9}{(x+5)^2}$ $\frac{6x^3+18x^2}{(x+2)^2}$

Chain Rule

Sometimes, you have a function to a power: $f(g(x)) = (x+3)^3$. We can use the chain rule to evaluate this. What we do is we take the derivative of the function and multiply it by the derivative of the inside: $f'(g(x)) * g'(x)$. So, for our example: $f(x) = x^3$ and $g(x) = (x+3)$.

The derivative of each is $f'(x) = 3x^2$ and $g'(x) = 1$. We substitute in to get: $3(x+3)^2 * 1$.

Try: $f(x) = (2x^2 + 8x)^4$ $f(x) = (9x - x^2)^6$
Ans $4(4x+8)(2x^2+8x)^3$
$6(9-2x)(9x-x^2)^5 = (54-12x)(9x-2x^2)^5$

Exponentials: e and ln

You can take the derivative of continuous functions – including those with a log and/or e in them. The rules are a little hard, but once you learn them, it's not too bad:

$f(x) = e^x \quad f'(x) = e^x$ (a favorite of mine)

$f(x) = e^{g(x)} \quad f'(x) = e^{g(x)} * g'(x)$

$f(x) = a^x \quad f'(x) = a^x(ln(a))$ (used rarely, if ever)

$f(x) = ln(x) \quad f'(x) = \frac{1}{x}$

$f(x) = ln(g(x)) \quad f'(x) = \frac{1}{x} * g'(x)$

$f(x) = log_a(x) \quad f'(x) = \frac{1}{xln(a)}$ (rarely used)

You can make these more complicated by including a function of x. How would we take the derivative in that case? Chain rule!

EX: $ln(3x)$ $f'(x) = \frac{1}{3x} * 3 = \frac{1}{x}$.

TRY: $ln(x^2)$, e^{2x} **ANS:** $\frac{2}{x}$ **and** $2e^{2x}$

Additional Resources

- Daniel Kleitman's Calculus for Beginners and Artists: `www-math.mit.edu/~djk/calculus_beginners`
- Dan Sloughter (online textbook): `math.furman.edu/~dcs/book`
- Calc refresher (Harvey Mudd Calc Tutorial): `www.math.hmc.edu/calculus/tutorials/`

Derivatives in Review

We can use derivatives to calculate rates of change and we're concerned how functions behave. We have a series of rules that can help us get there, even if we have multiple variables.

Table 6.1: List of Rules of Differentiation

Sum rule	$(f(x)+g(x))' = f'(x)+g'(x)$
Difference rule	$(f(x)-g(x))' = f'(x)-g'(x)$
Multiply by constant rule	$f'(ax) = af'(x)$
Product rule	$(f(x)g(x))' = f'(x)g(x)+f(x)g'(x)$
Quotient rule	$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$
Chain rule	$(g(f(x))' = g'(f(x))f'(x)$
Inverse function rule	$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$
Constant rule	$(a)' = 0$
Power rule	$(x^n)' = nx^{n-1}$
Exponential rule 1	$(e^x)' = e^x$
Exponential rule 2	$(a^x)' = a^x(\ln(a))$
Logarithm rule 1	$(\ln(x))' = \frac{1}{x}$
Logarithm rule 2	$(\log_a(x))' = \frac{1}{x(\ln(a))}$
Trigonometric rules	$(\sin(x))' = \cos(x)$
	$(\cos(x))' = -\sin(x)$
	$(\tan(x))' = 1+\tan^2(x)$
Piecewise rules	Treat each piece separately

However, there comes a time where we have the rate of change but we need to know what the original is. Next, we'll cover integrals – undoing derivatives!