PSET 5: Functions of several variables and optimization with several variables
SOLUTIONS

1 Find first partial derivatives

Find all of the first partial derivatives of each function.[1]

a. $f(x,y) = 3x + 2y^4$

Solution:
$f_x = 3$
$f_y = 8y^3$

b. $f(x,y) = \frac{1}{3}x^5 - 3x^3y^2 + 3xy^4$

Solution:
$f_x = \frac{5}{3}x^4 - 9x^2y^2 + 3y^4$
$f_y = -6x^3y + 12xy^3$

c. $g(x,y) = xe^{4y}$

Solution:
$g_x = e^{4y}$
$g_y = 4xe^{4y}$

d. $k(x,y) = \frac{x+y}{x-y}$

Solution:
$k_x = \frac{1(x-y)-1(x+y)}{(x-y)^2}$
$k_x = \frac{-2y}{(x-y)^2}$
$k_y = \frac{1(x-y)+1(x+y)}{(x-y)^2}$
$k_y = \frac{2x}{(x-y)^2}$

e. $h(x,y,z) = x^3e^{yz}$

Solution:
$h_x = 3x^2e^{yz}$
$h_y = x^3ze^{yz}$
$h_z = x^3ye^{yz}$

[1] Inspired by Grimmer HW6.3

2 Find the gradient

Find the gradient ∇f of the following functions and evaluate them at the given points.[2]

a. $f(x, y) = \sqrt{x^2 + y^2}, \quad (x, y) = (3, 4)$

   Working:
   $f_x = \frac{x}{\sqrt{x^2+y^2}}$
   $f_x(3, 4) = \frac{3}{5}$
   $f_y = \frac{y}{\sqrt{x^2+y^2}}$
   $f_y(3, 4) = \frac{4}{5}$

   Solution:
   $\nabla f(3, 4) = (\frac{3}{5}, \frac{4}{5})$

b. $f(x, y, z) = (x + z)e^{x-y}, \quad (x, y, z) = (1, 1, 1)$

   Working:
   $f_x = e^{x-y} + (x + z)e^{x-y}$
   $f_x(1, 1, 1) = 3$
   $f_y = -(x + z)e^{x-y}$
   $f_y(1, 1, 1) = -2$
   $f_z = e^{x-y}$
   $f_z(1, 1, 1) = 1$

   Solution:
   $\nabla f(1, 1, 1) = (3, -2, 1)$

[2] Inspired by Grimmer HW6.4

3 Find the Hessian

Find the Hessian H for the following functions.[3]

a. $g(x, y) = x^4 - 3x^2y^3$

Working:
$g_x = 4x^3 - 6xy^3$
$g_{xx} = 12x^2 - 6y^3$
$g_{xy} = -18xy^2$
$g_y = -9x^2y^2$
$g_{yx} = -18xy^2$
$g_{yy} = -18x^2y$

Solution:
$$H_g = \begin{bmatrix} 12x^2 - 6xy^3 & -18xy^2 \\ -18xy^2 & -18x^2y \end{bmatrix}$$

b. $f(x, y, z) = xyz - x^2$

Working:
$f_x = yz - 2x$
$f_{xx} = -2$
$f_{xy} = z$
$f_{xz} = y$
$f_y = xz$
$f_{yx} = z$
$f_{yy} = 0$
$f_{yz} = x$
$f_z = xy$
$f_{zx} = y$
$f_{zy} = x$
$f_{zz} = 0$

Solution:
$$H_f = \begin{bmatrix} -2 & z & y \\ z & 0 & x \\ y & x & 0 \end{bmatrix}$$

[3]Inspired by Grimmer HW7.3

4 Find the critical points

Find the local minimum values, local maximum values, and saddle point(s) of the function. Remember the process we discussed in class: Calculate the gradient, set it equal to zero to solve the system of equations, calculate the Hessian, and assess the Hessian at critical values. Be sure to show your work on each of these steps.[4]

a. $f(x, y) = x^4 + y^4 - 4xy + 2$

Working:
$f_x = 4x^3 - 4y$
$f_y = 4y^3 - 4x$
$\nabla f = (4x^3 - 4y, 4y^3 - 4x)$
$4x^3 - 4y = 0$
$x^3 = y$
$4y^3 - 4x = 0$
$y^3 = x$
$(x^3)^3 = x^3$
$x^3 = x$
$x^3 - x = 0$
$x(x^2 - 1) = 0$
$x(x + 1)(x - 1) = 0$
$x = -1, 0, 1$
Critical Points:
$(-1, -1),\ (0, 0),\ (1, 1)$
$H_f = \begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}$
$H_f(-1, -1) = \begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix}$
$det(H_f(-1, -1)) = 12^2 - (-4)^2 = 128 > 0$
and $12 > 0$, so $(-1, -1)$ is a local minimum
$f(-1, -1) = (-1)^4 + (-1)^4 - 4(-1)(-1) + 2 = 0$
$H_f(0, 0) = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix}$
$det(H_f(0, 0)) = 0 - (-4)^2 = -16 < 0$
so $(0, 0)$ is a saddle point
$H_f(1, 1) = \begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix}$
$det(H_f(1, 1)) = 12^2 - (-4)^2 = 128 > 0$
and $12 > 0$, so $(1, 1)$ is a local minimum
$f(1, 1) = 1^4 + 1^4 - 4(1)(1) + 2 = 0$

Solution:
f has a local minimum value of 0 at $(-1, -1)$, a local minimum value of 0 at $(1, 1)$, and a saddle point at $(0, 0)$.

[4] Inspired by Grimmer HW7.4

b. $k(x, y) = (1 + xy)(x + y)$

Working:
$k_x = y(x + y) + (1 + xy) = y^2 + 2xy + 1$
$k_y = x(x + y) + (1 + xy) = x^2 + 2xy + 1$
$\nabla f = (y^2 + 2xy + 1, x^2 + 2xy + 1)$
$y^2 + 2xy + 1 = 0$
$x^2 + 2xy + 1 = 0$
$2xy = -x^2 - 1$
$y = \frac{-x^2-1}{2x}$
$(\frac{-x^2-1}{2x})^2 + 2x(\frac{-x^2-1}{2x}) + 1 = 0$
$\frac{x^4+2x^2+1}{4x^2} - x^2 - 1 + 1 = 0$
$\frac{x^4+2x^2+1-4x^4}{4x^2} = 0$
$-3x^4 + 2x^2 + 1 = 0$
$-3x^4 + 3x^2 - x^2 + 1 = 0$
$-3x^2(x^2 - 1) - (x^2 - 1) = 0$
$(-3x^2 - 1)(x^2 - 1) = 0$
$x = \pm 1$
Critical points:
$(-1, 1), (1, -1)$
$H_f = \begin{bmatrix} 2y & 2x + 2y \\ 2x + 2y & 2x \end{bmatrix}$
$H_f(-1, 1) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$
$det(H_f(-1, 1)) = -4 < 0$
so $(-1, 1)$ is a saddle point
$H_f(1, -1) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$
$det(H_f(1, -1)) = -4 < 0$
so $(1, -1)$ is a saddle point

Solution:
k has a saddle point at $(-1, 1)$ and at $(1, -1)$.

5 Definite integrals

Solve the following definite integrals using the antiderivative method.[5]

For all these problems, the basic approach to compute the definite integral of $f(x)$ from a to b is by using the formula $F(b) - F(a)$, where $F(x)$ is the **antiderivative** of f.

a. $\int_6^8 x^3\,dx$

Working:
$F(x) = \frac{1}{4}x^4$
$F(x)|_6^8 = \frac{1}{4}(8^4) - \frac{1}{4}(6^4) = 700$

Solution:
700

b. $\int_{-1}^0 (3x^2 - 1)\,dx$

Working:
$F(x) = x^3 - x$
$F(x)|_{-1}^0 = (0^3 - 0) - ((-1)^3 - (-1)) = 0$

Solution:
0

c. $\int_2^4 e^y\,dy$

Working:
$F(y) = e^y$
$F(y)|_2^4 = e^4 - e^2 = e^2(e^2 - 1)$

Solution:
$e^4 - e^2 = e^2(e^2 - 1)$

d. $\int_3^3 \sqrt{x^5 + 2}\,dx$

Working:
$F(x)|_3^3 = F(3) - F(3) = 0$

Solution:
0

[5]Inspired by Gill 5.10 and Grimmer HW4.1

6 Indefinite integrals

Calculate the following indefinite integrals:[6]

a. $\int (x^2 - x^{-\frac{1}{2}})\, dx$

   Solution:
   $\frac{1}{3}x^3 - 2x^{\frac{1}{2}} + c = \frac{1}{3}x^3 - 2\sqrt{x} + c$

b. $\int 360t^6\, dt$

   Solution:
   $\frac{360}{7}t^7 + c$

c. $\int 2x \log(x^2)\, dx$

   Working:
   $u = \log(x^2) = 2\log(x)$
   $du = \frac{2}{x}dx$
   $dv = 2xdx$
   $v = x^2$
   $x^2 \log(x^2) - \int x^2 \frac{2}{x}\, dx$
   $x^2 \log(x^2) - \int 2x\, dx$
   $x^2 \log(x^2) - x^2 + c$

   Solution:
   $x^2 \log(x^2) - x^2 + c = x^2(\log(x^2) - 1) + c$

[6]Inspired by Gill 5.13 and 5.14

7 Determining convergence

Determine whether each integral is convergent or divergent. Evaluate those that are convergent.[7]

a. $\int_1^\infty \left(\frac{1}{3x}\right)^2 \, dx$

Working:
$\lim_{a\to\infty} \int_1^a \frac{1}{9}x^{-2} \, dx$
$\lim_{a\to\infty} -\frac{1}{9}x^{-1}|_1^a$
$\lim_{a\to\infty} -\frac{1}{9}a^{-1} + \frac{1}{9}1^{-1}$
$\lim_{a\to\infty} a^{-1} = 0$

Solution:
Converges to $\frac{1}{9}$

b. $\int_0^\infty e^{-x} \, dx$

Working:
$\lim_{a\to\infty} \int_0^a e^{-x} \, dx$
$\lim_{a\to\infty} -e^{-x}|_0^a$
$\lim_{a\to\infty} -e^{-a} + e^0$
$\lim_{a\to\infty} -e^{-a} = 0$

Solution:
Converges to 1

c. $\int_{-\infty}^0 x^3 \, dx$

Working:
$\lim_{a\to-\infty} \int_a^0 x^3 \, dx$
$\lim_{a\to-\infty} \frac{1}{4}x^4|_a^0$
$\lim_{a\to-\infty} \frac{1}{4}0^4 - \frac{1}{4}a^4$
$\lim_{a\to-\infty} a^4 = \infty$

Solution:
Diverges

d. $\int_0^1 \int_2^3 x^2y^3 \; dxdy$

Working:
$\int_0^1 \frac{1}{3}x^3y^3|_{x=2}^{x=3} \, dy$
$\frac{1}{3}\int_0^1 y^3(3^3 - 2^3) \, dy$
$\frac{19}{3}\int_0^1 y^3 \, dy$
$\frac{19}{3}(\frac{1}{4}y^4|_0^1$
$\frac{19}{12}(1^4 - 0^4)$

Solution:
Converges to $\frac{19}{12}$

[7] Inspired by Grimmer HW 4.3

e. $\int_0^2 \int_0^{\sqrt{1-x^2}} 2x^3y \; dydx$

Working:
Notice $\sqrt{1-x^2}$ is only defined for $x \in [0,1]$, so we should adjust the boundary for the integral accordingly. The answer is provided if you didn't make this adjustment at the end, and for grading purposes we accepted both solutions.
$\int_0^1 x^3y^2|_{y=0}^{y=\sqrt{1-x^2}} \, dx$
$\int_0^1 x^3(\sqrt{1-x^2}^2 - 0^2) \, dx$
$\int_0^1 x^3(1-x^2) \, dx$
$\int_0^1 x^3 - x^5 \, dx$
$\frac{1}{4}x^4 - \frac{1}{6}x^6|_0^1$
$(\frac{1}{4}(1^4) - \frac{1}{6}(1^6)) - (\frac{1}{4}(0^4) - \frac{1}{6}(0^6))$
$\frac{1}{4} - \frac{1}{6} = -\frac{1}{12}$

Solution:
Converges to $\frac{1}{12}$
If the boundary adjustment is not made, the math works out to $-\frac{20}{3}$

8 Applied integration

A group of three unidentified first-year graduate students at the University of Chicago are worn out after a week of math camp. Wanting to unwind, the students agree to not talk about math and decide to chat over some casual drinks at Medici.

After five shots of tequila each, two pitchers of beer, a bottle of wine, and a large Chicago-style pizza, the three students have had enough fun and decide to start the trip back home.

- Student A gets on a bike and starts pedaling away at a velocity of $v_A(t) = 2t^4 + t$, where t represents minutes. However, the student crashes into the side of an Uber and ends the journey after only 2 minutes.
- Student B has no bike, so starts running at a velocity of $v_B(t) = 4\sqrt{t}$. Sadly, after only 4 minutes, the student's legs give out and the student decides to sing a song, instead.
- Student C can't even stand up, so has no choice but to slowly crawl at a velocity of $v_C(t) = 2e^{-t}$. Student C steadily plods along for 20 minutes before falling asleep on the sidewalk.

Generally, if an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$. The Fundamental Theorem of Calculus then tells us that

Without using a calculator, use this formula to find the distance traveled by Students A, B, and C. (Assume, however unrealistic in may be, that all three students traveled in a straight line.) Who traveled the farthest? The least far?[8]

Working:

- Student A:
  $\int_0^2 2t^4 + t\,dt$
  $\frac{2}{5}t^5 + \frac{1}{2}t^2|_0^2$
  $(\frac{2}{5}(2^5) + \frac{1}{2}(2^2)) - \frac{2}{5}(0^5) + \frac{1}{2}(0^2))$
  $\frac{64}{5} + 2 = 14.8$

- Student B:
  $\int_0^4 4\sqrt{t}\,dt$
  $\frac{8}{3}t^{\frac{3}{2}}|_0^4$
  $\frac{8}{3}(4^{\frac{3}{2}}) - \frac{8}{3}(0^{\frac{3}{2}}) = \frac{64}{3} \approx 21.33$

- Student C:
  $\int_0^{20} 2e^{-t}\,dt$
  $-2e^{-t}|_0^{20}$
  $-2e^{-20} + 2e^0$
  $2(1 - e^{-20}) \approx 2$

Solution:

- Student A traveled $\frac{74}{5} = 14.8$ units.
- Student B traveled $\frac{64}{3} \approx 21.33$ units.
- Student C traveled $2(1 - e^{-20}) \approx 2$ units.
- Student B traveled the farthest.
- Student C traveled the least far.

[8] Inspired by Grimmer HW4.2