Pset 1: Linear equations, inequalities, sets and functions, quadratics

August 29, 2024

Simplify expressions

Simplify the following expressions as much as possible:[1]

a. $(-x^4y^2)^2$

1. Distribute exponents over products.

$$(-1)^2x^{(2\times4)}y^{(2\times2)}$$

2. Multiply 2 and 2 together.

$$(-1)^2x^{(2\times4)}y^4$$

3. Multiply 2 and 4 together.

$$(-1)^2x^8y^4$$

4. Evaluate $(-1)^2$.

$$x^8y^4$$

b. $9(3^0)$

1. Any nonzero number to the zero power is 1.

$$9(1)$$

2. Anything times 1 is the same value.

$$9$$

[1] Gill 1.1

c. $(2a^2)(4a^4)$

1. Combine products of like terms.

$$2a^2 \times 4a^4 = 2 \times 4a^{(2+4)}$$

2. Evaluate $2 + 4$.

$$2 \times 4a^6$$

3. Multiply 2 and 4 together.

$$8a^6$$

d. $\frac{x^4}{x^3}^3$

1. For all exponents, $\frac{a^n}{a^m} = a^{(n-m)}$.

$$x^{(4-3)}$$

2. Evaluate $4 - 3$.

$$x$$

3. Take this to the third power.

$$x^3$$

e. $(-2)^{4-7}$

1. Subtract 7 from 4.

$$(-2)^{-3}$$

2. In order to evaluate 2^{-3}, express as 2×2^2 and put in denominator.

$$-2 \times 2^2$$

3. Evaluate 2^2.

$$-2 \times 4$$

4. Multiply -2 and 4 together.

$$-8$$

5. Place in denominator.

$$\frac{1}{8}$$

f. $\left(\frac{1}{27b^3}\right)^{1/3}$

1. Separate component terms.

$$\frac{1}{27}^{1/3} \times \frac{1}{b^3}^{1/3}$$

2. Evaluate cube roots.

$$\frac{1}{3} \times \frac{1}{b}$$

3. Combine terms.

$$\frac{1}{3b}$$

g. $y^7 y^6 y^5 y^4$

1. Combine products of like terms.

$$y^{(7+6+5+4)}$$

2. Evaluate $7 + 6 + 5 + 4$.

$$y^{22}$$

h. $\frac{2a/7b}{11b/5a}$

1. Write as a single fraction by multiplying the numerator by the reciprocal of the denominator.

$$\frac{2a}{7b} \times \frac{5a}{11b}$$

2. Product property of exponents: $x^a \times x^b = x^{(a+b)}$.

$$\frac{5a \times 2a}{7b \times 11b} = \frac{5 \times 2a^{1+1}}{7 \times 11b^{1+1}}$$

3. Evaluate $1 + 1$.

$$\frac{5 \times 2a^2}{7 \times 11b^2}$$

4. Multiply scalars together.

$$\frac{10a^2}{77b^2}$$

i. $(z^2)^4$

1. Nested exponents rule: $(x^a)^b = x^{ab}$.

$$z^{2\times4}$$

2. Evaluate 2×4.

$$z^8$$

Root finding

Find the roots (solutions) to the following quadratic equations.[2]

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

a. $9x^2 - 3x - 12 = 0$

1. Factor the left-hand side.

$$3(x + 1)(3x - 4) = 0$$

2. Divide both sides by 3 to simplify the equation.

$$(x + 1)(3x - 4) = 0$$

3. Find the roots of each term in the product separately by solving for x.

$$\begin{aligned} x + 1 &= 0 & 3x &= 4 \\ x &= -1 & x &= \frac{4}{3} \end{aligned}$$

b. $x^2 - 2x - 16 = 0$

1. Complete the square.

$$\begin{aligned} x^2 - 2x - 16 &= 0 \\ x^2 - 2x &= 16 \\ x^2 - 2x + 1 &= 17 \\ (x - 1)^2 &= 17 \\ x - 1 &= \pm\sqrt{17} \\ x &= 1 \pm \sqrt{17} \end{aligned}$$

2. Quadratic formula.

   (a) Using the quadratic formula, solve for x.

$$\begin{aligned} x &= \frac{-(-2) \pm \sqrt{(-2)^2 - (4 \times 1 \times 16)}}{2 \times 1} \\ x &= \frac{2 \pm \sqrt{4 + 64}}{2} \\ x &= \frac{2 \pm \sqrt{68}}{2} \end{aligned}$$

[2] Gill 1.25

(b) Simplify the radical.

$$x = \frac{2 \pm \sqrt{2^2 \times 17}}{2}$$
$$x = \frac{2 \pm 2\sqrt{17}}{2}$$

(c) Factor the greatest common divisor.

$$x = 1 \pm \sqrt{17}$$

c. $6x^2 - 6x - 6 = 0$

1. Divide both sides by 6 to simplify the equation.

$$x^2 - x - 1 = 0$$

2. Using the quadratic formula, solve for x.

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - (4 \times 1 \times -1)}}{2 \times 1}$$
$$x = \frac{1 \pm \sqrt{1 - 4(-1)}}{2}$$
$$x = \frac{1 \pm \sqrt{1 + 4}}{2}$$
$$x = \frac{1 \pm \sqrt{5}}{2}$$

Systems of linear equations

Solve the following systems of equations for their unknown values. If there is no solution, indicate as such.

a. Two unknowns[3]

$$3x - 2y = 18$$
$$5x + 10y = -10$$

1. Via substitution:

(a) Solve for x in the first equation:

$$3x - 2y = 18$$
$$3x = 18 + 2y$$
$$x = \frac{2}{3}y + 6$$

[3] Gill 1.25

(b) Substitute $x = \frac{2}{3}y + 6$ into the second equation and solve for y:

$$\begin{aligned} 5\left(\frac{2}{3}y + 6\right) + 10y &= -10 \\ \frac{10}{3}y + 30 + 10y &= -10 \\ \frac{40}{3}y + 30 &= -10 \\ \frac{40}{3}y &= -40 \\ y &= -3 \end{aligned}$$

(c) Substitute $y = -3$ back into the first equation to find x:

$$\begin{aligned} 3x - 2(-3) &= 18 \\ 3x + 6 &= 18 \\ 3x &= 12 \\ x &= 4 \end{aligned}$$

(d) Final solution:

$$x = 4, \quad y = -3$$

b. Three unknowns[4]

$$\begin{aligned} 5x - 2y + 3z &= 20 \\ 2x - 4y - 3z &= -9 \\ x + 6y - 8z &= 21 \end{aligned}$$

1. Subtract $\frac{2}{5}$ of equation 1 from equation 2:

$$\begin{aligned} 5x - 2y + 3z &= 20 \\ 0x - \frac{16y}{5} - \frac{21z}{5} &= -17 \\ x + 6y - 8z &= 21 \end{aligned}$$

2. Multiply equation 2 by -5:

$$\begin{aligned} 5x - 2y + 3z &= 20 \\ 0x + 16y + 21z &= 85 \\ x + 6y - 8z &= 21 \end{aligned}$$

[4]Gill 1.25

3. Subtract $\frac{1}{5}$ of equation 1 from equation 3:

$$\begin{aligned} 5x - 2y + 3z &= 20 \\ 0x + 16y + 21z &= 85 \\ 0x + \frac{32y}{5} - \frac{43z}{5} &= 17 \end{aligned}$$

4. Multiply equation 3 by 5:

$$\begin{aligned} 5x - 2y + 3z &= 20 \\ 0x + 16y + 21z &= 85 \\ 0x + 32y - 43z &= 85 \end{aligned}$$

5. Swap equation 2 with equation 3:

$$\begin{aligned} 5x - 2y + 3z &= 20 \\ 0x + 32y - 43z &= 85 \\ 0x + 16y + 21z &= 85 \end{aligned}$$

6. Subtract $\frac{1}{2}$ of equation 2 from equation 3:

$$\begin{aligned} 5x - 2y + 3z &= 20 \\ 0x + 32y - 43z &= 85 \\ 0x + 0y + \frac{85z}{2} &= \frac{85}{2} \end{aligned}$$

7. Multiply equation 3 by $\frac{2}{85}$:

$$\begin{aligned} 5x - 2y + 3z &= 20 \\ 0x + 32y - 43z &= 85 \\ 0x + 0y + z &= 1 \end{aligned}$$

8. Add $43\times$ equation 3 to equation 2:

$$\begin{aligned} 5x - 2y + 3z &= 20 \\ 0x + 32y + 0z &= 128 \\ 0x + 0y + z &= 1 \end{aligned}$$

9. Divide equation 2 by 32:

$$\begin{aligned} 5x - 2y + 3z &= 20 \\ 0x + y + 0z &= 4 \\ 0x + 0y + z &= 1 \end{aligned}$$

10. Add 2× equation 2 to equation 1:

$$\begin{aligned} 5x + 0y + 3z &= 28 \\ 0x + y + 0z &= 4 \\ 0x + 0y + z &= 1 \end{aligned}$$

11. Subtract 3× equation 3 from equation 1:

$$\begin{aligned} 5x + 0y + 0z &= 25 \\ 0x + y + 0z &= 4 \\ 0x + 0y + z &= 1 \end{aligned}$$

12. Divide equation 1 by 5:

$$\begin{aligned} x + 0y + 0z &= 5 \\ 0x + y + 0z &= 4 \\ 0x + 0y + z &= 1 \end{aligned}$$

13. Final solution:

$$x = 5, \quad y = 4, \quad z = 1$$

c. An animal shelter has a total of 350 animals comprised of cats, dogs, and rabbits. If the number of rabbits is 5 less than one-half the number of cats, and there are 20 more cats than dogs, how many of each animal are at the shelter?[5]

1. Let x = number of cats
2. Let y = number of dogs
3. Let z = number of rabbits

This gives us the system of equations:

$$\begin{aligned} x + y + z &= 350 \\ z &= \frac{1}{2}x - 5 \\ x &= 20 + y \end{aligned}$$

1. Substitute $z = \frac{1}{2}x - 5$ into the first equation:

$$\begin{aligned} x + y + \frac{x}{2} - 5 &= 350 \\ \frac{3}{2}x + y - 5 &= 350 \end{aligned}$$

[5] OpenStax Algebra 7.2.54

2. Substitute $x = y + 20$ into the first equation:

$$\begin{aligned}\frac{3}{2}(y+20) + y - 5 &= 350\\ \frac{3}{2}y + 30 + y - 5 &= 350\\ \frac{5}{2}y + 25 &= 350\\ \frac{5}{2}y &= 325\\ y &= 130\end{aligned}$$

3. Substitute $y = 130$ into the third equation:

$$\begin{aligned}x &= y + 20\\ x &= 130 + 20\\ x &= 150\end{aligned}$$

4. Substitute $x = 150$ into the second equation:

$$\begin{aligned}z &= \frac{1}{2}x - 5\\ z &= \frac{1}{2}(150) - 5\\ z &= 70\end{aligned}$$

5. Final solution:

$$x = 150, \quad y = 130, \quad z = 70$$

There are 150 cats, 130 dogs, and 70 rabbits.

Work with sets

Using the sets

$$\begin{aligned}A &= \{2, 3, 7, 9, 13, 16\}\\ B &= \{x : 4 \leq x \leq 8 \text{ and } x \text{ is an integer}\}\\ C &= \{x : 2 < x < 25 \text{ and } x \text{ is prime}\}\\ D &= \{1, 4, 9, 16, 25, \ldots\}\end{aligned}$$

identify the following:[6]

1. $A \cup B$

$$E = \{2, 3, 4, 5, 6, 7, 8, 9, 13\}$$

Combine all integers between 4 and 8 inclusive with the numbers in set A.

[6] Grimmer HW1.1

2. $(A \cup B) \cap C$

$$F = \{3, 5, 7, 13\}$$

Since C contains only prime numbers greater than 2 and less than 25, we take all the prime numbers that are also included in E, but remember to drop out 2 since it is not included in C.

3. $C \cap D$

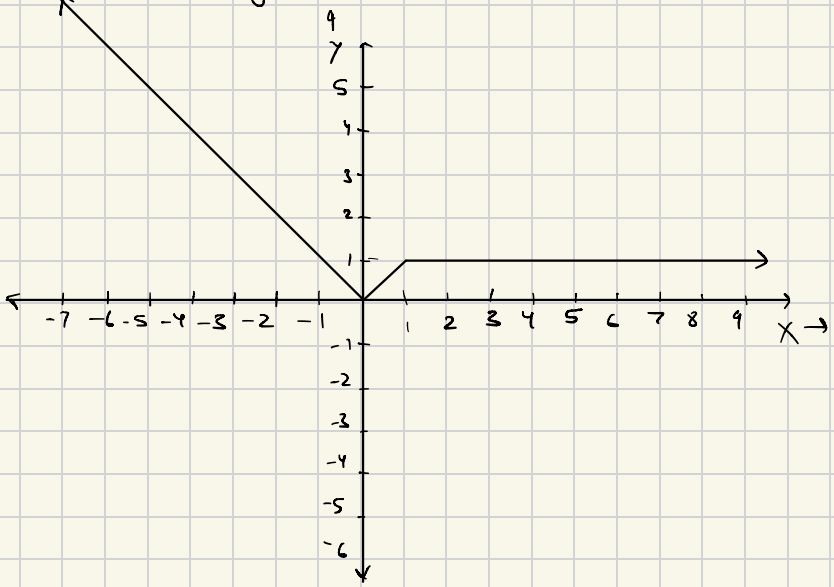
$$G = \emptyset$$

There are no prime numbers in D, so nothing is shared between C and D.

3)

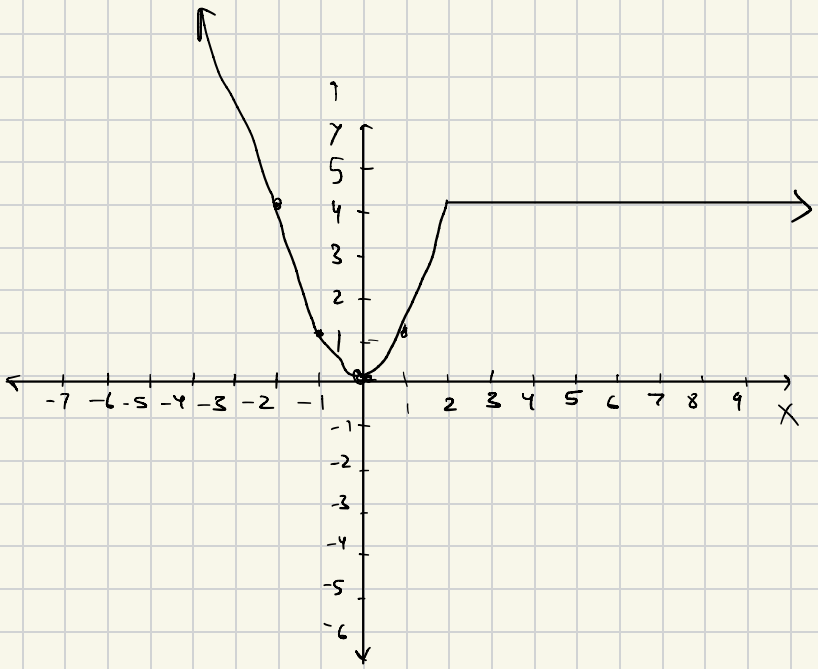
$$f(x) = \begin{cases} |x| & \text{if } x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}, \quad g(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 4 & \text{if } x \geq 2 \end{cases}$$

1) $f(x) = y$



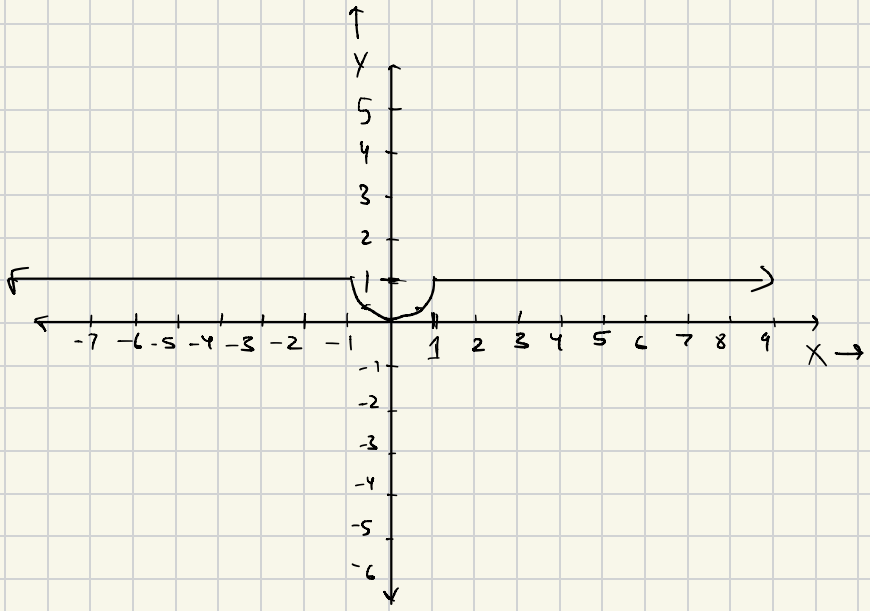
x	f(x)
2	1
1	1
0.5	0.5
0	0
-1	1
-2	2

$y = g(x)$

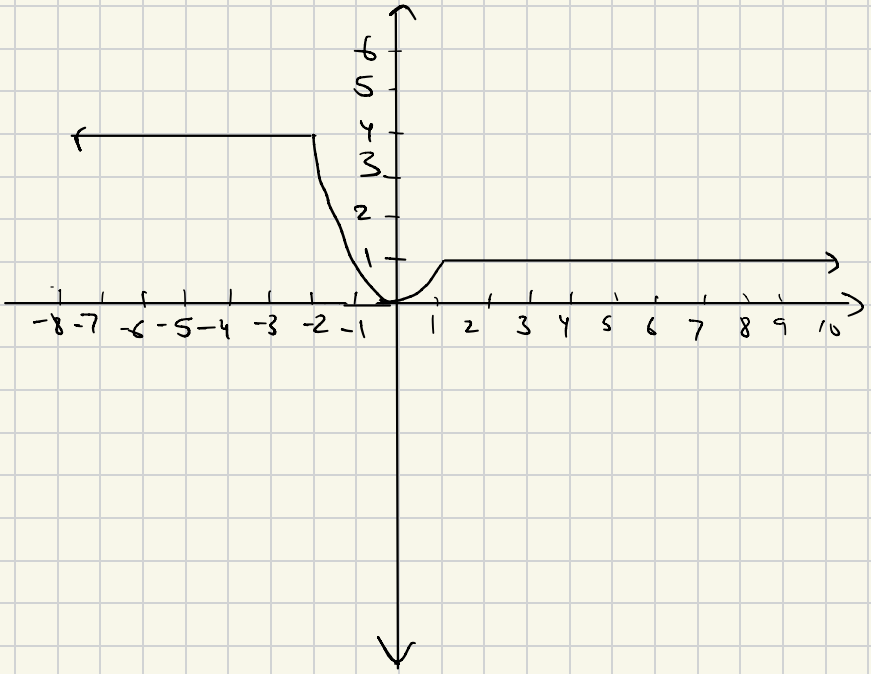


x	g(x)
3	4
2	4
1	1
0	0
-1	1
-2	4
-3	9

$y = f(g(x))$



x	g(x)	f(g(x))
3	9	1
2	4	1
1	1	1
0.5	0.25	0.25
0	0	0
-0.5	0.25	0.25
-1	1	1
-2	4	1
-3	9	1



x	f(x)	g(x)
2	1	1
1	1	1
0.5	0.5	0.25
0	0	0
-0.5	0.5	0.25
-1	1	1
-2	2	4
-3	3	4