Computational Math Camp

Problem Sets

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Contents

Overview

Contains problem sets for the 2019 Computational Math Camp.

Chapter 1

Linear equations, notation, sets, and functions

1.1 Simplify expressions

Simplify the following expressions as much as possible:

a. $(-x^4y^2)^2$

1. Distribute exponents over products.

$$(-1)^2x^{(2\times 4)}y^{(2\times 2)}$$

2. Multiply 2 and 2 together.

$$(-1)^2x^{(2\times 4)}y^4$$

3. Multiply 2 and 4 together.

$$(-1)^2x^8y^4$$

4. Evaluate $(-1)^2$.

$$x^8y^4$$

b. $9(3^0)$

1. Any nonzero number to the zero power is 1.

$$9(1)$$

2. Anything times 1 is the same value.

$$9$$

c. $(2a^2)(4a^4)$

1. Combine products of like terms.

$$2a^2 \times 4a^4 = 2 \times 4a^{(2+4)}$$

2. Evaluate $2+4$.

$$2 \times 4a^6$$

3. Multiply 2 and 4 together.

$$8a^6$$

d. $\dfrac{x^4}{x^3}$

1. For all exponents, $\dfrac{a^n}{a^m} = a^{(n-m)}$.

$$x^{(4-3)}$$

2. Evaluate $4-3$.

$$x$$

e. $(-2)^{7-4}$

1. Subtract 4 from 7.

$$(-2)^3$$

2. In order to evaluate 2^3 express 2^3 as 2×2^2.

$$-2 \times 2^2$$

3. Evaluate 2^2.

$$-2 \times 4$$

4. Multiply -2 and 4 together.

$$-8$$

f. $\left(\frac{1}{27b^3}\right)^{1/3}$

1. Separate component terms.

$$\frac{1}{27}^{1/3} \times \frac{1}{b^3}^{1/3}$$

2. Evaluate cube roots.

$$\frac{1}{3} \times \frac{1}{b}$$

3. Combine terms.

$$\frac{1}{3b}$$

g. $y^7 y^6 y^5 y^4$

1. Combine products of like terms.

$$y^{(7+6+5+4)}$$

2. Evaluate $7 + 6 + 5 + 4$.

$$y^{22}$$

h. $\frac{2a/7b}{11b/5a}$

1. Write as a single fraction by multiplying the numerator by the reciprocal of the denominator.

$$\frac{2a}{7b} \times \frac{5a}{11b}$$

2. Product property of exponents: $x^a \times x^b = x^{(a+b)}$

$$\frac{5a \times 2a}{7b \times 11b} = \frac{5 \times 2a^{1+1}}{7 \times 11b^{1+1}}$$

3. Evaluate $1 + 1$.

$$\frac{5 \times 2a^2}{7 \times 11b^2}$$

4. Multiple scalars together.

$$\frac{10a^2}{77b^2}$$

i. $(z^2)^4$

1. Nested exponents rule: $(x^a)^b = x^{ab}$

$$z^{2\times 4}$$

2. Evaluate 2×4

$$z^8$$

1.2 Simplify a (more complex) expression

Simplify the following expression:

$$(a + b)^2 + (a - b)^2 + 2(a + b)(a - b) - 3a^2$$

1. Expand $(a + b)^2$ with FOIL.

$$a^2 + 2ab + b^2 + (a - b)^2 + 2(a + b)(a - b) - 3a^2$$

2. Expand $(a - b)^2$ with FOIL.

$$a^2 + 2ab + b^2 + a^2 - 2ab + b^2 + 2(a + b)(a - b) - 3a^2$$

3. Multiply $a + b$ and $a - b$ together using FOIL.

$$a^2 + 2ab + b^2 + a^2 - 2ab + b^2 + 2(a^2 - b^2) - 3a^2$$

4. Distribute 2 over $a^2 - b^2$.

$$a^2 + 2ab + b^2 + a^2 - 2ab + b^2 + 2a^2 - 2b^2 - 3a^2$$

5. Group like terms.

$$(a^2 + a^2 + 2a^2 - 3a^2) + (b^2 + b^2 - 2b^2) + (2ab - 2ab)$$

6. Combine like terms.

$$a^2 + (b^2 + b^2 - 2b^2) + (2ab - 2ab)$$

7. Look for the difference of two identical terms.

$$a^2$$

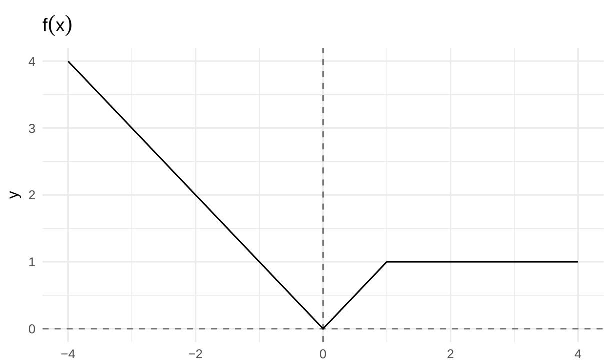
1.3 Graph sketching

Let the functions $f(x)$ and $g(x)$ be defined for all $x \in \Re$ by

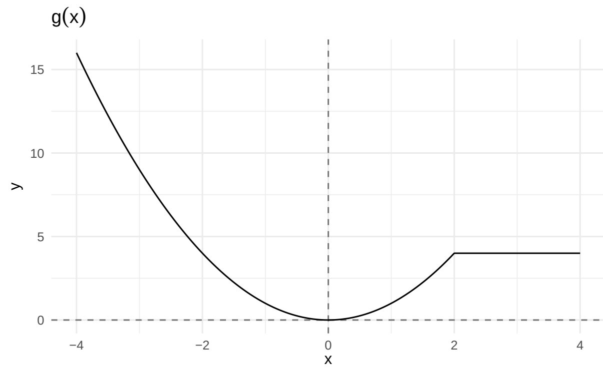
$$f(x) = \begin{cases} |x| & \text{if } x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}, \quad g(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 4 & \text{if } x \geq 2 \end{cases}$$

Sketch the graphs of:

1. $y = f(x)$



2. $y = g(x)$



3. $y = f(g(x))$

   To sketch the composite function, we first evaluate $g(x)$ for different values of x, and then evaluate $f(g(x))$ for different outputs of $g(x)$.

   - For $x \geq 2$, $g(x)$ is a constant value:

   $$\begin{aligned} x &\geq 2 \\ g(x) &= 4 \\ f(g(x)) &= f(4) = 1 \end{aligned}$$

   - For $x < 2$, $g(x)$ is not constant: $g(x) = x^2$. $f(x)$ evaluates differently depending on its input, so we have two cases based on the output of $g(x)$:

     – if $g(x) < 1$, $f(g(x)) = |g(x)| = |x^2| = x^2$. This is the case when:

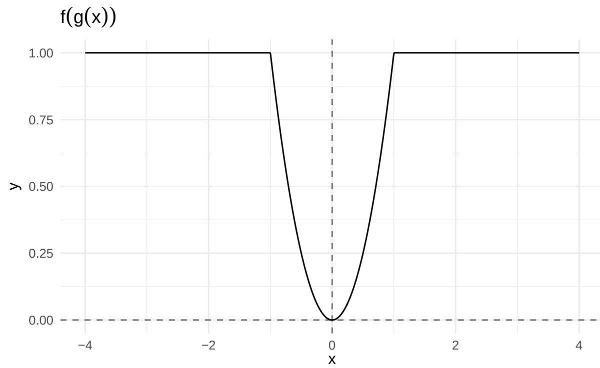
     $$\begin{aligned} g(x) &< 1 \\ x^2 < 1 \text{ and } x &< 2 \\ -1 < x &< 1 \end{aligned}$$

     – if $g(x) \geq 1$, $f(g(x)) = 1$. This is the case when:

     $$\begin{aligned} g(x) &\geq 1 \\ x^2 \geq 1 \text{ and } x &< 2 \\ x \leq -1 \text{ or } 1 \leq x &< 2 \end{aligned}$$

   - Therefore, $f(g(x))$ has the following values:

   $$f(g(x)) = \begin{cases} 1 & \text{if } x \leq -1 \\ x^2 & \text{if } -1 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$



4. $y = g(f(x))$

   To sketch the composite function, we first evaluate $f(x)$ for different values of x, and then evaluate $g(f(x))$ for different outputs of $f(x)$.

   - For $x \geq 1$, $f(x)$ is a constant value:

$$\begin{aligned} x &\geq 1 \\ f(x) &= 1 \\ g(f(x)) = f(1) &= 1^2 = 1 \end{aligned}$$

   - For $x < 1$, $f(x)$ is not constant: $f(x) = |x|$. $g(x)$ evaluates differently depending on its input, so we have two cases based on the output of $f(x)$:
     - if $f(x) < 2$, $g(f(x)) = f(x)^2 = |x|^2 = x^2$. This is the case when:

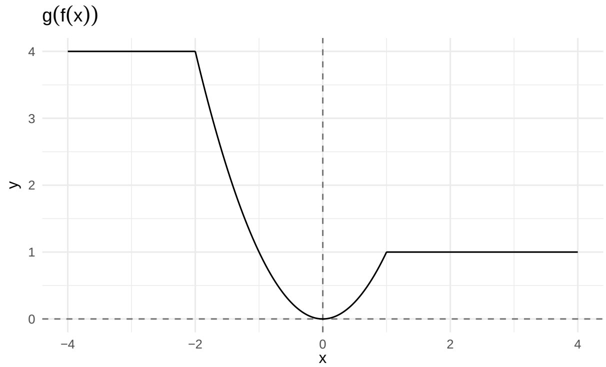
$$\begin{aligned} f(x) &< 2 \\ |x| < 2 \text{ and } x &< 1 \\ -2 < x &< 1 \end{aligned}$$

     - if $f(x) \geq 2$, $g(f(x)) = 4$. This is the case when:

$$\begin{aligned} f(x) &\geq 2 \\ |x| \geq 2 \text{ and } x &< 1 \\ x &\leq -2 \end{aligned}$$

   - Therefore, $g(f(x))$ has the following values:

$$g(f(x)) = \begin{cases} 4 & \text{if } x \leq -2 \\ x^2 & \text{if } -2 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$



1.4 Root finding

Find the roots (solutions) to the following quadratic equations.

Definition 1.1 (The quadratic formula)**.**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

a. $4x^2 - 1 = 17$

- Move terms so that x is alone on the left side of the equation.

$$\begin{aligned} 4x^2 - 1 &= 17 \\ 4x^2 &= 18 \\ x^2 &= \frac{18}{4} \\ x^2 &= \frac{9}{2} \\ x &= \pm\sqrt{\frac{9}{2}} \end{aligned}$$

b. $9x^2 - 3x - 12 = 0$

- Factor the left-hand side.

$$3(x + 1)(3x - 4) = 0$$

- Divide both sides by 3 to simplify the equation.

$$(x + 1)(3x - 4) = 0$$

- Find the roots of each term in the product separately by solving for x.

$$x+1=0 \qquad 3x=4$$
$$x=-1 \qquad x=\frac{4}{3}$$

c. $x^2-2x-16=0$

1. Complete the square

$$\begin{aligned} x^2-2x-16&=0\\ x^2-2x&=16\\ x^2-2x+1&=17\\ (x-1)^2&=17\\ x-1&=\pm\sqrt{17}\\ x&=1\pm\sqrt{17} \end{aligned}$$

2. Quadratic formula

   - Using the quadratic formula, solve for x

$$x=\frac{-(-2)\pm\sqrt{(-2)^2-(4\times1\times16)}}{2\times1}$$
$$x=\frac{2\pm\sqrt{4+64}}{2}$$
$$x=\frac{2\pm\sqrt{68}}{2}$$

   - Simplify the radical

$$x=\frac{2\pm\sqrt{2^2\times17}}{2}$$
$$x=\frac{2\pm2\sqrt{17}}{2}$$

   - Factor the greatest common divisor

$$x=1\pm\sqrt{17}$$

d. $6x^2-6x-6=0$

- Divide both sides by 6 to simplify the equation.

$$x^2 - x - 1 = 0$$

- Using the quadratic formula, solve for x

$$\begin{aligned}
x &= \frac{-(-1) \pm \sqrt{(-1)^2 - (4 \times 1 \times -1)}}{2 \times 1} \\
x &= \frac{1 \pm \sqrt{1 - 4(-1)}}{2} \\
x &= \frac{1 \pm \sqrt{1 + 4}}{2} \\
x &= \frac{1 \pm \sqrt{5}}{2}
\end{aligned}$$

e. $5 + 11x = -3x^2$

- Move everything to the left hand side.

$$3x^2 + 11x + 5 = 0$$

- Using the quadratic formula, solve for x

$$\begin{aligned}
x &= \frac{-11 \pm \sqrt{(11)^2 - (4 \times 3 \times 5)}}{2 \times 3} \\
x &= \frac{-11 \pm \sqrt{121 - 60}}{6} \\
x &= \frac{-11 \pm \sqrt{61}}{6}
\end{aligned}$$

1.5 Work with sets

Using the sets

$$\begin{aligned}
A &= \{2, 3, 7, 9, 13\} \\
B &= \{x : 4 \leq x \leq 8 \text{ and } x \text{ is an integer}\} \\
C &= \{x : 2 < x < 25 \text{ and } x \text{ is prime}\} \\
D &= \{1, 4, 9, 16, 25, \ldots\}
\end{aligned}$$

identify the following:

1. $A \cup B$

   $E = \{2, 3, 4, 5, 6, 7, 8, 9, 13\}$, combine all integers between 4 and 8 inclusive with the numbers in set A.

2. $(A \cup B) \cap C$

   $F = \{3, 5, 7, 13\}$, Since C is only prime numbers greater than 2 and less than 25, we take all the prime numbers that are also included in E, but remember to drop out 2 since it is not included in C.

3. $C \cap D$

   $G = \emptyset$, there are no prime numbers in D, so nothing is shared between C and D.

Chapter 2

Logarithms, sequences, and limits

2.1 Simplify logarithms

Express each of the following as a single logarithm:

a. $\log(x) + \log(y) - \log(z)$
b. $2\log(x) + 1$
c. $\log(x) - 2$

2.2 Sequences

Write down the first three terms of each of the following sequences. In each case, state whether the sequence is an arithmetic progression, a geometric progression, or neither.

a. $u_n = 4 + 3n$
b. $u_n = 5 - 6n$
c. $u_n = 4^n$
d. $u_n = 5 \times (-2)^n$
e. $u_n = n \times 3^n$

2.3 Find the limit

In each of the following cases, state whether the sequence $\{u_n\}$ tends to a limit, and find the limit if it exists:

a. $u_n = 1 + \frac{1}{2}n$
b. $u_n = 1 - \frac{1}{2}n$
c. $u_n = \left(\frac{1}{2}\right)^n$
d. $u_n = \left(-\frac{1}{2}\right)^n$

2.4 Determine convergence or divergence

Determine whether each of the following sequences converges or diverges. If it converges, find the limit.

a. $a_n = \frac{3+5n^2}{n+n^2}$
b. $a_n = \frac{(-1)^{n-1}n}{n^2+1}$

2.5 Find more limits

Given that

$$\lim_{x\to a} f(x) = -3, \quad \lim_{x\to a} g(x) = 0, \quad \lim_{x\to a} h(x) = 8$$

find the limits that exist. If the limit doesn't exist, explain why.

a. $\lim_{x\to a} [f(x) + h(x)]$
b. $\lim_{x\to a} [f(x)]^2 = (-3)^2$
c. $\lim_{x\to a} \sqrt[3]{h(x)}$
d. $\lim_{x\to a} \frac{1}{f(x)}$
e. $\lim_{x\to a} \frac{f(x)}{h(x)}$
f. $\lim_{x\to a} \frac{g(x)}{f(x)}$
g. $\lim_{x\to a} \frac{f(x)}{g(x)}$
h. $\lim_{x\to a} \frac{2f(x)}{h(x)-f(x)}$

2.6 Find even more limits

Find the limits of the following:

a. $\lim_{x\to -4} \frac{x^2+5x+4}{x^2+3x-4}$
b. $\lim_{x\to 4^-} \sqrt{16 - x^2}$

c. $\lim_{x \to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x}$

2.7 Check for discontinuities

Which of the following functions are continuous? If not, where are the discontinuities?

a. $f(x) = \dfrac{9x^3 - x}{(x-1)(x+1)}$

b. $f(x) = e^{-x^2}$

c. $f(y) = y^3 - y^2 + 1$

d. $f(x) = \begin{cases} x^3 + 1, & x > 0 \\ \frac{1}{2}\ x = 0 \\ -x^2, & x < 0 \end{cases}$