(More) Scalar and Vector Calculus

Computational Mathematics and Statistics Camp
University of Chicago
September 2018

1. Sketch the graph of a function (any function you like, no need to specify a functional form) that is:
   a. Continuous on $[0, 3]$ and has the following properties: an absolute minimum at 0, an absolute maximum at 3, a local maximum at 1 and a local minimum at 2.
   b. Do the same for another function with the following properties: 2 is a **critical number** (i.e. $f'(x) = 0$ or $f'(x)$ is undefined), but there is no local minimum and no local maximum.

2. Find the critical values of these functions:
   a. $f(x) = 5x^{3/2} - 4x$
   b. $s(t) = 3t^4 + 4t^3 - 6t^2$
   c. $f(r) = \dfrac{r}{r^2 + 1}$
   d. $h(x) = x \log(x)$

3. Find the absolute minimum and absolute maximum values of the functions on the given interval:
   a. $f(x) = 3x^2 - 12x + 5, [0, 3]$
   b. $f(t) = t\sqrt{4 - t^2}, [-1, 4]$
   c. $s(x) = x - \log(x), [1/2, 2]$
   d. $h(p) = 1 - e^{-p}, [0, 1000]$

4. Find all of the first partial derivatives of each function.
   a. $f(x, y) = 3x - 2y^4$
   b. $f(x, y) = x^5 + 3x^3y^2 + 3xy^4$
   c. $g(x, y) = xe^{3y}$
   d. $k(x, y) = \frac{x-y}{x+y}$
   e. $f(x, y, z) = \log(x + 2y + 3z)$
   f. $h(x, y, z) = x^2e^{yz}$

5. Calculate the following integrals:
   a. $\int_0^1 \int_2^3 x^2y^3 \; dxdy$
   b. $\int_2^3 \int_0^1 x^2y^3 \; dydx$
   c. $\int_0^1 \int_0^{\sqrt{1-x^2}} 2x^3y \; dydx$

6. Suppose we were interested in learning about how years of schooling affect the probability that a person turns out to vote. To simplify things, let's say we just have one observation of each variable. Let Y be our single observation of the dependent variable (whether or not a person turns out to vote) and X be our single observation of the independent variable, (the number of years of education that same person has). We believe that the process used to generate our data takes the following form:

$$Y = \beta X + \epsilon$$

   where ϵ is an error term. We include this error term because we think random occurrences in the world will mean our model produces estimates that are slightly wrong sometimes, but we believe that on average, this model accurately relates X to Y. We observe the values of X and Y, but what about β? How do we know the value of β that best approximates this relationship, i.e., what's the slope of this line?

There are different criteria we could use, but a popular choice is the method of least squares. In this process, we solve for the value of β that minimizes the sum of squared errors, ϵ^2, in our data. Using the tools of minimization we've been practicing, find the value of β that minimizes this quantity. (Hint: In this case there is only one observation, so the sum of squared errors is equal to the single error squared.)