Computational Math Camp

Problem Sets

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Contents

Overview

Contains problem sets for the 2019 Computational Math Camp.

Chapter 1

Linear equations, notation, sets, and functions

1.1 Simplify expressions

Simplify the following expressions as much as possible:

a. $(-x^4y^2)^2$

1. Distribute exponents over products.

$$(-1)^2x^{(2\times4)}y^{(2\times2)}$$

2. Multiply 2 and 2 together.

$$(-1)^2x^{(2\times4)}y^4$$

3. Multiply 2 and 4 together.

$$(-1)^2x^8y^4$$

4. Evaluate $(-1)^2$.

$$x^8y^4$$

b. $9(3^0)$

1. Any nonzero number to the zero power is 1.

$$9(1)$$

2. Anything times 1 is the same value.

$$9$$

c. $(2a^2)(4a^4)$

1. Combine products of like terms.

$$2a^2 \times 4a^4 = 2 \times 4a^{(2+4)}$$

2. Evaluate $2+4$.

$$2 \times 4a^6$$

3. Multiply 2 and 4 together.

$$8a^6$$

d. $\dfrac{x^4}{x^3}$

1. For all exponents, $\dfrac{a^n}{a^m} = a^{(n-m)}$.

$$x^{(4-3)}$$

2. Evaluate $4-3$.

$$x$$

e. $(-2)^{7-4}$

1. Subtract 4 from 7.

$$(-2)^3$$

2. In order to evaluate 2^3 express 2^3 as 2×2^2.

$$-2 \times 2^2$$

3. Evaluate 2^2.

$$-2 \times 4$$

4. Multiply -2 and 4 together.

$$-8$$

f. $\left(\dfrac{1}{27b^3}\right)^{1/3}$

1. Separate component terms.

$$\frac{1}{27}^{1/3} \times \frac{1}{b^3}^{1/3}$$

2. Evaluate cube roots.

$$\frac{1}{3} \times \frac{1}{b}$$

3. Combine terms.

$$\frac{1}{3b}$$

g. $y^7y^6y^5y^4$

1. Combine products of like terms.

$$y^{(7+6+5+4)}$$

2. Evaluate $7+6+5+4$.

$$y^{22}$$

h. $\dfrac{2a/7b}{11b/5a}$

1. Write as a single fraction by multiplying the numerator by the reciprocal of the denominator.

$$\frac{2a}{7b} \times \frac{5a}{11b}$$

2. Product property of exponents: $x^a \times x^b = x^{(a+b)}$

$$\frac{5a \times 2a}{7b \times 11b} = \frac{5 \times 2a^{1+1}}{7 \times 11b^{1+1}}$$

3. Evaluate $1 + 1$.

$$\frac{5 \times 2a^2}{7 \times 11b^2}$$

4. Multiple scalars together.

$$\frac{10a^2}{77b^2}$$

i. $(z^2)^4$

1. Nested exponents rule: $(x^a)^b = x^{ab}$

$$z^{2\times 4}$$

2. Evaluate 2×4

$$z^8$$

1.2 Simplify a (more complex) expression

Simplify the following expression:

$$(a + b)^2 + (a - b)^2 + 2(a + b)(a - b) - 3a^2$$

1. Expand $(a + b)^2$ with FOIL.

$$a^2 + 2ab + b^2 + (a - b)^2 + 2(a + b)(a - b) - 3a^2$$

2. Expand $(a - b)^2$ with FOIL.

$$a^2 + 2ab + b^2 + a^2 - 2ab + b^2 + 2(a + b)(a - b) - 3a^2$$

3. Multiply $a + b$ and $a - b$ together using FOIL.

$$a^2 + 2ab + b^2 + a^2 - 2ab + b^2 + 2(a^2 - b^2) - 3a^2$$

4. Distribute 2 over $a^2 - b^2$.

$$a^2 + 2ab + b^2 + a^2 - 2ab + b^2 + 2a^2 - 2b^2 - 3a^2$$

5. Group like terms.

$$(a^2 + a^2 + 2a^2 - 3a^2) + (b^2 + b^2 - 2b^2) + (2ab - 2ab)$$

6. Combine like terms.

$$a^2 + (b^2 + b^2 - 2b^2) + (2ab - 2ab)$$

7. Look for the difference of two identical terms.

$$a^2$$

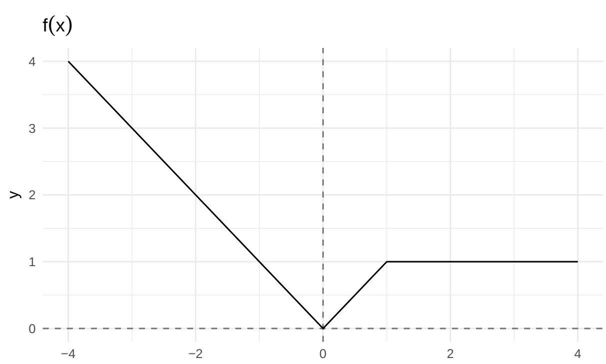
1.3 Graph sketching

Let the functions $f(x)$ and $g(x)$ be defined for all $x \in \Re$ by

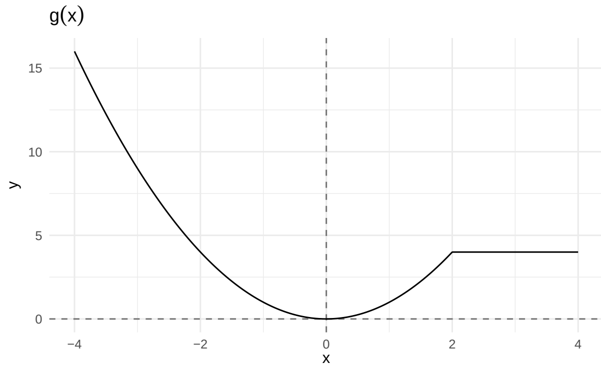
$$f(x) = \begin{cases} |x| & \text{if } x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}, \quad g(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 4 & \text{if } x \geq 2 \end{cases}$$

Sketch the graphs of:

1. $y = f(x)$



2. $y = g(x)$



3. $y = f(g(x))$

   To sketch the composite function, we first evaluate $g(x)$ for different values of x, and then evaluate $f(g(x))$ for different outputs of $g(x)$.

   - For $x \geq 2$, $g(x)$ is a constant value:

   $$\begin{aligned} x &\geq 2 \\ g(x) &= 4 \\ f(g(x)) &= f(4) = 1 \end{aligned}$$

   - For $x < 2$, $g(x)$ is not constant: $g(x) = x^2$. $f(x)$ evaluates differently depending on its input, so we have two cases based on the output of $g(x)$:

     – if $g(x) < 1$, $f(g(x)) = |g(x)| = |x^2| = x^2$. This is the case when:

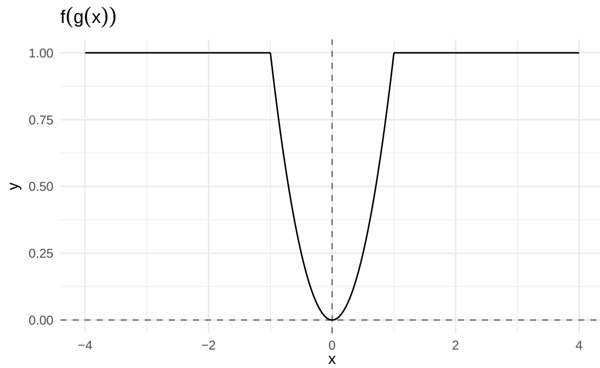
   $$\begin{aligned} g(x) &< 1 \\ x^2 &< 1 \text{ and } x < 2 \\ -1 &< x < 1 \end{aligned}$$

     – if $g(x) \geq 1$, $f(g(x)) = 1$. This is the case when:

   $$\begin{aligned} g(x) &\geq 1 \\ x^2 &\geq 1 \text{ and } x < 2 \\ x &\leq -1 \text{ or } 1 \leq x < 2 \end{aligned}$$

   - Therefore, $f(g(x))$ has the following values:

   $$f(g(x)) = \begin{cases} 1 & \text{if } x \leq -1 \\ x^2 & \text{if } -1 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$



4. $y = g(f(x))$

To sketch the composite function, we first evaluate $f(x)$ for different values of x, and then evaluate $g(f(x))$ for different outputs of $f(x)$.

- For $x \geq 1$, $f(x)$ is a constant value:

$$\begin{aligned} x &\geq 1 \\ f(x) &= 1 \\ g(f(x)) = f(1) &= 1^2 = 1 \end{aligned}$$

- For $x < 1$, $f(x)$ is not constant: $f(x) = |x|$. $g(x)$ evaluates differently depending on its input, so we have two cases based on the output of $f(x)$:
  - if $f(x) < 2$, $g(f(x)) = f(x)^2 = |x|^2 = x^2$. This is the case when:

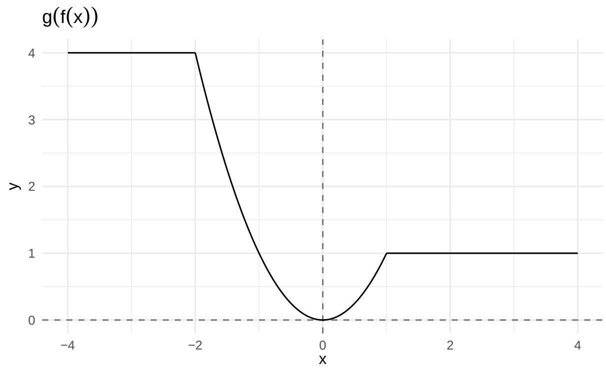
$$\begin{aligned} f(x) &< 2 \\ |x| < 2 \text{ and } x &< 1 \\ -2 < x &< 1 \end{aligned}$$

  - if $f(x) \geq 2$, $g(f(x)) = 4$. This is the case when:

$$\begin{aligned} f(x) &\geq 2 \\ |x| \geq 2 \text{ and } x &< 1 \\ x &\leq -2 \end{aligned}$$

- Therefore, $g(f(x))$ has the following values:

$$g(f(x)) = \begin{cases} 4 & \text{if } x \leq -2 \\ x^2 & \text{if } -2 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$



1.4 Root finding

Find the roots (solutions) to the following quadratic equations.

Definition 1.1 (The quadratic formula)**.**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

a. $4x^2 - 1 = 17$

- Move terms so that x is alone on the left side of the equation.

$$\begin{aligned} 4x^2 - 1 &= 17 \\ 4x^2 &= 18 \\ x^2 &= \frac{18}{4} \\ x^2 &= \frac{9}{2} \\ x &= \pm\sqrt{\frac{9}{2}} \end{aligned}$$

b. $9x^2 - 3x - 12 = 0$

- Factor the left-hand side.

$$3(x+1)(3x-4) = 0$$

- Divide both sides by 3 to simplify the equation.

$$(x+1)(3x-4) = 0$$

- Find the roots of each term in the product separately by solving for x.

$$\begin{aligned} x+1 &= 0 \\ x &= -1 \end{aligned} \qquad \begin{aligned} 3x &= 4 \\ x &= \frac{4}{3} \end{aligned}$$

c. $x^2 - 2x - 16 = 0$

1. Complete the square

$$\begin{aligned} x^2 - 2x - 16 &= 0 \\ x^2 - 2x &= 16 \\ x^2 - 2x + 1 &= 17 \\ (x-1)^2 &= 17 \\ x - 1 &= \pm\sqrt{17} \\ x &= 1 \pm \sqrt{17} \end{aligned}$$

2. Quadratic formula

   - Using the quadratic formula, solve for x

$$\begin{aligned} x &= \frac{-(-2) \pm \sqrt{(-2)^2 - (4 \times 1 \times 16)}}{2 \times 1} \\ x &= \frac{2 \pm \sqrt{4 + 64}}{2} \\ x &= \frac{2 \pm \sqrt{68}}{2} \end{aligned}$$

   - Simplify the radical

$$\begin{aligned} x &= \frac{2 \pm \sqrt{2^2 \times 17}}{2} \\ x &= \frac{2 \pm 2\sqrt{17}}{2} \end{aligned}$$

   - Factor the greatest common divisor

$$x = 1 \pm \sqrt{17}$$

d. $6x^2 - 6x - 6 = 0$

- Divide both sides by 6 to simplify the equation.

$$x^2 - x - 1 = 0$$

- Using the quadratic formula, solve for x

$$\begin{aligned}
x &= \frac{-(-1) \pm \sqrt{(-1)^2 - (4 \times 1 \times -1)}}{2 \times 1} \\
x &= \frac{1 \pm \sqrt{1 - 4(-1)}}{2} \\
x &= \frac{1 \pm \sqrt{1 + 4}}{2} \\
x &= \frac{1 \pm \sqrt{5}}{2}
\end{aligned}$$

e. $5 + 11x = -3x^2$

- Move everything to the left hand side.

$$3x^2 + 11x + 5 = 0$$

- Using the quadratic formula, solve for x

$$\begin{aligned}
x &= \frac{-11 \pm \sqrt{(11)^2 - (4 \times 3 \times 5)}}{2 \times 3} \\
x &= \frac{-11 \pm \sqrt{121 - 60}}{6} \\
x &= \frac{-11 \pm \sqrt{61}}{6}
\end{aligned}$$

1.5 Work with sets

Using the sets

$$\begin{aligned}
A &= \{2, 3, 7, 9, 13\} \\
B &= \{x : 4 \leq x \leq 8 \text{ and } x \text{ is an integer}\} \\
C &= \{x : 2 < x < 25 \text{ and } x \text{ is prime}\} \\
D &= \{1, 4, 9, 16, 25, \ldots\}
\end{aligned}$$

identify the following:

1. $A \cup B$

   $E = \{2, 3, 4, 5, 6, 7, 8, 9, 13\}$, combine all integers between 4 and 8 inclusive with the numbers in set A.

2. $(A \cup B) \cap C$

   $F = \{3, 5, 7, 13\}$, Since C is only prime numbers greater than 2 and less than 25, we take all the prime numbers that are also included in E, but remember to drop out 2 since it is not included in C.

3. $C \cap D$

   $G = \emptyset$, there are no prime numbers in D, so nothing is shared between C and D.

Chapter 2

Logarithms, sequences, and limits

2.1 Simplify logarithms

Express each of the following as a single logarithm:

a. $\log(x) + \log(y) - \log(z)$

- Multiplication rule of logarithms: $\log(x \times y) = \log(x) + \log(y)$
- Division rule of logarithms: $\log(\frac{x}{y}) = \log(x) - \log(y)$
- Applying the log rules, we combine logs that are added through multiplication and then combine logs that are subtracted with division.

$$\log(x) + \log(y) - \log(z)$$

$$\log(xy) - \log(z)$$

$$\log(\frac{xy}{z})$$

b. $2\log(x) + 1$

- Exponentiation rule of logarithms: $\log(x^y) = y\log(x)$
- $\log(e) = 1$

$$2\log(x) + 1$$

$$2\log(x) + \log(e)$$

$$\log(x^2) + \log(e)$$

$$\log(ex^2)$$

c. $\log(x) - 2$

- $\log(e) = 1$

$$\log(x) - 2$$

$$\log(x) - 2\log(e)$$

$$\log(x) - \log(e^2)$$

$$\log(\frac{x}{e^2})$$

2.2 Sequences

Write down the first three terms of each of the following sequences. In each case, state whether the sequence is an arithmetic progression, a geometric progression, or neither.

a. $u_n = 4 + 3n$

$$7, 10, 13$$

Arithmetic progression.

b. $u_n = 5 - 6n$

$$-1, -7, -13$$

Arithmetic progression.

c. $u_n = 4^n$

$$4, 16, 64$$

Geometric progression.

d. $u_n = 5 \times (-2)^n$

$$-10, 20, -40$$

Geometric progression.

e. $u_n = n \times 3^n$

$$3, 18, 81$$

Neither.

2.3 Find the limit

In each of the following cases, state whether the sequence $\{u_n\}$ tends to a limit, and find the limit if it exists:

a. $u_n = 1 + \frac{1}{2}n$

No limit ($u_n \to \infty$)

b. $u_n = 1 - \frac{1}{2}n$

No limit ($u_n \to \infty$)

c. $u_n = \left(\frac{1}{2}\right)^n$

Yes. $\lim\limits_{x\to\infty} u_n = 0$

d. $u_n = \left(-\frac{1}{2}\right)^n$

Yes. $\lim\limits_{x\to\infty} u_n = 0$

2.4 Determine convergence or divergence

Determine whether each of the following sequences converges or diverges. If it converges, find the limit.

a. $a_n = \frac{3+5n^2}{n+n^2}$

The sequence converges to 5. We can see this by factoring n^2 from both the numerator and denominator and then cancelling it out.

$$\lim_{n\to\infty} a_n = \frac{3+5n^2}{n+n^2} = \lim_{n\to\infty} \frac{n^2\left(\frac{3}{n^2}+5\right)}{n^2\left(\frac{1}{n}+1\right)} = \lim_{n\to\infty} \frac{\left(\frac{3}{n^2}+5\right)}{\left(\frac{1}{n}+1\right)} = \frac{\lim_{n\to\infty}\frac{3}{n^2}+5}{\lim_{n\to\infty}\frac{1}{n}+1} = \frac{0+5}{0+1} = 5$$

(This is slightly curt: Make sure you know how to show that the limit of $\frac{3}{n^2}$ approaches 0.) As $n \to \infty$, $\frac{3}{n} \to 0$ and $\frac{1}{n} \to 0$. Therefore, $a_n \to 5$.

Alternatively, you could split the fraction into two terms: one with a numerator of 3, and the other with a numerator of $5n^2$. The first fraction converges to 0. (Can you show that?) Factoring out an n from both sides of the second fraction, you're left with $\frac{5n}{n+1}$; $\frac{n}{n+1}$ converges to 1, giving you $5 \times 1 = 5$.

b. $a_n = \frac{(-1)^{n-1}n}{n^2+1}$

The sequence converges to 0. To see why, take the absolute value of the sequence, then factor out and cancel n from both sides of the fraction.

$$\lim_{n\to\infty}\left|\frac{(-1)^{n-1}n}{n^2+1}\right| = \lim_{n\to\infty}\frac{1^{n-1}n}{n^2+1} = \lim_{n\to\infty}\frac{1}{n+\frac{1}{n}} = \frac{\lim_{n\to\infty}1}{\lim_{n\to\infty}(n+\frac{1}{n})} = \frac{1}{\lim_{n\to\infty}n+0} = 0$$

2.5 Find more limits

Given that

$$\lim_{x\to a} f(x) = -3, \quad \lim_{x\to a} g(x) = 0, \quad \lim_{x\to a} h(x) = 8$$

find the limits that exist. If the limit doesn't exist, explain why.

a. $\lim\limits_{x\to a}[f(x)+h(x)] = -3+5 = 8$
b. $\lim\limits_{x\to a}[f(x)]^2 = (-3)^2 = 9$
c. $\lim\limits_{x\to a}\sqrt[3]{h(x)} = \sqrt[3]{8} = 2$
d. $\lim\limits_{x\to a}\frac{1}{f(x)} = -\frac{1}{3}$
e. $\lim\limits_{x\to a}\frac{f(x)}{h(x)} = -\frac{3}{8}$
f. $\lim\limits_{x\to a}\frac{g(x)}{f(x)} = \frac{0}{-3} = 0$
g. $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \frac{-3}{0}$ = Undefined – cannot divide by 0, no limit

h. $\lim_{x\to a} \frac{2f(x)}{h(x)-f(x)} = \frac{2\times -3}{8-(-3)} = -\frac{6}{11}$

2.6 Find even more limits

Find the limits of the following:

a. $\lim_{x\to -4} \frac{x^2+5x+4}{x^2+3x-4}$

$$\lim_{n\to -4} \frac{x^2+5x+4}{x^2+3x-4} = \lim_{n\to -4} \frac{(x+4)(x+1)}{(x+4)(x-1)} = \lim_{n\to -4} \frac{x+1}{x-1} = \frac{\lim_{n\to -4}(x+1)}{\lim_{n\to -4}(x-1)} = \frac{-3}{-5} = \frac{3}{5}$$

b. $\lim_{x\to 4^-} \sqrt{16-x^2}$

$$\begin{aligned}\lim_{n\to 4^-} \sqrt{16-x^2} &= \lim_{n\to 4^-} \sqrt{(4+x)(4-x)} \\ &= \lim_{n\to 4^-} \sqrt{4+x}\sqrt{4-x} \\ &= \lim_{n\to 4^-} \sqrt{4+x} \cdot \lim_{n\to 4^-} \sqrt{4-x} \\ &= \sqrt{8}\cdot\sqrt{0} \\ &= 0\end{aligned}$$

c. $\lim_{x\to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x}$

$$\begin{aligned}\lim_{n\to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x} &= \lim_{n\to -4} \frac{\frac{x+4}{4x}}{4+x} \\ &= \lim_{n\to -4} \frac{4+x}{4x}\cdot\frac{1}{4+x} \\ &= \lim_{n\to -4} \frac{1}{4x} \\ &= \frac{1}{4(-4)} \\ &= -\frac{1}{16}\end{aligned}$$

2.7 Check for discontinuities

Which of the following functions are continuous? If not, where are the discontinuities?

a. $f(x) = \dfrac{9x^3-x}{(x-1)(x+1)}$

- Discontinuous at $x = -1, +1$ (denominator would be 0, leaving the fraction undefined)

b. $f(x) = e^{-x^2}$

- Continuous for all real numbers.

c. $f(y) = y^3 - y^2 + 1$

- All polynomials are continuous.

d. $f(x) = \begin{cases} x^3 + 1, & x > 0 \\ \frac{1}{2} & x = 0 \\ -x^2, & x < 0 \end{cases}$

- Discontinuous at $x = 0$. This is a piecewise function. To be continuous $\lim_{x \to 0^+} f(x) = 0$. However in this function, $\lim_{x \to 0^+} f(x) = 1 \neq 0$.

Chapter 3

Differentiation

3.1 Find finite limits

Find the following finite limits:

a. $\lim_{x\to 4} x^2 - 6x + 4$

b. $\lim_{x\to 0} \left[\frac{x-25}{x+5}\right]$

c. $\lim_{x\to 4} \left[\frac{x^2}{3x-2}\right]$

d. $\lim_{x\to 1} \left[\frac{x^4-1}{x-1}\right]$

e. $\lim_{x\to -4} \left[\frac{x^2+5x+4}{x^2+3x-4}\right]$

f. $\lim_{x\to 4^-} \sqrt{16-x^2}$

g. $\lim_{x\to -1} \left[\frac{x-2}{x^2+4x-3}\right]$

h. $\lim_{x\to -4} \left[\frac{\frac{1}{4}+\frac{1}{x}}{4+x}\right]$

3.2 Find infinite limits

Find the following infinite limits:

> Hint: use **L'Hôpital's Rule** to switch from $\lim_{x\to\infty} \left(\frac{f(x)}{g(x)}\right)$ to $\lim_{x\to\infty} \left(\frac{f'(x)}{g'(x)}\right)$.

a. $\lim_{x \to \infty} \left[\frac{9x^2}{x^2+3} \right]$
b. $\lim_{x \to \infty} \left[\frac{3x-4}{x+3} \right]$
c. $\lim_{x \to \infty} \left[\frac{2^x-3}{2^x+1} \right]$
d. $\lim_{x \to \infty} \left[\frac{\log(x)}{x} \right]$
e. $\lim_{x \to \infty} \left[\frac{3^x}{x^3} \right]$
f. $\lim_{y \to \infty} \left[\frac{3e^y}{y^3} \right]$

3.3 Assessing continuity and differentiability

For each of the following functions, describe whether it is continuous and/or differentiable at the point of transition of its two formulas.

a.

$$f(x) = \begin{cases} +x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$

b.

$$f(x) = \begin{cases} +x^2+1, & x \geq 0 \\ -x^2-1, & x < 0 \end{cases}$$

c.

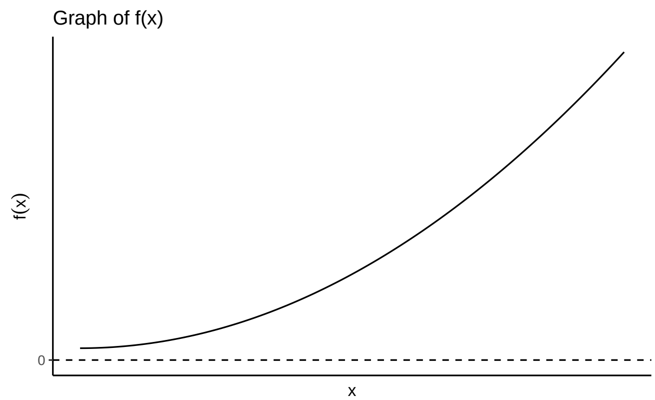
$$f(x) = \begin{cases} x^3, & x \leq 1 \\ x, & x > 1 \end{cases}$$

d.

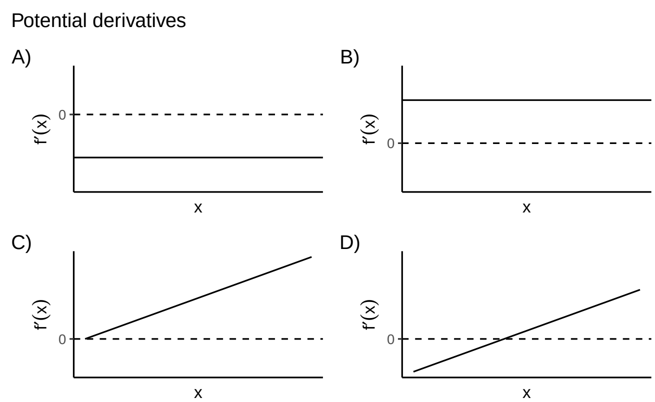
$$f(x) = \begin{cases} x^3, & x < 1 \\ 3x-2, & x \geq 1 \end{cases}$$

3.4 Possible derivative

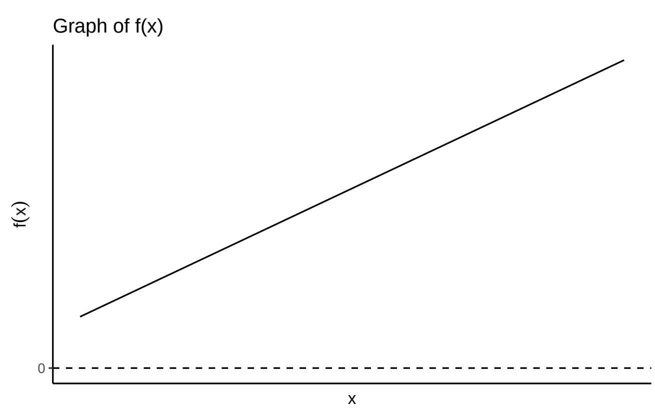
A friend shows you this graph of a function $f(x)$:



Which of the following could be a graph of $f'(x)$? For each graph, explain why or why not it might be the derivative of $f(x)$.



What if the figure below was the graph of $f(x)$? Which of the graphs might potentially be the derivative of $f(x)$ then?



3.5 Calculate derivatives

Differentiate the following functions:

a. $f(x) = 4x^3 + 2x^2 + 5x + 11$
b. $y = \sqrt{30}$
c. $h(t) = \log(9t + 1)$
d. $f(x) = \log(x^2 e^x)$
e. $h(y) = \left(\frac{1}{y^2} - \frac{3}{y^4}\right)(y + 5y^3)$
f. $g(t) = \frac{3t - 1}{2t + 1}$

3.6 Use the product and quotient rules

Differentiate the following using both the product and quotient rules:

$$f(x) = \frac{x^2 - 2x}{x^4 + 6}$$

3.7 Logarithms and exponential functions

Compute the derivative of each of the following functions:

a. $f(x) = xe^{3x}$
b. $f(x) = \frac{x}{e^x}$
c. $h(x) = \frac{x}{\log(x)}$

3.8 Composite functions

For each of the following pairs of functions $g(x)$ and $h(z)$, write out the composite function $g(h[z])$ and $h(g[x])$. In each case, describe the domain of the composite function.

a. $g(x) = x^2 + 4, \quad h(z) = 5z - 1$
b. $g(x) = x^3, \quad h(z) = (z - 1)(z + 1)$
c. $g(x) = 4x + 2, \quad h(z) = \frac{1}{4}(z - 2)$
d. $g(x) = \frac{1}{x}, \quad h(z) = z^2 + 1$

3.9 Chain rule

Use the chain rule to compute the derivative of the first three composite functions in the previous section from the derivatives of the two component functions. Then, compute each derivative directly using your expression for the composite function. Simplify and compare your answers.

a. $g(x) = x^2 + 4, \quad h(z) = 5z - 1$
b. $g(x) = x^3, \quad h(z) = (z-1)(z+1)$
c. $g(x) = 4x + 2, \quad h(z) = \frac{1}{4}(z-2)$