Scalar Calculus

Computational Mathematics and Statistics Camp
University of Chicago
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1. Find the following finite limits:

   a. $\lim\limits_{x \to 4} x^2 - 6x + 4$

$$\begin{aligned}\lim_{x \to 4} x^2 - 6x + 4 &= 4^2 - 6(4) + 4 \\ &= 16 - 24 + 4 \\ &= -4\end{aligned}$$

   b. $\lim\limits_{x \to 0} \left[\dfrac{x - 25}{x + 5} \right]$

$$\begin{aligned}\lim_{x \to 0} \left[\frac{x - 25}{x + 5} \right] &= \frac{0 - 25}{0 + 5} \\ &= \frac{-25}{5} \\ &= -5\end{aligned}$$

   c. $\lim\limits_{x \to 4} \left[\dfrac{x^2}{3x - 2} \right]$

$$\begin{aligned}\lim_{x \to 4} \left[\frac{x^2}{3x - 2} \right] &= \frac{4^2}{3(4) - 2} \\ &= \frac{16}{12 - 2} \\ &= \frac{16}{10} \\ &= \frac{8}{5}\end{aligned}$$

   d. $\lim\limits_{x \to 1} \left[\dfrac{x^4 - 1}{x - 1} \right]$

   The key here is to factor the initial expression in the numerator, then cancel terms out with the denominator:

$$\begin{aligned}\lim_{x \to 1} \left[\frac{x^4 - 1}{x - 1} \right] &= \lim_{x \to 1} \left[\frac{(x - 1)(x + 1)(x^2 + 1)}{x - 1} \right] \\ &= \lim_{x \to 1} [(x + 1)(x^2 + 1)] \\ &= (1 + 1)(1^2 + 1) \\ &= (2)(2) \\ &= 4\end{aligned}$$

   Alternatively, we can use L'Hôpital's Rule:

$$\begin{aligned}\lim_{x\to 1}\left[\frac{x^4-1}{x-1}\right] &= \lim_{x\to 1}\left[\frac{4x^3}{1}\right] \\ &= \frac{4(1)^3}{1} \\ &= 4\end{aligned}$$

e. $\lim\limits_{x\to -4}\left[\frac{x^2+5x+4}{x^2+3x-4}\right]$

The key here is to factor the initial expression:

$$\begin{aligned}\lim_{x\to -4}\frac{(x+4)(x+1)}{(x+4)(x-1)} &= \lim_{x\to -4}\frac{x+1}{x-1} \\ &= \frac{\lim_{x\to -4}(x+1)}{\lim_{x\to -4}(x-1)} \\ &= \frac{-3}{-5} \\ &= \frac{3}{5}\end{aligned}$$

f. $\lim\limits_{x\to 4^-}\sqrt{16-x^2}$

$$\begin{aligned}\lim_{x\to 4^-}\sqrt{16-x^2} &= \lim_{x\to 4^-}\sqrt{(4+x)(4-x)} \\ &= \lim_{x\to 4^-}\sqrt{4+x}\sqrt{4-x} \\ &= \lim_{x\to 4^-}\sqrt{4+x}\cdot\lim_{x\to 4^-}\sqrt{4-x} \\ &= \sqrt{8}*\sqrt{0} \\ &= 0\end{aligned}$$

A critical aspect of this limit, which allows for it to exist, is that it is a left-hand limit.

g. $\lim\limits_{x\to -1}\left[\frac{x-2}{x^2+4x-3}\right]$

$$\begin{aligned}\lim_{x\to -1}\frac{x-2}{x^2+4x-3} &= \frac{\lim_{x\to -1}(x-2)}{\lim_{x\to -1}(x^2+4x-3)} \\ &= \frac{-1-2}{(-1)^2+4(-1)-3} \\ &= \frac{-3}{-6} \\ &= \frac{1}{2}\end{aligned}$$

h. $\lim\limits_{x\to -4}\left[\frac{\frac{1}{4}+\frac{1}{x}}{4+x}\right]$

$$\begin{aligned}
\lim_{x\to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x} &= \lim_{x\to -4} \frac{\frac{x+4}{4x}}{4+x} \\
&= \lim_{x\to -4} \frac{4+x}{4x}\frac{1}{4+x} \\
&= \lim_{x\to -4} \frac{1}{4x} \\
&= \frac{1}{4(-4)} \\
&= -\frac{1}{16}
\end{aligned}$$

Alternatively, we can use L'Hôpital's Rule:

$$\begin{aligned}
\lim_{x\to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x} &= \lim_{x\to -4} \frac{-\frac{1}{x^2}}{1} \\
&= \lim_{x\to -4} (-\frac{1}{x^2}) \\
&= -\frac{1}{16}
\end{aligned}$$

2. Given that:

$$\lim_{x\to a} f(x) = -3 \qquad \lim_{x\to a} g(x) = 0 \qquad \lim_{x\to a} h(x) = 8$$

find the following limits. If the limit doesn't exist, explain why.

a. $\lim_{x\to a} [f(x) + h(x)]$

$$-3 + 8 = 5$$

b. $\lim_{x\to a} [f(x)]^2$

$$(-3)^2 = 9$$

c. $\lim_{x\to a} \left[\frac{f(x)}{h(x)}\right]$

$$\frac{-3}{8}$$

d. $\lim_{x\to a} \left[\frac{g(x)}{f(x)}\right]$

$$\frac{0}{-3} = 0$$

3. Find the following infinite limits:

Hint: use **L'Hôpital's Rule** to switch from $\lim_{x\to\infty} \left(\frac{f(x)}{g(x)}\right)$ to $\lim_{x\to\infty} \left(\frac{f'(x)}{g'(x)}\right)$.

a. $\lim_{x\to\infty}\left[\frac{9x^2}{x^2+3}\right]$

$$\begin{aligned}\lim_{x\to\infty}\left[\frac{9x^2}{x^2+3}\right] &= \lim_{x\to\infty}\left[\frac{18x}{2x}\right]\\ &= 9\end{aligned}$$

b. $\lim_{x\to\infty}\left[\frac{3x-4}{x+3}\right]$

$$\begin{aligned}\lim_{x\to\infty}\left[\frac{3x-4}{x+3}\right] &= \lim_{x\to\infty}\left[\frac{3}{1}\right]\\ &= 3\end{aligned}$$

c. $\lim_{x\to\infty}\left[\frac{2^x-3}{2^x+1}\right]$

Remember that $\frac{d}{dx}n^x = \log(n)n^x$:

$$\begin{aligned}\lim_{x\to\infty}\left[\frac{2^x-3}{2^x+1}\right] &= \lim_{x\to\infty}\left[\frac{\log(2)2^x}{\log(2)2^x}\right]\\ &= 1\end{aligned}$$

d. $\lim_{x\to\infty}\left[\frac{\log(x)}{x}\right]$

$$\begin{aligned}\lim_{x\to\infty}\left[\frac{\log(x)}{x}\right] &= \lim_{x\to\infty}\left[\frac{\frac{1}{x}}{1}\right]\\ &= \lim_{x\to\infty}\left[\frac{1}{x}\right]\\ &= \frac{1}{\infty}\\ &= 0\end{aligned}$$

e. $\lim_{x\to\infty}\left[\frac{3^x}{x^3}\right]$

The trick here is to repeatedly calculate the derivative of the numerator and denominators until there is no x term on the denominator. You end up calculating the third derivative, but L'Hôpital's Rule still applies.

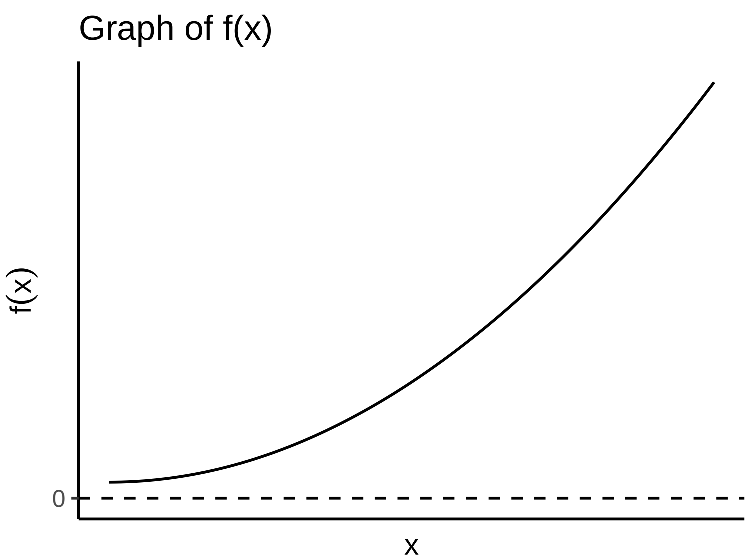
$$\begin{aligned}\lim_{x\to\infty}\left[\frac{3^x}{x^3}\right] &= \lim_{x\to\infty}\left[\frac{\log(3)3^x}{3x^2}\right]\\ &= \lim_{x\to\infty}\left[\frac{\log^2(3)3^x}{6x}\right]\\ &= \lim_{x\to\infty}\left[\frac{\log^3(3)3^x}{6}\right]\\ &= \frac{\log^3(3)3^\infty}{6}\\ &= \infty\end{aligned}$$

f. $\lim_{y\to\infty}\left[\frac{3e^y}{y^3}\right]$

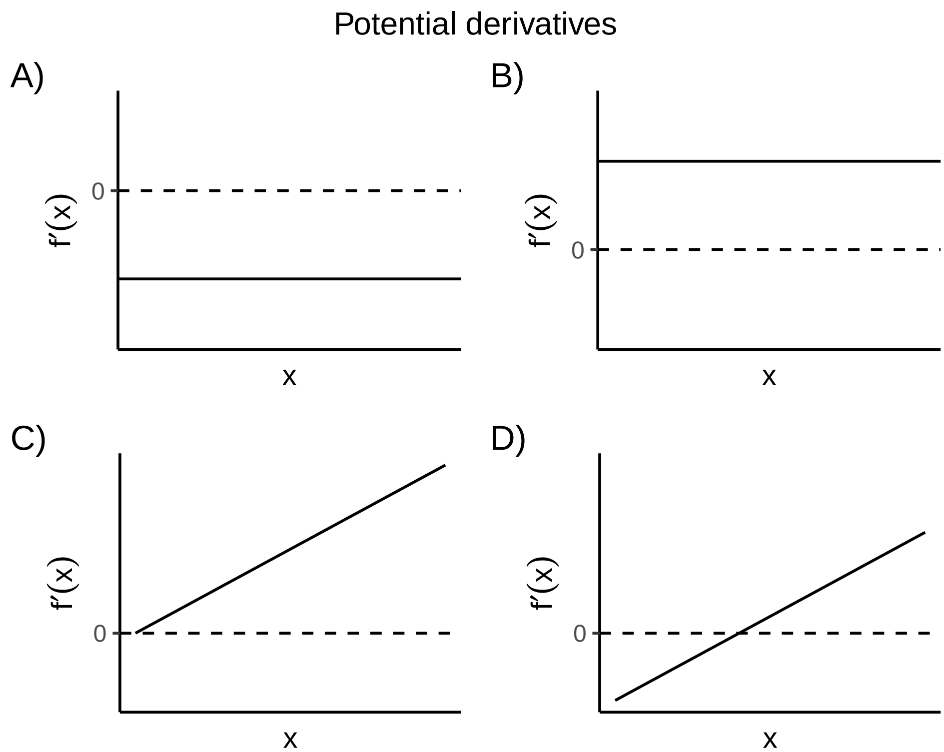
Same as above: repeatedly calculate the derivatives until the y term disappears in the denominator.

$$\begin{aligned}\lim_{x\to\infty}\left[\frac{3e^y}{y^3}\right] &= \lim_{x\to\infty}\left[\frac{3e^y}{3y^2}\right]\\ &= \lim_{x\to\infty}\left[\frac{3e^y}{6y}\right]\\ &= \lim_{x\to\infty}\left[\frac{3e^y}{6}\right]\\ &= \frac{3e^\infty}{6}\\ &= \infty\end{aligned}$$

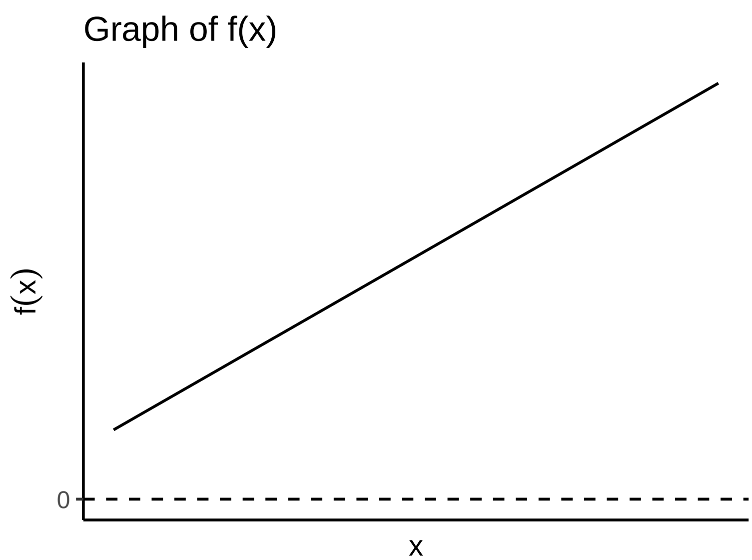
4. A friend shows you this graph of a function $f(x)$:



Which of the following could be a graph of $f'(x)$? For each graph, explain why or why not it might be the derivative of $f(x)$.



What if the figure below was the graph of $f(x)$? Which of the graphs might potentially be the derivative of $f(x)$ then?



a. A doesn't work because it is negative and the function we observe is increasing in x. B is constant so this also won't work, the function we observe gets larger at an increasing, not constant rate. C seems to be a plausible candidate because an upward sloping derivative would map to the behavior of the function we observe, that $g(x)$ gets large at an increasing rate. D does not work because it suggests the function would need to be decreasing over some interval and because, when we refer back to $g(x)$, there doesn't seem to be any local minimum, maximum or a saddle point despite the graph in D crossing 0.

b. Again, A doesn't work because it is negative and the function we observe is increasing in x. B seems to be plausible as the derivative, since $g(x)$ appears to increase at a constant rate, its derivative should be flat and greater than 0. C won't work because the slope of $g(x)$ is constant and does not increase in x. D doesn't work, again because it suggests the function would need to be decreasing at some point over the interval we observe.

5. Differentiate the following functions:

   a. $f(x) = 4x^3 + 2x^2 + 5x + 11$

   Power rule.

$$
\begin{aligned}
f(x) &= 4x^3 + 2x^2 + 5x + 11 \\
f'(x) &= 12x^2 + 4x + 5
\end{aligned}
$$

   b. $y = \sqrt{30}$

   Derivative of a constant is 0.

$$
\begin{aligned}
y &= \sqrt{30} \\
y' &= 0
\end{aligned}
$$

   c. $h(t) = \log(9t + 1)$

   Derivative of $\log(u)$ is $\frac{1}{u}$. Since u is a function in this problem, need to apply the chain rule to calculate the derivative of $9t + 1$ and multiply that by $\frac{1}{9t+1}$

$$
\begin{aligned}
h(t) &= \log(9t + 1) \\
h'(t) &= \frac{1}{9t+1} * 9
\end{aligned}
$$

   d. $f(x) = \log(x^2 e^x)$

   Derivative of a logarithm plus the chain rule.

$$
\begin{aligned}
f(x) &= \log(x^2 e^x) \\
f'(x) &= \frac{1}{x^2 e^x} * (2xe^x + e^x x^2) \\
&= \frac{2xe^x + e^x x^2}{x^2 e^x} \\
&= \frac{2}{x} + 1
\end{aligned}
$$

   e. $h(y) = \left(\frac{1}{y^2} - \frac{3}{y^4}\right)(y + 5y^3)$

   Simplify the expression first, then basic application of power rule.

$$
\begin{aligned}
h(y) &= \left(\frac{1}{y^2} - \frac{3}{y^4}\right)\left(y + 5y^3\right) \\
&= \frac{y}{y^2} + \frac{5y^3}{y^2} - \frac{3y}{y^4} - \frac{15y^3}{y^4} \\
&= \frac{1}{y} + 5y - \frac{3}{y^3} - \frac{15}{y} \\
&= 5y - \frac{14}{y} - \frac{3}{y^3} \\
h'(y) &= 5 + \frac{14}{y^2} + \frac{9}{y^4}
\end{aligned}
$$

f. $g(t) = \dfrac{3t-1}{2t+1}$

Quotient rule.

$$
\begin{aligned}
g(t) &= \frac{3t-1}{2t+1} \\
g'(t) &= \frac{(3)(2t+1) - (3t-1)(2)}{(2t+1)^2} \\
&= \frac{5}{(2t+1)^2}
\end{aligned}
$$

6. Differentiate the following using both the product and quotient rules:

$$
f(x) = \frac{x^2 - 2x}{x^4 + 6}
$$

a. First let's use the quotient rule:

$$
\begin{aligned}
h(x) &= \frac{f(x)}{g(x)} \\
f(x) &= x^2 - 2x \\
g(x) &= x^4 + 6 \\
f'(x) &= 2x - 2 \\
g'(x) &= 4x^3 \\
h'(x) &= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \\
&= \frac{(2x-2)(x^4+6) - (x^2-2x)(4x^3)}{(x^4+6)^2} \\
&= \frac{2x^5 + 12x - 2x^4 - 12 - 4x^5 + 8x^4}{(x^4+6)^2} \\
&= \frac{-2x^5 + 6x^4 + 12x - 12}{(x^4+6)^2}
\end{aligned}
$$

b. Now we can do the same thing with the product rule:

$$
\begin{aligned}
j(x) &= k(x)m(x) \\
k(x) &= x^2 - 2x \\
m(x) &= (x^4+6)^{-1} \\
k'(x) &= 2x - 2 \\
m'(x) &= -(x^4+6)^{-2}(4x^3) = -\frac{4x^3}{(x^4+6)^2} \\
j'(x) &= k(x)m'(x) + k'(x)m(x) \\
&= (x^2-2x)(-\frac{4x^3}{(x^4+6)^2}) + (2x-2)(x^4+6)^{-1} \\
&= -\frac{(x^2-2x)(4x^3)}{(x^4+6)^2} + \frac{2x-2}{x^4+6} \\
&= -\frac{4x^5-8x^4}{x^4+6)^2} + \frac{2x-2}{x^4+6} \\
&= -\frac{4x^5-8x^4}{x^4+6)^2} + \frac{2x-2}{x^4+6}\frac{x^4+6}{x^4+6} \\
&= -\frac{4x^5-8x^4}{(x^4+6)^2} + \frac{2x^5+12x-2x^4-12}{(x^4+6)^2} \\
&= \frac{2x^5+12x-2x^4-12-4x^5-8x^4}{(x^4+6)^2} \\
&= \frac{-2x^5+6x^4+12x-12}{(x^4+6)^2}
\end{aligned}
$$

The quotient rule is simply a derivation of the product rule combined with the chain rule:

$$
\begin{aligned}
h(x) &= \frac{f(x)}{g(x)} \\
&= f(x)g(x)^{-1}
\end{aligned}
$$

Apply product and chain rules:

$$
\begin{aligned}
h'(x) &= f'(x)g(x)^{-1} + f(x)(-1)g(x)^{-2}g'(x) \\
&= f'(x)g(x)g(x)^{-2} - f(x)g(x)^{-2}g'(x) \\
&= [f'(x)g(x) - f(x)g'(x)]g(x)^{-2} \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}
\end{aligned}
$$

which is the quotient rule.

7. Does a continuous, differentiable function exist on $[0,2]$ such that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leq 2\,\forall\, x$? Use the mean value theorem to explain your answer.

   First we set up the mean value theorem which states that, if a function is continuous and differentiable over some interval, then a c exists such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

   We plug in the values given by the problem and find, $f'(c) = \frac{f(2)-f(0)}{2-0} = \frac{4--1}{2} = \frac{5}{2}$.

   The problem states that the derivative of the function is less than or equal to 2 over this entire interval, but the mean value theorem tell us that that the derivative must equal 2.5 at some point. So by demonstrating this contradiction, we've shown that the earlier values could not have come from a continuous, differentiable function.

8. Solve the following definite integrals using the antiderivative method.

   For all these problems, the basic approach to compute the definite integral of $f(x)$ from a to b is by using the formula $F(b) - F(a)$, where $F(x)$ is the **antiderivative of f.

   a. $\int_6^8 x^3\,dx$

      Basic power rule. Or more so the reverse of the power rule for derivatives.

$$\begin{aligned}\int_6^8 x^3\,dx &= \left(\frac{1}{4}x^4\right)\Big|_6^8 \\ &= \frac{1}{4}8^4 - \frac{1}{4}6^4 \\ &= \frac{4096}{4} - \frac{1296}{4} \\ &= 1024 - 324 \\ &= 700\end{aligned}$$

   b. $\int_{-1}^0 (3x^2 - 1)\,dx$

$$\begin{aligned}\int_{-1}^0 (3x^2 - 1)\,dx &= x^3 - x\Big|_{-1}^0 \\ &= (0^3 - 0) - ((-1)^3 - (-1)) \\ &= 0 - (-1 + 1) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

   c. $\int_{-1}^1 (14 + x^2)\,dx$

$$\begin{aligned}\int_{-1}^1 (14 + x^2)\,dx &= 14x + \frac{1}{3}x^3\Big|_{-1}^1 \\ &= 14(1) + \frac{1}{3}(1)^3 - (14(-1) + \frac{1}{3}(-1)^3) \\ &= 14 + \frac{1}{3} - (-14 - \frac{1}{3}) \\ &= 14 + \frac{1}{3} + 14 + \frac{1}{3}) \\ &= 28\frac{2}{3}\end{aligned}$$

   d. $\int_1^2 \frac{1}{t^2}\,dt$

$$
\begin{aligned}
\int_1^2 \frac{1}{t^2}\, dt &= \left(-\frac{1}{t}\right)\Big|_1^2 \\
&= -\frac{1}{2} - -\frac{1}{1} \\
&= -\frac{1}{2} + 1 \\
&= \frac{1}{2}
\end{aligned}
$$

e. $\int_2^4 e^y\, dy$

$$
\begin{aligned}
\int_2^4 e^y\, dy &= e^y\Big|_2^4 \\
&= e^4 - e^2 \\
&\approx 47.209
\end{aligned}
$$

f. $\int_8^9 2^x\, dx$

As a general rule, the antiderivative of a^x (where a is a constant) is $\frac{a^x}{\ln a}$ plus a constant.

$$
\begin{aligned}
\int_8^9 2^x\, dx &= \frac{2^x}{\log 2}\Big|_8^9 \\
&= \frac{2^9}{\log 2} - (\frac{2^8}{\log 2}) \\
&= \frac{2^8}{\log 2} \\
&= \frac{256}{\log 2} \\
&\approx 369.33
\end{aligned}
$$

g. $\int_3^3 \sqrt{x^5 + 2}\, dx$

This question is a bit sneaky. Trying to find the antiderivative of this function would be far from trivial. However, we do not actually need to do so! Notice that we are evaluating the integral from 3 to 3. Whatever the antiderivative function F actually is, $F(3) - F(3) = 0$, and 0 is the solution. Intuitively, this should make sense - in effect, we are taking the area under the curve at a single point. In other words, we are taking the area of a line, and a line has no area.

9. A group of three unidentified first-year graduate students at the University of Chicago are worn out after a week of math camp. Wanting to unwind, the students agree to not talk about math and decide to chat over some casual drinks at Medici.

   After five shots of tequila each, two pitchers of beer, a bottle of wine, and a large Chicago-style pizza, the three students have had enough fun and decide to start the trip back home.

   Student A gets on a bike and starts pedaling away at a velocity of $v_A(t) = 2t^4 + t$, where t represents minutes. However, the student crashes into the side of an Uber and ends the journey after only 2 minutes.

   Student B has no bike, so starts running at a velocity of $v_B(t) = 4\sqrt{t}$. Sadly, after only 4 minutes, the student's legs give out and the student decides to sing a song, instead.

   Student C can't even stand up, so has no choice but to slowly crawl at a velocity of $v_C(t) = 2e^{-t}$. Student C steadily plods along for 20 minutes before falling asleep on the sidewalk.

Generally, if an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$. The Fundamental Theorem of Calculus then tells us that

$$\text{Total distance traveled} = \int_{t_1}^{t_2} v(t)\, dt$$
$$s(t_2) - s(t_1) = \int_{t_1}^{t_2} v(t)\, dt$$

Without using a calculator, use this formula to find the distance traveled by Students A, B, and C. (Assume, however unrealistic in may be, that all three students traveled in a straight line.) Who traveled the farthest? The least far?

a. For Student A:

$$\int_0^2 2t^4 + t = \frac{2}{5}t^5 + \frac{1}{2}t^2\Big|_0^2 = \frac{2}{5}(2^5) + \frac{1}{2}(2^2) - \left[\frac{2}{5}(0^5) + \frac{1}{2}(0^2)\right] = \frac{2(32)}{5} + \frac{4}{2} - [0+0] = \frac{64}{5} + 2 = \frac{74}{5}$$

b. For Student B:

$$\int_0^4 4\sqrt{t} = 4\left(\frac{2}{3}\right)t^{3/2}\Big|_0^4 = \frac{8}{3}(4^{3/2}) - \frac{8}{3}(0^{3/2}) = \frac{8}{3}\left(\sqrt{4^3}\right) - 0 = \frac{8}{3}\sqrt{64} = \frac{8}{3}(8) = \frac{64}{3}$$

c. For Student C:

$$\int_0^{20} 2e^{-t} = -2e^{-t}\Big|_0^{20} = -2e^{-20} - [-2e^0] = -2e^{-20} + 2(1) = -\frac{2}{e^{20}} + 2 \approx 0 + 2 = 2$$

Clearly, Student C had the shortest trip. We can also eyeball that Student B traveled just a bit more than 20 units of distance, while A went a little less than 15. If you want concrete numbers to perform the comparison, you can calculate the quotients or find a common denominator for $\frac{64}{3}$ and $\frac{74}{5}$. You'd see that Student A went $\frac{222}{15}$ while B went $\frac{320}{15}$. Student B went the farthest. (But nobody made it home.)

10. Calculate the following indefinite integrals:

a. $\int (x^2 - x^{-\frac{1}{2}})\, dx$

$$\int (x^2 - x^{-\frac{1}{2}})\, dx = \frac{1}{3}x^3 - 2\sqrt{x} + c$$

b. $\int 360t^6\, dt$

$$\int 360t^6\, dt = \frac{360}{7}t^7 + c$$

c. $\int 2x \log(x^2)\, dx$

This integral requires **integration by parts**. Recall that the rule states:

$$\begin{aligned} f(x) &= u \\ g(x) &= v \\ dv &= v'dx \\ du &= u'dx \end{aligned}$$

Based on the **product rule**:

$$\begin{aligned}\int \frac{d}{dx}(uv)\,dx &= \int uv'\,dx + \int vu'\,dx \\ uv &= \int u(v'\,dx) + \int v(u'\,dx) \\ &= \int udv + \int vdu\end{aligned}$$

Rearrange the formula:

$$\int udv = uv - \int vdu$$

Now apply this technique to the problem at hand:

$$\int 2x\log(x^2)\,dx$$

$$\begin{aligned}u &= \log(x^2) \\ du &= \frac{2}{x} \\ dv &= 2xdx \\ v &= x^2\end{aligned}$$

$$\begin{aligned}\int \log(x^2)2x\,dx &= \log(x^2)x^2 - \int (x)^2\frac{2}{x}dx \\ &= \log(x^2)x^2 - \int 2xdx \\ &= \log(x^2)x^2 - x^2 + c \\ &= x^2\log(x^2) - x^2 + c\end{aligned}$$