PSET 9: Answer Key
Classical statistical inference

1 Name + Info

Note: all homework uploads should have the questions identified. We'll be giving zero credit for submissions that don't follow this protocol as it adds considerable time to grading. Thank you!

- Name
- How long did this problem set take you?
- How difficult was this problem set? very easy 1 2 3 4 5 very challenging

2 Properties of estimators

a. Let $X_1, \ldots, X_n \sim \text{Poisson}(\lambda)$ and let $\hat{\lambda} = \frac{\sum_{i=1}^n X_i}{2n}$. Find the bias, standard error, and MSE of this estimator.[1]

Solution. $X_1, \ldots, X_n \overset{iid}{\sim} \text{Poisson}(\lambda)$, $\hat{\lambda} = \frac{\sum_{i=1}^n X_i}{2n} = \frac{\bar{X}}{2}$.

$$\mathbb{E}[\hat{\lambda}] = \frac{1}{2}\mathbb{E}[\bar{X}] = \frac{1}{2}\lambda, \qquad \text{Bias}(\hat{\lambda}) = \mathbb{E}[\hat{\lambda}] - \lambda = -\frac{\lambda}{2}.$$

$$\text{Var}(\hat{\lambda}) = \frac{1}{(2n)^2}\text{Var}\Big(\sum_{i=1}^n X_i\Big) = \frac{1}{4n^2}(n\lambda) = \frac{\lambda}{4n}, \quad \text{SE}(\hat{\lambda}) = \sqrt{\frac{\lambda}{4n}} = \frac{1}{2}\sqrt{\frac{\lambda}{n}}.$$

$$\text{MSE}(\hat{\lambda}) = \text{Var}(\hat{\lambda}) + \text{Bias}(\hat{\lambda})^2 = \frac{\lambda}{4n} + \frac{\lambda^2}{4}.$$

b. Let $X_1, \ldots, X_n \sim \text{Uniform}(0, \theta)$ and let $\hat{\theta} = 3\bar{X}_n$. Find the bias, standard error, and MSE of this estimator.[2]

Solution.
$X_1, \ldots, X_n \overset{iid}{\sim} \text{Unif}(0, \theta)$, $\hat{\theta} = 3\bar{X}$. Since $\mathbb{E}[\bar{X}] = \theta/2$ and $\text{Var}(\bar{X}) = \theta^2/(12n)$,

$$\mathbb{E}[\hat{\theta}] = 3 \cdot \frac{\theta}{2} = \frac{3\theta}{2}, \qquad \text{Bias}(\hat{\theta}) = \frac{3\theta}{2} - \theta = \frac{\theta}{2}.$$

$$\text{Var}(\hat{\theta}) = 9\,\text{Var}(\bar{X}) = \frac{9\theta^2}{12n} = \frac{3\theta^2}{4n}, \quad \text{SE}(\hat{\theta}) = \theta\sqrt{\frac{3}{4n}} = \frac{\theta}{2}\sqrt{\frac{3}{n}}.$$

[1] Inspired by Wasserman 6.6.1
[2] Inspired by Wasserman 6.6.3

$$\mathrm{MSE}(\hat{\theta}) = \frac{3\theta^2}{4n} + \left(\frac{\theta}{2}\right)^2 = \theta^2\left(\frac{3}{4n} + \frac{1}{4}\right).$$

3 Social newsing

A poll conducted in 2025 found that 12% of U.S. adults users get at least some news on TikTok. The standard error for this estimate was 2.4%, and a normal distribution may be used to model the sample proportion.[3]

a. Construct a 95% confidence interval for the fraction of U.S. adult users who get some news on TikTok, and interpret the confidence interval in context.

Solution.
Given $\hat{p} = 0.12$ with $\mathrm{SE} = 0.024$ and normal approximation, a 95% CI using $z_{1-\alpha/2}$ is

$$\hat{p} \pm z_{0.975} \cdot \mathrm{SE} = 0.12 \pm 1.96(0.024) = 0.12 \pm 0.04704 = (0.07296,\, 0.16704).$$

We are 95% confident that the true fraction of U.S. adults who get at least some news on TikTok lies between 7.3% and 16.7%.

4 (T/F) Social Newsing

Identify the following statements as true or false using the above information. Provide an explanation to justify each of your answers. Explain in approx 75 words.

a. The data provide statistically significant evidence that more than 16% of U.S. adults get some news through TikTok Use a significance level of $\alpha = 0.05$.

Solution.

False. Testing $H_0 : p \leq 0.16$ vs $H_a : p > 0.16$ with $z = (0.12 - 0.16)/0.024 = -1.67$. The one-sided p-value is $\Pr(Z \geq -1.67) \approx 0.95$, which is not significant at $\alpha = 0.05$. We do not have evidence to reject the null hypothesis that 16% or less of U.S. adults get some news through TikTok.

b. Since the standard error is 2.4%, we can conclude that 97.6% of all U.S. adults were included in the study.

Solution.

False. The standard error (2.4%) quantifies sampling variability of $\hat{p}$; it does not say anything about the fraction of the population surveyed. Coverage rates and sample inclusion are unrelated.

c. If we want to reduce the standard error of the estimate, we should collect less data.

Solution.

[3] Inspired byOI 4.8 and 4.10

False. For proportions, $\text{SE} = \sqrt{p(1-p)/n}$ decreases as n increases. Collecting more data reduces SE; collecting less data increases SE.

d. If we construct a 90% confidence interval for the percentage of U.S. adults who get some news through TikTok, this confidence interval will be wider than a corresponding 99% confidence interval.

Solution.

False. A 90% CI uses a smaller critical value than a 99% CI, so it is narrower. Higher confidence demands a wider interval to maintain coverage.

5 Dating on college campuses

A survey conducted on a reasonably random sample of 200 undergraduates asked, among many other questions, about the number of exclusive relationships these students have been in. The histogram below shows the distribution of the data from this sample.

The sample average is 3.2 with a standard deviation of 2.17.

Estimate the average number of exclusive relationships undergraduate students have been in using the Normal distribution and a 95% confidence interval and interpret this interval in context.[4]

Solution.

From the plot, the distribution of counts is right–skewed with a thin tail to 10–11. Because observations are counts, normality of the raw data is not likely. However, since $n = 200$, the sampling distribution of the mean is approximately Normal by the CLT, provided the sample is reasonably random and observations are independent. The tail inflates the sample variance relative to a Poisson with mean 3.2 (since $s^2 = 2.17^2 = 4.71 > 3.2$); using the sample s already accounts for this.

A one–sample $t_{1-\alpha/2,n-1}$ interval (population SD unknown) is appropriate:

$$\bar{x} = 3.2, \qquad s = 2.17, \qquad n = 200, \qquad \text{SE} = \frac{s}{\sqrt{n}} = \frac{2.17}{\sqrt{200}} = 0.15353.$$

With $df = 199$, $t_{0.975,199} = 1.972$. Hence

$$\bar{x} \pm t_{0.975,199}\,\text{SE} \;=\; 3.2 \pm 1.972(0.15353) \;=\; 3.2 \pm 0.3025 \;=\; (2.8975,\; 3.5025).$$

We are 95% confident that the population mean number of exclusive relationships among undergraduates lies between about 2.90 and 3.50. The histogram's right–skew indicates that a few students report many relationships, but with $n = 200$ these do not invalidate a normal approximation for the mean. This widens the interval via the larger s.

[4]Inspired byOI 4.15

6 Statistical significance

Determine whether the following statement is true or false, and explain your reasoning: "With large sample sizes, even small differences between the null value and the point estimate can be statistically significant."[5]

Solution.

True. As n grows, the standard error shrinks. For a fixed nonzero difference between the estimator and the null value, the test statistic increases in magnitude with n, often yielding small p-values. This is why statistical significance with large samples may not imply actual significance.

7 Sleep deprivation

New York is known as "the city that never sleeps". A random sample of 25 New Yorkers were asked how much sleep they get per night. Statistical summaries of these data are shown below. Do these data provide strong evidence that New Yorkers sleep less than 8 hours a night on average?[6]

n	$\bar{x}$	s	min	max
25	7.63	0.87	6.17	9.78

a. Write the hypotheses in symbols and in words.

Solution.

Hypotheses.

$$H_0 : \mu = 8 \quad \text{(mean sleep is 8 hours);} \qquad H_a : \mu < 8 \quad \text{(mean sleep is less than 8 hours).}$$

b. Calculate the test statistic, T, and the associated degrees of freedom.

Solution.

Using a one-sample t-test,

$$T = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{7.63 - 8}{0.87/\sqrt{25}} = \frac{-0.37}{0.174} = -2.126, \qquad \text{df} = n - 1 = 24.$$

c. Find and interpret the p-value in this context.

Solution.

$p = \Pr\{T_{24} \leq -2.126\} \approx 0.02\text{–}0.03$ (about 0.022).

If the true mean were 8 hours, observing a sample mean this low or lower is quite unlikely ($\sim 2\%$).

[5] Inspired byOI 4.47

[6] Inspired byOI 5.7

d. What is the conclusion of the hypothesis test?

Solution.

At $\alpha = 0.05$, we reject H_0 and conclude there is statistically significant evidence that New Yorkers sleep less than 8 hours on average.

e. If you were to construct a 90% confidence interval that corresponded to this hypothesis test, would you expect 8 hours to be in the interval?

Solution.

The two-sided 90% CI is

$$\bar{x} \pm t_{0.95,24} \frac{s}{\sqrt{n}} = 7.63 \pm 1.711 \cdot 0.174 = 7.63 \pm 0.297 = (7.333, 7.927).$$

Since 8 is not in this interval, the evidence is consistent with the one-sided rejection.

8 Interpreting public opinion polls

On June 28, 2012 the U.S. Supreme Court upheld the much debated 2010 healthcare law, declaring it constitutional. A Gallup poll released the day after this decision indicates that 46% of 1,012 Americans agree with this decision. At a 95% confidence level, this sample has a 3% margin of error. Based on this information, determine if the following statements are true or false, and explain your reasoning.[7]

a. We are 95% confident that between 43% and 49% of Americans in this sample support the decision of the U.S. Supreme Court on the 2010 healthcare law.

Solution.

False. The sample support is exactly 46%; a CI targets the population parameter, not the sample.

b. We are 95% confident that between 43% and 49% of Americans support the decision of the U.S. Supreme Court on the 2010 healthcare law.

Solution.

True. A correct 95% interpretation: we are 95% confident the true population support lies between 43% and 49%.

[7] Inspired by OI 6.6

c. If we considered many random samples of 1,012 Americans, and we calculated the sample proportions of those who support the decision of the U.S. Supreme Court, 95% of those sample proportions will be between 43% and 49%.

Solution.

False. Across many samples, 95% of constructed intervals will cover the true p. It is not correct to claim 95% of future sample proportions fall between the fixed bounds $(0.43, 0.49)$ from this specific sample.

d. The margin of error at a 90% confidence level would be higher than 3%.

Solution.

False. Lower confidence (e.g., 90%) uses a smaller critical value and thus a smaller margin of error than 3%.

9 Power and sample size.

Suppose you have two samples: one with a sample mean of 12, standard deviation of of 2.13 and n_1 of 87 and a second with a sample mean of 12.8, standard deviation of 3.25 and an n_2 of 89.

a. Is it statistically significant?

Solution.
Difference $\Delta = \bar{x}_2 - \bar{x}_1 = 0.8$.

$$\mathrm{SE}_\Delta = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = \sqrt{\frac{2.13^2}{87} + \frac{3.25^2}{89}} = \sqrt{0.05217 + 0.11812} = 0.4128.$$

$$t = \frac{0.8}{0.4128} = 1.94, \quad \mathrm{df} \approx \frac{(a+b)^2}{a^2/(n_1-1) + b^2/(n_2-1)} \approx 153,$$

where $a = s_1^2/n_1$, $b = s_2^2/n_2$. Two-sided $p \approx 0.054$ (one-sided $p \approx 0.027$).
Therefore, not significant at $\alpha = 0.05$ for a two-sided test.

b. What is the smallest n_2 you could have if you want to reject the null hypothesis at $\alpha < 0.05$?

Solution.
Smallest n_2 to reject at $\alpha = 0.05$ (two-sided), holding other quantities fixed. Require

$$\frac{|\Delta|}{\sqrt{a + \frac{s_2^2}{n_2}}} \geq 1.96 \iff a + \frac{s_2^2}{n_2} \leq \left(\frac{0.8}{1.96}\right)^2 = \frac{4}{24.01} \approx 0.16667.$$

With $a = \frac{2.13^2}{87} = 0.05217$,

$$\frac{s_2^2}{n_2} \leq 0.16667 - 0.05217 = 0.11449 \iff n_2 \geq \frac{3.25^2}{0.11449} = \frac{10.5625}{0.11449} \approx 92.29.$$

Thus the smallest integer n_2 is 93.

c. Darn it! Your n_2 is actually only 85! What should your goal be: smaller sample deviation by 10% or larger sample by 10? Explain your reasoning and provide mathematical support.

Solution.
With $n_2 = 85$, do we prioritize a 10% reduction in s_2 or +10 more observations?

$$\text{SE term for sample } 2 = \frac{s_2^2}{n_2}.$$

Variance reduction: $s_2 \mapsto 0.9s_2 \Rightarrow s_2^2$ scales by $0.9^2 = 0.81$ (a 19% drop). Sample-size increase: $n_2 : 85 \to 95 \Rightarrow$ factor $85/95 \approx 0.895$ (a 10.5% drop).
Numerically:

$$\text{Keep } n_2 = 85,\ s_2 \to 2.925: \quad \text{SE}_\Delta = \sqrt{0.05217 + \frac{2.925^2}{85}} = \sqrt{0.05217 + 0.10066} = 0.3910,\ t = 0.8/0.3910 = 2.046.$$
$$\text{Keep } s_2 = 3.25,\ n_2 \to 95: \quad \text{SE}_\Delta = \sqrt{0.05217 + \frac{3.25^2}{95}} = \sqrt{0.05217 + 0.11119} = 0.4042,\ t = 0.8/0.4042 = 1.98.$$

Reducing the standard deviation by 10% improves power more than adding 10 observations as the SE depends on s_2^2/n_2. The correct option is the smaller sample deviation goal.

10 AI and Resources statement

- Please list (in detail) all resources you used for this assignment. If you worked with people, list them here as well. It is not enough to say that you used a resource for help, you need to be specific on the link and *how* it was helpful. W/R/T gen AI tools (including GPT, etc.) you cannot use them to do work on your behalf – you cannot put in any of the questions, etc. You can ask for help on logic / sample problems. If you do use GPT or other AI tools, you need to provide a link to your chat transcript. Any suspected academic integrity violations will be immediately reported.

10.1 Survey Qs:

1. How prepared do you feel for the final exam? not prepared (1) 2 3 4 (5) very prepared
2. Which content area of the course was the easiest for you? Why? Please be specific.
3. Which content area of the course was the most challenging for you? Why? Please be specific.