Multivariate Distributions

Calculating conditional PDF

Let $f(x, y) = 15x^2y$ for $0 \leq x \leq y \leq 1$. Find $f(x|y)$.[1]
Solution:

$$\begin{aligned} f(y) &= \int_0^y f(x, y)\, dx \\ &= \int_0^y 15x^2 y\, dx \\ &= 15y \int_0^y x^2\, dx \\ &= 15y \frac{x^3}{3}\bigg|_0^y \\ &= \frac{15y^4}{3} \end{aligned}$$

$$\begin{aligned} f(x|y) &= \frac{f(x, y)}{f(y)} \\ &= \frac{15x^2 y}{15y^4/3} \\ &= \frac{3x^2}{y^3} \end{aligned}$$

[1] Grimmer HW12.4

Properties of a joint PDF

- **a.**

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} kx^2y^3\,dx\,dy = 1$$
$$\int_{0}^{6}\int_{0}^{6} kx^2y^3\,dx\,dy = 1$$
$$k\int_{0}^{6}\int_{0}^{6} x^2y^3\,dx\,dy = 1$$
$$k\int_{0}^{6} y^3\cdot\frac{x^3}{3}\Big|_0^6\,dy = 1$$
$$k\int_{0}^{6} y^3\cdot\left(\frac{6^3}{3}-0\right)\,dy = 1$$
$$72k\int_{0}^{6} y^3\,dy = 1$$
$$72k\cdot\frac{y^4}{4}\Big|_0^6 = 1$$
$$72k\left(\frac{6^4}{4}-0\right) = 1$$
$$72k\cdot 324 = 1$$
$$23328k = 1$$
$$k = \frac{1}{23328}$$

- **b.**

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy \\
&= \int_{-\infty}^{\infty} \frac{x^2y^3}{23328}\,dy \\
&= \int_{0}^{6} \frac{x^2y^3}{23328}\,dy \\
&= \frac{x^2}{23328}\int_{0}^{6} y^3\,dy \\
&= \frac{x^2}{23328}\cdot\frac{y^4}{4}\Big|_0^6 \\
&= \frac{x^2}{23328}\cdot\frac{6^4}{4} \\
&= \frac{x^2}{72}
\end{aligned}$$

- **c.**

$$\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)\,dx \\
&= \frac{1}{23328}\int_0^6 x^2 y^3\,dx \\
&= \frac{1}{23328}\cdot \left.\frac{x^3}{3}\right|_0^6 y^3 \\
&= \frac{1}{23328}\left(\frac{216}{3} - 0\right) y^3 \\
&= \frac{1}{23328}\ (72)\ y^3 \\
&= \frac{y^3}{324}
\end{aligned}$$

- **d.**

$$\begin{aligned}
E[X] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\,dx\,dy \\
&= \frac{1}{23328}\int_0^6\int_0^6 x\cdot x^2 y^3\,dx\,dy \\
&= \frac{1}{23328}\times \int_0^6 x^3\,dx \times \int_0^6 y^3\,dy \\
&= \frac{1}{23328}\times \left.\frac{x^4}{4}\right|_0^6 \times \left.\frac{y^4}{4}\right|_0^6 \\
&= \frac{1}{23328}\left(\frac{1296}{4} - 0\right)\left(\frac{1296}{4} - 0\right) \\
&= \frac{1}{23328}\ (324)\ (324) \\
&= 4.5
\end{aligned}$$

e. Find $E[Y]$

Solution:

$$
\begin{aligned}
E[Y] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)\,dx\,dy \\
&= \frac{1}{23328}\int_0^6\int_0^6 y\cdot x^2y^3\,dx\,dy \\
&= \frac{1}{23328}\times\int_0^6 x^2\,dx\times\int_0^6 y^4\,dy \\
&= \frac{1}{23328}\times\left.\frac{x^3}{3}\right|_0^6\times\left.\frac{y^5}{5}\right|_0^6 \\
&= \frac{1}{23328}\left(\frac{216}{3}-0\right)\left(\frac{6^5}{5}-0\right) \\
&= \frac{1}{23328}(72)\left(\frac{7776}{5}\right) \\
&= 4.8
\end{aligned}
$$

f. Find $Var(X)$

Solution: To find $Var(X)$, we first need to calculate $E[X^2]$:

$$
\begin{aligned}
E[X^2] &= \frac{1}{23328}\int_0^6\int_0^6 x^2\cdot x^2y^3\,dx\,dy \\
&= \frac{1}{23328}\times\int_0^6 x^4\,dx\times\int_0^6 y^3\,dy \\
&= \frac{1}{23328}\times\left.\frac{x^5}{5}\right|_0^6\times\left.\frac{y^4}{4}\right|_0^6 \\
&= \frac{1}{23328}\left(\frac{7776}{5}\right)(324) \\
&= 21.6
\end{aligned}
$$

With $E[X^2]$ determined, we can now calculate $Var(X)$:

$$
\begin{aligned}
Var(X) &= E[X^2]-E[X]^2 \\
&= 21.6-(4.5)^2 \\
&= 21.6-20.25 \\
&= 1.35
\end{aligned}
$$

g. Find $Var(Y)$

Solution: Again, to find $Var(Y)$, we first need $E[Y^2]$:

$$
\begin{aligned}
E[Y^2] &= \frac{1}{23328}\int_0^6\int_0^6 y^2 \cdot x^2y^3\,dx\,dy \\
&= \frac{1}{23328} \times \int_0^6 x^2\,dx \times \int_0^6 y^5\,dy \\
&= \frac{1}{23328} \times \frac{x^3}{3}\bigg|_0^6 \times \frac{y^6}{6}\bigg|_0^6 \\
&= \frac{1}{23328}(72)(7776) \\
&= 24
\end{aligned}
$$

With $E[Y^2]$ in hand, we can now calculate $Var(Y)$:

$$
\begin{aligned}
Var(Y) &= E[Y^2] - E[Y]^2 \\
&= 24 - (4.8)^2 \\
&= 24 - 23.04 \\
&= 0.96
\end{aligned}
$$

h. Find $Cov(X, Y)$

Solution: To find $Cov(X, Y)$, we first need $E[XY]$:

$$
\begin{aligned}
E[XY] &= \frac{1}{23328}\int_0^6\int_0^6 xy \cdot x^2y^3\,dx\,dy \\
&= \frac{1}{23328} \times \int_0^6 x^3\,dx \times \int_0^6 y^4\,dy \\
&= \frac{1}{23328} \times \frac{x^4}{4}\bigg|_0^6 \times \frac{y^5}{5}\bigg|_0^6 \\
&= \frac{1}{23328}\left(\frac{1296}{4}\right)\left(\frac{7776}{5}\right) \\
&= 21.6
\end{aligned}
$$

Now, we calculate $Cov(X, Y)$:

$$\begin{aligned}
Cov(X,Y) &= E[XY] - E[X]E[Y] \\
&= 21.6 - (4.5)(4.8) \\
&= 21.6 - 21.6 \\
&= 0
\end{aligned}$$

i. Are X and Y independent?

Solution: X and Y are independent because $f_{XY}(x,y) = f_X(x)f_Y(y)$ (definition of independence). In other words, the product of the marginal densities of X and Y is equal to the joint density of X and Y:

$$f_X(x)f_Y(y) = \frac{x^2}{72} \times \frac{y^3}{324} = \frac{x^2y^3}{23328} = f_{XY}(x,y)$$

However, we **cannot** say that X and Y are independent simply because the covariance is zero. While it is true that independent variables have a covariance of zero, it is not necessarily true that variables with a covariance of zero are independent.

j. What is the PDF of X conditional on Y, $f_{X|Y}(x|y)$?

Solution: We've previously shown that X and Y are independent. This implies that $f(x) = f(x|y)$, so the answer is the same as the marginal distribution of x from part (b):

$$f(x|y) = f(x) = \frac{x^2}{72}$$

k. What is the PDF of Y conditional on X, $f_{Y|X}(y|x)$?

Solution: Since we have already shown that X and Y are independent, we can refer back to the answer to part (c):

$$f(y|x) = f(y) = \frac{y^3}{324}$$

Properties of Joint Random Variables

Suppose the following:

- $E[D] = 10$
- $E[F] = 4$
- $E[DF] = 8$
- $Var(D) = 60$
- $Var(F) = 60$

a. What is $Cov(D, F)$?

Solution:

$$\begin{aligned} Cov(D, F) &= E[DF] - E[D]E[F] \\ &= 8 - (4 \times 10) \\ &= -32 \end{aligned}$$

b. What is the correlation between D and F?

Solution:

$$\begin{aligned} Cor(D, F) &= \frac{Cov(D, F)}{\sqrt{Var(F)Var(D)}} \\ &= \frac{-32}{\sqrt{60 \times 60}} \\ &= -0.5333 \end{aligned}$$

c. Suppose you multiplied F by 2 to generate a new variable, H. What is $Cov(D, H)$?

Solution: Multiplying F by 2 increases the magnitude of the covariance between D and H.

$$\begin{aligned} Cov(D, H) &= E[DH] - E[D]E[H] \\ E[DH] &= E[D \times 2F] = 2E[DF] = 16 \\ E[H] &= E[2F] = 2E[F] = 8 \\ Cov(D, H) &= 16 - (8 \times 10) = -64 \end{aligned}$$

d. What is $Cor(D, H)$? How does this compare to your answer to Part (b)?

Solution:

$$\begin{aligned}
Var(H) &= Var(2F) = 2^2 Var(F) = 4 \times 60 = 240 \\
Cor(D, H) &= \frac{Cov(D, H)}{\sqrt{Var(D)Var(H)}} \\
&= \frac{-64}{\sqrt{60 \times 240}} \\
&= -0.5333
\end{aligned}$$

This is the same as $Cor(D, F)$. In other words, multiplying one of the variables by a constant leaves the correlation between the two variables unchanged, even though the covariance changes.

e. Suppose instead that $Var(D) = 30$. How would this change $Cor(D, F)$?

Solution: The magnitude of the correlation between the variables increases as $Var(D)$ decreases:

$$\begin{aligned}
Cor(D, F) &= \frac{Cov(D, F)}{\sqrt{Var(F)Var(D)}} \\
&= \frac{-32}{\sqrt{60 \times 30}} \\
&= -0.7542
\end{aligned}$$

Continuous Bayes' Theorem

Previously, we used Bayes' theorem to link the conditional probability of discrete events A given B to the probability of B given A. There is an analogous Bayes' theorem that relates the conditional densities of random variables X and θ (below):

[2]

$$f(\theta \mid X) = \frac{f(X \mid \theta)f(\theta)}{\int f(X \mid \theta)f(\theta)d\theta}$$

Solution:

Recall the definition of the conditional distribution of two random variables:

[2] Grimmer HW12.5

$$f_{\theta|X}(\theta \mid X) = \frac{f(\theta, X)}{f_X(X)}$$

Remember via the "chain rule" of probability that $f(\theta, X) = f(X \mid \theta) f_\theta(\theta)$, and via our rule for marginalization, $f_X(X) = \int f_{X|\theta}(X \mid \theta) f_\theta(\theta) d\theta$. Substitute these equalities in and we have proven the statement:

$$\begin{aligned} f_{\theta|X}(\theta \mid X) &= \frac{f(\theta, X)}{f_X(X)} \\ &= \frac{f(X \mid \theta) f_\theta(\theta)}{\int f_{X|\theta}(X \mid \theta) f(\theta) d\theta} \end{aligned}$$

Submission of Practice Questions

Submit practice questions for the final exam here: https://forms.gle/CPo9FMQgQRPePDfN7. Note that we need at least 10 people to submit before there's enough to circulate!