

PSET 4: Linear algebra

- Name
- How long did this problem set take you?
- How difficult was this problem set? very easy 1 2 3 4 5 very challenging

1 Basic matrix arithmetic

If

$$\mathbf{a} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

find:¹

- a. $\mathbf{a} - \mathbf{b}$
- b. $-4\mathbf{b}$
- c. $3\mathbf{a} + 3\mathbf{b}$

2 More complex matrix arithmetic

Suppose

$$\mathbf{x} = \begin{bmatrix} 3 \\ 2q \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} p+2 \\ -5 \\ 3r \end{bmatrix}$$

.

If $\mathbf{x} = 2\mathbf{y}$, find p, q, r .²

3 Check for linear dependence

Which of the following sets of vectors are linearly dependent?³

In each part, you can denote each vector as $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively.

- a. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$
- b. $\begin{bmatrix} 13 \\ 7 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \\ 8 \end{bmatrix}$

¹inspired by Pemberton and Rau 11.1.2

²inspired by Pemberton and Rau 11.1.3

³inspired by Pemberton and Rau 11.1.4

c. $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

4 Vector length

Find the length of the following vectors:⁴

- a. $(4, 2)$
- b. $(2, 2, 2)$
- c. $(4, 2, 3, 1)$
- d. $(0, 0, 0, 0, 3)$

5 Law of cosines

The **law of cosines** states:

$$\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}$$

where θ is the angle from \mathbf{w} to \mathbf{v} measured in radians. Of importance, $\arccos()$ is the inverse of $\cos()$:

$$\theta = \arccos\left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}\right)$$

For each of the following pairs of vectors, calculate the angle between them. Report your answers in both radians and degrees. To convert between radians and degrees:⁵

$$\text{Degrees} = \text{Radians} \times \frac{180^\circ}{\pi}$$

- a. $\mathbf{v} = (1, 0), \quad \mathbf{w} = (2, 2)$
- b. $\mathbf{v} = (4, 1), \quad \mathbf{w} = (2, -8)$
- c. $\mathbf{v} = (1, 1, 0), \quad \mathbf{w} = (1, 2, 2)$

6 Matrix algebra

Using the matrices below, calculate the following. Some may not be defined; if that is the case, say so.⁶

$$\mathbf{A} = \begin{bmatrix} 3 \\ -2 \\ 9 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 7 & -1 & 5 \\ 0 & 2 & -4 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 3 & 1 \\ 3 & 4 \\ 3 & -7 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 0 & -4 \\ -2 & 1 & -6 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 4 & 1 & -5 \\ 0 & 7 & 7 \\ 2 & -3 & 0 \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} 2 & -8 & -5 \\ -3 & 7 & -4 \\ 1 & 0 & 3 \\ 1 & 2 & 6 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} 9 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

⁴inspired by Simon and Blume 10.10

⁵inspired by Simon and Blume 10.12

⁶inspired by Grimmer HW5.3

$$\mathbf{L} = \begin{bmatrix} 5 & 0 & 3 & 1 \end{bmatrix} \quad \mathbf{M} = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$$

- a. $\mathbf{A} + \mathbf{B}$
- b. $-\mathbf{G}$
- c. \mathbf{D}'
- d. $\mathbf{C} + \mathbf{D}$
- e. $\mathbf{A}'\mathbf{B}$
- f. \mathbf{BC}
- g. \mathbf{FB}
- h. $\mathbf{E} - 5\mathbf{I}_3$
- i. \mathbf{M}^2

7 Matrix inversion

Invert each of the following matrices by hand (you can use a calculator or computer to check your solution, but be sure to show your work). Verify you have the correct inverse by calculating $\mathbf{XX}^{-1} = \mathbf{I}$. Not all of the matrices may be invertible - if not, show why.⁷

- a. $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$
- b. $\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$
- c. $\begin{bmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{bmatrix}$

8 Dummy encoding for categorical variables

Ordinary least squares regression is a common method for obtaining regression parameters relating a set of explanatory variables with a continuous outcome of interest. The vector $\hat{\mathbf{b}}$ that contains the intercept and the regression slope is calculated by the equation:

$$\hat{\mathbf{b}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$$

If an explanatory variable is nominal (i.e. ordering does not matter) with more than two classes (e.g. {White, Black, Asian, Mixed, Other}), the variable must be modified to include in the regression model. A common technique known as **dummy encoding** converts the column into a series of $n - 1$ binary (0/1) columns where each column represents a single class and n is the total number of unique classes in the original column. Explain why this method converts the column into $n - 1$ columns, rather than n columns, in terms of linear algebra. **Reminder: \mathbf{X} contains both the dummy encoded columns as well as a column of 1s representing the intercept.**⁸

⁷inspired by Simon and Blume 8.19

⁸inspired by Benjamin Soltoff

9 Solve the system of equations

Solve the following systems of equations for x, y, z , either via matrix inversion or substitution:⁹

a. System #1

$$\begin{aligned}x + y + 2z &= 2 \\ 3x - 2y + z &= 1 \\ y - z &= 3\end{aligned}$$

b. System #2

$$\begin{aligned}x - y + 2z &= 2 \\ 4x + y - 2z &= 10 \\ x + 3y + z &= 0\end{aligned}$$

10 Multiplying by 0

When it comes to real numbers, we know that if $xy = 0$, then either $x = 0$ or $y = 0$ or both. One might believe that a similar idea applies to matrices, but one would be wrong. Prove that if the matrix product $\mathbf{AB} = \mathbf{0}$ (by which we mean a matrix of appropriate dimensionality made up entirely of zeroes), then it is not necessarily true that either $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$. Hint: in order to prove that something is not always true, simply identify one example where $\mathbf{AB} = \mathbf{0}$, $\mathbf{A}, \mathbf{B} \neq \mathbf{0}$.¹⁰

10.1 AI and Resources statement

- Please list (in detail) all resources you used for this assignment. If you worked with people, list them here as well. It is not enough to say that you used a resource for help, you need to be specific on the link and *how* it was helpful. W/R/T gen AI tools (including GPT, etc.) you cannot use them to do work on your behalf – you cannot put in any of the questions, etc. You can ask for help on logic / sample problems. If you do use GPT or other AI tools, you need to provide a link to your chat transcript. Any suspected academic integrity violations will be immediately reported.

⁹inspired by Gill 4.19

¹⁰inspired by Grimmer HW5.5