Functions of several variables and optimization with several variables

Find first partial derivatives

Find all of the first partial derivatives of each function.[1]

a. $f(x, y) = 3x - 2y^4$

b. $f(x, y) = x^5 + 3x^3y^2 + 3xy^4$

c. $g(x, y) = xe^{3y}$

d. $k(x, y) = \frac{x-y}{x+y}$

e. $h(x, y, z) = x^2e^{yz}$

Find the gradient

Find the gradient ∇f of the following functions and evaluate them at the given points.[2]

a. $f(x, y) = \sqrt{x^2 + y^2}, \quad (x, y) = (3, 4)$

b. $f(x, y, z) = (x + z)e^{x-y}, \quad (x, y, z) = (1, 1, 1)$

Find the Hessian

Find the Hessian H for the following functions.[3]

a. $g(x, y) = x^4 - 3x^2y^3$

b. $f(x, y, z) = xyz - x^2$

Find the critical points

Find the local minimum values, local maximum values, and saddle point(s) of the function. Remember the process we discussed in class: Calculate the gradient, set it equal to zero to solve the system of equations, calculate the Hessian, and assess the Hessian at critical values. Be sure to show your work on each of these steps.[4]

a. $f(x, y) = x^4 + y^4 - 4xy + 2$

b. $k(x, y) = (1 + xy)(x + y)$

Definite integrals

Solve the following definite integrals using the antiderivative method.[5]

[1] Grimmer HW6.3
[2] Grimmer HW6.4
[3] Grimmer HW7.3
[4] Grimmer HW7.4
[5] Gill 5.10 and Grimmer HW4.1

For all these problems, the basic approach to compute the definite integral of $f(x)$ from a to b is by using the formula $F(b) - F(a)$, where $F(x)$ is the **antiderivative** of f.

a. $\int_6^8 x^3\,dx$

b. $\int_{-1}^0 (3x^2 - 1)\,dx$

c. $\int_0^1 x^{\frac{3}{7}}\,dx$

d. $\int_1^2 \dfrac{1}{t^2}\,dt$

e. $\int_2^4 e^y\,dy$

f. $\int_8^9 2^x\,dx$

g. $\int_3^3 \sqrt{x^5 + 2}\,dx$

Applied integration

A group of three unidentified first-year graduate students at the University of Chicago are worn out after a week of math camp. Wanting to unwind, the students agree to not talk about math and decide to chat over some casual drinks at Medici.

After five shots of tequila each, two pitchers of beer, a bottle of wine, and a large Chicago-style pizza, the three students have had enough fun and decide to start the trip back home.

- Student A gets on a bike and starts pedaling away at a velocity of $v_A(t) = 2t^4 + t$, where t represents minutes. However, the student crashes into the side of an Uber and ends the journey after only 2 minutes.
- Student B has no bike, so starts running at a velocity of $v_B(t) = 4\sqrt{t}$. Sadly, after only 4 minutes, the student's legs give out and the student decides to sing a song, instead.
- Student C can't even stand up, so has no choice but to slowly crawl at a velocity of $v_C(t) = 2e^{-t}$. Student C steadily plods along for 20 minutes before falling asleep on the sidewalk.

Generally, if an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$. The Fundamental Theorem of Calculus then tells us that

$$\text{Total distance traveled} = \int_{t_1}^{t_2} v(t)\,dt$$
$$s(t_2) - s(t_1) = \int_{t_1}^{t_2} v(t)\,dt$$

Without using a calculator, use this formula to find the distance traveled by Students A, B, and C. (Assume, however unrealistic in may be, that all three students traveled in a straight line.) Who traveled the farthest? The least far?[6]

Indefinite integrals

Calculate the following indefinite integrals:[7]

a. $\int (x^2 - x^{-\frac{1}{2}})\,dx$

b. $\int 360t^6\,dt$

[6] Grimmer HW4.2
[7] Gill 5.13 and 5.14

c. $\int 2x \log(x^2)\, dx$

Determining convergence

Determine whether each integral is convergent or divergent. Evaluate those that are convergent.[8]

a. $\int_1^\infty \left(\frac{1}{3x}\right)^2\, dx$

b. $\int_0^\infty \cos(x)\, dx$

c. $\int_0^\infty e^{-x}\, dx$

d. $\int_{-\infty}^0 x^3\, dx$

More integrals

Calculate the following integrals:[9]

a. $\int_0^1 \int_2^3 x^2 y^3\ dxdy$

b. $\int_2^3 \int_0^1 x^2 y^3\ dydx$

c. $\int_0^1 \int_0^{\sqrt{1-x^2}} 2x^3 y\ dydx$

[8] Grimmer HW 4.3
[9] Grimmer HW7.5