# Linear algebra

- Name
- How long did this problem set take you?
- How difficult was this problem set? very easy 1 2 3 4 5 very challenging

## Basic matrix arithmetic

If

$$\mathbf{a} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

find:[1]

a. $\mathbf{a} + \mathbf{b}$

$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

b. $-4\mathbf{b}$

$$-4\mathbf{b} = -4 \times \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ -12 \end{bmatrix}$$

c. $3\mathbf{a} - 4\mathbf{b}$

$$3\mathbf{a} - 4\mathbf{b} = 3 \times \begin{bmatrix} 2 \\ 2 \end{bmatrix} - 4 \times \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$$

## More complex matrix arithmetic

Suppose

$$\mathbf{x} = \begin{bmatrix} 3 \\ 2q \\ 6 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} p+2 \\ -5 \\ 3r \end{bmatrix}$$

.

[1] Pemberton and Rau 11.1.2

If $\mathbf{x} = 2\mathbf{y}$, find $p, q, r$.[2]

**Solution:** We can calculate each element of the vector independently, given our knowledge of the relationship between $\mathbf{x}$ and $\mathbf{y}$.

$$\begin{aligned} 3 &= 2(p+2) \\ 3 &= 2p+4 \\ -1 &= 2p \\ -\frac{1}{2} &= p \end{aligned}$$

$$\begin{aligned} 2q &= 2(-5) \\ 2q &= -10 \\ q &= -5 \end{aligned}$$

$$\begin{aligned} 6 &= 2(3r) \\ 6 &= 6r \\ 1 &= r \end{aligned}$$

So $p = -\frac{1}{2}, q = -5, r = 1$.

## Check for linear dependence

Which of the following sets of vectors are linearly dependent?[3]

In each part, you can denote each vector as $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively.

a. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
   Yes: $\mathbf{a} + \mathbf{b} - \mathbf{c} = \mathbf{0}$

b. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$
   Yes: $\mathbf{a} - 2\mathbf{b} + \mathbf{c} = \mathbf{0}$

c. $\begin{bmatrix} 13 \\ 7 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \\ 8 \end{bmatrix}$
   Yes: $0\mathbf{a} + 1\mathbf{b} + 0\mathbf{c} = \mathbf{0}$

d. $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$
   Linearly independent.

## Vector length

Find the length of the following vectors:[4]

[2] Pemberton and Rau 11.1.3
[3] Pemberton and Rau 11.1.4
[4] Simon and Blume 10.10

a. $(3, 4)$

$$\begin{aligned}\sqrt{3^2 + 4^2} &= \sqrt{9 + 16} \\ &= \sqrt{25} \\ &= 5\end{aligned}$$

b. $(0, -3)$

$$\begin{aligned}\sqrt{0^2 + (-3)^2} &= \sqrt{0 + 9} \\ &= \sqrt{9} \\ &= 3\end{aligned}$$

c. $(1, 1, 1)$

$$\begin{aligned}\sqrt{1^2 + 1^2 + 1^2} &= \sqrt{1 + 1 + 1} \\ &= \sqrt{3}\end{aligned}$$

d. $(1, 2, 3)$

$$\begin{aligned}\sqrt{1^2 + 2^2 + 3^2} &= \sqrt{1 + 4 + 9} \\ &= \sqrt{14}\end{aligned}$$

e. $(1, 2, 3, 4)$

$$\begin{aligned}\sqrt{1^2 + 2^2 + 3^2 + 4^2} &= \sqrt{1 + 4 + 9 + 16} \\ &= \sqrt{30} \\ &\approx 5.47726\end{aligned}$$

f. $(3, 0, 0, 0, 0)$

$$\begin{aligned}\sqrt{3^2 + 0^2 + 0^2 + 0^2 + 0^2} &= \sqrt{9 + 0 + 0 + 0 + 0} \\ &= \sqrt{3} \\ &= 3\end{aligned}$$

## Law of cosines

The **law of cosines** states:

$$\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \, \|\mathbf{w}\|}$$

where $\theta$ is the angle from $\mathbf{w}$ to $\mathbf{v}$ measured in radians. Of importance, arccos() is the inverse of cos():

$$\theta = \arccos\left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \, \|\mathbf{w}\|}\right)$$

For each of the following pairs of vectors, calculate the angle between them. Report your answers in both radians and degrees. To convert between radians and degrees:[5]

[5] Simon and Blume 10.12

$$\text{Degrees} = \text{Radians} \times \frac{180^o}{\pi}$$

a. $\mathbf{v} = (1,0), \quad \mathbf{w} = (2,2)$

$$\begin{aligned}
\mathbf{v}\cdot\mathbf{w} &= (1)(2) + (0)(2) \\
&= 2 + 0 \\
&= 2 \\
\|\mathbf{v}\| &= \sqrt{1^2 + 0^2} \\
&= \sqrt{1 + 0} \\
&= \sqrt{1} \\
&= 1 \\
\|\mathbf{w}\| &= \sqrt{2^2 + 2^2} \\
&= \sqrt{4 + 4} \\
&= \sqrt{8} \\
&= \sqrt{2^2 \times 2} \\
&= 2\sqrt{2} \\
\theta &= \arccos\left(\frac{2}{1(2\sqrt{2})}\right) \\
&= \frac{\pi}{4} \\
&= 45^\circ
\end{aligned}$$

b. $\mathbf{v} = (4,1), \quad \mathbf{w} = (2,-8)$

$$\begin{aligned}
\mathbf{v}\cdot\mathbf{w} &= (4)(2) + (1)(-8) \\
&= 8 + (-8) \\
&= 0 \\
\|\mathbf{v}\| &= \sqrt{4^2 + 1^2} \\
&= \sqrt{16 + 1} \\
&= \sqrt{17} \\
&= 1 \\
\|\mathbf{w}\| &= \sqrt{2^2 + (-8)^2} \\
&= \sqrt{4 + 64} \\
&= \sqrt{68} \\
&= \sqrt{2^2 \times 17} \\
&= 2\sqrt{17} \\
\theta &= \arccos\left(\frac{0}{1(2\sqrt{17})}\right) \\
&= \frac{\pi}{2} \\
&= 90^\circ
\end{aligned}$$

Note: you could stop after solving $\mathbf{v}\cdot\mathbf{w}$, because the denominator will be irrelevant.

c. $\mathbf{v} = (1, 1, 0), \quad \mathbf{w} = (1, 2, 2)$

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{w} &= (1)(1) + (1)(2) + (0)(1) \\
&= 1 + 2 + 0 \\
&= 3 \\
\|\mathbf{v}\| &= \sqrt{1^2 + 1^2 + 0^2} \\
&= \sqrt{1 + 1 + 0} \\
&= \sqrt{2} \\
\|\mathbf{w}\| &= \sqrt{1^2 + 2^2 + 2^2} \\
&= \sqrt{1 + 4 + 4} \\
&= \sqrt{9} \\
\theta &= \arccos\left(\frac{3}{\sqrt{2}(\sqrt{9})}\right) \\
&= \arccos\left(\frac{3}{\sqrt{2 \times 9}}\right) \\
&= \arccos\left(\frac{3}{\sqrt{18}}\right) \\
&= \arccos\left(\frac{3}{3\sqrt{2}}\right) \\
&= \arccos\left(\frac{\sqrt{2}}{2}\right) \\
&= \frac{\pi}{4} \\
&= 45^\circ
\end{aligned}$$

# Matrix algebra

Using the matrices below, calculate the following. Some may not be defined; if that is the case, say so.[6]

$$\mathbf{A} = \begin{bmatrix} 3 \\ -2 \\ 9 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 7 & -1 & 5 \\ 0 & 2 & -4 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 3 & 1 \\ 3 & 4 \\ 3 & -7 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 0 & -4 \\ -2 & 1 & -6 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 4 & 1 & -5 \\ 0 & 7 & 7 \\ 2 & -3 & 0 \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} 2 & -8 & -5 \\ -3 & 7 & -4 \\ 1 & 0 & 3 \\ 1 & 2 & 6 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} 9 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} 5 & 0 & 3 & 1 \end{bmatrix} \quad \mathbf{M} = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$$

a. $\mathbf{A} + \mathbf{B}$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 3 + 8 \\ -2 + 0 \\ 9 + (-1) \end{bmatrix} = \begin{bmatrix} 11 \\ -2 \\ 8 \end{bmatrix}$$

[6] Grimmer HW5.3

b. $-\mathbf{G}$

$$-\mathbf{G} = (-1)\begin{bmatrix} 2 & -8 & -5 \\ -3 & 7 & -4 \\ 1 & 0 & 3 \\ 1 & 2 & 6 \end{bmatrix} = \begin{bmatrix} -2 & 8 & 5 \\ 3 & -7 & 4 \\ -1 & 0 & -3 \\ -1 & -2 & -6 \end{bmatrix}$$

c. $\mathbf{D}'$

$$\mathbf{D}' = \begin{bmatrix} 3 & 3 & 3 \\ 1 & 4 & -7 \end{bmatrix}$$

d. $\mathbf{C} + \mathbf{D}$

$\mathbf{C} + \mathbf{D}$ does not exist. The matricies are not the same dimensions.

e. $\mathbf{A}'\mathbf{B}$

This is a $1 \times 3$ matrix multiplied by a $3 \times 1$ matrix, resulting in a $1 \times 1$ matrix (aka a **dot product**).

$$\mathbf{A}'\mathbf{B} = 3(8) + (-2)(0) + 9(-1) = 24 + 0 - 9 = 15$$

f. $\mathbf{BC}$

$\mathbf{BC}$ does not exist. The matricies are non-conformable.

g. $\mathbf{FB}$

$$\begin{aligned} \mathbf{FB} &= \begin{bmatrix} 4 & 1 & -5 \\ 0 & 7 & 7 \\ 2 & -3 & 0 \end{bmatrix}\begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 4(8) & + & 1(0) & + & (-5)(-1) \\ 0(8) & + & 7(0) & + & 7(-1) \\ 2(8) & + & (-3)(0) & + & 0(-1) \end{bmatrix} \\ &= \begin{bmatrix} 32 + 0 + 5 \\ 0 + 0 - 7 \\ 16 + 0 + 0 \end{bmatrix} \\ &= \begin{bmatrix} 37 \\ -7 \\ 16 \end{bmatrix} \end{aligned}$$

h. $\mathbf{E} - 5\mathbf{I_3}$

$$\begin{aligned} \mathbf{E} - 5\mathbf{I}_3 &= \begin{bmatrix} 5 & 2 & 3 \\ 1 & 0 & -4 \\ -2 & 1 & -6 \end{bmatrix} - (5)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 2 & 3 \\ 1 & 0 & -4 \\ -2 & 1 & -6 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2 & 3 \\ 1 & -5 & -4 \\ -2 & 1 & -11 \end{bmatrix} \end{aligned}$$

i. $\mathbf{M}^2$

Recall that $\mathbf{M}^2 = \mathbf{MM}$, so we must pre-multiply the matrix by itself.

$$\begin{aligned}
\mathbf{M}^2 &= \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 1 \times 1 + -1 \times -1 & 1 \times -1 + -1 \times 3 \\ -1 \times 1 + 3 \times -1 & -1 \times -1 + 3 \times 3 \end{bmatrix} \\
&= \begin{bmatrix} 1+1 & -1+(-3) \\ -1-3 & 1+9 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -4 \\ -4 & 10 \end{bmatrix}
\end{aligned}$$

# Matrix inversion

Invert each of the following matricies by hand (you can use a calculator or computer to check your solution, but be sure to show your work). Verify you have the correct inverse by calculating $\mathbf{XX}^{-1} = \mathbf{I}$. Not all of the matricies may be invertible - if not, show why.[7]

a. $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

**Solution:** Recall the rule for inverting $2 \times 2$ matricies:

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

$$\begin{aligned}
\mathbf{X}^{-1} &= |\mathbf{X}|^{-1} \begin{bmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{bmatrix} \\
&= \frac{1}{|\mathbf{X}|} \begin{bmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{bmatrix}
\end{aligned}$$

Given this rule, first calculate the determinant of the matrix.

$$\begin{aligned}
|\mathbf{X}| &= (2 \times 1) - (1 \times 1) \\
&= 2 - 1 \\
&= 1
\end{aligned}$$

Now we can easily solve for the inverse:

$$\begin{aligned}
\mathbf{X}^{-1} &= \frac{1}{1} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}
\end{aligned}$$

b. $\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$

**Solution:** Solve for the determinant

[7] Simon and Blume 8.19

$$
\begin{aligned}
|\mathbf{X}| &= (2 \times -2) - (1 \times -4) \\
&= -4 - (-4) \\
&= 0
\end{aligned}
$$

At this point we are done. The matrix has a determinant of zero, making it singular. Singular matricies cannot be inverted.

c. $\begin{bmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{bmatrix}$

**Solution:** With a $3 \times 3$ matrix, we need to apply Gauss-Jordan elimination to obtain the inverse.

1. Setup the augmented matrix with the identity matrix

$$\left[\begin{array}{rrr|rrr} 2 & 4 & 0 & 1 & 0 & 0 \\ 4 & 6 & 3 & 0 & 1 & 0 \\ -6 & -10 & 0 & 0 & 0 & 1 \end{array}\right]$$

2. Swap row 1 with row 3

$$\left[\begin{array}{rrr|rrr} -6 & -10 & 0 & 0 & 0 & 1 \\ 4 & 6 & 3 & 0 & 1 & 0 \\ 2 & 4 & 0 & 1 & 0 & 0 \end{array}\right]$$

3. Add $\frac{2}{3}\times$ row 1 to row 2

$$\left[\begin{array}{rrr|rrr} -6 & -10 & 0 & 0 & 0 & 1 \\ 0 & -2/3 & 3 & 0 & 1 & 2/3 \\ 2 & 4 & 0 & 1 & 0 & 0 \end{array}\right]$$

4. Add $\frac{1}{3}\times$ row 1 to row 3

$$\left[\begin{array}{rrr|rrr} -6 & -10 & 0 & 0 & 0 & 1 \\ 0 & -2/3 & 3 & 0 & 1 & 2/3 \\ 0 & 2/3 & 0 & 1 & 0 & 1/3 \end{array}\right]$$

5. Add row 2 to row 3

$$\left[\begin{array}{rrr|rrr} -6 & -10 & 0 & 0 & 0 & 1 \\ 0 & -2/3 & 3 & 0 & 1 & 2/3 \\ 0 & 0 & 3 & 1 & 1 & 1 \end{array}\right]$$

6. Divide row 3 by 3

$$\left[\begin{array}{rrr|rrr} -6 & -10 & 0 & 0 & 0 & 1 \\ 0 & -2/3 & 3 & 0 & 1 & 2/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{array}\right]$$

7. Subtract $3\times$ row 3 from row 2

$$\left[\begin{array}{rrr|rrr} -6 & -10 & 0 & 0 & 0 & 1 \\ 0 & -2/3 & 0 & -1 & 0 & -1/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{array}\right]$$

8. Multiply row 2 by $-\frac{3}{2}$

$$\left[\begin{array}{ccc|ccc} -6 & -10 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{array}\right]$$

9. Add $10\times$ row 2 to row 1

$$\left[\begin{array}{ccc|ccc} -6 & 0 & 0 & 15 & 0 & 6 \\ 0 & 1 & 0 & 3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{array}\right]$$

10. Divide row 1 by $-6$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5/2 & 0 & -1 \\ 0 & 1 & 0 & 3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{array}\right]$$

11. The inverse of the original matrix is the right part of the augmented matrix.

$$\begin{bmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} -5/2 & 0 & -1 \\ 3/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

12. Factor out common terms

$$\begin{bmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{bmatrix}^{-1} = \frac{1}{6}\begin{bmatrix} -15 & 0 & -6 \\ 9 & 0 & 3 \\ 2 & 2 & 2 \end{bmatrix}$$

# Dummy encoding for categorical variables

Ordinary least squares regression is a common method for obtaining regression parameters relating a set of explanatory variables with a continuous outcome of interest. The vector $\hat{\mathbf{b}}$ that contains the intercept and the regression slope is calculated by the equation:

$$\hat{\mathbf{b}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$$

If an explanatory variable is nominal (i.e. ordering does not matter) with more than two classes (e.g. $\{\text{White}, \text{Black}, \text{Asian}, \text{Mixed}, \text{Other}\}$), the variable must be modified to include in the regression model. A common technique known as **dummy encoding** converts the column into a series of $n-1$ binary (0/1) columns where each column represents a single class and $n$ is the total number of unique classes in the original column. Explain why this method converts the column into $n-1$ columns, rather than $n$ columns, in terms of linear algebra. **Reminder: X contains both the dummy encoded columns as well as a column of 1s representing the intercept.**[8]

**Solution:** In order to calculate $\hat{\mathbf{b}}$, we must be able to calculate $(\mathbf{X}'\mathbf{X})^{-1}$. And we can only invert $\mathbf{X}'\mathbf{X}$ if the matrix is **nonsingular**. What could make a matrix singular? If at least one column is **linearly dependent** (i.e. its value can be produced by linear combinations of other columns in the matrix), then the matrix will not be **full rank**. A square matrix that is not full rank will produce a determinant of 0, which as you'll recall in the case of a $2 \times 2$ matrix would require division by zero.

[8] Benjamin Soltoff

$$\mathbf{X}^{-1} = \frac{1}{0} \begin{bmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{bmatrix}$$

So $\mathbf{X'X}$ must be full rank in order to invert it. How does this effect our one-hot encoding scheme? If we were to convert the explanatory variable into $n$ binary variables, the matrix $X$ is nonsingular. That is, any of the columns in $\mathbf{X}$ can be represented as a linear combination of the other columns.

This leads to the problem of what happens when we calculate $\mathbf{X'X}$. Suppose

$$\mathbf{X} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

It's transpose is

$$\mathbf{X'} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So

$$\mathbf{X'X} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

The problem is that $\mathbf{X'X}$ is still non-invertible. The determinant of $\mathbf{X'X}$ is 0. Notice that the first column $\mathbf{x_1}$ is a linear combination of $\mathbf{x_2} + \mathbf{x_3}$. In fact, $\mathbf{X}$ being invertible is a necessary condition for $\mathbf{X'X}$ being invertible.

# Solve the system of equations

Solve the following systems of equations for $x, y, z$, either via matrix inversion or substitution:[9]

a. System #1

$$\begin{aligned} x + y + 2z &= 2 \\ 3x - 2y + z &= 1 \\ y - z &= 3 \end{aligned}$$

- Using matrix inversion:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \quad \mathbf{y} = [2, 1, 3]' \quad \mathbf{x} = [x, y, z]$$

$$\begin{aligned} \mathbf{Ax} &= \mathbf{y} \\ \mathbf{A^{-1}y} &= \mathbf{x} \end{aligned}$$

You can use (a lot) of Gauss-Jordan elimination to invert the matrix. Or just use R.

[9] Gill 4.19

```r
A <- matrix(c(1, 1, 2, 3, -2, 1, 0, 1, -1),
            nrow = 3,
            ncol = 3, byrow = TRUE)

y <- c(2, 1, 3)

solve(A)      # inverse of A
```

```
##      [,1] [,2] [,3]
## [1,]  0.1  0.3  0.5
## [2,]  0.3 -0.1  0.5
## [3,]  0.3 -0.1 -0.5
```

```r
solve(A, y)   # inverse of A times y = x
```

```
## [1]  2  2 -1
```

- Using Gauss-Jordan elimination

We start with the augmented matrix representation:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 3 & -2 & 1 & 1 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

### Step-by-step Gauss-Jordan Elimination

1. Keep Row 1 as is (pivot is already 1).

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 3 & -2 & 1 & 1 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

2. Row 2 = Row 2 - 3 * Row 1

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -5 & -5 & -5 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

3. Row 2 = Row 2 / (-5)

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

4. Row 1 = Row 1 - Row 2, Row 3 = Row 3 - Row 2

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & 2 \end{array}\right]$$

5. Row 3 = Row 3 / (-2)

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

6. Row 1 = Row 1 - Row 3, Row 2 = Row 2 - Row 3

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Result

$$x = 2,\ y = 2,\ z = -1$$

- Using substitution

  1. $1\times$ third row added to the second row and $2\times$ third row added to the first row.

  $$\begin{aligned} x + 3y &= 8 \\ 3x - y &= 4 \\ y - z &= 3 \end{aligned}$$

  2. $-3\times$ first row added to the second row.

  $$\begin{aligned} x + 3y &= 8 \\ -10y &= -20 \\ y - z &= 3 \end{aligned}$$

  3. Solve for $y$ and $z$.

  $$\begin{aligned} -10y = -20 &\implies y = 2 \\ y - z = 3 &\implies z = -1 \end{aligned}$$

  4. Substitute $y$ into the first equation.

  $$x + 3(2) = 8 \implies x = 2$$

  Final solution: $x = 2, y = 2, z = -1$.

a. System #2

$$\begin{aligned} x - y + 2z &= 2 \\ 4x + y - 2z &= 10 \\ x + 3y + z &= 0 \end{aligned}$$

- Using Gauss-Jordan Elimination

We start with the augmented matrix representation:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 4 & 1 & -2 & 10 \\ 1 & 3 & 1 & 0 \end{array}\right]$$

1. Keep Row 1 as is (pivot is already 1).

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 4 & 1 & -2 & 10 \\ 1 & 3 & 1 & 0 \end{array}\right]$$

2. Row 2 = Row 2 - 4 * Row 1, Row 3 = Row 3 - Row 1

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 5 & -10 & 2 \\ 0 & 4 & -1 & -2 \end{array}\right]$$

3. Row 2 = Row 2 / 5

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 1 & -2 & \frac{2}{5} \\ 0 & 4 & -1 & -2 \end{array}\right]$$

4. Row 3 = Row 3 - 4 * Row 2, Row 1 = Row 1 + Row 2

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{12}{5} \\ 0 & 1 & -2 & \frac{2}{5} \\ 0 & 0 & 7 & -\frac{18}{5} \end{array}\right]$$

5. Row 3 = Row 3 / 7

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{12}{5} \\ 0 & 1 & 0 & -\frac{22}{35} \\ 0 & 0 & 1 & -\frac{18}{35} \end{array}\right]$$

Result

$$x = \frac{12}{5},\ y = -\frac{22}{35},\ z = -\frac{18}{35}$$

- Using matrix inversion

```r
A <- matrix(c(1, -1, 2, 4, 1, -2, 1, 3, 1),
            nrow = 3,
            ncol = 3, byrow = TRUE)

y <- c(2, 10, 0)

solve(A)      # inverse of A
```

```
##            [,1]        [,2]         [,3]
## [1,]  0.2000000  0.20000000 1.387779e-17
## [2,] -0.1714286 -0.02857143 2.857143e-01
## [3,]  0.3142857 -0.11428571 1.428571e-01
```

```r
solve(A, y)   # inverse of A times y = x
```

```
## [1]  2.4000000 -0.6285714 -0.5142857
```

- Using substitution

  1. Add row 1 to row 2:

$$
\begin{aligned}
x - y + 2z &= 2 \\
5x &= 12 \\
x + 3y + z &= 0
\end{aligned}
$$

  2. Solve for $x$:

$$5x = 12 \implies x = \frac{12}{5}$$

  3. Plug in $x = \frac{12}{5}$ and add row 1 multiplied by 3 to row 3:

$$
\begin{aligned}
\frac{12}{5} - y + 2z &= 2 \\
4\left(\frac{12}{5}\right) + 7z &= 6
\end{aligned}
$$

  4. Solve for $z$:

$$4\left(\frac{12}{5}\right) + 7z = 6 \implies z = -\frac{18}{35}$$

  5. Solve for $y$:

$$\frac{12}{5} - y + 2\left(-\frac{18}{35}\right) = 2 \implies y = -\frac{22}{35}$$

  Final solution:

$$x = \frac{12}{5},\ y = -\frac{22}{35},\ z = -\frac{18}{35}$$

## Multiplying by 0

When it comes to real numbers, we know that if $xy = 0$, then either $x = 0$ or $y = 0$ or both. One might believe that a similar idea applies to matricies, but one would be wrong. Prove that if the matrix product $\mathbf{AB} = \mathbf{0}$ (by which we mean a matrix of appropriate dimensionality made up entirely of zeroes), then it is not necessarily true that either $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$. Hint: in order to prove that something is not always true, simply identify one example where $\mathbf{AB} = \mathbf{0},\ \mathbf{A}, \mathbf{B} \neq \mathbf{0}$.[10]

**Solution:** Generally speaking, it is easy to show that something is *not* necessarily true. All that is needed is a single counterexample! And in this case, there are infinitely many counterexamples. Here's one:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 1(1) + 1(-1) & 1(1) + 1(-1) \\ 1(-1) + 1(1) & 1(-1) + 1(1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

[10] Grimmer HW5.5