PSET 5: Functions of several variables and optimization with several variables

Note: all homework uploads should be as a PDF *and* have the questions identified. We'll be giving zero credit for submissions that don't follow this protocol as it adds considerable time to grading. Thank you!

- Name

- How long did this problem set take you?

- How difficult was this problem set? very easy 1 2 3 4 5 very challenging

1 Find first partial derivatives

Find all of the first partial derivatives of each function.[1]

a. $f(x,y) = 3x + 2y^4$

b. $f(x,y) = \frac{1}{3}x^5 - 3x^3y^2 + 3xy^4$

c. $g(x,y) = xe^{4y}$

d. $k(x,y) = \frac{x+y}{x-y}$

e. $h(x,y,z) = x^3e^{yz}$

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[1]Inspired by Grimmer HW6.3

2   Find the gradient

Find the gradient ∇f of the following functions and evaluate them at the given points.[2]

   a. $f(x, y) = \sqrt{x^2 + y^2}, \quad (x, y) = (3, 4)$

   b. $f(x, y, z) = (x + z)e^{x-y}, \quad (x, y, z) = (1, 1, 1)$

3   Find the Hessian

Find the Hessian H for the following functions.[3]

   a. $g(x, y) = x^4 - 3x^2y^3$

   b. $f(x, y, z) = xyz - x^2$

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4   Find the critical points

Find the local minimum values, local maximum values, and saddle point(s) of the function. Remember the process we discussed in class: Calculate the gradient, set it equal to zero to solve the system of equations, calculate the Hessian, and assess the Hessian at critical values. Be sure to show your work on each of these steps.[4]

   a. $f(x, y) = x^4 + y^4 - 4xy + 2$

   b. $k(x, y) = (1 + xy)(x + y)$

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[2] Inspired by Grimmer HW6.4
[3] Inspired by Grimmer HW7.3
[4] Inspired by Grimmer HW7.4

5   Definite integrals

Solve the following definite integrals using the antiderivative method.[5]

For all these problems, the basic approach to compute the definite integral of $f(x)$ from a to b is by using the formula $F(b) - F(a)$, where $F(x)$ is the **antiderivative** of f.

a. $\int_6^8 x^3 \, dx$

b. $\int_{-1}^0 (3x^2 - 1) \, dx$

c. $\int_2^4 e^y \, dy$

d. $\int_3^3 \sqrt{x^5 + 2} \, dx$

6   Indefinite integrals

Calculate the following indefinite integrals:[6]

a. $\int (x^2 - x^{-\frac{1}{2}}) \, dx$

b. $\int 360 t^6 \, dt$

c. $\int 2x \log(x^2) \, dx$

7   Determining convergence

Determine whether each integral is convergent or divergent. Evaluate those that are convergent.[7]

a. $\int_1^\infty \left(\frac{1}{3x}\right)^2 \, dx$

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b. $\int_0^\infty e^{-x} \, dx$

c. $\int_{-\infty}^0 x^3 \, dx$

d. $\int_0^1 \int_2^3 x^2 y^3 \, dx dy$

e. $\int_0^2 \int_0^{\sqrt{1-x^2}} 2x^3 y \, dy dx$

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[5]Inspired by Gill 5.10 and Grimmer HW4.1
[6]Inspired by Gill 5.13 and 5.14
[7]Inspired by Grimmer HW 4.3

8 Applied integration

A group of three unidentified first-year graduate students at the University of Chicago are worn out after a week of math camp. Wanting to unwind, the students agree to not talk about math and decide to chat over some casual drinks at Medici.

After five shots of tequila each, two pitchers of beer, a bottle of wine, and a large Chicago-style pizza, the three students have had enough fun and decide to start the trip back home.

- Student A gets on a bike and starts pedaling away at a velocity of $v_A(t) = 2t^4 + t$, where t represents minutes. However, the student crashes into the side of an Uber and ends the journey after only 2 minutes.
- Student B has no bike, so starts running at a velocity of $v_B(t) = 4\sqrt{t}$. Sadly, after only 4 minutes, the student's legs give out and the student decides to sing a song, instead.
- Student C can't even stand up, so has no choice but to slowly crawl at a velocity of $v_C(t) = 2e^{-t}$. Student C steadily plods along for 20 minutes before falling asleep on the sidewalk.

Generally, if an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$. The Fundamental Theorem of Calculus then tells us that

Without using a calculator, use this formula to find the distance traveled by Students A, B, and C. (Assume, however unrealistic in may be, that all three students traveled in a straight line.) Who traveled the farthest? The least far?[8]

9 AI and Resources statement

- Please list (in detail) all resources you used for this assignment. If you worked with people, list them here as well. It is not enough to say that you used a resource for help, you need to be specific on the link and *how* it was helpful. W/R/T gen AI tools (including GPT, etc.) you cannot use them to do work on your behalf – you cannot put in any of the questions, etc. You can ask for help on logic / sample problems. If you do use GPT or other AI tools, you need to provide a link to your chat transcript. Any suspected academic integrity violations will be immediately reported.

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[8]Inspired by Grimmer HW4.2