

# PSET 4: Linear algebra - SOLUTIONS

## 1 Basic matrix arithmetic

Given:

$$\mathbf{a} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

a.  $\mathbf{a} - \mathbf{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

b.  $-4\mathbf{b} = -4 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ -4 \end{bmatrix}$

c.  $3\mathbf{a} + 3\mathbf{b} = 3 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} + \begin{bmatrix} 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 15 \\ 9 \end{bmatrix}$

## 2 More complex matrix arithmetic

Given:

$$\mathbf{x} = \begin{bmatrix} 3 \\ 2q \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} p+2 \\ -5 \\ 3r \end{bmatrix}$$

If  $\mathbf{x} = 2\mathbf{y}$ , then:

$$\begin{bmatrix} 3 \\ 2q \\ 4 \end{bmatrix} = 2 \begin{bmatrix} p+2 \\ -5 \\ 3r \end{bmatrix} = \begin{bmatrix} 2(p+2) \\ 2(-5) \\ 2(3r) \end{bmatrix} = \begin{bmatrix} 2p+4 \\ -10 \\ 6r \end{bmatrix}$$

Equating components:

$$3 = 2p + 4 \Rightarrow 2p = -1 \Rightarrow p = -\frac{1}{2} \tag{1}$$

$$2q = -10 \Rightarrow q = -5 \tag{2}$$

$$4 = 6r \Rightarrow r = \frac{2}{3} \tag{3}$$

**Solution:**  $p = -\frac{1}{2}$ ,  $q = -5$ ,  $r = \frac{2}{3}$ ,  $r^2 = \frac{4}{9}$

## 3 Check for linear dependence

a.  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

Form matrix and calculate determinant:

$$\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = 1(45 - 48) - 4(18 - 24) + 7(12 - 15) = -3 + 24 - 21 = 0$$

Since the determinant is 0, these vectors are **linearly dependent**.

$$\text{b. } \begin{bmatrix} 13 \\ 7 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \\ 8 \end{bmatrix}$$

Since one vector is the zero vector, the set is automatically **linearly dependent**.

$$\text{c. } \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 1 & -1 & 3 \end{vmatrix} = 1(-6 + 1) - 2(6 + 1) + 2(-2 + 2) = -5 - 14 + 0 = -19 \neq 0$$

Since the determinant is non-zero, these vectors are **linearly independent**.

## 4 Vector length

- a.  $\|(4, 2)\| = \sqrt{4^2 + 2^2} = \sqrt{16 + 4} = \sqrt{20} = 2\sqrt{5}$
- b.  $\|(2, 2, 2)\| = \sqrt{2^2 + 2^2 + 2^2} = \sqrt{12} = 2\sqrt{3}$
- c.  $\|(4, 2, 3, 1)\| = \sqrt{4^2 + 2^2 + 3^2 + 1^2} = \sqrt{16 + 4 + 9 + 1} = \sqrt{30}$
- d.  $\|(0, 0, 0, 0, 3)\| = \sqrt{0^2 + 0^2 + 0^2 + 0^2 + 3^2} = \sqrt{9} = 3$

## 5 Law of cosines

Using  $\theta = \arccos\left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right)$

- a.  $\mathbf{v} = (1, 0), \mathbf{w} = (2, 2)$

$$\mathbf{v} \cdot \mathbf{w} = 1(2) + 0(2) = 2 \quad (4)$$

$$\|\mathbf{v}\| = 1, \quad \|\mathbf{w}\| = \sqrt{8} = 2\sqrt{2} \quad (5)$$

$$\cos(\theta) = \frac{2}{1 \cdot 2\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad (6)$$

$$\theta = \frac{\pi}{4} \text{ radians} = 45^\circ \quad (7)$$

- b.  $\mathbf{v} = (4, 1), \mathbf{w} = (2, -8)$

$$\mathbf{v} \cdot \mathbf{w} = 4(2) + 1(-8) = 8 - 8 = 0 \quad (8)$$

$$\|\mathbf{v}\| = \sqrt{17}, \quad \|\mathbf{w}\| = \sqrt{68} = 2\sqrt{17} \quad (9)$$

$$\cos(\theta) = \frac{0}{\sqrt{17} \cdot 2\sqrt{17}} = 0 \quad (10)$$

$$\theta = \frac{\pi}{2} \text{ radians} = 90^\circ \quad (11)$$

c.  $\mathbf{v} = (1, 1, 0)$ ,  $\mathbf{w} = (1, 2, 2)$

$$\mathbf{v} \cdot \mathbf{w} = 1(1) + 1(2) + 0(2) = 3 \quad (12)$$

$$\|\mathbf{v}\| = \sqrt{2}, \quad \|\mathbf{w}\| = \sqrt{9} = 3 \quad (13)$$

$$\cos(\theta) = \frac{3}{\sqrt{2} \cdot 3} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad (14)$$

$$\theta = \frac{\pi}{4} \text{ radians} = 45^\circ \quad (15)$$

## 6 Matrix algebra

a.  $\mathbf{A} + \mathbf{B} = \begin{bmatrix} 3 \\ -2 \\ 9 \end{bmatrix} + \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 11 \\ -2 \\ 8 \end{bmatrix}$

b.  $-\mathbf{G} = -\begin{bmatrix} 2 & -8 & -5 \\ -3 & 7 & -4 \\ 1 & 0 & 3 \\ 1 & 2 & 6 \end{bmatrix} = \begin{bmatrix} -2 & 8 & 5 \\ 3 & -7 & 4 \\ -1 & 0 & -3 \\ -1 & -2 & -6 \end{bmatrix}$

c.  $\mathbf{D}' = \begin{bmatrix} 3 & 1 \\ 3 & 4 \\ 3 & -7 \end{bmatrix}' = \begin{bmatrix} 3 & 3 & 3 \\ 1 & 4 & -7 \end{bmatrix}$

d.  $\mathbf{C} + \mathbf{D}$ : Not defined (C is  $2 \times 3$ , D is  $3 \times 2$ )

e.  $\mathbf{A}'\mathbf{B} = \begin{bmatrix} 3 & -2 & 9 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 24 + 0 - 9 \end{bmatrix} = \begin{bmatrix} 15 \end{bmatrix}$

f.  $\mathbf{BC}$ : Not defined (B is  $3 \times 1$ , C is  $2 \times 3$ )

g.  $\mathbf{FB} = \begin{bmatrix} 4 & 1 & -5 \\ 0 & 7 & 7 \\ 2 & -3 & 0 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 32 + 0 + 5 \\ 0 + 0 - 7 \\ 16 + 0 + 0 \end{bmatrix} = \begin{bmatrix} 37 \\ -7 \\ 16 \end{bmatrix}$

h.  $\mathbf{E} - 5\mathbf{I}_3 = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 0 & -4 \\ -2 & 1 & -6 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 \\ 1 & -5 & -4 \\ -2 & 1 & -11 \end{bmatrix}$

i.  $\mathbf{M}^2 = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -4 & 10 \end{bmatrix}$

## 7 Matrix inversion

a.  $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$\det = 2(1) - 1(1) = 1 \neq 0$ , so invertible.

$$\mathbf{X}^{-1} = \frac{1}{1} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

Verification:  $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b.  $\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$

$$\det = 2(-2) - 1(-4) = -4 + 4 = 0$$

**Not invertible** (determinant is zero).

$$\text{c. } \begin{bmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{bmatrix}$$

$$\det = 2(0 + 30) - 4(0 + 18) + 0 = 60 - 72 = -12 \neq 0, \text{ so invertible.}$$

Using cofactor method:

$$\mathbf{X}^{-1} = \frac{1}{-12} \begin{bmatrix} 30 & 0 & 12 \\ -18 & 0 & -6 \\ -4 & -4 & -4 \end{bmatrix} = \begin{bmatrix} -5/2 & 0 & -1 \\ 3/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

## 8 Dummy encoding for categorical variables

When we have a categorical variable with  $n$  classes, creating  $n$  dummy variables would result in perfect multicollinearity. This is because the sum of all dummy variables would always equal 1 (the intercept column), making the matrix  $\mathbf{X}'\mathbf{X}$  singular (determinant = 0).

If  $\mathbf{X}'\mathbf{X}$  is singular, then  $(\mathbf{X}'\mathbf{X})^{-1}$  does not exist, and we cannot compute  $\hat{\mathbf{b}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$ .

By using only  $n - 1$  dummy variables, we avoid perfect multicollinearity. The omitted category serves as the reference group, and its effect is captured in the intercept term, ensuring that  $\mathbf{X}'\mathbf{X}$  is invertible.

## 9 Solve the system of equations

a. System #1

$$x + y + 2z = 2 \tag{16}$$

$$3x - 2y + z = 1 \tag{17}$$

$$y - z = 3 \tag{18}$$

From equation 3:  $y = z + 3$

Substituting into equation 1:  $x + (z + 3) + 2z = 2 \Rightarrow x = -3z - 1$

Substituting into equation 2:  $3(-3z - 1) - 2(z + 3) + z = 1$

$$-9z - 3 - 2z - 6 + z = 1 \tag{19}$$

$$-10z = 10 \tag{20}$$

$$z = -1 \tag{21}$$

Therefore:  $y = -1 + 3 = 2$ ,  $x = -3(-1) - 1 = 2$

**Solution:**  $x = 2, y = 2, z = -1$

b. System #2

$$x - y + 2z = 2 \tag{22}$$

$$4x + y - 2z = 10 \tag{23}$$

$$x + 3y + z = 0 \tag{24}$$

Adding equations 1 and 2:  $5x = 12 \Rightarrow x = \frac{12}{5}$

From equation 3:  $z = -\frac{12}{5} - 3y$

Substituting into equation 1:  $\frac{12}{5} - y + 2\left(-\frac{12}{5} - 3y\right) = 2$

$$\frac{12}{5} - y - \frac{24}{5} - 6y = 2 \quad (25)$$

$$-7y = 2 + \frac{12}{5} = \frac{22}{5} \quad (26)$$

$$y = -\frac{22}{35} \quad (27)$$

$$z = -\frac{12}{5} - 3\left(-\frac{22}{35}\right) = -\frac{18}{35}$$

**Solution:**  $x = \frac{12}{5}, y = -\frac{22}{35}, z = -\frac{18}{35}$

## 10 Multiplying by 0

To prove that  $\mathbf{AB} = \mathbf{0}$  does not necessarily imply  $\mathbf{A} = \mathbf{0}$  or  $\mathbf{B} = \mathbf{0}$ , I provide a counterexample:

Let  $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  and  $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

Then:

$$\mathbf{AB} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$$

However, neither  $\mathbf{A}$  nor  $\mathbf{B}$  is the zero matrix. This proves that matrix multiplication does not follow the same zero-product property as real numbers.