Computational Math Camp

Problem Sets

Benjamin Soltoff

2019-09-09

Contents

Overview

Contains problem sets for the 2019 Computational Math Camp.

Chapter 1

Linear equations, notation, sets, and functions

1.1 Simplify expressions

Simplify the following expressions as much as possible:

a. $(-x^4y^2)^2$

1. Distribute exponents over products.

$$(-1)^2x^{(2\times4)}y^{(2\times2)}$$

2. Multiply 2 and 2 together.

$$(-1)^2x^{(2\times4)}y^4$$

3. Multiply 2 and 4 together.

$$(-1)^2x^8y^4$$

4. Evaluate $(-1)^2$.

$$x^8y^4$$

b. $9(3^0)$

1. Any nonzero number to the zero power is 1.

$$9(1)$$

2. Anything times 1 is the same value.

$$9$$

c. $(2a^2)(4a^4)$

1. Combine products of like terms.

$$2a^2 \times 4a^4 = 2 \times 4a^{(2+4)}$$

2. Evaluate $2 + 4$.

$$2 \times 4a^6$$

3. Multiply 2 and 4 together.

$$8a^6$$

d. $\dfrac{x^4}{x^3}$

1. For all exponents, $\dfrac{a^n}{a^m} = a^{(n-m)}$.

$$x^{(4-3)}$$

2. Evaluate $4 - 3$.

$$x$$

e. $(-2)^{7-4}$

1. Subtract 4 from 7.

$$(-2)^3$$

2. In order to evaluate 2^3 express 2^3 as 2×2^2.

$$-2 \times 2^2$$

3. Evaluate 2^2.

$$-2 \times 4$$

4. Multiply -2 and 4 together.

$$-8$$

f. $\left(\frac{1}{27b^3}\right)^{1/3}$

1. Separate component terms.

$$\frac{1}{27}^{1/3} \times \frac{1}{b^3}^{1/3}$$

2. Evaluate cube roots.

$$\frac{1}{3} \times \frac{1}{b}$$

3. Combine terms.

$$\frac{1}{3b}$$

g. $y^7y^6y^5y^4$

1. Combine products of like terms.

$$y^{(7+6+5+4)}$$

2. Evaluate $7 + 6 + 5 + 4$.

$$y^{22}$$

h. $\frac{2a/7b}{11b/5a}$

1. Write as a single fraction by multiplying the numerator by the reciprocal of the denominator.

$$\frac{2a}{7b} \times \frac{5a}{11b}$$

2. Product property of exponents: $x^a \times x^b = x^{(a+b)}$

$$\frac{5a \times 2a}{7b \times 11b} = \frac{5 \times 2a^{1+1}}{7 \times 11b^{1+1}}$$

3. Evaluate $1 + 1$.

$$\frac{5 \times 2a^2}{7 \times 11b^2}$$

4. Multiple scalars together.

$$\frac{10a^2}{77b^2}$$

i. $(z^2)^4$

1. Nested exponents rule: $(x^a)^b = x^{ab}$

$$z^{2\times 4}$$

2. Evaluate 2×4

$$z^8$$

1.2 Simplify a (more complex) expression

Simplify the following expression:

$$(a + b)^2 + (a - b)^2 + 2(a + b)(a - b) - 3a^2$$

1. Expand $(a + b)^2$ with FOIL.

$$a^2 + 2ab + b^2 + (a - b)^2 + 2(a + b)(a - b) - 3a^2$$

2. Expand $(a - b)^2$ with FOIL.

$$a^2 + 2ab + b^2 + a^2 - 2ab + b^2 + 2(a + b)(a - b) - 3a^2$$

3. Multiply $a + b$ and $a - b$ together using FOIL.

$$a^2 + 2ab + b^2 + a^2 - 2ab + b^2 + 2(a^2 - b^2) - 3a^2$$

4. Distribute 2 over $a^2 - b^2$.

$$a^2 + 2ab + b^2 + a^2 - 2ab + b^2 + 2a^2 - 2b^2 - 3a^2$$

5. Group like terms.

$$(a^2 + a^2 + 2a^2 - 3a^2) + (b^2 + b^2 - 2b^2) + (2ab - 2ab)$$

6. Combine like terms.

$$a^2 + (b^2 + b^2 - 2b^2) + (2ab - 2ab)$$

7. Look for the difference of two identical terms.

$$a^2$$

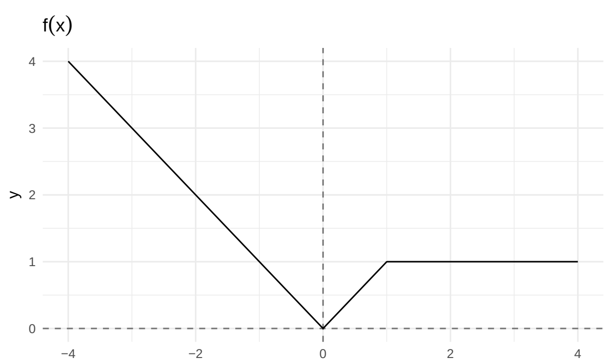
1.3 Graph sketching

Let the functions $f(x)$ and $g(x)$ be defined for all $x \in \Re$ by

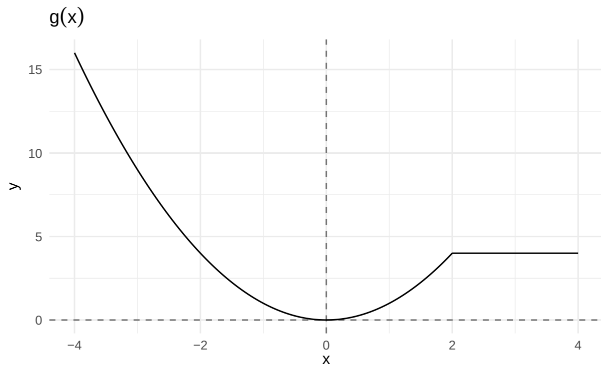
$$f(x) = \begin{cases} |x| & \text{if } x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}, \quad g(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 4 & \text{if } x \geq 2 \end{cases}$$

Sketch the graphs of:

1. $y = f(x)$



2. $y = g(x)$



3. $y = f(g(x))$

   To sketch the composite function, we first evaluate $g(x)$ for different values of x, and then evaluate $f(g(x))$ for different outputs of $g(x)$.

   - For $x \geq 2$, $g(x)$ is a constant value:

   $$\begin{aligned} x &\geq 2 \\ g(x) &= 4 \\ f(g(x)) &= f(4) = 1 \end{aligned}$$

   - For $x < 2$, $g(x)$ is not constant: $g(x) = x^2$. $f(x)$ evaluates differently depending on its input, so we have two cases based on the output of $g(x)$:

     – if $g(x) < 1$, $f(g(x)) = |g(x)| = |x^2| = x^2$. This is the case when:

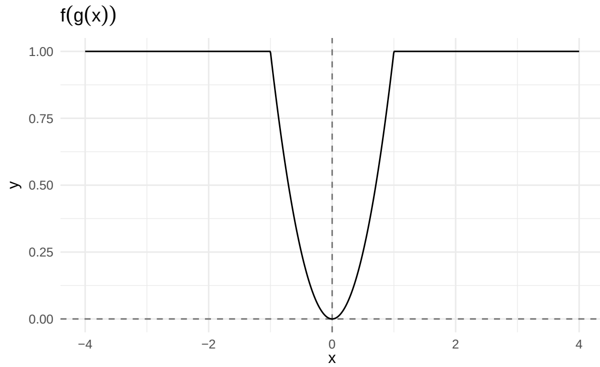
     $$\begin{aligned} g(x) &< 1 \\ x^2 < 1 \text{ and } x &< 2 \\ -1 < x &< 1 \end{aligned}$$

     – if $g(x) \geq 1$, $f(g(x)) = 1$. This is the case when:

     $$\begin{aligned} g(x) &\geq 1 \\ x^2 \geq 1 \text{ and } x &< 2 \\ x \leq -1 \text{ or } 1 \leq x &< 2 \end{aligned}$$

   - Therefore, $f(g(x))$ has the following values:

   $$f(g(x)) = \begin{cases} 1 & \text{if } x \leq -1 \\ x^2 & \text{if } -1 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$



4. $y = g(f(x))$

   To sketch the composite function, we first evaluate $f(x)$ for different values of x, and then evaluate $g(f(x))$ for different outputs of $f(x)$.

   - For $x \geq 1$, $f(x)$ is a constant value:

$$\begin{aligned} x &\geq 1 \\ f(x) &= 1 \\ g(f(x)) &= f(1) = 1^2 = 1 \end{aligned}$$

   - For $x < 1$, $f(x)$ is not constant: $f(x) = |x|$. $g(x)$ evaluates differently depending on its input, so we have two cases based on the output of $f(x)$:
     - if $f(x) < 2$, $g(f(x)) = f(x)^2 = |x|^2 = x^2$. This is the case when:

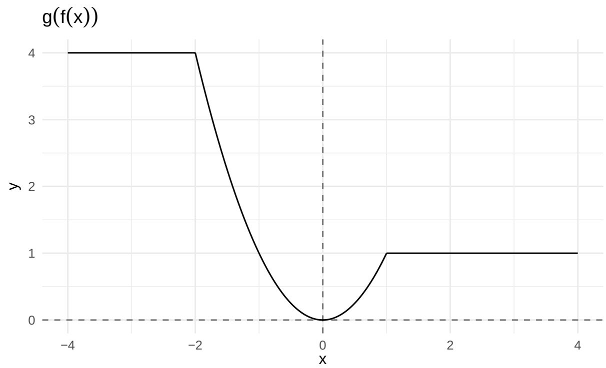
$$\begin{aligned} f(x) &< 2 \\ |x| &< 2 \text{ and } x < 1 \\ -2 &< x < 1 \end{aligned}$$

     - if $f(x) \geq 2$, $g(f(x)) = 4$. This is the case when:

$$\begin{aligned} f(x) &\geq 2 \\ |x| &\geq 2 \text{ and } x < 1 \\ x &\leq -2 \end{aligned}$$

   - Therefore, $g(f(x))$ has the following values:

$$g(f(x)) = \begin{cases} 4 & \text{if } x \leq -2 \\ x^2 & \text{if } -2 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$



1.4 Root finding

Find the roots (solutions) to the following quadratic equations.

Definition 1.1 (The quadratic formula)**.**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

a. $4x^2 - 1 = 17$

- Move terms so that x is alone on the left side of the equation.

$$\begin{aligned} 4x^2 - 1 &= 17 \\ 4x^2 &= 18 \\ x^2 &= \frac{18}{4} \\ x^2 &= \frac{9}{2} \\ x &= \pm\sqrt{\frac{9}{2}} \end{aligned}$$

b. $9x^2 - 3x - 12 = 0$

- Factor the left-hand side.

$$3(x + 1)(3x - 4) = 0$$

- Divide both sides by 3 to simplify the equation.

$$(x + 1)(3x - 4) = 0$$

- Find the roots of each term in the product separately by solving for x.

$$\begin{aligned} x+1&=0 & 3x&=4 \\ x&=-1 & x&=\frac{4}{3} \end{aligned}$$

c. $x^2-2x-16=0$

1. Complete the square

$$\begin{aligned} x^2-2x-16&=0 \\ x^2-2x&=16 \\ x^2-2x+1&=17 \\ (x-1)^2&=17 \\ x-1&=\pm\sqrt{17} \\ x&=1\pm\sqrt{17} \end{aligned}$$

2. Quadratic formula

   - Using the quadratic formula, solve for x

$$\begin{aligned} x&=\frac{-(-2)\pm\sqrt{(-2)^2-(4\times 1\times 16)}}{2\times 1} \\ x&=\frac{2\pm\sqrt{4+64}}{2} \\ x&=\frac{2\pm\sqrt{68}}{2} \end{aligned}$$

   - Simplify the radical

$$\begin{aligned} x&=\frac{2\pm\sqrt{2^2\times 17}}{2} \\ x&=\frac{2\pm 2\sqrt{17}}{2} \end{aligned}$$

   - Factor the greatest common divisor

$$x=1\pm\sqrt{17}$$

d. $6x^2-6x-6=0$

- Divide both sides by 6 to simplify the equation.

$$x^2 - x - 1 = 0$$

- Using the quadratic formula, solve for x

$$\begin{aligned}
x &= \frac{-(-1) \pm \sqrt{(-1)^2 - (4 \times 1 \times -1)}}{2 \times 1} \\
x &= \frac{1 \pm \sqrt{1 - 4(-1)}}{2} \\
x &= \frac{1 \pm \sqrt{1 + 4}}{2} \\
x &= \frac{1 \pm \sqrt{5}}{2}
\end{aligned}$$

e. $5 + 11x = -3x^2$

- Move everything to the left hand side.

$$3x^2 + 11x + 5 = 0$$

- Using the quadratic formula, solve for x

$$\begin{aligned}
x &= \frac{-11 \pm \sqrt{(11)^2 - (4 \times 3 \times 5)}}{2 \times 3} \\
x &= \frac{-11 \pm \sqrt{121 - 60}}{6} \\
x &= \frac{-11 \pm \sqrt{61}}{6}
\end{aligned}$$

1.5 Work with sets

Using the sets

$$\begin{aligned}
A &= \{2, 3, 7, 9, 13\} \\
B &= \{x : 4 \leq x \leq 8 \text{ and } x \text{ is an integer}\} \\
C &= \{x : 2 < x < 25 \text{ and } x \text{ is prime}\} \\
D &= \{1, 4, 9, 16, 25, \ldots\}
\end{aligned}$$

identify the following:

1. $A \cup B$

   $E = \{2, 3, 4, 5, 6, 7, 8, 9, 13\}$, combine all integers between 4 and 8 inclusive with the numbers in set A.

2. $(A \cup B) \cap C$

   $F = \{3, 5, 7, 13\}$, Since C is only prime numbers greater than 2 and less than 25, we take all the prime numbers that are also included in E, but remember to drop out 2 since it is not included in C.

3. $C \cap D$

   $G = \emptyset$, there are no prime numbers in D, so nothing is shared between C and D.

Chapter 2

Logarithms, sequences, and limits

2.1 Simplify logarithms

Express each of the following as a single logarithm:

a. $\log(x) + \log(y) - \log(z)$

- Multiplication rule of logarithms: $\log(x \times y) = \log(x) + \log(y)$
- Division rule of logarithms: $\log(\frac{x}{y}) = \log(x) - \log(y)$
- Applying the log rules, we combine logs that are added through multiplication and then combine logs that are subtracted with division.

$$\log(x) + \log(y) - \log(z)$$

$$\log(xy) - \log(z)$$

$$\log(\frac{xy}{z})$$

b. $2\log(x) + 1$

- Exponentiation rule of logarithms: $\log(x^y) = y\log(x)$
- $\log(e) = 1$

$$2\log(x) + 1$$

$$2\log(x) + \log(e)$$

$$\log(x^2) + \log(e)$$

$$\log(ex^2)$$

c. $\log(x) - 2$

- $\log(e) = 1$

$$\log(x) - 2$$

$$\log(x) - 2\log(e)$$

$$\log(x) - \log(e^2)$$

$$\log(\frac{x}{e^2})$$

2.2 Sequences

Write down the first three terms of each of the following sequences. In each case, state whether the sequence is an arithmetic progression, a geometric progression, or neither.

a. $u_n = 4 + 3n$

$$7, 10, 13$$

Arithmetic progression.

b. $u_n = 5 - 6n$

$$-1, -7, -13$$

Arithmetic progression.

c. $u_n = 4^n$

$$4, 16, 64$$

Geometric progression.

d. $u_n = 5 \times (-2)^n$

$$-10, 20, -40$$

Geometric progression.

e. $u_n = n \times 3^n$

$$3, 18, 81$$

Neither.

2.3 Find the limit

In each of the following cases, state whether the sequence $\{u_n\}$ tends to a limit, and find the limit if it exists:

a. $u_n = 1 + \frac{1}{2}n$

No limit $(u_n \to \infty)$

b. $u_n = 1 - \frac{1}{2}n$

No limit $(u_n \to \infty)$

c. $u_n = \left(\frac{1}{2}\right)^n$

Yes. $\lim_{x \to \infty} u_n = 0$

d. $u_n = \left(-\frac{1}{2}\right)^n$

Yes. $\lim_{x \to \infty} u_n = 0$

2.4 Determine convergence or divergence

Determine whether each of the following sequences converges or diverges. If it converges, find the limit.

a. $a_n = \frac{3+5n^2}{n+n^2}$

The sequence converges to 5. We can see this by factoring n^2 from both the numerator and denominator and then cancelling it out.

$$\lim_{n\to\infty} a_n = \frac{3+5n^2}{n+n^2} = \lim_{n\to\infty} \frac{n^2\left(\frac{3}{n^2}+5\right)}{n^2\left(\frac{1}{n}+1\right)} = \lim_{n\to\infty} \frac{\left(\frac{3}{n^2}+5\right)}{\left(\frac{1}{n}+1\right)} = \frac{\lim_{n\to\infty}\frac{3}{n^2}+5}{\lim_{n\to\infty}\frac{1}{n}+1} = \frac{0+5}{0+1} = 5$$

(This is slightly curt: Make sure you know how to show that the limit of $\frac{3}{n^2}$ approaches 0.) As $n \to \infty$, $\frac{3}{n} \to 0$ and $\frac{1}{n} \to 0$. Therefore, $a_n \to 5$.

Alternatively, you could split the fraction into two terms: one with a numerator of 3, and the other with a numerator of $5n^2$. The first fraction converges to 0. (Can you show that?) Factoring out an n from both sides of the second fraction, you're left with $\frac{5n}{n+1}$; $\frac{n}{n+1}$ converges to 1, giving you $5 \times 1 = 5$.

b. $a_n = \frac{(-1)^{n-1}n}{n^2+1}$

The sequence converges to 0. To see why, take the absolute value of the sequence, then factor out and cancel n from both sides of the fraction.

$$\lim_{n\to\infty}\left|\frac{(-1)^{n-1}n}{n^2+1}\right| = \lim_{n\to\infty}\frac{1^{n-1}n}{n^2+1} = \lim_{n\to\infty}\frac{1}{n+\frac{1}{n}} = \frac{\lim_{n\to\infty}1}{\lim_{n\to\infty}(n+\frac{1}{n})} = \frac{1}{\lim_{n\to\infty}n+0} = 0$$

2.5 Find more limits

Given that

$$\lim_{x\to a} f(x) = -3, \quad \lim_{x\to a} g(x) = 0, \quad \lim_{x\to a} h(x) = 8$$

find the limits that exist. If the limit doesn't exist, explain why.

a. $\lim\limits_{x\to a}[f(x)+h(x)] = -3+5 = 8$

b. $\lim\limits_{x\to a}[f(x)]^2 = (-3)^2 = 9$

c. $\lim\limits_{x\to a}\sqrt[3]{h(x)} = \sqrt[3]{8} = 2$

d. $\lim\limits_{x\to a}\frac{1}{f(x)} = -\frac{1}{3}$

e. $\lim\limits_{x\to a}\frac{f(x)}{h(x)} = -\frac{3}{8}$

f. $\lim\limits_{x\to a}\frac{g(x)}{f(x)} = \frac{0}{-3} = 0$

g. $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \frac{-3}{0}$ = Undefined – cannot divide by 0, no limit

h. $\lim_{x \to a} \frac{2f(x)}{h(x)-f(x)} = \frac{2\times -3}{8-(-3)} = -\frac{6}{11}$

2.6 Find even more limits

Find the limits of the following:

a. $\lim_{x \to -4} \frac{x^2+5x+4}{x^2+3x-4}$

$$\lim_{n \to -4} \frac{x^2+5x+4}{x^2+3x-4} = \lim_{n \to -4} \frac{(x+4)(x+1)}{(x+4)(x-1)} = \lim_{n \to -4} \frac{x+1}{x-1} = \frac{\lim_{n \to -4}(x+1)}{\lim_{n \to -4}(x-1)} = \frac{-3}{-5} = \frac{3}{5}$$

b. $\lim_{x \to 4^-} \sqrt{16-x^2}$

$$\begin{aligned} \lim_{n \to 4^-} \sqrt{16-x^2} &= \lim_{n \to 4^-} \sqrt{(4+x)(4-x)} \\ &= \lim_{n \to 4^-} \sqrt{4+x}\sqrt{4-x} \\ &= \lim_{n \to 4^-} \sqrt{4+x} \cdot \lim_{n \to 4^-} \sqrt{4-x} \\ &= \sqrt{8} \cdot \sqrt{0} \\ &= 0 \end{aligned}$$

c. $\lim_{x \to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x}$

$$\begin{aligned} \lim_{n \to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x} &= \lim_{n \to -4} \frac{\frac{x+4}{4x}}{4+x} \\ &= \lim_{n \to -4} \frac{4+x}{4x} \cdot \frac{1}{4+x} \\ &= \lim_{n \to -4} \frac{1}{4x} \\ &= \frac{1}{4(-4)} \\ &= -\frac{1}{16} \end{aligned}$$

2.7 Check for discontinuities

Which of the following functions are continuous? If not, where are the discontinuities?

a. $f(x) = \dfrac{9x^3 - x}{(x-1)(x+1)}$

- Discontinuous at $x = -1, +1$ (denominator would be 0, leaving the fraction undefined)

b. $f(x) = e^{-x^2}$

- Continuous for all real numbers.

c. $f(y) = y^3 - y^2 + 1$

- All polynomials are continuous.

d. $f(x) = \begin{cases} x^3 + 1, & x > 0 \\ \frac{1}{2}\ x = 0 \\ -x^2, & x < 0 \end{cases}$

- Discontinuous at $x = 0$. This is a piecewise function. To be continuous $\lim_{x \to 0^+} f(x) = 0$. However in this function, $\lim_{x \to 0^+} f(x) = 1 \neq 0$.

Chapter 3

Differentiation

3.1 Find finite limits

Find the following finite limits:

a. $\lim_{x \to 4} x^2 - 6x + 4$

$$\begin{aligned}
\lim_{x \to 4} x^2 - 6x + 4 &= 4^2 - 6(4) + 4 \\
&= 16 - 24 + 4 \\
&= -4
\end{aligned}$$

b. $\lim_{x \to 0} \left[\frac{x - 25}{x + 5} \right]$

$$\begin{aligned}
\lim_{x \to 0} \left[\frac{x - 25}{x + 5} \right] &= \frac{0 - 25}{0 + 5} \\
&= \frac{-25}{5} \\
&= -5
\end{aligned}$$

c. $\lim_{x \to 4} \left[\frac{x^2}{3x - 2} \right]$

$$\begin{aligned}\lim_{x\to 4}\left[\frac{x^2}{3x-2}\right] &= \frac{4^2}{3(4)-2}\\ &= \frac{16}{12-2}\\ &= \frac{16}{10}\\ &= \frac{8}{5}\end{aligned}$$

d. $\lim_{x\to 1}\left[\frac{x^4-1}{x-1}\right]$

The key here is to factor the initial expression in the numerator, then cancel terms out with the denominator:

$$\begin{aligned}\lim_{x\to 1}\left[\frac{x^4-1}{x-1}\right] &= \lim_{x\to 1}\left[\frac{(x-1)(x+1)(x^2+1)}{x-1}\right]\\ &= \lim_{x\to 1}[(x+1)(x^2+1)]\\ &= (1+1)(1^2+1)\\ &= (2)(2)\\ &= 4\end{aligned}$$

Alternatively, we can use L'Hôpital's Rule:

$$\begin{aligned}\lim_{x\to 1}\left[\frac{x^4-1}{x-1}\right] &= \lim_{x\to 1}\left[\frac{4x^3}{1}\right]\\ &= \frac{4(1)^3}{1}\\ &= 4\end{aligned}$$

e. $\lim_{x\to -4}\left[\frac{x^2+5x+4}{x^2+3x-4}\right]$

The key here is to factor the initial expression:

$$\begin{aligned}\lim_{x\to -4}\frac{(x+4)(x+1)}{(x+4)(x-1)} &= \lim_{x\to -4}\frac{x+1}{x-1}\\ &= \frac{\lim\limits_{x\to -4}(x+1)}{\lim\limits_{x\to -4}(x-1)}\\ &= \frac{-3}{-5}\\ &= \frac{3}{5}\end{aligned}$$

f. $\lim\limits_{x\to 4^-}\sqrt{16-x^2}$

$$\begin{aligned}\lim_{x\to 4^-}\sqrt{16-x^2} &= \lim_{x\to 4^-}\sqrt{(4+x)(4-x)}\\ &= \lim_{x\to 4^-}\sqrt{4+x}\sqrt{4-x}\\ &= \lim_{x\to 4^-}\sqrt{4+x}\cdot\lim_{x\to 4^-}\sqrt{4-x}\\ &= \sqrt{8}*\sqrt{0}\\ &= 0\end{aligned}$$

A critical aspect of this limit, which allows for it to exist, is that it is a left-hand limit.

g. $\lim\limits_{x\to -1}\left[\frac{x-2}{x^2+4x-3}\right]$

$$\begin{aligned}\lim_{x\to -1}\frac{x-2}{x^2+4x-3} &= \frac{\lim\limits_{x\to -1}(x-2)}{\lim\limits_{x\to -1}(x^2+4x-3)}\\ &= \frac{-1-2}{(-1)^2+4(-1)-3}\\ &= \frac{-3}{-6}\\ &= \frac{1}{2}\end{aligned}$$

h. $\lim\limits_{x\to -4}\left[\frac{\frac{1}{4}+\frac{1}{x}}{4+x}\right]$

$$\begin{aligned}
\lim_{x\to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x} &= \lim_{x\to -4} \frac{\frac{x+4}{4x}}{4+x} \\
&= \lim_{x\to -4} \frac{4+x}{4x}\frac{1}{4+x} \\
&= \lim_{x\to -4} \frac{1}{4x} \\
&= \frac{1}{4(-4)} \\
&= -\frac{1}{16}
\end{aligned}$$

Alternatively, we can use L'Hôpital's Rule:

$$\begin{aligned}
\lim_{x\to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x} &= \lim_{x\to -4} \frac{-\frac{1}{x^2}}{1} \\
&= \lim_{x\to -4}(-\frac{1}{x^2}) \\
&= -\frac{1}{16}
\end{aligned}$$

3.2 Find infinite limits

Find the following infinite limits:

Hint: use **L'Hôpital's Rule** to switch from $\lim_{x\to\infty}\left(\frac{f(x)}{g(x)}\right)$ to $\lim_{x\to\infty}\left(\frac{f'(x)}{g'(x)}\right)$.

a. $\lim_{x\to\infty}\left[\frac{9x^2}{x^2+3}\right]$

$$\begin{aligned}
\lim_{x\to\infty}\left[\frac{9x^2}{x^2+3}\right] &= \lim_{x\to\infty}\left[\frac{18x}{2x}\right] \\
&= 9
\end{aligned}$$

b. $\lim_{x\to\infty}\left[\frac{3x-4}{x+3}\right]$

$$\begin{aligned}
\lim_{x\to\infty}\left[\frac{3x-4}{x+3}\right] &= \lim_{x\to\infty}\left[\frac{3}{1}\right] \\
&= 3
\end{aligned}$$

c. $\lim_{x\to\infty}\left[\frac{2^x-3}{2^x+1}\right]$

Remember that $\frac{d}{dx}n^x = \log(n)n^x$:

$$\begin{aligned}\lim_{x\to\infty}\left[\frac{2^x-3}{2^x+1}\right] &= \lim_{x\to\infty}\left[\frac{\log(2)2^x}{\log(2)2^x}\right]\\ &= 1\end{aligned}$$

d. $\lim_{x\to\infty}\left[\frac{\log(x)}{x}\right]$

$$\begin{aligned}\lim_{x\to\infty}\left[\frac{\log(x)}{x}\right] &= \lim_{x\to\infty}\left[\frac{\frac{1}{x}}{1}\right]\\ &= \lim_{x\to\infty}\left[\frac{1}{x}\right]\\ &= \frac{1}{\infty}\\ &= 0\end{aligned}$$

e. $\lim_{x\to\infty}\left[\frac{3^x}{x^3}\right]$

The trick here is to repeatedly calculate the derivative of the numerator and denominators until there is no x term on the denominator. You end up calculating the third derivative, but L'Hôpital's Rule still applies.

$$\begin{aligned}\lim_{x\to\infty}\left[\frac{3^x}{x^3}\right] &= \lim_{x\to\infty}\left[\frac{\log(3)3^x}{3x^2}\right]\\ &= \lim_{x\to\infty}\left[\frac{\log^2(3)3^x}{6x}\right]\\ &= \lim_{x\to\infty}\left[\frac{\log^3(3)3^x}{6}\right]\\ &= \frac{\log^3(3)3^\infty}{6}\\ &= \infty\end{aligned}$$

f. $\lim_{y\to\infty}\left[\frac{3e^y}{y^3}\right]$

Same as above: repeatedly calculate the derivatives until the y term disappears in the denominator.

$$\begin{aligned}
\lim_{x\to\infty}\left[\frac{3e^y}{y^3}\right] &= \lim_{x\to\infty}\left[\frac{3e^y}{3y^2}\right] \\
&= \lim_{x\to\infty}\left[\frac{3e^y}{6y}\right] \\
&= \lim_{x\to\infty}\left[\frac{3e^y}{6}\right] \\
&= \frac{3e^\infty}{6} \\
&= \infty
\end{aligned}$$

3.3 Assessing continuity and differentiability

For each of the following functions, describe whether it is continuous and/or differentiable at the point of transition of its two formulas.

a.

$$f(x) = \begin{cases} +x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$

Solution:

$$f'(x) = \begin{cases} 2x, & x \geq 0 \\ -2x, & x < 0 \end{cases}$$

As x converges to 0 from both above and below, $f'(0)$ converges to 0, so the function is continuous and differentiable.

b.

$$f(x) = \begin{cases} +x^2+1, & x \geq 0 \\ -x^2-1, & x < 0 \end{cases}$$

Solution: This function is not continuous (and thus not differentiable). As x converges to 0 from above, $f(x)$ tends to 1, whereas x tends to 0 from below, $f(x)$ converges to -1.

c.

$$f(x) = \begin{cases} x^3, & x \leq 1 \\ x, & x > 1 \end{cases}$$

Solution: This function is continuous, since $\lim_{x\to 1} f(x) = 1$ no matter how the limit is taken. However it is not differentiable since

$$f'(x) = \begin{cases} 3x^2, & x \leq 1 \\ 1, & x > 1 \end{cases}$$

$\lim_{x\to1^+} f'(x) = 1$, whereas $\lim_{x\to1^-} f'(x) = 3$. The function is not smooth and continuous at $f(1)$.

d.

$$f(x) = \begin{cases} x^3, & x < 1 \\ 3x - 2, & x \geq 1 \end{cases}$$

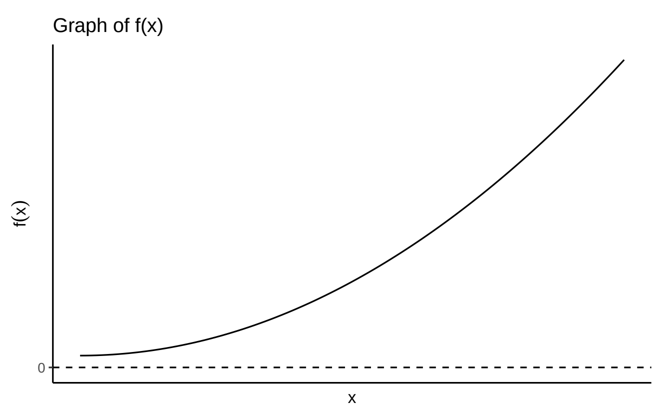
Solution: This function is continuous since $f(1)$ tends to 1 from either direction. Likewise, this function is continuous because

$$f'(x) = \begin{cases} 3x^2, & x \leq 1 \\ 3, & x > 1 \end{cases}$$

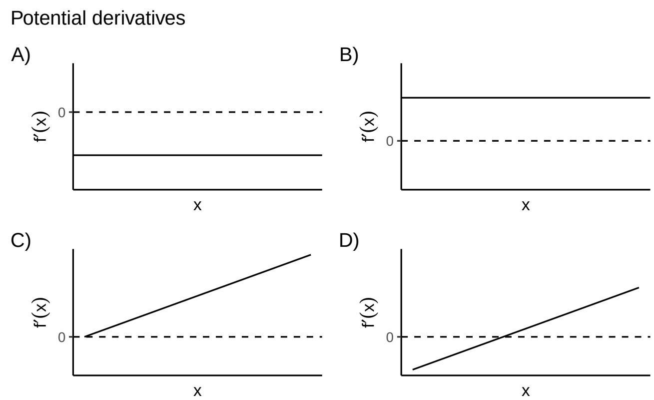
and $\lim_{x\to1} f'(x) = 3$ from either direction.

3.4 Possible derivative

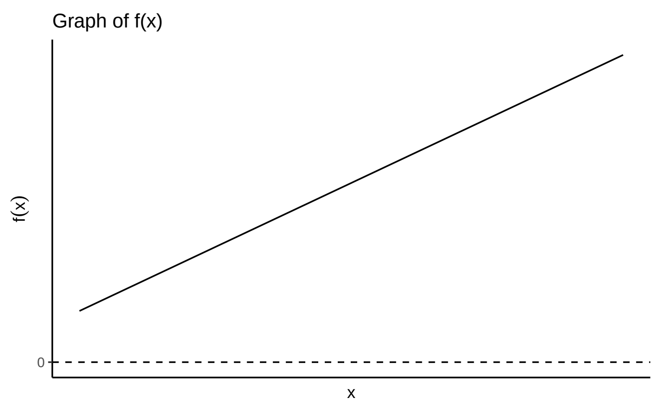
A friend shows you this graph of a function $f(x)$:



Which of the following could be a graph of $f'(x)$? For each graph, explain why or why not it might be the derivative of $f(x)$.



What if the figure below was the graph of $f(x)$? Which of the graphs might potentially be the derivative of $f(x)$ then?



Solution:

a. A doesn't work because it is negative and the function we observe is increasing in x. B is constant so this also won't work, the function we observe gets larger at an increasing, not constant rate. C seems to be a plausible candidate because an upward sloping derivative would map to the behavior of the function we observe, that $g(x)$ gets large at an increasing rate. D does not work because it suggests the function would need to be decreasing over some interval and because, when we refer back to $g(x)$, there doesn't seem to be any local minimum, maximum or a saddle point despite the graph in D crossing 0.

b. Again, A doesn't work because it is negative and the function we observe is increasing in x. B seems to be plausible as the derivative, since $g(x)$ appears to increase at a constant rate, its derivative should be flat and greater than 0. C won't work because the slope of $g(x)$ is constant and does not increase in x. D doesn't work, again because it suggests the function would need to be decreasing at some point over the interval we observe.

3.5 Calculate derivatives

Differentiate the following functions:

a. $f(x) = 4x^3 + 2x^2 + 5x + 11$

Solution: Power rule.

$$\begin{aligned} f(x) &= 4x^3 + 2x^2 + 5x + 11 \\ f'(x) &= 12x^2 + 4x + 5 \end{aligned}$$

b. $y = \sqrt{30}$

Solution: Derivative of a constant is 0.

$$\begin{aligned} y &= \sqrt{30} \\ y' &= 0 \end{aligned}$$

c. $h(t) = \log(9t + 1)$

Solution: Derivative of $\log(u)$ is $\frac{1}{u}$. Since u is a function in this problem, need to apply the chain rule to calculate the derivative of $9t + 1$ and multiply that by $\frac{1}{9t+1}$

$$\begin{aligned} h(t) &= \log(9t + 1) \\ h'(t) &= \frac{1}{9t + 1} * 9 \end{aligned}$$

d. $f(x) = \log(x^2 e^x)$

Solution: Derivative of a logarithm plus the chain rule.

$$\begin{aligned} f(x) &= \log(x^2 e^x) \\ f'(x) &= \frac{1}{x^2 e^x} * (2xe^x + e^x x^2) \\ &= \frac{2xe^x + e^x x^2}{x^2 e^x} \\ &= \frac{2}{x} + 1 \end{aligned}$$

e. $h(y) = \left(\frac{1}{y^2} - \frac{3}{y^4}\right)(y + 5y^3)$

Solution: Simplify the expression first, then basic application of power rule.

$$\begin{aligned} h(y) &= \left(\frac{1}{y^2} - \frac{3}{y^4}\right)\left(y + 5y^3\right) \\ &= \frac{y}{y^2} + \frac{5y^3}{y^2} - \frac{3y}{y^4} - \frac{15y^3}{y^4} \\ &= \frac{1}{y} + 5y - \frac{3}{y^3} - \frac{15}{y} \\ &= 5y - \frac{14}{y} - \frac{3}{y^3} \\ h'(y) &= 5 + \frac{14}{y^2} + \frac{9}{y^4} \end{aligned}$$

f. $g(t) = \frac{3t-1}{2t+1}$

Solution: Quotient rule.

$$\begin{aligned} g(t) &= \frac{3t-1}{2t+1} \\ g'(t) &= \frac{(3)(2t+1) - (3t-1)(2)}{(2t+1)^2} \\ &= \frac{5}{(2t+1)^2} \end{aligned}$$

3.6 Use the product and quotient rules

Differentiate the following using both the product and quotient rules:

$$f(x) = \frac{x^2 - 2x}{x^4 + 6}$$

Solution:

a. First let's use the quotient rule:

$$
\begin{aligned}
h(x) &= \frac{f(x)}{g(x)} \\
f(x) &= x^2 - 2x \\
g(x) &= x^4 + 6 \\
f'(x) &= 2x - 2 \\
g'(x) &= 4x^3 \\
h'(x) &= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \\
&= \frac{(2x-2)(x^4+6) - (x^2-2x)(4x^3)}{(x^4+6)^2} \\
&= \frac{2x^5 + 12x - 2x^4 - 12 - 4x^5 + 8x^4}{(x^4+6)^2} \\
&= \frac{-2x^5 + 6x^4 + 12x - 12}{(x^4+6)^2}
\end{aligned}
$$

b. Now we can do the same thing with the product rule:

$$
\begin{aligned}
j(x) &= k(x)m(x) \\
k(x) &= x^2 - 2x \\
m(x) &= (x^4+6)^{-1} \\
k'(x) &= 2x - 2 \\
m'(x) &= -(x^4+6)^{-2}(4x^3) = -\frac{4x^3}{(x^4+6)^2} \\
j'(x) &= k(x)m'(x) + k'(x)m(x) \\
&= (x^2-2x)(-\frac{4x^3}{(x^4+6)^2}) + (2x-2)(x^4+6)^{-1} \\
&= -\frac{(x^2-2x)(4x^3)}{(x^4+6)^2} + \frac{2x-2}{x^4+6} \\
&= -\frac{4x^5 - 8x^4}{x^4+6)^2} + \frac{2x-2}{x^4+6} \\
&= -\frac{4x^5 - 8x^4}{x^4+6)^2} + \frac{2x-2}{x^4+6}\frac{x^4+6}{x^4+6} \\
&= -\frac{4x^5 - 8x^4}{(x^4+6)^2} + \frac{2x^5 + 12x - 2x^4 - 12}{(x^4+6)^2} \\
&= \frac{2x^5 + 12x - 2x^4 - 12 - 4x^5 - 8x^4}{(x^4+6)^2} \\
&= \frac{-2x^5 + 6x^4 + 12x - 12}{(x^4+6)^2}
\end{aligned}
$$

The quotient rule is simply a derivation of the product rule combined with the chain rule:

$$\begin{aligned} h(x) &= \frac{f(x)}{g(x)} \\ &= f(x)g(x)^{-1} \end{aligned}$$

Apply product and chain rules:

$$\begin{aligned} h'(x) &= f'(x)g(x)^{-1} + f(x)(-1)g(x)^{-2}g'(x) \\ &= f'(x)g(x)g(x)^{-2} - f(x)g(x)^{-2}g'(x) \\ &= [f'(x)g(x) - f(x)g'(x)]g(x)^{-2} \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \end{aligned}$$

which is the quotient rule.

3.7 Logarithms and exponential functions

Compute the derivative of each of the following functions:

a. $f(x) = xe^{3x}$

Solution: Use the product rule to split the function into component functions.

$$g(x) = x, \quad h(x)e^{3x}$$

Use the chain rule to solve $h'(x)$.

$$\begin{aligned} g(x) &= x & h(x) &= e^{3x} \\ g'(x) &= 1 & h'(x) &= 3e^{3x} \end{aligned}$$

$$\begin{aligned} f(x) &= g'(x)h(x) + g(x)h'(x) \\ &= 1(e^{3x}) + x(3e^{3x}) \\ &= e^{3x} + 3xe^{3x} \\ &= e^{3x}(3x + 1) \end{aligned}$$

b. $f(x) = \frac{x}{e^x}$

Solution: Use the product rule.

$$g(x) = x, \quad h(x) = \frac{1}{e^x}$$

Use the chain rule to solve $h'(x)$.

$$\begin{aligned} g(x) &= x & h(x) &= \frac{1}{e^x} \\ g'(x) &= 1 & h'(x) &= -e^{-x} \end{aligned}$$

$$\begin{aligned} f(x) &= g'(x)h(x) + g(x)h'(x) \\ &= 1(\frac{1}{e^x}) + x(-e^{-x}) \\ &= \frac{1}{e^x} - xe^{-x} \\ &= \frac{1}{e^x} - \frac{x}{e^x} \\ &= \frac{1-x}{e^x} \end{aligned}$$

c. $h(x) = \frac{x}{\log(x)}$

Solution: Use the quotient rule.

$$g(x) = x, \quad h(x) = \log(x)$$

$$\begin{aligned} f(x) &= x & g(x) &= \log(x) \\ f'(x) &= 1 & g'(x) &= \frac{1}{x} \end{aligned}$$

$$\begin{aligned} \left[\frac{f(x)}{g(x)}\right]' &= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \\ &= \frac{1(\log(x)) - x(\frac{1}{x})}{[\log(x)]^2} \\ &= \frac{\log(x) - 1}{[\log(x)]^2} \end{aligned}$$

3.8 Composite functions

For each of the following pairs of functions $g(x)$ and $h(z)$, write out the composite function $g(h[z])$ and $h(g[x])$. In each case, describe the domain of the composite function.

a. $g(x) = x^2 + 4, \quad h(z) = 5z - 1$

Solution:

$$\begin{aligned} g(h[z]) &= (5z-1)^2 + 4 \\ h(g[x]) &= 5(x^2+4) - 1 \\ &= 5x^2 + 20 - 1 \\ &= 5x^2 + 19 \end{aligned}$$

- Domain of $g(h[z])$ $x \in \Re$
- Domain of $h(g[x])$ $x \in \Re$

b. $g(x) = x^3, \quad h(z) = (z-1)(z+1)$

Solution:

$$\begin{aligned} g(h[z]) &= [(z-1)(z+1)]^3 \\ &= (z-1)^3(z+1)^3 \\ h(g[x]) &= (x^3-1)(x^3+1) \end{aligned}$$

- Domain of $g(h[z])$ $x \in \Re$
- Domain of $h(g[x])$ $x \in \Re$

c. $g(x) = 4x + 2, \quad h(z) = \frac{1}{4}(z-2)$

Solution:

$$\begin{aligned} g(h[z]) &= 4\left[\frac{1}{4}(z-2)\right] + 2 \\ &= (z-2) + 2 \\ &= z \\ h(g[x]) &= \frac{1}{4}([4x+2] - 2) \\ &= \frac{1}{4}(4x) \\ &= x \end{aligned}$$

- Domain of $g(h[z])$ $x \in \Re$
- Domain of $h(g[x])$ $x \in \Re$

d. $g(x) = \frac{1}{x}, \quad h(z) = z^2 + 1$

Solution:

$$
\begin{aligned}
g(h[z]) &= \frac{1}{z^2+1} \\
h(g[x]) &= \left(\frac{1}{x}\right)^2 + 1 \\
&= \frac{1}{x^2} + 1
\end{aligned}
$$

- Domain of $g(h[z])$ $x \in \Re$
- Domain of $h(g[x])$ $x \in \Re : x \neq 0$

3.9 Chain rule

Use the chain rule to compute the derivative of the first three composite functions in the previous section from the derivatives of the two component functions. Then, compute each derivative directly using your expression for the composite function. Simplify and compare your answers.

a. $g(x) = x^2 + 4, \quad h(z) = 5z - 1$

Solution:

- Using component functions and the chain rule

$$g'(x) = 2x \quad h'(z) = 5$$

$$
\begin{aligned}
\frac{d}{dz}\{g(h[z])\} &= g'(h[z])h'(z) \\
&= 2(5z-1)(5) \\
&= 2(25z-5) \\
&= 50z - 10 \\
\frac{d}{dx}\{h(g[x])\} &= h'(g[x])g'(x) \\
&= 5(2x) \\
&= 10x
\end{aligned}
$$

- Using the composite function

$$\begin{aligned}
g(h[z]) &= (5z-1)^2+4 \\
&= 25z^2-10z+1+4 \\
&= 25z^2-10z+5 \\
\frac{d}{dz}g(h[z]) &= 50z-10 \\
h(g[x]) &= 5x^2+19 \\
\frac{d}{dx}h(g[x]) &= 10x
\end{aligned}$$

b. $g(x) = x^3, \quad h(z) = (z-1)(z+1)$

Solution:

- Using component functions and the chain rule

$$g'(x) = 3x^2 \quad h'(z) = 2z$$

$$\begin{aligned}
\frac{d}{dz}\{g(h[z])\} &= g'(h[z])h'(z) \\
&= 3[(z-1)(z+1)]^2(2z) \\
&= 3(z^2-1)^2(2z) \\
&= 6z(z^2-1)^2 \\
\frac{d}{dx}\{h(g[x])\} &= h'(g[x])g'(x) \\
&= 2(x^3)(3x^2) \\
&= 6x^5
\end{aligned}$$

- Using the composite function

$$\begin{aligned}
g(h[z]) &= (z-1)^3(z+1)^3 \\
&= (z-1)(z-1)(z-1)(z+1)(z+1)(z+1) \\
&= z^6-3z^4+3z^2-1 \\
\frac{d}{dz}g(h[z]) &= 6z^5-12z^3+6z \\
&= 6z(z^4-2z^2+1) \\
&= 6z(z^2-1)^2 \\
h(g[x]) &= (x^3-1)(x^3+1) \\
&= x^6-1 \\
\frac{d}{dx}h(g[x]) &= 6x^5
\end{aligned}$$

c. $g(x) = 4x + 2, \quad h(z) = \frac{1}{4}(z - 2)$

Solution:

- Using component functions and the chain rule

$$g'(x) = 4 \quad h'(z) = \frac{1}{4}$$

$$\begin{aligned}
\frac{d}{dz}\{g(h[z])\} &= g'(h[z])h'(z) \\
&= 4(\frac{1}{4}) \\
&= 1 \\
\frac{d}{dx}\{h(g[x])\} &= h'(g[x])g'(x) \\
&= \frac{1}{4}(4) \\
&= 1
\end{aligned}$$

- Using the composite function

$$\begin{aligned}
g(h[z]) &= z \\
\frac{d}{dz}g(h[z]) &= 1 \\
h(g[x]) &= x \\
\frac{d}{dx}h(g[x]) &= 1
\end{aligned}$$

Chapter 4

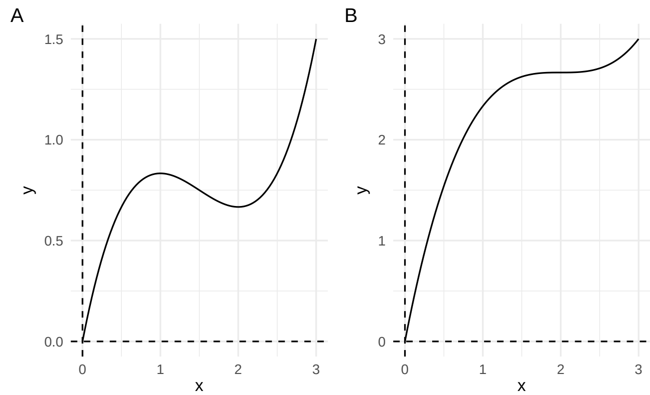
Critical points and approximation

4.1 Sketch a function

Sketch the graph of a function (any function you like, no need to specify a functional form) that is:

a. Continuous on $[0, 3]$ and has the following properties: an absolute minimum at 0, an absolute maximum at 3, a local maximum at 1 and a local minimum at 2.
b. Do the same for another function with the following properties: 2 is a **critical number** (i.e. $f'(x) = 0$ or $f'(x)$ is undefined), but there is no local minimum and no local maximum.

Solution: There are many, many (in fact, uncountably infinitely many) correct answers to this question, but they will all have a few characteristics in common. For the first function, the highest value of the function must be produced by $x = 0$, and the lowest value of the function must be produced by $x = 3$. Furthermore, the graph must change from moving up to moving down at $x = 2$ and from moving down to moving up at $x = 1$. For the second graph, there simply must be a saddle point at $x = 2$ - for $x = 2$ to be a critical point, it must be a local minimum, a local maximum, or a saddle point, but we've specified that there are no local minima and no local maxima - and the graph must not change from increasing to decreasing or vice versa at any point.



4.2 Find critical values

Find the critical values of these functions:

a. $f(x) = 5x^{3/2} - 4x$

Solution: First, find the derivative of the function.

$$f'(x) = \frac{3}{2}(5)x^{3/2-1} - 4 = \frac{15}{2}x^{1/2} - 4 = \frac{15\sqrt{x}}{2} - 4$$

If we set this derivative equal to 0 and solve, we get the following critical point:

$$\begin{aligned}
\frac{15\sqrt{x}}{2} - 4 &= 0 \\
\frac{15\sqrt{x}}{2} &= 4 \\
15\sqrt{x} &= 8 \\
\sqrt{x} &= \frac{8}{15} \\
x &= \left(\frac{8}{15}\right)^2 \\
x &= \frac{8^2}{15^2} \\
x &= \frac{64}{225}
\end{aligned}$$

b. $s(t) = 3t^4 + 4t^3 - 6t^2$

Solution: The derivative of $s(t)$ requires simple power rule:

$$s'(t) = 4(3)t^{4-1} + 3(4)t^{3-1} - 2(6)t^{2-1} = 12t^3 + 12t^2 - 12t = 12t(t^2 + t - 1)$$

If we set this equal to zero, we immediately see that $t = 0$ is a critical point. However, we cannot "eyeball" if/where $(t^2 + t - 1) = 0$. For that, we will need to use the quadratic formula. In case you don't remember, the quadratic formula helps us find the roots of a quadratic function – the points at which the function equals 0. Think of a generic quadratic equation, $f(x) = ax^2 + bx + c$, where a, b, c are the coefficients or constants to each term. Then,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In our case with $t^2 + t - 1$, $a = 1, b = 1, c = -1$. Let's plug these into the formula.

$$\begin{aligned}
t &= \frac{-1 \pm \sqrt{1^2 - 4(1)(-1}}{2(1)} \\
&= \frac{-1 \pm \sqrt{1+4}}{2} \\
&= \frac{-1 \pm \sqrt{5}}{2} \\
&= \frac{-1 - \sqrt{5}}{2}, \frac{-1 + \sqrt{5}}{2}
\end{aligned}$$

Great. We have three critical points: $t = \frac{-1-\sqrt{5}}{2}, 0, \frac{-1+\sqrt{5}}{2}$.

c. $f(r) = \dfrac{r}{r^2 + 1}$

Solution: We can use the quotient rule to find the derivative.

$$f'(r) = \frac{(r)'(r^2+1) - r(r^2+1)'}{(r^2+1)^2} = \frac{(1)(r^2+1) - r(2r)}{(r^2+1)^2} = \frac{r^2 + 1 - 2r^2}{(r^2+1)^2} = \frac{1 - r^2}{(r^2+1)^2}$$

To find the critical values, set the numerator of $s'(r)$ equal to 0. (Remember that since we are dealing with a fraction, we only need to find where the numerator equals 0. Also, given the nature of the denominator, we don't have to worry about whether the function is ever undefined.)

$$1 - r^2 = 0 \Longrightarrow r^2 = 1 \Longrightarrow r = \pm 1$$

d. $h(x) = x\log(x)$

Solution: The function requires product rule to differentiate.

$$h'(x) = x \cdot (\log x)' + x' \cdot \log x = x \cdot \frac{1}{x} + 1 \cdot \log x = 1 + \log x$$

Now set this derivative equal to zero and solve.

$$\begin{aligned} 1 + \log x &= 0 \\ \log x &= -1 \\ x &= e^{-1} \\ x &= \frac{1}{e} \end{aligned}$$

4.3 Find absolute minimum/maximum values

Find the absolute minimum and absolute maximum values of the functions on the given interval:

a. $f(x) = 3x^2 - 12x + 5, [0, 3]$

Solution: First, we must identify the critical points; to do this, we must find $f'(x)$. By our rules of derivation, $f'(x) = 6x - 12$. The critical points will then be where $f'(x) = 6x - 12 = 0$. Solving for x then gives $x = 2$. As this is a continuous function over the given interval, the absolute minimum and absolute maximum values must be at critical points or the endpoints of the interval. In this case, the set of candidate values of x is then {0, 2, 3}. Evaluating the function at these points gives $f(0) = 5$, $f(2) = -7$, and $f(3) = -4$. So the absolute maximum occurs where $x = 0, f(x) = 5$ and the absolute minimum occurs where $x = 2, f(x) = -7$. It is a useful check of our work to make sure that $x = 2$ gives a local minimum - after all, since $x = 2$ is not one of the endpoints, in order to be the absolute minimum it must be a local minimum as well. To see whether it is a local minimum, we evaluate $f''(x)$ where $x = 2$. $f''(x) = (6x - 12)' = 6$, so $f''(2) = 6 > 0$. Since the second derivative is positive, the first derivative must be increasing at $x = 2$ - in other words, it must be moving from negative to positive - and $x = 2$ is indeed a local minimum.

b. $f(t) = t\sqrt{4 - t^2}, [-1, 4]$

Solution: As before, we must first identify critical points. To find $f'(t)$, we must use both the power rule and the chain rule. The power rule tells us that $f'(t) = t(\sqrt{4 - t^2})' + (t)'\sqrt{4 - t^2}$. By the chain rule, the derivative of $\sqrt{4 - t^2}$ is $\frac{1}{2}(4 - t^2)^{-\frac{1}{2}}(4 - t^2)' = \frac{1}{2}(4 - t^2)^{-\frac{1}{2}}(-2t) = -t(4 - t^2)^{-\frac{1}{2}}$. So the derivative of $f(t)$ is $f'(t) = -t^2(4 - t^2)^{-\frac{1}{2}} + \sqrt{4 - t^2}$. Setting this equal to zero and adding $t^2(4-t^2)^{-\frac{1}{2}}$ to both sides gives $t^2(4-t^2)^{-\frac{1}{2}} = \sqrt{(4-t^2)}$. Multiplying both sides by $\sqrt{4-t^2}$ then gives $t^2 = 4-t^2$; solving for t then gives $t = \pm\sqrt{2}$. Of course, $-\sqrt{2}$ is outside of our domain, so we can ignore it and instead only investigate $\sqrt{2}$.

To see something about the behavior of the function at this point, we have to take the second derivative. Omitting the steps involved (it would be good practice to see if you can get the same answer!), $f''(t) = -t^3(4-t^2)^{-\frac{3}{2}} - 3t(4-t^2)^{-\frac{1}{2}}$. At $t = \sqrt{2}$, this will be negative, so $t = \sqrt{2}$ produces a local maximum.

Note that this function is not actually defined over the entire interval provided - if $t > 2$, then we'd have to take the square root of a negative number. So the *effective* endpoints of the interval, for the purpose of finding the absolute minimum and maximum, are -1 and 2. So our candidates for minimum and maximum are where $t \in \{-1, \sqrt{2}, 2\}$. Plugging in, we get $f(-1) = -\sqrt{3}$, $f(\sqrt{2}) = 2$, and $f(2) = 0$. So the absolute maximum occurs where $t = \sqrt{2}$, and the absolute minimum occurs at $t = -1$.

c. $s(x) = x - \log(x), [1/2, 2]$

Solution: This one should be less painful than the previous problem. First, we need to find $s'(x)$. Remember, the derivative of the natural log of x is just $\frac{1}{x}$! So $s'(x) = 1 - \frac{1}{x}$. Setting this equal to 0, we get $0 = 1 - \frac{1}{x}$, which means $1 = \frac{1}{x}$, or $x = 1$. Let's check whether this is a minimum or a maximum. $s''(x) = \frac{1}{x^2}$, so $s''(1) = 1 > 0$. Since the second derivative is positive at $x = 1$, $x = 1$ should produce a local minimum. Plugging in $x = 1$ and the endpoints of the interval, we get $s(1/2) = 1.19$, $s(1) = 1$, and $s(2) = 1.31$. $x = 1$ is therefore not only produces a local minimum, but it produces the absolute minimum. The absolute maximum occurs at one of the endpoints, where $x = 2$.

d. $h(p) = 1 - e^{-p}, [0, 1000]$

Solution: The procedure should be getting familiar by now. First, we find the derivative of $h(p)$, giving us $h'(p) = e^{-p}$. However, we can make an interesting observation when we set this equal to 0, namely that e^{-p} never equals 0! Its limit as p goes to infinity is 0, but it is not 0 for any finite p, let alone one in our interval. So we have no critical points, and the endpoints will give us the absolute minimum and maximum over the interval. Plugging in, we get $h(0) = 0$ and $h(1000)$ is very close to 1, so over our interval, $p = 0$ produces the absolute minimum and $p = 1000$ produces the absolute maximum.

4.4 A function with no local minima/maxima

Demonstrate that the function $f(x) = x^5 + x^3 + x + 1$ has no local maximum and no local minimum.

Solution: This proof might seem hard to approach, so let's just see what happens when we try to find a local minimum or maximum. First, as usual, we have to find the derivative, and we find that $f'(x) = 5x^4 + 3x^2 + 1$. Next, we have to set this equal to zero and solve for x. After looking at the equation $5x^4 + 3x^2 + 1 = 0$, though, we might make an important observation - namely, that this has no solutions! There are two ways we could show this fact. First, we could create a variable $y = x^2$, rewrite the equation as $5y^2 + 3y + 1 = 0$, and then observe that the quadratic equation gives us no solutions. Second, and perhaps more elegantly, we can observe that $x^4 \geq 0$ and $x^2 \geq 0$ for all x. Therefore $5x^4 + 3x^2 + 1 \geq 5(0) + 3(0) + 1 = 1$. So $f'(x) \geq 1 > 0$ for all x. Thus, we see that the derivative never equals zero, and the function has no critical points. But all local maxima and local minima occur at critical points, so the function cannot have a local maximum or local minimum.

4.5 Approximate root-finding

Show that the equation

$$x^7 - 6x + 4 = 0$$

has a root between 0 and 1.

a. Find an initial approximation by ignoring the term x^7.

   Solution: If we ignore x^7, we can solve for the root as

$$\begin{aligned} -6x + 4 &= 0 \\ -6x &= -4 \\ x &= \frac{4}{6} = \frac{2}{3} \end{aligned}$$

b. Use Newton's method to find the root correct to 3 decimal places.

   Solution: Recall that the first derivative of the function is $f'(x) = x^6 - 6$. Assume a starting value of $x_0 = 0.7$.

$$
\begin{aligned}
x_0 &= 0.7 \\
x_1 &= x_0 - \frac{x_0^7 - 6x_0 + 4}{7x_0^6 - 6} \\
x_1 &= 0.7 - (0.0227271) \\
x_1 &= 0.677273
\end{aligned}
$$

$$
\begin{aligned}
x_2 &= x_1 - \frac{x_1^7 - 6x_1 + 4}{7x_1^6 - 6} \\
x_2 &= 0.677273 - (-0.000324455) \\
x_2 &= 0.677597
\end{aligned}
$$

$$
\begin{aligned}
x_3 &= x_2 - \frac{x_2^7 - 6x_2 + 4}{7x_2^6 - 6} \\
x_3 &= 0.677597 - (-5.92353»10^-8) \\
x_3 &= 0.677597
\end{aligned}
$$

4.6 Apply the mean value theorem

Does a continuous, differentiable function exist on $[0, 2]$ such that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leq 2\ \forall x$? Use the mean value theorem to explain your answer.

Solution: First we set up the mean value theorem which states that, if a function is continuous and differentiable over some interval, then a c exists such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

We plug in the values given by the problem and find, $f'(c) = \frac{f(2)-f(0)}{2-0} = \frac{4--1}{2} = \frac{5}{2}$.

The problem states that the derivative of the function is less than or equal to 2 over this entire interval, but the mean value theorem tell us that that the derivative must equal 2.5 at some point. So by demonstrating this contradiction, we've shown that the earlier values could not have come from a continuous, differentiable function.

Chapter 5

Matrix algebra and systems of linear equations

5.1 Basic matrix arithmetic

If

$$\mathbf{a} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

find:

a. $\mathbf{a} + \mathbf{b}$
b. $3\mathbf{a}$
c. $-4\mathbf{b}$
d. $3\mathbf{a} - 4\mathbf{b}$

5.2 More complex matrix arithmetic

Suppose

$$\mathbf{x} = \begin{bmatrix} 3 \\ 2q \\ 6 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} p+2 \\ -5 \\ 3r \end{bmatrix}$$

.

If $\mathbf{x} = 2\mathbf{y}$, find p, q, r.

5.3 Check for linear dependence

Which of the following sets of vectors are linearly dependent?

In each part, you can denote each vector as $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively.

a. $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$

b. $\begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}$

c. $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}$

d. $\begin{bmatrix}13\\7\\9\\2\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}3\\-2\\5\\8\end{bmatrix}$

e. $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\-2\\-1\end{bmatrix}, \begin{bmatrix}2\\1\\3\end{bmatrix}$

f. $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\-2\\4\end{bmatrix}, \begin{bmatrix}2\\1\\3\end{bmatrix}$

5.4 Vector length

Find the length of the following vectors:

a. $(3, 4)$
b. $(0, -3)$
c. $(1, 1, 1)$
d. $(3, 3)$
e. $(-1, -1)$
f. $(1, 2, 3)$
g. $(2, 0)$
h. $(1, 2, 3, 4)$
i. $(3, 0, 0, 0, 0)$

5.5 Law of cosines

The **law of cosines** states:

$$\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \, \|\mathbf{w}\|}$$

where θ is the angle from $\mathbf{w}$ to $\mathbf{v}$ measured in radians. Of importance, arccos() is the inverse of cos():

$$\theta = \arccos\left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \, \|\mathbf{w}\|}\right)$$

For each of the following pairs of vectors, calculate the angle between them. Report your answers in both radians and degrees. To convert between radians and degrees:

$$\text{Degrees} = \text{Radians} \times \frac{180^o}{\pi}$$

a. $\mathbf{v} = (1, 0), \quad \mathbf{w} = (2, 2)$
b. $\mathbf{v} = (4, 1), \quad \mathbf{w} = (2, -8)$
c. $\mathbf{v} = (1, 1, 0), \quad \mathbf{w} = (1, 2, 1)$

5.6 Matrix algebra

Using the matrices below, calculate the following. Some may not be defined; if that is the case, say so.

$$\mathbf{A} = \begin{bmatrix} 3 \\ -2 \\ 9 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 7 & -1 & 5 \\ 0 & 2 & -4 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 3 & 1 \\ 3 & 4 \\ 3 & -7 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 0 & -4 \\ -2 & 1 & -6 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 4 & 1 & -5 \\ 0 & 7 & 7 \\ 2 & -3 & 0 \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} 2 & -8 & -5 \\ -3 & 7 & -4 \\ 1 & 0 & 3 \\ 1 & 2 & 6 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} 9 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} 5 & 0 & 3 & 1 \end{bmatrix} \quad \mathbf{M} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

a. $\mathbf{A} + \mathbf{B}$
b. $-\mathbf{G}$
c. $\mathbf{D}'$
d. $\mathbf{C} + \mathbf{D}$
e. $3\mathbf{C} - 2\mathbf{D}'$
f. $\mathbf{A}' \cdot \mathbf{B}$

g. $\mathbf{CB}$
h. $\mathbf{BC}$
i. $\mathbf{FB}$
j. $\mathbf{EF}$
k. $\mathbf{K} \cdot \mathbf{L}'$
l. $\mathbf{G}'$
m. $\mathbf{E} - 5\mathbf{I_3}$
n. $\mathbf{M}^2$

5.7 Additive property of matrix transposition

Prove the additive property of matrix transposition:

$$(\mathbf{X} + \mathbf{Y})' = \mathbf{X}' + \mathbf{Y}'$$