Pset 9 Answer Keys

September 11, 2024

1 Identifying the PDF

$$\begin{aligned}
E[X] &= \frac{a+b}{2} \\
&= \frac{200,000 + 250,000}{2} \\
&= \frac{450,000}{2} \\
&= 225,000
\end{aligned}$$

$$\begin{aligned}
Var[X] &= \frac{(b-a)^2}{12} \\
&= \frac{(250,000 - 200,000)^2}{12} \\
&= \frac{50,000^2}{12} \\
&= 50000^2/12 \\
Var[X] &= \sigma^2 \\
sd(X) &= \sigma \\
&= \sqrt{50000^2/12} = 14433.756
\end{aligned}$$

2 Calculating ideal points

- **a.** In order to express $E(U(L))$ as a function of μ_l, $Var(L)$, and v, we need to find a form of the equation that contains those variables and no others. We know that

$$
\begin{aligned}
E(U(L)) &= E(-(L-v)^2) \\
&= E(-(L^2 - 2vL + v^2)) \\
&= E(-L^2 + 2vL - v^2) \\
&= \int_{-\infty}^{\infty} (-l^2 + 2vl - v^2) f(l)\, dl \\
&= \int_{-\infty}^{\infty} -l^2 f(l)\, dl + \int_{-\infty}^{\infty} 2vl f(l)\, dl + \int_{-\infty}^{\infty} -v^2 f(l)\, dl \\
&= -\int_{-\infty}^{\infty} l^2 f(l)\, dl + 2v \int_{-\infty}^{\infty} l f(l)\, dl - v^2 \int_{-\infty}^{\infty} f(l)\, dl \\
&= -E(L^2) + 2vE(L) - v^2 \\
&= -E(L^2) + 2v\mu_l - v^2
\end{aligned}
$$

Notice, however, that $E(L^2) = Var(L) + E(L)^2$, which can be determined by adding $E(L)^2$ to both sides of the definition of $Var(L)$. Therefore

$$
\begin{aligned}
E(U(L)) &= -E(L^2) + 2v\mu_l - v^2 \\
&= -Var(L) - \mu_l^2 + 2v\mu_l - v^2
\end{aligned}
$$

Bob can be considered risk averse because his expected utility is inversely related to the variance of L; as the variance of potential policy outcomes increases – or, in other words, as the candidate becomes a riskier bet – the less utility he receives. This problem illustrates one way in which the concepts we've been talking about, including random variables, expected value, and variance, can be applied to formal or game theoretic analysis.

- **b.** To see who Bob will choose, we first calculate the utility he derives from each of the potential recipients of his vote.

$$
\begin{aligned}
E(U(L_S)) &= -Var(L_S) - E(L_S)^2 + 2vE(L_S) - v^2 \\
&= -6 - (1^2) + 2(1)(1) - (1^2) \\
&= -6
\end{aligned}
$$

$$
\begin{aligned}
E(U(L_K)) &= -Var(L_K) - E(L_K)^2 + 2vE(L_K) - v^2 \\
&= -1 - (3^2) + 2(1)(3) - (1^2) \\
&= -1 - 9 + 6 - 1 \\
&= -5
\end{aligned}
$$

Notice that $E(U(L_K)) > E(U(L_S))$. Therefore Bob will vote for Leslie Knope. Although his ideal point is closer to the expected value of Schrute's policies, the greatly reduced uncertainty surrounding Knope's potential policies is sufficient to convince Bob to vote for her instead. More generally, risk aversion can in some cases induce individuals to vote for candidates whose mean potential policies are not the closest to those individuals' ideal points.

3 Parliamentary elections

- **a.** Recall that a valid PDF must sum up to 1. So we have to find a k such that the integral of the PDF equals 1.

$$\begin{aligned}
\int_0^5 kx^3 dx &= 1 \\
k\int_0^5 x^3 dx &= 1 \\
k\left[\frac{x^4}{4}\right]\Bigg|_0^5 &= 1 \\
k\left[\frac{5^4}{4} - \frac{0^4}{4}\right] &= 1 \\
\frac{625k}{4} &= 1 \\
k &= \frac{4}{625}
\end{aligned}$$

- **b.**

$$\int_{-\infty}^{x} \frac{4x^3}{625} dx = \int_0^x \frac{4x^3}{625} dx = \frac{4}{625}\int_0^x x^3 dx = \frac{4}{625}\left[\frac{x^4}{4}\Big|_0^x\right] = \frac{x^4}{625}$$

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x^4}{625} & \text{if } 0 < x < 5 \\ 1 & \text{if } x > 5 \end{cases}$$

Note that, since technically $-\infty < x < \infty$, you should define for CDF for values of x above and below the bounds are you primarily interested in.

- **c.**

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x \cdot \frac{4}{625} \cdot x^3 dx \\
&= \int_0^5 x \cdot \frac{4}{625} \cdot x^3 dx \\
&= \frac{4}{625} \int_0^5 x^4 dx \\
&= \frac{4}{625} \left[\frac{x^5}{5} \Big|_0^5 \right] \\
&= \frac{4}{625} \cdot 625 \\
&= 4
\end{aligned}
$$

$$
\begin{aligned}
E(X^2) &= \int_{-\infty}^{\infty} x^2 \cdot \frac{4}{625} \cdot x^3 dx \\
&= \int_0^5 x^2 \cdot \frac{4}{625} \cdot x^3 dx \\
&= \frac{4}{625} \int_0^5 x^5 dx \\
&= \frac{4}{625} \left[\frac{x^6}{6} \Big|_0^5 \right] \\
&= \frac{4}{625} \cdot \frac{15625}{6} \\
&= 25 \cdot \frac{2}{3} \\
&= \frac{50}{3}
\end{aligned}
$$

$$
Var(X) = E(X^2) - [E(X)]^2 = \frac{50}{3} - 4^2 = \frac{50}{3} - 16 = \frac{2}{3}
$$

- **d.**

$$\begin{aligned}
\int_{-\infty}^{m} \frac{4}{625} \cdot x^3 dx &= \frac{1}{2} \\
\int_{0}^{m} \frac{4}{625} \cdot x^3 dx &= \frac{1}{2} \\
\frac{4}{625} \int_{0}^{m} x^3 dx &= \frac{1}{2} \\
\frac{4}{625} \left[\frac{x^4}{4} \right] \Big|_0^m &= \frac{1}{2} \\
\frac{4}{625} \cdot \frac{m^4}{4} &= \frac{1}{2} \\
m^4 &= \frac{625}{2} \\
m &= \sqrt[4]{\frac{625}{2}} \\
&= \frac{5}{\sqrt[4]{2}} \approx 4.2045
\end{aligned}$$

- **e.** Zero. If the random variable is continuous, the probability of any one particular outcome is zero.

- **f.** The probability that the government remains in power between 2 and 4 years is 0.384.

$$F(4) - F(2) = \frac{4^4}{625} - \frac{2^4}{625} = \frac{256 - 16}{625} = \frac{240}{625} = 0.384$$

- **g.** This is the probability that the government survives between 0 and 1 year, $F(1)$, plus the probability that the government survives between 4 and 5 years, $F(5) - F(4) = 1 - F(4)$.

$$\begin{aligned}
F(1) + [1 - F(4)] &= \frac{1^4}{625} + \left[1 - \frac{4^4}{625}\right] = \frac{1}{625} + \left[1 - \frac{256}{625}\right] \\
&= \frac{1}{625} + \frac{369}{625} = \frac{370}{625} = 0.592
\end{aligned}$$

  The probability that the government remains in power for less than one year or more than 4 years is 0.592.

4 Calculating the CDF

- **a.** Z is only defined for the interval when greater than 0, so to find the CDF we assess $P(Z \leq z)$.

$$\begin{aligned} F(z) &= \int_0^z \gamma \exp(-\gamma z) dz \\ &= -\exp(-\gamma z)|_0^z \\ &= -\exp(-\gamma z) - -\exp(0) \\ &= 1 - \exp(-\gamma z) \end{aligned}$$

- **b.**
$$\begin{aligned} F(12) - F(7) &= (1 - \exp(-12\gamma)) - (1 - \exp(-7\gamma)) \\ &= \exp(-7\gamma) - \exp(-12\gamma) \end{aligned}$$

- **c.**
$$\begin{aligned} .1 &= \int_0^q f(z) dz \\ &= F(q) - F(0) \\ .1 &= (1 - \exp(-3q)) - (1 - \exp(0)) \\ .1 &= 1 - \exp(-3q) \\ \exp(-3q) &= .9 \\ -3q &= \log(.9) \\ q &= -\frac{\log(.9)}{3} \\ q &\approx 0.035 \end{aligned}$$

- **d.** We would expect a random draw from this distribution to be less than .5 with a probability equal to .78.

$$\begin{aligned} \int_0^{.5} f(z) dz &= F(.5) - F(0) \\ &= (1 - \exp(-3 \times .5)) - (1 - \exp(0)) \\ &= 1 - \exp(-1.5) \\ &= 0.7768698 \end{aligned}$$

5 Working with normal random variables

- **a.** For $X \sim N(0, 1)$, we can calculate:

$$\Pr(X \leq 1.5) = \Phi(1.5) = 0.9331928$$

where $\Phi(x)$ represents the cumulative distribution function (CDF) of the standard normal distribution.

Similarly, for $\Pr(X \leq -1)$:

$$\Pr(X \leq -1) = \Phi(-1) = 0.1586553$$

The values for $\Phi(1.5)$ and $\Phi(-1)$ can be looked up in a standard normal table (Z table) or computed numerically using a calculator.

- **b.** The random variable $\frac{Y-1}{2}$ is obtained by normalizing Y, which has mean 1 and variance 4. First, we shift the mean by subtracting 1, then we scale by dividing by 2 (the standard deviation of Y).

  Thus, $\frac{Y-1}{2}$ follows a standard normal distribution, $N(0,1)$. Therefore, the probability density function (PDF) of $\frac{Y-1}{2}$ is:

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

  which is the PDF of a standard normal distribution.