Problem set 8 Answer Keys

September 10, 2024

1 PMF vs CMF

Note, this is intentionally missing the bounds for x – we can infer that x must have a range that is 8 units long, otherwise the requirements for the cmf won't be satisfied. However, we could choose any eight values

2 Conversion of temperatures

If X is the temperature in Celsius, the temperature in Fahrenheit is $Y = 32 + \frac{9X}{5}$. Therefore,

$$E[Y] = 32 + \frac{9E[X]}{5} = 32 + \frac{9 \times 10}{5} = 32 + 18 = 50$$

Also,

$$Var[Y] = Var[32 + \frac{9X}{5}]$$

We know Variance of a constant is 0 and any constants multiplied to (Handy Variance rule for any constants a, b):

$$V(a + bX) = b2V(X)$$

Applying that out equation becomes -

$$Var(Y) = \left(\frac{9}{5}\right)^2 Var(X)$$

$$Var(Y) = \left(\frac{9}{5}\right)^2 10^2$$

$$Var(Y) = \left(\frac{9}{5}10\right)^2$$

$$Std(Y) = \left(\frac{9}{5}10\right) = 18$$

Thus, the standard deviation of Y is $\frac{9}{5} \times 10 = 18$. Hence, a normal day in Fahrenheit is one for which the temperature is in the range $[32, 68]$.

3 Getting a traffic ticket

You drive to work 5 days a week for a full year (50 weeks), and with probability $p = 0.02$ you get a traffic ticket on any given day, independent of other days. Let X be the total number of tickets you get in the year.

- **a.** The PMF of X is the binomial PMF with parameters $p = 0.02$ and $n = 250$. The mean is $\mathbb{E}[X] = np = 250 \times 0.02 = 5$. The desired probability is

$$\Pr(X = 5) = \binom{250}{5}(0.02)^5(0.98)^{245} = 0.1773$$

- **b.** The Poisson approximation has parameter $\lambda = np = 5$, so the probability in (a) is approximated by

$$e^{-\lambda}\frac{\lambda^5}{5!} = 0.1755$$

4 The unbirthday song

The number of people P needed before you encounter a person whose birthday is today is a geometric random variable with parameter $p = \frac{1}{365}$. Thus, the PMF is

$$\begin{aligned}\Pr(P = k) &= (1-p)^k p, \quad k = 1, 2, \dots \\ &= \left(1 - \frac{1}{365}\right)^k \times \frac{1}{365}\end{aligned}$$

The expected value is

$$\begin{aligned}\mathbb{E}[P] &= \frac{1-p}{p} \\ &= \frac{1 - 1/365}{1/365} \\ &= 365\end{aligned}$$

The variance is

$$\begin{aligned}Var(P) &= \frac{1-p}{p^2} \\ &= \frac{1 - 1/365}{(1/365)^2} \\ &= \frac{364/365}{(1/365)^2} \\ &= 132860\end{aligned}$$

5 Properties of variance

A useful property to know here is that $\mathbb{E}[\mathbb{E}[X]] = \mathbb{E}[X]$.

$$\begin{aligned}
Var(X) &= \mathbb{E}[(X - \mathbb{E}[X])^2] \\
&= \mathbb{E}[X^2 - (2\mathbb{E}[X] \times X) + \mathbb{E}[X]^2] \\
&= \mathbb{E}[X^2] - 2\mathbb{E}[\mathbb{E}[X] \times X] + \mathbb{E}[\mathbb{E}[X]^2] \\
&= \mathbb{E}[X^2] - 2\mathbb{E}[X]\mathbb{E}[X] + \mathbb{E}[X]^2 \\
&= \mathbb{E}[X^2] - 2\mathbb{E}[X]^2 + \mathbb{E}[X]^2 \\
&= \mathbb{E}[X^2] - \mathbb{E}[X]^2
\end{aligned}$$

6 Calculate an exact probability

With a Poisson variable, the pdf supplies probability of a given number (count) of events, i.e. $\Pr(X = x)$. From the prompt, we know that $\Pr(X = 0) = .135$. Using this, we can solve for λ, which is the average rate of "success" in a given interval, (i.e. the average number of times an event occurs in a given interval). Here, the event is an error occurring and the interval is the page.

$$\begin{aligned}
f(x; \lambda) &= \frac{\lambda^x \exp(-\lambda)}{x!} \\
f(x = 0; \lambda) = \frac{\lambda^0 \exp(-\lambda)}{0!} &= .135 \\
\frac{\exp(-\lambda)}{1} &= .135 \\
\frac{1}{\exp(\lambda)} &= .135 \\
\exp(\lambda) &= \frac{1}{.135} \\
\exp(\lambda) &= 7.407 \\
\lambda &= 2.002
\end{aligned}$$

Now that we know the value of λ, we know that on average, a page contains 2.002 errors. (Note: If we want to change the interval to say, 10 pages, we just adjust λ accordingly. If there are 2.002 errors per page, there are 2.002*10=20.02 errors per 10 pages.) Now we can solve for $\Pr(X = 1)$.

$$f(x = 1; \lambda) = \frac{2.002^1 \exp(-2.002)}{1!} = .27$$

7 Obtaining requests for information

- **a.** $\Pr(X \leq 3) \approx .43$

  $\Pr(X \leq 3) = \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2) + \Pr(X = 3)$

  $\Pr(X = 0) = e^{-4}\frac{4^0}{0!} = 0.01831564$

  $\Pr(X = 1) = e^{-4}\frac{4^1}{1!} = 0.07326256$

  $\Pr(X = 2) = e^{-4}\frac{4^2}{2!} = 0.1465251$

  $\Pr(X = 3) = e^{-4}\frac{4^3}{3!} = 0.1953668$

  $\Pr(X \leq 3) = 0.01831564 + 0.07326256 + 0.1465251 + 0.1953668 = 0.4334701$

- **b.** $\Pr(10 < X < 13) \approx .003$

  $$\begin{aligned}\Pr(10 < X < 13) &= \Pr(X = 11) + \Pr(X = 12)\\ \Pr(X = 11) &= e^{-4}\frac{4^{11}}{11!} = 0.001924537\\ \Pr(X = 12) &= e^{-4}\frac{4^{12}}{12!} = 0.0006415123\\ \Pr(10 < X < 13) &= 0.001924537 + 0.0006415123 = 0.002566049\end{aligned}$$

- **c.** $\Pr(X > 5) = 1 - \Pr(X \leq 5) \approx 0.22$

  We already know $\Pr(X \leq 3) = .43$ so now we only need to find $\Pr(X = 4)$ and $\Pr(X = 5)$.

  $$\begin{aligned}\Pr(X \leq 3) &= 0.4334701\\ \Pr(X = 4) &= e^{-4}\frac{4^4}{4!} = 0.1953668\\ \Pr(X = 5) &= e^{-4}\frac{4^5}{5!} = 0.1562935\\ \Pr(X \leq 5) &= 0.4334701 + 0.1953668 + 0.1562935 = 0.7851304\\ \Pr(X > 5) &= 1 - 0.7851304 = 0.2148696\end{aligned}$$

- **d.** Since *Responsive* is dichotomous, $\mathbb{E}[\text{Responsive}] = \Pr(\text{Responsive} = 1) = \Pr(X \leq 5)$. From the previous question we know that $\Pr(5 < X) = 0.2148696$ and $\Pr(X \leq 5) = 0.7851304$. So $\mathbb{E}[\text{Responsive}] = 0.7851304$.

- **e.** *Responsive* is distributed Bernoulli. $Var(\text{Bernoulli}) = p(1-p)$. In this case $p = 0.7851304$. So $Var(\text{Responsive}) = (0.7851304 \times (1 - 0.7851304)) = 0.1687007$.

8 Modeling electoral outcomes

- **a.** The election outcomes are distributed according to a binomial distribution so we find the expected value given our two parameters, n and θ.

$$\begin{aligned}\mathbb{E}[\text{Binomial}] &= n \times \theta \\ &= 4 \times .55 \\ &= 2.2\end{aligned}$$

- **b.** $Pr(k = 0) \approx 0.04100625$

$$\begin{aligned}\Pr(k = 0) &= \binom{4}{0} .55^0 (1 - .55)^4 \\ &= 1 \times 1 \times 0.04100625 \\ &= 0.04100625\end{aligned}$$

- **c.** $Pr(k > 2) \approx 0.39$

$$\begin{aligned}\Pr(k = 3) &= \binom{4}{3} .55^3 (1 - .55)^1 = 4 \times 0.07486875 = 0.299475 \\ \Pr(k = 4) &= \binom{4}{4} .55^4 (1 - .55)^0 = 1 \times 0.09150625 = 0.09150625 \\ \Pr(2 < k) &= 0.299475 + 0.09150625 = 0.3909813\end{aligned}$$

- **d.** We should take this bet as we will net an expected return of \$6.32 from it. We can think of *bet* as a variable with the associated values:

$$\text{bet} = \begin{cases} -15 & \text{if Republican majority;} \\ 20 & \text{otherwise.} \end{cases}$$

$$\begin{aligned}\mathbb{E}[\text{bet}] &= \mathbb{E}[\text{bet}|\text{Republican Majority}] + \mathbb{E}[\text{bet}|\text{Otherwise}] \\ &= -15 \times \Pr(\text{Republican Majority}) + 20 \times \Pr(\text{Otherwise}) \\ &= -15 * (0.3909813) + 20 * (1 - 0.3909813) \\ &= -5.864719 + 12.18037 \\ &\approx 6.3\end{aligned}$$

- **e.** We should take this bet as well since we will net an expected \$9.18. We can think of bet_2 as a variable that takes the following values:

$$\text{bet}_2 = \begin{cases} 100 & \text{if Republican majority;} \\ 50 & \text{if tie;} \\ -200 & \text{otherwise.} \end{cases}$$

$$
\begin{aligned}
\Pr(\text{Rep Majority}) &= 0.3909813 \\
\Pr(\text{Tie}) &= \binom{4}{2}.55^2(1-.55)^2 = 6 \times 0.06125625 = 0.3675375 \\
\Pr(\text{Dem Majority}) &= 1 - \Pr(\text{Rep Majority}) - \Pr(\text{Tie}) \\
&= 1 - 0.3909813 - 0.3675375 \\
&= 0.2414812 \\
\mathbb{E}[\text{bet}_2] &= \mathbb{E}[\text{bet}_2|\text{Rep Majority}] + \mathbb{E}[\text{bet}_2|\text{Tie}] + \mathbb{E}[\text{bet}_2|\text{Otherwise}] \\
&= 100 \times \Pr(\text{Rep Majority}) + 50 \times \Pr(\text{Tie}) \\
&\quad + -200 \times \Pr(\text{Republicans don't win majority}) \\
&= 100 \times 0.3909813 + 50 \times 0.3675375 - 200 \times 0.2414812 \\
&\approx 9.2
\end{aligned}
$$