PSET 8, Multivariate Dist.

2. $f(x|y) = \dfrac{f(x,y)}{f(y)}$

$f(y) = \int_{x=0}^{y} f(x,y)\,dx$

$= \int_0^y 5x^2y^2\,dx$

$= \frac{5}{3}x^3y^2 \Big|_0^y$

$= \frac{5}{3}y^5 \quad (0 \le y \le 1)$

$\therefore f(x|y) = 5x^2y^2 / \frac{5}{3}y^5$

$= 5x^2y^2 \cdot \dfrac{3}{5y^5}$

$= 3x^2y^{-3}$

$(0 \le x \le y \le 1)$

3. a. $\int_0^6 \int_0^6 kx^3y^2\,dx\,dy = k \cdot \int_0^6 x^3 dx \cdot \int_0^6 y^2 dy$

$= k \cdot \frac{x^4}{4}\Big|_0^6 \cdot \frac{y^3}{3}\Big|_0^6$

$= k \cdot \frac{6^4}{4} \cdot \frac{6^3}{3}$

$\therefore k \cdot \frac{6^4}{4} \cdot \frac{6^3}{3} = 1 \quad \therefore k = \frac{1}{23328}$

b. $f_X(x) = \int_0^6 f_{X,Y}(x,y)\,dy$

$= k \cdot x^3 \cdot \int_0^6 y^2\,dy$

$= kx^3 \cdot \frac{1}{3} \cdot 6^3$

$= \frac{x^3}{324}$

c. $f_Y(y) = \int_0^6 f_{X,Y}(x,y)\,dx$

$= k \cdot y^2 \cdot \int_0^6 x^3\,dx$

$= k \cdot y^2 \cdot \frac{6^4}{4}$

$= \frac{y^2}{72}$

d. $E[X] = \int_0^6 x \cdot f_X(x)\, dx$

$= \int_0^6 \frac{x^4}{324}\, dx$

$= \frac{1}{324} \cdot \frac{1}{5} x^5 \Big|_0^6$

$= \frac{24}{5} = 4.8$

e. $E[Y] = \int_0^6 y \cdot f_Y(y)\, dy$

$= \int_0^6 y \cdot \frac{y^2}{72}\, dy$

$= \frac{1}{72} \cdot \frac{1}{4} y^4 \Big|_0^6$

$= \frac{9}{2} = 4.5$

f. $Var(X) = E[X^2] - E[X]^2$

$E[X^2] = \int_0^6 x^2 \cdot \frac{x^3}{324}\, dx$

$= \frac{1}{324} \cdot \frac{1}{6} x^6 \Big|_0^6$

$= 24$

$\therefore Var(X) = 24 - 4.8^2 = \frac{24}{25} = 0.96$

g. $Var(Y) = E[Y^2] - E[Y]^2$

$E[Y^2] = \int_0^6 y^2 \cdot \frac{y^2}{72}\, dy$

$= \frac{1}{72} \cdot \frac{1}{5} y^5 \Big|_0^6$

$= \frac{108}{5} = 21.6$

Var(Y) = 21.6 - 4.5 * 4.5 = 27/20 = 1.35

h. $Cov(X,Y) = E[XY] - E[X]E[Y]$

$E[XY] = \iint xy \cdot f_{X,Y}(x,y)\,dx\,dy$

$= k \cdot \int_0^6 x^3 \cdot x\,dx \cdot \int_0^6 y^2 \cdot y\,dy$

$= k \cdot \frac{1}{5}x^5 \Big|_0^6 \cdot \frac{1}{4}y^4 \Big|_0^6$

$= \frac{108}{5} = 21.6$

$\therefore Cov(X,Y) = 21.6 - 4.8 \times 4.5 = 0$

i. $f_{X,Y}(x,y) = kx^3y^2 = \frac{x^3}{324} \cdot \frac{y^2}{72} = f_X(x) \cdot f_Y(y)$

$\therefore$ Independent

j. $f_{X|Y}(x|y) = f_X(x) = \frac{x^3}{324}$ given independency

k. $f_{Y|X}(y|x) = f_Y(y) = \frac{y^2}{72}$ given independency

4. a. $Cov(D,F) = E[DF] - E[D] \cdot E[F]$

$= 10 - 4 \times 8$

$= -22$

b. $Cor(D,F) = \frac{Cov(D,F)}{\sqrt{Var(D) \cdot Var(F)}} = \frac{-22}{\sqrt{30 \times 60}} = -\frac{22}{30\sqrt{2}} \approx -0.518$

c. $Cov(D,H) = Cov(D,2F) = 2\,Cov(D,F) = -44$

d. $Cor(D,H) = \frac{Cov(D,H)}{\sqrt{Var(D) \cdot Var(H)}} = \frac{-44}{\sqrt{30 \times 60 \times 4}} = -\frac{44}{60\sqrt{2}} \approx -0.518$

$Var(H) = Var(2F) = 4\,Var(F)$

Same as (b), multiplying a variable by a positive constant leaves correlation unchanged.

e. $\text{Cor}(D,F) = \dfrac{\text{Cov}(D,F)}{\sqrt{\text{Var}(D)\,\text{Var}(F)}} = \dfrac{-22}{\sqrt{40\times 60}} = -\dfrac{22}{20\sqrt{6}} \approx -0.449$

5. By definition, $f(\theta|x) = \dfrac{f(\theta,x)}{f_X(x)}$

Rewrite the joint distribution, $f(\theta,x) = f(x|\theta)f(\theta)$

Plug into the conditional, $f(\theta|x) = \dfrac{f(x|\theta)f(\theta)}{f_X(x)}$

Express $f_X(x)$, $f_X(x) = \int f(x,\theta)\,d\theta$

$= \int f(x|\theta)f(\theta)\,d\theta.$

Substitute back, $f(\theta|x) = \dfrac{f(x|\theta)f(\theta)}{\int f(x|\theta)f(\theta)\,d\theta}$