PSET 4: Linear algebra - SOLUTIONS

1 Basic matrix arithmetic

Given:

$$\mathbf{a} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

a. $\mathbf{a} - \mathbf{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

b. $-4\mathbf{b} = -4 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ -4 \end{bmatrix}$

c. $3\mathbf{a} + 3\mathbf{b} = 3 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} + \begin{bmatrix} 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 15 \\ 9 \end{bmatrix}$

2 More complex matrix arithmetic

Given:

$$\mathbf{x} = \begin{bmatrix} 3 \\ 2q \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} p+2 \\ -5 \\ 3r \end{bmatrix}$$

If $\mathbf{x} = 2\mathbf{y}$, then:

$$\begin{bmatrix} 3 \\ 2q \\ 4 \end{bmatrix} = 2 \begin{bmatrix} p+2 \\ -5 \\ 3r \end{bmatrix} = \begin{bmatrix} 2(p+2) \\ 2(-5) \\ 2(3r) \end{bmatrix} = \begin{bmatrix} 2p+4 \\ -10 \\ 6r \end{bmatrix}$$

Equating components:

$$3 = 2p + 4 \Rightarrow 2p = -1 \Rightarrow p = -\frac{1}{2} \tag{1}$$

$$2q = -10 \Rightarrow q = -5 \tag{2}$$

$$4 = 6r \Rightarrow r = \frac{2}{3} \tag{3}$$

Solution: $p = -\frac{1}{2}$, $q = -5$, $r = \frac{2}{3}$, $r^2 = \frac{4}{9}$

3 Check for linear dependence

a. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

Form matrix and calculate determinant:

$$\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = 1(45-48) - 4(18-24) + 7(12-15) = -3 + 24 - 21 = 0$$

Since the determinant is 0, these vectors are **linearly dependent**.

b. $\begin{bmatrix}13\\7\\9\\2\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}3\\-2\\5\\8\end{bmatrix}$

Since one vector is the zero vector, the set is automatically **linearly dependent**.

c. $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\-2\\-1\end{bmatrix}, \begin{bmatrix}2\\1\\3\end{bmatrix}$

$$\begin{vmatrix}1 & 2 & 2\\2 & -2 & 1\\1 & -1 & 3\end{vmatrix} = 1(-6+1) - 2(6+1) + 2(-2+2) = -5 - 14 + 0 = -19 \neq 0$$

Since the determinant is non-zero, these vectors are **linearly independent**.

4 Vector length

a. $\|(4,2)\| = \sqrt{4^2+2^2} = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$

b. $\|(2,2,2)\| = \sqrt{2^2+2^2+2^2} = \sqrt{12} = 2\sqrt{3}$

c. $\|(4,2,3,1)\| = \sqrt{4^2+2^2+3^2+1^2} = \sqrt{16+4+9+1} = \sqrt{30}$

d. $\|(0,0,0,0,3)\| = \sqrt{0^2+0^2+0^2+0^2+3^2} = \sqrt{9} = 3$

5 Law of cosines

Using $\theta = \arccos\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right)$

a. $\mathbf{v} = (1,0)$, $\mathbf{w} = (2,2)$

$$\mathbf{v}\cdot\mathbf{w} = 1(2) + 0(2) = 2 \tag{4}$$
$$\|\mathbf{v}\| = 1, \quad \|\mathbf{w}\| = \sqrt{8} = 2\sqrt{2} \tag{5}$$
$$\cos(\theta) = \frac{2}{1\cdot 2\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \tag{6}$$
$$\theta = \frac{\pi}{4} \text{ radians} = 45\text{ř} \tag{7}$$

b. $\mathbf{v} = (4,1)$, $\mathbf{w} = (2,-8)$

$$\mathbf{v}\cdot\mathbf{w} = 4(2) + 1(-8) = 8 - 8 = 0 \tag{8}$$
$$\|\mathbf{v}\| = \sqrt{17}, \quad \|\mathbf{w}\| = \sqrt{68} = 2\sqrt{17} \tag{9}$$
$$\cos(\theta) = \frac{0}{\sqrt{17}\cdot 2\sqrt{17}} = 0 \tag{10}$$
$$\theta = \frac{\pi}{2} \text{ radians} = 90\text{ř} \tag{11}$$

c. $\mathbf{v} = (1, 1, 0)$, $\mathbf{w} = (1, 2, 2)$

$$\mathbf{v} \cdot \mathbf{w} = 1(1) + 1(2) + 0(2) = 3 \tag{12}$$
$$\|\mathbf{v}\| = \sqrt{2}, \quad \|\mathbf{w}\| = \sqrt{9} = 3 \tag{13}$$
$$\cos(\theta) = \frac{3}{\sqrt{2} \cdot 3} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \tag{14}$$
$$\theta = \frac{\pi}{4} \text{ radians} = 45ř \tag{15}$$

6 Matrix algebra

a. $\mathbf{A} + \mathbf{B} = \begin{bmatrix} 3 \\ -2 \\ 9 \end{bmatrix} + \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 11 \\ -2 \\ 8 \end{bmatrix}$

b. $-\mathbf{G} = -\begin{bmatrix} 2 & -8 & -5 \\ -3 & 7 & -4 \\ 1 & 0 & 3 \\ 1 & 2 & 6 \end{bmatrix} = \begin{bmatrix} -2 & 8 & 5 \\ 3 & -7 & 4 \\ -1 & 0 & -3 \\ -1 & -2 & -6 \end{bmatrix}$

c. $\mathbf{D}' = \begin{bmatrix} 3 & 1 \\ 3 & 4 \\ 3 & -7 \end{bmatrix}' = \begin{bmatrix} 3 & 3 & 3 \\ 1 & 4 & -7 \end{bmatrix}$

d. $\mathbf{C} + \mathbf{D}$: Not defined (C is 2×3, D is 3×2)

e. $\mathbf{A}'\mathbf{B} = \begin{bmatrix} 3 & -2 & 9 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} = 24 + 0 - 9 = 15$

f. $\mathbf{BC}$: Not defined (B is 3×1, C is 2×3)

g. $\mathbf{FB} = \begin{bmatrix} 4 & 1 & -5 \\ 0 & 7 & 7 \\ 2 & -3 & 0 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 32 + 0 + 5 \\ 0 + 0 - 7 \\ 16 + 0 + 0 \end{bmatrix} = \begin{bmatrix} 37 \\ -7 \\ 16 \end{bmatrix}$

h. $\mathbf{E} - 5\mathbf{I_3} = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 0 & -4 \\ -2 & 1 & -6 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 \\ 1 & -5 & -4 \\ -2 & 1 & -11 \end{bmatrix}$

i. $\mathbf{M}^2 = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -4 & 10 \end{bmatrix}$

7 Matrix inversion

a. $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

det $= 2(1) - 1(1) = 1 \neq 0$, so invertible.

$\mathbf{X}^{-1} = \frac{1}{1} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

Verification: $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$

$\det = 2(-2) - 1(-4) = -4 + 4 = 0$

Not invertible (determinant is zero).

c. $\begin{bmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{bmatrix}$

$\det = 2(0 + 30) - 4(0 + 18) + 0 = 60 - 72 = -12 \neq 0$, so invertible.

Using cofactor method:

$$\mathbf{X}^{-1} = \frac{1}{-12} \begin{bmatrix} 30 & 0 & 12 \\ -18 & 0 & -6 \\ -4 & -4 & -4 \end{bmatrix} = \begin{bmatrix} -5/2 & 0 & -1 \\ 3/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

8 Dummy encoding for categorical variables

When we have a categorical variable with n classes, creating n dummy variables would result in perfect multicollinearity. This is because the sum of all dummy variables would always equal 1 (the intercept column), making the matrix $\mathbf{X}'\mathbf{X}$ singular (determinant = 0).

If $\mathbf{X}'\mathbf{X}$ is singular, then $(\mathbf{X}'\mathbf{X})^{-1}$ does not exist, and we cannot compute $\hat{\mathbf{b}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$.

By using only $n - 1$ dummy variables, we avoid perfect multicollinearity. The omitted category serves as the reference group, and its effect is captured in the intercept term, ensuring that $\mathbf{X}'\mathbf{X}$ is invertible.

9 Solve the system of equations

a. System #1

$$x + y + 2z = 2 \tag{16}$$
$$3x - 2y + z = 1 \tag{17}$$
$$y - z = 3 \tag{18}$$

From equation 3: $y = z + 3$

Substituting into equation 1: $x + (z + 3) + 2z = 2 \Rightarrow x = -3z - 1$

Substituting into equation 2: $3(-3z - 1) - 2(z + 3) + z = 1$

$$-9z - 3 - 2z - 6 + z = 1 \tag{19}$$
$$-10z = 10 \tag{20}$$
$$z = -1 \tag{21}$$

Therefore: $y = -1 + 3 = 2$, $x = -3(-1) - 1 = 2$

Solution: $x = 2, y = 2, z = -1$

b. System #2

$$x - y + 2z = 2 \tag{22}$$
$$4x + y - 2z = 10 \tag{23}$$
$$x + 3y + z = 0 \tag{24}$$

Adding equations 1 and 2: $5x = 12 \Rightarrow x = \frac{12}{5}$

From equation 3: $z = -\frac{12}{5} - 3y$

Substituting into equation 1: $\frac{12}{5} - y + 2\left(-\frac{12}{5} - 3y\right) = 2$

$$\frac{12}{5} - y - \frac{24}{5} - 6y = 2 \tag{25}$$

$$-7y = 2 + \frac{12}{5} = \frac{22}{5} \tag{26}$$

$$y = -\frac{22}{35} \tag{27}$$

$z = -\frac{12}{5} - 3\left(-\frac{22}{35}\right) = -\frac{18}{35}$

Solution: $x = \frac{12}{5}, y = -\frac{22}{35}, z = -\frac{18}{35}$

10 Multiplying by 0

To prove that $\mathbf{AB} = \mathbf{0}$ does not necessarily imply $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$, I provide a counterexample:

Let $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

Then:

$$\mathbf{AB} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$$

However, neither $\mathbf{A}$ nor $\mathbf{B}$ is the zero matrix. This proves that matrix multiplication does not follow the same zero-product property as real numbers.