Scalar Calculus

Computational Mathematics and Statistics Camp
University of Chicago
September 2018

1. Find the following finite limits:

   a. $\lim_{x \to 4} x^2 - 6x + 4$
   b. $\lim_{x \to 0} \left[\frac{x - 25}{x + 5} \right]$
   c. $\lim_{x \to 4} \left[\frac{x^2}{3x - 2} \right]$
   d. $\lim_{x \to 1} \left[\frac{x^4 - 1}{x - 1} \right]$
   e. $\lim_{x \to -4} \left[\frac{x^2 + 5x + 4}{x^2 + 3x - 4} \right]$
   f. $\lim_{x \to 4^-} \sqrt{16 - x^2}$
   g. $\lim_{x \to -1} \left[\frac{x - 2}{x^2 + 4x - 3} \right]$
   h. $\lim_{x \to -4} \left[\frac{\frac{1}{4} + \frac{1}{x}}{4 + x} \right]$

2. Given that:

$$\lim_{x \to a} f(x) = -3 \qquad \lim_{x \to a} g(x) = 0 \qquad \lim_{x \to a} h(x) = 8$$

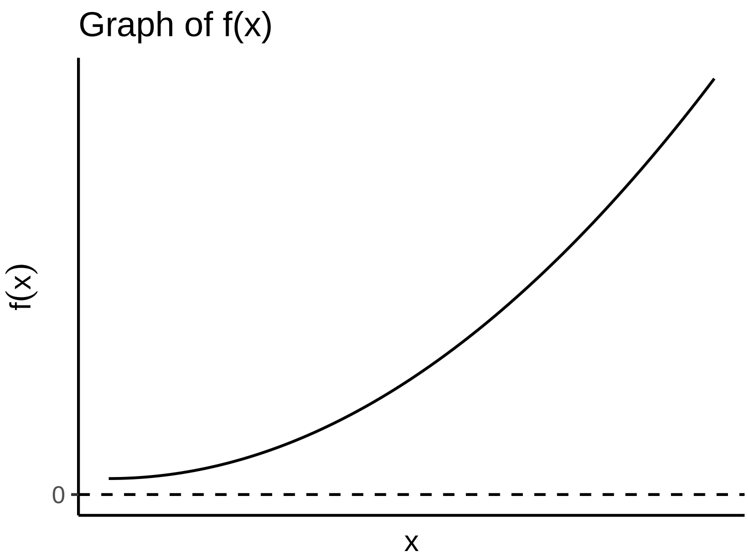
   find the following limits. If the limit doesn't exist, explain why.

   a. $\lim_{x \to a} [f(x) + h(x)]$
   b. $\lim_{x \to a} [f(x)]^2$
   c. $\lim_{x \to a} \left[\frac{f(x)}{h(x)} \right]$
   d. $\lim_{x \to a} \left[\frac{g(x)}{f(x)} \right]$

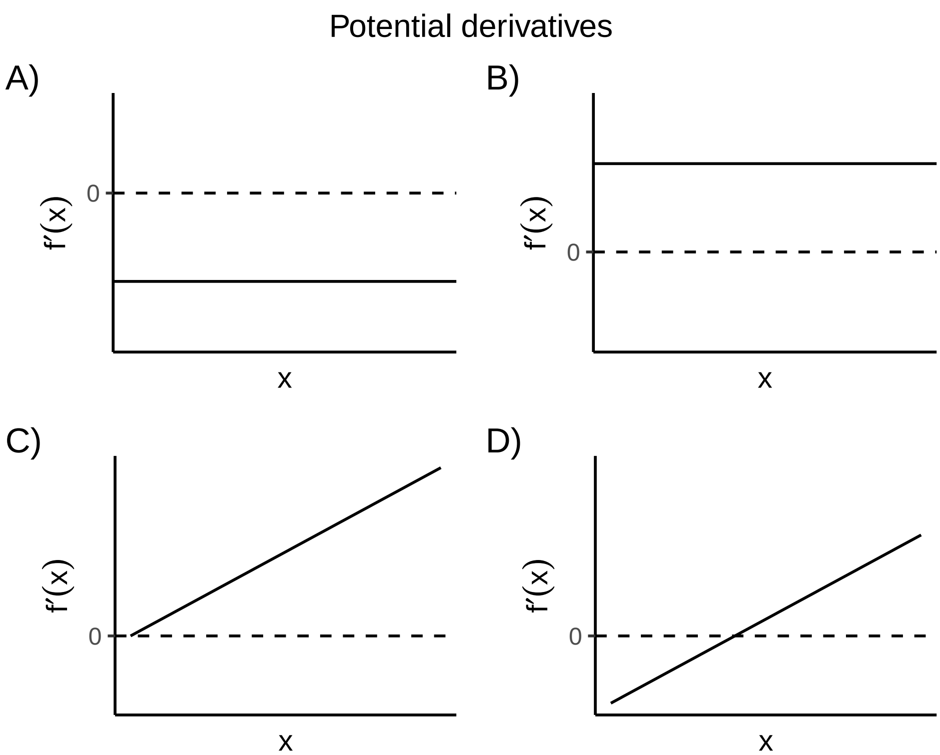
3. Find the following infinite limits:

   a. $\lim_{x \to \infty} \left[\frac{9x^2}{x^2 + 3} \right]$
   b. $\lim_{x \to \infty} \left[\frac{3x - 4}{x + 3} \right]$
   c. $\lim_{x \to \infty} \left[\frac{2^x - 3}{2^x + 1} \right]$
   d. $\lim_{x \to \infty} \left[\frac{\log(x)}{x} \right]$
   e. $\lim_{x \to \infty} \left[\frac{3^x}{x^3} \right]$
   f. $\lim_{y \to \infty} \left[\frac{3e^y}{y^3} \right]$

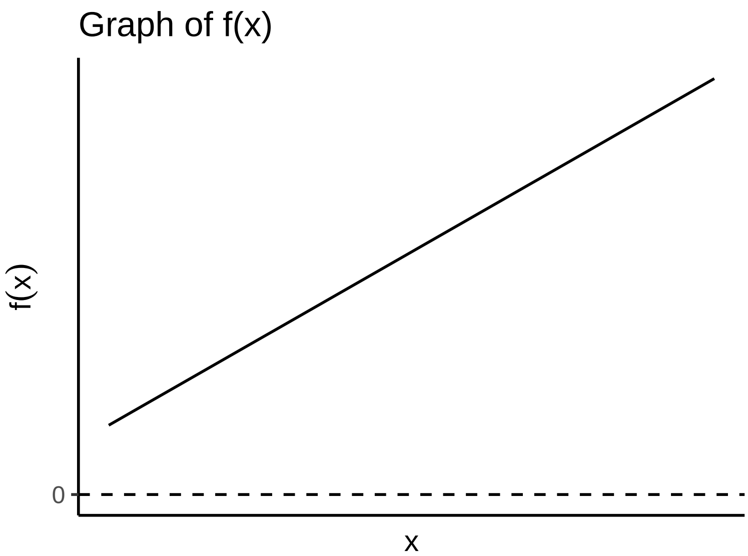
4. A friend shows you this graph of a function $f(x)$:



Which of the following could be a graph of $f'(x)$? For each graph, explain why or why not it might be the derivative of $f(x)$.



What if the figure below was the graph of $f(x)$? Which of the graphs might potentially be the derivative of $f(x)$ then?



5. Differentiate the following functions:
    a. $f(x) = 4x^3 + 2x^2 + 5x + 11$
    b. $y = \sqrt{30}$
    c. $h(t) = \log(9t + 1)$
    d. $f(x) = \log(x^2 e^x)$
    e. $h(y) = \left(\frac{1}{y^2} - \frac{3}{y^4}\right)(y + 5y^3)$
    f. $g(t) = \frac{3t - 1}{2t + 1}$
6. Differentiate the following using both the product and quotient rules:
$$f(x) = \frac{x^2 - 2x}{x^4 + 6}$$
7. Does a continuous, differentiable function exist on $[0, 2]$ such that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leq 2 \, \forall \, x$? Use the mean value theorem to explain your answer.
8. Solve the following definite integrals using the antiderivative method.
    a. $\int_6^8 x^3 \, dx$
    b. $\int_{-1}^0 (3x^2 - 1) \, dx$
    c. $\int_{-1}^1 (14 + x^2) \, dx$
    d. $\int_1^2 \frac{1}{t^2} \, dt$
    e. $\int_2^4 e^y \, dy$
    f. $\int_8^9 2^x \, dx$
    g. $\int_3^3 \sqrt{x^5 + 2} \, dx$
9. A group of three unidentified first-year graduate students at the University of Chicago are worn out after a week of math camp. Wanting to unwind, the students agree to not talk about math and decide to chat over some casual drinks at Medici.

    After five shots of tequila each, two pitchers of beer, a bottle of wine, and a large Chicago-style pizza, the three students have had enough fun and decide to start the trip back home.

    Student A gets on a bike and starts pedaling away at a velocity of $v_A(t) = 2t^4 + t$, where t represents minutes. However, the student crashes into the side of an Uber and ends the journey after only 2 minutes.

Student B has no bike, so starts running at a velocity of $v_B(t) = 4\sqrt{t}$. Sadly, after only 4 minutes, the student's legs give out and the student decides to sing a song, instead.

Student C can't even stand up, so has no choice but to slowly crawl at a velocity of $v_C(t) = 2e^{-t}$. Student C steadily plods along for 20 minutes before falling asleep on the sidewalk.

Generally, if an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$. The Fundamental Theorem of Calculus then tells us that

$$\begin{aligned} \text{Total distance traveled} &= \int_{t_1}^{t_2} v(t)\,dt \\ s(t_2) - s(t_1) &= \int_{t_1}^{t_2} v(t)\,dt \end{aligned}$$

Without using a calculator, use this formula to find the distance traveled by Students A, B, and C. (Assume, however unrealistic in may be, that all three students traveled in a straight line.) Who traveled the farthest? The least far?

10. Calculate the following indefinite integrals:

   a. $\int (x^2 - x^{-\frac{1}{2}})\,dx$
   b. $\int 360t^6\,dt$
   c. $\int 2x \log(x^2)\,dx$