Sample Space and Probability

The following laws of set algebra may be useful:

Commutative Law

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Associative Law

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

Distributive Law

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

De Morgan's Law

$$(A \cap B)^c = A^c \cup B^c$$

1 Sets

Consider rolling a six-sided die. Let A be the set of outcomes where the roll is an even number. Let B be the set of outcomes where the roll is greater than 3. Calculate the sets on both sides of De Morgan's Laws.

$$(A \cup B)^c = A^c \cap B^c, \quad (A \cap B)^c = A^c \cup B^c$$

Solution: We have

$$A = \{2, 4, 6\}, \quad B = \{4, 5, 6\}$$

so

$$A \cup B = \{2, 4, 5, 6\}, \quad (A \cup B)^c = \{1, 3\}$$

and

$$A^c \cup B^c = \{1,3,5\} \cap \{1,2,3\} = \{1,3\}$$

Similarly, we have $A \cap B = \{4,6\}$, and

$$(A \cap B)^c = \{1,2,3,5\}$$

2 Ghostbusting

Twenty ghostbusters are on their annual camping retreat. Two of them, Abe and Betty, have discovered that another pair, Candace and Dan, are in fact ghosts posing as ghostbusters. Abe and Betty hatch a plan: When all 20 campers are sitting in a circle around the campfire, Abe will fire his proton pack at Candace, and Betty will simultaneously fire her proton pack at Dan, annihilating the ghosts. However, if two proton streams cross, it means the end of all life on Earth.

If the ghostbusters are arranged randomly around the fire, what are the chances that Abe and Betty will cross streams?

Solution: The chances are $\frac{1}{3}$.

There are 20 ghostbusters, but we only care about four of them - Abe, Betty, Candace, and Dan. The position of the other 16 won't affect the possible crossing of the streams. Fix Abe's spot at the campfire. Betty has three places to sit (east, west, south). There are $3 \times 2 \times 1 = 6$ seating possibilities. In two of these arrangements, the proton streams cross, so the probability of disaster is $\frac{2}{6} = \frac{1}{3}$.

3 Calculate Probabilities in a Sample Space S

Events A and B are contained within a sample space S. Given that $\Pr(A) = 0.5$, $\Pr(B) = 0.3$ and $\Pr(A \cap B) = 0.1$, find:

1. $\Pr(A \cup B)$

$$\Pr(A \cup B) = \Pr(A) + P(B) - \Pr(A \cap B) = 0.5 + 0.3 - 0.1 = 0.7$$

2. $\Pr(A \cap B^c)$

$$\Pr(A \cap B^c) = \Pr(A) - \Pr(A \cap B) = 0.5 - 0.1 = 0.4$$

3. $\Pr[(A \cap B^c) \cup (B \cap A^c)]$

$$\Pr[(A \cap B^c) \cup (B \cap A^c)] = 0.6$$

4 Silly Campaigns

A political campaign in New Haven decides to conduct an ”experiment” on the effectiveness of door knocking. They neglect to ensure teams knock on different doors. Team 1 reports knocking on 70% of doors, and Team 2 on 40%. Every house was contacted, some by both teams. How many houses had both teams knock?

Solution: Let A be the event Team 1 knocks and B the event Team 2 knocks. We know $\Pr(A \cup B) = 1$ and $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$. Thus, $\Pr(A \cap B) = 0.1$. Since there are 120 houses, $0.1 \times 120 = 12$ houses had both teams knock.

5 Rolling the Dice

We roll two fair 6-sided dice. Each of the 36 outcomes is equally likely.

1. Find the probability that doubles are rolled.

$$\Pr(\text{doubles}) = \frac{6}{36} = \frac{1}{6}$$

2. Given a sum of 4 or less, find the conditional probability of doubles.

$$\Pr(\text{doubles}|\text{sum} \leq 4) = \frac{2}{6} = \frac{1}{3}$$

3. Find the probability that at least one die roll is a 6.

$$\Pr(\text{at least one } 6) = \frac{11}{36}$$

4. Given different numbers on the dice, find the conditional probability that one roll is a 6.

$$\Pr(\text{at least one } 6|\text{different numbers}) = \frac{1}{3}$$

6 A Two-Envelope Puzzle

The release of two out of three prisoners has been announced, but the identity is secret. One prisoner considers asking the guard who the other released prisoner is, but fears knowing this will reduce his own chances of release from 2/3 to 1/2. What is wrong with this reasoning?

Solution: The reasoning is incorrect because it does not account for all possible outcomes in the probabilistic model. The posterior probability of release does not change based on which prisoner is named.

Let A, B, and C be the prisoners, and suppose that all prisoners are equally likely to be released. The probability remains 2/3 for the asking prisoner regardless of the guard's response.

7 Survey Says

A survey has 52% respondents 50 or older and 48% respondents under 50. Within the survey, on a particular question, 9.5% of the 50-plus population agrees strongly while 1.7% of under 50 respondents agree strongly.

1. What is the probability that someone selected at random is 50 or older?

$$P(\text{50 or older}) = 0.52$$

2. The selected individual strongly agrees with the survey question. Now what is the likelihood that the person is 50 or older? Explain your reasoning and **SHOW ALL YOUR WORK**.

   Solution:

   Using Bayes' Theorem:

$$P(\text{50 or older — strongly agrees}) = \frac{P(\text{strongly agrees — 50 or older})P(\text{50 or older})}{P(\text{strongly agrees})}$$

   First, calculate the total probability that a randomly selected individual strongly agrees:

$$P(\text{strongly agrees}) = P(\text{strongly agrees — 50 or older})P(\text{50 or older}) + P(\text{strongly agrees — under 50})P$$

$$P(\text{strongly agrees}) = (0.095 \times 0.52) + (0.017 \times 0.48) = 0.0494 + 0.00816 = 0.05756$$

   Now apply Bayes' Theorem:

$$P(\text{50 or older — strongly agrees}) = \frac{0.095 \times 0.52}{0.05756} = \frac{0.0494}{0.05756} \approx 0.8584$$

   So, the probability that the individual is 50 or older given that they strongly agree is approximately 85.84%.

3. Are the two answers above the same or different? Explain.

   Solution:

   The two answers are different. The probability that someone is 50 or older (52%) is not the same as the probability that someone is 50 or older given that they strongly agree (85.84%). This is because strong agreement with the survey question is more likely among those 50 or older, as seen in the higher percentage (9.5%) of 50+ respondents strongly agreeing compared to the under-50 group (1.7%).

4. (For fun, no points) What is the survey question?

   Possible Answer: "Do you strongly agree with the statement: 'People over 50 have more life experience than those under 50'?"