Computational Math Camp

Problem Sets

Benjamin Soltoff

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Contents

Overview

Contains problem sets for the 2019 Computational Math Camp.

Chapter 1

Linear equations, notation, sets, and functions

1.1 Simplify expressions

Simplify the following expressions as much as possible:

a. $(-x^4y^2)^2$

1. Distribute exponents over products.

$$(-1)^2x^{(2\times4)}y^{(2\times2)}$$

2. Multiply 2 and 2 together.

$$(-1)^2x^{(2\times4)}y^4$$

3. Multiply 2 and 4 together.

$$(-1)^2x^8y^4$$

4. Evaluate $(-1)^2$.

$$x^8y^4$$

b. $9(3^0)$

1. Any nonzero number to the zero power is 1.

$$9(1)$$

2. Anything times 1 is the same value.

$$9$$

c. $(2a^2)(4a^4)$

1. Combine products of like terms.

$$2a^2 \times 4a^4 = 2 \times 4a^{(2+4)}$$

2. Evaluate $2+4$.

$$2 \times 4a^6$$

3. Multiply 2 and 4 together.

$$8a^6$$

d. $\dfrac{x^4}{x^3}$

1. For all exponents, $\dfrac{a^n}{a^m} = a^{(n-m)}$.

$$x^{(4-3)}$$

2. Evaluate $4-3$.

$$x$$

e. $(-2)^{7-4}$

1. Subtract 4 from 7.

$$(-2)^3$$

2. In order to evaluate 2^3 express 2^3 as 2×2^2.

$$-2 \times 2^2$$

3. Evaluate 2^2.

$$-2 \times 4$$

4. Multiply -2 and 4 together.

$$-8$$

f. $\left(\dfrac{1}{27b^3}\right)^{1/3}$

1. Separate component terms.

$$\frac{1}{27}^{1/3} \times \frac{1}{b^3}^{1/3}$$

2. Evaluate cube roots.

$$\frac{1}{3} \times \frac{1}{b}$$

3. Combine terms.

$$\frac{1}{3b}$$

g. $y^7y^6y^5y^4$

1. Combine products of like terms.

$$y^{(7+6+5+4)}$$

2. Evaluate $7 + 6 + 5 + 4$.

$$y^{22}$$

h. $\dfrac{2a/7b}{11b/5a}$

1. Write as a single fraction by multiplying the numerator by the reciprocal of the denominator.

$$\frac{2a}{7b} \times \frac{5a}{11b}$$

2. Product property of exponents: $x^a \times x^b = x^{(a+b)}$

$$\frac{5a \times 2a}{7b \times 11b} = \frac{5 \times 2a^{1+1}}{7 \times 11b^{1+1}}$$

3. Evaluate $1 + 1$.

$$\frac{5 \times 2a^2}{7 \times 11b^2}$$

4. Multiple scalars together.

$$\frac{10a^2}{77b^2}$$

i. $(z^2)^4$

1. Nested exponents rule: $(x^a)^b = x^{ab}$

$$z^{2\times 4}$$

2. Evaluate 2×4

$$z^8$$

1.2 Simplify a (more complex) expression

Simplify the following expression:

$$(a + b)^2 + (a - b)^2 + 2(a + b)(a - b) - 3a^2$$

1. Expand $(a + b)^2$ with FOIL.

$$a^2 + 2ab + b^2 + (a - b)^2 + 2(a + b)(a - b) - 3a^2$$

2. Expand $(a - b)^2$ with FOIL.

$$a^2 + 2ab + b^2 + a^2 - 2ab + b^2 + 2(a + b)(a - b) - 3a^2$$

3. Multiply $a + b$ and $a - b$ together using FOIL.

$$a^2 + 2ab + b^2 + a^2 - 2ab + b^2 + 2(a^2 - b^2) - 3a^2$$

4. Distribute 2 over $a^2 - b^2$.

$$a^2 + 2ab + b^2 + a^2 - 2ab + b^2 + 2a^2 - 2b^2 - 3a^2$$

5. Group like terms.

$$(a^2 + a^2 + 2a^2 - 3a^2) + (b^2 + b^2 - 2b^2) + (2ab - 2ab)$$

6. Combine like terms.

$$a^2 + (b^2 + b^2 - 2b^2) + (2ab - 2ab)$$

7. Look for the difference of two identical terms.

$$a^2$$

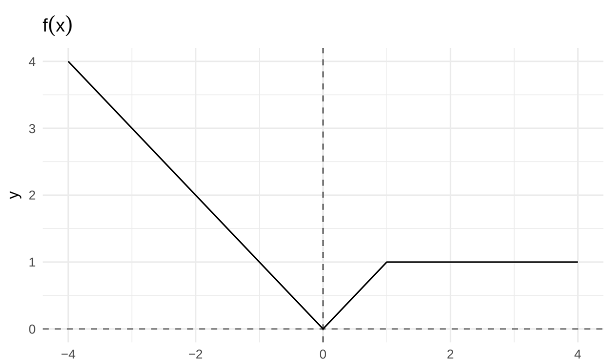
1.3 Graph sketching

Let the functions $f(x)$ and $g(x)$ be defined for all $x \in \Re$ by

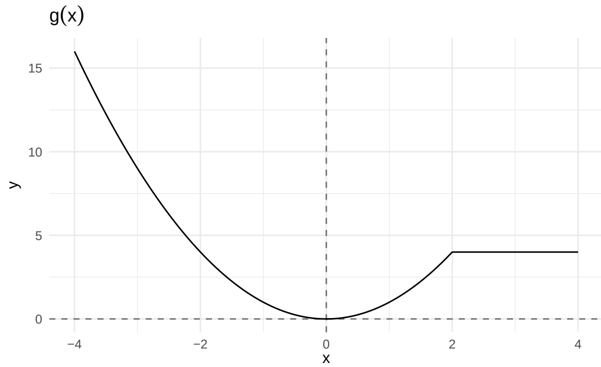
$$f(x) = \begin{cases} |x| & \text{if } x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}, \quad g(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 4 & \text{if } x \geq 2 \end{cases}$$

Sketch the graphs of:

1. $y = f(x)$



2. $y = g(x)$



3. $y = f(g(x))$

   To sketch the composite function, we first evaluate $g(x)$ for different values of x, and then evaluate $f(g(x))$ for different outputs of $g(x)$.

   - For $x \geq 2$, $g(x)$ is a constant value:

   $$\begin{aligned} x &\geq 2 \\ g(x) &= 4 \\ f(g(x)) &= f(4) = 1 \end{aligned}$$

   - For $x < 2$, $g(x)$ is not constant: $g(x) = x^2$. $f(x)$ evaluates differently depending on its input, so we have two cases based on the output of $g(x)$:

     – if $g(x) < 1$, $f(g(x)) = |g(x)| = |x^2| = x^2$. This is the case when:

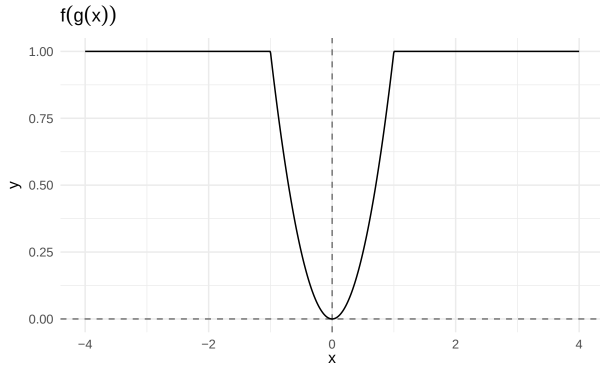
     $$\begin{aligned} g(x) &< 1 \\ x^2 &< 1 \text{ and } x < 2 \\ -1 &< x < 1 \end{aligned}$$

     – if $g(x) \geq 1$, $f(g(x)) = 1$. This is the case when:

     $$\begin{aligned} g(x) &\geq 1 \\ x^2 &\geq 1 \text{ and } x < 2 \\ x \leq -1 &\text{ or } 1 \leq x < 2 \end{aligned}$$

   - Therefore, $f(g(x))$ has the following values:

   $$f(g(x)) = \begin{cases} 1 & \text{if } x \leq -1 \\ x^2 & \text{if } -1 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$



4. $y = g(f(x))$

   To sketch the composite function, we first evaluate $f(x)$ for different values of x, and then evaluate $g(f(x))$ for different outputs of $f(x)$.

   - For $x \geq 1$, $f(x)$ is a constant value:

   $$\begin{aligned} x &\geq 1 \\ f(x) &= 1 \\ g(f(x)) = f(1) &= 1^2 = 1 \end{aligned}$$

   - For $x < 1$, $f(x)$ is not constant: $f(x) = |x|$. $g(x)$ evaluates differently depending on its input, so we have two cases based on the output of $f(x)$:
     - if $f(x) < 2$, $g(f(x)) = f(x)^2 = |x|^2 = x^2$. This is the case when:

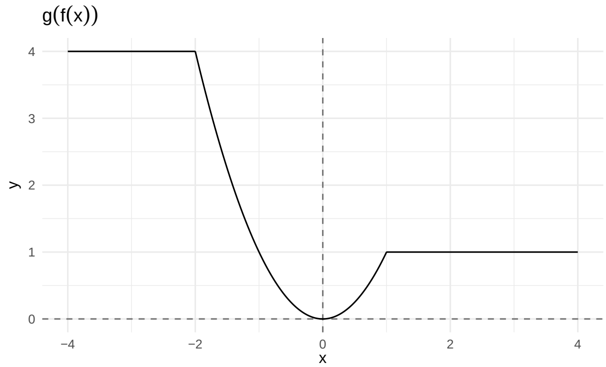
     $$\begin{aligned} f(x) &< 2 \\ |x| < 2 \text{ and } x &< 1 \\ -2 < x &< 1 \end{aligned}$$

     - if $f(x) \geq 2$, $g(f(x)) = 4$. This is the case when:

     $$\begin{aligned} f(x) &\geq 2 \\ |x| \geq 2 \text{ and } x &< 1 \\ x &\leq -2 \end{aligned}$$

   - Therefore, $g(f(x))$ has the following values:

   $$g(f(x)) = \begin{cases} 4 & \text{if } x \leq -2 \\ x^2 & \text{if } -2 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$



1.4 Root finding

Find the roots (solutions) to the following quadratic equations.

Definition 1.1 (The quadratic formula)**.**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

a. $4x^2 - 1 = 17$

- Move terms so that x is alone on the left side of the equation.

$$\begin{aligned} 4x^2 - 1 &= 17 \\ 4x^2 &= 18 \\ x^2 &= \frac{18}{4} \\ x^2 &= \frac{9}{2} \\ x &= \pm\sqrt{\frac{9}{2}} \end{aligned}$$

b. $9x^2 - 3x - 12 = 0$

- Factor the left-hand side.

$$3(x+1)(3x-4) = 0$$

- Divide both sides by 3 to simplify the equation.

$$(x+1)(3x-4) = 0$$

- Find the roots of each term in the product separately by solving for x.

$$\begin{aligned} x+1 &= 0 \\ x &= -1 \end{aligned} \qquad \begin{aligned} 3x &= 4 \\ x &= \frac{4}{3} \end{aligned}$$

c. $x^2 - 2x - 16 = 0$

1. Complete the square

$$\begin{aligned} x^2 - 2x - 16 &= 0 \\ x^2 - 2x &= 16 \\ x^2 - 2x + 1 &= 17 \\ (x-1)^2 &= 17 \\ x - 1 &= \pm\sqrt{17} \\ x &= 1 \pm \sqrt{17} \end{aligned}$$

2. Quadratic formula

   - Using the quadratic formula, solve for x

$$\begin{aligned} x &= \frac{-(-2) \pm \sqrt{(-2)^2 - (4 \times 1 \times 16)}}{2 \times 1} \\ x &= \frac{2 \pm \sqrt{4+64}}{2} \\ x &= \frac{2 \pm \sqrt{68}}{2} \end{aligned}$$

   - Simplify the radical

$$\begin{aligned} x &= \frac{2 \pm \sqrt{2^2 \times 17}}{2} \\ x &= \frac{2 \pm 2\sqrt{17}}{2} \end{aligned}$$

   - Factor the greatest common divisor

$$x = 1 \pm \sqrt{17}$$

d. $6x^2 - 6x - 6 = 0$

- Divide both sides by 6 to simplify the equation.

$$x^2 - x - 1 = 0$$

- Using the quadratic formula, solve for x

$$\begin{aligned}
x &= \frac{-(-1) \pm \sqrt{(-1)^2 - (4 \times 1 \times -1)}}{2 \times 1} \\
x &= \frac{1 \pm \sqrt{1 - 4(-1)}}{2} \\
x &= \frac{1 \pm \sqrt{1 + 4}}{2} \\
x &= \frac{1 \pm \sqrt{5}}{2}
\end{aligned}$$

e. $5 + 11x = -3x^2$

- Move everything to the left hand side.

$$3x^2 + 11x + 5 = 0$$

- Using the quadratic formula, solve for x

$$\begin{aligned}
x &= \frac{-11 \pm \sqrt{(11)^2 - (4 \times 3 \times 5)}}{2 \times 3} \\
x &= \frac{-11 \pm \sqrt{121 - 60}}{6} \\
x &= \frac{-11 \pm \sqrt{61}}{6}
\end{aligned}$$

1.5 Work with sets

Using the sets

$$\begin{aligned}
A &= \{2, 3, 7, 9, 13\} \\
B &= \{x : 4 \leq x \leq 8 \text{ and } x \text{ is an integer}\} \\
C &= \{x : 2 < x < 25 \text{ and } x \text{ is prime}\} \\
D &= \{1, 4, 9, 16, 25, \ldots\}
\end{aligned}$$

identify the following:

1. $A \cup B$

   $E = \{2, 3, 4, 5, 6, 7, 8, 9, 13\}$, combine all integers between 4 and 8 inclusive with the numbers in set A.

2. $(A \cup B) \cap C$

   $F = \{3, 5, 7, 13\}$, Since C is only prime numbers greater than 2 and less than 25, we take all the prime numbers that are also included in E, but remember to drop out 2 since it is not included in C.

3. $C \cap D$

   $G = \emptyset$, there are no prime numbers in D, so nothing is shared between C and D.

Chapter 2

Logarithms, sequences, and limits

2.1 Simplify logarithms

Express each of the following as a single logarithm:

a. $\log(x) + \log(y) - \log(z)$

- Multiplication rule of logarithms: $\log(x \times y) = \log(x) + \log(y)$
- Division rule of logarithms: $\log(\frac{x}{y}) = \log(x) - \log(y)$
- Applying the log rules, we combine logs that are added through multiplication and then combine logs that are subtracted with division.

$$\log(x) + \log(y) - \log(z)$$

$$\log(xy) - \log(z)$$

$$\log(\frac{xy}{z})$$

b. $2\log(x) + 1$

- Exponentiation rule of logarithms: $\log(x^y) = y\log(x)$
- $\log(e) = 1$

$$2\log(x) + 1$$

$$2\log(x) + \log(e)$$

$$\log(x^2) + \log(e)$$

$$\log(ex^2)$$

c. $\log(x) - 2$

- $\log(e) = 1$

$$\log(x) - 2$$

$$\log(x) - 2\log(e)$$

$$\log(x) - \log(e^2)$$

$$\log(\frac{x}{e^2})$$

2.2 Sequences

Write down the first three terms of each of the following sequences. In each case, state whether the sequence is an arithmetic progression, a geometric progression, or neither.

a. $u_n = 4 + 3n$

$$7, 10, 13$$

Arithmetic progression.

b. $u_n = 5 - 6n$

$$-1, -7, -13$$

Arithmetic progression.

c. $u_n = 4^n$

$$4, 16, 64$$

Geometric progression.

d. $u_n = 5 \times (-2)^n$

$$-10, 20, -40$$

Geometric progression.

e. $u_n = n \times 3^n$

$$3, 18, 81$$

Neither.

2.3 Find the limit

In each of the following cases, state whether the sequence $\{u_n\}$ tends to a limit, and find the limit if it exists:

a. $u_n = 1 + \frac{1}{2}n$

No limit ($u_n \to \infty$)

b. $u_n = 1 - \frac{1}{2}n$

No limit ($u_n \to \infty$)

c. $u_n = \left(\frac{1}{2}\right)^n$

Yes. $\lim_{x \to \infty} u_n = 0$

d. $u_n = \left(-\frac{1}{2}\right)^n$

Yes. $\lim_{x \to \infty} u_n = 0$

2.4 Determine convergence or divergence

Determine whether each of the following sequences converges or diverges. If it converges, find the limit.

a. $a_n = \frac{3+5n^2}{n+n^2}$

The sequence converges to 5. We can see this by factoring n^2 from both the numerator and denominator and then cancelling it out.

$$\lim_{n\to\infty} a_n = \frac{3+5n^2}{n+n^2} = \lim_{n\to\infty} \frac{n^2\left(\frac{3}{n^2}+5\right)}{n^2\left(\frac{1}{n}+1\right)} = \lim_{n\to\infty} \frac{\left(\frac{3}{n^2}+5\right)}{\left(\frac{1}{n}+1\right)} = \frac{\lim_{n\to\infty} \frac{3}{n^2}+5}{\lim_{n\to\infty} \frac{1}{n}+1} = \frac{0+5}{0+1} = 5$$

(This is slightly curt: Make sure you know how to show that the limit of $\frac{3}{n^2}$ approaches 0.) As $n \to \infty$, $\frac{3}{n} \to 0$ and $\frac{1}{n} \to 0$. Therefore, $a_n \to 5$.

Alternatively, you could split the fraction into two terms: one with a numerator of 3, and the other with a numerator of $5n^2$. The first fraction converges to 0. (Can you show that?) Factoring out an n from both sides of the second fraction, you're left with $\frac{5n}{n+1}$; $\frac{n}{n+1}$ converges to 1, giving you $5 \times 1 = 5$.

b. $a_n = \frac{(-1)^{n-1}n}{n^2+1}$

The sequence converges to 0. To see why, take the absolute value of the sequence, then factor out and cancel n from both sides of the fraction.

$$\lim_{n\to\infty} \left|\frac{(-1)^{n-1}n}{n^2+1}\right| = \lim_{n\to\infty} \frac{1^{n-1}n}{n^2+1} = \lim_{n\to\infty} \frac{1}{n+\frac{1}{n}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty}(n+\frac{1}{n})} = \frac{1}{\lim_{n\to\infty} n+0} = 0$$

2.5 Find more limits

Given that

$$\lim_{x\to a} f(x) = -3, \quad \lim_{x\to a} g(x) = 0, \quad \lim_{x\to a} h(x) = 8$$

find the limits that exist. If the limit doesn't exist, explain why.

a. $\lim_{x\to a}[f(x) + h(x)] = -3 + 5 = 8$
b. $\lim_{x\to a}[f(x)]^2 = (-3)^2 = 9$
c. $\lim_{x\to a} \sqrt[3]{h(x)} = \sqrt[3]{8} = 2$
d. $\lim_{x\to a} \frac{1}{f(x)} = -\frac{1}{3}$
e. $\lim_{x\to a} \frac{f(x)}{h(x)} = -\frac{3}{8}$
f. $\lim_{x\to a} \frac{g(x)}{f(x)} = \frac{0}{-3} = 0$
g. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{-3}{0}$ = Undefined – cannot divide by 0, no limit

h. $\lim_{x \to a} \frac{2f(x)}{h(x)-f(x)} = \frac{2\times -3}{8-(-3)} = -\frac{6}{11}$

2.6 Find even more limits

Find the limits of the following:

a. $\lim_{x \to -4} \frac{x^2+5x+4}{x^2+3x-4}$

$$\lim_{n \to -4} \frac{x^2+5x+4}{x^2+3x-4} = \lim_{n \to -4} \frac{(x+4)(x+1)}{(x+4)(x-1)} = \lim_{n \to -4} \frac{x+1}{x-1} = \frac{\lim_{n \to -4}(x+1)}{\lim_{n \to -4}(x-1)} = \frac{-3}{-5} = \frac{3}{5}$$

b. $\lim_{x \to 4^-} \sqrt{16-x^2}$

$$\begin{aligned}
\lim_{n \to 4^-} \sqrt{16-x^2} &= \lim_{n \to 4^-} \sqrt{(4+x)(4-x)} \\
&= \lim_{n \to 4^-} \sqrt{4+x}\sqrt{4-x} \\
&= \lim_{n \to 4^-} \sqrt{4+x} \cdot \lim_{n \to 4^-} \sqrt{4-x} \\
&= \sqrt{8} \cdot \sqrt{0} \\
&= 0
\end{aligned}$$

c. $\lim_{x \to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x}$

$$\begin{aligned}
\lim_{n \to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x} &= \lim_{n \to -4} \frac{\frac{x+4}{4x}}{4+x} \\
&= \lim_{n \to -4} \frac{4+x}{4x} \cdot \frac{1}{4+x} \\
&= \lim_{n \to -4} \frac{1}{4x} \\
&= \frac{1}{4(-4)} \\
&= -\frac{1}{16}
\end{aligned}$$

2.7 Check for discontinuities

Which of the following functions are continuous? If not, where are the discontinuities?

a. $f(x) = \dfrac{9x^3 - x}{(x-1)(x+1)}$

- Discontinuous at $x = -1, +1$ (denominator would be 0, leaving the fraction undefined)

b. $f(x) = e^{-x^2}$

- Continuous for all real numbers.

c. $f(y) = y^3 - y^2 + 1$

- All polynomials are continuous.

d. $f(x) = \begin{cases} x^3 + 1, & x > 0 \\ \frac{1}{2} & x = 0 \\ -x^2, & x < 0 \end{cases}$

- Discontinuous at $x = 0$. This is a piecewise function. To be continuous $\lim_{x \to 0^+} f(x) = 0$. However in this function, $\lim_{x \to 0^+} f(x) = 1 \neq 0$.

Chapter 3

Differentiation

3.1 Find finite limits

Find the following finite limits:

a. $\lim_{x \to 4} x^2 - 6x + 4$

$$\begin{aligned} \lim_{x \to 4} x^2 - 6x + 4 &= 4^2 - 6(4) + 4 \\ &= 16 - 24 + 4 \\ &= -4 \end{aligned}$$

b. $\lim_{x \to 0} \left[\frac{x - 25}{x + 5} \right]$

$$\begin{aligned} \lim_{x \to 0} \left[\frac{x - 25}{x + 5} \right] &= \frac{0 - 25}{0 + 5} \\ &= \frac{-25}{5} \\ &= -5 \end{aligned}$$

c. $\lim_{x \to 4} \left[\frac{x^2}{3x - 2} \right]$

$$\begin{aligned}\lim_{x\to 4}\left[\frac{x^2}{3x-2}\right] &= \frac{4^2}{3(4)-2}\\ &= \frac{16}{12-2}\\ &= \frac{16}{10}\\ &= \frac{8}{5}\end{aligned}$$

d. $\lim_{x\to 1}\left[\frac{x^4-1}{x-1}\right]$

The key here is to factor the initial expression in the numerator, then cancel terms out with the denominator:

$$\begin{aligned}\lim_{x\to 1}\left[\frac{x^4-1}{x-1}\right] &= \lim_{x\to 1}\left[\frac{(x-1)(x+1)(x^2+1)}{x-1}\right]\\ &= \lim_{x\to 1}[(x+1)(x^2+1)]\\ &= (1+1)(1^2+1)\\ &= (2)(2)\\ &= 4\end{aligned}$$

Alternatively, we can use L'Hôpital's Rule:

$$\begin{aligned}\lim_{x\to 1}\left[\frac{x^4-1}{x-1}\right] &= \lim_{x\to 1}\left[\frac{4x^3}{1}\right]\\ &= \frac{4(1)^3}{1}\\ &= 4\end{aligned}$$

e. $\lim_{x\to -4}\left[\frac{x^2+5x+4}{x^2+3x-4}\right]$

The key here is to factor the initial expression:

$$\begin{aligned}
\lim_{x\to-4} \frac{(x+4)(x+1)}{(x+4)(x-1)} &= \lim_{x\to-4} \frac{x+1}{x-1} \\
&= \frac{\lim\limits_{x\to-4}(x+1)}{\lim\limits_{x\to-4}(x-1)} \\
&= \frac{-3}{-5} \\
&= \frac{3}{5}
\end{aligned}$$

f. $\lim\limits_{x\to4^-} \sqrt{16-x^2}$

$$\begin{aligned}
\lim_{x\to4^-} \sqrt{16-x^2} &= \lim_{x\to4^-} \sqrt{(4+x)(4-x)} \\
&= \lim_{x\to4^-} \sqrt{4+x}\sqrt{4-x} \\
&= \lim_{x\to4^-} \sqrt{4+x} \cdot \lim_{x\to4^-} \sqrt{4-x} \\
&= \sqrt{8} * \sqrt{0} \\
&= 0
\end{aligned}$$

A critical aspect of this limit, which allows for it to exist, is that it is a left-hand limit.

g. $\lim\limits_{x\to-1} \left[\frac{x-2}{x^2+4x-3}\right]$

$$\begin{aligned}
\lim_{x\to-1} \frac{x-2}{x^2+4x-3} &= \frac{\lim\limits_{x\to-1}(x-2)}{\lim\limits_{x\to-1}(x^2+4x-3)} \\
&= \frac{-1-2}{(-1)^2+4(-1)-3} \\
&= \frac{-3}{-6} \\
&= \frac{1}{2}
\end{aligned}$$

h. $\lim\limits_{x\to-4} \left[\frac{\frac{1}{4}+\frac{1}{x}}{4+x}\right]$

$$\begin{aligned}
\lim_{x\to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x} &= \lim_{x\to -4} \frac{\frac{x+4}{4x}}{4+x} \\
&= \lim_{x\to -4} \frac{4+x}{4x}\frac{1}{4+x} \\
&= \lim_{x\to -4} \frac{1}{4x} \\
&= \frac{1}{4(-4)} \\
&= -\frac{1}{16}
\end{aligned}$$

Alternatively, we can use L'Hôpital's Rule:

$$\begin{aligned}
\lim_{x\to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x} &= \lim_{x\to -4} \frac{-\frac{1}{x^2}}{1} \\
&= \lim_{x\to -4} (-\frac{1}{x^2}) \\
&= -\frac{1}{16}
\end{aligned}$$

3.2 Find infinite limits

Find the following infinite limits:

Hint: use **L'Hôpital's Rule** to switch from $\lim_{x\to\infty}\left(\frac{f(x)}{g(x)}\right)$ to $\lim_{x\to\infty}\left(\frac{f'(x)}{g'(x)}\right)$.

a. $\lim_{x\to\infty}\left[\frac{9x^2}{x^2+3}\right]$

$$\begin{aligned}
\lim_{x\to\infty}\left[\frac{9x^2}{x^2+3}\right] &= \lim_{x\to\infty}\left[\frac{18x}{2x}\right] \\
&= 9
\end{aligned}$$

b. $\lim_{x\to\infty}\left[\frac{3x-4}{x+3}\right]$

$$\begin{aligned}
\lim_{x\to\infty}\left[\frac{3x-4}{x+3}\right] &= \lim_{x\to\infty}\left[\frac{3}{1}\right] \\
&= 3
\end{aligned}$$

c. $\lim_{x\to\infty}\left[\frac{2^x-3}{2^x+1}\right]$

Remember that $\frac{d}{dx}n^x = \log(n)n^x$:

$$\begin{aligned}\lim_{x\to\infty}\left[\frac{2^x-3}{2^x+1}\right] &= \lim_{x\to\infty}\left[\frac{\log(2)2^x}{\log(2)2^x}\right]\\ &= 1\end{aligned}$$

d. $\lim_{x\to\infty}\left[\frac{\log(x)}{x}\right]$

$$\begin{aligned}\lim_{x\to\infty}\left[\frac{\log(x)}{x}\right] &= \lim_{x\to\infty}\left[\frac{\frac{1}{x}}{1}\right]\\ &= \lim_{x\to\infty}\left[\frac{1}{x}\right]\\ &= \frac{1}{\infty}\\ &= 0\end{aligned}$$

e. $\lim_{x\to\infty}\left[\frac{3^x}{x^3}\right]$

The trick here is to repeatedly calculate the derivative of the numerator and denominators until there is no x term on the denominator. You end up calculating the third derivative, but L'Hôpital's Rule still applies.

$$\begin{aligned}\lim_{x\to\infty}\left[\frac{3^x}{x^3}\right] &= \lim_{x\to\infty}\left[\frac{\log(3)3^x}{3x^2}\right]\\ &= \lim_{x\to\infty}\left[\frac{\log^2(3)3^x}{6x}\right]\\ &= \lim_{x\to\infty}\left[\frac{\log^3(3)3^x}{6}\right]\\ &= \frac{\log^3(3)3^\infty}{6}\\ &= \infty\end{aligned}$$

f. $\lim_{y\to\infty}\left[\frac{3e^y}{y^3}\right]$

Same as above: repeatedly calculate the derivatives until the y term disappears in the denominator.

$$\begin{aligned}
\lim_{x\to\infty}\left[\frac{3e^y}{y^3}\right] &= \lim_{x\to\infty}\left[\frac{3e^y}{3y^2}\right] \\
&= \lim_{x\to\infty}\left[\frac{3e^y}{6y}\right] \\
&= \lim_{x\to\infty}\left[\frac{3e^y}{6}\right] \\
&= \frac{3e^\infty}{6} \\
&= \infty
\end{aligned}$$

3.3 Assessing continuity and differentiability

For each of the following functions, describe whether it is continuous and/or differentiable at the point of transition of its two formulas.

a.

$$f(x) = \begin{cases} +x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$

Solution:

$$f'(x) = \begin{cases} 2x, & x \geq 0 \\ -2x, & x < 0 \end{cases}$$

As x converges to 0 from both above and below, $f'(0)$ converges to 0, so the function is continuous and differentiable.

b.

$$f(x) = \begin{cases} +x^2 + 1, & x \geq 0 \\ -x^2 - 1, & x < 0 \end{cases}$$

Solution: This function is not continuous (and thus not differentiable). As x converges to 0 from above, $f(x)$ tends to 1, whereas x tends to 0 from below, $f(x)$ converges to -1.

c.

$$f(x) = \begin{cases} x^3, & x \leq 1 \\ x, & x > 1 \end{cases}$$

Solution: This function is continuous, since $\lim_{x\to1} f(x) = 1$ no matter how the limit is taken. However it is not differentiable since

$$f'(x) = \begin{cases} 3x^2, & x \leq 1 \\ 1, & x > 1 \end{cases}$$

$\lim_{x \to 1^+} f'(x) = 1$, whereas $\lim_{x \to 1^-} f'(x) = 3$. The function is not smooth and continuous at $f(1)$.

d.

$$f(x) = \begin{cases} x^3, & x < 1 \\ 3x - 2, & x \geq 1 \end{cases}$$

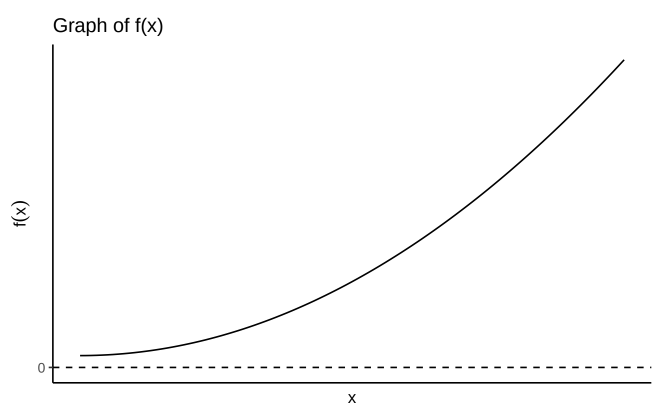
Solution: This function is continuous since $f(1)$ tends to 1 from either direction. Likewise, this function is continuous because

$$f'(x) = \begin{cases} 3x^2, & x \leq 1 \\ 3, & x > 1 \end{cases}$$

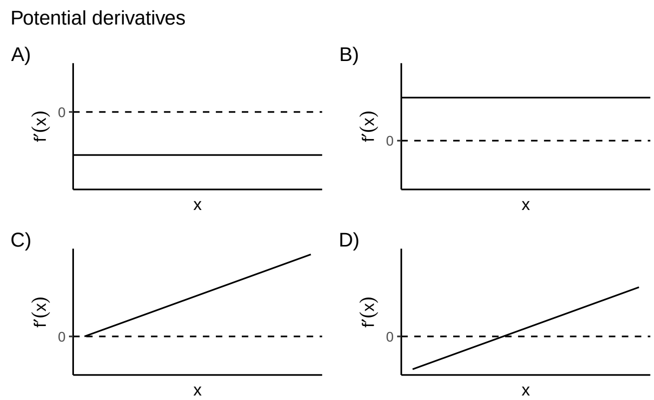
and $\lim_{x \to 1} f'(x) = 3$ from either direction.

3.4 Possible derivative

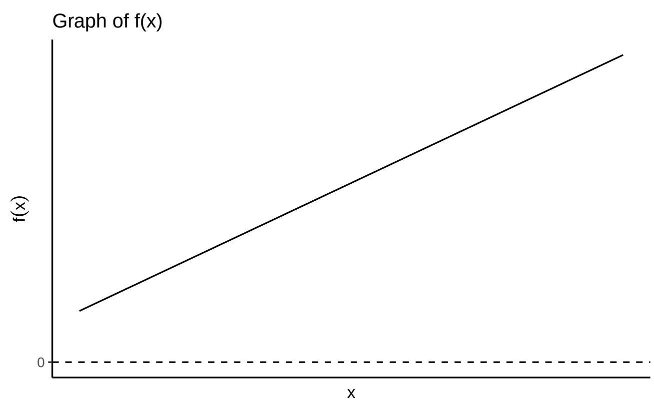
A friend shows you this graph of a function $f(x)$:



Which of the following could be a graph of $f'(x)$? For each graph, explain why or why not it might be the derivative of $f(x)$.



What if the figure below was the graph of $f(x)$? Which of the graphs might potentially be the derivative of $f(x)$ then?



Solution:

a. A doesn't work because it is negative and the function we observe is increasing in x. B is constant so this also won't work, the function we observe gets larger at an increasing, not constant rate. C seems to be a plausible candidate because an upward sloping derivative would map to the behavior of the function we observe, that $g(x)$ gets large at an increasing rate. D does not work because it suggests the function would need to be decreasing over some interval and because, when we refer back to $g(x)$, there doesn't seem to be any local minimum, maximum or a saddle point despite the graph in D crossing 0.

b. Again, A doesn't work because it is negative and the function we observe is increasing in x. B seems to be plausible as the derivative, since $g(x)$ appears to increase at a constant rate, its derivative should be flat and greater than 0. C won't work because the slope of $g(x)$ is constant and does not increase in x. D doesn't work, again because it suggests the function would need to be decreasing at some point over the interval we observe.

3.5 Calculate derivatives

Differentiate the following functions:

a. $f(x) = 4x^3 + 2x^2 + 5x + 11$

Solution: Power rule.

$$\begin{aligned} f(x) &= 4x^3 + 2x^2 + 5x + 11 \\ f'(x) &= 12x^2 + 4x + 5 \end{aligned}$$

b. $y = \sqrt{30}$

Solution: Derivative of a constant is 0.

$$\begin{aligned} y &= \sqrt{30} \\ y' &= 0 \end{aligned}$$

c. $h(t) = \log(9t + 1)$

Solution: Derivative of $\log(u)$ is $\frac{1}{u}$. Since u is a function in this problem, need to apply the chain rule to calculate the derivative of $9t + 1$ and multiply that by $\frac{1}{9t+1}$

$$\begin{aligned} h(t) &= \log(9t + 1) \\ h'(t) &= \frac{1}{9t + 1} * 9 \end{aligned}$$

d. $f(x) = \log(x^2 e^x)$

Solution: Derivative of a logarithm plus the chain rule.

$$\begin{aligned} f(x) &= \log(x^2 e^x) \\ f'(x) &= \frac{1}{x^2 e^x} * (2xe^x + e^x x^2) \\ &= \frac{2xe^x + e^x x^2}{x^2 e^x} \\ &= \frac{2}{x} + 1 \end{aligned}$$

e. $h(y) = \left(\frac{1}{y^2} - \frac{3}{y^4}\right)(y + 5y^3)$

Solution: Simplify the expression first, then basic application of power rule.

$$\begin{aligned}
h(y) &= \left(\frac{1}{y^2} - \frac{3}{y^4}\right)\left(y + 5y^3\right) \\
&= \frac{y}{y^2} + \frac{5y^3}{y^2} - \frac{3y}{y^4} - \frac{15y^3}{y^4} \\
&= \frac{1}{y} + 5y - \frac{3}{y^3} - \frac{15}{y} \\
&= 5y - \frac{14}{y} - \frac{3}{y^3} \\
h'(y) &= 5 + \frac{14}{y^2} + \frac{9}{y^4}
\end{aligned}$$

f. $g(t) = \frac{3t-1}{2t+1}$

Solution: Quotient rule.

$$\begin{aligned}
g(t) &= \frac{3t-1}{2t+1} \\
g'(t) &= \frac{(3)(2t+1) - (3t-1)(2)}{(2t+1)^2} \\
&= \frac{5}{(2t+1)^2}
\end{aligned}$$

3.6 Use the product and quotient rules

Differentiate the following using both the product and quotient rules:

$$f(x) = \frac{x^2 - 2x}{x^4 + 6}$$

Solution:

a. First let's use the quotient rule:

$$\begin{aligned}
h(x) &= \frac{f(x)}{g(x)} \\
f(x) &= x^2 - 2x \\
g(x) &= x^4 + 6 \\
f'(x) &= 2x - 2 \\
g'(x) &= 4x^3 \\
h'(x) &= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \\
&= \frac{(2x-2)(x^4+6) - (x^2-2x)(4x^3)}{(x^4+6)^2} \\
&= \frac{2x^5 + 12x - 2x^4 - 12 - 4x^5 + 8x^4}{(x^4+6)^2} \\
&= \frac{-2x^5 + 6x^4 + 12x - 12}{(x^4+6)^2}
\end{aligned}$$

b. Now we can do the same thing with the product rule:

$$\begin{aligned}
j(x) &= k(x)m(x) \\
k(x) &= x^2 - 2x \\
m(x) &= (x^4+6)^{-1} \\
k'(x) &= 2x - 2 \\
m'(x) &= -(x^4+6)^{-2}(4x^3) = -\frac{4x^3}{(x^4+6)^2} \\
j'(x) &= k(x)m'(x) + k'(x)m(x) \\
&= (x^2-2x)(-\frac{4x^3}{(x^4+6)^2}) + (2x-2)(x^4+6)^{-1} \\
&= -\frac{(x^2-2x)(4x^3)}{(x^4+6)^2} + \frac{2x-2}{x^4+6} \\
&= -\frac{4x^5 - 8x^4}{x^4+6)^2} + \frac{2x-2}{x^4+6} \\
&= -\frac{4x^5 - 8x^4}{x^4+6)^2} + \frac{2x-2}{x^4+6}\frac{x^4+6}{x^4+6} \\
&= -\frac{4x^5 - 8x^4}{(x^4+6)^2} + \frac{2x^5 + 12x - 2x^4 - 12}{(x^4+6)^2} \\
&= \frac{2x^5 + 12x - 2x^4 - 12 - 4x^5 - 8x^4}{(x^4+6)^2} \\
&= \frac{-2x^5 + 6x^4 + 12x - 12}{(x^4+6)^2}
\end{aligned}$$

The quotient rule is simply a derivation of the product rule combined with the chain rule:

$$\begin{aligned} h(x) &= \frac{f(x)}{g(x)} \\ &= f(x)g(x)^{-1} \end{aligned}$$

Apply product and chain rules:

$$\begin{aligned} h'(x) &= f'(x)g(x)^{-1} + f(x)(-1)g(x)^{-2}g'(x) \\ &= f'(x)g(x)g(x)^{-2} - f(x)g(x)^{-2}g'(x) \\ &= [f'(x)g(x) - f(x)g'(x)]g(x)^{-2} \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \end{aligned}$$

which is the quotient rule.

3.7 Logarithms and exponential functions

Compute the derivative of each of the following functions:

a. $f(x) = xe^{3x}$

Solution: Use the product rule to split the function into component functions.

$$g(x) = x, \quad h(x)e^{3x}$$

Use the chain rule to solve $h'(x)$.

$$\begin{aligned} g(x) &= x & h(x) &= e^{3x} \\ g'(x) &= 1 & h'(x) &= 3e^{3x} \end{aligned}$$

$$\begin{aligned} f(x) &= g'(x)h(x) + g(x)h'(x) \\ &= 1(e^{3x}) + x(3e^{3x}) \\ &= e^{3x} + 3xe^{3x} \\ &= e^{3x}(3x+1) \end{aligned}$$

b. $f(x) = \frac{x}{e^x}$

Solution: Use the product rule.

$$g(x) = x, \quad h(x) = \frac{1}{e^x}$$

Use the chain rule to solve $h'(x)$.

$$\begin{aligned} g(x) &= x & h(x) &= \frac{1}{e^x} \\ g'(x) &= 1 & h'(x) &= -e^{-x} \end{aligned}$$

$$\begin{aligned} f(x) &= g'(x)h(x) + g(x)h'(x) \\ &= 1(\frac{1}{e^x}) + x(-e^{-x}) \\ &= \frac{1}{e^x} - xe^{-x} \\ &= \frac{1}{e^x} - \frac{x}{e^x} \\ &= \frac{1-x}{e^x} \end{aligned}$$

c. $h(x) = \frac{x}{\log(x)}$

Solution: Use the quotient rule.

$$g(x) = x, \quad h(x) = \log(x)$$

$$\begin{aligned} f(x) &= x & g(x) &= \log(x) \\ f'(x) &= 1 & g'(x) &= \frac{1}{x} \end{aligned}$$

$$\begin{aligned} \left[\frac{f(x)}{g(x)}\right]' &= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \\ &= \frac{1(\log(x)) - x(\frac{1}{x})}{[\log(x)]^2} \\ &= \frac{\log(x) - 1}{[\log(x)]^2} \end{aligned}$$

3.8 Composite functions

For each of the following pairs of functions $g(x)$ and $h(z)$, write out the composite function $g(h[z])$ and $h(g[x])$. In each case, describe the domain of the composite function.

a. $g(x) = x^2 + 4, \quad h(z) = 5z - 1$

Solution:

$$\begin{aligned} g(h[z]) &= (5z-1)^2 + 4 \\ h(g[x]) &= 5(x^2+4) - 1 \\ &= 5x^2 + 20 - 1 \\ &= 5x^2 + 19 \end{aligned}$$

- Domain of $g(h[z])$ $x \in \Re$
- Domain of $h(g[x])$ $x \in \Re$

b. $g(x) = x^3, \quad h(z) = (z-1)(z+1)$

Solution:

$$\begin{aligned} g(h[z]) &= [(z-1)(z+1)]^3 \\ &= (z-1)^3(z+1)^3 \\ h(g[x]) &= (x^3-1)(x^3+1) \end{aligned}$$

- Domain of $g(h[z])$ $x \in \Re$
- Domain of $h(g[x])$ $x \in \Re$

c. $g(x) = 4x + 2, \quad h(z) = \frac{1}{4}(z-2)$

Solution:

$$\begin{aligned} g(h[z]) &= 4\left[\frac{1}{4}(z-2)\right] + 2 \\ &= (z-2) + 2 \\ &= z \\ h(g[x]) &= \frac{1}{4}([4x+2] - 2) \\ &= \frac{1}{4}(4x) \\ &= x \end{aligned}$$

- Domain of $g(h[z])$ $x \in \Re$
- Domain of $h(g[x])$ $x \in \Re$

d. $g(x) = \frac{1}{x}, \quad h(z) = z^2 + 1$

Solution:

$$\begin{aligned} g(h[z]) &= \frac{1}{z^2+1} \\ h(g[x]) &= \left(\frac{1}{x}\right)^2 + 1 \\ &= \frac{1}{x^2} + 1 \end{aligned}$$

- Domain of $g(h[z])$ $x \in \Re$
- Domain of $h(g[x])$ $x \in \Re : x \neq 0$

3.9 Chain rule

Use the chain rule to compute the derivative of the first three composite functions in the previous section from the derivatives of the two component functions. Then, compute each derivative directly using your expression for the composite function. Simplify and compare your answers.

a. $g(x) = x^2 + 4, \quad h(z) = 5z - 1$

Solution:

- Using component functions and the chain rule

$$g'(x) = 2x \quad h'(z) = 5$$

$$\begin{aligned} \frac{d}{dz}\{g(h[z])\} &= g'(h[z])h'(z) \\ &= 2(5z-1)(5) \\ &= 2(25z-5) \\ &= 50z-10 \\ \frac{d}{dx}\{h(g[x])\} &= h'(g[x])g'(x) \\ &= 5(2x) \\ &= 10x \end{aligned}$$

- Using the composite function

$$\begin{aligned}
g(h[z]) &= (5z-1)^2 + 4 \\
&= 25z^2 - 10z + 1 + 4 \\
&= 25z^2 - 10z + 5 \\
\frac{d}{dz} g(h[z]) &= 50z - 10 \\
h(g[x]) &= 5x^2 + 19 \\
\frac{d}{dx} h(g[x]) &= 10x
\end{aligned}$$

b. $g(x) = x^3, \quad h(z) = (z-1)(z+1)$

Solution:

- Using component functions and the chain rule

$$g'(x) = 3x^2 \quad h'(z) = 2z$$

$$\begin{aligned}
\frac{d}{dz}\{g(h[z])\} &= g'(h[z])h'(z) \\
&= 3[(z-1)(z+1)]^2(2z) \\
&= 3(z^2-1)^2(2z) \\
&= 6z(z^2-1)^2 \\
\frac{d}{dx}\{h(g[x])\} &= h'(g[x])g'(x) \\
&= 2(x^3)(3x^2) \\
&= 6x^5
\end{aligned}$$

- Using the composite function

$$\begin{aligned}
g(h[z]) &= (z-1)^3(z+1)^3 \\
&= (z-1)(z-1)(z-1)(z+1)(z+1)(z+1) \\
&= z^6 - 3z^4 + 3z^2 - 1 \\
\frac{d}{dz} g(h[z]) &= 6z^5 - 12z^3 + 6z \\
&= 6z(z^4 - 2z^2 + 1) \\
&= 6z(z^2-1)^2 \\
h(g[x]) &= (x^3-1)(x^3+1) \\
&= x^6 - 1 \\
\frac{d}{dx} h(g[x]) &= 6x^5
\end{aligned}$$

c. $g(x) = 4x + 2, \quad h(z) = \frac{1}{4}(z - 2)$

Solution:

- Using component functions and the chain rule

$$g'(x) = 4 \quad h'(z) = \frac{1}{4}$$

$$\begin{aligned}
\frac{d}{dz}\{g(h[z])\} &= g'(h[z])h'(z) \\
&= 4(\frac{1}{4}) \\
&= 1 \\
\frac{d}{dx}\{h(g[x])\} &= h'(g[x])g'(x) \\
&= \frac{1}{4}(4) \\
&= 1
\end{aligned}$$

- Using the composite function

$$\begin{aligned}
g(h[z]) &= z \\
\frac{d}{dz}g(h[z]) &= 1 \\
h(g[x]) &= x \\
\frac{d}{dx}h(g[x]) &= 1
\end{aligned}$$

Chapter 4

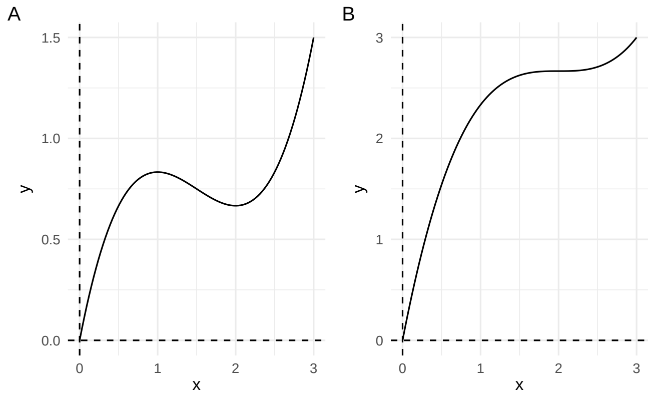
Critical points and approximation

4.1 Sketch a function

Sketch the graph of a function (any function you like, no need to specify a functional form) that is:

a. Continuous on $[0, 3]$ and has the following properties: an absolute minimum at 0, an absolute maximum at 3, a local maximum at 1 and a local minimum at 2.
b. Do the same for another function with the following properties: 2 is a **critical number** (i.e. $f'(x) = 0$ or $f'(x)$ is undefined), but there is no local minimum and no local maximum.

Solution: There are many, many (in fact, uncountably infinitely many) correct answers to this question, but they will all have a few characteristics in common. For the first function, the highest value of the function must be produced by $x = 0$, and the lowest value of the function must be produced by $x = 3$. Furthermore, the graph must change from moving up to moving down at $x = 2$ and from moving down to moving up at $x = 1$. For the second graph, there simply must be a saddle point at $x = 2$ - for $x = 2$ to be a critical point, it must be a local minimum, a local maximum, or a saddle point, but we've specified that there are no local minima and no local maxima - and the graph must not change from increasing to decreasing or vice versa at any point.



4.2 Find critical values

Find the critical values of these functions:

a. $f(x) = 5x^{3/2} - 4x$

Solution: First, find the derivative of the function.

$$f'(x) = \frac{3}{2}(5)x^{3/2-1} - 4 = \frac{15}{2}x^{1/2} - 4 = \frac{15\sqrt{x}}{2} - 4$$

If we set this derivative equal to 0 and solve, we get the following critical point:

$$\begin{aligned}
\frac{15\sqrt{x}}{2} - 4 &= 0 \\
\frac{15\sqrt{x}}{2} &= 4 \\
15\sqrt{x} &= 8 \\
\sqrt{x} &= \frac{8}{15} \\
x &= \left(\frac{8}{15}\right)^2 \\
x &= \frac{8^2}{15^2} \\
x &= \frac{64}{225}
\end{aligned}$$

b. $s(t) = 3t^4 + 4t^3 - 6t^2$

Solution: The derivative of $s(t)$ requires simple power rule:

$$s'(t) = 4(3)t^{4-1} + 3(4)t^{3-1} - 2(6)t^{2-1} = 12t^3 + 12t^2 - 12t = 12t(t^2 + t - 1)$$

If we set this equal to zero, we immediately see that $t = 0$ is a critical point. However, we cannot "eyeball" if/where $(t^2 + t - 1) = 0$. For that, we will need to use the quadratic formula. In case you don't remember, the quadratic formula helps us find the roots of a quadratic function – the points at which the function equals 0. Think of a generic quadratic equation, $f(x) = ax^2 + bx + c$, where a, b, c are the coefficients or constants to each term. Then,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In our case with $t^2 + t - 1$, $a = 1, b = 1, c = -1$. Let's plug these into the formula.

$$\begin{aligned} t &= \frac{-1 \pm \sqrt{1^2 - 4(1)(-1}}{2(1)} \\ &= \frac{-1 \pm \sqrt{1 + 4}}{2} \\ &= \frac{-1 \pm \sqrt{5}}{2} \\ &= \frac{-1 - \sqrt{5}}{2}, \frac{-1 + \sqrt{5}}{2} \end{aligned}$$

Great. We have three critical points: $t = \frac{-1-\sqrt{5}}{2}, 0, \frac{-1+\sqrt{5}}{2}$.

c. $f(r) = \dfrac{r}{r^2 + 1}$

Solution: We can use the quotient rule to find the derivative.

$$f'(r) = \frac{(r)'(r^2 + 1) - r(r^2 + 1)'}{(r^2 + 1)^2} = \frac{(1)(r^2 + 1) - r(2r)}{(r^2 + 1)^2} = \frac{r^2 + 1 - 2r^2}{(r^2 + 1)^2} = \frac{1 - r^2}{(r^2 + 1)^2}$$

To find the critical values, set the numerator of $s'(r)$ equal to 0. (Remember that since we are dealing with a fraction, we only need to find where the numerator equals 0. Also, given the nature of the denominator, we don't have to worry about whether the function is ever undefined.)

$$1 - r^2 = 0 \Longrightarrow r^2 = 1 \Longrightarrow r = \pm 1$$

d. $h(x) = x\log(x)$

Solution: The function requires product rule to differentiate.

$$h'(x) = x \cdot (\log x)' + x' \cdot \log x = x \cdot \frac{1}{x} + 1 \cdot \log x = 1 + \log x$$

Now set this derivative equal to zero and solve.

$$\begin{aligned} 1 + \log x &= 0 \\ \log x &= -1 \\ x &= e^{-1} \\ x &= \frac{1}{e} \end{aligned}$$

4.3 Find absolute minimum/maximum values

Find the absolute minimum and absolute maximum values of the functions on the given interval:

a. $f(x) = 3x^2 - 12x + 5, [0, 3]$

Solution: First, we must identify the critical points; to do this, we must find $f'(x)$. By our rules of derivation, $f'(x) = 6x - 12$. The critical points will then be where $f'(x) = 6x - 12 = 0$. Solving for x then gives $x = 2$. As this is a continuous function over the given interval, the absolute minimum and absolute maximum values must be at critical points or the endpoints of the interval. In this case, the set of candidate values of x is then $\{0, 2, 3\}$. Evaluating the function at these points gives $f(0) = 5$, $f(2) = -7$, and $f(3) = -4$. So the absolute maximum occurs where $x = 0, f(x) = 5$ and the absolute minimum occurs where $x = 2, f(x) = -7$. It is a useful check of our work to make sure that $x = 2$ gives a local minimum - after all, since $x = 2$ is not one of the endpoints, in order to be the absolute minimum it must be a local minimum as well. To see whether it is a local minimum, we evaluate $f''(x)$ where $x = 2$. $f''(x) = (6x - 12)' = 6$, so $f''(2) = 6 > 0$. Since the second derivative is positive, the first derivative must be increasing at $x = 2$ - in other words, it must be moving from negative to positive - and $x = 2$ is indeed a local minimum.

b. $f(t) = t\sqrt{4 - t^2}, [-1, 4]$

Solution: As before, we must first identify critical points. To find $f'(t)$, we must use both the power rule and the chain rule. The power rule tells us that $f'(t) = t(\sqrt{4 - t^2})' + (t)'\sqrt{4 - t^2}$. By the chain rule, the derivative of $\sqrt{4 - t^2}$ is $\frac{1}{2}(4 - t^2)^{-\frac{1}{2}}(4 - t^2)' = \frac{1}{2}(4 - t^2)^{-\frac{1}{2}}(-2t) = -t(4 - t^2)^{-\frac{1}{2}}$. So the derivative of $f(t)$ is $f'(t) = -t^2(4 - t^2)^{-\frac{1}{2}} + \sqrt{4 - t^2}$. Setting this equal to zero and adding $t^2(4-t^2)^{-\frac{1}{2}}$ to both sides gives $t^2(4-t^2)^{-\frac{1}{2}} = \sqrt{(4-t^2)}$. Multiplying both sides by $\sqrt{4-t^2}$ then gives $t^2 = 4-t^2$; solving for t then gives $t = \pm\sqrt{2}$. Of course, $-\sqrt{2}$ is outside of our domain, so we can ignore it and instead only investigate $\sqrt{2}$.

To see something about the behavior of the function at this point, we have to take the second derivative. Omitting the steps involved (it would be good practice to see if you can get the same answer!), $f''(t) = -t^3(4-t^2)^{-\frac{3}{2}} - 3t(4-t^2)^{-\frac{1}{2}}$. At $t = \sqrt{2}$, this will be negative, so $t = \sqrt{2}$ produces a local maximum.

Note that this function is not actually defined over the entire interval provided - if $t > 2$, then we'd have to take the square root of a negative number. So the *effective* endpoints of the interval, for the purpose of finding the absolute minimum and maximum, are -1 and 2. So our candidates for minimum and maximum are where $t \in \{-1, \sqrt{2}, 2\}$. Plugging in, we get $f(-1) = -\sqrt{3}$, $f(\sqrt{2}) = 2$, and $f(2) = 0$. So the absolute maximum occurs where $t = \sqrt{2}$, and the absolute minimum occurs at $t = -1$.

c. $s(x) = x - \log(x), [1/2, 2]$

Solution: This one should be less painful than the previous problem. First, we need to find $s'(x)$. Remember, the derivative of the natural log of x is just $\frac{1}{x}$! So $s'(x) = 1 - \frac{1}{x}$. Setting this equal to 0, we get $0 = 1 - \frac{1}{x}$, which means $1 = \frac{1}{x}$, or $x = 1$. Let's check whether this is a minimum or a maximum. $s''(x) = \frac{1}{x^2}$, so $s''(1) = 1 > 0$. Since the second derivative is positive at $x = 1$, $x = 1$ should produce a local minimum. Plugging in $x = 1$ and the endpoints of the interval, we get $s(1/2) = 1.19$, $s(1) = 1$, and $s(2) = 1.31$. $x = 1$ is therefore not only produces a local minimum, but it produces the absolute minimum. The absolute maximum occurs at one of the endpoints, where $x = 2$.

d. $h(p) = 1 - e^{-p}, [0, 1000]$

Solution: The procedure should be getting familiar by now. First, we find the derivative of $h(p)$, giving us $h'(p) = e^{-p}$. However, we can make an interesting observation when we set this equal to 0, namely that e^{-p} never equals 0! Its limit as p goes to infinity is 0, but it is not 0 for any finite p, let alone one in our interval. So we have no critical points, and the endpoints will give us the absolute minimum and maximum over the interval. Plugging in, we get $h(0) = 0$ and $h(1000)$ is very close to 1, so over our interval, $p = 0$ produces the absolute minimum and $p = 1000$ produces the absolute maximum.

4.4 A function with no local minima/maxima

Demonstrate that the function $f(x) = x^5 + x^3 + x + 1$ has no local maximum and no local minimum.

Solution: This proof might seem hard to approach, so let's just see what happens when we try to find a local minimum or maximum. First, as usual, we have to find the derivative, and we find that $f'(x) = 5x^4 + 3x^2 + 1$. Next, we have to set this equal to zero and solve for x. After looking at the equation $5x^4 + 3x^2 + 1 = 0$, though, we might make an important observation - namely, that this has no solutions! There are two ways we could show this fact. First, we could create a variable $y = x^2$, rewrite the equation as $5y^2 + 3y + 1 = 0$, and then observe that the quadratic equation gives us no solutions. Second, and perhaps more elegantly, we can observe that $x^4 \geq 0$ and $x^2 \geq 0$ for all x. Therefore $5x^4 + 3x^2 + 1 \geq 5(0) + 3(0) + 1 = 1$. So $f'(x) \geq 1 > 0$ for all x. Thus, we see that the derivative never equals zero, and the function has no critical points. But all local maxima and local minima occur at critical points, so the function cannot have a local maximum or local minimum.

4.5 Approximate root-finding

Show that the equation

$$x^7 - 6x + 4 = 0$$

has a root between 0 and 1.

a. Find an initial approximation by ignoring the term x^7.

   Solution: If we ignore x^7, we can solve for the root as

$$\begin{aligned} -6x + 4 &= 0 \\ -6x &= -4 \\ x &= \frac{4}{6} = \frac{2}{3} \end{aligned}$$

b. Use Newton's method to find the root correct to 3 decimal places.

   Solution: Recall that the first derivative of the function is $f'(x) = x^6 - 6$. Assume a starting value of $x_0 = 0.7$.

$$
\begin{aligned}
x_0 &= 0.7 \\
x_1 &= x_0 - \frac{x_0^7 - 6x_0 + 4}{7x_0^6 - 6} \\
x_1 &= 0.7 - (0.0227271) \\
x_1 &= 0.677273
\end{aligned}
$$

$$
\begin{aligned}
x_2 &= x_1 - \frac{x_1^7 - 6x_1 + 4}{7x_1^6 - 6} \\
x_2 &= 0.677273 - (-0.000324455) \\
x_2 &= 0.677597
\end{aligned}
$$

$$
\begin{aligned}
x_3 &= x_2 - \frac{x_2^7 - 6x_2 + 4}{7x_2^6 - 6} \\
x_3 &= 0.677597 - (-5.92353» 10^{-}8) \\
x_3 &= 0.677597
\end{aligned}
$$

4.6 Apply the mean value theorem

Does a continuous, differentiable function exist on $[0, 2]$ such that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leq 2 \; \forall x$? Use the mean value theorem to explain your answer.

Solution: First we set up the mean value theorem which states that, if a function is continuous and differentiable over some interval, then a c exists such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

We plug in the values given by the problem and find, $f'(c) = \frac{f(2)-f(0)}{2-0} = \frac{4--1}{2} = \frac{5}{2}$.

The problem states that the derivative of the function is less than or equal to 2 over this entire interval, but the mean value theorem tell us that that the derivative must equal 2.5 at some point. So by demonstrating this contradiction, we've shown that the earlier values could not have come from a continuous, differentiable function.

Chapter 5

Matrix algebra and systems of linear equations

5.1 Basic matrix arithmetic

If

$$\mathbf{a} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

find:

a. $\mathbf{a} + \mathbf{b}$

$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2+1 \\ 2+3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

b. $3\mathbf{a}$

$$3\mathbf{a} = 3 \times \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

c. $-4\mathbf{b}$

$$-4\mathbf{b} = -4 \times \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ -12 \end{bmatrix}$$

d. $3\mathbf{a} - 4\mathbf{b}$

$$3\mathbf{a} - 4\mathbf{b} = 3 \times \begin{bmatrix} 2 \\ 2 \end{bmatrix} - 4 \times \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 12 \end{bmatrix} = \begin{bmatrix} 6-4 \\ 6-12 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$$

5.2 More complex matrix arithmetic

Suppose

$$\mathbf{x} = \begin{bmatrix} 3 \\ 2q \\ 6 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} p+2 \\ -5 \\ 3r \end{bmatrix}$$

.

If $\mathbf{x} = 2\mathbf{y}$, find p, q, r.

Solution: We can calculate each element of the vector independently, given our knowledge of the relationship between $\mathbf{x}$ and $\mathbf{y}$.

$$\begin{aligned} 3 &= 2(p+2) \\ 3 &= 2p+4 \\ -1 &= 2p \\ -\frac{1}{2} &= p \end{aligned}$$

$$\begin{aligned} 2q &= 2(-5) \\ 2q &= -10 \\ q &= -5 \end{aligned}$$

$$\begin{aligned} 6 &= 2(3r) \\ 6 &= 6r \\ 1 &= r \end{aligned}$$

So $p = -\frac{1}{2}, q = -5, r = 1$.

5.3 Check for linear dependence

Which of the following sets of vectors are linearly dependent?

In each part, you can denote each vector as $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively.

a. $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$

Yes: $\mathbf{a}+\mathbf{b}-\mathbf{c}=\mathbf{0}$

b. $\begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}$

Linearly independent.

c. $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}$

Yes: $\mathbf{a}-2\mathbf{b}+\mathbf{c}=\mathbf{0}$

d. $\begin{bmatrix}13\\7\\9\\2\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}3\\-2\\5\\8\end{bmatrix}$

Yes: $0\mathbf{a}+1\mathbf{b}+0\mathbf{c}=\mathbf{0}$

e. $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\-2\\-1\end{bmatrix}, \begin{bmatrix}2\\1\\3\end{bmatrix}$

Linearly independent.

f. $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\-2\\4\end{bmatrix}, \begin{bmatrix}2\\1\\3\end{bmatrix}$

Yes: $2\mathbf{a}+\mathbf{b}-2\mathbf{c}=\mathbf{0}$

5.4 Vector length

Find the length of the following vectors:

a. $(3,4)$

$$\begin{aligned}\sqrt{3^2+4^2} &= \sqrt{9+16}\\ &= \sqrt{25}\\ &= 5\end{aligned}$$

b. $(0, -3)$

$$\begin{aligned}\sqrt{0^2 + (-3)^2} &= \sqrt{0+9} \\ &= \sqrt{9} \\ &= 3\end{aligned}$$

c. $(1, 1, 1)$

$$\begin{aligned}\sqrt{1^2 + 1^2 + 1^2} &= \sqrt{1+1+1} \\ &= \sqrt{3}\end{aligned}$$

d. $(3, 3)$

$$\begin{aligned}\sqrt{3^2 + 3^2} &= \sqrt{9+9} \\ &= \sqrt{18} \\ &= \sqrt{3^2 \times 2} \\ &= 3\sqrt{2}\end{aligned}$$

e. $(-1, -1)$

$$\begin{aligned}\sqrt{(-1)^2 + (-1)^2} &= \sqrt{1+1} \\ &= \sqrt{2}\end{aligned}$$

f. $(1, 2, 3)$

$$\begin{aligned}\sqrt{1^2 + 2^2 + 3^2} &= \sqrt{1+4+9} \\ &= \sqrt{14}\end{aligned}$$

g. $(2, 0)$

$$\begin{aligned}\sqrt{2^2 + 0^2} &= \sqrt{4+0} \\ &= \sqrt{4} \\ &= 2\end{aligned}$$

h. $(1, 2, 3, 4)$

$$\begin{aligned}\sqrt{1^2 + 2^2 + 3^2 + 4^2} &= \sqrt{1+4+9+16} \\ &= \sqrt{30} \\ &\approx 5.47726\end{aligned}$$

i. $(3, 0, 0, 0, 0)$

$$\begin{aligned}\sqrt{3^2+0^2+0^2+0^2+0^2} &= \sqrt{9+0+0+0+0} \\ &= \sqrt{3} \\ &= 3\end{aligned}$$

5.5 Law of cosines

The **law of cosines** states:

$$\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \, \|\mathbf{w}\|}$$

where θ is the angle from $\mathbf{w}$ to $\mathbf{v}$ measured in radians. Of importance, arccos() is the inverse of cos():

$$\theta = \arccos\left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \, \|\mathbf{w}\|}\right)$$

For each of the following pairs of vectors, calculate the angle between them. Report your answers in both radians and degrees. To convert between radians and degrees:

$$\text{Degrees} = \text{Radians} \times \frac{180^o}{\pi}$$

a. $\mathbf{v} = (1, 0), \quad \mathbf{w} = (2, 2)$

$$
\begin{aligned}
\mathbf{v} \cdot \mathbf{w} &= (1)(2) + (0)(2) \\
&= 2 + 0 \\
&= 2 \\
\|\mathbf{v}\| &= \sqrt{1^2 + 0^2} \\
&= \sqrt{1 + 0} \\
&= \sqrt{1} \\
&= 1 \\
\|\mathbf{w}\| &= \sqrt{2^2 + 2^2} \\
&= \sqrt{4 + 4} \\
&= \sqrt{8} \\
&= \sqrt{2^2 \times 2} \\
&= 2\sqrt{2} \\
\theta &= \arccos\left(\frac{2}{1(2\sqrt{2})}\right) \\
&= \frac{\pi}{4} \\
&= 45^\circ
\end{aligned}
$$

b. $\mathbf{v} = (4, 1), \quad \mathbf{w} = (2, -8)$

$$
\begin{aligned}
\mathbf{v} \cdot \mathbf{w} &= (4)(2) + (1)(-8) \\
&= 8 + (-8) \\
&= 0 \\
\|\mathbf{v}\| &= \sqrt{4^2 + 1^2} \\
&= \sqrt{16 + 1} \\
&= \sqrt{17} \\
&= 1 \\
\|\mathbf{w}\| &= \sqrt{2^2 + (-8)^2} \\
&= \sqrt{4 + 64} \\
&= \sqrt{68} \\
&= \sqrt{2^2 \times 17} \\
&= 2\sqrt{17} \\
\theta &= \arccos\left(\frac{0}{1(2\sqrt{17})}\right) \\
&= \frac{\pi}{2} \\
&= 90^\circ
\end{aligned}
$$

Note: you could stop after solving $\mathbf{v} \cdot \mathbf{w}$, because the denominator will be irrelevant.

c. $\mathbf{v} = (1, 1, 0), \quad \mathbf{w} = (1, 2, 1)$

$$\begin{aligned}
\mathbf{v}\cdot\mathbf{w} &= (1)(1)+(1)(2)+(0)(1)\\
&= 1+2+0\\
&= 3\\
\|\mathbf{v}\| &= \sqrt{1^2+1^2+0^2}\\
&= \sqrt{1+1+0}\\
&= \sqrt{2}\\
\|\mathbf{w}\| &= \sqrt{1^2+2^2+1^2}\\
&= \sqrt{1+4+1}\\
&= \sqrt{6}\\
\theta &= \arccos\left(\frac{3}{\sqrt{2}(\sqrt{6})}\right)\\
&= \arccos\left(\frac{3}{\sqrt{2\times 6}}\right)\\
&= \arccos\left(\frac{3}{\sqrt{12}}\right)\\
&= \arccos\left(\frac{3}{\sqrt{2^2\times 3}}\right)\\
&= \arccos\left(\frac{3}{2\sqrt{3}}\right)\\
&= \arccos\left(\frac{3\sqrt{3}}{2\sqrt{3}\sqrt{3}}\right)\\
&= \arccos\left(\frac{3\sqrt{3}}{2\times 3}\right)\\
&= \arccos\left(\frac{\sqrt{3}}{2}\right)\\
&= \frac{\pi}{6}\\
&= 30^\circ
\end{aligned}$$

5.6 Matrix algebra

Using the matrices below, calculate the following. Some may not be defined; if that is the case, say so.

$$\mathbf{A}=\begin{bmatrix}3\\-2\\9\end{bmatrix}\quad \mathbf{B}=\begin{bmatrix}8\\0\\-1\end{bmatrix}\quad \mathbf{C}=\begin{bmatrix}7&-1&5\\0&2&-4\end{bmatrix}\quad \mathbf{D}=\begin{bmatrix}3&1\\3&4\\3&-7\end{bmatrix}\quad \mathbf{E}=\begin{bmatrix}5&2&3\\1&0&-4\\-2&1&-6\end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 4 & 1 & -5 \\ 0 & 7 & 7 \\ 2 & -3 & 0 \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} 2 & -8 & -5 \\ -3 & 7 & -4 \\ 1 & 0 & 3 \\ 1 & 2 & 6 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} 9 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} 5 & 0 & 3 & 1 \end{bmatrix} \quad \mathbf{M} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

a. $\mathbf{A} + \mathbf{B}$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 3+8 \\ -2+0 \\ 9+(-1) \end{bmatrix} = \begin{bmatrix} 11 \\ -2 \\ 8 \end{bmatrix}$$

b. $-\mathbf{G}$

$$-\mathbf{G} = (-1)\begin{bmatrix} 2 & -8 & -5 \\ -3 & 7 & -4 \\ 1 & 0 & 3 \\ 1 & 2 & 6 \end{bmatrix} = \begin{bmatrix} -2 & 8 & 5 \\ 3 & -7 & 4 \\ -1 & 0 & -3 \\ -1 & -2 & -6 \end{bmatrix}$$

c. $\mathbf{D}'$

$$\mathbf{D}' = \begin{bmatrix} 3 & 3 & 3 \\ 1 & 4 & -7 \end{bmatrix}$$

d. $\mathbf{C} + \mathbf{D}$

$\mathbf{C} + \mathbf{D}$ does not exist. The matricies are not the same dimensions.

e. $3\mathbf{C} - 2\mathbf{D}'$

$$\begin{aligned} 3\mathbf{C} - 2\mathbf{D}' &= (3)\begin{bmatrix} 7 & -1 & 5 \\ 0 & 2 & -4 \end{bmatrix} - (2)\begin{bmatrix} 3 & 3 & 3 \\ 1 & 4 & -7 \end{bmatrix} \\ &= \begin{bmatrix} 21 & -3 & 15 \\ 0 & 6 & -12 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ 2 & 8 & -14 \end{bmatrix} \\ &= \begin{bmatrix} 15 & -9 & 9 \\ -2 & -2 & 2 \end{bmatrix} \end{aligned}$$

f. $\mathbf{A}'\mathbf{B}$

This is a 1×3 matrix multiplied by a 3×1 matrix, resulting in a 1×1 matrix (aka a **dot product**).

$$\mathbf{A}'\mathbf{B} = 3(8) + (-2)(0) + 9(-1) = 24 + 0 - 9 = 15$$

g. **CB**

$$
\begin{aligned}
\mathbf{CB} &= \begin{bmatrix} 7 & -1 & 5 \\ 0 & 2 & -4 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 7(8) & + & (-1)(0) & + & 5(-1) \\ 0(8) & + & 2(0) & + & (-4)(-1) \end{bmatrix} \\
&= \begin{bmatrix} 56+0-5 \\ 0+0+4 \end{bmatrix} \\
&= \begin{bmatrix} 51 \\ 4 \end{bmatrix}
\end{aligned}
$$

h. **BC**

BC does not exist. The matricies are non-conformable.

i. **FB**

$$
\begin{aligned}
\mathbf{FB} &= \begin{bmatrix} 4 & 1 & -5 \\ 0 & 7 & 7 \\ 2 & -3 & 0 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 4(8) & + & 1(0) & + & (-5)(-1) \\ 0(8) & + & 7(0) & + & 7(-1) \\ 2(8) & + & (-3)(0) & + & 0(-1) \end{bmatrix} \\
&= \begin{bmatrix} 32+0+5 \\ 0+0-7 \\ 16+0+0 \end{bmatrix} \\
&= \begin{bmatrix} 37 \\ -7 \\ 16 \end{bmatrix}
\end{aligned}
$$

j. **EF**

(Vertical and horizontal lines are added in the work here to clearly delineate each term.)

$$\mathbf{EF} = \left[\begin{array}{ccccc|ccccc|ccccc} 5(4) & + & 2(0) & + & 3(2) & 5(1) & + & 2(7) & + & 3(-3) & 5(-5) & + & 2(7) & + & 3(0) \\ \hline 1(4) & + & 0(0) & + & (-4)(2) & 1(1) & + & 0(7) & + & (-4)(-3) & 1(-5) & + & 0(7) & + & (-4)(0) \\ \hline -2(4) & + & 1(0) & + & (-6)(2) & -2(1) & + & 1(7) & + & (-6)(-3) & -2(-5) & + & 1(7) & + & (-6)(0) \end{array}\right]$$

$$= \begin{bmatrix} 20+0+6 & 5+14-9 & -25+14+0 \\ 4+0-8 & 1+0+12 & -5+0+0 \\ -8+0-12 & -2+7+18 & 10+7+0 \end{bmatrix}$$

$$= \begin{bmatrix} 26 & 10 & -11 \\ -4 & 13 & -5 \\ -20 & 23 & 17 \end{bmatrix}$$

k. $\mathbf{KL}'$

Non-conformable operation. You cannot multiply a 4×1 matrix with another 4×1 matrix.

l. $\mathbf{G}'$

$$\mathbf{G}' = \begin{bmatrix} 2 & -8 & -5 \\ -3 & 7 & -4 \\ 1 & 0 & 3 \\ 1 & 2 & 6 \end{bmatrix}' = \begin{bmatrix} 2 & -3 & 1 & 1 \\ -8 & 7 & 0 & 2 \\ -5 & -4 & 3 & 6 \end{bmatrix}$$

m. $\mathbf{E} - 5\mathbf{I_3}$

$$\mathbf{E} - 5\mathbf{I}_3 = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 0 & -4 \\ -2 & 1 & -6 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 2 & 3 \\ 1 & 0 & -4 \\ -2 & 1 & -6 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 3 \\ 1 & -5 & -4 \\ -2 & 1 & -11 \end{bmatrix}$$

n. $\mathbf{M}^2$

Recall that $\mathbf{M}^2 = \mathbf{MM}$, so we must pre-multiply the matrix by itself.

$$
\begin{aligned}
\mathbf{M}^2 &= \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 1 \times 1 + -1 \times 1 & 1 \times -1 + -1 \times 3 \\ 1 \times 1 + 3 \times 1 & 1 \times -1 + 3 \times 3 \end{bmatrix} \\
&= \begin{bmatrix} 1 + (-1) & -1 + (-3) \\ 1 + 3 & -1 + 9 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -4 \\ 4 & 8 \end{bmatrix}
\end{aligned}
$$

5.7 Additive property of matrix transposition

Prove the additive property of matrix transposition:

$$(\mathbf{X} + \mathbf{Y})' = \mathbf{X}' + \mathbf{Y}'$$

Solution: This is a relatively simple proof, and there are a lot of different ways to do it. The most straightforward way is simply to write out each of the two matrices and observe that they are identical. Without loss of generality, suppose $\mathbf{X}$ and $\mathbf{Y}$ are $m \times n$ matrices. We know that

$$\mathbf{X} + \mathbf{Y} = \begin{bmatrix} x_{11} + y_{11} & \dots & x_{1n} + y_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} + y_{m1} & \dots & x_{mn} + y_{mn} \end{bmatrix}$$

Then it must be that

$$(\mathbf{X} + \mathbf{Y})' = \begin{bmatrix} x_{11} + y_{11} & \dots & x_{m1} + y_{m1} \\ \vdots & \ddots & \vdots \\ x_{1n} + y_{1n} & \dots & x_{mn} + y_{mn} \end{bmatrix}$$

Now let's consider the right hand side of the equation. Since

$$\mathbf{X}' = \begin{bmatrix} x_{11} & \dots & x_{m1} \\ \vdots & \ddots & \vdots \\ x_{1n} & \dots & x_{mn} \end{bmatrix} \qquad \mathbf{Y} = \begin{bmatrix} y_{11} & \dots & y_{m1} \\ \vdots & \ddots & \vdots \\ y_{1n} & \dots & y_{mn} \end{bmatrix}$$

we know that

$$\mathbf{X}' + \mathbf{Y}' = \begin{bmatrix} x_{11} + y_{11} & \dots & x_{m1} + y_{m1} \\ \vdots & \ddots & \vdots \\ x_{1n} + y_{1n} & \dots & x_{mn} + y_{mn} \end{bmatrix}$$

which is the same as $(\mathbf{X} + \mathbf{Y})'$.

Chapter 6

Systems of linear equations and determinants

6.1 Calculate the determinant

Recall the general rule for calculating the determinant of an $n \times n$ matrix:

$$|\mathbf{X}| = \sum_{j=1}^{n} (-1)^{i+j} x_{ij} |\mathbf{X}_{[ij]}|$$

where the ijth **minor** of $\mathbf{X}$ for x_{ij}, $|\mathbf{X}_{[ij]}|$, is the determinant of the $(n-1) \times (n-1)$ submatrix that results from taking the ith row and jth column out. The **cofactor** of $\mathbf{X}$ is the minor signed as $(-1)^{i+j} x_{ij} |\mathbf{X}_{[ij]}|$. To calculate the determinant we cycle recursively through the columns and take sums with a formula that multiplies the cofactor by the determining value.

Given this rule, obtain the determinant of the following matrix. You can do this the hard way, or the easy(ier) way. I encourage you to think a bit before starting the calculations for the determinant - how can you make the problem easier?

$$\begin{bmatrix} 6 & 6 & 1 & 0 \\ 0 & 4 & 0 & 1 \\ 4 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Solution: The best way to tackle this problem is to start the recursive operation on a row with the most zeros - this is because x_{ij} will be 0, so the resulting submatricies will drop out of the equation. Here I choose to start on the second row:

$$\begin{aligned}
\begin{vmatrix} 6 & 6 & 1 & 0 \\ 0 & 4 & 0 & 1 \\ 4 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{vmatrix} &= (-1)^{2+1}(0)\begin{vmatrix} 6 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} + (-1)^{2+2}(4)\begin{vmatrix} 6 & 1 & 0 \\ 4 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} \\
&\quad + (-1)^{2+3}(0)\begin{vmatrix} 6 & 6 & 0 \\ 4 & 2 & 1 \\ 1 & 1 & 0 \end{vmatrix} + (-1)^{2+4}(1)\begin{vmatrix} 6 & 6 & 1 \\ 4 & 2 & 1 \\ 1 & 1 & 0 \end{vmatrix} \\
&= 4\begin{vmatrix} 6 & 1 & 0 \\ 4 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} + 1\begin{vmatrix} 6 & 6 & 1 \\ 4 & 2 & 1 \\ 1 & 1 & 0 \end{vmatrix}
\end{aligned}$$

Now that two of the terms have dropped out, we repeat the process on the remaining two submatricies. Again, we want to start with rows that have the most zeros, so for both submatricies I start on the third row.

$$\begin{aligned}
&= 4\begin{vmatrix} 6 & 1 & 0 \\ 4 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} + 1\begin{vmatrix} 6 & 6 & 1 \\ 4 & 2 & 1 \\ 1 & 1 & 0 \end{vmatrix} \\
&= 4\left[(-1)^{3+1}(1)\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + (-1)^{3+2}(0)\begin{vmatrix} 6 & 0 \\ 4 & 1 \end{vmatrix} + (-1)^{3+3}(0)\begin{vmatrix} 6 & 1 \\ 4 & 1 \end{vmatrix}\right] \\
&\quad + 1\left[(-1)^{3+1}(1)\begin{vmatrix} 6 & 1 \\ 2 & 1 \end{vmatrix} + (-1)^{3+2}(1)\begin{vmatrix} 6 & 1 \\ 4 & 1 \end{vmatrix} + (-1)^{3+3}(0)\begin{vmatrix} 6 & 6 \\ 4 & 2 \end{vmatrix}\right] \\
&= 4\left[1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}\right] + 1\left[1\begin{vmatrix} 6 & 1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 6 & 1 \\ 4 & 1 \end{vmatrix}\right] \\
&= 4\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 6 & 1 \\ 2 & 1 \end{vmatrix} - \begin{vmatrix} 6 & 1 \\ 4 & 1 \end{vmatrix}
\end{aligned}$$

Now that the submatricies are all 2×2, we can use the standard formula to calculate the determinants:

$$\begin{aligned}
&= 4(1 \times 1 - 0 \times 1) + (6 \times 1 - 1 \times 2) - (6 \times 1 - 1 \times 4) \\
&= 4(1 - 0) + (6 - 2) - (6 - 4) \\
&= 4 + 4 - 2 \\
&= 6
\end{aligned}$$

6.2 Matrix inversion

Invert each of the following matricies. Verify you have the correct inverse by calculating $\mathbf{X}\mathbf{X}^{-1} = \mathbf{I}$. Not all of the matricies may be invertible - if not, show why.

a. $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

Solution: Recall the rule for inverting 2×2 matricies:

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

$$\begin{aligned} \mathbf{X}^{-1} &= |\mathbf{X}|^{-1} \begin{bmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{bmatrix} \\ &= \frac{1}{|\mathbf{X}|} \begin{bmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{bmatrix} \end{aligned}$$

Given this rule, first calculate the determinant of the matrix.

$$\begin{aligned} |\mathbf{X}| &= (2 \times 1) - (1 \times 1) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

Now we can easily solve for the inverse:

$$\begin{aligned} \mathbf{X}^{-1} &= \frac{1}{1} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \end{aligned}$$

b. $\begin{bmatrix} 4 & 5 \\ 2 & 4 \end{bmatrix}$

Solution:

1. Solve for the determinant

$$\begin{aligned} |\mathbf{X}| &= (4 \times 4) - (5 \times 2) \\ &= 16 - 10 \\ &= 6 \end{aligned}$$

2. Solve for the inverse

$$\begin{aligned}
\mathbf{X}^{-1} &= \frac{1}{6}\begin{bmatrix} 4 & -5 \\ -2 & 4 \end{bmatrix} \\
&= \begin{bmatrix} \frac{4}{6} & -\frac{5}{6} \\ -\frac{2}{6} & \frac{4}{6} \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{3} & -\frac{5}{6} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}
\end{aligned}$$

c. $\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$

Solution: Solve for the determinant

$$\begin{aligned}
|\mathbf{X}| &= (2 \times -2) - (1 \times -4) \\
&= -4 - (-4) \\
&= 0
\end{aligned}$$

At this point we are done. The matrix has a determinant of zero, making it singular. Singular matricies cannot be inverted.

d. $\begin{bmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{bmatrix}$

Solution: With a 3×3 matrix, we need to apply Gauss-Jordan elimination to obtain the inverse.

1. Setup the augmented matrix with the identity matrix

$$\left[\begin{array}{rrr|rrr} 2 & 4 & 0 & 1 & 0 & 0 \\ 4 & 6 & 3 & 0 & 1 & 0 \\ -6 & -10 & 0 & 0 & 0 & 1 \end{array}\right]$$

2. Swap row 1 with row 3

$$\left[\begin{array}{rrr|rrr} -6 & -10 & 0 & 0 & 0 & 1 \\ 4 & 6 & 3 & 0 & 1 & 0 \\ 2 & 4 & 0 & 1 & 0 & 0 \end{array}\right]$$

3. Add $\frac{2}{3}\times$ row 1 to row 2

$$\left[\begin{array}{rrr|rrr} -6 & -10 & 0 & 0 & 0 & 1 \\ 0 & -2/3 & 3 & 0 & 1 & 2/3 \\ 2 & 4 & 0 & 1 & 0 & 0 \end{array}\right]$$

4. Add $\frac{1}{3}\times$ row 1 to row 3

$$\left[\begin{array}{rrr|rrr} -6 & -10 & 0 & 0 & 0 & 1 \\ 0 & -2/3 & 3 & 0 & 1 & 2/3 \\ 0 & 2/3 & 0 & 1 & 0 & 1/3 \end{array}\right]$$

5. Add row 2 to row 3

$$\left[\begin{array}{rrr|rrr} -6 & -10 & 0 & 0 & 0 & 1 \\ 0 & -2/3 & 3 & 0 & 1 & 2/3 \\ 0 & 0 & 3 & 1 & 1 & 1 \end{array}\right]$$

6. Divide row 3 by 3

$$\left[\begin{array}{rrr|rrr} -6 & -10 & 0 & 0 & 0 & 1 \\ 0 & -2/3 & 3 & 0 & 1 & 2/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{array}\right]$$

7. Subtract $3\times$ row 3 from row 2

$$\left[\begin{array}{rrr|rrr} -6 & -10 & 0 & 0 & 0 & 1 \\ 0 & -2/3 & 0 & -1 & 0 & -1/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{array}\right]$$

8. Multiply row 2 by $-\frac{3}{2}$

$$\left[\begin{array}{rrr|rrr} -6 & -10 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{array}\right]$$

9. Add $10\times$ row 2 to row 1

$$\left[\begin{array}{rrr|rrr} -6 & 0 & 0 & 15 & 0 & 6 \\ 0 & 1 & 0 & 3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{array}\right]$$

10. Divide row 1 by -6

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -5/2 & 0 & -1 \\ 0 & 1 & 0 & 3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{array}\right]$$

11. The inverse of the original matrix is the right part of the augmented matrix.

$$\begin{bmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} -5/2 & 0 & -1 \\ 3/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

12. Factor out common terms

$$\begin{bmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{bmatrix}^{-1} = \frac{1}{6}\begin{bmatrix} -15 & 0 & -6 \\ 9 & 0 & 3 \\ 2 & 2 & 2 \end{bmatrix}$$

6.3 One-hot encoding for categorical variables

Ordinary least squares regression is a common method for obtaining regression parameters relating a set of explanatory variables with a continuous outcome of interest. The vector $\hat{\mathbf{b}}$ that contains the intercept and the regression slope is calculated by the equation:

$$\hat{\mathbf{b}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$$

If an explanatory variable is nominal (i.e. ordering does not matter) with more than two classes (e.g. $\{\text{White}, \text{Black}, \text{Asian}, \text{Mixed}, \text{Other}\}$), the variable must be modified to include in the regression model. A common technique known as **one-hot encoding** converts the column into a series of $n-1$ binary (0/1) columns where each column represents a single class and n is the total number of unique classes in the original column. Explain why this method converts the column into $n-1$ columns, rather than n columns, in terms of linear algebra. **Reminder: $\mathbf{X}$ contains both the one-hot encoded columns as well as a column of 1s representing the intercept.**

Solution: In order to calculate $\hat{\mathbf{b}}$, we must be able to calculate $(\mathbf{X}'\mathbf{X})^{-1}$. And we can only invert $\mathbf{X}'\mathbf{X}$ if the matrix is **nonsingular**. What could make a matrix singular? If at least one column is **linearly dependent** (i.e. its value can be produced by linear combinations of other columns in the matrix), then the matrix will not be **full rank**. A square matrix that is not full rank will produce a determinant of 0, which as you'll recall in the case of a 2×2 matrix would require division by zero.

$$\mathbf{X}^{-1} = \frac{1}{0}\begin{bmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{bmatrix}$$

So $\mathbf{X}'\mathbf{X}$ must be full rank in order to invert it. How does this effect our one-hot encoding scheme? If we were to convert the explanatory variable into n binary variables, the matrix X is nonsingular. That is, any of the columns in $\mathbf{X}$ can be represented as a linear combination of the other columns.

This leads to the problem of what happens when we calculate $\mathbf{X}'\mathbf{X}$. Suppose

$$\mathbf{X} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

It's transpose is

$$\mathbf{X}' = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So

$$\mathbf{X}'\mathbf{X} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

The problem is that $\mathbf{X}'\mathbf{X}$ is still non-invertible. The determinant of $\mathbf{X}'\mathbf{X}$ is 0. Notice that the first column $\mathbf{x_1}$ is a linear combination of $\mathbf{x_2} + \mathbf{x_3}$. In fact, $\mathbf{X}$ being invertible is a necessary condition for $\mathbf{X}'\mathbf{X}$ being invertible.

6.4 Solve the system of equations

Solve the following systems of equations for x, y, z, either via matrix inversion or substitution:

a. System #1

$$\begin{aligned} x + y + 2z &= 2 \\ 3x - 2y + z &= 1 \\ y - z &= 3 \end{aligned}$$

- Using matrix inversion:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 3 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix} \quad \mathbf{y} = [2, 1, 3]' \quad \mathbf{x} = [x, y, z]$$

$$\mathbf{A}\mathbf{x} = \mathbf{y}$$
$$\mathbf{A}^{-1}\mathbf{y} = \mathbf{x}$$

You can use (a lot) of Gauss-Jordan elimination to invert the matrix. Or I can just use R.

```
##      [,1] [,2] [,3]
## [1,]    1    1    2
## [2,]    3   -2    1
## [3,]    0    1   -1
## [1] 2 1 3
##      [,1] [,2] [,3]
## [1,]  0.1  0.3  0.5
## [2,]  0.3 -0.1  0.5
## [3,]  0.3 -0.1 -0.5
## [1]  2  2 -1
```

- Using systems of equations
  1. 1 x third row added to second row and 2 x third row added to first row.

$$\begin{aligned} x + 3y &= 8 \\ 3x - y &= 4 \\ y - z &= 3 \end{aligned}$$

  2. -3 x first row added to second row

$$\begin{aligned} x + 3y &= 8 \\ -10y &= -20 \\ y - z &= 3 \end{aligned}$$

  3. Solve for y and z

$$\begin{aligned} -10y = -20 &\rightarrow y = 2 \\ y - z = 3 &\rightarrow z = -1 \end{aligned}$$

  4. Substitute y into the first equation

$$x + 3(2) = 8 \rightarrow x = 2$$

$$x = 2, y = 2, z = -1$$

b. System #2

$$\begin{aligned} 2x + 3y - z &= -8 \\ x + 2y - z &= 2 \\ -x - 4y + z &= -6 \end{aligned}$$

- Using matrix inversion

```
##      [,1] [,2] [,3]
## [1,]    2    3   -1
## [2,]    1    2   -1
## [3,]   -1   -4    1
## [1] -8  2 -6
##      [,1] [,2] [,3]
## [1,]    1 -0.5  0.5
## [2,]    0 -0.5 -0.5
## [3,]    1 -2.5 -0.5
## [1] -12   2 -10
```

- Using systems of equations
  1. Add third row to first and second rows

$$\begin{aligned} x - y &= -14 \\ -2y &= -4 \\ -x - 4y + z &= -6 \end{aligned}$$

  2. Solve for y

$$-2y = -4 \rightarrow y = 2$$

  3. Substitution

$$\begin{aligned} x - 2 = -14 &\rightarrow x = -12 \\ -(-12) - 4(2) + z = -6 &\rightarrow z = -10 \end{aligned}$$

$$x = -12, y = 2, z = -10$$

c. System #3

$$\begin{aligned} x - y + 2z &= 2 \\ 4x + y - 2z &= 10 \\ x + 3y + z &= 0 \end{aligned}$$

- Using matrix inversion

```
##      [,1] [,2] [,3]
## [1,]    1   -1    2
## [2,]    4    1   -2
## [3,]    1    3    1
## [1]  2 10  0
##        [,1]    [,2]     [,3]
## [1,]  0.200  0.2000 1.39e-17
## [2,] -0.171 -0.0286 2.86e-01
## [3,]  0.314 -0.1143 1.43e-01
## [1]  2.400 -0.629 -0.514
```

- Using systems of equations

  1. Add row 1 to row 2

$$\begin{aligned} x - y + 2z &= 2 \\ 5x &= 12 \\ x + 3y + z &= 0 \end{aligned}$$

  2. Solve for x

$$5x = 12 \rightarrow x = \frac{12}{5}$$

  3. Plug in $x = 2$ and add row 1 x 3 to row 3

$$\begin{aligned} \frac{12}{5} - y + 2z &= 2 \\ 4\left(\frac{12}{5}\right) + 7z &= 6 \end{aligned}$$

  4. Solve for z

$$4\left(\frac{12}{5}\right) + 7z = 6 \rightarrow z = -\frac{18}{35}$$

  5. Solve for y

$$\frac{12}{5} - y + 2\left(-\frac{18}{35}\right) = 2 \rightarrow y = -\frac{22}{35}$$

$$x = \frac{12}{5}, y = -\frac{22}{35}, z = -\frac{18}{35}$$

6.5 Multiplying by 0

When it comes to real numbers, we know that if $xy = 0$, then either $x = 0$ or $y = 0$ or both. One might believe that a similar idea applies to matricies, but one would be wrong. Prove that if the matrix product $\mathbf{AB} = \mathbf{0}$ (by which we mean a matrix of appropriate dimensionality made up entirely of zeroes), then it is not necessarily true that either $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$. Hint: in order to prove that something is not always true, simply identify one example where $\mathbf{AB} = \mathbf{0},\ \mathbf{A}, \mathbf{B} \neq \mathbf{0}$.

Solution: Generally speaking, it is easy to show that something is *not* necessarily true. All that is needed is a single counterexample! And in this case, there are infinitely many counterexamples. Here's one:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 1(1) + 1(-1) & 1(1) + 1(-1) \\ 1(-1) + 1(1) & 1(-1) + 1(1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Chapter 7

Functions of several variables and optimization with several variables

7.1 Find first partial derivatives

Find all of the first partial derivatives of each function.

a. $f(x, y) = 3x - 2y^4$

Solution: The partial derivatives here involve straightforward power rule. As you do these partial derivatives, get used to seeing all other variables that you're not interested in as constants. It will become more natural over time.

$$\frac{\partial f}{\partial x} = 3 \qquad \frac{\partial f}{\partial y} = -8y^3$$

b. $f(x, y) = x^5 + 3x^3y^2 + 3xy^4$

Solution: More power rule.

$$\frac{\partial f}{\partial x} = 5x^4 + 9x^2y^2 + 3y^4 \qquad \frac{\partial f}{\partial y} = 6x^3y + 12xy^3$$

c. $g(x, y) = xe^{3y}$

Solution: Now, we have to start dealing with the chain rule. Note that even though there is technically a product involving x and y here, at no point does either variable show up in both parts of the product. As such, we don't need product rule.

The partial derivative with respect to x is straightforward. The function is linear with respect to x, so the partial derivative is e^{3y}.

The partial derivative with respect to y is a bit more complicated.

$$\frac{\partial g}{\partial y} = xe^{3y} \cdot \frac{\partial}{\partial y}(3y) = xe^{3y} \cdot 3 = 3xe^{3y}$$

d. $k(x, y) = \frac{x-y}{x+y}$

Solution: We require the use of quotient rule here. (Or as we showed in a previous homework, you can rewrite this as a product.) Recall that the quotient rule for some generic $h(x) = \frac{f(x)}{g(x)}$ is

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

So, we can apply this formula to derive the function with respect to each variable to get:

$$\begin{aligned}
\frac{\partial k}{\partial x} &= \frac{(\frac{\partial}{\partial x}(x-y))(x+y) - (x-y)(\frac{\partial}{\partial x}(x+y))}{(x+y)^2} \\
&= \frac{(1)(x+y) - (x-y)(1)}{(x+y)^2} \\
&= \frac{x+y-x+y}{(x+y)^2} \\
&= \frac{2y}{(x+y)^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial k}{\partial y} &= \frac{(\frac{\partial}{\partial y}(x-y))(x+y) - (x-y)(\frac{\partial}{\partial y}(x+y))}{(x+y)^2} \\
&= \frac{(-1)(x+y) - (x-y)(1)}{(x+y)^2} \\
&= \frac{-x-y-x+y}{(x+y)^2} \\
&= -\frac{2x}{(x+y)^2}
\end{aligned}$$

e. $f(x, y, z) = \log(x + 2y + 3z)$

Solution: This requires relatively straightforward chain rule.

$$\frac{\partial f}{\partial x} = \frac{1}{x+2y+3z} \cdot \frac{\partial}{\partial x}(x+2y+3z) = \frac{1}{x+2y+3z}(1) = \frac{1}{x+2y+3z}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x+2y+3z} \cdot \frac{\partial}{\partial y}(x+2y+3z) = \frac{1}{x+2y+3z}(2) = \frac{2}{x+2y+3z}$$

$$\frac{\partial f}{\partial z} = \frac{1}{x+2y+3z} \cdot \frac{\partial}{\partial z}(x+2y+3z) = \frac{1}{x+2y+3z}(3) = \frac{3}{x+2y+3z}$$

f. $h(x, y, z) = x^2 e^{yz}$

Solution: This requires some chain rule, as well.

$$\frac{\partial h}{\partial x} = 2x^{(2-1)}e^{yz} = 2xe^{yz}$$

$$\frac{\partial h}{\partial y} = x^2 e^{yz} \cdot \frac{\partial}{\partial y}(yz) = x^2 e^{yz} \cdot z = x^2 z e^{yz}$$

$$\frac{\partial h}{\partial z} = x^2 e^{yz} \cdot \frac{\partial}{\partial z}(yz) = x^2 e^{yz} \cdot y = x^2 y e^{yz}$$

7.2 Find the gradient

Find the gradient ∇f of the following functions and evaluate them at the given points.

a. $f(x, y) = \sqrt{x^2 + y^2}, \quad (x, y) = (3, 4)$

Solution: Let's first rewrite the function so that it is easier to differentiate.

$$f(x, y) = \sqrt{x^2 + y^2} = (x^2 + y^2)^{1/2}$$

Now, taking the partial derivatives with respect to x and y comes more naturally.

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{1}{2}(x^2+y^2)^{-1/2} \cdot \frac{\partial}{\partial x}(x^2+y^2) \\
&= \frac{1}{2}(x^2+y^2)^{-1/2} \cdot (2x) \\
&= x(x^2+y^2)^{-1/2} \\
&= \frac{x}{\sqrt{x^2+y^2}}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{1}{2}(x^2+y^2)^{-1/2} \cdot \frac{\partial}{\partial y}(x^2+y^2) \\
&= \frac{1}{2}(x^2+y^2)^{-1/2} \cdot (2y) \\
&= y(x^2+y^2)^{-1/2} \\
&= \frac{y}{\sqrt{x^2+y^2}}
\end{aligned}$$

So, the gradient of this function is:

$$\nabla f(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$$

And if we evaluate it at the given point, we get:

$$\nabla f(3,4) = \left(\frac{3}{\sqrt{3^2+4^2}}, \frac{4}{\sqrt{3^2+4^2}}\right) = \left(\frac{3}{\sqrt{9+16}}, \frac{4}{\sqrt{9+16}}\right) = \left(\frac{3}{\sqrt{25}}, \frac{4}{\sqrt{25}}\right) = \left(\frac{3}{5}, \frac{4}{5}\right)$$

b. $f(x,y,z) = (x+z)e^{x-y}, \quad (x,y,z) = (1,1,1)$

Solution: This question is slightly more involved; the partial derivative with respect to x will require product rule since x appears in both factors of the product. Recall that the product rule for a generic $h(x) = j(x)k(x)$ is $h'(x) = f'(x)g(x) + f(x)g'(x)$. The partial derivative with respect to y will require the chain rule.

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \left(\frac{\partial}{\partial x}(x+z)\right) e^{x-y} + (x+z)\left(\frac{\partial}{\partial x} e^{x-y}\right) \\
&= (1) \cdot e^{x-y} + (x+z) \cdot e^{x-y} \\
&= e^{x-y} + (x+z) \cdot e^{x-y} \\
&= e^{x-y}(x+z+1)
\end{aligned}$$

(The partial derivative above technically requires chain rule where you take the derivative of $x - y$ with respect to x, but that's just 1, so we omit that work here.)

$$\begin{aligned}\frac{\partial f}{\partial y} &= (x+z)\cdot e^{x-y}\cdot\left(\frac{\partial}{\partial y}(x-y)\right)\\ &= (x+z)\cdot e^{x-y}\cdot(-1)\\ &= -e^{x-y}(x+z)\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{\partial}{\partial z}(xe^{x-y} + ze^{x-y})\\ &= \frac{\partial}{\partial z}(xe^{x-y}) + \frac{\partial}{\partial z}(ze^{x-y})\\ &= 0 + e^{x-y}\\ &= e^{x-y}\end{aligned}$$

The gradient of this function, which we can also write as a vertical vector, is

$$\nabla f(x,y,z) = \begin{bmatrix} e^{x-y}(x+z+1) \\ -e^{x-y}(x+z) \\ e^{x-y} \end{bmatrix}$$

When we evaluate the gradient at the given value, we obtain

$$\nabla f(1,1,1) = \begin{bmatrix} e^{1-1}(1+1+1) \\ -e^{1-1}(1+1) \\ e^{1-1} \end{bmatrix} = \begin{bmatrix} e^0(3) \\ -e^0(2) \\ e^0) \end{bmatrix} = \begin{bmatrix} 1\cdot 3 \\ -1\cdot 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

7.3 Find the Hessian

Find the Hessian H for the following functions.

a. $g(x,y) = x^4 - 3x^2y^3$

Solution: In order to find any second partial derivatives, we need the first partial derivatives. These are straightforward. (Here, we'll use the other partial derivative notation just to keep things interesting and to get you accustomed to it.)

$$g_x = 4x^3 - 6xy^3 \qquad g_y = -9x^2y^2$$

Now, we find the second partial derivatives, which are also pretty simple to find.

$$g_{xx} = 12x^2 - 6y^3 \qquad g_{xy} = -18xy^2 \qquad g_{yx} = -18xy^2 \qquad g_{yy} = -18x^2y$$

Note that $f_{xy} = f_{yx}$. We now have everything we need for the Hessian.

$$H = \begin{bmatrix} 12x^2 - 6y^3 & -18xy^2 \\ -18xy^2 & -18x^2y \end{bmatrix}$$

b. $f(x, y, z) = xyz - x^2$

Solution: Finding this Hessian initially seems daunting because it involves three variables. Fortunately, the second derivatives are all really simple. We start by finding first partial derivatives.

$$f_x = yz - 2x \qquad f_y = xz \qquad f_z = xy$$

Now, we look for the second partial derivatives. Just to keep things orderly, let's start with f_{xx}, f_{yy}, and f_{zz}.

$$f_{xx} = -2 \qquad f_{yy} = 0 \qquad f_{zz} = 0$$

Then we can look for the other second partial derivatives that involve two different variables.

$$f_{xy} = \frac{\partial}{\partial y}(yz - 2x) = z \qquad f_{yz} = \frac{\partial}{\partial z}(xz) = x \qquad f_{zx} = \frac{\partial}{\partial x}(xy) = y$$

$$f_{yx} = \frac{\partial}{\partial x}(xz) = z \qquad f_{zy} = \frac{\partial}{\partial y}(xy) = x \qquad f_{xz} = \frac{\partial}{\partial z}(yz - 2x) = y$$

We have found all second derivatives, so we can now produce the Hessian.

$$H = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} = \begin{bmatrix} -2 & z & y \\ z & 0 & x \\ y & x & 0 \end{bmatrix}$$

7.4 Find the critical points

Find the local minimum values, local maximum values, and saddle point(s) of the function. Remember the process we discussed in class: Calculate the gradient, set it equal to zero to solve the system of equations, calculate the Hessian, and assess the Hessian at critical values. Be sure to show your work on each of these steps.

a. $f(x, y) = x^4 + y^4 - 4xy + 2$

Solution: The first step, as the problem indicates, is to determine the gradient of the function. Taking the first derivatives here is quite straightforward.

$$\begin{aligned}
\frac{\partial}{\partial x}(x^4 + y^4 - 4xy + 2) &= 4x^3 + 0 - 4y + 0 \\
&= 4x^3 - 4y \\
\frac{\partial}{\partial y}(x^4 + y^4 - 4xy + 2) &= 0 + 4y^3 - 4x + 0 \\
&= 4y^3 - 4x \\
\nabla f(x, y) &= (4x^3 - 4y, 4y^3 - 4x)
\end{aligned}$$

Now we must set this gradient equal to the zero vector. So we know $4x^3 - 4y = 0$ and $4y^3 - 4x = 0$. The standard method for solving a system of equations is to solve one equation for one variable in terms of the other(s) and substitute that value into the other equations. In this case, let's choose to solve the second equation for x in terms of y. So we have $4y^3 - 4x = 0$. Dividing both sides by 4 gives $y^3 - x = 0$, and adding x to both sides gives $y^3 = x$. Now let's plug this into the other equation. So we have $4(y^3)^3 - 4y = 4y^9 - 4y = 0$. Dividing by 4 gives $y^9 - y = 0$. There are a few ways to go about looking at this equation to get a value for y, but let's take the most rigorous approach. First, let us try to simplify a bit; *if y is not equal to 0*, we can divide by y to get $y^8 - 1 = 0$, or $y^8 = 1$. So taking the square root of both sides gives $y^4 = \pm 1$, but it obviously cannot be -1, so $y^4 = 1$. Doing so again gives $y^2 = \pm 1$, but again, it cannot be that $y^2 = -1$, so $y^2 = 1$. Taking the square root one last time gives $y = \pm 1$. Both of these values are feasible. Since we know that $x = y^3$, we know that (1,1) and (-1, -1) are critical points of this function.

Are these the only ones, though? Well, not necessarily. Recall that in finding those critical points, we had to divide by y, which we cannot do when $y = 0$. In other words, we made an assumption that y was not 0 - which means we only found the non-zero values of y that produce critical points. Is there a critical point where $y = 0$? In this case, yes - if both y and x are zero, the gradient is zero as well. So (0,0) is also a critical point.

Now let's calulate the Hessian.

$$H(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x}(4x^3-4y) & \frac{\partial}{\partial x}(4y^3-4x) \\ \frac{\partial}{\partial y}(4x^3-4y) & \frac{\partial}{\partial y}(4y^3-4x) \end{bmatrix} = \begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}$$

Now we simply have to plug in the x and y values at the critical points and apply the second derivative test.

$$H(1,1) = H(-1,-1) = \begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix}$$

So for the critical points $(1,1)$ and $(-1,-1)$, $AC - B^2 = 12 * 12 - (-4)^2 = 128 > 0$ and $A = 12 > 0$. So the Hessian at these points is positive definite, and the points $(1,1)$ and $(-1,-1)$ are local minima.

$$H(0,0) = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix}$$

So for the critical point $(0,0)$, $AC - B^2 = 0 * 0 - (-4)^2 = -16 < 0$, so the Hessian is indefinite and the point $(0,0)$ is a saddle point.

b. $k(x,y) = (1+xy)(x+y)$

Solution: As in the first problem, we begin by finding the gradient. To simplify the derivation process, note that $(1+xy)(x+y) = x^2y+xy^2+x+y$

$$\frac{\partial}{\partial x}(x^2y+xy^2+x+y) = 2xy + y^2 + 1$$
$$\frac{\partial}{\partial y}(x^2y+xy^2+x+y) = x^2 + 2xy + 1$$
$$\nabla f(x,y) = (y^2+2xy+1, x^2+2xy+1)$$

Now we need to set this gradient equal to the zero vector and solve the system of equations (sorry for the awful algebra here!). The standard method of solving one equation for one variable in terms of the other and plugging back into the other equation will work here (and I will write out the algebra below), but there is a much easier way of doing it. Notice that if $x^2 + 2xy + 1 = 0$ and $y^2 + 2xy + 1 = 0$, then it must be the case that $x^2+2xy+1 = y^2+2xy+1$. Now we can subtract $2xy+1$ from both sides to get $x^2 = y^2$, or $x = \pm y$. This gives us two cases to consider: either $x = y$ or $x = -y$. Let's start with $x = y$. Plugging this value of x into the first equation gives us $y^2 + 2yy + 1 = 3y^2 + 1 = 0$. But then $3y^2 = -1$, which is impossible for any real value of y. So now let's look at $x = -y$. Then the first equation gives us $y^2 + 2(-y)y + 1 = -y^2 + 1 = 0$, which means that $y^2 = 1$, or $y = \pm 1$. If $y = 1$, we know $x = -y = -1$, and if $y = -1$, $x = -y = 1$. So $(-1, 1)$ and $(1, -1)$ are critical points of this function (you can confirm for yourselves that these points are solutions to the other equation as well).

The longer way to do it is as follows. First, let's solve the first equation for x in terms of y. So we begin with $y^2 + 2xy + 1 = 0$. Subtract $2xy$ to get $y^2 + 1 = -2xy$. Clearly, we will want to divide by y, but first, let's check what happens when $y = 0$. If $y = 0$, then $y^2 + 1 = 1$ and $-2xy = 0$ for all x - so there are no solutions. So now, let's divide both sides of $y^2 + 1 = -2xy$ by $-2y$. Then we have $x = -\frac{y^2+1}{2y}$. Plugging this value into the second equation gives us $(-\frac{y^2+1}{2y})^2 + 2y(-\frac{y^2+1}{2y} + 1 = 0$. Distributing out the first term and cancelling out the $2y$ in the second leaves us with $\frac{y^4+2y^2+1}{4y^2} - y^2 = 0$. Now let's multiply both sides by $4y^2$. Then we have $y^4 + 2y^2 + 1 - 4y^4 = -3y^4 + 2y^2 + 1 = 0$. We can then apply the quadratic formula to get

$$y^2 = \frac{-2 \pm \sqrt{2^2 - 4(-3)(1)}}{2(-3} = \frac{-2 \pm \sqrt{16}}{-6} = \frac{-2 \pm 4}{-6}$$

Clearly, if $y^2 = \frac{-2+4}{-6} = -1/3$, y has no real values. So we must have $y^2 = \frac{-2-4}{-6} = 1$, or $y = \pm 1$, as before, with the same x values following.

Now that we have established $(-1, 1)$ and $(1, -1)$ as critical points, we must find the Hessian.

$$H(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x}(y^2 + 2xy + 1) & \frac{\partial}{\partial x}(x^2 + 2xy + 1) \\ \frac{\partial}{\partial y}(y^2 + 2xy + 1) & \frac{\partial}{\partial y}(x^2 + 2xy + 1) \end{bmatrix} = \begin{bmatrix} 2y & 2x + 2y \\ 2y + 2x & 2x \end{bmatrix}$$

Then we have

$$H(-1, 1) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

and

$$H(1, -1) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$

In both cases, $AC - B^2 = 2(-2) - 0^2 = -4 < 0$, so the Hessian is indefinite and these are saddle points. So there are no local minima or maxima.

7.5 Least squares regression

Suppose we were interested in learning about how years of schooling affect the probability that a person turns out to vote. To simplify things, let's say we just have one observation of each variable. Let Y be our single observation of the dependent variable (whether or not a person turns out to vote) and X be our single observation of the independent variable, (the number of years of education that same person has). We believe that the process used to generate our data takes the following form:

$$Y = \beta X + \epsilon$$

where ϵ is an error term. We include this error term because we think random occurrences in the world will mean our model produces estimates that are slightly wrong sometimes, but we believe that on average, this model accurately relates X to Y. We observe the values of X and Y, but what about β? How do we know the value of β that best approximates this relationship, i.e., what's the slope of this line?

There are different criteria we could use, but a popular choice is the method of least squares. In this process, we solve for the value of β that minimizes the sum of squared errors, ϵ^2, in our data. Using the tools of minimization we've been practicing, find the value of β that minimizes this quantity. (Hint: In this case there is only one observation, so the sum of squared errors is equal to the single error squared.)

Solution: First we must identify the quantity we are trying to minimize, the sum of squared errors. Using some simple algebra, we get:

$$\begin{aligned} Y &= \beta X + \epsilon \\ Y - \beta X &= \epsilon \\ (Y - \beta X)^2 &= \epsilon^2 \end{aligned}$$

If we have many data points, there would be a summation sign in front of both sides of this equation, but since there is only one data point here, we have the quantity of interest. We want to find the value of β that minimizes $(Y - \beta X)^2$, the squared error. Let's call this function $f(\beta)$.

The first step to minimizing a function is finding its first derivative with respect to the variable of interest, which is β in this case. For that we will need the chain rule, since this is a nested function. Define $f(\beta)$ as $h(g(\beta))$, where $h(u) = u^2$ and $g(u) = (Y - uX)$ for any u. So, by the chain rule,

$$\begin{aligned} f'(\beta) &= h'(g(\beta)) * g'(\beta) \\ &= 2(Y - \beta X) * -X \\ &= -2X(Y - \beta X) \end{aligned}$$

With our first derivative in hand, we need to solve for β^*, the value of β at which $f'(\beta) = 0$. Again, some algebra:

$$\begin{aligned} f'(\beta) = -2X(Y - \beta^* X) &= 0 \\ -2XY + 2X^2\beta^* &= 0 \\ -XY + X^2\beta^* &= 0 \\ X^2\beta^* &= XY \\ \beta^* &= \frac{XY}{X^2} \\ \beta^* &= \frac{Y}{X} \end{aligned}$$

Is this a minimum or a maximum? Remember, the first derivative will equal 0 in both cases. To find out, we perform the second derivative test.

$$\begin{aligned} f''(\beta) &= \frac{df'(\beta)}{d\beta} \\ &= \frac{d}{d\beta} - 2X(Y - \beta X) \\ &= \frac{d}{d\beta} - 2XY + 2X^2\beta \\ &= 2X^2 \end{aligned}$$

We know that the quantity $2X^2 > 0$, so the second derivative is positive, so we have found a minimum. We have set no bounds on the domain, and there is only one minimum (i.e. only one solution for β^*), so we are done. $\frac{Y}{X}$ is the value of β that minimizes the squared error. For any other value of β, the squared distance between the true outcome Y, and the predicted outcome, βX, would be larger, which means our model would not fit as well.

7.6 Least squares regression, refined

Following on the previous exercise, suppose we showed a colleague our model of voter turnout and she complained. “What a lame model”, our colleague said, “You definitely have to include an intercept term.” So in this problem we’ll follow our colleague’s advice and do just that.

Let Y be our single observation of the dependent variable (whether or not a person turned out to vote) and X_1 be our single observation of an independent variable, *education*, the number of years of schooling for this individual. Now though, we're also going to include an intercept term, β_0 in our model along with β_1 a coefficient that's associated with X_1.

This produces the following model for which we want to find the values of both β_0 and β_1 that minimize the sum of square errors.

$$Y = \beta_0 + \beta_1 X_1 + \epsilon$$

where ϵ is an error term.

Use the method of least squares to solve for the values of β_0, β_1 that minimizes the sum of squared errors in the our data. Using the tools of multivariate minimization we've been practicing, find the values of β_0 and β_1 that minimize this quantity.

Solution:

$$\begin{aligned}
Y &= \beta_0 + \beta_1 X_1 + \epsilon \\
Y - (\beta_0 + \beta_1 X_1) &= \epsilon \\
(Y - (\beta_0 + \beta_1 X_1))^2 &= \epsilon^2
\end{aligned}$$

$$\begin{aligned}
f(\beta_0, \beta_1 | x_i, y_i) &= (Y - \beta_0 - \beta_1 X_1)^2 \\
\frac{\partial f(\beta_0, \beta_1 | x_i, y_i)}{\partial \beta_0} &= -2(Y - \beta_0 - \beta_1 X_1) \\
&= -2Y + 2\beta_0 + 2\beta_1 X_1 \\
0 &= 2Y - 2\beta_0 - 2\beta_1 X_1 \\
\hat{\beta}_0 &= Y - \beta_1 X_1 \\
\frac{\partial f(\beta_0, \beta_1 | x_i, y_i)}{\partial \beta_1} &= -2X_1(Y - \beta_0 - \beta_1 X_1) \\
&= -2YX_1 + 2\beta_0 X_1 + 2\beta_1 X_1^2 \\
0 &= 2YX_1 - 2\beta_0 X_1 - 2\beta_1 X_1^2 \\
\beta_1 X_1^2 &= YX_1 - \beta_0 X_1
\end{aligned}$$

We then sub in $\hat{\beta}_0$ for β_0 and continue:

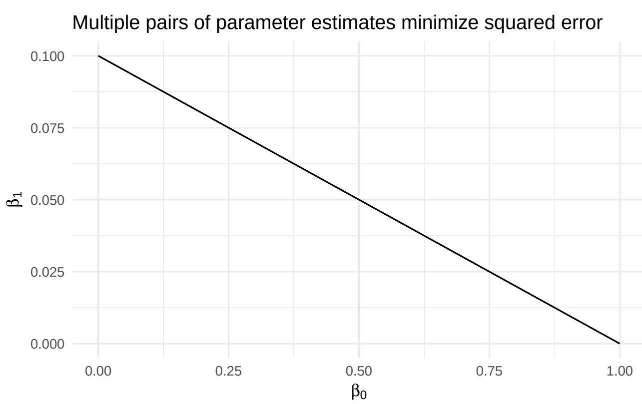
$$\begin{aligned}
\beta_1 X_1^2 &= YX_1 - (Y - \beta_1 X_1)X_1 \\
\beta_1 X_1^2 &= YX_1 - YX_1 + \beta_1 X_1^2 \\
\beta_1 X_1^2 &= \beta_1 X_1^2 \\
\beta_1 X_1^2 - \beta_1 X_1^2 &= 0 \\
0 &= 0
\end{aligned}$$

So, what's going on? Why can't we find a solution for $\hat{\beta}_1$?

The problem is that we are trying to use two parameters to describe a single observation of data. This is a case of a problem that occurs when solving systems of equations known as **underdetermination** (i.e., we have two unknown parameters, β_0 and β_1, but only one equation with which to estimate them).

The intuition behind this problem is that, given our data, we don't have enough information to uniquely identify two parameters. With one observation we could minimize the squared error by *either*:

1. Fitting the point exactly with our intercept term, β_0
2. Fitting the point exactly with the parameter on education β_1 or
3. Fitting the point with some linear combination of values for β_0 and β_1. Assuming this individual voted (y=1) and had 10 years of education (x=10), the line shows all the pairs of parameter estimates that would minimize the squared error with the two edge cases as end points (i.e., explain entirely with the intercept term or with the parameter attached to education) and a number of combinations of parameter values that also work in between (e.g., $\beta_0 = .5$ and $\beta_1 = .05$).



Since we have so many different combinations of parameter values that accomplish our goal, to minimize the squared error, equally well, no unique solution exists for this parameter vector. When this occurs we say our model is not identified.

If we start to get more observations than parameters we want to estimate (i.e., we have an overdetermined system of equations), this answer will change and we'll get unique solutions for each.

1. This problem illustrates, in a very simple manner, one symptom of underdetermination: perfect multicollinearity. Because we have a linearly dependent $\hat{\beta} = (X'X)^{-1}X'Y$.

$$
\begin{aligned}
X &= \begin{bmatrix} 1 & X_1 \end{bmatrix} \\
X'X &= \begin{bmatrix} 1 \\ X_1 \end{bmatrix} \begin{bmatrix} 1 & X_1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & X_1 \\ X_1 & X_1^2 \end{bmatrix}
\end{aligned}
$$

   Where we can show linear dependence by multiplying the first row in the matrix by X_1 to get an exact copy of the second row. We'll see that perfect multicollinearity can also occur even when we have more observations than variables, if we can write some variables as a linear combination of other variables.

2. If you got values for $\hat{\beta}_0$ and $\hat{\beta}_1$ on this problem you did something wrong (e.g., assumed more than 1 observation, made an algebra mistake).

Chapter 8

Integration and integral calculus

8.1 Definite integrals

Solve the following definite integrals using the antiderivative method.

For all these problems, the basic approach to compute the definite integral of $f(x)$ from a to b is by using the formula $F(b) - F(a)$, where $F(x)$ is the **antiderivative** of f.

a. $\int_6^8 x^3\,dx$
b. $\int_{-1}^0 (3x^2 - 1)\,dx$
c. $\int_0^1 x^{\frac{3}{7}}\,dx$
d. $\int_1^2 \left(\frac{3}{x^4} + 2\right)\,dx$
e. $\int_{-1}^1 (14 + x^2)\,dx$
f. $\int_1^2 \dfrac{1}{t^2}\,dt$
g. $\int_2^4 e^y\,dy$
h. $\int_8^9 2^x\,dx$
i. $\int_3^3 \sqrt{x^5 + 2}\,dx$

8.2 Applied integration

A group of three unidentified first-year graduate students at the University of Chicago are worn out after a week of math camp. Wanting to unwind, the students agree to not talk about math and decide to chat over some casual drinks at Medici.

After five shots of tequila each, two pitchers of beer, a bottle of wine, and a large Chicago-style pizza, the three students have had enough fun and decide to start the trip back home.

Student A gets on a bike and starts pedaling away at a velocity of $v_A(t) = 2t^4 + t$, where t represents minutes. However, the student crashes into the side of an Uber and ends the journey after only 2 minutes.

Student B has no bike, so starts running at a velocity of $v_B(t) = 4\sqrt{t}$. Sadly, after only 4 minutes, the student's legs give out and the student decides to sing a song, instead.

Student C can't even stand up, so has no choice but to slowly crawl at a velocity of $v_C(t) = 2e^{-t}$. Student C steadily plods along for 20 minutes before falling asleep on the sidewalk.

Generally, if an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$. The Fundamental Theorem of Calculus then tells us that

$$\begin{aligned} \text{Total distance traveled} &= \int_{t_1}^{t_2} v(t)\,dt \\ s(t_2) - s(t_1) &= \int_{t_1}^{t_2} v(t)\,dt \end{aligned}$$

Without using a calculator, use this formula to find the distance traveled by Students A, B, and C. (Assume, however unrealistic in may be, that all three students traveled in a straight line.) Who traveled the farthest? The least far?

8.3 Indefinite integrals

Calculate the following indefinite integrals:

a. $\int (x^2 - x^{-\frac{1}{2}})\,dx$
b. $\int 360t^6\,dt$
c. $\int 2x\log(x^2)\,dx$

8.4 Determining convergence

Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

a. $\int_1^\infty \left(\frac{1}{3x}\right)^2 \, dx$
b. $\int_0^\infty \cos(x) \, dx$
c. $\int_0^\infty e^{-x} \, dx$
d. $\int_{-\infty}^0 x^3 \, dx$

8.5 More integrals

Calculate the following integrals:

a. $\int_0^1 \int_2^3 x^2y^3 \; dxdy$
b. $\int_2^3 \int_0^1 x^2y^3 \; dydx$
c. $\int_0^1 \int_0^{\sqrt{1-x^2}} 2x^3y \; dydx$