# PSET 1: Linear equations, inequalities, sets and functions, quadratics (SOLUTIONS)

## 1 Simplify expressions

a. $(-x^2y^3)^3 = (-1)^3(x^2)^3(y^3)^3 = -x^6y^9$

b. $9(3^0) = 9(1) = 9$

c. $(3^2a^2)^2(6a^4) = (9a^2)^2(6a^4) = 81a^4 \cdot 6a^4 = 486a^8$

d. $\left(\frac{x^3}{x^4}\right)^3 = \left(x^{3-4}\right)^3 = (x^{-1})^3 = x^{-3} = \frac{1}{x^3}$

e. $(-2)^{(4-9)} = (-2)^{-5} = \frac{1}{(-2)^5} = -\frac{1}{32}$

f. $\left(\frac{1}{27b^4}\right)^{1/3} = \frac{1^{1/3}}{(27b^4)^{1/3}} = \frac{1}{3b^{4/3}}$

g. $y^5y^6y^5y^2 = y^{5+6+5+2} = y^{18}$

h. $\frac{13a/7b}{13b/2a} = \frac{13a}{7b} \cdot \frac{2a}{13b} = \frac{26a^2}{91b^2} = \frac{2a^2}{7b^2}$ (since 26 and 91 are divisible by 13)

## 2 Simplify a (more complex) expression

This expression follows the pattern of a perfect square: $(x+y)^2 = x^2 + y^2 + 2xy$. Let $x = (a+b)$ and $y = (a-b)$.

$$\begin{aligned}
&(a+b)^2 + (a-b)^2 + 2(a+b)(a-b) - 3a^2 \\
&= ((a+b) + (a-b))^2 - 3a^2 \\
&= (a+b+a-b)^2 - 3a^2 \\
&= (2a)^2 - 3a^2 \\
&= 4a^2 - 3a^2 = a^2
\end{aligned}$$

## 3 Graph sketching

The graphs for $f(x)$, $g(x)$, $f(g(x))$, and $g(f(x))$ are shown below.

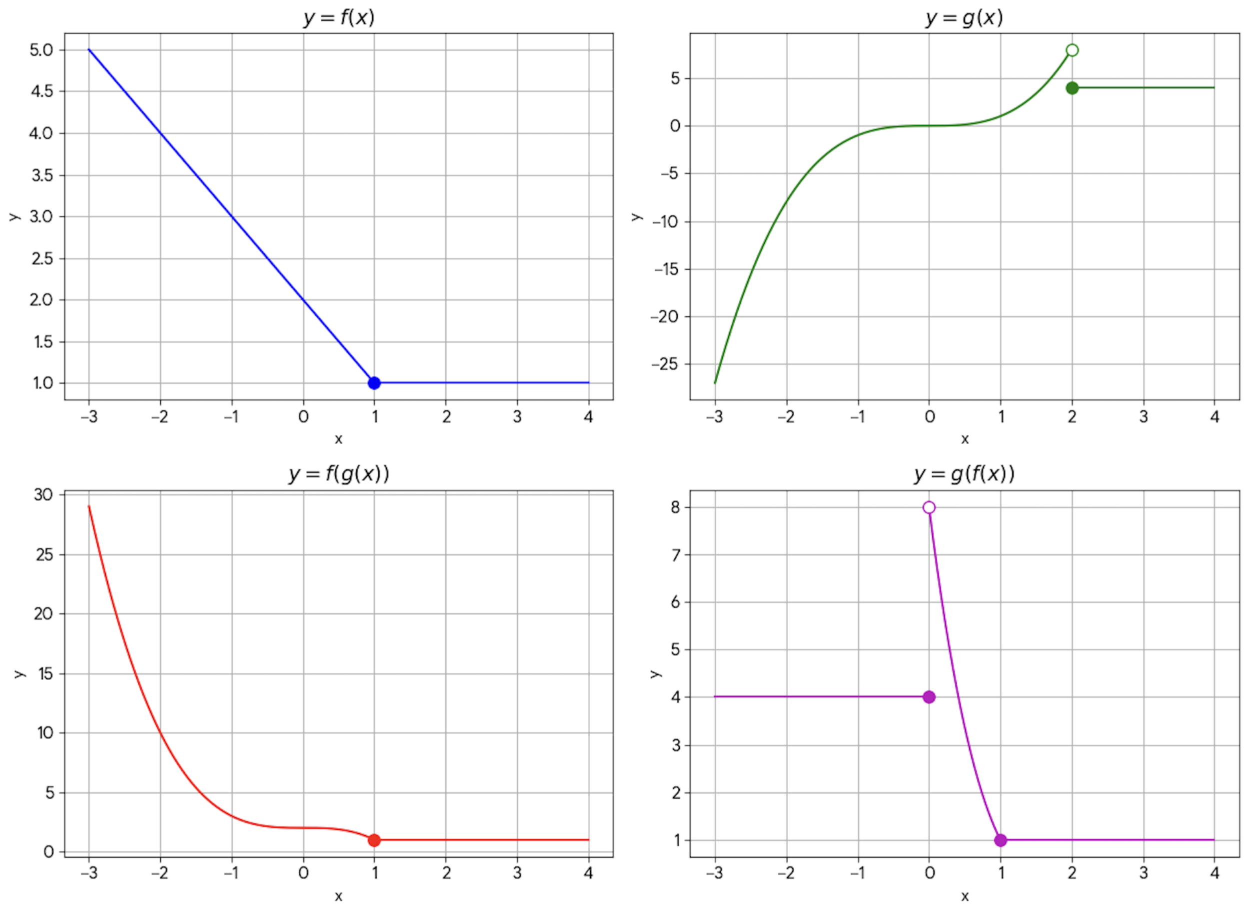


Figure 1: Sketches of the functions.

# 4 Root finding

a. $9x^2 - 3x - 12 = 0$. Using the quadratic formula with $a = 9, b = -3, c = -12$:

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(9)(-12)}}{2(9)} = \frac{3 \pm \sqrt{9 + 432}}{18} = \frac{3 \pm \sqrt{441}}{18} = \frac{3 \pm 21}{18}$$

The roots are $x_1 = \dfrac{3+21}{18} = \dfrac{24}{18} = \dfrac{4}{3}$ and $x_2 = \dfrac{3-21}{18} = \dfrac{-18}{18} = -1$.

b. $x^2 - 2x - 16 = 0$. Using the quadratic formula with $a = 1, b = -2, c = -16$:

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-16)}}{2(1)} = \frac{2 \pm \sqrt{4 + 64}}{2} = \frac{2 \pm \sqrt{68}}{2} = \frac{2 \pm 2\sqrt{17}}{2}$$

The roots are $x = 1 \pm \sqrt{17}$.

c. $6x^2 - 6x - 6 = 0$. First, divide by 6 to simplify: $x^2 - x - 1 = 0$. Using the quadratic formula with $a = 1, b = -1, c = -1$:

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{1 + 4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

The roots are $x = \dfrac{1 \pm \sqrt{5}}{2}$ (the golden ratio and its conjugate).

# 5 Systems of linear equations

a. The given system of equations is:

$$3x - 2y = 17$$
$$5x - 10y = -10$$

First, simplify the second equation by dividing by 5:

$$x - 2y = -2 \implies x = 2y - 2$$

Now, substitute this expression for $x$ into the first equation:

$$3(2y - 2) - 2y = 17$$

$$6y - 6 - 2y = 17$$

$$4y = 23 \implies y = \frac{23}{4}$$

Finally, substitute the value of $y$ back to find $x$:

$$x = 2\left(\frac{23}{4}\right) - 2 = \frac{23}{2} - \frac{4}{2} = \frac{19}{2}$$

The solution is $x = \frac{19}{2}, y = \frac{23}{4}$.

b. The system is:

$$(1) \quad 5x - 2y + 3z = 9$$
$$(2) \quad 2x - 4y - 3z = -9$$
$$(3) \quad x + 6y - 8z = 24$$

**Step 1: Eliminate $z$ to create a 2-variable system.** Add equation (1) and (2):

$$(5x + 2x) + (-2y - 4y) + (3z - 3z) = 9 - 9 \implies 7x - 6y = 0 \quad (4)$$

Multiply equation (1) by 8 and equation (3) by 3, then add them:

$$(40x - 16y + 24z) + (3x + 18y - 24z) = 72 + 72 \implies 43x + 2y = 144 \quad (5)$$

**Step 2: Solve the new system for $x$ and $y$.** From equation (4), we get $y = \frac{7}{6}x$. Substitute this into equation (5):

$$43x + 2\left(\frac{7}{6}x\right) = 144 \implies 43x + \frac{7}{3}x = 144 \implies \frac{129x + 7x}{3} = 144 \implies \frac{136}{3}x = 144$$

$$x = \frac{144 \cdot 3}{136} = \frac{432}{136} = \frac{54}{17}$$

Now find $y$: $y = \frac{7}{6}\left(\frac{54}{17}\right) = \frac{63}{17}$.

**Step 3: Substitute back to find $z$.** Using equation (1):

$$5\left(\frac{54}{17}\right) - 2\left(\frac{63}{17}\right) + 3z = 9 \implies \frac{270 - 126}{17} + 3z = 9 \implies \frac{144}{17} + 3z = \frac{153}{17}$$

$$3z = \frac{153 - 144}{17} = \frac{9}{17} \implies z = \frac{3}{17}$$

The solution is $x = \frac{54}{17}, y = \frac{63}{17}, z = \frac{3}{17}$.

c. Let $c$ be the number of cats, $d$ the number of dogs, and $r$ the number of rabbits. The problem gives the following system of equations:

$$c + d + r = 124$$
$$r = 2d - 4$$
$$c = d + 76$$

Substitute the expressions for $r$ and $c$ from the second and third equations into the first equation:

$$(d + 76) + d + (2d - 4) = 124$$

Now, solve for $d$:

$$4d + 72 = 124$$
$$4d = 52 \implies d = 13$$

Use the value of $d$ to find $c$ and $r$:

$$c = 13 + 76 = 89$$
$$r = 2(13) - 4 = 26 - 4 = 22$$

Thus, there are **89 cats, 13 dogs, and 22 rabbits**.

# 6 Work with sets

First, we list the elements of each set: $A = \{2, 3, 7, 9, 12, 13\}$ $B = \{6, 8, 10, 12\}$ $C = \{3, 5, 7, 11, 13, 17, 19, 23\}$ $D = \{1, 4, 9, 16, 25, \ldots\}$ (the set of perfect squares)

1. $A \cup B = \{2, 3, 6, 7, 8, 9, 10, 12, 13\}$
2. $(A \cup B) \cap C = \{3, 7, 13\}$
3. $C \cap D = \emptyset$. There is no number that is both a prime and a perfect square.

## 6.1 AI and Resources statement

As an AI, I generated these solutions based on my training data and internal computational abilities. I did not use external resources or collaborate with any individuals.