Functions of several variables and optimization with several variables

PSET 5/6

Note: all homework uploads should be as a PDF *and* have the questions identified. We'll be giving zero credit for submissions that don't follow this protocol as it adds considerable time to grading. Thank you!

Find first partial derivatives

Find all of the first partial derivatives of each function.[1]

a. $f(x, y) = 3x - 2y^4$

Solution: The partial derivatives here involve straightforward power rule. As you do these partial derivatives, get used to seeing all other variables that you're not interested in as constants. It will become more natural over time.

$$\frac{\partial f}{\partial x} = 3 \qquad \frac{\partial f}{\partial y} = -8y^3$$

b. $f(x, y) = x^5 + 3x^3y^2 + 3xy^4$

Solution: More power rule.

$$\frac{\partial f}{\partial x} = 5x^4 + 9x^2y^2 + 3y^4 \qquad \frac{\partial f}{\partial y} = 6x^3y + 12xy^3$$

c. $g(x, y) = xe^{3y}$

Solution: Now, we have to start dealing with the chain rule. Note that even though there is technically a product involving x and y here, at no point does either variable show up in both parts of the product. As such, we don't need product rule.

The partial derivative with respect to x is straightforward. The function is linear with respect to x, so the partial derivative is e^{3y}.

The partial derivative with respect to y is a bit more complicated.

[1] Grimmer HW6.3

$$\frac{\partial g}{\partial y} = xe^{3y} \cdot \frac{\partial}{\partial y}(3y) = xe^{3y} \cdot 3 = 3xe^{3y}$$

d. $k(x, y) = \frac{x-y}{x+y}$

Solution: We require the use of quotient rule here. (Or as we showed in a previous homework, you can rewrite this as a product.) Recall that the quotient rule for some generic $h(x) = \frac{f(x)}{g(x)}$ is

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

So, we can apply this formula to derive the function with respect to each variable to get:

$$\begin{aligned}
\frac{\partial k}{\partial x} &= \frac{(\frac{\partial}{\partial x}(x-y))(x+y) - (x-y)(\frac{\partial}{\partial x}(x+y))}{(x+y)^2} \\
&= \frac{(1)(x+y) - (x-y)(1)}{(x+y)^2} \\
&= \frac{x+y-x+y}{(x+y)^2} \\
&= \frac{2y}{(x+y)^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial k}{\partial y} &= \frac{(\frac{\partial}{\partial y}(x-y))(x+y) - (x-y)(\frac{\partial}{\partial y}(x+y))}{(x+y)^2} \\
&= \frac{(-1)(x+y) - (x-y)(1)}{(x+y)^2} \\
&= \frac{-x-y-x+y}{(x+y)^2} \\
&= -\frac{2x}{(x+y)^2}
\end{aligned}$$

e. $h(x, y, z) = x^2 e^{yz}$

Solution: This requires some chain rule, as well.

$$\frac{\partial h}{\partial x} = 2x^{(2-1)} e^{yz} = 2xe^{yz}$$

$$\frac{\partial h}{\partial y} = x^2 e^{yz} \cdot \frac{\partial}{\partial y}(yz) = x^2 e^{yz} \cdot z = x^2 z e^{yz}$$

$$\frac{\partial h}{\partial z} = x^2 e^{yz} \cdot \frac{\partial}{\partial z}(yz) = x^2 e^{yz} \cdot y = x^2 y e^{yz}$$

Find the gradient

Find the gradient ∇f of the following functions and evaluate them at the given points.[2]

a. $f(x, y) = \sqrt{x^2 + y^2}, \quad (x, y) = (3, 4)$

Solution: Let's first rewrite the function so that it is easier to differentiate.

$$f(x, y) = \sqrt{x^2 + y^2} = (x^2 + y^2)^{1/2}$$

Now, taking the partial derivatives with respect to x and y comes more naturally.

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot \frac{\partial}{\partial x}(x^2 + y^2) \\
&= \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot (2x) \\
&= x(x^2 + y^2)^{-1/2} \\
&= \frac{x}{\sqrt{x^2 + y^2}}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot \frac{\partial}{\partial y}(x^2 + y^2) \\
&= \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot (2y) \\
&= y(x^2 + y^2)^{-1/2} \\
&= \frac{y}{\sqrt{x^2 + y^2}}
\end{aligned}$$

So, the gradient of this function is:

$$\nabla f(x, y) = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right)$$

And if we evaluate it at the given point, we get:

$$\nabla f(3, 4) = \left(\frac{3}{\sqrt{3^2 + 4^2}}, \frac{4}{\sqrt{3^2 + 4^2}} \right) = \left(\frac{3}{\sqrt{9 + 16}}, \frac{4}{\sqrt{9 + 16}} \right) = \left(\frac{3}{\sqrt{25}}, \frac{4}{\sqrt{25}} \right) = \left(\frac{3}{5}, \frac{4}{5} \right)$$

b. $f(x, y, z) = (x + z)e^{x-y}, \quad (x, y, z) = (1, 1, 1)$

Solution: This question is slightly more involved; the partial derivative with respect to x will require product rule since x appears in both factors of the product. Recall that the product rule for a generic $h(x) = j(x)k(x)$ is $h'(x) = f'(x)g(x) + f(x)g'(x)$. The partial derivative with respect to y will require the chain rule.

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \left(\frac{\partial}{\partial x}(x + z) \right) e^{x-y} + (x + z) \left(\frac{\partial}{\partial x} e^{x-y} \right) \\
&= (1) \cdot e^{x-y} + (x + z) \cdot e^{x-y} \\
&= e^{x-y} + (x + z) \cdot e^{x-y} \\
&= e^{x-y}(x + z + 1)
\end{aligned}$$

[2] Grimmer HW6.4

(The partial derivative above technically requires chain rule where you take the derivative of $x - y$ with respect to x, but that's just 1, so we omit that work here.)

$$\begin{aligned}\frac{\partial f}{\partial y} &= (x+z)\cdot e^{x-y}\cdot\left(\frac{\partial}{\partial y}(x-y)\right)\\ &= (x+z)\cdot e^{x-y}\cdot(-1)\\ &= -e^{x-y}(x+z)\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{\partial}{\partial z}(xe^{x-y}+ze^{x-y})\\ &= \frac{\partial}{\partial z}(xe^{x-y})+\frac{\partial}{\partial z}(ze^{x-y})\\ &= 0+e^{x-y}\\ &= e^{x-y}\end{aligned}$$

The gradient of this function, which we can also write as a vertical vector, is

$$\nabla f(x,y,z) = \begin{bmatrix} e^{x-y}(x+z+1) \\ -e^{x-y}(x+z) \\ e^{x-y} \end{bmatrix}$$

When we evaluate the gradient at the given value, we obtain

$$\nabla f(1,1,1) = \begin{bmatrix} e^{1-1}(1+1+1) \\ -e^{1-1}(1+1) \\ e^{1-1} \end{bmatrix} = \begin{bmatrix} e^{0}(3) \\ -e^{0}(2) \\ e^{0}) \end{bmatrix} = \begin{bmatrix} 1\cdot 3 \\ -1\cdot 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

Find the Hessian

Find the Hessian H for the following functions.[3]

a. $g(x,y) = x^4 - 3x^2y^3$

Solution: In order to find any second partial derivatives, we need the first partial derivatives. These are straightforward. (Here, we'll use the other partial derivative notation just to keep things interesting and to get you accustomed to it.)

$$g_x = 4x^3 - 6xy^3 \qquad g_y = -9x^2y^2$$

Now, we find the second partial derivatives, which are also pretty simple to find.

$$g_{xx} = 12x^2 - 6y^3 \qquad g_{xy} = -18xy^2 \qquad g_{yx} = -18xy^2 \qquad g_{yy} = -18x^2y$$

Note that $f_{xy} = f_{yx}$. We now have everything we need for the Hessian.

$$H = \begin{bmatrix} 12x^2 - 6y^3 & -18xy^2 \\ -18xy^2 & -18x^2y \end{bmatrix}$$

[3]Grimmer HW7.3

b. $f(x,y,z) = xyz - x^2$

Solution: Finding this Hessian initially seems daunting because it involves three variables. Fortunately, the second derivatives are all really simple. We start by finding first partial derivatives.

$$f_x = yz - 2x \qquad f_y = xz \qquad f_z = xy$$

Now, we look for the second partial derivatives. Just to keep things orderly, let's start with f_{xx}, f_{yy}, and f_{zz}.

$$f_{xx} = -2 \qquad f_{yy} = 0 \qquad f_{zz} = 0$$

Then we can look for the other second partial derivatives that involve two different variables.

$$f_{xy} = \frac{\partial}{\partial y}(yz - 2x) = z \qquad f_{yz} = \frac{\partial}{\partial z}(xz) = x \qquad f_{zx} = \frac{\partial}{\partial x}(xy) = y$$

$$f_{yx} = \frac{\partial}{\partial x}(xz) = z \qquad f_{zy} = \frac{\partial}{\partial y}(xy) = x \qquad f_{xz} = \frac{\partial}{\partial z}(yz - 2x) = y$$

We have found all second derivatives, so we can now produce the Hessian.

$$H = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} = \begin{bmatrix} -2 & z & y \\ z & 0 & x \\ y & x & 0 \end{bmatrix}$$

Find the critical points

Find the local minimum values, local maximum values, and saddle point(s) of the function. Remember the process we discussed in class: Calculate the gradient, set it equal to zero to solve the system of equations, calculate the Hessian, and assess the Hessian at critical values. Be sure to show your work on each of these steps.[4]

a. $f(x,y) = x^4 + y^4 - 4xy + 2$

Solution: The first step, as the problem indicates, is to determine the gradient of the function. Taking the first derivatives here is quite straightforward.

$$\begin{aligned} \frac{\partial}{\partial x}(x^4 + y^4 - 4xy + 2) &= 4x^3 + 0 - 4y + 0 \\ &= 4x^3 - 4y \\ \frac{\partial}{\partial y}(x^4 + y^4 - 4xy + 2) &= 0 + 4y^3 - 4x + 0 \\ &= 4y^3 - 4x \\ \nabla f(x,y) &= (4x^3 - 4y, 4y^3 - 4x) \end{aligned}$$

Now we must set this gradient equal to the zero vector. So we know $4x^3 - 4y = 0$ and $4y^3 - 4x = 0$. The standard method for solving a system of equations is to solve one equation for one variable in terms of the other(s) and substitute that value into the other equations. In this case, let's choose to

[4]Grimmer HW7.4

solve the second equation for x in terms of y. So we have $4y^3 - 4x = 0$. Dividing both sides by 4 gives $y^3 - x = 0$, and adding x to both sides gives $y^3 = x$. Now let's plug this into the other equation. So we have $4(y^3)^3 - 4y = 4y^9 - 4y = 0$. Dividing by 4 gives $y^9 - y = 0$. There are a few ways to go about looking at this equation to get a value for y, but let's take the most rigorous approach. First, let us try to simplify a bit; *if y is not equal to 0*, we can divide by y to get $y^8 - 1 = 0$, or $y^8 = 1$. So taking the square root of both sides gives $y^4 = \pm 1$, but it obviously cannot be -1, so $y^4 = 1$. Doing so again gives $y^2 = \pm 1$, but again, it cannot be that $y^2 = -1$, so $y^2 = 1$. Taking the square root one last time gives $y = \pm 1$. Both of these values are feasible. Since we know that $x = y^3$, we know that (1,1) and (-1, -1) are critical points of this function.

Are these the only ones, though? Well, not necessarily. Recall that in finding those critical points, we had to divide by y, which we cannot do when $y = 0$. In other words, we made an assumption that y was not 0 - which means we only found the non-zero values of y that produce critical points. Is there a critical point where $y = 0$? In this case, yes - if both y and x are zero, the gradient is zero as well. So (0,0) is also a critical point.

Now let's calulate the Hessian.

$$H(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x}(4x^3 - 4y) & \frac{\partial}{\partial x}(4y^3 - 4x) \\ \frac{\partial}{\partial y}(4x^3 - 4y) & \frac{\partial}{\partial y}(4y^3 - 4x) \end{bmatrix} = \begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}$$

Now we simply have to plug in the x and y values at the critical points and apply the second derivative test.

$$H(1,1) = H(-1,-1) = \begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix}$$

So for the critical points $(1,1)$ and $(-1,-1)$, $AC - B^2 = 12 * 12 - (-4)^2 = 128 > 0$ and $A = 12 > 0$. So the Hessian at these points is positive definite, and the points $(1,1)$ and $(-1,-1)$ are local minima.

$$H(0,0) = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix}$$

So for the critical point $(0,0)$, $AC - B^2 = 0 * 0 - (-4)^2 = -16 < 0$, so the Hessian is indefinite and the point $(0,0)$ is a saddle point.

b. $k(x,y) = (1 + xy)(x + y)$

Solution: As in the first problem, we begin by finding the gradient. To simplify the derivation process, note that $(1 + xy)(x + y) = x^2y + xy^2 + x + y$

$$\frac{\partial}{\partial x}(x^2y + xy^2 + x + y) = 2xy + y^2 + 1$$
$$\frac{\partial}{\partial y}(x^2y + xy^2 + x + y) = x^2 + 2xy + 1$$
$$\nabla f(x,y) = (y^2 + 2xy + 1, x^2 + 2xy + 1)$$

Now we need to set this gradient equal to the zero vector and solve the system of equations (sorry for the awful algebra here!). The standard method of solving one equation for one variable in terms of the other and plugging back into the other equation will work here (and I will write out the algebra below), but there is a much easier way of doing it. Notice that if $x^2 + 2xy + 1 = 0$ and $y^2 + 2xy + 1 = 0$, then it must be the case that $x^2 + 2xy + 1 = y^2 + 2xy + 1$. Now we can subtract $2xy + 1$ from both sides to get $x^2 = y^2$, or $x = \pm y$. This gives us two cases to consider: either $x = y$ or $x = -y$. Let's start with $x = y$. Plugging this value of x into the first equation gives us $y^2 + 2yy + 1 = 3y^2 + 1 = 0$. But then $3y^2 = -1$, which is impossible for any real value of y. So now let's look at $x = -y$. Then the first equation gives us $y^2 + 2(-y)y + 1 = -y^2 + 1 = 0$, which means that $y^2 = 1$, or $y = \pm 1$. If $y = 1$, we know $x = -y = -1$, and if $y = -1$, $x = -y = 1$. So $(-1, 1)$ and $(1, -1)$ are critical points of this function (you can confirm for yourselves that these points are solutions to the other equation as well).

The longer way to do it is as follows. First, let's solve the first equation for x in terms of y. So we begin with $y^2 + 2xy + 1 = 0$. Subtract $2xy$ to get $y^2 + 1 = -2xy$. Clearly, we will want to divide by y, but first, let's check what happens when $y = 0$. If $y = 0$, then $y^2 + 1 = 1$ and $-2xy = 0$ for all x - so there are no solutions. So now, let's divide both sides of $y^2 + 1 = -2xy$ by $-2y$. Then we have $x = -\frac{y^2+1}{2y}$. Plugging this value into the second equation gives us $(-\frac{y^2+1}{2y})^2 + 2y(-\frac{y^2+1}{2y} + 1 = 0$. Distributing out the first term and cancelling out the $2y$ in the second leaves us with $\frac{y^4+2y^2+1}{4y^2} - y^2 = 0$. Now let's multiply both sides by $4y^2$. Then we have $y^4 + 2y^2 + 1 - 4y^4 = -3y^4 + 2y^2 + 1 = 0$. We can then apply the quadratic formula to get

$$y^2 = \frac{-2 \pm \sqrt{2^2 - 4(-3)(1)}}{2(-3} = \frac{-2 \pm \sqrt{16}}{-6} = \frac{-2 \pm 4}{-6}$$

Clearly, if $y^2 = \frac{-2+4}{-6} = -1/3$, y has no real values. So we must have $y^2 = \frac{-2-4}{-6} = 1$, or $y = \pm 1$, as before, with the same x values following.

Now that we have established $(-1, 1)$ and $(1, -1)$ as critical points, we must find the Hessian.

$$H(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x}(y^2 + 2xy + 1) & \frac{\partial}{\partial x}(x^2 + 2xy + 1) \\ \frac{\partial}{\partial y}(y^2 + 2xy + 1) & \frac{\partial}{\partial y}(x^2 + 2xy + 1) \end{bmatrix} = \begin{bmatrix} 2y & 2x + 2y \\ 2y + 2x & 2x \end{bmatrix}$$

Then we have

$$H(-1, 1) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

and

$$H(1, -1) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$

In both cases, $AC - B^2 = 2(-2) - 0^2 = -4 < 0$, so the Hessian is indefinite and these are saddle points. So there are no local minima or maxima.

Definite integrals

Solve the following definite integrals using the antiderivative method.[5]

For all these problems, the basic approach to compute the definite integral of $f(x)$ from a to b is by using the formula $F(b) - F(a)$, where $F(x)$ is the **antiderivative** of f.

a. $\int_6^8 x^3\,dx$

   Solution: Basic power rule. Or more so the reverse of the power rule for derivatives.

$$\begin{aligned}\int_6^8 x^3\,dx &= \left(\frac{1}{4}x^4\right)\Big|_6^8 \\ &= \frac{1}{4}8^4 - \frac{1}{4}6^4 \\ &= \frac{4096}{4} - \frac{1296}{4} \\ &= 1024 - 324 \\ &= 700\end{aligned}$$

b. $\int_{-1}^0 (3x^2 - 1)\,dx$

   Solution:

$$\begin{aligned}\int_{-1}^0 (3x^2 - 1)\,dx &= x^3 - x\Big|_{-1}^0 \\ &= (0^3 - 0) - ((-1)^3 - (-1)) \\ &= 0 - (-1 + 1) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

c. $\int_0^1 x^{\frac{3}{7}}\,dx$

   Solution: As a general rule, we can say that the antiderivative of x^n is, for values of n other than -1, $\frac{x^{n+1}}{n+1}$ plus a constant that we can safely ignore when doing definite integrals of this type. Then this integral evaluates as $\frac{x^{10/7}}{10/7}\Big|_0^1 = \frac{1^{10/7}}{10/7} - \frac{0^{10/7}}{10/7} = \frac{1}{10/7} = \frac{7}{10}$.

d. $\int_1^2 \frac{1}{t^2}\,dt$

   Solution:

$$\begin{aligned}\int_1^2 \frac{1}{t^2}\,dt &= \left(-\frac{1}{t}\right)\Big|_1^2 \\ &= -\frac{1}{2} - -\frac{1}{1} \\ &= -\frac{1}{2} + 1 \\ &= \frac{1}{2}\end{aligned}$$

[5] Gill 5.10 and Grimmer HW4.1

e. $\int_2^4 e^y\, dy$

Solution:

$$\begin{aligned}\int_2^4 e^y\, dy &= e^y\Big|_2^4 \\ &= e^4 - e^2 \\ &\approx 47.209\end{aligned}$$

f. $\int_8^9 2^x\, dx$

Solution: As a general rule, the antiderivative of a^x (where a is a constant) is $\frac{a^x}{\ln a}$ plus a constant.

$$\begin{aligned}\int_8^9 2^x\, dx &= \frac{2^x}{\log 2}\Big|_8^9 \\ &= \frac{2^9}{\log 2} - (\frac{2^8}{\log 2}) \\ &= \frac{2^8}{\log 2} \\ &= \frac{256}{\log 2} \\ &\approx 369.33\end{aligned}$$

g. $\int_3^3 \sqrt{x^5+2}\, dx$

Solution: This question is a bit sneaky. Trying to find the antiderivative of this function would be far from trivial. However, we do not actually need to do so! Notice that we are evaluating the integral from 3 to 3. Whatever the antiderivative function F actually is, $F(3) - F(3) = 0$, and 0 is the solution. Intuitively, this should make sense - in effect, we are taking the area under the curve at a single point. In other words, we are taking the area of a line, and a line has no area.

Applied integration

A group of three unidentified first-year graduate students at the University of Chicago are worn out after a week of math camp. Wanting to unwind, the students agree to not talk about math and decide to chat over some casual drinks at Medici.

After five shots of tequila each, two pitchers of beer, a bottle of wine, and a large Chicago-style pizza, the three students have had enough fun and decide to start the trip back home.

- Student A gets on a bike and starts pedaling away at a velocity of $v_A(t) = 2t^4 + t$, where t represents minutes. However, the student crashes into the side of an Uber and ends the journey after only 2 minutes.
- Student B has no bike, so starts running at a velocity of $v_B(t) = 4\sqrt{t}$. Sadly, after only 4 minutes, the student's legs give out and the student decides to sing a song, instead.
- Student C can't even stand up, so has no choice but to slowly crawl at a velocity of $v_C(t) = 2e^{-t}$. Student C steadily plods along for 20 minutes before falling asleep on the sidewalk.

Generally, if an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$. The Fundamental Theorem of Calculus then tells us that

$$\begin{aligned} \text{Total distance traveled} &= \int_{t_1}^{t_2} v(t)\,dt \\ s(t_2) - s(t_1) &= \int_{t_1}^{t_2} v(t)\,dt \end{aligned}$$

Without using a calculator, use this formula to find the distance traveled by Students A, B, and C. (Assume, however unrealistic in may be, that all three students traveled in a straight line.) Who traveled the farthest? The least far?[6]

Solution: For Student A:

$$\int_0^2 2t^4 + t = \frac{2}{5}t^5 + \frac{1}{2}t^2\bigg|_0^2 = \frac{2}{5}(2^5) + \frac{1}{2}(2^2) - \left[\frac{2}{5}(0^5) + \frac{1}{2}(0^2)\right] = \frac{2(32)}{5} + \frac{4}{2} - [0+0] = \frac{64}{5} + 2 = \frac{74}{5}$$

For Student B:

$$\int_0^4 4\sqrt{t} = 4\left(\frac{2}{3}\right)t^{3/2}\bigg|_0^4 = \frac{8}{3}(4^{3/2}) - \frac{8}{3}(0^{3/2}) = \frac{8}{3}\left(\sqrt{4^3}\right) - 0 = \frac{8}{3}\sqrt{64} = \frac{8}{3}(8) = \frac{64}{3}$$

For Student C:

$$\int_0^{20} 2e^{-t} = -2e^{-t}\bigg|_0^{20} = -2e^{-20} - [-2e^0] = -2e^{-20} + 2(1) = -\frac{2}{e^{20}} + 2 \approx 0 + 2 = 2$$

Clearly, Student C had the shortest trip. We can also eyeball that Student B traveled just a bit more than 20 units of distance, while A went a little less than 15. If you want concrete numbers to perform the comparison, you can calculate the quotients or find a common denominator for $\frac{64}{3}$ and $\frac{74}{5}$. You'd see that Student A went $\frac{222}{15}$ while B went $\frac{320}{15}$. Student B went the farthest. (But nobody made it home.)

[6] Grimmer HW4.2

Indefinite integrals

Calculate the following indefinite integrals:[7]

a. $\int (x^2 - x^{-\frac{1}{2}})\, dx$

$$\int (x^2 - x^{-\frac{1}{2}})\, dx = \frac{1}{3}x^3 - 2\sqrt{x} + c$$

b. $\int 360t^6\, dt$

$$\int 360t^6\, dt = \frac{360}{7}t^7 + c$$

c. $\int 2x \log(x^2)\, dx$

Solution: This integral requires **integration by parts**.

$$\int p'(x)q(x)dx = p(x)q(x) - \int p(x)q'(x)dx$$

$$\begin{aligned} p(x) &= x^2 \\ p'(x) &= 2x\, dx \\ q(x) &= \log(x^2) \\ q'(x) &= \frac{2}{x}\, dx \end{aligned}$$

$$\begin{aligned} \int p'(x)q(x)dx &= p(x)q(x) - \int p(x)q'(x)dx \\ &= x^2 \log(x^2) - \int x^2 \frac{2}{x}\, dx \\ &= x^2 \log(x^2) - \int 2x\, dx \\ &= x^2 \log(x^2) - x^2 + c \end{aligned}$$

Determining convergence

Determine whether each integral is convergent or divergent. Evaluate those that are convergent.[8]

a. $\int_1^\infty \left(\frac{1}{3x}\right)^2\, dx$

Solution: To find whether this integral converges and what it converges to, we must evaluate the limit $\lim_{t\to\infty} \int_1^t \frac{1}{9x^2}\, dx$. By standard rules of integration, this integral evaluates to $-\frac{1}{9x}\Big|_1^t = -\frac{1}{9t} - (-\frac{1}{9}) = \frac{1}{9}(1 - \frac{1}{t})$. As t goes to infinity, $\frac{1}{t}$ goes to 0. Therefore $\lim_{t\to\infty} \frac{1}{9}(1 - \frac{1}{t}) = \frac{1}{9}(1) = \frac{1}{9}$, and the integral converges to this value.

[7] Gill 5.13 and 5.14
[8] Grimmer HW 4.3

b. $\int_0^\infty \cos(x)\,dx$

Solution: Following the same procedure as before, we must evaluate the limit $\lim_{t\to\infty}\int_0^t \cos(x)\,dx$. This integral evaluates to $\sin(x)\Big|_0^t = \sin(t)-\sin(0)$. Since $\sin(0)=0$, this is equal to $\sin(t)$. But $\lim_{t\to\infty}\sin(t)$ does not exist - the value of the sine function oscillates continuously along the [-1,1] interval, and does not converge to any value as t goes to infinity. Therefore the integral diverges.

c. $\int_0^\infty e^{-x}\,dx$

Solution: Again, the limit we need to evaluate is $\lim_{t\to\infty}\int_0^t e^{-x}\,dx$. This integral evaluates to $-e^{-x}\Big|_0^t$ - technically, we would probably use the substitution rule here, but it is easy to see that this is the correct antiderivative even without it. This evaluation equals $-e^{-t}-(-e^0) = -e^{-t}+1$. As t goes to infinity, e^{-t} goes to 0. Then $\lim_{t\to\infty}(1-e^{-t}) = 1-0 = 1$, so this integral converges to 1.

d. $\int_{-\infty}^0 x^3\,dx$

Solution: The integral we need to evaluate here is slightly different, because we are dealing with an infinite lower bound on the integral rather than an infinite upper bound. The limit we must evaluate is $\lim_{t\to-\infty}\int_t^0 x^3\,dx$. This integral evaluates to $\frac{x^4}{4}\Big|_t^0 = 0-\frac{t^4}{4} = -\frac{t^4}{4}$. But it is obvious that the limit of t^4 does not exist as t goes to negative infinity; therefore the integral does not converge.

More integrals

Calculate the following integrals:[9]

a. $\int_0^1\int_2^3 x^2y^3\ dxdy$

$$\begin{aligned}\int_0^1\int_2^3 x^2y^3dxdy &= \int_0^1[\frac{1}{3}x^3y^3|_{x=2}^{x=3}]dy \\ &= \int_0^1[\frac{1}{3}3^3y^3-\frac{1}{4}2^3y^3]dy = \int_0^1\frac{19}{3}y^3dy \\ &= \frac{19}{12}y^4|_{y=0}^{y=1} = \frac{19}{12}\end{aligned}$$

b. $\int_2^3\int_0^1 x^2y^3\ dydx$

$$\begin{aligned}\int_2^3\int_0^1 x^2y^3dydx &= \int_2^3[\frac{1}{4}x^2y^4|_{y=0}^{y=1}]dx \\ &= \int_2^3[\frac{1}{4}1^4x^2-\frac{1}{4}0^4x^2]dx = \int_2^3\frac{1}{4}x^2dx \\ &= \frac{1}{12}x^3|_{x=2}^{x=3} = \frac{19}{12}\end{aligned}$$

[9]Grimmer HW7.5

A and B demonstrate the order of integration doesn't matter, you should get the same answer either way.

c. $\int_0^1 \int_0^{\sqrt{1-x^2}} 2x^3y \ dydx$

$$\begin{aligned}\int_0^1 \int_0^{\sqrt{1-x^2}} 2x^3ydydx &= \int_0^1 x^3y^2|_{y=0}^{y=\sqrt{1-x^2}} dx \\ &= \int_0^1 x^3(1-x^2)dx = \int_0^1 x^3 - x^5 dx \\ &= \frac{1}{4}x^4 - \frac{1}{6}x^6|_{x=0}^{x=1} = \frac{1}{12}\end{aligned}$$