PS and Sociology Math Prefresher

Math Camp

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Day TWO: Limits, Sequences, and Sets

Sequences

A sequence is an ordered list of numbers. They can be infinite or finite, but all are *countable*.

Lingo

We refer to the elements by their position in the sequence – the third element would be x_3. We can talk about the entire sequence as being generated by some equation or formula and represent it accordingly. So, if we take each element to the third, our sequence would be $\{1, 2^3, 3^3, ...\}$ and we could reference the sequence as $\{i^3\}_{i=1}^{\infty}$

Sequences and Series

Series

A series is the *sum* of a sequence. (Book reasoning unhelpful here – we care because you'll be adding probabilities in class).

Summation

We may have a large or otherwise complicated series of numbers to add. For example, suppose we wanted to add the numbers from 1-10. We could write the list out, (1,2, 3, ... , 10) or we could use the summation operator:

$$\sum_{n=1}^{10} n$$

Summation

Using sums can help us when we're summing a large number of items or when summing more complex functions.
Summation rules:

1. $\sum_{n=1}^{k} c = kc$ where c is a constant
2. $\sum_{n=1}^{k} n = \frac{k(k+1)}{2}$
3. $\sum_{n=1}^{k} 4 + 3n = \sum_{n=1}^{k} 4 + 3\sum_{n=1}^{k} n = 4k + 3\frac{k(k+1)}{2}$

Try:
$\sum_{n=1}^{5} 6$ Ans: $6 * 5 = 30$ $\qquad$ $\sum_{n=1}^{4} 2n + 3$ Ans:
$2 * (4 * 5)/2 + 3 * 4 = 32$

Limits

The sums on the previous page had *limits*: you can add elements and get an answer. We say that these series **converge** while those series that just keep getting bigger and bigger and bigger (or smaller and smaller and smaller) **diverge**.

$$lim_{N \to \infty} \sum_{i=1}^{N} x_i = S$$

Translation: The limit of the sum x_i from i to N as N approaches infinity is S.
We can talk in the same way about sequences $lim_{i \to \infty} x_i = L$.

Limits

Limits are also useful in calculus – when we take a derivative, we are essentially asking: "What is the slope of the line at this infinitesimally tiny part of the line"?

A second way to ask this is to look at what happens to the slope as as the distance between points approaches zero. However, to actually calculate the derivative, we need to first be sure that the point is differentiable. Again, we will use limits. (useful in PS 405 and Soc 401-1)

Behaving Badly

Must be *continuous* on the *interval* to be differentiable

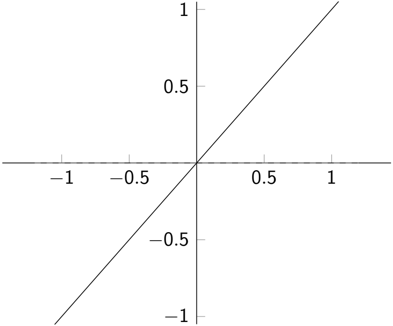
Some functions not differentiable or not differentiable at a certain point–not “well behaved” functions. To be continuous, must satisfy the following:

- Limit exists
- Limit from left equals limit from right
- These limits equal the value at the point

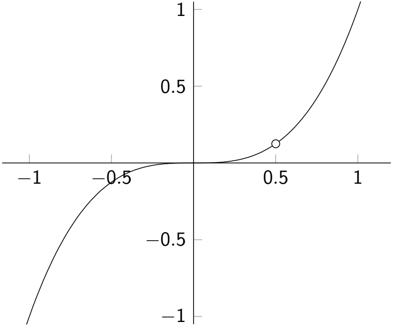
Continuity

Continuous function: draw without picking up a pencil

YES



NO



Limits

For us, you should be able to:

- Plug a value in to the limit and see what you get out (check for dividing by zero).
- Recognize an indeterminate limit: $\frac{0}{0}$ or $\frac{\infty}{\infty}$

We don't go beyond here for our calculations, however the book has a nice explanation on limits and the bounds of limits and l'Hopital's rule. This is nifty, but more firepower than we need.

Try: $lim_{x \to 4} \frac{x+2}{x+5}$ $\qquad$ $lim_{x \to 4} \frac{x^2-4}{x+2}$
Ans: 6/9, simplifies to 2/3 $\qquad$ simplifies to x-2; limit is 2.

SETS! & Matrices!

Sets: Numbers

Recall from yesterday:

- N: Natural numbers $\{(0), 1, 2, 3, ...\}$
- Z: Integers (negative and positive including zero) $\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$
- Q: Rational numbers (q for quotient, rational numbers, e.g. expressed as a fraction)
- R: Real numbers (positive, negative, zero, integers, fractions/rational); any point on the number line
- I: Imaginary numbers ($i = \sqrt{-1}$)
- C: Complex numbers ($a + bi$)

Notes:
Subscript: Z_+ only the positive or Q_- negative elements of the set
Superscript: N^2 dimensions of the space

Sets: displaying sets

- Roster notation (most common): $\{x_1, x_2, ...\}$
- Set builder notation (recipe): $\{x|x \in N, x < 4\}$

Try: $\{x \in Z, x = x^2\}$ "x in the set of (positive and negative)

integers such that x is equal to its own square" The only number that satisfies this is $\{1\}$

Sets, details

There are types of sets:

- Finite (finite number of elements) or infinite (no limit)
- Countable or uncountable (elements can be counted or not (e.g $\{0, 1\}$ is countable)

Sets, details

We care about sets and how elements are contained within them, and how the sets are shaped. We'll use this information in probability. It's also helpful when thinking about the possible responses and individuals who may fall in your dataset.

Open

Open sets–essentially the boundary is a little fuzzy. This is the set version of open brackets (). The more technical definition has to do with an 'epsilon ball' where, you can always nudge a little closer to the boundary of the set without actually reaching it.

Closed

Closed sets have a clear and firm boundary. This is the set version of closed brackets [ ]. Here, you can stand right on the edge of the set.

Open and closed sets matter because this affects how we think about the contents of sets – what we call the 'elements'.

Sets, details

- (Proper) Subset: elements of some sets are contained within a second set, e.g. even integers and integers. (proper subset: $\subset$, subset: $\subseteq$)
- Cardinality: number of elements
  - Singleton: one element
- Empty set: no elements.
  - $A = \emptyset$ is VERY different from $A = \{\emptyset\}$
- Ordered and unordered sets: ordered set has significance in order (preference ranking) while in unordered, no significance in ranking (e.g. membership roster)
- Universal set (the universe: all elements)

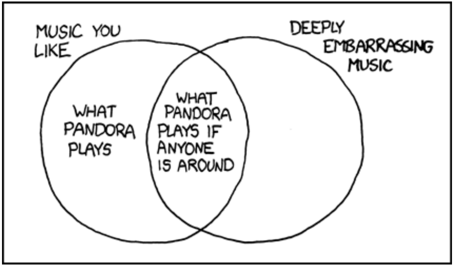
Sets: Union and Intersection

Suppose $A = \{4, hat\}$ and $B = \{hat, 7\}$

Union ($\cup$)
Union is the combination of elements in either set (OR): $A \cup B = \{4, hat, 7\}$

Intersection ($\cap$)
Intersection is the collection of elements present in both sets (AND): $A \cap B = \{hat\}$ (Def: $A \cap B = X : X \in A$ and $X \in B$)
Sometimes the intersection is empty.

Sets, details



Source: XKCD

Sets: Disjoint and Partitions

Disjoint

Disjoint sets are those where the intersection is empty.

Partition

Sometimes we may wish to partition a set – to do this, we want to ensure that we *cover* the space (all the elements in the set are assigned to a separate subset), but we also want to make sure that we don't double assign. A proper partition is one where the collection of sets are disjoint and their union is the entire set.

Example: How to partition countries? Could do by continents, population size, etc – but pay attention that, say Turkey, isn't assigned to two regions. Or that you don't forget Malta!

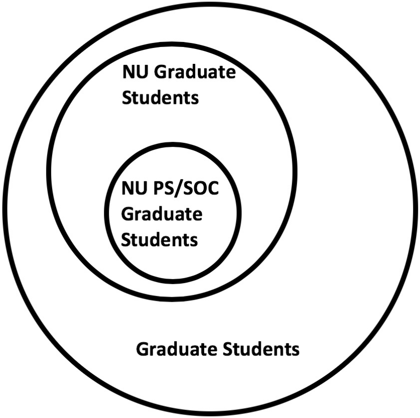
Sets: cont'd

Difference Difference between A and B, $A \backslash B$ ("A difference B") is the set containing all the elements of A that are not also in B. $x \in A$ but $x \notin B$

Complement Complement A' or A^c contains the elements that are not contained in A. $x \in A^c$ if $x \notin A$

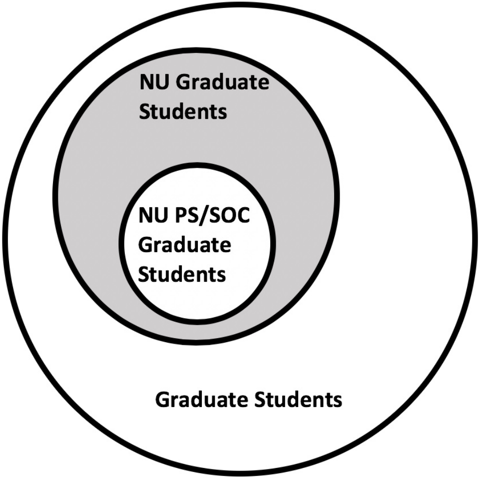
Sets: cont'd

Find NU GS$\setminus$PS/SOC Grad Students



Sets: cont'd

NU GS\PS/SOC Grad Students: AKA, where are the NU grad students who aren't in PS/SOC



Sets: Partition and Cartesian Product

- Mutually exclusive sets: intersection is empty (e.g. even and odd numbers)
- Partition: collection whose subsets form the universal set (e.g. citizens who are over/under 18). Numerous ways to partition (try)
- Cartesian product: set containing all possible ordered pairs (a,b) where a is from set A and b is from set B for any two sets.
    - AxB where $A = \{apple, banana, kiwi\}$ and $B = \{2, 4\}$.
      $AxB =$\{ \{apple, 2\}, \{apple, 4\}, \{banana,2\}, \{banana,4\}, \{kiwi, 2\}, \{kiwi, 4\} \}

Sets: Review

- Unions: OR, $\cup$
- Intersections: AND, $\cap$
- Ordered/Unordered
- Complements (not inside) c, written A^c, for example.
- Subsets and proper subsets (contained within): $\subset, \subseteq$
- Cardinality (number of elements)

Sets: Sample Spaces, applied

Suppose you have a deck of ten cards, numbered 1-10, and are dealt four cards. Answer the following questions, labelling them with the appropriate terms we've discussed so far:

- How many hands are possible? **this is the *universe* and we would calculate it using $\binom{10}{4}$**
- Does order matter? **No!**
- How would it affect the sample space? Would we have more or fewer possible hands? **Permutation vs Combination: we would have many more possible hands**
- Now, you can swap one card. How many ways could you do it? Does it matter what cards you have? **You could trade in any one of your 4 cards. In this scenario, it does not matter**

Sets: Sample Spaces, applied

Suppose you have a deck of ten cards, numbered 1-10, and are dealt four cards. Answer the following questions, labelling them with the appropriate terms we've discussed so far:

- Finally, what if I told you that all 4 of my cards were even numbered. Would you be surprised? How would you know whether to be surprised? We will discuss this, and many other interesting things!, in class! soon!

MATRICES!

Matrices: A basic introduction

- Not the most fun you'll ever have
- Not that scary once you get the hang of it
- A way of organizing things so you can do different types of operations on a large structure
- Can refer to a matrix as just the elements, or give it a name, like $[A]$ or $\mathbf{A}$

Matrices: overview

Filled with rows and columns – we refer to them by the number of rows and columns (e.g. 3 x 4 matrix). It can be a really efficient way to deal with data and to perform operations (like you would with a linear regression! woohoo!). It's also how your data will be organized any time you use a spreadsheet.

$$[A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The above matrix is a square matrix (same number of rows and columns) and each element is subscripted by its respective row and column number. Sometimes matrices are subscripted so you know their size. E.g $[A]_{2x2}$.

Matrices: Try it!

Suppose we have the following matrix. What are the dimensions of matrix [B] below, and what value is b_{23}?

$$\begin{bmatrix} 3 & 2 & 8 & 11 & 14 & 19 \\ 9 & 81 & 21 & 31 & 41 & 1 \\ 13 & 7 & 6 & 4 & 5 & 20 \end{bmatrix}$$

Dimensions are 3 x 6; b_{23} is 21.

Elements of Matrices: the diagonal

The elements along the diagonal often are important in matrices. We typically focus upon the diagonal that starts in the upper left and goes down to the lower right.

$$[A] = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \\ 0 & 0 & 6 \end{bmatrix} [B] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} [C] = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 5 & 4 & 6 \end{bmatrix} [D] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Types of Matrices

We won't get into the nitty gritty details of all sorts of different matrices but it might be helpful to know that there are 'special' matrices:

- Vector matrices (only one row (row vector) or column (column vector)
- Submatrix (subset of matrix)
- Triangular matrix: part of matrix is zeros – all bottom triangle zeros is upper triangular, all upper triangle zeros is lower triangular. (focus: where are the numbers?)
- Diagonal matrix: only the diagonal is non-zero
- Zero matrix: everything is zeros!
- Identity matrix: most important! all zeros except on diagonal AND diagonal is only ones...this is the matrix version of multiplying by 1
- Transpose (AKA transposition matrix): This is where you flip all the rows/columns. Meaning, if something was row 3, col 2, it will now be row 2, col 3. Denoted $[A]^T$. Done by 'reflecting' over the main diagonal (so the diagonal stays the same)

Identifying Matrices

What kinds are the matrices below? Also, notice their dimensions – they are all square. Square matrices tend to make the math nicer (it's all relative)

$$[A] = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \\ 0 & 0 & 6 \end{bmatrix} \, [B] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \, [C] = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 5 & 4 & 6 \end{bmatrix} \, [D] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A: Upper triangular, B: Zero matrix, C: Lower triangular, D: Identity matrix; Fun fact! A and C are the transpose of each other

Adding matrices

This is our last stop before things get too weird. Adding matrices (and subtracting) works exactly like you think it would: you need two matrices that have the same dimensions as each other. You then add the elements together (or subtract, as applicable). The final matrix has the same dimensions as the first two and everything is well and good.

$$\begin{bmatrix} 1 & 13 & 15 \\ 2 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 17 & 23 \\ 3 & 7 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 13 & 15 \\ 2 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 9 & 9 \\ 1 & 1 & 1 \end{bmatrix}$$

Multiplying Matrices

This might make your brain hurt a little, but this is the part where matrices get weird and complicated.

What you do is you take two matrices and to multiply the matrices you MULTIPLY things AND ADD THEM (!). To do this, the order really matters (you will not necessarily get the same thing if you do matrix A times matrix B if you were to do B times A, for example (although you might)). Thus, you need to have the matrices in the right order and the number of COLUMNS in the first matrix to be multiplied must equal the number of ROWS in the second matrix. (yes, that's right).

We'll start with two matrices, A is 1 x 2 and B is 2 x 3. Notice that it doesn't matter that they aren't exactly the same dimensions – only the middle two elements. The final matrix will have the outer numbers for the dimensions. Here, it will be 1 x 3.

Multiplying Matrices

$$[A] = \begin{bmatrix} 7 & 8 \end{bmatrix} \, [B] = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \end{bmatrix}$$

Let's multiply! You take the row of the first matrix, multiply it by the COLUMN (hence the need to match) of the second matrix, ADD the sum of these products, and that goes into the first cell of the 'final' matrix. Then you do the same thing for the next column. So, here, it would be:

$$[C] = \begin{bmatrix} 7*2+8*1 & 7*4+8*3 & 7*6+8*5 \end{bmatrix}$$

Note this is a row vector: $[C] = \begin{bmatrix} 22 & 52 & 82 \end{bmatrix}$

Multiplying Matrices: bigger matrices

If you have multiple rows in your initial matrix, you just do the same process over again, following the same procedure for each row. Your final matrix will have dimensions determined in the same way. For example, if you have a 2 x 3 and a 3 x 3, you'll have a 2 x 3 as your resulting matrix.

$$[A] = \begin{bmatrix} 7 & 8 \\ 1 & 2 \end{bmatrix} [B] = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \end{bmatrix}$$

$$[C] = \begin{bmatrix} 7*2+8*1 & 7*4+8*3 & 7*6+8*5 \\ 1*2+2*1 & 1*4+2*3 & 1*6+2*5 \end{bmatrix} = \begin{bmatrix} 22 & 52 & 82 \\ 4 & 10 & 16 \end{bmatrix}$$

Multiplying Matrices: Practice

$$[A] = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 2 & 8 \end{bmatrix} [B] = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \\ 0 & 0 & 6 \end{bmatrix} [C] = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 5 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix} [D] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Try the following:

- **A x B**
- **B x A**
- **A x C**
- **B x D**

Answers

- **A x B** $\begin{bmatrix} 1 & 9 & 41 \\ 2 & 10 & 46 \end{bmatrix}$
- **B x A** Not possible: 3 x 3 and 2 x 3 (middle numbers must match)
- **A x C** Not possible: 2 x 3 and 4 x 3 (middle numbers must match)
- **B x D** B (D is the identity matrix so you always get back whatever you multiplied it by)

Matrices: What you really need to know

- Identify a matrix (is it a matrix?)
- Determine dimensions of a matrix
- Add/Multiply simple matrices
- Understand that there's a whole rich world out there waiting (lurking?) for you re: matrices

Questions? How are we doing??