Functions and Notation

Computational Mathematics and Statistics Camp
University of Chicago
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1. Simplify the following expressions as much as possible:

   a. $(-x^4y^2)^2$

      1. Distribute exponents over products.
      $$(-1)^2x^{(2\times 4)}y^{(2\times 2)}$$
      2. Multiply 2 and 2 together.
      $$(-1)^2x^{(2\times 4)}y^4$$
      3. Multiply 2 and 4 together.
      $$(-1)^2x^8y^4$$
      4. Evaluate $(-1)^2$.
      $$x^8y^4$$

   b. $9(3^0)$

      1. Any nonzero number to the zero power is 1.
      $$9(1)$$
      2. Anything times 1 is the same value.
      $$9$$

   c. $(2a^2)(4a^4)$

      1. Combine products of like terms.
      $$2a^2 \times 4a^4 = 2 \times 4a^{(2+4)}$$
      2. Evaluate $2+4$.
      $$2 \times 4a^6$$
      3. Multiply 2 and 4 together.
      $$8a^6$$

d. $\dfrac{x^4}{x^3}$

1. For all exponents, $\dfrac{a^n}{a^m} = a^{(n-m)}$.

$$x^{(4-3)}$$

2. Evaluate $4 - 3$.

$$x$$

e. $(-2)^{7-4}$

1. Subtract 4 from 7.

$$(-2)^3$$

2. In order to evaluate 2^3 express 2^3 as 2×2^2.

$$-2 \times 2^2$$

3. Evaluate 2^2.

$$-2 \times 4$$

4. Multiply -2 and 4 together.

$$-8$$

f. $\left(\dfrac{1}{27b^3}\right)^{1/3}$

1. Separate component terms.

$$\frac{1}{27}^{1/3} \times \frac{1}{b^3}^{1/3}$$

2. Evaluate cube roots.

$$\frac{1}{3} \times \frac{1}{b}$$

3. Combine terms.

$$\frac{1}{3b}$$

g. $y^7y^6y^5y^4$

1. Combine products of like terms.

$$y^{(7+6+5+4)}$$

2. Evaluate $7 + 6 + 5 + 4$.

$$y^{22}$$

h. $\dfrac{2a/7b}{11b/5a}$

1. Write as a single fraction by multiplying the numerator by the reciprocal of the denominator.

$$\frac{2a}{7b} \times \frac{5a}{11b}$$

2. Product property of exponents: $x^a \times x^b = x^{(a+b)}$

$$\frac{5a \times 2a}{7b \times 11b} = \frac{5 \times 2a^{1+1}}{7 \times 11b^{1+1}}$$

3. Evaluate $1 + 1$.

$$\frac{5 \times 2a^2}{7 \times 11b^2}$$

4. Multiple scalars together.

$$\frac{10a^2}{77b^2}$$

i. $(z^2)^4$

1. Nested exponents rule: $(x^a)^b = x^{ab}$

$$z^{2\times 4}$$

2. Evaluate 2×4

$$z^8$$

2. Simplify the following expression:

$$(a + b)^2 + (a - b)^2 + 2(a + b)(a - b) - 3a^2$$

1. Expand $(a + b)^2$ with FOIL.

$$a^2 + 2ab + b^2 + (a - b)^2 + 2(a + b)(a - b) - 3a^2$$

2. Expand $(a - b)^2$ with FOIL.

$$a^2 + 2ab + b^2 + a^2 - 2ab + b^2 + 2(a + b)(a - b) - 3a^2$$

3. Multiply $a + b$ and $a - b$ together using FOIL.

$$a^2 + 2ab + b^2 + a^2 - 2ab + b^2 + 2(a^2 - b^2) - 3a^2$$

4. Distribute 2 over $a^2 - b^2$.

$$a^2 + 2ab + b^2 + a^2 - 2ab + b^2 + 2a^2 - 2b^2 - 3a^2$$

5. Group like terms.

$$(a^2 + a^2 + 2a^2 - 3a^2) + (b^2 + b^2 - 2b^2) + (2ab - 2ab)$$

6. Combine like terms.

$$a^2 + (b^2 + b^2 - 2b^2) + (2ab - 2ab)$$

7. Look for the difference of two identical terms.

$$a^2$$

3. Which of the following functions are continuous? If not, where are the discontinuities?

a. $f(x) = \dfrac{9x^3 - x}{(x-1)(x+1)}$

- Discontinuous at $x = -1, +1$ (denominator would be 0, leaving the fraction undefined)

b. $g(y, z) = \dfrac{6y^4z^3 + 3y^2z - 56}{12y^5 - 3zy + 18z}$

- Ratio of polynomials is always continuous unless the denominator is 0. In this case, there are infinite combinations of y and z that would make the function discontinuous. $y, z = 0$ is the simplest but we can calculate the whole range:

$$12y^5 - 3zy + 18z = 0$$

$$4y^5 - yz + 6z = 0$$

$$yz - 6z = 4y^5$$

$$z(y - 6) = 4y^5$$

$$z = \frac{4y^5}{(y-6)}$$

- So the function $g(y, z)$ is discontinuous for all $y \neq 6, z = \dfrac{4y^5}{(y-6)}$.

c. $f(x) = e^{-x^2}$

- Continuous for all real numbers.

d. $f(y) = y^3 - y^2 + 1$

- All polynomials are continuous.

e. $f(x) = \begin{cases} x^3 + 1, & x > 0 \\ \frac{1}{2} & x = 0 \\ -x^2, & x < 0 \end{cases}$

- Discontinuous at $x = 0$. This is a piecewise function. To be continuous $\lim_{x\to 0^+} f(x) = 0$. However in this function, $\lim_{x\to 0^+} f(x) = 1 \neq 0$.

4. Express each of the following as a single logarithm:

a. $\log(x) + \log(y) - \log(z)$

- Multiplication rule of logarithms: $\log(x \times y) = \log(x) + \log(y)$
- Division rule of logarithms: $\log(\frac{x}{y}) = \log(x) - \log(y)$
- Applying the log rules, we combine logs that are added through multiplication and then combine logs that are subtracted with division.

$$\log(x) + \log(y) - \log(z)$$

$$\log(xy) - \log(z)$$

$$\log(\frac{xy}{z})$$

b. $2\log(x) + 1$

- Exponentiation rule of logarithms: $\log(x^y) = y\log(x)$
- $\log(e) = 1$

$$2\log(x) + 1$$

$$2\log(x) + \log(e)$$

$$\log(x^2) + \log(e)$$

$$\log(ex^2)$$

c. $\log(x) - 2$

- $\log(e) = 1$

$$\log(x) - 2$$

$$\log(x) - 2\log(e)$$

$$\log(x) - \log(e^2)$$

$$\log(\frac{x}{e^2})$$

5. Find the roots (solutions) to the following quadratic equations. **Hint: Remember the quadratic formula.**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

a. $4x^2 - 1 = 17$

- Move terms so that x is alone on the left side of the equation.

$$\begin{aligned} 4x^2 - 1 &= 17 \\ 4x^2 &= 18 \\ x^2 &= \frac{18}{4} \\ x^2 &= \frac{9}{2} \\ x &= \pm\sqrt{\frac{9}{2}} \end{aligned}$$

b. $9x^2 - 3x - 12 = 0$

- Factor the left-hand side.

$$3(x+1)(3x-4) = 0$$

- Divide both sides by 3 to simplify the equation.

$$(x+1)(3x-4) = 0$$

- Find the roots of each term in the product separately by solving for x.

$$\begin{aligned} x + 1 &= 0 \\ x &= -1 \end{aligned} \qquad \begin{aligned} 3x &= 4 \\ x &= \frac{4}{3} \end{aligned}$$

c. $x^2 - 2x - 16 = 0$

1. Complete the square

$$\begin{aligned} x^2 - 2x - 16 &= 0 \\ x^2 - 2x &= 16 \\ x^2 - 2x + 1 &= 17 \\ (x-1)^2 &= 17 \\ x - 1 &= \pm\sqrt{17} \\ x &= 1 \pm \sqrt{17} \end{aligned}$$

2. Quadratic formula

- Using the quadratic formula, solve for x

$$\begin{aligned}
x &= \frac{-(-2) \pm \sqrt{(-2)^2 - (4 \times 1 \times 16)}}{2 \times 1} \\
x &= \frac{2 \pm \sqrt{4 + 64}}{2} \\
x &= \frac{2 \pm \sqrt{68}}{2}
\end{aligned}$$

- Simplify the radical

$$\begin{aligned}
x &= \frac{2 \pm \sqrt{2^2 \times 17}}{2} \\
x &= \frac{2 \pm 2\sqrt{17}}{2}
\end{aligned}$$

- Factor the greatest common divisor

$$x = 1 \pm \sqrt{17}$$

d. $6x^2 - 6x - 6 = 0$

- Divide both sides by 6 to simplify the equation.

$$x^2 - x - 1 = 0$$

- Using the quadratic formula, solve for x

$$\begin{aligned}
x &= \frac{-(-1) \pm \sqrt{(-1)^2 - (4 \times 1 \times -1)}}{2 \times 1} \\
x &= \frac{1 \pm \sqrt{1 - 4(-1)}}{2} \\
x &= \frac{1 \pm \sqrt{1 + 4}}{2} \\
x &= \frac{1 \pm \sqrt{5}}{2}
\end{aligned}$$

e. $5 + 11x = -3x^2$

- Move everything to the left hand side.

$$3x^2 + 11x + 5 = 0$$

- Using the quadratic formula, solve for x

$$\begin{aligned}
x &= \frac{-11 \pm \sqrt{(11)^2 - (4 \times 3 \times 5)}}{2 \times 3} \\
x &= \frac{-11 \pm \sqrt{121 - 60}}{6} \\
x &= \frac{-11 \pm \sqrt{61}}{6}
\end{aligned}$$