CS393 — Nixx Vorbancoeur HW 01

$\boxed{\text{I}}$ $(\vec{P}(x_+), \vec{P}(x_-)) = (\hat{x}, -\hat{x})$ Map $(\hat{x}, -\hat{x})$ to Points on bloch sphere

Assume: $\hat{x} = \vec{P}(x_+)$ & $\vec{P}(x_+) = |x_+\rangle$

proof: $|x_+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ → (a) Pauli X eigenstate:

$(|x_+\rangle, |x_-\rangle) = \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}, \frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$

using def of single qbit state on bloch sphere: → (b) Single qbit bloch sphere:

$|\psi(\vec{P})\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$

$1/\sqrt{2} = \cos(\theta/2)$

$\pi/4 = \theta/2$

$\pi/2 = \theta$

using θ:

$1/\sqrt{2} = e^{i\phi}\sin(\pi/4)$

$\frac{1}{\sqrt{2}} = e^{i\phi}\frac{\sqrt{2}}{2}$

$\frac{2}{\sqrt{2}^2} = e^{i\phi}$

$1 = e^{i\phi}$

$\phi = 0$

(c) Polar – Cartesian representation

$(1, \theta(\psi), \phi(\psi)) =$

$x = \sin\theta(\psi)\cos\phi(\psi)$

$y = \sin\theta(\psi)\sin\phi(\psi)$

$z = \cos\theta(\psi)$

$\theta = \pi/2, \ \phi = 0$

$(\sin(\pi/2)\cos(0), \sin(\pi/2)\sin(0), \cos(\pi/2)) = (1, 0, 0)$

unit $\hat{x} = (1, 0, 0)$

$\therefore \ |x_+\rangle = \vec{P}(x_+) = \hat{x}$

Assume: $\vec{P}(x_-) = -\hat{x} = |x_-\rangle$

use (a): $|x_-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

use (b): $\cos(\theta/2) = 1/\sqrt{2}$

$\theta = \pi/2$

$e^{i\phi}\sqrt{2}/2 = -1/\sqrt{2}$

$e^{i\phi} = -1$

$\phi = \pi$

use (c)

$(\sin(\pi/2)\cos(\pi), \sin(\pi/2)\sin(\pi), \cos(\pi/2))$

$(1\cdot -1, 1\cdot 0, 0) = (-1, 0, 0)$

$-\hat{x} = (-1, 0, 0)$ $\therefore \ |x_-\rangle = P(x_-) = |x_-\rangle$

$\therefore (\vec{P}(x_+), \vec{P}(x_-)) = (\hat{x}, -\hat{x})$

$\boxed{\mathrm{I}}$ $(\vec{P}(y_+), \vec{P}(y_-)) = (\hat{y}, -\hat{y})$

$\vec{P}(y_+) = \hat{y}$ $\qquad$ $\vec{P}(y_-) = -\hat{y}$

$P(y_+) = |y_+\rangle$ $\qquad$ $P(y_-) = |y_-\rangle$

Pauli Y Eigenbase
$(|y_+\rangle, |y_-\rangle) =$
$(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}} i|1\rangle,$
$\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}} i|1\rangle)$

$|y_+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}} i|1\rangle$ $\qquad$ $|y_-\rangle = \frac{1}{\sqrt{2}}|0\rangle - i|1\rangle$

using single qbit state $|\psi(\vec{p})\rangle$ definition:

$\frac{1}{\sqrt{2}} = \cos(\theta/2)$ $\qquad$ $\frac{1}{\sqrt{2}} i = e^{i\phi} \sin(\theta/2)$

$\theta = \pi/2$ $\qquad$ $i/\sqrt{2} = e^{i\phi} \sqrt{2}/2$

$i = e^{i\phi}$ $\qquad$ $\frac{2}{\sqrt{2}} \cdot \frac{i}{\sqrt{2}} = e^{i\phi}$

$\phi = \pi/2$ $\qquad$ $\theta = \pi/2$

use Cartesian conversion:

$(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$

$\sin(\pi/2)\cos(\pi/2), 1 \cdot 1, 0$

$|y_+\rangle = (0, 1, 0)$

unit length $\hat{y}$: $(0, 1, 0)$

$\therefore P(y_+) = |y_+\rangle = \hat{y}$

$|y_-\rangle = \frac{1}{\sqrt{2}}|0\rangle - i|1\rangle$

$\frac{1}{\sqrt{2}} = \cos(\theta/2)$ $\qquad$ $-i/\sqrt{2} = \sin(\theta/2)$

$\theta = \pi/2$ $\qquad$ $-i/\sqrt{2} = \sqrt{2}/2\, e^{i\phi}$

$-i = e^{i\phi}$

$\cancel{-i = \cos(\phi) + i\sin(\phi)}$

$-i = e^{i\phi}$

$\phi = 3\pi/2$ $\qquad$ $\theta = \pi/2$

$\sin(\theta/2)\cos(3\pi/2), \sin(\theta/2)\sin(3\pi/2)$

$(0, -1, 0)$

$|y_-\rangle = (0, -1, 0)$

$-\hat{y} = (0, -1, 0)$

$\therefore P(y_-) = |y_-\rangle = -\hat{y}$

$$\therefore (\vec{P}(y_+), \vec{P}(y_-)) = (\hat{y}, -\hat{y})$$

$\boxed{\text{II}}$ Prop. 3.3.5 : CNOT operator is Hermitian unitary

AKA: $CNOT^* = CNOT = CNOT^{-1}$

& $CNOT^2 = I_2$

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(a) $CNOT^*$ (adjoint)

$$CNOT^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Assume:

CNOT = CNOT Adjoint

Via (a),

$$CNOT^* \text{ (complex conjugate)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = CNOT \checkmark$$

Assume:

$CNOT = CNOT^{-1}$

for $CNOT^{-1}$

- ensure invertible: rref(CNOT) = I → is invertible

det(CNOT) = -1

$$CNOT^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = CNOT \checkmark$$

$\therefore CNOT^* = CNOT = CNOT^{-1}$

Assume $CNOT^2 = I_{2\times2}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\begin{bmatrix}1\\2\end{bmatrix}$ $\begin{bmatrix}2\\3\end{bmatrix}$

$\boxed{\text{III}}$ 3.3.9 – output states $\mathcal{U}|\vec{b}\rangle$, $\vec{b} \in \{0,1\}$

$\mathcal{U}_0$: $H \otimes H \otimes I$
H on 0 & 1, iden on 2

$\mathcal{U}_1$: $I \otimes CNOT$
Iden on 0, CNOT on 1 & 2
// Flip 2 when 1==1

$\mathcal{U}_2$: $H \otimes I \otimes I$
H on 0, iden on 1 & 2

in

$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ $\quad I = |0\rangle$

$|000\rangle$

$\mathcal{U}_0$

$\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \otimes |0\rangle$

$= \frac{1}{2}(|00\rangle+|01\rangle+|10\rangle+|11\rangle) \otimes |0\rangle$

$= \frac{1}{2}(|000\rangle+|010\rangle+|100\rangle+|110\rangle)$

$\mathcal{U}_1$

$\frac{1}{2}(|000\rangle+|010\rangle+|100\rangle+|110\rangle)$

$= \frac{1}{2}(|000\rangle+|011\rangle+|100\rangle+|111\rangle)$

// Factor: $\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)(|00\rangle+|11\rangle)$ → apply operation H ~~to qbit 1, split.~~

$\mathcal{U}_2$

$H|0\rangle$

$|000\rangle$ $\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \otimes (|00\rangle)$

$= \frac{1}{\sqrt{2}}(|000\rangle+|100\rangle)$

$H|0\rangle$

$|011\rangle$ $\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \otimes |11\rangle$

$= \frac{1}{\sqrt{2}}(|011\rangle+|111\rangle)$

$H|1\rangle$

$|0\rangle - |1\rangle$

$|100\rangle$ $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes |00\rangle$

$\frac{1}{\sqrt{2}}(|000\rangle-|100\rangle)$

$H|1\rangle$

$|111\rangle$ $\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)|11\rangle$

$= \frac{1}{\sqrt{2}}(|011\rangle-|111\rangle)$

$\frac{1}{2}\left[\frac{1}{\sqrt{2}}(|000\rangle+|100\rangle)\right]+\left[\frac{1}{\sqrt{2}}(|011\rangle+|111\rangle)\right]+\left[\frac{1}{\sqrt{2}}(|000\rangle-|100\rangle)\right]+\left[\frac{1}{\sqrt{2}}(|011\rangle-|111\rangle)\right]$

$=\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)\left[(|000\rangle+|100\rangle)+(|011\rangle+|111\rangle)+(|000\rangle-|100\rangle)+(|011\rangle-|111\rangle)\right]$

$= \frac{1}{2\sqrt{2}}\left[2|000\rangle+(|100\rangle-|100\rangle)+2|011\rangle+(|111\rangle-|111\rangle)\right]$

$= \frac{1}{2\sqrt{2}}(2|000\rangle+2|011\rangle) = \frac{2}{2\sqrt{2}}(|000\rangle+|011\rangle) = \boxed{\frac{1}{\sqrt{2}}(|000\rangle+|011\rangle)}$

$|001\rangle$ $\mathcal{U}_0$: $H|0\rangle \otimes H|0\rangle \otimes I \otimes |1\rangle$

$\frac{1}{2}(|00\rangle+|01\rangle+|10\rangle+|11\rangle)\otimes|1\rangle$

$\frac{1}{2}(|001\rangle+|011\rangle+|101\rangle+|111\rangle)$

$\mathcal{U}_1$: $I|0\rangle \otimes |0\rangle\, CNOT\, |1\rangle$

$= \frac{1}{2}(|001\rangle+|010\rangle+|101\rangle+|110\rangle)$
a b c d

$\mathcal{U}_1$: ⓐ $|001\rangle = H|0\rangle \otimes |01\rangle = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes|01\rangle$

$\frac{1}{\sqrt{2}}(|001\rangle+|101\rangle)$

ⓑ $|010\rangle = H|0\rangle \otimes |10\rangle = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes|10\rangle$

$= \frac{1}{\sqrt{2}}(|010\rangle+|110\rangle)$

ⓒ $|101\rangle = H|1\rangle \otimes |01\rangle = \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\otimes|01\rangle$

$= \frac{1}{\sqrt{2}}(|001\rangle-|101\rangle)$

ⓓ $|110\rangle = H|1\rangle \otimes |10\rangle = \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\otimes|10\rangle$

$= \frac{1}{\sqrt{2}}(|010\rangle-|110\rangle)$

$=\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)\left[2|001\rangle+(|101\rangle-|101\rangle)+2|010\rangle+(|110\rangle-|110\rangle)\right] = \frac{2}{2\sqrt{2}}(|001\rangle+|010\rangle)$

$= \frac{1}{\sqrt{2}}(|001\rangle+|010\rangle)$

$U_0: H\otimes H\otimes I$ $\quad U_1: I \otimes CNOT$ $\quad U_2: H\otimes I \otimes I$

$|010\rangle = H|0\rangle \otimes H|1\rangle \otimes I|0\rangle$

$\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle) \otimes |0\rangle$

$= \frac{1}{2}(|00\rangle-|01\rangle+|10\rangle-|11\rangle) \otimes |0\rangle$

$= \frac{1}{2}(|000\rangle-|010\rangle+|100\rangle-|110\rangle)$

$I|\alpha\rangle \otimes (|\beta\rangle\ CNOT\ |\gamma\rangle)$

$= \frac{1}{2}(|000\rangle-|011\rangle+|100\rangle-|111\rangle)$

U_2: ⓐ $\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes|00\rangle = \frac{1}{\sqrt{2}}(|000\rangle+|100\rangle)$ ⓑ $-\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes|11\rangle = -\frac{1}{\sqrt{2}}(|011\rangle+|111\rangle)$

ⓒ $\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\otimes|00\rangle = \frac{1}{\sqrt{2}}(|000\rangle-|100\rangle)$ ⓓ $-\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\otimes|11\rangle = -\frac{1}{\sqrt{2}}(|011\rangle-|111\rangle)$

$\frac{1}{2}\left[\frac{1}{\sqrt{2}}(|000\rangle+|100\rangle)-\frac{1}{\sqrt{2}}(|011\rangle+|111\rangle)+\frac{1}{\sqrt{2}}(|000\rangle-|100\rangle)-\frac{1}{\sqrt{2}}(|011\rangle-|111\rangle)\right]$

$(\frac{1}{2})(\frac{1}{\sqrt{2}})\left[(|000\rangle+|100\rangle)-(|011\rangle+|111\rangle)+(|000\rangle-|100\rangle)-(|011\rangle-|111\rangle)\right]$

$= \frac{1}{2\sqrt{2}}\left[|000\rangle+|100\rangle-|011\rangle-|111\rangle+|000\rangle-|100\rangle-|011\rangle+|111\rangle\right]$

$2|000\rangle+(|100\rangle-|100\rangle)-2|011\rangle+(-|111\rangle+|111\rangle)$

$\frac{1}{\sqrt{2}}(|000\rangle-|011\rangle)$

$|011\rangle = H|0\rangle \otimes H|1\rangle \otimes I|1\rangle$

$\frac{1}{2}(|00\rangle-|01\rangle+|10\rangle-|11\rangle)\otimes|1\rangle$

$= \frac{1}{2}(|001\rangle-|011\rangle+|101\rangle-|111\rangle)$

$I|\alpha\rangle \otimes (|\beta\rangle\ CNOT\ |\gamma\rangle)$

$= \frac{1}{2}(|001\rangle-|010\rangle+|101\rangle-|110\rangle)$

U_2: ~~4 term~~ $H|0\rangle\otimes|01\rangle = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes|01\rangle$

$= \frac{1}{\sqrt{2}}(|001\rangle+|101\rangle)$

$-H|0\rangle\otimes|10\rangle = -\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes|10\rangle$

$= -\frac{1}{\sqrt{2}}(|010\rangle+|110\rangle)$

$H|1\rangle\otimes|01\rangle = \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\otimes|01\rangle$

$= \frac{1}{\sqrt{2}}(|001\rangle-|101\rangle)$

$-H|1\rangle\otimes|10\rangle = -\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\otimes|10\rangle$

$= -\frac{1}{\sqrt{2}}(|010\rangle-|110\rangle)$

$(\frac{1}{2})(\frac{1}{\sqrt{2}})\left[|001\rangle+|101\rangle-|010\rangle-|110\rangle+|001\rangle-|101\rangle-|010\rangle+|110\rangle\right]$

$2|001\rangle+(|101\rangle-|101\rangle)-2|010\rangle+(-|010\rangle+|110\rangle)$

$= \frac{1}{\sqrt{2}}(|001\rangle-|010\rangle)$

$H|\alpha\rangle \otimes H|\beta\rangle \otimes I|\gamma\rangle$

$|100\rangle$ $H|1\rangle \otimes H|0\rangle \otimes |0\rangle$

$\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \otimes |0\rangle$

$\frac{1}{2}(|00\rangle+|01\rangle-|10\rangle-|11\rangle) \otimes |0\rangle = \frac{1}{2}(|000\rangle+|010\rangle-|100\rangle-|110\rangle)$

$I|\alpha\rangle \otimes |\beta\rangle\ CNOT\ |\gamma\rangle$

$\frac{1}{2}(|000\rangle+|011\rangle-|100\rangle-|111\rangle)$

$H|\alpha\rangle \otimes I|\beta\rangle \otimes I|\gamma\rangle$

$\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \otimes |00\rangle = \frac{1}{\sqrt{2}}(|000\rangle+|100\rangle)$

$+|011\rangle$

$\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \otimes |11\rangle = \frac{1}{\sqrt{2}}(|011\rangle+|111\rangle)$

$-|100\rangle$

$-\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle) \otimes |00\rangle = -\frac{1}{\sqrt{2}}(|000\rangle-|100\rangle)$

$-|111\rangle$

$-\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle) \otimes |11\rangle = -\frac{1}{\sqrt{2}}(|011\rangle-|111\rangle)$

$(\frac{1}{2})(\frac{1}{\sqrt{2}})[|000\rangle+|100\rangle+|011\rangle+|111\rangle-|000\rangle+|100\rangle-|011\rangle+|111\rangle] = (|000\rangle-|000\rangle)+2|$

$= \frac{1}{2\sqrt{2}}[(|000\rangle-|000\rangle)+2|100\rangle+(|011\rangle-|011\rangle)+2|111\rangle] = \frac{1}{\sqrt{2}}(|100\rangle+|111\rangle)$

$|101\rangle$ $H|1\rangle \otimes H|0\rangle \otimes I|1\rangle$

$\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \otimes |1\rangle$

$\frac{1}{2}(|00\rangle+|01\rangle-|10\rangle-|11\rangle) \otimes |1\rangle = \frac{1}{2}(|001\rangle+|011\rangle-|101\rangle-|111\rangle)$

CNOT

$\frac{1}{2}(|001\rangle+|010\rangle-|101\rangle-|110\rangle)$

$|001\rangle$

$\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \otimes |01\rangle = \frac{1}{\sqrt{2}}(|001\rangle+|101\rangle)$

$+|010\rangle$

$\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \otimes |10\rangle = \frac{1}{\sqrt{2}}(|010\rangle+|110\rangle)$

$-|101\rangle$ $\frac{1}{\sqrt{2}}(-|0\rangle+|1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}(-|001\rangle$

$-|110\rangle$

$-\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle) \otimes |01\rangle = -\frac{1}{\sqrt{2}}(|001\rangle-|101\rangle)$

$-\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle) \otimes |10\rangle = -\frac{1}{\sqrt{2}}(|010\rangle-|110\rangle)$

$\frac{1}{2\sqrt{2}}[|001\rangle+|101\rangle+|010\rangle+|110\rangle-|001\rangle+|101\rangle-|010\rangle+|110\rangle]$

$(|001\rangle-|001\rangle)+2|101\rangle+(|010\rangle-|010\rangle)+2|110\rangle$

$= \frac{1}{\sqrt{2}}(|101\rangle+|110\rangle)$

$\frac{1}{2\sqrt{2}}(|001\rangle+|101\rangle+|010\rangle+|110\rangle-(|001\rangle-|101\rangle)-(|010\rangle-|110\rangle))$

$H|\alpha\rangle \otimes H|\beta\rangle \otimes I|\gamma\rangle \qquad I|\alpha\rangle \otimes |\beta\rangle\, CNOT\, |\gamma\rangle \qquad H|\alpha\rangle \otimes I|\beta\rangle \otimes I|\gamma\rangle$

$|110\rangle \quad H|1\rangle \otimes H|1\rangle \otimes I|0\rangle$

CNOT

$\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes |0\rangle$

$= \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle) \otimes |0\rangle = \frac{1}{2}(|000\rangle - |010\rangle - |100\rangle + |110\rangle) \;\vdots\; \frac{1}{2}(|000\rangle - |011\rangle - |100\rangle + |111\rangle)$

$|000\rangle = H|0\rangle \otimes |00\rangle \qquad -|011\rangle = -\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |11\rangle \qquad -|100\rangle = -\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |00\rangle \qquad = -\frac{1}{\sqrt{2}}(|011\rangle + |111\rangle)$

$= \frac{1}{\sqrt{2}}(|000\rangle + |100\rangle)$

$-|100\rangle = -\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes |00\rangle \qquad |111\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes |11\rangle$

$= -\frac{1}{\sqrt{2}}(|000\rangle - |100\rangle) \qquad = \frac{1}{\sqrt{2}}(|011\rangle - |111\rangle)$

$\frac{1}{2\sqrt{2}}(|000\rangle + |100\rangle - |011\rangle - |111\rangle - |000\rangle + |100\rangle + |011\rangle - |111\rangle)$

$((|000\rangle - |000\rangle) + 2|100\rangle + (-|011\rangle + |011\rangle) - 2|111\rangle)$

$\frac{1}{\sqrt{2}}(|100\rangle - |111\rangle)$

$|111\rangle \quad H|1\rangle \otimes H|1\rangle \otimes I|1\rangle$

CNOT

$\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes |1\rangle$

$\frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle) \otimes |1\rangle = \frac{1}{2}(|001\rangle - |011\rangle - |101\rangle + |111\rangle) \;\vdots\; \frac{1}{2}(|001\rangle - |010\rangle - |101\rangle + |110\rangle)$

$|001\rangle: \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |01\rangle \qquad -|010\rangle = -\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |10\rangle \qquad -|101\rangle: -\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes |01\rangle$

$\frac{1}{\sqrt{2}}(|001\rangle + |101\rangle) \qquad = -\frac{1}{\sqrt{2}}(|010\rangle + |110\rangle) \qquad = -\frac{1}{\sqrt{2}}(|001\rangle - |101\rangle)$

$|110\rangle: \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes |10\rangle \qquad \frac{1}{2\sqrt{2}}[|001\rangle + |101\rangle - |010\rangle - |110\rangle - |001\rangle + |101\rangle + |010\rangle - |110\rangle]$

$= \frac{1}{\sqrt{2}}(|010\rangle - |110\rangle)$

$[(|001\rangle - |001\rangle) + 2|101\rangle + (-|010\rangle + |010\rangle) - 2|110\rangle]$

$= \frac{1}{\sqrt{2}}(|101\rangle - |110\rangle)$

$\boxed{IV}$ 5.1.5 p. 209 HH, operator V, eigenstate $|\psi\rangle$

① show applying V to $|\psi\rangle$ means Global phase shift

$V|\psi\rangle = \lambda|\psi\rangle$

→ Assume: - V is a unitary operator
- $|\psi\rangle$ is an eigenstate of V
- $V|\psi\rangle = \lambda|\psi\rangle$

$$V|\psi\rangle = e^{i\phi}|\psi\rangle$$

$$\lambda = e^{i\phi}$$

property of eigenvectors: $|\lambda|^2 = 1$

$$|e^{i\phi}|^2 = \cos^2(\phi) + i\sin^2(\phi)$$

$$|z|^2 = a^2 + b^2 i$$

$$\cos^2(\phi) + \sin^2(\phi) = 1$$

$$\therefore |e^{i\phi}|^2 = |\lambda|^2$$

[Phase Kickback:

$$U_f|0\rangle|x_-\rangle = |0\rangle X^{f(0)}|x_-\rangle$$
$$= (-1)^{f(0)}|0\rangle|x_-\rangle$$
$$U_f|1\rangle|x_-\rangle = |0\rangle X^{f(1)}|x_-\rangle$$
$$= (-1)^{f(1)}|0\rangle|x_-\rangle$$

[Eigenstate:

$$V|\psi\rangle = e^{i\phi}|\psi\rangle$$

[Global phase shift

↳ Scalar that doesn't affect the state $|\psi\rangle$

↳ "unobservable", don't change measurement probabilities

ie: Probability of $|0\rangle$: $|\alpha|^2$ phase shift:

$$|e^{i\phi}\alpha|^2 = |e^{i\phi}|^2|\alpha|^2$$
$$= |1||\alpha|^2$$
$$\cos^2(\phi) + \sin^2(\phi) = 1$$
$$1^2 = 1$$

② Controlled V operator $C(V)$ w/ qbit 0 as a control to state $|x_+\rangle|\psi\rangle$, Phase kickback to amplitude of $|1\rangle$ in qbit 0

$$|x_+\rangle|\psi\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |\psi\rangle$$

$$|x_+\rangle|\psi\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle|\psi\rangle + |1\rangle|\psi\rangle)$$

use $C(V)$:

$$\tfrac{1}{\sqrt{2}}(|0\rangle|\psi\rangle + |1\rangle V \cdot |\psi\rangle) \rightarrow$$

Control on qbit 0:

$$C(V)|0\rangle|\psi\rangle = |0\rangle \otimes I|\psi\rangle = |0\rangle|\psi\rangle$$
$$C(V)|1\rangle|\psi\rangle = |1\rangle \otimes V|\psi\rangle$$

$$C(V)|x_+\rangle|\psi\rangle = (|0\rangle|\psi\rangle + |1\rangle e^{i\phi}|\psi\rangle)$$
$$= (|0\rangle + |1\rangle e^{i\phi})|\psi\rangle$$
$$\tfrac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle)|\psi\rangle$$

amplitude of $|0\rangle$: $1/\sqrt{2}$

amplitude of $|1\rangle$: $e^{i\phi}/\sqrt{2}$ ✓

$\boxed{IV}$ ③ Spherical coordinates on bloch sphere before & after $C(V)$: on $|x_+\rangle|\psi\rangle$

Bloch Sphere:
$|\psi_P(\vec{p})\rangle =$
$\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$

$|x_+\rangle = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$
$= |0\rangle/\sqrt{2} + |1\rangle/\sqrt{2}$
$\therefore \cos(\theta/2) = 1/\sqrt{2} \quad \theta = \pi/2$
$e^{i\phi}\sin(\theta/2) = 1/\sqrt{2}$
$e^{i\phi}\sin(\pi/4) = 1/\sqrt{2}$
$\phi = 0 \qquad (\theta, \phi) = (\pi/2, 0)$

after $C(V)$: $1/\sqrt{2}(|0\rangle + e^{i\phi}|1\rangle)$

$\cos(\theta/2) = 1/\sqrt{2}, \ \theta = \pi/2 \quad (\theta, \phi) = (\pi/2, \phi)$

$\cos(\phi) + i\sin(\phi)$, phase ϕ