HW 05 – Mat 320

Nyxx Vorbonceur
11/24/25

$\boxed{1}$ Inverse Comb filter $\nearrow$ P.77 $\quad y_t = x_t - R^L x_{t-L} \quad R = 0.999 \quad L = 6$

$y_t = e^{i\omega} - R^L e^{iL\omega}$

ⓐ $\mathcal{H}(z) = 1 - R^L z^{-L}$

$= 1 - 0.99401 z^{L}$

ⓑ $H(\omega) = 1 - R^L e^{iL\omega} = \cancel{1-R}\ 1-$

ⓒ $|H(\omega)| = 1 - R^L(\cos(L\omega) - i\sin(L\omega)) = 1 - 0.99401(\cos(6\omega) - i\sin(6\omega))$

ⓓ $\mathcal{H}(z) = 1 - R^6 z^{-6} \dfrac{z^6}{z^6} = \dfrac{z^6 - R^6}{z^6} \qquad z^6 = R^6 e^{i\cdot 6\theta}$

~~$e^{i6\theta} = \cos(6\omega) + i\sin(6\theta)$~~

// Zeroes @ $z^L = R^L$

$z = R e^{i 2\pi/L \cdot k}$

$z = R \cdot e^{i\frac{2\pi}{6}k} = Re^{i\frac{\pi}{3}k} \qquad \dfrac{2\pi}{\pi/3} = 6$ roots

$k=0 \to R \qquad k=2 \to Re^{i2\pi/3} \qquad k=4\ Re^{i4\pi/3}$

$k=1 = Re^{i\pi/3} \qquad k=3 = Re^{i\pi} \qquad k=5 = Re^{i5\pi/3}$

$\mathcal{H}(z) = (z-R)(z-Re^{i\pi/3})(z-Re^{i2\pi/3})(z-e^{i\pi})\cancel{(Re^{i4\pi/3})}$
$(z-Re^{i4\pi/3})(z-Re^{i5\pi/3})$

$\hookrightarrow \mathcal{H}(z) = \prod_{k=0}^{5} (z - Re^{i\pi/3\cdot k})$

~~$\mathcal{H}(z)^2 = \left[\dfrac{(z^6-R^6)}{z^6}\right]^2 = \dfrac{(z^6-R^6)^2}{z^{12}}$~~

$\dfrac{d}{d\omega}\mathcal{H}(\omega)^2 = \dfrac{d}{d\omega}(1 - R^6 \cdot e^{i6\omega})^2 = \dfrac{d}{d\omega}(1 - R^6\cdot\cos(6\omega) - i\sin(6\omega))^2$

$\cancel{d/d\omega}(1 - R^6\cos(6\omega) - i\sin(6\omega)) = \cancel{(1+R^6\cos(6\omega))}$

$= 1 - R^6\cos(6\omega) + iR^6\sin(6\omega)$ // Conjugate : $(a+bi)(a-bi) = a^2+b^2$

$\mathcal{H}(\omega)^* = 1 - R^6\cos(6\omega) - iR^6\sin(6\omega)$

$(1 - R^6\cos(6\omega))^2 + (R^6\sin(6\omega))^2$

$(1 - 2R^6\cos(6\omega) + R^{12}\cos^2(6\omega)) + (R^{12}\sin^2(6\omega))$

$1 - 2R^6\cos(6\omega) + R^{12}(\cos^2(6\omega) + \sin^2(6\omega))$

$d/d\omega(1 - 2R^6\cos(6\omega) + R^{12}) = d/d\omega(1 - 2R^6\cos(u) + R^{12})$ // $d/d\omega(\cos(6\omega)) \cdot d/d\omega(6\omega)$

// Chain $\to \cos(6\omega) = -\sin(6\omega)\cdot 6$

$-2R^6 \cdot (-\sin(6\omega)\cdot 6) = 12R^6\sin(6\omega)$

Min $\to \sin(6\omega) = 0$

$6\omega = \pi n$

$\omega = \pi/6\, n$

1 (d) factor: $H(z) = (z-R)(z-Re^{j\pi/3})(z-Re^{j2\pi/3})(z-e^{j\pi})(z-Re^{j4\pi/3})(z-Re^{j5\pi/3})$

$H(z)^{2'} = 12R^6 \sin(\omega)$, critical @ $\pi/6n$

$H(z)^{2''} = 72R^6 \cos(\omega)$, min @ $2\pi n$, Max @ $\pi/6n$

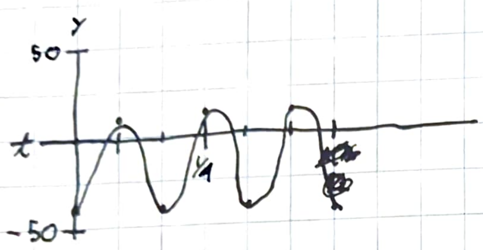
$\omega \in [0, \pi]$ (fraction of sample rate, dB):

$w(0)$: $(0, -44.5)$ $\quad w(\pi/6)$: $(1/12, 5.99)$ $\quad w(\pi/3)$: $(1/6, -44.5)$

$w(\pi/2)$: $(1/4, 5.99)$ $\quad w(2\pi/3)$: $(1/3, -44.5)$ $\quad w(5\pi/6)$: $(5/12, 5.99)$

$w(\pi)$: $(1/2, -44.5)$

(e)



(f) $H(\omega) = 1 - R^6 e^{-j6\omega} = 1 - R^6\cos(6\omega) - jR^6\sin(6\omega) = 1 - R^6\cos(6\omega) + jR^6\sin(6\omega)$

$\theta_\omega = \tan^{-1}\left(\frac{R^6 \sin(6\omega)}{1 - R^6 e^{-j6\omega}}\right)$

Phase @ Min $2\pi n$: 0 $\quad$ Phase response at extrema makes

Max $\pi/6n$: 0 $\quad$ sense because $H(z)' = 1$

$\boxed{2}$ $f_s = 44100\ Hz$

$\boxed{I}$ $B = 2\phi$, $\cos(\phi) = 2 - \frac{1}{2}(R - \frac{1}{R})$

$\cos(\theta) = \frac{2R}{1+R^2}\cos(\psi)$

$\theta = \frac{2\pi\psi}{f_s}$ $A_0 = (1-R)^2 \sin(\theta)$

ⓐ $B = 10\ Hz$ $\psi = 1000\ Hz$

$\theta = \frac{20\pi}{441} = 0.1425$

$\phi = B/2 = 5\ Hz$ $\frac{2\pi \cdot 5}{44100} = \frac{\pi}{4410}$

$\cos(\phi) = 2 - \frac{1}{2}(R - \frac{1}{R})$

$2(2 - \cos(\phi)) = R - \frac{1}{R} \cdot \frac{R}{R} = 2(2\cos(\phi)) = \frac{R^2+1}{R}$

$2R(2\cos(\phi)) - R^2 + 1 = 0 \rightarrow -R^2 + 2R(2 - \cos(\phi)) + 1 = 0$

$= R^2 + 2R + 1 = 0$

$R = 1.007, \boxed{.9993}$

$A_0 = (1 - 0.9993^2) \cdot \sin(\frac{20\pi}{441}) = 2.022E-4$

$Y_t = A_0 X_t + 2R\cos(\theta) Y_{t-1} - R^2 \cdot Y_{t-2}$

$A_0 = 2.022E-4$
$R = 0.9993$
$\theta = \frac{20\pi}{441}$

$2R\cos(\theta) = 1.979$

$Y_t = 2.022E-4\, X_t + 1.979\, Y_{t-1} - 0.9986\, Y_{t-2}$

δ_t: Y_0: $2.022E-4$ $Y_1 = 3.999E-4$ $Y_2 = 5.8934E-4$

$\boxed{II}$ $B = 2(1-R) \rightarrow$ ~~$B-2 = -2R \rightarrow B+1 = R$~~

$B/2 = 1 - R$ $R = 0.9998$

$\theta = 0.1425$

$1 - B/2 = R$ $A_0 = 3.219E-5$

$\boxed{B_{Hz} / \frac{f_s}{2} = d f_s}$

$Y_t = 3.219E-5\, X_t + 1.979\, Y_{t-1} - 0.9997\, Y_{t-2}$

δ_t: $Y_0 = 3.219E-5$ $Y_1 = 6.373E-5$ $Y_2 = 9.397E-5$

2 (b) $B = 20\ Hz$ $\psi = 5000\ Hz$
$= \cancel{\ldots}$ $\cancel{\ldots} = 100\pi/441 = \psi$
I $= \pi/2205$

$B/2 = \phi$ $R^2 - (2\cdot(2-\cos(\phi)) + 1 = 0$

$\phi = \cancel{\ldots}$
$\pi/2205$ $\cancel{R = 0.9993}$ $R = 0.9985$

$\cancel{\ldots}$ $\cos(\theta) = \frac{2R}{1+R}\cos(\psi)$ $\cos(\theta) = 0.7563$
$\theta = 0.7132$

$A_0 = (1-R^2)\cdot \sin(\theta)$
$\cancel{\ldots}$ $= 1.861E{-3}$ $Y_t = 1.861E{-3} X_t + 1.5104\, Y_{t-1} - 0.9972\, Y_{t-2}$

δ_t: $Y_0 = 1.861E{-3}$ $Y_1 = 2.812E{-3}$ $Y_2 = 2.391E{-3}$

II $B = 2(1-R)$ $R = 0.99929$ $\cos(\theta) = 0.7563$
$\theta = 0.71279$

$A_0 = 9.314E{-4}$ $2\cdot R\cdot \cos(\theta) = 1.512$

$Y_t = 9.314E{-4} X_t + 1.512\, Y_{t-1} - 0.9986\, Y_{t-2}$

δ_0: $Y_0 = 9.314E{-4}$ $Y_1 = 1.408E{-3}$ $Y_2 = 1.199E{-3}$

$\boxed{3}$ $y_t = \sqrt{2}\,(0.99^t)\sin\left(\pi/4\,(t+1)\right)$

6.6: $y_t = R^t\left[\dfrac{\sin(\theta(t+1))}{\sin(\theta)}\right]$

$y_t = \delta_t + (2R\cos\theta)\,y_{t-1} - R^2 y_{t-2}$

$R = 0.99 \qquad \theta = \pi/4$

$\cos(\pi/4) = \sqrt{2}/2$

$= 2\cdot\cos(\theta)\cdot R = 2\cdot\dfrac{\sqrt{2}}{2}\cdot 0.99$

$\sqrt{2}(0.99)$

$\therefore\ y_t = \delta_t + 0.99\sqrt{2}\,y_{t-1} - 0.9801\,y_{t-2}$

$\boxed{4}$ $R \approx e^{-B/2}$, show very close to eqn 3.6:

$R \approx 1 - B/2$

$$\left[\begin{array}{l} e^x = 1 + x + \dfrac{x^n}{n!}\cdots \\ e^x = \displaystyle\sum_{k=0}^{\infty}\frac{x^k}{k!}\end{array}\right]$$

$$e^{-B/2} \approx 1 + \frac{(-B/2)^1}{1} + \frac{(-B/2)^2}{2} + \frac{(-B/2)^3}{6}$$

$$\approx 1 - \frac{B}{2} + \frac{B^2}{8} - \frac{B^3}{48}\cdots$$

→ first-order approximation

↳ terms after k=1 are negligibly impactful on the pole radius approximation for very small bandwidths.

$\therefore$ can reasonably be approximated as

$$e^{-B/2} \approx 1 - B/2$$

Impulse response R^t can be written as $e^{t\ln(R)}$

For $R \approx 1$ (pole close to unit circle), $\ln(R) \approx R-1$*

Using above $R \approx 1 - B/2$, $\ln(R) \approx (1 - B/2) - 1$

$$\ln(R) \approx -B/2 \quad \therefore \quad e^{-B/2\,t} \approx e^{t\ln(R)}$$

*Using taylor series expansion of ln around 1:

$\ln(1+u) = u - \dfrac{u^2}{2} + \dfrac{u^3}{3}\cdots \rightarrow \ln(R) = u$ → first-order approximation

$u = x-1 \qquad \ln(R) = R-1$

$\boxed{5}$ (a) For what range of values of R & θ is the discrepancy greatest between the true peak frequency θ_0 & pole angle θ?

- As R is very close to 1, $\theta_0(\psi)$ is very close to θ

peak frequency (ψ)

$$\cos(\psi) = \frac{1+R^2}{2R} \cos(\theta)$$

∴ the discrepancy occurs as R moves away from 1. Can also vary by pole angle θ but is primarily affected by value of R.

(b) When θ is small, θ_0 is shifted to higher or lower frequencies?

- As described on p.91, $\cos(\psi)$ & $\cos(\theta)$ are used to make it easier to see the directly proportional relationship of θ_0 & θ.

When θ is small $\cos(\psi)$, θ_0 is shifted to lower frequencies.