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[Nixx Varboncoeur
HW 03]

Nyquist Frequency = ~~F_{max}~~ $f_s/2$

Sample rate $f_s = \frac{1}{T_s}$ Hz

Sample size = T_s

$\boxed{1}$ $f_s = 44{,}100$

find smallest possible alias → exact same set of samples
for

Angular Sampling frequency (ω_s) = $2\pi / T_s$

ⓐ 23000 Hz
$n = 1$
$|2300 - 44100|$
$\boxed{= 21{,}100}$

~~44100 = ...~~ find smallest frequency (stepping down via)
$|f - n \cdot f_s| <$ nyquist

or $(f_s/2) - f =$ smallest step (Δ_s)

$f - \Delta_s =$ Alias

ⓑ 45000 Hz
$|45000 - 44100|$
$\boxed{= 900 \text{ Hz}}$

Alias should always be in the range $[0, f_s/2]$, or will fold

ⓒ 1000 Hz
$|1000 - 44100| = \boxed{43100 \text{ Hz}}$ (would fold) to 1000
~~= 43100 Hz~~ → not < 22050 (nyquist frequency)

Since already below nyquist, find alias of 1000 as sample rate

ⓓ 96000 Hz
$n=1$ $|96000 - 44100| = 88200 > f_s$ ×
$n=2$ $|96000 - 88200| = \boxed{7800 \text{ Hz}}$

[2] SNR ~~RMS~~

$B=10$ $\{-340, 223.45, 190.6, -48.2\}$

error dist: $[0, 1/2]$

[Average Error]

$[a,b]: \frac{1}{b-a}\int_a^b f(x)dx$

RMS Error $= 1/\sqrt{12}$

SNR $= \frac{2^{B-1}}{1/\sqrt{12}} = 2^{B-1}\sqrt{12} = \sqrt{4}\cdot\sqrt{3}\, 2^{B-1} = 2\sqrt{3}\cdot 2^{B-1}$

$\boxed{=\sqrt{3}\, 2^{B}} = \sqrt{3}\ 2^{10} = \boxed{\sqrt{3}\cdot 1024} \approx 64.98$ dB

RMS$_t$ $[-2^9, 2^9-1] = [-512, 511]$ $B=9$

For Quantization, Round to nearest integer ($\mathbb{Z}$) in the range

RMS error: Square each error val, then sqrt ~~the~~ ~~square~~ the avg

$-340 \to -340$ err $= 0$

$223.45 \to 223$ err $= 0.45$

$190.60 \to 191$ err $= +0.40$

$-48.2 \to -48$ err $= +0.20$

err $= |Y_i - X_i|$, Y_i = analog, X_i = rounded

$\sqrt{(0^2 + 0.45^2 + 0.4^2 + 0.2^2)/4} = 1.00625$ E-1

$\sqrt{0.10625} = 3.172140$E-1 // rounding using sig figs

RMS err $= 3.17214 \times 10^{-1}$

$\boxed{3}$ Use a Phasor Sum for:

$$2\cos(20\pi t + \pi/3) + 3\cos(20\pi t + \pi/4) = A\cos(20\pi t + \phi)$$

Find A & ϕ as decimal approximations

I Phasor form: (complex exponential)

$r\cos\theta = r\cdot e^{j\theta}$

II Cartesian form:

$r\cdot e^{j\theta} = r(\cos(\theta) + j\sin(\theta))$

ⓐ^{I} $2\cos(20\pi t + \pi/3) = 2e^{j(20\pi t + \pi/3)}$

$= 2(e^{j20\pi t}\cdot e^{j\pi/3})$

ⓑ^{I} $3\cos(20\pi t + \pi/4) = 2e^{j(20\pi t + \pi/4)}$

$= 3(e^{j20\pi t}\cdot e^{j\pi/4})$

ⓐ^{II} Convert Imaginary

$2\cdot e^{j\pi/3} = 2(\cos(\pi/3) + j\sin(\pi/3)) = 2(1/2 + j\sqrt{3}/2) = 1 + j\sqrt{3}$

ⓑ^{II} $3e^{j\pi/4} = 3(\cos(\pi/4) + j\sin(\pi/4)) = 3(\sqrt{2}/2 + j\sqrt{2}/2) = 3\sqrt{2}/2 + j\,3\sqrt{2}/2$

I + II: $2\cos(20\pi t + \pi/3) + 3\cos(20\pi t + \pi/4)$

$=$

$2(e^{j20\pi t}\cdot e^{j\pi/3}) + 3(e^{j20\pi t}\cdot e^{j\pi/4})$

$= e^{j20\pi t}(2e^{j\pi/3} + 3e^{j\pi/4})$

$= e^{j20\pi t}(2(1/2 + j\sqrt{3}/2)) + (3(\sqrt{2}/2 + j\sqrt{2}/2))$

$= e^{j20\pi t}(1 + j\sqrt{3} + 3\sqrt{2}/2 + j\,3\sqrt{2}/2)$

Combine imaginary

$(1 + \frac{3\sqrt{2}}{2}) + (j\sqrt{3} + j\,3\sqrt{2}/2)$

$e^{j20\pi t}(3.121 + j3.853)$

III Polar form:

$a + bj$: Magnitude $(A) = \sqrt{a^2 + b^2}$ (radius)

phase (ϕ) (angle θ) $= \tan^{-1}(b/a)$

III $\sqrt{3.121^2 + 3.853^2} = 4.959 = A$

$\tan^{-1}\left(\frac{3.853}{3.121}\right) = 0.89 = \phi$

$e^{j20\pi t}(4.959e^{j0.89}) = 4.959e^{j20\pi t + 0.89}$

And finally, ~~Back to~~ undoing phasor form:

IV $re^{j\theta} = r\cdot\cos\theta$

IV $4.959\cos(20\pi t + 0.89)$

$A = 4.959$ $\phi = 0.89$

$\boxed{4}$ Basis$(B_4) = \{\vec{u}_0, \vec{u}_1, \vec{u}_2, \vec{u}_3\}$, u_k is Sampled Phasor

$e^{\dot{s}\frac{2\pi}{4}kt}$, $t = 0,1,2,3$ $\quad \omega = \frac{2\pi}{4}$

Matrix $A = \vec{u}_0 = \cancel{\cos(\frac{2\pi}{4}kt)}+$ $\vec{u}_0 = \langle 1,1,1,1\rangle$

$$\begin{array}{c} \begin{array}{cccc} \vec{u}_0 & \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{array} \\ A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \dot{s} & -1 & -\dot{s} \\ 1 & -1 & 1 & -1 \\ 1 & -\dot{s} & -1 & \dot{s} \end{bmatrix} \end{array}$$

$e^{\dot{s}\cdot\frac{2\pi}{4}\cdot 0\cdot 0}$ $t=0$ $\quad t=1 \quad t=2 \quad t=3$

$k=1$ $\quad \vec{u}_1 = e^{\dot{s}\cdot 0}$, $e^{\dot{s}\cdot\frac{2\pi}{4}\cdot 1\cdot 1}$, $e^{\dot{s}\frac{2\pi}{4}\cdot 2}$, $e^{\dot{s}\frac{2\pi}{4}\cdot 3}$

$e^{\dot{s}\pi/2}$, $e^{\dot{s}\pi}$, $e^{\dot{s}3\pi/2}$

$\vec{u}_1 = \langle 1, 0+\dot{s}, -1+0, 0+-\dot{s}\rangle$

$= \langle 1, \dot{s}, -1, -\dot{s}\rangle$

$t=0 \quad t=1 \quad t=2 \quad t=3$

$\vec{u}_2$ $\quad e^{\dot{s}\cdot 0}$, $e^{\dot{s}\frac{2\pi}{4}\cdot 2\cdot 1}$ $\quad e^{\dot{s}\frac{2\pi}{4}\cdot 4}$ $\quad e^{\dot{s}\frac{2\pi}{4}\cdot 2\cdot 3}$

$k=2$ $\quad e^{\dot{s}\pi}$ $\quad e^{\dot{s}2\pi}$ $\quad e^{\dot{s}3\pi}$

$\vec{u}_2 = \langle 1, -1+0\dot{s}, 1+0\dot{s}, -1+0\dot{s}\rangle$

$= \langle 1, -1, 1, -1\rangle$

$\vec{u}_3$ $\quad t=0 \quad t=1 \quad t=2 \quad t=3$

$k=3$ $\quad e^{\dot{s}\cdot 0}$ $\quad e^{\dot{s}\frac{2\pi}{4}3\cdot 1}$ $\quad e^{\dot{s}\frac{2\pi}{4}\cdot 3\cdot 2}$ $\quad e^{\dot{s}\frac{2\pi}{4}\cdot 9}$

$e^{\dot{s}3\pi/2}$ $\quad e^{\dot{s}3\pi}$ $\quad e^{\dot{s}\frac{9\pi}{2}}$

$= \langle 1, 0+-\dot{s}, -1+0\dot{s}, 0+\dot{s}\rangle$

$= \langle 1, -\dot{s}, -1, \dot{s}\rangle$

$\det(A) = -16\dot{s}$ (a)

(b) $\det(A)$ == backwards differences product of Second Column, $\vec{u}_1$

$\vec{u}_1 = \begin{bmatrix} 1 \\ \dot{s} \\ -1 \\ -\dot{s} \end{bmatrix}$ $\quad \prod_{0\le \dot{s} \le j \le 3} z_j - z_{\dot{s}}$

// nested for loop

$\dot{s}$ j	$z_j - z_{\dot{s}}$
$(0,1)$	$= \dot{s} - 1$
$(0,2)$	$= -1-1 = -2$
$(0,3)$	$= -\dot{s} - 1$
$(1,2)$	$= -1 - \dot{s}$
$(1,3)$	$= -\dot{s} - \dot{s} = -2\dot{s}$
$(2,3)$	$= -\dot{s} - -1 = 1 - \dot{s}$

Product

$(\dot{s}-1)(-2)(-\dot{s}-1)(-1-\dot{s})(-2\dot{s})(1-\dot{s})$

$= -16\dot{s} = \det(A)$ ✓

(c) $A^{-1} \rightarrow$ Conjugate transpose scalar n, $\xrightarrow{}$ dimension

$A^{-1} = \frac{1}{4}\cdot A^*$

$$A^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\dot{s} & -1 & \dot{s} \\ 1 & -1 & 1 & -1 \\ 1 & \dot{s} & -1 & -\dot{s} \end{bmatrix} \qquad A^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\dot{s} & -1 & \dot{s} \\ 1 & -1 & 1 & -1 \\ 1 & \dot{s} & -1 & -\dot{s} \end{bmatrix}$$

$A \cdot A^{-1} = I$ ✓

(d) $\vec{a} = \langle 1,2,3,4\rangle$

$A^{-1}\cdot\vec{a} = \langle \frac{5}{2}, -\frac{1}{2}+\frac{1}{2}\dot{s}, -\frac{1}{2}, -\frac{1}{2}-\frac{1}{2}\dot{s}\rangle$

Coeffs: $a_0 = \frac{5}{2}$ $\quad a_1 = (-1+\dot{s})\frac{1}{2}$ $\quad a_2 = -\frac{1}{2}$ $\quad a_3 = -\frac{1}{2}(1+\dot{s})$

$\boxed{4}$ e

Dot Product

$c\vec{u}\cdot\vec{v} = c(u\cdot v)$

$u\cdot cv = c(v\cdot v)$

$\vec{a} = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$

Dot w/ $\vec{u}_j$

$X = \vec{a}_0\vec{u}_0 + \vec{a}_1\vec{u}_1 + \vec{a}_2\vec{u}_2 + \vec{a}_3\vec{u}_3$

$\cancel{\vec{u}\_k\cdot\vec{x} = a\_k(u\_k\cdot u\_k)}$

$\vec{u}_j\cdot\vec{X} = a_j(\vec{u}_j\cdot\vec{u}_j)$

$\vec{a}_j = \dfrac{\vec{u}_j\cdot\vec{x}}{(\vec{u}_j\cdot\vec{u}_j)}$

$[\vec{u}_j\cdot\vec{x}_j]$

$\vec{u}_0$

$k=0$ $\langle |1\cdot 1| + |1^2| + |1^2| + |1^2| \rangle$

$k=1$ $|1^2| + |j^2| + |-1^2| + |-j^2|$

$1 + 1 + 1 + 1$

$k=2 = 4$

$k=3 = 4$

each $|u_j| = 4$ ✓

$a_0 = 10/4 = 5/2$

$a_1 = \dfrac{-2+2j}{4} = (-1+j)/2$

$a_2 = -2/4 = -1/2$

$a_3 = \dfrac{-2+2j}{4} = (-1+j)/2$

✓ matches d.

$u_0\cdot x_0 \quad (1\cdot1)+(1\cdot2)+(1\cdot3)+(1\cdot4) = 10$

$u_1\cdot x_1 = (1\cdot1)+(j\cdot2)+(-1\cdot3)+(-j\cdot4) = 1+2j-3-4j$
$= -2+2j$

$u_2\cdot x = (1\cdot1)+(-1\cdot2)+(1\cdot3)+(-\cdot4) = 1-2+3-4$
$= 4-6 = -2$

$u_3\cdot x = (1\cdot1)+(-j\cdot2)+(-1\cdot3)+(j\cdot4) = \cancel{4}\,1-2j-3+4j$
$= -2+2j$

(5) find DFT of

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\sum_{t=0}^{N-1} X_t e^{-j\frac{2\pi}{N}k \cdot t} \qquad N=4$$

$t=0$, $\left.\begin{array}{l} 1 \cdot e^0 \\ 2 \cdot e^0 \\ 3 \cdot e^0 \\ 4 \cdot e^0 \end{array}\right\} = 10$

$t=1$ $\quad e^{j\frac{2\pi}{4}k \cdot t} = e^{j\pi/2}$

$\vec{x} \cdot [1, -j, -1, 1]$

$= -2+2j$

$t=2$ $\quad e^{j\pi} = [1,-1,1,-1]$

$\vec{x} \cdot [1,-1,1,-1]$

$= -2$

$t=3$ $\quad \vec{x} \cdot [1, j, -1, -j]$

$= -2+2j$