Fourier Cosine Option pricing method and implementation in Premia

Bowen Zhang*

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In this article we will give a brief introduction on COS option pricing method for European and American options, with different underlying processes. Our contribution for Premia project is based on papers [1] and [2]. These papers can be find at:

http://ta.twi.tudelft.nl/mf/users/oosterle/index.html.

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This article is organised as follows. As a start, an introduction on COS pricing algorithm for European and early–exercise options are given in Section 1 and Section 2, respectively. A 4–point Richardson extrapolation scheme is applied to approximate the value of American option from Bermudan option prices with different number of early-exercise dates. In section 3, we goes through different underlying processes which we have implemented, together with our pricing method, in the Premia project. At the end, we give a few explanation words concerning our codes/programmes.

1 COS Pricing Method for European options

COS pricing method recovers the conditional density by its characteristic function through Fourier Cosine expansions. It can be applied for all processes where the characterisitc function of the underlying is available, which

*Delft University of Technology, Delft Institute of Applied Mathematics, Delft, the Netherlands, email: bowen.zhang@tudelft.nl

includes all affine processes. The method performs impressively especially when the underlying follows the Lévy processes. It starts from the risk-neutral valuation formula

$$v(x, t_0) = e^{-r\Delta t} \int_{-\infty}^{\infty} v(y, T) f(y|x) dy,$$

where $v(x,t)$ is the option value, and x, y can be any increasing functions of the underlying at t_0 and T, respectively. We truncate the integration range, so that

$$v(x, t_0) \approx e^{-r\Delta t} \int_{a}^{b} v(y, T) f(y|x) dy. \tag{1}$$

with $|\int_{\mathbb{R}} f(y|x)dy - \int_a^b f(y|x)dy| < TOL$. Error analysis of the various approximations is given in [1, 2].

The conditional density function of the underlying is then approximated by means of the characteristic function via a truncated Fourier cosine expansion, as follows:

$$f(y|x) \approx \frac{2}{b-a} {\sum_{k=0}^{N-1}}' Re(\phi(\frac{k\pi}{b-a}; x) \exp{(-i\frac{ak\pi}{b-a})}) \cos{(k\pi\frac{y-a}{b-a})}, \tag{2}$$

where Re means taking the real part of the expression in brackets, and $\phi(\omega; x)$ is the characteristic function of $f(y|x)$ defined as:

$$\phi(\omega; x) = \mathbb{E}(e^{i\omega y}|x). \tag{3}$$

The prime at the sum symbol in (2) indicates that the first term in the expansion is multiplied by one-half. Replacing $f(y|x)$ by its approximation (2) in (1) and interchanging integration and summation, gives us the COS algorithm to approximate the value of a European option:

$$v(x, t_0) = e^{-r\Delta t} {\sum_{k=0}^{N-1}}' Re(\phi(\frac{k\pi}{b-a}; x) e^{-ik\pi\frac{a}{b-a}}) V_k, \tag{4}$$

where

$$V_k = \frac{2}{b-a} \int_a^b v(y, T) \cos{(k\pi\frac{y-a}{b-a})} dy \tag{5}$$

is the Fourier cosine coefficient of $v(y,T)$, which is available in closed form for several European option payoff functions.

Formula (4) can be directly applied to calculate the value of a European option, and it also forms the basis for pricing Bermudan options.

The COS algorithm exhibits an exponential convergence rate for all processes whose conditional density $f(y|x) \in C^{\infty}((a,b) \subset \mathbb{R})$. The size of the integration interval $[a,b]$ can be determined with help of the cumulants [1].

2 COS Pricing Method for Bermudan and American Options

The pricing formula for a Bermudan option with M exercise dates, with $m = M, M-1, \ldots, 2$, is divided into a stage in which a *continuation value* is computed, and a stage where this value is compared to the payoff $g(x, t_{m-1}) \equiv v(x, T)$, as shown below

$$\begin{cases} c(x, t_{m-1}) = e^{-r\Delta t} \int_{\mathbb{R}} v(y, t_m) f(y|x) dy, \\ v(x, t_{m-1}) = \max\left(g(x, t_{m-1}), c(x, t_{m-1})\right), \end{cases} \tag{6}$$

, followed by a final computation,

$$v(x, t_0) = e^{-r\Delta t} \int_{\mathbb{R}} v(y, t_1) f(y|x) dy. \tag{7}$$

In this description, we have $x := \ln\left(S(t_{m-1})/K\right)$ and $y := \ln\left(S(t_m)/K\right)$, and $v(x,t), c(x,t), g(x,t)$ are the option value, the continuation value and the payoff at time t, respectively. For vanilla options

$$g(x, t) = \max\left[\alpha K(e^x - 1), 0\right], \quad \alpha = \begin{cases} 1 & \text{for a call,} \\ -1 & \text{for a put.} \end{cases} \tag{8}$$

Practically, for each time step we first determine an early-exercise point, x_m^*, for which $c(x_m^*, t_m) = g(x_m^*, t_m)$ by means of the Newton method. If x_m^* lies outside interval $[a, b]$ we set x_m^* equal to the nearest boundary point. At each time step, t_m, we can split the Fourier cosine coefficients $V_k(t_m)$ into two parts:

$$V_k(t_m) = C_k(a, x_m^*, t_m) + G_k(x_m^*, b), \qquad \text{for a call,} \tag{9}$$

$$V_k(t_m) = G_k(a, x_m^*) + C_k(x_m^*, b, t_m), \qquad \text{for a put.} \tag{10}$$

for $m = M-1, M-2, \ldots, 1$, and

$$V_k(t_M) = G_k(0, b) \qquad \text{for a call,}$$

$$V_k(t_M) = G_k(a, 0) \qquad \text{for a put.}$$

Here,

$$G_k(x_1, x_2) = \frac{2}{b-a} \int_{x_1}^{x_2} g(x, t_m) cos(k\pi \frac{x-a}{b-a}) dx, \tag{11}$$

$$C_k(x_1,x_2,t_m) = \frac{2}{b-a}\int_{x_1}^{x_2} \hat{c}(x,t_m)cos(k\pi\frac{x-a}{b-a})dx, \tag{12}$$

with

$$c(x,t_m) = e^{-r\Delta t}{\sum}'^{N-1}_{k=0} Re(\phi(\frac{k\pi}{b-a};x)e^{-ik\pi\frac{a}{b-a}})V_k(t_{m+1}),$$

from the Fourier cosine expansion.

$G_k(x_1,x_2)$ and $C_k(x_1,x_2,t_m)$ are the Fourier Cosine coefficients of the payoff and continuation value, repectively. $G_k(x_1,x_2)$ is known analytically. With (8), it follows for a put, with $x_2 \le 0$, that

$$G_k(x_1,x_2) = \frac{2}{b-a}\int_{x_1}^{x_2} K(1-e^x)\cos\left(k\pi\frac{x-a}{b-a}\right)dx, \tag{13}$$

and for a call, with $x_1 \ge 0$, that

$$G_k(x_1,x_2) = \frac{2}{b-a}\int_{x_1}^{x_2} K(e^x-1)\cos\left(k\pi\frac{x-a}{b-a}\right)dx, \tag{14}$$

The fact that $x^*_m \le 0$, for put options, and $x^*_m \ge 0$, for call options, $\forall t \in \mathcal{T}$, gives

$$G_k(x_1,x_2) = \frac{2}{b-a}\alpha K\left[\chi_k(x_1,x_2) - \psi_k(x_1,x_2)\right], \quad \alpha = \begin{cases} 1 & \text{for a call,} \\ -1 & \text{for a put,} \end{cases} \tag{15}$$

with

$$\chi_k(x_1,x_2) := \int_{x_1}^{x_2} e^x\cos\left(k\pi\frac{x-a}{b-a}\right)dx, \tag{16}$$

$$\psi_k(x_1,x_2) := \int_{x_1}^{x_2} \cos\left(k\pi\frac{x-a}{b-a}\right)dx. \tag{17}$$

These integrals admit analytic solutions.

We now derive the formulas for the $C_k(x_1,x_2,t_m)$. For Lévy processes, applying (4) and (6) matrix

$$C(x_1,x_2,t_m) \equiv C_k(x_1,x_2,t_m))_{j=0}^{N-1}$$

can be rewritten as

$$C_k(x_1,x_2,t_m) := e^{-r\Delta t}{\sum_{j=0}^{N-1}}' Re\Big(\phi_A\left(\frac{j\pi}{b-a}\right)V_j(t_{m+1})\cdot\mathcal{M}_{k,j}(x_1,x_2)\Big), \tag{18}$$

with

$$M_{k,j}(x_1, x_2) = \frac{2}{b-a} \int_{x_1}^{x_2} e^{ij\pi \frac{x-a}{b-a}} cos(k\pi \frac{x-a}{b-a}) dx \tag{19}$$

and for Lévy processes, $\phi_A(u, \tau)$ comes from the function

$$\phi(u; \tau) = e^{iux_0} \phi_A(u, \tau), \tag{20}$$

Matrix $M_{k,j}(x_1, x_2)$ can be split into the sum of a Toeplitz and a Hankel matrix where Fast Fourier Transform can be applied. If we denote $D(vector)$ as the discrete Fourier transform of the vector, whereas D^{-1} stands for the inverse discrete Fourier transform. Then we have

$$C(x_1, x_2, t_m) = e^{-r\Delta t} Im(M_s u + M_c u)/\pi$$

where Im means taking the imaginary part of the expression in brackets. $M_s u$ represents the first N elements of $D^{-1}(D(m_s) \cdot D(u_s))$ and $M_c u$ denotes the computation of the first N elements of $D^{-1}(D(m_c) \cdot sgn \cdot D(u_s))$, in reversed order, see [2].

In this description, we have

$$sgn = [1, -1, 1, -1, \dots]^T, \; m_s = [m_0, m_{-1}, \cdots, m_{1-N}, 0, m_{N-1}, \cdots, m_1]^T,$$

$$m_c = [m_{2N-1}, m_{2N-2}, \cdots, m_1, m_0]^T, \; u_s = [u_0, u_1, \cdots, u_{N-1}, 0, \cdots, 0]^T,$$

with elements

$$m_j = \frac{(x_2 - x_1)}{b-a} \pi i, \quad \text{if} \quad j = 0,$$

$$m_j = \frac{\exp(ij\frac{(x_2-a)\pi}{b-a}) - \exp(ij\frac{(x_1-a)\pi}{b-a})}{j}, \quad \text{if} \quad j \neq 0.$$

Finally, $u_j = \phi(\frac{j\pi}{(b-a)}) V_j(t_{m+1})$ and $u_0 = \frac{1}{2}\phi(0) V_0(t_{m+1})$.

For all time steps, $m = M-1, \cdots, 1$, approximation of $V_k(t_m)$ is recovered from (9) or (10). Option value $v(x, t_0)$ is obtained by inserting $V_k(t_1)$ into (7), and then, applying (4) with T replaced by t_1.

Let $v(M)$ denote the value of a Bermudan option with M early exercise dates. Then the following 4–point Richardson extrapolation scheme is used to estimate the value of an American option.

$$v_{AM}(d) = \frac{1}{21}(64v(2^{d+3}) - 56v(2^{d+2}) + 14v(2^{d+1}) - v(2^d)) \tag{21}$$

where $v_{AM}(d)$ denotes the approximated value of the American option.

3 Underlying Asset Processes

3.1 Black-Scholes model

In BS model, the underlying price follows a Geometric Brownian motion with constant drift and volatility, which satisfies the following stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{22}$$

where W_t is a Wiener process or Brownian motion and μ and σ are model parameters for the drift and volatility. It follows from this that the return is a Log-normal distribution with expected value $\mathbb{E}(S_t) = e^{\mu t} S_0$ and variance $\mathbb{V}ar(S_t) = e^{2\mu t} S_0^2 (e^{\sigma^2 t} - 1)$.

3.2 CGMY model

The CGMY process, as defined in [5], is a generalisation of the Variance Gamma process with the following characteristic function:

$$\phi_{CGMY}(\omega, t) = \exp(tC\Gamma(-Y)[(M - i\omega)^Y - M^Y + (G + i\omega)^Y - G^Y]). \tag{23}$$

Four parameters need to be calibrated to market data: Parameter $Y : Y < 2$ controls whether the CGMY process has finite or infinite activity. Parameter $C : C > 0$ controls the kurtosis of the distribution and non-negative parameters G, M give control over the rate of exponential decay on the right and left tails of the density, respectively.

3.3 Heston model

In the Heston stochastic volatility model, the underlying and the volatility are modeled by the following stochastic differential equations,

$$\begin{aligned} dx_t &= (r - \frac{1}{2}\mu_t)dt + \sqrt{\mu_t} dW_{1,t}, \\ d\mu_t &= \lambda(\bar{\mu} - \mu_t)dt + \eta\sqrt{\mu_t} dW_{2,t}, \end{aligned} \tag{24}$$

where x_t and μ_t denote the log–asset price process and the process of its volatility, respectively. Parameters $\lambda, \bar{\mu}, \eta$ represent the speed of mean–recursion, the long–term mean value of variance and the volatility of volatility, respectively. Moreover, $W_{1,t}$ and $W_{2,t}$ are Brownian motions, correlated with correlation coefficient ρ.

For the log-asset price in the Heston model an analytic characteristic function can be found, which reads:

$$\begin{aligned}\phi(\omega, \Delta t, \mu_0) = \quad & \exp \quad (i\omega r \Delta t + \frac{\mu_0}{\eta^2}(\frac{1-e^{-D\Delta t}}{1-Ge^{-D\Delta t}})(\lambda - i\rho\eta\omega - D)) \cdot \\ & \exp \quad (\frac{\lambda\bar{\mu}}{\eta^2}(\Delta t(\lambda - i\rho\eta\omega - D) - 2\log(\frac{1-Ge^{-D\Delta t}}{1-G})))\end{aligned}$$

with $D = \sqrt{(\lambda - i\eta\rho\omega)^2 + (\omega^2 + i\omega)\eta^2}$ and $G = \lambda - i\eta\rho\omega - D/\lambda - i\eta\rho\omega + D$.

As for the value of D, we take the square root whose real part is non-negative.

4 Premia Implementation

Oour codes only compute the option prices.

4.1 European Options

For European options we have implemented the method for the BS, CGMY and Heston models. For all these models, the user has to give the input values of the following parameters, which are common to all models:

- The initial underlying price $S0$.
- The strike price K.
- The maturity of the option T.
- The interest rate r.
- The dividend rate q.

Moreover, every model has some specific parameters:

For the BS model, the user needs to specify the volatility σ. For the CGMY model, the user needs to specify the values for C, G, M, Y. For the Heston model the values of the following parameters need to be entered:

- Initial volatility u0.
- Long term mean value of variance u.

- Speed of mean recursion λ.
- Volatility of volatility η.
- Covariance Coefficient between the processes of underlying and volatility ρ.

Our C–codes are as follows:

- BS_Euro.c is the code of COS pricing method for European option under BS model. Our default value of N is set as 128, where N is the number of terms inside Fourier Cosine expansion. With $N = 128$, we could achieve an error of order 1e–14.
- CGMY_Euro.c is the code of COS pricing method for European option under CGMY model. Our default value of N is set as 128. With $N = 128$, the error is of order 1e–10 or less for $Y \in [0.5, 2]$. For $Y < 0.5$ we suggest the user to put a larger value of N.
- Heston_Euro.c is the code of COS pricing method for European option under Heston model. Our default value of N is set as 256. With $N = 256$, we could achieve an error less than 1e–10 for long maturity and less than 1e–7 for short maturity.

4.2 American Options

For American options we need to specify the number of early exercise dates M, where the Bermudan option prices with $M, 2M, 4M, 8M$ are used in the 4–point Richardsion extrapolation scheme to get the American option values. Our C–codes are as follows:

- BS_Ame.c is the code of COS pricing method for American Put Option under BS model. Our default value of N and M are 256 and 8, which returns an error of order 1e–7 or less.
- CGMY_Ame.c is the code of COS pricing method for American Put Option under CGMY model. Our default value of N and M are 256 and 8, which returns an error of order 1e–7 or less.

Taking into consideration both efficiency and accuracy, please do NOT change the value of N and M.

References

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