Pricing of Bermudan Swaption in the Libor Market Model by Kolodko & Schoenmakers iterative construction

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March 1, 2012

Contents

Premia 14

1 LIBOR market model

This part is based on [S05].

1.1 Swap

Let us consider a set of dates $\{T_0, T_1, ..., T_k\}$ with $0 < T_0 < T_1 < ... < T_k < T$.
At time t, knowing the numeraires $B_j(t) = B(t, T_j)$, the LIBOR rates are defined by the formula

$$Lj(t) := \left(\frac{B_j(t)}{B_{j+1}(t)} - 1\right).\delta_j^{-1} \quad , T_0 \leq t \leq T_j$$

with $\delta_j := T_{j+1} - T_j$ (we then take a fixed δ). The value of a swap on those rates with strike θ, start-date T_s and end-date T_e is :

$$Swap(T_s, T_e, \theta) = (\sum_{j=s}^{e-1} B_{j+1}(T_s)\delta_j(L_j(T_s) - \theta))^+$$

1.2 LIBOR rates dynamics

In the LIBOR market model, LIBOR rates dynamics is given under the spot measure by the formula :

$$dL_i = \sum_{j=m(t)}^{i} \frac{\delta L_i L_j \gamma_i.\gamma_j}{1 + \delta L_j} dt + L_i \gamma_i.dW^*$$

where γ is a k dimensional deterministic function and W^* is in an $\mathbb{R}^k$ valued brownian motion.

The choice of γ is a calibration problem.
We here consider γ of the type

$$\gamma_i(t) = cg(T_i - t)e_i$$

with c constant and :

$$g(s) = g_\infty + (1 - g_\infty + as)e^{-bs}$$

where a, b and g_∞ are constants and e_i $\mathbb{R}^k$ vectors.

The correlation matrix $\rho = (\rho_{i,j})_{i,j} = (e_i.e_j)_{i,j}$, is taken equal to $\rho_{i,j} = exp(-\phi|i-j|)$. We may reduce the problem's dimension by principal component analysis.

1.3 Swaptions

Using that formula, we can simulate the LIBOR rates by a log-Euler scheme to price a european swaption (i.e an option on a swap) by Monte-Carlo methods.

An approximate Black formula (based on the exact formula in the Swap Model case) for european swaption with strike θ, start-date in $\{T_p, T_{p+1}, ..., T_q\}$ and end-date T_{q+1} is:

$$Swaption \simeq B_{p,q}(t).(S_{p,q}(t).N(d_+(t)) - \theta.N(d_-(t)))$$

$$d_\pm = \frac{ln(\frac{S_{p,q}(t)}{\theta})}{\sigma.\sqrt{T_p - t}} \pm \frac{\sigma.\sqrt{T_p - t}}{2}$$

$$(\sigma^B_{p,q}(t))^2 = \sum_{l,l'=p}^{q-1} \frac{v_l(t).v_{l'}(t).L_l(t).L_{l'}(t)}{S_{p,q}(t)^2} \int_t^{T_p} \gamma_l(s)^T.\gamma_{l'}(s)ds$$

$$v_l = \frac{\delta.B_{l+1}}{B_{p,q}}$$

We aim to price the Bermudan version of the swaption.

2 Discrete optimal stopping

2.1 Framework

Let $(Z_i)_{0\leq i\leq k}$ be a discrete non-negative random reward process with state space $\mathbb{R}$ adapted to some fitration $\mathbb{F} = (\mathcal{F}_i)_{0\leq i\leq k}$. For a fixed i, an $\mathbb{F}$-stopping time is called optimal stopping time in the discrete set of exercice dates $\{i, ..., k\}$, if

$$Y_i^* := \sup_{\tau\in\{i,...,k\}} E^i Z_\tau =: E^i Z_{\tau_i^*}$$

The process Y^* called the Snell envelope process is a supermartingale and the supremum is taken over all $\mathbb{F}$-stopping times τ with values in the set $\{i, ..., k\}$.
The family $(\tau_i^*)_{0\leq i\leq k}$ is called an optimal stopping family.

As a first simple example, we consider the die game :
A player is allowed to roll a die at most k times. At each turn, he has the option to earn the face value of the die or to pursue the game.
If (Z_i) represent the dice's result at the i^{th} attempt, then Y_0^* is the right price for the game.

2.2 Application to finance

Consider $L(t)$ a $\mathbb{R}^d$ valued random process on a finite time interval $[0, T]$ adapted to a filtration $(\mathcal{F}(t))_{0\leq t\leq T}$ and a set of dates $\mathbb{T} := \{\mathcal{T}_0, ..., \mathcal{T}_n\}$ with $0 \leq \mathcal{T}_0 < \mathcal{T}_1 < ... < \mathcal{T}_n \leq T$.
An option issued at time $t = 0$ to obtain a cash flow $C_{\mathcal{T}_\tau} := C(\mathcal{T}_\tau, L(\mathcal{T}_\tau))$ at date $\mathcal{T}_\tau \in \mathbb{T}$ chosen by the option holder is called a Bermudan style derivative.
With respect to a pricing mesure P connected with some pricing numeraire B, the value of the Bermudan derivative at $t = 0$ is

$$V(0) := B(0) \sup_{\tau\in\{0,...,n\}} E^{\mathcal{F}(t=0)} \frac{C_{\mathcal{T}_\tau}}{B(\mathcal{T}_\tau)}$$

where the supremum argument is taken over all $\mathbb{F}$-stopping times with $\mathcal{T}_\tau$ in the set $\{\mathcal{T}_i, ..., \mathcal{T}_k\}$.

Defining $Z_i = \frac{C_{\mathcal{T}_i}}{B(\mathcal{T}_i)}$, Y_0^* gives the price of the Bermudan option.

<u>Remarks :</u>
- We here consider the case where the exercise dates $\mathcal{T}_i$ and the rates end dates T_i conincide, and we will denote them by T_i from now on.
- In some cases, the Bermudan version of a derivated may be trivialy priced. For example, for a call with log-normal underlying, choosing the last exercice date to exercise is always optimal. The hump shape of the European prices as a function of maturity shows that there is some interest in computing optimal exercice dates for Bermudan derivatives. This is the case in particular for swaptions in the LIBOR market model and basket put with log-normal underlying.
We also must keep in mind that the Bermudan derivative is more expensive that all European derivatives with maturities $\mathcal{T}_0$, $\mathcal{T}_1$ ... $\mathcal{T}_n$ because one can choose arbitrairement the exercice date.

3 Methods

3.1 Backward dynamic programming

This is the canonical method to solve discrete optimal stopping problems. The Snell envelope is constructed as follows :
We have trivially $Y_k^* = Z_k$, and for $0 \leq j < k$ we have $Y_j^* = \max(Z_j, E^{\mathcal{F}(j)} Y_{j+1}^*)$.
An optimal stopping family is represented by :
$\tau_i^* = \inf\{j, i \leq j \leq k : Y_j^* \leq Z_j\}$

The 2 following methods are respectively described in [AND99] & [KOL05]. The 3 following methods correspond of the chapter 5 of [SCH05] : Pricing of Bermudan Style Libor Derivatives.

3.2 Andersen algorithm

This method is described in [A99] and [S05]. We aim to find an optimal exercice frontier for the payoff : $(H(T_i))_{0 \leq i \leq k}$.

This frontier is found by backward programming using Monte-Carlo simulations.

This method is exact in dimension 1 and gives good results in multi-dimensional cases such as Bermudan swaption.

We can save time searching for a H with less degrees of freedom, for instance a piecewise affine H.

We propose 4 alternative strategies (H is computed in respect with the strategy):
<u>strategy 1 :</u> exercice if $Z_i > H(T_i)$
<u>strategy 2 :</u> exercice if $Z_i > H(T_i)$ and $Z_i > BE(T_i)$
<u>strategy 3 :</u> exercice if $Z_i > H(T_i) + BE(T_i)$
<u>strategy 4 :</u> exercice if $Z_i > H(T_i)$ and $Z_i > NE(T_i)$
<u>strategy 5 :</u> exercice if $Z_i > H(T_i) + NE(T_i)$
where, $BE(T_i)$ denotes the most expensive of all european derivates at time T_i and maturities $T_j, i \leq j \leq k$ and $NE(T_i)$ is the value of the european derivative with maturity T_{i+1} at time T_i

Strategies 1 & 2 are mentionned in [A99]. We can find for instance strategies 1, 2 & 3 in [S05] and all strategies in [SS03].

3.3 Kolodko & Schoenmakers iterative construction of the optimal Bermudan stopping time

This method is described [KS05] and [S05].
A one step improvement upon a given family of stopping times
Considering a family of integer valued stopping indexes $(\tau_i)_{0\leq i\leq k}$, let us define :

$$\tilde{Y}_i := \max_{p:i\leq p\leq k} E^{\mathcal{F}(i)} Z_{\tau_p}$$

Let us consider a new family of integer valued stopping indexes :

$$\hat{\tau}_i := \inf\{j : i \leq j \leq k, \tilde{Y}_j \leq Z_{\tau_j}\}$$

and the process $\hat{Y}_i := E^{\mathcal{F}(i)} Z_{\hat{\tau}_i}$.
Then, we have $Y_i \leq \tilde{Y}_i \leq \hat{Y}_i \leq Y_i^*, 0 \leq i \leq k$. See Th 3.1 in [KS05].

Iterative construction of the optimal stopping time and the Snell envelope process
Let us start with the family $\tau_i^{(0)} \equiv i$ and define the sequence $\tau^{(n)}$ by $\tau^{(n+1)} = \hat{\tau}^{(n)}$.
Then, the process $Y_i^{(m)} := E^{\mathcal{F}_i} Z_{T_{\tau_i^{(m)}}}$ converges to the Snell envelope (and $Y_0^{(m)}$ to the price). See Th 4.2 in [KS05]. In fact $Y_i^{(m)}$ equals Y_i^* as $m \geq k-i$.
A simple way to understand this convergence is the fact that this methods is in a way a non backward reformulation of backward dynamic programming.

Remarks :
- The efficiency of K & S's algo is based on convergence's speed. Indeed, the cost is exponential in m. In practice, we won't exceed m =2.
- $Y_0^{(1)}$ is the price with the strategy : exercice in T_i if the payoff is bigger than all europeans with maturities T_{i+1}, ..., T_k (as in Andersen strategy 2).

3.4 Dual approach

This method is described [KS04] and [S05].rough upper bound
We first notice that

$$V(0) := B(0) \sup_{\tau\in\{0,\ldots,n\}} E^{\mathcal{F}(\tau)} \frac{C_{T_\tau}}{B(T_\tau)} \leq B(0) E^{\mathcal{F}(0)} \sup_{t\in\{0,\ldots,n\}} \frac{C_{T_t}}{B(T_t)}$$

Notice that in this formula, the first supremum argument is taken over all $\mathbb{F}$-stopping times with T_τ in the set $\{T_i, ..., T_k\}$ and the second supremum argument is taken simply over all $\{i, ..., k\}$.
This rough upper bound for the price can be easily computed by Monte-Carlo simulation.
It would be the expected payoff of the Bermudan option for somebody who could guess (as an oracle) the optimal exercice date.

tight upper bound
If Y is a lower estimator process of the Snell envelope, an upper estimator is given by the formula $Y_i^{up} = E \sup_{i\le j\le k}(Z_j - \sum_{i=1}^{j} Y_i + \sum_{i=1}^{j} E^{\mathcal{F}_{i-1}} Y_i)$.

The closer Y is from Y^*, the closer is Y^{up} from Y^* as shown in [S05] or [KS05].
So, the computation of Y_0^{up} from Y given by Andersen's or Kolodko & Schoenmakers' method gives a good tight upper approximation for the price.

As described in [SCH05], we simulate both an upper and a lower estimator for Y^{up}.
upper : $Y_i^{up^{up}} = E \sup_{i\le j\le k}(Z_j - \sum_{i=1}^{j} Y_i + \sum_{i=1}^{j} \frac{1}{K} \sum_{q=1}^{K} \xi_{i,q})$
where $\xi_{i,q}$ are K copies of Y_i under the conditional measure $P(L_{i-1}, .)$.
lower : $Y_i^{up_{low}} = E(Z_{\hat{j}_{max}} - \sum_{i=1}^{\hat{j}_{max}} Y_i + \sum_{i=1}^{\hat{j}_{max}} \frac{1}{K} \sum_{q=1}^{K} \xi_{i,q})$
where $\hat{j}_{max}$ is an estimation of $\mathrm{argmax}_{i\le j\le k}(Z_j - \sum_{i=1}^{j} Y_i + \sum_{i=1}^{j} \frac{1}{K} \sum_{q=1}^{K} \xi_i, q)$ computed during the simulation of $Y_i^{up^{up}}$ and the two collections of $\xi_{i,q}$ are simulated independently.

A computation of a combined estimator $Y_0^{up,\alpha} = \alpha Y_0^{up^{up}} + (1-\alpha) Y_0^{up_{low}}$ for the upper estimator is described in [S05].

In order to compute a suitable α, we use linear regression, to search for $c_u, c_l, \beta_u, \beta_l$ minimizing

$$\sum_{i=2}^{6} \left(\frac{\log(Y_0^{up^{up}} - < Y_0^{up} >) - (\log(c_u) - \beta_u \log(2^i))}{\log(Y_0^{up^{up}} - < Y_0^{up} >)}\right)^2$$

$$\sum_{i=2}^{6} \left(\frac{\log(< Y_0^{up} > - Y_0^{up_{low}}) - (\log(c_l) - \beta_l \log(2^i))}{\log(< Y_0^{up} > - Y_0^{up_{low}})}\right)^2$$

where $< Y_0^{up} > = Y_0^{up,\alpha=\frac{1}{2}} = \frac{Y_0^{up^{up}} + Y_0^{up_{low}}}{2}$.
Let $\alpha = \frac{c_l}{c_u + c_l}$.
We can simply get c_u and c_l as $c_u =< Y_0^{up^{up}} - < Y_0^{up} >>$ and $c_l =<< Y_0^{up} > - Y_0^{up_{low}} >$.
Computing α in this way give good hopes that $Y_0^{up,\alpha}$ by a tight estimator

of Y_0^{up}.
The time for a precise computation of this upper bound needs is bigger than the one for Andersen or Kolodko & Schoenmakers' methods for the lower bounds but is reasonable.

4 Implementation

4.1 Program structure

This C++ program is structured as follows.
In a mother class called LMM (for LIBOR market model) is given all the information concerning the LIBOR rates : log-Euler Scheme and Black approximation formula.
Derived classes contain the algorithms used to find bermudan price approximation as explained in this table.

class	derived from	contents
LMM	-	all information about rates : LIBOR's rates dynamic, Black formula
Y1	LMM	K&S's algo (1 iteration)
Y2	LMM	K&S's algo (2 iterations)
Andersen	LMM	Andersen's algo
Andersenhat	Andersen	Andersen's algo + 1 iteration of K&S's algo
Y1up	LMM	upper estimator with K&S's algo (1 iteration)
YAup	Andersen	upper estimator with Andersen's algo

4.2 Simple and more refined versions

We wrote a simple version with γ equal to a constant using a 1-dimentional coefficient σ and a low log-Euler scheme precision : $\Delta_t = \delta$. In this simple version both simulation of LIBOR's rates dynamic and Black formula are simpler and faster. So, all pricing algorithms terminate in under 1 minute on a 2 GHz PC.
The simple version has the same parameters as Sonke Blunck's program (Premia 2005) : rates' float initial value : 6 %, strike : 6 %, $\sigma = 0.2$, $\delta = 0.5$ (6 months), swaption maturity : 1 year, swap maturity : 4 years.
We here give a table comparing our program's results with S. Blunck 's program's results.

algorithm	price estimator(SD)
Premia, S Blunck - Andersen (strategy 1)	157
Premia, S Blunck - Andersen (strategy 2)	157
our results :	
rough lower estimator	123
rough upper estimator	198
Andersen (strategy 1)	157.0 (0.1)
Andersen (strategy 2)	157.0 (0.1)
K & S (1 iteration)	156.0 (0.1)
K & S (2 iterations)	156.7 (0.4)
upper estimator with Andersen's algo	156.5 (1.7)
upper estimator with K & S's algo	157.5 (1.2)

We also wrote a refined version (see An important remark) to show how Kolodko & Schoenmakers' method is better than Andersen's in case the information on rates is richer (OTM case, high multi-dimensional case) in respect with [KS05].
We so can compare results of the refined version with the ones in [KS05]. We didn't get very tight result because of the long time needed to compute the results.
Refined version parameters : $k = 40$, $\delta = 0.25$, Principal component analysis : $d = 10$
$c = 0.2$, $a = 1.5$, $b = 3.5$, $g_\infty = 0.5$, $\phi = 0.0413$.

	price in [KS05] (SD)	price with our program (SD)
K & S's algo (1 iteration)	104.2 (0.1)	104 (1.)
K & S's algo (2 iterations)	110.5 (0.6)	112 (2.)
Andersen's algo (strat 1)	102.8 (.6)	100 (1.5)

4.3 An important remark

We could have put the methods in the mother class and the information on the model in a derived class. Then, it would have been easier to change the model for instance pricing Bermudan basket put instead of Bermudan swaption. Our choice has the advantage to show clearly how works each method.
With our choice, changing the field of the mother class Libor in order to use a simpler model or to price something else as Bermudan basket put or the dice game remains quite easy.
Indeed, please note that the 2 versions (simple & refined) have very small differenties. In fact, the parameter σ in the simple version is replaced by the parameters a, b, c, ϕ, g_∞ in the refined version. A matrix as a field is added in class LMM and the rates dynamics and the Black formula are changed.

The user may write a customised version quickly.

4.4 Basket put pricer

It is also quite easy to change the program into a basket put (on a log-normal underlying) pricer by changing parameters, dynamics and Black formula (for a basket put there is an approximation Black & Scholes formula given in [BKS05]).
We wrote a basket put version and compared its results with [BKS05] and other Premia programs.

algorithm	price (SD)
in the paper [BKS05], lower	2.400 (0.005)
in the paper [BKS05], upper	2.406 (0.004)
Premia (Longstaff Schwarz)	$\simeq$ 2.40
Premia (LS Importance Sampling)	$\simeq$ 2.40
our results :	
K & S's algo (1 iteration)	2.361 (0.005)
K & S's algo (2 iterations)	2.40 ($<$ 0.01)
Andersen's algo (strat 1)	2.403 (0.002)
upper estimator with Andersen's algo	2.403 (0.003)
upper estimator with K & S's algo	2.403 (0.002)

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