Estimating the Delta of Options by the Likelihood Ratio Method

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Abstract

We estimate the delta of options in exponential Lévy models by the likelihood ratio method. The latter needs the probability density functions of Lévy increments and their derivatives. These densities are often known through their charateristic functions. Hence, as proposed by Glasserman and Liu (see [1, 2]), we use saddlepoint approximations.

1 Preliminaries

A real Lévy process X is characterized by its generating triplet (γ, σ^2, ν). Where $(\gamma, \sigma) \in \mathbb{R} \times \mathbb{R}^+$, and ν is a Radon measure satisfying

$$\int_{\mathbb{R}} \left(1 \wedge x^2\right) \nu(dx) < \infty.$$

By Lévy-Itô decomposition X can be written in this form

$$X_t = \gamma t + \sigma B_t + X_t^l + \lim_{\epsilon \downarrow 0} \widetilde{X}_t^\epsilon, \tag{1.1}$$

where

$$X_t^l = \int_{|x|>1, s\in[0,t]} x J_X(dx \times ds) \equiv \sum_{0\leq s\leq t}^{|\Delta X_s|\geq 1} \Delta X_s$$

$$\begin{aligned} \widetilde{X}_t^\epsilon &= \int_{\epsilon\leq|x|\leq 1, s\in[0,t]} x(J_X(dx\times ds) - \nu(dx)dt) \\ &\equiv \int_{\epsilon\leq|x|\leq 1, s\in[0,t]} x\widetilde{J}_X(dx\times ds) \\ &\equiv \sum_{0\leq s\leq t}^{\epsilon\leq|\Delta X_s|<1} \Delta X_s - t\int_{\epsilon\leq|x|\leq 1} x\nu(dx), \end{aligned}$$

Where J is a Poisson measure on $\mathbb{R} \times [0, \infty)$ with rate $\nu(dx)dt$ and B is a standard Brownian motion. In Lévy-Khinchine representation X, we characterize X by its characteristic function. That means

$$\mathbb{E}e^{iuX_t} = e^{t\varphi(u)} \quad \forall u \in \mathbb{R},$$

where φ is given by

$$\varphi(u) = i\gamma u - \frac{\sigma^2 u^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{1}_{|x|\leq 1})\nu(dx). \tag{1.2}$$

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2 Likelihood ratio method

In this section we will evaluate the delta of an option using the likelihood ration method. For details about this method and the evaluation of greeks see [1]. Let $(S_t)_{t\geq 0]}$ be the price of a security, the process S behaves as the exponential of a Lévy process

$$S_t = S_0 e^{X_t}, \quad \forall t \geq 0.$$

We are interested in options with discounted payoff of the form

$$V(S) = V(S_{t_1}, \ldots, S_{t_m}),$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_m$. We let $\Delta_i = t_i - t_{i-1}$ for $i \in \{1, \ldots, m\}$. Thus S is a function of $X = (X_1, \ldots, X_m)$, where $X_i = X_{t_i} - X_{t_{i-1}}$. Note that the r.v. $(X_i)_{1\leq i\leq m}$ are independent and for any $i \in \{1, \ldots, m\}$ X_i has the same distribution as X_{Δ_i}. When estimating $\mathbb{E}(V(S))$, we first simulate $(X_1, \ldots, X_m)$ and map these to S to evaluate $V(S)$. We suppose that for any $t > 0$, X_t has a probability density function denoted by f_t. Then the joint density of $X = (X_1, \ldots, X_m)$ is

$$f(x) = f_{\Delta_1}(x_1) \ldots f_{\Delta_m}(x_m) \equiv f_1(x_1) \ldots f_m(x_m),$$

and might write the expected payoff as

$$\mathbb{E}(V(S)) = \int V(s) f(x) dx, \tag{2.3}$$

where $s = (s_1, \ldots, s_m)$ and $x = (x_1, \ldots, x_m)$. Recall that for $i \in \{1, \ldots, m\}$

$$S_{t_i} = S_0 e^{X_{t_i}} \equiv S_{t_{i-1}} e^{X_i}.$$

So

$$\begin{aligned}
\mathbb{E}(V(S)) &= \int V(S_0 e^{x_1}, s_2, \ldots, s_m) f_1(x_1) \ldots f_m(x_m) dx_1 \ldots dx_m \\
&= \int V(e^{x_1}, s_2, \ldots, s_m) f_1(x_1 - \log(S_0)) f_2(x_2) \ldots f_m(x_m) dx_1 \ldots dx_m.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\partial}{\partial S_0}\mathbb{E}(V(S)) &= -\frac{1}{S_0}\int V(e^{x_1}, s_2, \ldots, s_m) f_1'(x_1 - \log(S_0)) f_2(x_2) \ldots f_m(x_m) dx_1 \ldots dx_m \\
&= -\frac{1}{S_0}\int V(s_1, \ldots, s_m) f_1'(x_1) f_2(x_2) \ldots f_m(x_m) dx_1 \ldots dx_m \\
&= -\frac{1}{S_0}\int V(s) f(x) \frac{f_1'(x_1)}{f_1(x_1)} dx \\
&= \mathbb{E}(V(S) S_f(X)),
\end{aligned}$$

where S_f called the score function, is given by

$$S_f(x) = -\frac{f_1'(x_1)}{S_0 f_1(x_1)}.$$

3 Saddlepoint approximations

We will use saddlepoint approximations to approximate the probability density functions and derivatives of their logarithm (to approximate the score function defined in the previous section). Suppose that a random variable X has a probability density function f and a cumulant generating function

$$K(s) \quad = \quad \mathbb{E}e^{sX}.$$

The function is assumed to be finite of the neighborhood of the origin. By [3] (p.27)

$$f(x) \quad = \quad \frac{1}{\sqrt{2\pi K''(\hat{s})}} e^{K(\hat{s})-\hat{s}x}\left(1+\frac{\lambda_4(\hat{s})}{8}+\frac{1}{8}\lambda_4(\hat{s})-\frac{5}{24}\left(\lambda_3(\hat{s})\right)^2+\dots\right),$$

where $\hat{s}$ (called the saddlepoint approximation) satisfies $K'(\hat{s})=x$, and

$$\lambda_n(s)=\frac{K^{(n)}(s)}{\left(K''(s)\right)^{\frac{n}{2}}}.$$

So $f(x)$ can be approximated by

$$\frac{1}{\sqrt{2\pi K''(\hat{s})}} e^{K(\hat{s})-\hat{s}x}\left(1+\frac{\lambda_4(\hat{s})}{8}+\frac{1}{8}\lambda_4(\hat{s})-\frac{5}{24}\left(\lambda_3(\hat{s})\right)^2\right), \tag{3.4}$$

or (less precise) by

$$\frac{1}{\sqrt{2\pi K''(\hat{s})}} e^{K(\hat{s})-\hat{s}x}. \tag{3.5}$$

If X has a Gaussian distribution, (3.5) is exactly the probability density function of X at x.

To approximate the score function, we have to take the logarithm of (3.4) and differentiate. To do so, we must evalute $\frac{\partial\hat{s}}{\partial x}$. But we have $K'(\hat{s})=x$. Thus

$$\frac{\partial\hat{s}}{\partial x}K''(\hat{s})=1.$$

Therefore

$$\frac{\partial\hat{s}}{\partial x}=\frac{1}{K''(\hat{s})}. \tag{3.6}$$

But we have

$$\begin{aligned}\log\left(f(x)\right) \quad &= \quad \log\left(\frac{1}{\sqrt{2\pi K''(\hat{s})}} e^{K(\hat{s})-\hat{s}x}\right)+\log\left(1+\frac{\lambda_4(\hat{s})}{8}+\frac{1}{8}\lambda_4(\hat{s})-\frac{5}{24}\left(\lambda_3(\hat{s})\right)^2\right)\\ &\equiv \quad g_1(x)+g_2(x),\end{aligned}$$

where g_1 (resp. g_2) is the first (resp. the second) term at right of the penultimate equality. So

$$\frac{f'(x)}{f(x)}=g_1'(x)+g_2'(x), \tag{3.7}$$

where, using (3.6), we have

$$\begin{aligned} g_1'(x) \quad &= \quad -\left(\frac{K^{(3)}(\hat{s})}{2\left(K''(\hat{s})\right)^2}+\hat{s}\right)\\ g_2'(x) \quad &= \quad \frac{\frac{1}{8}\left(K^{(5)}(\hat{s})K''(\hat{s})-2K^{(3)}(\hat{s})K^{(4)}(\hat{s})\right)-\frac{5}{12}\left(K^{(4)}(\hat{s})-\frac{3\left(K^{(3)}(\hat{s})\right)^2}{2K''(\hat{s})}\right)K^{(3)}(\hat{s})}{g_2(x)\left(K''(\hat{s})\right)^4}.\end{aligned}$$

4 Jump-diffusion case

If $\nu(\mathbb{R}) < \infty$, X is called a finite activity Lévy process. If in addition $\sigma > 0$, the process X is called a *jump-diffusion* process, and can be written in this form

$$X_t = \gamma_0 t + \sigma B_t + \sum_{i=1}^{N_t} Y_i, \tag{4.8}$$

where N is a Poisson process with rate $\lambda = \nu(\mathbb{R})$, $(Y_i)_{i\geq 1}$ are i.i.d. random variables with common distribution $\frac{\nu(dx)}{\nu(\mathbb{R})}$ and

$$\gamma_0 = \gamma - \int_{|x|\leq 1} x\nu(dx). \tag{4.9}$$

If the $(\Delta_i)_i\, 1 \leq i \leq m$ (defined in Section 2) are equal to $\frac{T}{m}$, we have for any $i \in \{1, \ldots, m\}$ (see Section 2)

$$X_1 = \gamma_0 \frac{T}{m} + \sigma B_{\frac{T}{m}} + \sum_{i=1}^{N_{\frac{T}{m}}} Y_i.$$

The constant T is the maturity of the option. We can, thus, approximate f_1, the probability density function of X_1, by the probability density function of $\gamma_0 \frac{T}{m} + \sigma B_{\frac{T}{m}}$. This gives the following approximation of the score function.

$$S_f(x) = \frac{x - \gamma_0 \frac{T}{m}}{\sigma^2 \frac{T}{m} S_0}. \tag{4.10}$$

Though the error generated by such approximation should be studied, numerically we obtain good results quickly.

References

[1] Glasserman, P., Liu Z.: Estimating greeks in simulating Lévy-driven models, Working paper, Columbia University (2008). 1, 2

[2] Glasserman, P., Liu, Z.: Sensitivity estimates from characteristic functions, Working Paper, Columbia University (2008). 1

[3] Jensen, J.: Saddlepoint Approximations, Oxford University Press, Oxford, UK (1995). 3