FFT-BASED BACKWARD INDUCTION METHOD FOR OPTION PRICING UNDER LÉVY PROCESSES

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ABSTRACT. We describe "The Fourier Space Time-stepping method" (FST-method) for pricing barrier and American options for a wide class of Lévy processes. The method uses a backward induction approach and the Fast Fourier Transform algorithm for the efficient computation of the convolution with the probability density. The method implemented into Premia 13 is based on the one developed in Jackson et al. (2008).

1. Introduction

In recent years more and more attention has been given to stochastic models of financial markets which depart from the traditional Black-Scholes model. At this moment a wide range of models is available. One of the tractable empirical models are jump diffusions or, more generally, Lévy processes. We concentrate on the one-dimensional case. For an introduction on these models applied to finance, we refer to Cont and Tankov (2004).

By now, there exist several large groups of relatively universal numerical methods for pricing of American and barrier options under exponential Lévy processes. The number of publications is huge, and, therefore, an exhaustive list is virtually impossible. We concentrate on the one-dimensional case.

Existing numerical methods in literature can be categorized into three groups: Monte Carlo simulation, partial-(integro) differential equation (PIDE) methods, and backward induction methods. We will consider the last group.

The backward induction methods are based on the fact that the risk-neutral valuation formula for the European option can be seen as a convolution of the payoff function with the transition density. The key idea is to set up a time lattice and view the option as of European type between two adjacent dates. Hence, the backward induction method requires the transition density to be known in closed-form, which is the case in e.g. the Black-Scholes model and Merton's jump-diffusion model. The approximation proposed by Geske and Johnson (1984) uses the discretization of the time parameter and the backward induction for pricing American options in the GBM model. The

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method was extended in Boyarchenko and Levendorskiĭ (2002) for some Lévy models, and its applications can be founded e.g. in Kudryavtsev and Levendorskiĭ (2006) and Levendorskiĭ et al. (2006). If there is no an explicit formula for the probability density, it can be recovered by inverting the characteristic function, so the method can be used for a wide range of Lévy models.

Since convolutions can be handled very efficiently by means of the Fast Fourier Transform (FFT), an overall complexity of the method is $O(mn\ln n)$, where m and n are the numbers of points on the grid in time and space, respectively. The FFT-based backward induction method was applied in Jackson et al. (2008), see also Lord et al. (2008).

First, we will consider the model case of a down-and-out put option, without rebate. The down-and-out call options and up-and-out calls and puts can be reduced to the model case by suitable change of numeraire and/or the direction on the real axis. The method is applicable to American options as well, and, after straightforward modifications, to barrier options with a rebate (hence, to digitals as well).

2. Backward induction method for the down-and-out barrier options

Let T, K, H be the maturity, strike and barrier, and the stock price $S_t = e^{X_t}$ is an exponential Lévy process under a chosen risk-neutral measure. The riskless rate is assumed constant. Set $h = \ln H$. Then the payoff at maturity is $\mathbf{1}_{(h,+\infty)}(X_T)G(X_T)$, where $G(x) = (K - e^x)_+$, and the no-arbitrage price of the barrier option at time $t < T$ and $X_t = x > h$ is given by

$$V(x,t) = V(T,H;G;t,x) = E^{t,x}\left[e^{-r(T-t)}\mathbf{1}_{\underline{X}_T > h}G(X_T)\right], \tag{2.1}$$

where $\underline{X}_t = \inf_{0\le s\le t} X_t$ is the infimum process. Further, without loss of generality, we will assume that $h = 0$.

In Section A.1, we remind the definitions of the Lévy density $F(dy)$ of the Lévy process X_t and characteristic exponent $\psi(\xi)$, and explicit formulas for ψ and L, the infinitesimal generator of X_t. We also list several classes of Lévy processes used in empirical studies of financial markets.

We approximate $V(X_t,t)$ in (2.1) by the price $f(X_t,t)$ of the down-and-out barrier option in the corresponding discrete time model with equally spaced dates t_k, $k = 0,1,\dots,m$, where $t_0 = 0$, $t_m = T$. Set $\Delta\tau := T/m$. We have

$$f(x,t_m) = (K - e^x)_+,\ x > 0, \tag{2.2}$$

and for all k,

$$f(x,t_k) = 0,\ x \le 0. \tag{2.3}$$

For $k = m-1, m-2, \dots,$ and $x > 0$, the price $f(x,t_k)$ can be found as the price of the European option with the terminal payoff $f(X_{t_{k+1}}, t_{k+1})$ and the expiry date t_{k+1}:

$$f(x,t_k) = E[e^{-r\Delta\tau}f(X_{t_{k+1}}) \mid X_{t_k} = x],\ x > h. \tag{2.4}$$

If an explicit formula for the probability density $p_{\Delta\tau}$ of $X_{\Delta\tau}$ under EMM is known (e.g. GBM or NIG model), we can use it to write (2.4) in the form

$$f(x,t_k) = e^{-r\Delta\tau}\int_{-\infty}^{+\infty} p_{\Delta\tau}(y)f(x+y,t_{k+1})dy, x > 0. \tag{2.5}$$

In the general case, $p_{\Delta\tau}$ can be expressed in terms of the characteristic exponent $\psi(\xi)$, by using the Fourier transform

$$p_{\Delta\tau}(x) = (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{-ix\xi-\Delta\tau\psi(\xi)}d\xi. \tag{2.6}$$

Now, we can rewrite (2.5) by using (2.6), and we obtain for $x > 0$

$$f(x,t_k) = e^{-r\Delta t}(2\pi)^{-1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x+y,t_{k+1})\exp[-iy\xi-\Delta\tau\psi(\xi)]d\xi dy.$$

We change a variable $z = x + y$. If $f(x,t_{k+1})$ is absolutely integrable, we can change the order of the integration:

$$\begin{aligned} f(x,t_k) &= e^{-r\Delta t}(2\pi)^{-1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(z,t+\Delta\tau)e^{-i(z-x)\xi}\exp[-\Delta\tau\psi(\xi)]d\xi dz \\ &= (2\pi)^{-1}\int_{-\infty}^{+\infty}\hat{f}(\xi,t_{k+1})\exp[ix\xi-\Delta\tau(r+\psi(\xi))]d\xi, x > 0, \end{aligned} \tag{2.7}$$

where $\hat{f}(\xi,t_{k+1})$ is the Fourier transform of a function $f(z,t_{k+1})$ in the first variable. The integral operator in the RHS of the formula (2.7) can be represented as a pseudo-differential operator (PDO) with the symbol

$$\Psi_{\Delta\tau}(\xi) = \exp[-\Delta\tau(r+\psi(\xi))]. \tag{2.8}$$

Recall that a PDO $A = a(D)$ with the symbol $a(\xi)$ acts as follows:

$$Au(x) = (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{ix\xi}a(\xi)\hat{u}(\xi)d\xi, \tag{2.9}$$

where $\hat{u}$ is the Fourier transform of a function u:

$$\hat{u}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi}u(x)dx.$$

Thus, in terms of PDO, we can rewrite the algorithm in the following form.

$$\begin{aligned} f(x,t_k) &= \Psi_{\Delta\tau}(D)f(x,t_{k+1}), x > 0 \\ f(x,t_k) &= 0, x \le 0. \end{aligned}$$

Note that the inverse Fourier transform in (2.9) is defined in the classical sense only if the symbol $a(\xi)$ and function $\hat{u}(\xi)$ are sufficiently nice. In general, one defines the (inverse) Fourier transform, $F_{x\to\xi}$ (the subscript means that a function defined on the x–space becomes a function defined on the dual ξ-space), and the inverse Fourier transform, $F^{-1}_{\xi\to x}$ by duality; in many cases, in particular, in the context of pricing of put and call options, one can use the classical definition of the integral but integrate in (2.9) not over the real line but along an appropriate line in the complex plane. It means that we integrate the function with an appropriate exponential weight. See, e.g., Boyarchenko and Levendorskiǐ (2002) for details and examples.

To improve the convergence, one can introduce new function:

$$f_\omega(x,t_k) = e^{\omega x}f(x,t_k), k = 0,1,\ldots,m, \tag{2.10}$$

where $\omega \in \mathbf{R}$ is chosen in a such way that $f_\omega(x, t_k)$ is absolutely integrable. Then taking into account that $e^{\omega x}\Psi_{\Delta\tau}(D)e^{-\omega x} = \Psi_{\Delta\tau}(D + i\omega)$, we obtain the following backward recursion for $f_\omega(x, t_k)$.

$$\begin{aligned} f_\omega(x, t_k) &= \Psi_{\Delta\tau}(D + i\omega)f_\omega(x, t_{k+1}), x > 0 \\ f_\omega(x, t_k) &= 0, x \leq 0. \end{aligned}$$

Formally, the action of a PDO A with the constant symbol $a(\xi)$ can be described as the composition

$$Au(x) = F^{-1}_{\xi \to x}a(\xi)F_{x\to\xi}u(x) \tag{2.11}$$

If the functions u and a are represented as arrays suitable for application of the Fast Fourier Transform and inverse Fast Fourier Transfrom algorithms (FFT and iFFT), then (2.11) can be programmed as $Au = iFFT(a. * (FFT(u)))$.

3. Backward induction for American put

We consider the American put on a stock which pays no dividends; the generalization to the case of a dividend-paying stock and the American call is straightforward. (Moreover, as it is well-known, changing the direction on the line, the unknown function, the riskless rate and the process, one can reduce the pricing problem for the American call to the pricing problem for the American put).

Let $V(t, S_t)$ be the price of American put with the strike price K and the terminal date T. Set $x = \ln(S/K)$, $G(x) = K(1-e^x)_+$ and $v(t, x) = V(t, Ke^x)$. Again we divide $[0, T]$ into n subperiods by points $t_j = j\Delta t, j = 0, 1, \ldots, m$, where $\Delta\tau = T/m$, and denote by $f(x, t)$, the price of the corresponding option which can be exercised before expiry but only on dates $t_1 < t_2 < \cdots < t_m$ fixed in advance; h_j denotes the approximation to the early exercise boundary at time t_j. Such the approximation to $v(x, t)$, the *Geske-Johnson approximation*, can be interpreted as an approximation of an American option by the corresponding Bermudan option.

In the Gaussian case, the explicit analytical formula for the joint distribution of the stock prices $(S_{t_1}, \ldots, S_{t_m})$ is known, and so the application of the following procedure is straightforward.

For $j = m$, when $t_m = T$, $f(x, t_m)$ is the terminal payoff: $f(x, t_m) = g(x)$. At time t_j, $j = m-1, m-2, \ldots, 1$, the option owner chooses the optimal exercise boundary h_j as the solution to the equation

$$G(x) = e^{-r\Delta\tau}E[f(X_{t_{j+1}}, t_{j+1})|X_{t_j} = x], \tag{3.1}$$

In (3.1), the LHS is the payoff at the current level of the log-price, x, and the RHS is the discounted expected value of keeping the option alive. If an analytical formula for the probability density $p_{\Delta\tau}$ of the process X_t under EMM is known, as is the case with NIG, or Black-Scholes model, then (3.1) can be solved numerically, and after the optimal exercise boundary h_j is found, $f(x, t_j)$ is calculated as

$$f(x, t_j) = \begin{cases} e^{-r\Delta\tau}\int_{-\infty}^{+\infty} p_{\Delta\tau}(y)f(x + y, t_{j+1})dy, & x > h_j, \\ G(x), & x \leq h_j. \end{cases}$$

In the general case, $p_{\Delta\tau}$ can be expressed in terms of the characteristic exponent $\psi(\xi)$, by using the formula (2.6). To speed up the computation of the convolutions again we use Fast Fourier Transform (see details in Jakson et al. (2008)) and introduce weighted functions $f_\omega(x,t_j)$ (see (2.10). Now we can write the formula for $f_\omega(x,t_j)$ as follows,

$$f_\omega(x,t_j) = \max\{\Psi_{\Delta\tau}(D+i\omega)f(x,t_{j+1}), e^{\omega x}G(x)\}.$$

3.1. **Approximation of $\Psi(D)$ using Fast Fourier Transform.** Let d be the step in x-space, ζ–the step in ξ-space, and $M = 2^m$ the number of the points on the grid; decreasing d and increasing (even faster) M, we obtain a sequence of approximations to the option price. An approximation for the $\Psi(D)$ operator action can be efficiently computed by using the Fast Fourier Transform (FFT). Consider the algorithm (the discrete Fourier transform (DFT)) defined by

$$G_l = DFT[g](l) = \sum_{k=0}^{M-1} g_k e^{2\pi ikl/M}, \quad l = 0, ..., M-1. \tag{3.2}$$

(It differs in sign in front of i from the algorithm fft in MATLAB). The DFT maps m complex numbers (the g_k's) into m complex numbers (the G_l's). The formula for the inverse DFT which recovers the set of g_k's exactly from G_l's is:

$$g_k = iDFT[G](k) = \frac{1}{M}\sum_{l=0}^{M-1} G_l e^{-2\pi ikl/M}, \quad k = 0, ..., M-1. \tag{3.3}$$

In our case, the data consist of a real-valued array $\{g_k\}_{k=0}^{M}$. The resulting transform satisfies $G_{M-l} = \bar{G}_l$. Since this complex-valued array has real values G_0 and $G_{M/2}$, and $M/2-1$ other independent complex values $G_1, ..., G_{M/2-1}$, then it has the same "degrees of freedom" as the original real data set. In this case, it is inefficient to use full complex FFT algorithm. The main idea of FFT of real functions is to pack the real input array cleverly, without extra zeros, into a complex array of half of length. Then a complex FFT can be applied to this shorter length; the trick is then to get the required values from this result (see Press, W. et al (1992) for technical details). To distinguish DFT of real functions we will use notation RDFT.

Fix the space step $d > 0$ and number of the space points $M = 2^m$. Define the partitions of normalized log-price domain $[-\frac{Md}{2}; \frac{Md}{2})$ by points $x_k = -\frac{Md}{2} + kd$, $k = 0, ..., M-1$, and frequency domain $[-\frac{\pi}{d}; \frac{\pi}{d}]$ by points $\xi_l = \frac{2\pi l}{dM}$, $l = -M/2, ..., M/2$. Then the Fourier transform of a function g on the real line can be approximated as follows:

$$\hat{g}(\xi_l) \approx \int_{-Md/2}^{Md/2} e^{-ix\xi_l} g(x)dx \approx \sum_{k=0}^{M-1} g(x_k)e^{-ix_k\xi_l}d = de^{i\pi l}\sum_{k=0}^{M-1} g(x_k)e^{-2\pi ikl/M},$$

and finally,

$$\hat{g}(\xi_l) \approx de^{i\pi l}\overline{RDFT[g](l)}, \quad l = 0, ..., M/2. \tag{3.4}$$

Here $\overline{z}$ denotes the complex conjugate of z. Now, we approximate $\Psi_{\Delta\tau}(D+i\omega)$. Using the notation $p(\xi) = \Psi_{\Delta\tau}(\xi+i\omega)$, we can approximate

$$(\Psi_{\Delta\tau}(D+i\omega)g)(x_k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix_k\xi}p(\xi)\hat{g}(\xi)d\xi$$

by

$$\frac{2}{M}\operatorname{Re}\sum_{l=1}^{M/2-1} e^{-2\pi ikl/M}p(-\xi_l)RDFT[g]_l+\frac{1}{M}(RDFT[g]_0+\operatorname{Re}p(-\xi_{M/2})RDFT[g]_{M/2})$$

Finally,

$$(\Psi_{\Delta\tau}(D+i\omega)g)(x_k)\approx iRDFT[\overline{p}.*RDFT[g]](k),\quad k=0,...,M-1. \tag{3.5}$$

Note that real-FFT is two times faster than FFT.

Remark: Approximation of the Fourier transform (resp., inverse Fourier transform) using FFT (resp., iFFT) involves two types of errors: truncation error and discretization error. For an RLPE, the truncation error for FFT can be made small very easily because the put option price decays exponentially as $x\to+\infty$ (we may consider the put options only because of the call-put parity) and, after that, the discretization error can be controlled by decreasing the step d of the grid in x-space, equivalently, increasing the number of points $M=2^m$ of the grid. Now, consider the inverse Fourier transform. Assuming that the truncated region on the line $\operatorname{Im}\xi=\omega$ is of the form $[-\Lambda+i\omega,\Lambda+i\omega]$, and denoting the step of the uniform grid by ζ, we have $d\zeta/(2\pi)=1/M$, hence, $\Lambda=2^{m-1}\zeta=\pi/d$. It follows that if we keep the truncated region in x-space fixed and decrease d (equivalently, increase m) then we can control the truncation error of iFFT but not the discretization error. To control the latter, we need to increase the truncated region in x-space and, in addition, increase M by a larger factor.

4. Implementation to the Premia 13

We implemented FST-method for three types of options (barrier, American and digital) under the Variance Gamma model (see Example A.3). One can use the routine for the other types of Lévy processes by replacing the corresponding part with the computation of the characteristic exponent.

Note that in the program implemented to Premia 13 one can manage by three parameters of the algorithm: the space step d, the scale of logprice range L and the number of time steps N. Parameter L controls the size of the truncated region in x-space (see **Remark**); it corresponds to the region $(-L\ln(2)/d;L\ln(2)/d)$. The typical values of the parameter are $L=1$, $L=2$ and $L=4$. To improve the results one should decrease d and/or increase N, when L is fixed.

Appendix A. Basic facts

A.1. **Lévy processes: a short reminder.** A Lévy process is a process with stationary independent increments (for details, see e.g. Sato (1999)). A Lévy process may have a Gaussian component and/or pure jump component. The latter is characterized by the density of jumps, which is called the Lévy density. We denote it by $F(dy)$. A Lévy process can be completely specified by its characteristic exponent, ψ, definable from the equality $E[e^{i\xi X(t)}]=e^{-t\psi(\xi)}$ (we confine ourselves to the one-dimensional case). The characteristic exponent is given by the Lévy-Khintchine formula:

$$\psi(\xi)=\frac{\sigma^2}{2}\xi^2-i\mu\xi+\int_{-\infty}^{+\infty}(1-e^{i\xi y}+i\xi y\mathbf{1}_{|y|\leq 1})F(dy), \tag{A.1}$$

where σ^2 is the variance of the Gaussian component, and $F(dy)$ satisfies

$$\int_{\mathbf{R}\setminus\{0\}} \min\{1, y^2\} F(dy) < +\infty. \tag{A.2}$$

If the jump component is a process of finite variation, equivalently, if

$$\int_{\mathbf{R}\setminus\{0\}} \min\{1, |y|\} F(dy) < +\infty, \tag{A.3}$$

then (A.1) can be simplified

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty} (1 - e^{i\xi y}) F(dy), \tag{A.4}$$

with a different μ, and the new μ is the drift of the Gaussian component.

Assume that under a risk-neutral measure chosen by the market, the stock has the dynamics $S_t = e^{X_t}$. Then we must have $E[e^{X_t}] < +\infty$, and, therefore, ψ must admit the analytic continuation into a strip $\operatorname{Im}\xi \in (-1, 0)$ and continuous continuation into the closed strip $\operatorname{Im}\xi \in [-1, 0]$. Further, if the riskless rate, r, is constant, and the stock does not pay dividends, then the discounted price process must be a martingale. Equivalently, the following condition must hold

$$r + \psi(-i) = 0, \tag{A.5}$$

which can be used to express μ via the other parameters of the Lévy process:

$$\mu = r - \frac{\sigma^2}{2} + \int_{-\infty}^{+\infty} (1 - e^y + y\mathbf{1}_{|y|\le 1}) F(dy). \tag{A.6}$$

Example A.1. **[Tempered stable Lévy processes]** The characteristic exponent of a pure jump KoBoL process of order $\nu \in (0, 2), \nu \neq 1$ is given by

$$\psi(\xi) = -i\mu\xi + c\Gamma(-\nu)[\lambda_+^\nu - (\lambda_+ + i\xi)^\nu + (-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu], \tag{A.7}$$

where $c > 0$, $\mu \in \mathbf{R}$, and $\lambda_- < -1 < 0 < \lambda_+$. Formula (A.7) is derived in Boyarchenko and Levendorskiǐ (2000, 2002) from the Lévy-Khintchine formula with the Lévy densities of negative and positive jumps, $F_\mp(dy)$, given by

$$F_\mp(dy) = ce^{\lambda_\pm y}|y|^{-\nu-1}dy; \tag{A.8}$$

in the first two papers, the name extended Koponen family was used. Later, the same class of processes was used in Carr et al. (2002) under the name CGMY-model. Note that Boyarchenko and Levendorskiǐ (2000, 2002b) consider a more general version with $c_\pm$ instead of c, as well as the case $\nu = 1$ and cases of different exponents $\nu_\pm$.

Example A.2. **[Normal Inverse Gaussian processes]** A normal inverse Gaussian process (NIG) can be described by the characteristic exponent of the form (see Barndorff-Nielsen (1998))

$$\psi(\xi) = -i\mu\xi + \delta[(\alpha^2 - (\beta + i\xi)^2)^{1/2} - (\alpha^2 - \beta^2)^{1/2}], \tag{A.9}$$

where $\alpha > |\beta| > 0$, $\delta > 0$ and $\mu \in \mathbf{R}$.

Example A.3. [Variance Gamma processes] The Lévy density of a Variance Gamma process is of the form (A.8) with $\nu = 0$, and the characteristic exponent is given by (see Madan et al. (1998))

$$\psi(\xi) = -i\mu\xi + c[\ln(\lambda_+ + i\xi) - \ln\lambda_+ + \ln(-\lambda_- - i\xi) - \ln(-\lambda_-)], \tag{A.10}$$

where $c > 0$, $\mu \in \mathbf{R}$, and $\lambda_- < -1 < 0 < \lambda_+$.

Example A.4. [Kou model] If $F_\mp(dy)$ are given by exponential functions on negative and positive axis, respectively:

$$F_\mp(dy) = c_\pm(\pm\lambda_\pm)e^{\lambda_\pm y},$$

where $c_\pm \geq 0$ and $\lambda_- < 0 < \lambda_+$, then we obtain Kou model. The characteristic exponent of the process is of the form

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \frac{ic_+\xi}{\lambda_+ + i\xi} + \frac{ic_-\xi}{\lambda_- + i\xi}. \tag{A.11}$$

The version with one-sided jumps is due to Das and Foresi (1996), the two-sided version was introduced in Duffie, Pan and Singleton (2000), see also S.G.Kou (2002).

References

[1] Barndorff-Nielsen, O. E., 1998, “Processes of Normal Inverse Gaussian Type", *Finance and Stochastics*, 2, 41–68.

[2] Boyarchenko, S. I., and S. Z. Levendorskiǐ, 2000, “Option pricing for truncated Lévy processes", *International Journal of Theoretical and Applied Finance*, 3, 549–552.

[3] Boyarchenko, S. I., and S. Z. Levendorskiǐ, 2002b, *Non-Gaussian Merton-Black-Scholes theory*, World Scientific, New Jercey, London, Singapore, Hong Kong.

[4] Carr, P., H. Geman, D.B. Madan, and M. Yor, 2002, “The fine structure of asset returns: an empirical investigation", *Journal of Business*, 75, 305-332.

[5] Cont, R., and P.Tankov, 2004, *Financial modelling with jump processes*, Chapman & Hall/CRC Press.

[6] Das, S., and S. Foresi, 1996, “Exact solutions for bond and option pricing with systematic risk", *Review of Derivatives Research*, 1, 7–24.

[7] Duffie, D., J.Pan, and K. Singleton, 2000, “Transform analysis and options pricing for affine jump diffusions", *Econometrica*, 68, 1343–1376.

[8] R. Geske and H. Johnson, 1984, “The American put valued analytically”, *J. of Finance*, 39, 1511–1542.

[9] Jackson, K. R., S. Jaimungal, and V. Surkov, 2008, “Fourier space time-stepping for option pricing with Lévy models”, *Journal of Computational Finance*, 12 (2), 1–28.

[10] S.G. Kou, 2002, “A jump-diffusion model for option pricing", *Management Science*, 48, 1086-1101

[11] Kudryavtsev, O.E., and S.Z. Levendorskiǐ, 2006, “Pricing of first touch digitals under normal inverse Gaussian processes”, *International Journal of Theoretical and Applied Finance,* Vol. 9, No. 6, 915–949.

[12] Levendorskii, S., Kudryavtsev, O., and V.Zherder, 2006 “The relative efficiency of numerical methods for pricing American options under Lévy processes", *Journal of Computational Finance*, Vol. 9. No 2, Winter 2005/06.

[13] Lord, R., F. Fang, F. Bervoets, and C. W. Oosterlee, 2008, “ A fast and accurate FFTbased method for pricing early-exercise options under Lévy processes”, *SIAM Journal on Scientific Computing*, 30 (4), 1678–1705.

[14] Madan, D.B., Carr, P., and E. C. Chang, 1998, “The variance Gamma process and option pricing", *European Finance Review*, 2, 79–105.

[15] Press,W., Flannery,B., Teukolsky, S. and W. Vetterling, 1992, *Numerical recipes in C: The Art of Scientific Computing*, Cambridge Univ. Press, available at www.nr.com.

[16] Sato, K., 1999, *Lévy processes and infinitely divisible distributions*, Cambridge University Press, Cambridge.