

# Pricing and hedging American-style options: a simple simulation-based approach, by Y.Wang and R.Caflisch

Ismail Laachir

March 1, 2012

## Contents

1	Discrete optimal stopping problem	1
2	Regression method to price American option: Longstaff-Schwartz algorithm	2
3	Modified Longstaff-Schwartz algorithm to estimate the greeks	3

## Premia 14

### 1 Discrete optimal stopping problem

Let us consider a discrete non-negative random process  $(H_i)_{0 \leq i \leq k}$  adapted to a filtration  $(\mathcal{F}_i)_{0 \leq i \leq k}$ . We suppose that this process is a function of another underlying process  $X_i$ :  $H_i = h_i(X_i)$  for some function  $h_i$ .

We are interested in the problem of finding a stopping time that maximizes the  $\mathcal{F}_i$ -conditional expectation  $\mathbb{E}_i[H_\tau]$  over stopping time  $\tau$  taking value in  $\{i, \dots, k\}$

$$\sup_{\tau \in \{i, \dots, k\}} \mathbb{E}_i[H_\tau] \quad (1)$$

We call an *optimal stopping time* for this problem an  $\mathcal{F}_i$ -stopping time  $\tau_i^*$  such that

$$\mathbb{E}_i[H_{\tau_i^*}] = \sup_{\tau \in \{i, \dots, k\}} \mathbb{E}_i[H_\tau]$$

We define the process  $Q_i^* := \mathbb{E}_i[H_{\tau_i^*}]$ . The process  $(Q_i^*)_{0 \leq i \leq k}$  is called the *Snell envelope* process of  $(H_i)_{0 \leq i \leq k}$ . It can be constructed using the well known *backward dynamic programming* algorithm. In fact, by definition we have  $Q_k^* = H_k$

$$\begin{cases} Q_k^* = H_k \\ Q_i^* = \max \left( H_i, \mathbb{E}_i [Q_{i+1}^*] \right) \end{cases}, \text{ for } 0 \leq i \leq k-1 \quad (2)$$

Then  $\tau_i^*$  can be represented by

$$\begin{aligned} \tau_i^* &= \inf \left\{ j, i \leq j \leq k : H_j \geq \mathbb{E}_j [Q_{j+1}^*] \right\} \\ &= \inf \left\{ j, i \leq j \leq k : H_j \geq \mathbb{E}_j [H_{\tau_{j+1}^*}] \right\} \end{aligned}$$

or

$$\begin{cases} \tau_k^* = k \\ \tau_i^* = i \mathbb{I}_A + \tau_{i+1}^* \mathbb{I}_{A^c} \end{cases} \quad (3)$$

where  $A = \{H_i \geq \mathbb{E}_{i+1}[H_{\tau_{i+1}^*}]\}$

## 2 Regression method to price American option: Longstaff-Schwartz algorithm

The optimal stopping time family  $(\tau_i^*)_i$  (or equivalently the price process  $(Q_i^*)$ ) can be estimated via plain Monte Carlo simulation of the backward dynamic program (3). But this raises the problem of how to estimate the conditional expectation  $\mathbb{E}_{i+1}[H_{\tau_{i+1}^*}]$ . Using Monte Carlo method to estimate this latter implies nested simulations, which makes the algorithm potentially very slow.

In fact, to estimate  $Q_i^* = \max \left( H_i, \mathbb{E}_i [Q_{i+1}^*] \right)$ , we need the conditional expectation  $\mathbb{E}_i [Q_{i+1}^*]$ . We estimate this latter by sampling conditional on the state at time  $i$ , hence the nested simulations.

One possible way to circumvent this problem is the well known Longstaff-Schwartz algorithm. We note the price given by this algorithm by  $Q_i^{LS}$ .

The idea is to estimate the conditional expectations  $\mathbb{E}_i [Q_{i+1}^{LS}]$  using some regression methods, without any nested simulation. In fact, in the case of a Markovian setting, the conditional expectation is a function of the underlying process  $X_i$

$$\mathbb{E}_i [Q_{i+1}^{LS}] = \phi_i(X_i)$$

The function  $\phi_i$  is then approximated by an orthogonal projection  $\langle \alpha_i, g_i \rangle$ . We then choose  $\alpha_i$  as the one that minimize the second order moment

$$\begin{cases} \min_{\alpha_i \in \mathbb{R}^d} \mathbb{E} \left[ \left( Q_{i+1}^{LS} - \langle \alpha_i, g_i \rangle (X_i) \right)^2 \right] \\ \alpha_i \in \mathbb{R}^d \end{cases}$$

Using Monte Carlo method, this minimization problem can be rewritten, for a set of  $M$  simulated paths  $\{w_1, w_2, \dots, w_M\}$ :

$$\begin{cases} \min_{\alpha_i \in \mathbb{R}^d} \sum_{m=1}^M \left[ Q_{i+1}^{LS}(w_m) - \langle \alpha_i, g_i \rangle (X_i(w_m)) \right]^2 \\ \alpha_i \in \mathbb{R}^d \end{cases}$$

Then  $Q_i^{LS}$  is estimated by  $Q_i^{LS}(w_m) = \max(h_i(X_i(w_m)), \langle \alpha_i, g_i \rangle (X_i(w_m)))$ .

The stopping time given by this method is

$$\tau_i^{LS} = \inf \{j, i \leq j \leq k : h_j(X_j) \geq \langle \alpha_j, g_j \rangle (X_j)\}$$

for  $1 \leq i \leq k$ . At initial time  $i = 0$ , the continuation value is just the plain expectation  $\mathbb{E}_0[Q_1^{LS}]$ , so we don't need regression to evaluate it. The option price at initial time is then given by  $Q_0^{LS} = \max(H_0, \mathbb{E}_0[Q_1^{LS}])$ .

However, this method gives only an approximation of the price, and no information about the sensitivities of American option, also called the greeks. The method proposed by [Wang, Caflisch 2009], based on the Least Squares approximation of conditional expectations by [Longstaff, Schwartz 2001], gives an estimation of these sensitivities.

### 3 Modified Longstaff-Schwartz algorithm to estimate the greeks

The approach proposed in [Wang, Caflisch 2009] is based on the least squares regression method in Longstaff-Schwartz algorithm (LS). We will note this algorithm (MLSM) as in [Wang, Caflisch 2009]. The key idea is to use a random set of initial stock price and approximate the option price function at initial date by an additional regression. This means that instead of having  $X_0$  as deterministic, we consider that it has some predefined distribution.

Hence, by using the Longstaff-Schwartz algorithm, we get the option price at initial time 0 on this random set of initial stock price, then by an additional regression we can have an option price functional approximation of the form  $Q_0^{MLSM}(X) = \langle \alpha_0, g_0 \rangle (X)$ . Therefore, to obtain option price and its the sensitivities to  $X$ , we just evaluate the function  $\langle \alpha_0, g_0 \rangle$  at the actual initial stock price  $X_0$  and use the explicit derivative of

this basis function ie  $\frac{\partial \langle \alpha_0, g_0 \rangle}{\partial X}$ .

We can get for instance the option price, the delta and the gamma by

$$\begin{aligned} Q_0^{MLSM}(X_0) &= \langle \alpha_0, g_0 \rangle(X_0) \\ \Delta_0^{MLSM}(X_0) &= \frac{\partial Q_0^{MLSM}}{\partial X}(X_0) = \frac{d \langle \alpha_0, g_0 \rangle}{dX}(X_0) \\ \Gamma_0^{MLSM}(X_0) &= \frac{\partial^2 Q_0^{MLSM}}{\partial X^2}(X_0) = \frac{d^2 \langle \alpha_0, g_0 \rangle}{dX^2}(X_0) \end{aligned}$$

As noted in [Wang, Caffisch 2009], this algorithm can be interpreted as starting the stock price process from a date before initial time 0.

## Numerical implementation

- Taking into account the previous remark, we generate the random set of initial stock price from an initial distribution of the form

$$X(w, t = 0) = X_0 e^{-\alpha \sigma^2 + \sigma \sqrt{\alpha} w}, \quad w \sim \mathcal{N}(0, 1)$$

where  $\sigma$ ,  $T$  are the corresponding stock volatility and option maturity.  $\alpha$  should be used to adjust the variance of the distribution. We note that the mean of  $X(w, t = 0)$  is  $X_0$ .

- For the regression step, we used the least squares algorithm implemented in the Premia Numerical Library (PNL) via the function `pnl_basis_fit_ls`. It permits to use canonical, Hermite and Tchebychev polynomials as regression basis and provides also the derivatives value.

## References

- [Wang, Caflisch 2009] Wang, Y., Caflisch, R. (2009) Pricing and Hedging American-Style Options: A Simple Simulation-Based Approach *The Journal of Computational Finance*, 2009. 3, 4
- [Longstaff, Schwartz 2001] Longstaff, F. A., and Schwartz, E. S. (2001) Valuing American options by simulation: a simple least-square approach *Review of Financial Studies*, 14, 113–147. 3