Exponential Moments of the Discrete Maximum of a Lévy process

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Premia 14

Abstract

To estimate the lookback option price we will use a method elaborated by Feng and Linetsky (see [1]). This method consist of estimating the exponential moments of the discrete maximum by using Hilbert transform.

1 Preliminaries

Assume that the spot S_t follows an exponential Levy process (i.e $S_t = e^{X_t}$) where X_t is a Levy process defined by the characteristic triplet(μ, σ, ν). Hence its characteristic is obtained by:

$$\mathbb{E}(e^{i\xi X_t}) = e^{-t\psi(\xi)}$$

where

$$\psi(\xi) = \frac{\sigma^2\xi^2}{2} - i\mu\xi - \int_{\mathbb{R}} e^{i\xi y} - 1 - i\xi y 1_{|y|<1}\nu(dy)$$

One denotes M_N as the discreetly observed maximum of the Levy process is given by X_t

$$M_N = \max_{j\leq N} X_j$$

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Since the interest is focused on the estimation of $\mathbb{E}(e^{sM_N})$, let Z_t be the process defined by

$$Z_t = e^{sX_t + t\psi(-is)}$$

then Z_t is a positive martingal with an expectation equal to 1 which is used to define an equivalent measure probability $\mathbb{P}^*$ as following, for $t \leq T$

$$\frac{d\mathbb{P}^*}{d\mathbb{P}}|_{\mathcal{F}_t} = Z_t$$

Under this new measure X_t remains a Levy process with a characteristic function $\phi_t^*(\xi) = \frac{\phi_t(\xi - is)}{\phi_t(-is)}$. This measure transform (called the Esscher transform) allows one to write

$$\begin{aligned}
\mathbb{E}(e^{sM_N}) &= \mathbb{E}(e^{sM_N - sX_N} e^{sX_N}) \\
&= e^{-T\psi(-is)} \mathbb{E}(Z_t e^{s(M_N - X_N)}) \\
&= e^{-T\psi(-is)} \mathbb{E}^*(e^{s(M_N - X_N)})
\end{aligned}$$

It is easier to consider $M_N - X_N$. Indeed, one can compute its distribution recursively.

$$\begin{aligned}
M_j - X_j &= \max(M_{j-1} - X_j, 0) \\
&= \max((M_{j-1} - X_{j-1}) - (X_j - X_{j-1}), 0)
\end{aligned}$$

$\tilde{f}_j$ is the distribution density of $M_j - X_j$ and p_Δ^* the transition probability density of the process X_Δ under $\mathbb{P}^*$. One notices clearly that $(M_{j-1} - X_{j-1})$ and $(X_j - X_{j-1})$ are independent according to the proprieties of the Levy process. Due to this independence, a recurrence relation can be easily established between $(M_{j-1} - X_{j-1})$ and $(X_j - X_{j-1})$ in order to compute by convolution $\tilde{f}_{j+1}$.
$\tilde{f}_j$ can be therefore decomposed into 2 parts, $\tilde{f}_j(x) = 1_{]0,\infty[}(x).f_j(x) + C_j.\delta_0(x)$. Thus, $\tilde{f}_0(x) = 1_{]0,\infty[}(x).f_0(x) + C_0.\delta_0(x)$ with $f_0(x) = 0$ and $C_0 = 1$

$$f_0 = 0, C_0 = 1$$

$$f_j(x) = C_{j-1} p_\Delta^*(-x) + \int f_{j-1}(y).1_{]0,\infty[}(y) p_\Delta^*(y - x) dy$$

$$C_j = 1 - \int f_j(x).1_{]0,\infty[}(x) dx$$

After N computations, one will be able to determine

$$\begin{aligned}
\mathbb{E}(e^{sM_N}) &= \phi_T(-is) \mathbb{E}^*(e^{s(M_N - X_N)}) \\
&= \phi_T(-is)(C_N + \int (e^{sx}) 1_{]0,\infty[}(x).f_N(x) dx)
\end{aligned}$$

2 Computing the exponential moment in the Fourier space

By passing to the Fourier space, one avoids using the transition probability density p^*_Δ which is often difficult to approach and will use instead the characteristic function ϕ^*. But first, it is important to denote that

$$g_j(x) = 1_{]0,\infty[}(x)(C_{j-1}e^{sx}p^*_\Delta(-x) + \int f_{j-1}(y)e^{sx-sy}p^*_\Delta(y-x)dy)$$

which leads to obtain

$$\phi_T(-is)\mathbb{E}^*(e^{s(M_N-X_N)}) = \phi_T(-is)(C_N + \int g_N(x)dx)$$

The Hilbert Transformation is also introduced and defined by

$$\mathcal{H}(f)(x) = \frac{1}{\pi}P.V.\int_{\mathbb{R}} \frac{f(y)}{x-y}dy$$

where $P.V.$ is the principal value. Among its properties, there is:

$$\mathcal{F}(sign(x)f(x))(\xi) = i\mathcal{H}(\hat{f})(\xi)$$

which if considered from a different perspective, gives almost everywhere

$$1_{]0,\infty[} = \frac{1}{2}(sign(x)+1)$$

Thus, one obtains immediately

$$\mathcal{F}(f.1_{]0,\infty[}) = \frac{1}{2}\hat{f}(\xi) + \frac{i}{2}\mathcal{H}(\hat{f})(\xi)$$

And then, the recurrence becomes

$$c_0 = 1, \hat{f}_0 = 0$$

$$\hat{f}_j(\xi) = \frac{1}{2}\phi^*_\Delta(-\xi)(c_{j-1} + \hat{f}_{j-1}(\xi)) + \frac{i}{2}\mathcal{H}(\phi^*_\Delta(-\eta)(c_{j-1} + \hat{f}_{j-1}(\eta)))(\xi)$$

$$\hat{g}_j(\xi) = \frac{1}{2}\phi^*_\Delta(-\xi+is)(c_{j-1} + \hat{g}_{j-1}(\xi)) + \frac{i}{2}\mathcal{H}(\phi^*_\Delta(-\eta+is)(c_{j-1} + \hat{g}_{j-1}(\eta)))(\xi)$$

$$c_j = 1 - \hat{f}_j(0)$$

This will give finally

$$\mathbb{E}(e^{sM_N}) = \phi_T(-is)(c_N + \hat{g}_N(0))$$

3 Approximation of the Hilbert transform

For our numerical computation, one will estimate the Hilbert transform by

$$\mathcal{H}(f)(x) = \Sigma_{m=-M}^{M} f(mh) \frac{1-\cos[\pi\frac{(x-mh)}{h}]}{\pi\frac{(x-mh)}{h}}$$

It is important to note that if $x = m_0 h$ then

$$\mathcal{H}(f)(x) = \Sigma_{m=-M, m\neq m_0}^{M} f(mh) \frac{1-(-1)^{m_0-m}}{\pi(m_0-m)}$$

So one will approximate the $(\hat{f}_j)_{j\leq N}$ and $(\hat{g}_j)_{j\leq N}$ on the grid $kh; -M \leq k \leq M$ and the final algorithm which will be implemented becomes

$$\begin{aligned}
\hat{f}_{j,h,M}(hk) &= \frac{1}{2}\phi_\Delta^*(-kh)(c_{j-1} + \hat{f}_{j-1}(kh)) \\
&+ \frac{i}{2\pi}\Sigma_{m=-M,m\neq k}^{M} \phi_\Delta^*(-mh)(c_{j-1} + \hat{f}_{j-1}(mh))\frac{1-(-1)^{k-m}}{\pi(k-m)}
\end{aligned}$$

$$\begin{aligned}
\hat{g}_{j,h,M}(hk) &= \frac{1}{2}\phi_\Delta^*(-kh+is)(c_{j-1} + \hat{g}_{j-1}(kh)) \\
&+ \frac{i}{2\pi}\Sigma_{m=-M,m\neq k}^{M} \phi_\Delta^*(-mh+is)(c_{j-1} + \hat{g}_{j-1}(mh))\frac{1-(-1)^{k-m}}{\pi(k-m)}
\end{aligned}$$

Let's consider

$$H_M = \frac{1}{2}\begin{pmatrix}
1 & \frac{2i}{\pi} & 0 & \frac{2i}{3\pi} & 0 & \frac{2i}{5\pi} & \dots & 0 & \frac{2i}{2M+1\pi} \\
-\frac{2i}{\pi} & 1 & \frac{2i}{\pi} & 0 & \frac{2i}{3\pi} & 0 & \ddots & \ddots & \vdots \\
0 & \frac{2i}{\pi} & 1 & \frac{2i}{\pi} & 0 & \frac{2i}{3\pi} & \ddots & \ddots & \vdots \\
-\frac{2i}{3\pi} & 0 & \frac{2i}{\pi} & 1 & \frac{2i}{\pi} & 0 & \ddots & \ddots & \vdots \\
0 & \frac{2i}{3\pi} & 0 & \frac{2i}{\pi} & 1 & \frac{2i}{\pi} & \ddots & \ddots & \vdots \\
-\frac{2i}{5\pi} & 0 & \frac{2i}{3\pi} & 0 & \frac{2i}{\pi} & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
-\frac{2i}{2M+1\pi} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1
\end{pmatrix}$$

$$\Phi_{\Delta,F}^* = \begin{pmatrix}
\phi_\Delta^*(Mh) & 0 & \dots & \dots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \phi_\Delta^*(0) & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \dots & \dots & 0 & \phi_\Delta^*(-Mh)
\end{pmatrix}$$

$$\Phi^*_{\Delta,G} = \begin{pmatrix} \phi^*_\Delta(Mh+is) & 0 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \phi^*_\Delta(0+is) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \phi^*_\Delta(-Mh+is) \end{pmatrix}$$

$$F_j = \begin{pmatrix} \hat{f}_{j,h,M}(-Mh) \\ \vdots \\ \hat{f}_{j,h,M}(kh) \\ \vdots \\ \hat{f}_{j,h,M}(Mk) \end{pmatrix}$$

$$G_j = \begin{pmatrix} \hat{g}_{j,h,M}(-Mh) \\ \vdots \\ \hat{g}_{j,h,M}(kh) \\ \vdots \\ \hat{g}_{j,h,M}(Mk) \end{pmatrix}$$

$$C_j = c_j. \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

Then the algorithm becomes

$$F_j = H_M[\Phi^*_{\Delta,F}(C_{j-1} + F_{j-1})]$$

$$G_j = H_M[\Phi^*_{\Delta,G}(C_{j-1} + G_{j-1})]$$

$$c_j = 1 - (F_j)_{M+1,1} = 1 - \hat{f}_{j,h,M}(0)$$

H_M is a Toeplitz matrix, it is therefore possible to transform it into a circulant

Matrix K_{4M+1}

$$K_{4M+1}(\frac{V}{2}) = \frac{1}{2}\begin{pmatrix} \begin{bmatrix} 1 & \frac{2i}{\pi} & 0 & \frac{2i}{3\pi} & \cdots & \frac{2i}{2M+1\pi} \\ -\frac{2i}{\pi} & 1 & \frac{2i}{\pi} & 0 & \ddots & \ddots \\ 0 & \frac{-2i}{\pi} & 1 & \frac{2i}{\pi} & \ddots & \ddots \\ -\frac{2i}{3\pi} & 0 & \frac{-2i}{\pi} & 1 & \ddots & \ddots \\ 0 & \frac{2i}{3\pi} & 0 & \frac{-2i}{\pi} & \ddots & \ddots \\ -\frac{2i}{2M+1\pi} & \ddots & \ddots & \ddots & \ddots & 1 \end{bmatrix} & \begin{bmatrix} -\frac{2i}{2M+1\pi} & \cdots & -\frac{2i}{3\pi} & 0 & -\frac{2i}{\pi} \\ \frac{2i}{2M+1\pi} & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \\ \begin{bmatrix} \frac{2i}{2M+1\pi} & -\frac{2i}{2M+1\pi} & \ddots & \ddots & \ddots -\frac{2i}{\pi} \\ 0 & \frac{2i}{2M+1\pi} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & -\frac{2i}{2M+1\pi} \\ \frac{2i}{\pi} & \cdots & \cdots & 0 & \frac{2i}{2M+1\pi} \end{bmatrix} & \begin{bmatrix} 1 & \frac{2i}{\pi} & \ddots & \ddots & \ddots \\ \frac{-2i}{\pi} & 1 & \frac{2i}{\pi} & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & \frac{2i}{\pi} \\ -\frac{2i}{2M+1\pi} & 0 & \cdots & \frac{-2i}{\pi} & 1 \end{bmatrix} \end{pmatrix}$$

where

$$V = \begin{pmatrix} 1 \\ -\frac{2i}{\pi} \\ 0 \\ -\frac{2i}{3\pi} \\ \vdots \\ -\frac{2i}{2M+1\pi} \\ \frac{2i}{2M+1\pi} \\ \vdots \\ \frac{2i}{3\pi} \\ 0 \\ \frac{2i}{\pi} \end{pmatrix}$$

Thanks to the Fast Fourier Transform (FFT), the product matrix-vector can be easily computed in $O(M\log_2(M))$ operations instead of $O(M^2)$. Indeed,

$$FFT^{-1}(FFT(\frac{V}{2}) \circ FFT(X)) = K_{4M+1}(\frac{V}{2})X$$

So to calculate $H_M X$, one computes

$$\frac{1}{2}\left(\begin{array}{cc}
\begin{bmatrix}
1 & \frac{2i}{\pi} & 0 & \frac{2i}{3\pi} & \cdots & \frac{2i}{2M+1\pi} \\
-\frac{2i}{\pi} & 1 & \frac{2i}{\pi} & 0 & \ddots & \ddots \\
0 & \frac{-2i}{\pi} & 1 & \frac{2i}{\pi} & \ddots & \ddots \\
-\frac{2i}{3\pi} & 0 & \frac{-2i}{\pi} & 1 & \ddots & \ddots \\
0 & \frac{2i}{3\pi} & 0 & \frac{-2i}{\pi} & \ddots & \ddots \\
-\frac{2i}{2M+1\pi} & \ddots & \ddots & \ddots & \ddots & 1
\end{bmatrix} &
\begin{bmatrix}
-\frac{2i}{2M+1\pi} & \cdots & -\frac{2i}{3\pi} & 0 & -\frac{2i}{\pi} \\
\frac{2i}{2M+1\pi} & \ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix} \\
\begin{bmatrix}
\frac{2i}{2M+1\pi} & -\frac{2i}{2M+1\pi} & \ddots & \ddots & \ddots -\frac{2i}{\pi} \\
0 & \frac{2i}{2M+1\pi} & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & -\frac{2i}{2M+1\pi} \\
\frac{2i}{\pi} & \cdots & \cdots & 0 & \frac{2i}{2M+1\pi}
\end{bmatrix} &
\begin{bmatrix}
1 & \frac{2i}{\pi} & \ddots & \ddots & \ddots \\
\frac{-2i}{\pi} & 1 & \frac{2i}{\pi} & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & \frac{2i}{\pi} \\
-\frac{2i}{2M+1\pi} & 0 & \cdots & \frac{-2i}{\pi} & 1
\end{bmatrix}
\end{array}\right)
\begin{pmatrix} x_1 \\ \vdots \\ x_{2M+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

References

[1] Feng, L. et V. Linetsky. Computing Exponential Moments of the Discrete Maximum of a Levy process and Look-back Options. Finance and Stochastics 13(4), 501-529 (2009). 1

References