Föllmer-Schweizer Decomposition and Application

El Hadj Aly DIA*

Abstract

In this note we will present the important results of [1]. These results can be used to valuate European options in models where the log price of the underlying asset is a process with independent increments.

1 Introduction

Let $\left(\Omega, \mathcal{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ be a probability space. We consider an underlying asset price given by $S_{t_k} = S_0 \exp(X_{t_k})$ for $k = \{0, \ldots, N\}$, where $t_0, \ldots, t_N$ are the trading dates and the process X is a process with independent increments. We will write k instead of t_k in the sequel. Denote by H the payoff of the option with underlying asset S, the Variance-Optimal pricing and hedging problem consist in finding an initial endowment $V_0 \in \mathbb{R}$ and an optimal strategy $\varphi = (\varphi_k)_{1\leq k\leq N}$ which minimizes

$$\mathbb{E}\left(V_T^N - H\right)^2 \quad with \quad V_T^N = V_0 + \sum_{k=1}^{N} \varphi_k \Delta S_{t_k}. \tag{1.1}$$

The reason of such framework is explained in [1]. We will introduce some definitions and assumptions used which will be used in the sequel. For more details see [1].

Definition 1.1 *We say that S satisfies the non-degeneracy condition (ND) if there exists a constant $\delta \in]0,1[$ such that*

$$\left(\mathbb{E}\left[\Delta S_k / \mathcal{F}_{k-1}\right]\right)^2 \leq \delta \mathbb{E}\left[(\Delta S_k)^2 / \mathcal{F}_{k-1}\right],$$

$\mathbb{P}$ a.s. for $k = 1, \ldots, N$.

Definition 1.2 *We define the discrete cumulant generating function as*

$$m: \ D \times \{0, \ldots, N\} \to \mathbb{C} \ with \ m(z,k) = \mathbb{E}e^{z\Delta X_k},$$

where $D = \{z \in \mathbb{C}, \ \mathbb{E}\exp(z\Delta X_N) < \infty\}$.

Assumption I *S satisfies the non-degeneracy condition.*

Assumption II *1. ΔX_k is never deterministic for any $k = 01, \ldots, N$.*

2. $2 \in D$.

*INRIA Paris-Rocquencourt, domaine de Voluceau, BP 105, 78153 Le Chesnay Cedex France (dia.eha@gmail.com).

2 Discrete Föllmer-Schweizer Decomposition

In this section we will derive discrete Föllmer-Schweizer decomposition for some kind of payoffs. More details can be found in [1].

Proposition 2.1 *Under Assumption II, let $z \in D$ fixed, such that $2Re(z) \in D$. Then $H(z) = S_N^z$ admits a discrete Föllmer-Schweizer decomposition*

$$\begin{cases} H(z)_n = H(z)_0 + \sum_{k=1}^{n} \xi(z)_k \Delta S_k + L(z)_n \\ H(z)_N = H(z) = S_N^z, \end{cases}$$

where

$$\begin{aligned} H(z)_n &= h(z,n) S_n^z, \quad \forall n \in \{0, \dots, N\} \\ \xi(z)_n &= g(z,n) h(z,n) S_{n-1}^{z-1}, \quad \forall n \in \{1, \dots, N\} \\ L(z)_n &= H(z)_n - H(z)_0 - \sum_{k=1}^{n} \xi(z)_k \Delta S_k, \quad \forall n \in \{0, \dots, N\}, \end{aligned}$$

and $g(z,n)$, $h(z,n)$ are defined by

$$\begin{aligned} h(z,n) &= \prod_{i=n+1}^{N} (m(z,i) - g(z,i)(m(1,i) - 1)) \\ g(z,n) &= \frac{m(z+1,n) - m(1,n)m(z,n)}{m(2,n) - m(1,n)^2} \end{aligned}$$

Consider now that

$$H = f(S_N), \quad with \quad f(s) = \int_{\mathbb{C}} s^z \Pi(dz),$$

where Π is a finite complex measure in the sense of Rudin [2], Section 6.1. For examples we have

$$\begin{aligned} (s-K)^+ - s &= \frac{1}{2\pi i} \in_{R-iB}^{R+iB} s^z \frac{K^{1-z}}{z(z-1)} dz, \text{ for arbitrary } 0 < R < 1,\ s > 0,\ K > 0 \\ (K-s)^+ &= \frac{1}{2\pi i} \in_{R-iB}^{R+iB} s^z \frac{K^{1-z}}{z(z-1)} dz, \text{ for arbitrary } R < 0,\ s > 0,\ K > 0 \end{aligned}$$

Set $I_0 = supp\Pi \cap \mathbb{R}$.

Assumption III *1. I_0 is compact.*

2. $2I_0 \subset D$.

Proposition 2.2 *We suppose the validity of Assumptions II and III. Any contingent claim $H = f(S_N)$ admits the real discrete Föllmer-Schweizer decomposition given by*

$$\begin{cases} H_n = H_0 + \sum_{k=1}^{n} \xi_k^H \Delta S_k + L_n^H \\ H_N = H \end{cases}$$

where

$$\begin{aligned} H_n &= \int_{\mathbb{C}} H(z)_n \Pi(dz) \\ \xi_n^H &= \int_{\mathbb{C}} \xi(z)_n \Pi(dz) \\ L_n^H &= \int_{\mathbb{C}} L(z)_n \Pi(dz) = H_n - H_0 - \sum_{k=1}^{n} \xi_k^H \Delta S_k. \end{aligned}$$

The processes (H_n), (ξ_n^H) and (L_n^H) are real-valued.

The fundamental result of [1] is given by the following Theorem.

Theorem 2.3 *We suppose the validity of Assumptions II and III. Let $H = f(S_N)$. A solution to the optimal problem* (1.1) *is given by (V_0^*, φ^*) with $V_0^* = H_0$ and φ^* is determined by*

$$\varphi_n^* = \xi_n^H + \lambda_n \left(H_{n-1} - H_0 - \sum_{i=1}^{n-1} \varphi_i^* \Delta S_i \right),$$

where

$$\lambda_n = \frac{1}{S_{n-1}} \frac{m(1,n) - 1}{m(2,n) - 2m(1,n) + 1}.$$

Moreover the solution is unique (up to a null set).

References

[1] Goute, S., Oudjane N., Russo F.: Variance optimal hedging for discrete time process with independent increments. Application to electricity markets, Working paper (2010).

[2] Rudin, W.: Real and complex analysis, third edition. New York: McGraw-Hill (1987).

[3] Schweizer, M.: Variance-optimal hedging in discrete time, Mathematics of Operations Research, 20, 1-32 (1995).