FUNCTIONAL STOCHASTIC APPROXIMATION SCHEME FOR AMERICAN OPTION PRICING

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Abstract. The aim of this work is to discuss the application of the work presented in [BRS05a] to option pricing. The kernel-based stochastic gradient algorithm avoids any discretization of the state space and any projection on a subvector space of function basis. Thus, it approximates the optimum without any restriction.
This algorithm is especially interesting for the pricing of multi-dimensional American options (cf. [BRS05b]).

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1. Basic Principle

This method acts on a dynamic programming algorithm corresponding to the expected returns associated to the American option discretized in a Bermudean option. Its fundamental principle relies on the fact that the expected return satisfies a fixed point equation in a functional space. The solution of this fixed point equation can then be characterized through variational inequalities leading to a perturbed gradient algorithm in this functional space. The principles of this method have been introduced in [BRS05a], and further applied to option pricing in [BRS05b]. At the end of the algorithm, the expected returns are given as sums of kernel functions.

2. Pricing method

2.1. **Framework.** The price of a Bermudean option with evenly spaced exercise dates $\{t_0, t_1, \dots, t_N = T\}$, maturity T, discount factor $\alpha = B(t_1, t_2)$ and initial stock price x is given by

$$J_0(x) = \max_{\tau \in \{t_0, t_1, \dots, t_N\}} \mathbb{E}\left[\alpha^\tau g(X_\tau) \mid X_{t_0} = x\right], \tag{1}$$

where the price process X is a Markov chain $\left\{X_{t_j} \in S, 0 \le j \le N\right\}$ with $S = \mathbb{R}^d$ the multi-dimensional state space. The intrinsic value of the option is $g : S \to \mathbb{R}^+$.

Let us now introduce the dynamic programming counterpart of (1).

$$J_{N+1}(x) = 0, \tag{2a}$$

$$J_j(x) = \max\left(g(x), \alpha\mathbb{E}\left[J_{j+1}(X_{t_{j+1}}) \mid X_{t_j} = x\right]\right), \ \forall\, 0 \le j \le N. \tag{2b}$$

We can equivalently write the equations (2) with the so-called Q-functions :

$$Q_j(x) = \alpha\mathbb{E}\left[J_{j+1}(X_{t_{j+1}}) \mid X_{t_j} = x\right],$$

i.e. the expected payoff if we do not exercise the option. Hence it comes:

$$\forall 0 \le j \le N, \quad J_j(x) = \max\left(g(x), Q_j(x)\right). \tag{3}$$

Date: January 2006.
This work was completely supported by Électricité de France Recherche et Développement, while the four authors were involved under different contracts in the OSIRIS Division at Clamart, France, between November 2004 and December 2005.

Equation (2) now reads :

$$Q_N(x) = 0 \tag{4a}$$

$$Q_j(x) = \alpha \mathbb{E}\left[\max\left(g\left(X_{t_{j+1}}\right), Q_{j+1}\left(X_{t_{j+1}}\right)\right) \mid X_{t_j} = x\right], \ \forall\, 0 \leq j \leq N-1. \tag{4b}$$

Equations (4) can of course be rewritten as an infinite horizon stochastic dynamic program, by letting the state be defined by $(j, x) \in \{0 \leq j \leq N\} \times S$, and defining the associated nonhomogeneous Markov chain. Finally, the function $\hat{Q} : \{0 \leq j \leq N\} \times S \to \mathbb{R}^N$ defined by $\hat{Q}(j, x) = Q_j(x)$ verifies the following fixed point equation:

$$\hat{Q}(j,x) = \alpha \mathbb{E}\left[\max\left(\hat{g}\left(\hat{V}\right), \hat{Q}\left(\hat{V}\right)\right) \mid V = (j,x)\right], \tag{5}$$

with

$$\mathcal{L}\left(\hat{V} \middle| V = (j,x)\right) = \begin{cases} \left(j+1, \mathcal{L}\left(X_{t_{j+1}} | X_{t_j} = x\right)\right), & \text{if } j \leq N, \\ (j+1, V^\infty), & \text{else, with } V^\infty = +\infty, \end{cases}$$

and for all $x \in S$, $\hat{g}(j,x) = g(x)$, and $\hat{g}(j, V^\infty) = 0$.

2.2. **Algorithm.** Equation (5) allows us to propose a perturbed gradient algorithm to solve the fixed point problem equivalent to (4). It leads to the following algorithm:

Algorithm 2.1. Step -1 : initialize $Q_j^0(\cdot) = 0$ for all $0 \leq j \leq N-1$,
Step $k \geq 0$:

- Draw $X_{t_j}^k, \forall 1 \leq j \leq N$ independently from the past drawings, starting from $X_{t_0}^k = x$ and with respect to the law of the Markov chain X ;
- Update:

$$\begin{cases} Q_N^{k+1}(\cdot) &= 0, \\ Q_{N-1}^{k+1}(\cdot) &= Q_{N-1}^k(\cdot) + \rho_{N-1}^k \, \Delta_{N-1}^k \, K_{N-1}^k(X_{t_{N-1}}^k, \cdot), \\ &\vdots \\ Q_j^{k+1}(\cdot) &= Q_j^k(\cdot) + \rho_j^k \, \Delta_j^k \, K_j^k(X_{t_j}^k, \cdot), \\ &\vdots \\ Q_0^{k+1}(\cdot) &= Q_0^k(\cdot) + \rho_0^k \, \Delta_0^k \, K_0^k(X_{t_0}^k, \cdot). \end{cases}$$

where

$$\Delta_j^k = \alpha \max\left(g\left(X_{t_{j+1}}^k\right), Q_{t_{j+1}}^{k+1}\left(X_{t_{j+1}}^k\right)\right) - Q_j^k(X_{t_j}^k).$$

where K-functions are kernels, i.e. bounded mappings from $S \times S \to \mathbb{R}$, with $K(x, \cdot)$ non null on a subset of S centered on x. A typical choice of these kernels is the Gaussian one:

$$K_j^k(x,y) = \exp\left\{\left(\frac{x-y}{\varepsilon_j^k}\right)^2\right\}.$$

where ε_k decreases to zero when k goes to infinity.

Let us note $(Q^k) = (Q_j^k)_{0 \leq j \leq N}$. Barty et al. proved, under several assumptions, that the sequence $(Q^k)_{k \in \mathbb{N}}$ strongly converges to Q^*, the solution of the fixed point equation (5). Some of these assumptions, that are especially important in practice, are discussed below, in subsection 2.3.

Steps ρ_j^k and radius of the kernels are decreasing scalar sequences, whose decreasing speeds are ruled by relations discussed in subsection 2.3.

As one can see, we are working directly in the infinite dimension state space to which the solution belongs. In spite of the infinite dimension, this method remains numerically tractable since in order to compute Q^{k+1} one only needs to keep in memory $\left\{Q^k, \Delta^k, X^{k+1}\right\}$. Using the previous notation of Δ^k it holds that:

$$Q_j^{k+1}(\cdot) = \sum_{i=0}^{k} \rho_j^i \ \Delta_j^i \ K_j^i(X_{t_j}^{i+1}, \cdot) + Q_j^0(\cdot), \quad \forall 0 \leq j \leq N.$$

2.3. **Choice of the steps.** Let us assume for the discussion that for all j, $\rho_j^k = \rho^k$, and $\varepsilon_j^k = \varepsilon^k$. Recall that ρ^k is the multiplying factor in the temporal difference and let us denote by ε^k the bandwidth of kernel $K^k(\cdot,\cdot)$. For the sequence $Q^k(\cdot)$ to converge, one needs the following relations :

$$\sum_{k\in\mathbb{N}} \rho^k \varepsilon^k = \infty, \quad \sum_{k\in\mathbb{N}} (\rho^k)^2 \varepsilon^k < \infty, \quad \sum_{k\in\mathbb{N}} b\rho^k \varepsilon^k (\varepsilon^k)^{1/d} < \infty,$$

Let us choose :

$$\rho^k \simeq \frac{1}{k^a}, \quad \varepsilon^k \simeq \frac{1}{k^b}.$$

Then we have the following relations :

$$\left\{ \begin{array}{lcl} 0 & \leq & a+b \leq 1, \\ 1 & < & a+2b, \\ 1 & < & a+b(1+\frac{1}{d}) \end{array} \right.$$

It defines a triangle in the plane (a,b), which shrinks to a line when $d \to \infty$ To choose a and b as the coordinates of the barycentre of the triangle seems to be a good compromise between robustness and speed of convergence. It is given by:

$$\left\{ \begin{array}{lcccl} a & = & \frac{3+d}{3(d+2)} & = & \frac{1}{3} + \frac{1}{3(d+2)}, \\ b & = & \frac{2d+2}{3(d+2)} & = & \frac{2}{3} - \frac{2}{3(d+2)} \end{array} \right.$$

When d becomes high, $(a,b) \to (\frac{1}{3}, \frac{2}{3})$. Note that $\frac{2}{3}$, as a power step for a stochastic algorithm, is often reffered to in [PJ92] as an appropriate choice.

2.4. **Acceleration of the rate of convergence by averaging.** A good way to develop optimal algorithms (in the sense of the convergence rate) has been studied in [PJ92]. It is based on the idea of averaging the iterates.
Since the 1960s, the essential step was reached on the basis of the idea: a slow algorithm having less than optimal convergence rate must be averaged.

We replace the update equation (2.1) by the following two-step update for all $0 \leq j \leq N$:

$$\left\{ \begin{array}{lcl} Q_j^{k+1}(\cdot) & = & Q_j^k(\cdot) + \rho_j^k \Delta_j^k K_j^k(X_{t_j}^k, \cdot), \\ \\ \bar{Q}_j^{k+1}(\cdot) & = & \frac{1}{k+1} \sum_{l=1}^{k+1} Q_j^l(\cdot). \end{array} \right.$$

We could also write the more practical update equation:

$$\bar{Q}_j^{k+1}(\cdot) = \bar{Q}_j^k(\cdot) + \frac{1}{k+1}(Q_j^{k+1}(\cdot) - \bar{Q}_j^k(\cdot)). \tag{6}$$

It has been shown in [PJ92] that the variance of the residue decreases like $\sqrt{k}$.

In practice, the best method is to begin averaging when the iterates have already done a great part of the approximation. Moreover, since the form of our iterates in a kernel-based stochastic gradient algorithm is the following :

$$Q_j^{k+1}(\cdot) = \sum_{i=0}^{k} \rho_j^i \Delta_j^i K_j^i(X_{t_j}^i, \cdot) + Q_j^0(\cdot),$$

we can rewrite equation (6) in the following way :

$$\bar{Q}_j^{k+1}(\cdot) = \begin{cases} \bar{Q}_j^k(\cdot) + \rho_j^k \Delta_j^k K_j^k(X_{t_j}^k, \cdot) & \text{if } k < k_0 \\ \bar{Q}_j^k(\cdot) + \left(\frac{k_{max}-k+1}{k_{max}-k_0+1}\right) \rho_j^k \Delta_j^k K_j^k(X_{t_j}^k, \cdot) & \text{if } k \geq k_0 \end{cases}$$

where k_{max} is the total number of iterations desired.

3. Numerical Applications

We applied algorithm 2.1 to a multi-dimensional option pricing problem. In the two-dimensional case, the problem is the following :

- Price processes follow a risk-neutral discretized Black-Scholes dynamic:

$$X_{t+\Delta t} \sim X_t \, exp\left((r - \sigma^2/2)\Delta t + \sqrt{\Delta t} N(0, \sigma^2)\right). \tag{7}$$

- The intrinsic value of the option is:

$$g(X_1, X_2) = (S - min(X_1, X_2))_+,$$

  where S is the strike price.
- Time horizon is set to 7/12 years,
- $X_1^0 = X_2^0 = 40$, $\sigma_1 = 0.20$, $\sigma_2 = 0.30$, $\rho = 0.5$ the correlation rate bewteen the prices and $r = 0.04879$ continuously compounded corresponding to 5% effective per annum.

Results are shown in Table 1. We chose the exercise dates to be every month, it means that we have 7 possible exercise dates here.

Strike Price	Reference [Boy88]	Averaged Kernel Method	Relative Error
35	1.423	1.354	0.048
40	3.892	3.841	0.013
45	7.689	7.636	0.007

Table 1. Results of convergence after 10000 iterations for two-dimensional option pricing compared with [Boy88], here $dt = 1$ month.

We finally present a higher dimensional example: Price processes follow the same dynamic as described in equation (7). The intrinsic value of the option is :

$$g(x_1, \ldots, x_d) = (max(x_1, \ldots, x_d) - S)_+ \ .$$

Results in dimension 3 are shown in Table 2. Every price is computed in approximately 15 seconds.

$S = 35.0,\ \rho = 0,\ \sigma_1 = 0.2,\ \sigma_2 = 0.3,\ \sigma_3 = 0.5$, reference : 8.59		
Iterations	Averaged Kernel Method	Variance
500	6.56	0.11
1 000	7.24	0.09
5 000	8.22	0.08
20 000	8.47	0.03
50 000	8.55	0.03
100 000	8.59	0.02

$S = 40.0,\ \rho = 0,\ \sigma_1 = 0.2,\ \sigma_2 = 0.3,\ \sigma_3 = 0.5$, reference : 3.84		
Iterations	Averaged Kernel Method	Variance
500	2.98	0.19
1 000	3.31	0.12
5 000	3.72	0.06
20 000	3.80	0.04
50 000	3.83	0.04
100 000	3.83	0.02

$S = 45.0,\ \rho = 0,\ \sigma_1 = 0.2,\ \sigma_2 = 0.3,\ \sigma_3 = 0.5$, reference : 0.90		
Iterations	Averaged Kernel Method	Variance
500	0.76	0.08
1 000	1.19	0.08
5 000	1.15	0.04
20 000	0.94	0.03
50 000	0.92	0.02
100 000	0.91	0.02

TABLE 2. Estimates of the value of an option in dimension 3, compared with results of Barraquand and Martineau [BM95].

REFERENCES

[BM95] J. Barraquand and D. Martineau. Numerical valuation of high dimensional multivariate american securities. *Journal of Financial and Quantitative Analysis*, pages vol.30, no 3, pp.383–405, 1995. 5

[Boy88] P. Boyle. A lattice framework for option pricing with two state variables. *Journal of Financial and Quantitative Analysis*, pages 23(1):1–12, 1988. 4

[BRS05a] K. Barty, J.-S. Roy, and C. Strugarek. A stochastic gradient type algorithm for closed loop problems. *submitted to Mathematical Programming*, 2005. 1

[BRS05b] Barty, K., Roy, J.-S., and Strugarek, C. (2005b). Temporal difference learning with kernels for pricing american-style options. *submitted*. http://www.optimization-online.org/DB_HTML/2005/05/1133.html. 1

[BY03] M. Broadie and Y. Yamamoto. Application of the fast gauss transform to option pricing. *Management Science*, 49(8):1071–1088, 2003.

[PJ92] B. T. Polyak and A. B. Juditsky. Acceleration of stochastic approximation by averaging. *SIAM Journal on Control and Optimization*, pages 838 – 855, 1992. 3

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