Pricing derivatives on realized variance. Implementation

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1. Realized variance and volatility derivatives

Let S_t denote the value of a stock or stock index at time t. We assume that the variance/volatility product starts at time zero and ends at time T. For $t \in [0;T]$ let $Q(t)$ denote the realised variance of returns over the time interval $[0;t]$.

$$Q(t) = u^2 \sum_{0 < t_n \leq t} \left[\log \frac{S_{t_n}}{S_{t_{n-1}}} \right]^2,$$

where u^2 is a constant conversion factor, the value is chosen to obtain desired scaling. For example, to express Realized Variance in annual basis points one chooses

$$u^2 := \frac{252}{N} * 10000.$$

Now consider swaps and options written on variance Q and volatility $Q^{1/2}$.

A variance swap is an instrument which allows investors to trade future realized (or historical) volatility against current implied volatility.

A *variance swap* with maturity T and strike K^2 (in volatility points) pays the holder

$$VarS(K,T) = Q(T) - K^2.$$

The quantity

$$m_t = \mathbf{E}[Q(t)]$$

is called "fair strike of a variance swap". For the variance swap with strike price K^2 and maturity T in the future its fair price at the time 0 is

$$m_T - K^2.$$

A $[0,T]$ *variance call* with strike K^2 (in volatility points) pays the holder at time T

$$VarCall(K,T) = \left(Q(T) - K^2 \right)^+.$$

Volatility swaps and options on volatility are derivatives written on $Q^{1/2}(T)$:

Volatility swap with fixed strike K pays the holder

$$VolS(K,T) = Q^{1/2}(T) - K,$$

and $[0,T]$ *volatility call* with strike K pays the holder at time T

$$VolCall(K,T) = \left(Q^{1/2}(T) - K \right)^+$$

One may use two different approaches for pricing variance or volatility derivatives. First, the expectation of realized variance may be computed under assumption that the price process of underlying is modelled by some certain parametric process like in Black-Scholes or Lévy-based models for derivative pricing. Second, the price of variance derivative may be approximated by some portfolio (the so called replicating portfolio) of some amount of underlying and derivatives on it.

In the Section 2 we present pricing formulae for variance/volatility derivatives obtained by using Laplace transform of the distribution. The formulae are appliable if the distribution of quadratic variation Q is known in a closed form.

In the Section 3 we apply general formulae to several models, namely Tempered Stable Lévy model (TSL, subsection 3.1), Heston and Bates models (subsection 3.2) and double Heston model (subsection 3.3)

Alternatively, we can use available prices of other contracts to build a replication portfolio. The basic points of the theory of replication are provided in the Section 4 together with the pricing formulae for variance/volatility contracts.

2. Integral representation

2.1. **Options on variance.** A call-type option on quadratic variation with strike K and maturity t pays at maturity the sum

$$(Q(t) - K)^+.$$

The time zero value of this option using risk neutral valuation is given by

$$f(K, t) = \mathbf{E}\left[e^{-rt}(Q(t) - K)^+\right].$$

Consider the Laplace transform of the process $Q(t)$:

$$\Phi(v, t) = \mathbf{E}[e^{-vQ(t)}]$$

We define the Laplace transform of option price with respect to the strike by

$$\gamma(v, t) = \int_0^{+\infty} e^{-vK} f(K, t) dK,$$

and we have after integration that

$$\gamma(v, t) = e^{-rt}\left[\frac{\Phi(v, t) - 1}{v^2} + \frac{m_t}{v}\right],$$

where

$$m_t = \mathbf{E}[Q(t)]$$

is "the fair strike of a variance swap".

Option price may be computed now by inverting the Laplace transform. Under certain conditions for the function $\Phi(v, t)$ we can reduce the task to the Fourier inversion problem by changing variable:

$$v = \sigma + i\xi, \quad \sigma > 0, \xi \in \mathbf{R}.$$

Then we have

$$\gamma(\sigma + i\xi, t) = e^{-rt} \left[\frac{\Phi(\sigma + i\xi, t) - 1}{(\sigma + i\xi)^2} + \frac{m_t}{\sigma + i\xi} \right].$$

According to general formula for Laplace transform inversion,

$$
\begin{aligned}
f(K, t) &= \frac{e^{-rt}}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{vK} \gamma(v, t) dv = \frac{e^{-rt}}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{vK} \left(\frac{\Phi(v, t) - 1}{v^2} + \frac{m_t}{v} \right) dv \quad (1) \\
&= \frac{e^{-rt}}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{vK} \frac{\Phi(v, t)}{v^2} dv - \frac{e^{-rt}}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{vK}}{v^2} + m_t \frac{e^{-rt}}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{vK}}{v} dv \\
&= \frac{e^{-rt}}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{vK} \frac{\Phi(v, t)}{v^2} dv - e^{-rt} K + e^{-rt} m_t \\
&= e^{-rt}(m_t - K) + e^{-rt} \frac{e^{\sigma K}}{2\pi} \int_{-\infty}^{+\infty} e^{i\xi K} \frac{\Phi(\sigma + i\xi, t)}{(\sigma + i\xi)^2} d\xi.
\end{aligned}
$$

$$(2)$$

Consider the latter integral and approximate it by discrete sum:

$$e^{-rt} \frac{e^{\sigma K}}{2\pi} h \sum_{n=0}^{N-1} e^{i\xi_n K} \frac{\Phi(\sigma + i\xi_n, t)}{(\sigma + i\xi_n)^2} w_n,$$

where $\xi_n = -A + nh$, $n = 0, 1, 2, .., N - 1$ for some large $A > 0$ and w_n is a weghting multiplyer chosen according to some quadrature rule (for example, $w_n = (3 - (-1)^n - \delta_n)/3$ for Simpson's rule).

To apply FFT, we also discetize the set of possible strikes values by using $K_j = K_0 + j\eta$, $j = 0, 1, .., N - 1$. Substitute:

$$e^{-rt} \frac{e^{\sigma K_j}}{2\pi} h \sum_{n=0}^{N-1} e^{i(-A+nh)(K_0+j\eta)} \frac{\Phi(\sigma + i\xi_n, t)}{(\sigma + i\xi_n)^2} w_n =$$

$$= e^{-rt} \frac{e^{(\sigma - iA)K_j}}{2\pi} h \sum_{n=0}^{N-1} e^{injh\eta} e^{inhK_0} \frac{\Phi(\sigma + i\xi_n, t)}{(\sigma + i\xi_n)^2} w_n$$

We chose h and η so that $h\eta = \frac{2\pi}{N}$ and obtain finally

$$e^{-rt} \frac{e^{(\sigma - iA)K_j}}{2\pi} h \sum_{n=0}^{N-1} e^{inj\frac{2\pi}{N}} e^{inhK_0} \frac{\Phi(\sigma + i\xi_n, t)}{(\sigma + i\xi_n)^2} w_n$$

This descrete sum may easily be computed by FFT algorithm.

2.2. Volatility products.

To compute the fair price of *Volatility swap* we use the following integral representation

$$\sqrt{x} = \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} \frac{1 - e^{-xt}}{t^{3/2}} dt.$$

Due to this formula the fair strike for *Volatility swap* with maturity T may be computed as [7]

$$\mathbf{E}[\sqrt{Q(T)}] = \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} \frac{1 - \Phi(x, T)}{x^{3/2}} dx \tag{3}$$

where $\Phi(x, T) = \mathbf{E}[e^{-xQ(T)}]$ is the Laplace transform of the distribution of $Q(T)$ [6][8].

Notice that the integral has a weak singularity at $x = 0$ and converges slowly as $x \to +\infty$. To simplify computation we approximate the integral over small neighborhood $[0; \epsilon]$ of zero point by the term $\mathbf{E}[Q(T)] \cdot 2\sqrt{\epsilon}$. To compute the integral over $[\epsilon; +\infty)$ we change variable $y = \frac{1}{\sqrt{x}}$ and comes to the integral

$$2 \int_0^{\frac{1}{\sqrt{\epsilon}}} \left(1 - \Phi(\frac{1}{y^2}, T)\right) dy$$

3. Pricing under certain models

3.1. **Tempered Stable Lévy model (TSL).** Let the log of stock price be modelled by Lévy process X_t of TSL/CGMY class with risk-neutral Lévy measure

$$k(x) = c_- \frac{\exp(\lambda_- x)}{|x|^{1+\alpha_-}} \mathbf{1}_{x<0} + c_+ \frac{\exp(-\lambda_+ x)}{x^{1+\alpha_+}} \mathbf{1}_{x>0}.$$

Variance swap

The fair strike for the Variance Swap with maturity T is given by the closed formula [4]

$$\mathbf{E}[Q(T)] = \left(\frac{c_+ \Gamma(2 - \alpha_+)}{\lambda_+^{2-\alpha_+}} + \frac{c_- \Gamma(2 - \alpha_-)}{\lambda_-^{2-\alpha_-}}\right).$$

Alternatively, one may use the replicating portfolio of vanilla options on spot (10) or (14) (as in [5],[2]).

Other contracts

Variance call price may be computed by inverting the Laplace transform (2) while the fair strike for volatility swap is given by (3). In both cases the crucial point is to compute the Laplace transform of distribution $\Phi(v, t)$.

For TSL/CGMY processes we can use explicit formula for Lévy density expression to compute

$$\Phi(v, t) = \exp\left[-t \int_0^{+\infty} (1 - e^{-vx^2}) \left(c_+ \frac{e^{-\lambda_+ x}}{x^{1+\alpha_+}} + c_- \frac{e^{-\lambda_- x}}{x^{1+\alpha_-}}\right) dx\right] =$$

$$= \exp\left[-c_+ t \int_0^{+\infty} (1 - e^{-vx^2}) \frac{e^{-\lambda_+ x} dx}{x^{1+\alpha_+}} - c_- t \int_0^{+\infty} (1 - e^{-vx^2}) \frac{e^{-\lambda_- x} dx}{x^{1+\alpha_-}}\right]$$

Consider the integral

$$J(v, \lambda, \alpha) := \int_0^{+\infty} (1 - e^{-vx^2}) e^{-\lambda x} x^{-1-\alpha} dx,$$

then

$$\log \Phi(v, t) = -c_+ t J(v, \lambda_+, \alpha_+) - c_- t J(v, \lambda_-, \alpha_-).$$

Integrating by parts (several times) we come to:

$$J(v, \lambda, \alpha) = \left(\frac{2v}{\alpha} + \frac{\lambda^2}{\alpha(1-\alpha)} \right) \int_0^{+\infty} x^{1-\alpha} e^{-\lambda x - vx^2} dx + \frac{2v\lambda}{\alpha(1-\alpha)} \int_0^{+\infty} x^{-\alpha} e^{-\lambda x - vx^2} dx$$

$$= \left(\frac{2v}{\alpha} + \frac{\lambda^2}{\alpha(1-\alpha)} \right) \mathcal{I}(\lambda, v, 1-\alpha) + \frac{2v\lambda}{\alpha(1-\alpha)} \mathcal{I}(\lambda, v, 2-\alpha) - \frac{\lambda^\alpha}{\alpha(1-\alpha)} \Gamma(2-\alpha),$$

where

$$\mathcal{I}(\lambda, v, s) := \int_0^{+\infty} x^s e^{-\lambda x - vx^2} dx.$$

For $\alpha \in (0, 1)$, $s = 1 - \alpha > 0$ so that integrand in $\mathcal{I}(\lambda, v, s)$ has no singularity at $x = 0$, but for $\alpha \in (1, 2)$, $s = 1 - \alpha \in (-1, 0)$ so that the integrand is integrable but has a singularity at $x = 0$. In the latter case, for computational reason, we integrate by part again and obtain for $\alpha > 1$:

$$J(v, \lambda, \alpha) = \frac{\lambda(2v(3-2\alpha)+\lambda^2)}{\alpha(1-\alpha)(2-\alpha)} \mathcal{I}(\lambda, v, 2-\alpha) + \frac{2v}{\alpha(2-\alpha)} \left(2v + \frac{\lambda^2}{1-\alpha} \right) \mathcal{I}(\lambda, v, 3-\alpha) -$$

$$- \frac{\lambda^\alpha}{\alpha(1-\alpha)} \Gamma(2-\alpha),$$

Thus, the computation of $\Phi(v, t)$ reduces to the computation of several integrals of the type $\mathcal{I}(\lambda, v, s)$. To do this, we apply directly some quadrature rule for $\Im v$ being small and contour integration for $\Im v$ large.

Exactly, for $v = \sigma + i\xi$ we have:

$$\mathcal{I}(\lambda, v, s) := \int_0^{+\infty} x^s e^{-\lambda x - \sigma x^2} e^{-i\xi x^2} dx. \tag{4}$$

For large ξ the integrand oscillates fast and hence quadrature methods converge slowly. To avoid this difficulties, we change the line of integration in the complex plain from the real axis to $\mathcal{L}_+ = \{z | z = (1-i)t, t \in [0, +\infty)\}$ for $\xi > 0$, and to $\mathcal{L}_- = \{z | z = (1+i)t, t \in [0, +\infty)\}$ for $\xi < 0$.

We obtain, after change of variable, for $\xi > 0$,

$$\mathcal{I}(\lambda, v, s) := (1-i)^{s+1} \int_0^{+\infty} t^s e^{-\lambda t - 2\xi t^2} e^{i(\lambda t + 2\sigma t^2)} dt, \tag{5}$$

and for $\xi < 0$,

$$\mathcal{I}(\lambda, v, s) := (1+i)^{s+1} \int_0^{+\infty} t^s e^{-\lambda t + 2\xi t^2} e^{-i(\lambda t + 2\sigma t^2)} dt. \tag{6}$$

For large ξ, the integrand decays very fast and provides better rate of convergence. But for small ξ there is no sense to make such transformation. In practice we use (5) and (6) for $|\xi| > 0.8\sigma$, and (4) for other values of ξ.

3.2. **Heston and Bates Models.** In this subsection we follow the results by Sepp [9]. We consider the Heston model with return jumps

$$
\begin{aligned}
dS(t) &= (r - d - \gamma m_j)S(t)dt + \sqrt{V(t)}S(t)dW^s(t) + (e^{J^s} - 1)dN(t), \quad S(0) = S; \\
dV(t) &= \kappa(\theta - V(t))dt + \sigma\sqrt{V(t)}dW^v(t)), \quad V(0) = v_0,
\end{aligned}
$$

where r and d are deterministic risk-free interest and dividend yields, respectively, θ is a long-term variance, κ is a mean-reverting rate, σ is a volatility of variance, $W^s(t)$ and $W^v(t)$ are correlated Wiener processes with constant correlation parameter ρ, $N(t)$ is a Poisson process with constant intensity γ. We assume that amplitudes of return jumps J^s have normal distribution with mean ν and standard deviation δ, and

$$
m^j = \exp\left(\nu + \frac{\delta^2}{2}\right) - 1
$$

is the compensator. To garantee the positiveness of the variance $V(t)$ we assume that

$$
2\kappa\theta - \sigma^2 > 0. \tag{7}
$$

With $\gamma = 0$ we come to a standard Heston model.

Variance Swap

The fair strike for Variance Swap with maturity T is computed by the closed formula

$$
\mathbf{E}[Q(T)] = \theta + (v_0 - \theta)\frac{1 - e^{-kT}}{kT} + \gamma(\nu^2 + \delta^2).
$$

Alternatively, one may use the replicating portfolio of vanilla options on spot prices (10) or (14).

Volatility Swap and other contracts

Variance call price may be computed by inverting the Laplace transform (2) while the fair strike for volatility swap is given by (3). In both cases the crucial point is to compute the Laplace transform of distribution $\Phi(\xi, t)$.

For SVJ model

$$
\Phi(\xi, t) = \exp(A(\xi, t) + v_0 B(\xi, t) + \Gamma(\xi, t)),
$$

where

$$A(\xi,t) = -\frac{\kappa\theta}{\sigma^2}\left[\psi_+\tau + 2\ln\left(\frac{\psi_- + \psi_+ e^{-\zeta t}}{2\zeta}\right)\right],$$

$$B(\xi,t) = -2\xi\frac{1 - e^{-\zeta t}}{\psi_- + \psi_+ e^{-\zeta t}},$$

$$\Gamma(\xi,t) = \gamma t\left(\frac{\exp\frac{-\nu^2\xi}{2\xi\delta^2+1}}{\sqrt{2\xi\delta^2+1}} - 1\right),$$

$$\psi_\pm = \mp\kappa + \zeta, \quad \zeta = \sqrt{\kappa^2 + 2\sigma^2\xi}.$$

3.3. **Double Heston Model.** Double Heston is the model with stochastic variance and stochastic "variance of variance":

$$\frac{dS(t)}{S(t)} = \sqrt{V(t)}dW^s(t), \quad S(0) = S;$$

$$dV(t) = \kappa(V'(t) - V(t))dt + \sigma_1\sqrt{V(t)}dW^1(t)), \quad V(0) = z_1,$$

$$dV'(t) = c(z_3 - V'(t))dt + \sigma_2\sqrt{V'(t)}dW^1(t)), \quad V'(0) = z_2,$$

where W^s, W^1 and W^2 are correlated Wiener processes with correlation parameters

$$<W^s, W^1> = \rho_1, \quad <W^s, W^2> = \rho_2, \quad <W^1, W^2> = \rho.$$

The model is a special case of a double mean reverting model proposed by Buehler [3].

Variance Swap

The fair strike for Variance Swap with maturity T is given by the closed formula

$$\mathbf{E}[Q(T)] = z_3 + \left(\frac{z_1 - z_3}{\kappa T} - \frac{z_2 - z_3}{(\kappa - c)T}\right)(1 - e^{-\kappa T}) + \frac{\kappa}{c}\frac{z_2 - z_3}{(\kappa - c)T}(1 - e^{-cT}).$$

4. Replication

In this section we mainly follow the theory of replication by Carr and Lee. Also, one may find the construction of replicating portfolio in [JPMorgan] and [Buehler].

Assume that S_t is modelled by some diffusion process of the type:

$$\frac{dS_t}{S_t} = \mu(t, S_t, ...)dt + \sigma(t, S_t, ...)dW_t$$

where the drift μ and the volatility σ are either deterministic or stochastic and W_t is a Wiener process.

Then the process of Realized Variance is determined by

$$Q_{[0,T]} = \frac{1}{T}\int_0^T \sigma^2(t, S_t, ...)dt = \frac{1}{T}[\log S, \log S]_T$$

where $[\log S, \log S]$ denotes the quadratic variation of $\log S$.

Denote by $\hat{S} = \frac{S}{B}$ the discounted spot prices process, where B refers to the deterministic money market account. It is important to note that $[\log S, \log S] = [\log \hat{S}, \log \hat{S}]$ when rates are deterministic. The continuty of $\hat{S}$ together with Ito's formula yeilds:

$$\log \hat{S}_t = \int_0^t \frac{1}{\hat{S}_u} d\hat{S}_u - \frac{1}{2} [\log \hat{S}, \log \hat{S}]_t$$

for all $t \in [0; T]$. Define for all $t \in [0; T]$

$$\pi_t = \frac{1}{2} [\log \hat{S}, \log \hat{S}]_t + \log \hat{S}_t = \int_0^t \frac{1}{\hat{S}_u} d\hat{S}_u)$$

To price a variance swap we take the risk-neutral expectation of $\frac{\pi_T}{B_T}$:

$$\mathbf{E}\left[\frac{\pi_T}{B_T}\right] = \mathbf{E}\left[\frac{1}{2B_T}[\log \hat{S}, \log \hat{S}]_T + \frac{1}{B_T} \log \hat{S}_T\right] = \mathbf{E}\left[\int_0^T \frac{1}{B_T \hat{S}_u} d\hat{S}_u\right] = 0$$

since $\hat{S}$ is assumed to be martingale under the risk-neutral measure. Hence,

$$\mathbf{E}\left[\frac{1}{B_T} Q_{[0;T]}\right] = -\frac{2}{T} \mathbf{E}\left[\frac{1}{B_T} \log \hat{S}_T\right]$$

Thus, annualized realized variance can be replicated with a static position in a log contract on the discounted stock price. Since in general it is not possible to trade log contracts, we represent the price of a variance swap using standard put and call options by the following argument:

It is known that for any twice differentiable payoff $f(s)$ the following decomposition holds:

$$\begin{aligned}
f(s) &= f(s_0) + f'(s_0)(s - s_0) && (8) \\
&+ \int_0^{s_0} f''(k)(k-s)^+ dk + \int_{s_0}^{+\infty} f''(k)(s-k)^+ dk \\
&= [f(s_0) - f'(s_0)s_0] + f'(s_0)s \quad (\text{"Bond" and "stock" position}) \\
&+ \int_0^{s_0} f''(k)(k-s)^+ dk \quad (\text{"Put" position}) \\
&+ \int_{s_0}^{+\infty} f''(k)(s-k)^+ dk. \quad (\text{"Call" position})
\end{aligned}$$

Variance swap

Let $F_T = S_0 B_T$ be a forward price and S_T be a spot price. We have for $f(y) = \log y$ and $s_0 = F_T$

$$\mathbf{E}\left[\frac{1}{B_T} \log \frac{S_T}{F_T}\right] = -\left[\int_0^1 \frac{1}{y^2} Put(y) dy + \int_1^{+\infty} \frac{1}{y^2} Call(y) dy\right]$$

where y is a forward moneyness. Hence,

$$\mathbf{E}\left[\frac{1}{B_T} Q_{[0;T]}\right] = \frac{2}{T}\left[\int_0^1 \frac{1}{y^2} Put(y) dy + \int_1^{+\infty} \frac{1}{y^2} Call(y) dy\right] \qquad (9)$$

Since we have no infinitely many puts and calls at the market, we apply some quadrature rule to approximate integrals in the r.h.s. of (9). The simplest discrete formula for Variance Swap pricing may be written as:

$$VarS_0 \approx \frac{2}{T}\left[\sum_1^{N_{puts}}\frac{P_0(K_i^{put})}{(K_i^{put})^2}(K_i^{put}-K_{i-1}^{put})+\sum_1^{N_{calls}}\frac{C_0(K_i^{call})}{(K_i^{call})^2}(K_i^{call}-K_{i-1}^{call})\right]-\frac{1}{B_T}(K^{vs})^2, \tag{10}$$

where

$VarS_0$ is a fair Present Value of Variance Swap,

K^{vs} is a strike of the swap (in volatility terms),

K_i^{put}, K_i^{call} are the strikes of vanilla options in percentage of underlying forward price with $K_0 = 0$.

Another approach is to approximate (8) by piecewise linear functions as described below.

Consider the function

$$H(s) := f(s) - (f(s_0) + f'(s_0)(s - s_0)).$$

Select two sequences of strikes: $K_0 = K_0^p > K_1^p > ...$ and $K_0 = K_0^c < K_1^c < ..$ with $\lim_{n\to\infty} K_n^p = 0$ and $\lim_{n\to\infty} K_n^c = \infty$ respectively, then an approximation H_T^{sup} of the function H from above, $H_T^{sup} > H$, is given by

$$H_T^{sup} := \sum_{n=1}^{\infty} w_p^n(K_n^p - s)^+ + \sum_{n=1}^{\infty} w_c^n(s - K_n^c)^+ \tag{11}$$

with

$$w_c^n = \frac{H(K_n^c) - H(K_{n-1}^c)}{K_n^c - K_{n-1}^c} - \sum_{k=1}^{n-1} w_k^c \tag{12}$$

and

$$w_p^n = \frac{H(K_n^p) - H(K_{n-1}^p)}{K_n^p - K_{n-1}^p} - \sum_{k=1}^{n-1} w_k^p \tag{13}$$

Since we have no infinitely many contracts trading at the market, we postulate that for some large strike K^* the value of the respective call is practically zero and will remain zero for the life of the contract we want to price. We hence assume that there is no probability mass beyond this "zero price strike" K^*. Then (11) gives a super-replication price and super-hedging position for all convex payoffs.

Taking expectation we come to

$$VarS_0 \approx \frac{2}{T}\left[\sum_1^{N_{puts}} w_p^n P_0(K_n^{put}) + \sum_1^{N_{calls}} w_c^n C_0(K_n^{call})\right] - \frac{1}{B_T}(K^{vs})^2. \tag{14}$$

with weights w_p^n, w_c^n given by (12) (13).

Volatility swap

Analogously, we can replicate the volatility swap by the portfolio of options on realized variance.

Assume that we have a set of prices of calls and puts on realized variance.

Apply (8) with realized variance $Q_{[0;T]}$ in place of s, $f(y) = \sqrt{y}$ and fair strike K_0 of Variance Swap in place of s_0. Then we obtain

$s_0 = K_0$, $f(s_0) = \sqrt{K_0}$, $f'(s) = \frac{1}{2\sqrt{s}}$, $f'(s_0) = \frac{1}{2\sqrt{K_0}}$.

Then the "bond-stock" position is

$$B(s) = \sqrt{K_0} + \frac{1}{2\sqrt{K_0}}(s - K_0) = \frac{1}{2}\sqrt{K_0} + \frac{1}{2\sqrt{K_0}}s.$$

Similarly to the Variance Swap case, the Put and Call positions may be approximated in two ways (10) or (14) with put-like and call-like options on Realized Variance in places of Vanilla puts and calls.

From Carr-Lee, the replicating portfolio for volatility swap holds

$u\sqrt{\pi/2}/F_0$ straddles at strike $K = F_0$

$u\sqrt{\frac{\pi}{8K^3F_0}}\left[I_1\left(\log\sqrt{K/F_0}\right) - I_0\left(\log\sqrt{K/F_0}\right)\right]dK$ calls at strikes $K > F_0$

$u\sqrt{\frac{\pi}{8K^3F_0}}\left[I_0\left(\log\sqrt{K/F_0}\right) - I_1\left(\log\sqrt{K/F_0}\right)\right]dK$ puts at strikes $K < F_0$.

Options on Realized Variance and Volatility

Assume that the realized volatility has a displaced lognormal distribution. Then the price of variance and volatility calls are given by

$$VarCall(K,T) = VarS \cdot N(d + 2s) - e^{-rT}K^2 \cdot N(d)$$

$$VolCall(K,T) = VolS \cdot N(d + s) - e^{-rT}K \cdot N(d)$$

where

$$
\begin{aligned}
VarS &= VarS(K,T) \quad \text{is a fair price of a variance swap;} \\
d &= (m - \log K)/s; \\
s &= \sqrt{-rT + \log(VarS/VolS^2)}; \\
m &= 1.5rT + \log(VolS^2/\sqrt{VarS}); \\
VolS &= VolS(K,T) \quad \text{is a fair price of a volatility swap.}
\end{aligned}
$$

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