Simulation of Lookback Options under Infinite Activity Lévy Model

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1 Preliminaries

A real Lévy process X is characterized by its generating triplet (γ, σ^2, ν). Where $(\gamma, \sigma) \in \mathbb{R} \times \mathbb{R}^+$, and ν is a Radon measure satisfying

$$\int_{\mathbb{R}} \left(1 \wedge x^2\right) \nu(dx) < \infty$$

By Lévy-Itô decomposition X can be written in this form

$$X_t = \gamma t + \sigma B_t + X_t^l + \lim_{\epsilon \downarrow 0} \widetilde{X}_t^\epsilon \tag{1.1}$$

With

$$X_t^l = \int_{|x|>1, s\in[0,t]} x J_X(dx \times ds) \equiv \sum_{0\le s\le t}^{|\Delta X_s|\ge 1} \Delta X_s$$

$$\begin{aligned} \widetilde{X}_t^\epsilon &= \int_{\epsilon\le|x|\le 1, s\in[0,t]} x(J_X(dx \times ds) - \nu(dx)dt) \\ &\equiv \int_{\epsilon\le|x|\le 1, s\in[0,t]} x\widetilde{J}_X(dx \times ds) \\ &\equiv \sum_{0\le s\le t}^{\epsilon\le|\Delta X_s|<1} \Delta X_s - t\int_{\epsilon\le|x|\le 1} x\nu(dx) \end{aligned}$$

Where J is a Poisson measure on $\mathbb{R}\times[0,\infty)$ with rate $\nu(dx)dt$ and B is a standard Brownian motion. In Lévy-Khinchine representation X, we characterize X by its characteristic function. That means

$$\mathbb{E}e^{iuX_t} = e^{t\varphi(u)} \quad \forall u \in \mathbb{R}$$

where φ is given by

$$\varphi(u) = i\gamma u - \frac{\sigma^2 u^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{1}_{|x|\leq 1})\nu(dx) \tag{1.2}$$

For any $\epsilon \in (0,1)$ we define the process R^ϵ by

$$R_t^\epsilon = -\widetilde{X}_t^\epsilon + \lim_{\delta \downarrow 0} \widetilde{X}_t^\delta \tag{1.3}$$

and X^ϵ by

$$X_t^\epsilon = \gamma t + \sigma B_t + X_t^l + \widetilde{X}_t^\epsilon \tag{1.4}$$

Then

$$X_t = X_t^\epsilon + \mathbb{R}_t^\epsilon \tag{1.5}$$

We set

$$\begin{aligned}
M_t &= \sup_{0\leq s\leq t} X_s \\
M_t^{\epsilon,X} &= \sup_{0\leq s\leq t} X_s^\epsilon \\
m_t^{\epsilon,X} &= \inf_{0\leq s\leq t} X_s^\epsilon \\
\hat{M}_t^\epsilon &= \sup_{0\leq s\leq t} (X_s^\epsilon + \sigma_\epsilon W_s)
\end{aligned}$$

Where W is a standard Brownian motion independent of X, and $\sigma(\epsilon) = \sqrt{\int_{|x|<\epsilon} x^2 \nu(dx)}$.

2 Simulation method

We focus on the simulation of a lookback option with maturity T, where the Levy process is infinite activity without Brownian part. Our goal is to simulate M_T. In fact we can not simulate M_T, we will then approximated by M_T^ϵ or $\hat{(M)}_T^\epsilon$. This introduces a bias. Denote by J the Poisson measure on $\mathbb{R} \times [0, \infty)$

of intensity $\nu(dx)dt$, then for $t \geq 0$, we have

$$\begin{aligned}
X_t^\epsilon &= X_t - R_t^\epsilon \\
&= \gamma t + \int_{|x|>1, s\in[0,t]} x J_X(dx \times ds) + \int_{\epsilon\leq|x|\leq 1, s\in[0,t]} x J_X(dx \times ds) \\
&= \left(\gamma - \int_{\epsilon\leq|x|\leq 1} x\nu(dx)\right) t + \int_{|x|>\epsilon, s\in[0,t]} x J_X(dx \times ds) \\
&= \left(\gamma - \int_{\epsilon\leq|x|\leq 1} x\nu(dx)\right) t + \int_{x>\epsilon, s\in[0,t]} x J_X(dx \times ds) \\
&\quad + \int_{x<-\epsilon, s\in[0,t]} x J_X(dx \times ds) \\
&= \gamma_0^\epsilon t + \sum_{i=1}^{N_t^+} Y_i^+ - \sum_{i=1}^{N_t^-} Y_i^-
\end{aligned}$$

Where $\gamma_0^\epsilon = \gamma - \int_{\epsilon\leq|x|\leq 1} x\nu(dx)$, the r.v. $\left(Y_i^+\right)_{i\geq 1}$ are i.i.d. with common law $\frac{\nu_\epsilon^+(dx)}{\nu(\epsilon,+\infty)}$, the r.v. $\left(Y_i^-\right)_{i\geq 1}$ are i.i.d. with common law $\frac{\nu_\epsilon^-(-dx)}{\nu(-\infty,\epsilon)}$. The measures ν_ϵ^+ and ν_ϵ^- correspond respectively to ν restricted on $(0,+\infty)$ and on $(-\infty,0)$. The process X^ϵ is a compound Poisson process. So to simulate M_T^ϵ, it suffices to simulate the instants of jump of X^ϵ and the corresponding jump. The random variable $\hat{(M)}_T^\epsilon$ must be approximated by its discrete version in the case of look-back options. The number of discretization points in this case is greater than in the case of classic jump-diffusion model. The Probem that arises is because the numbers of jumps on $[0,T]$ is relatively large, how to quickly simulate the size of the jumps. The simulation of the instants of jump is relatively simple, we will focus on simulation of jumps, including $\left(Y_i^+\right)_{i\geq 1}$. Simulation of $\left(Y_i^-\right)_{i\geq 1}$ will be identical. Let $\lambda_+^\epsilon = \nu(\epsilon,\infty)$. The cumulative distribution function of Y_1^+ cannot be determined explicitly, and hence the inverse distribution function either. So one way to simulate Y_1^+ is to use a rejection method. This is time consuming, especially since it will make on average $\lambda_+^\epsilon T$ simulations. The alternative is to make a *discrete inversion* of the cdf, F_+, of Y_1^+. We have, for all $x > \epsilon$

$$F_+(x) = \frac{1}{\lambda_+^\epsilon} \int_\epsilon^x \nu(dx)$$

We will define a positive real A in order to have $\nu(A,+\infty)$ very small, in order of 10^{-16} for example (that is what we choose in our simulations). We suppose then that the r.v. Y_1^+ is in $[\epsilon, A]$. Set for any $k \in \{0,\ldots,n\}$

$$\begin{aligned}
x_k &= k\frac{A-\epsilon}{n} + \epsilon \\
y_k &= \frac{F_+(t_k)}{F_+(A)}
\end{aligned}$$

Where n is the number of the discretization points on $[\epsilon, A]$. Note that $y_0 = 0$. How do we compute $(F_+(x_k))_{1 \leq k \leq n}$? Notice that for any $k \in \{1, \dots, n\}$, we have

$$F_+(x_k) = \sum_{j=1}^{k} (F_+(x_j) - F_+(x_{j-1}))$$

with

$$(F_+(x_j) - F_+(x_{j-1})) = \int_{t_{j-1}}^{t_j} \nu(dx)$$

Depending on the Lévy measure, we will define some approximation method for the integrale $\int_{t_{j-1}}^{t_j} \nu(dx)$. We define the function G_+ by, for any $y \in [0,1]$

$$G_+(y) = x$$

where x is the unique real satisfying $\frac{F_+(x)}{F_+(A)} = y$. Let $y \in [0,1]$, to compute $G_+(y)$, we use the following method. We have to find first the integer $k > 1$ satifying $y_{k-1} \leq y < y_k$. Then we have

$$yF_+(A) = y_{k-1} + \int_{x_{k-1}}^{G_+(y)} \nu(dy)$$

We must approximate the above integrale depending on $G_+(y)$, and express the latter as a function of y. We will call G_+, the *discrete inverse function* of F_+. When n and A are going to the infinity, we the inverse function of F_+. For our simulations, we suppose that Y_1^+ is equal in distribution to $G_+(U)$, where U is a uniform r.v. on $[0,1]$. We will use as control variate, $e^{X_T^\epsilon}$. its expected value is known with an error whoch we can control.

3 Estimation of the inverse cdf of the jumps

We will, for some popular models, estimate the function G_+. The models that we consider in this section are VG, CGMY and NIG. Our method can work for any other model.

3.1 The Variance-Gamma case

Let G be a gamma process with de parameters $(\mu, \kappa) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ (see [5]), satisfying $G_0 = 0$ and for any $t \geq 0$ and $h > 0$, $G_{t+h} - G_t$ have a gamma distribution with parameters $\left(h\frac{\mu^2}{\kappa}, \frac{\kappa}{\mu}\right)$. In fact in financial applications $\mu = 1$, and the process $(W_{G_t})_{t \geq 0}$ is a VG processus VG with parameter (θ, σ, κ). Its characteristic exponent is given by

$$\varphi(u) = \log\left(\left(1 - i\theta\kappa u + \frac{\sigma^2}{2}\kappa u^2\right)^{-\frac{1}{\kappa}}\right)$$

The process $(W_{G_t})_{t\geq 0}$, can be defined by its Lévy measure ν. Indeed

$$\nu(dx) = C\frac{e^{-Mx}}{x}\mathbb{1}_{x>0}dx + C\frac{e^{-G|x|}}{|x|}\mathbb{1}_{x<0}dx$$

Where

$$\begin{aligned} C &= \frac{1}{\kappa} \\ M &= \frac{1}{\sigma}\sqrt{\frac{2}{\kappa}+\frac{\theta^2}{\sigma^2}} - \frac{\theta}{\sigma^2} \\ G &= \frac{1}{\sigma}\sqrt{\frac{2}{\kappa}+\frac{\theta^2}{\sigma^2}} + \frac{\theta}{\sigma^2} \end{aligned}$$

This is a particular case of the CGMY process (by taking $Y = 0$, see [3]). The pad of Y_1^+ is then

$$f_+(x) = \frac{C}{\lambda_+^\epsilon}\frac{e^{-Mx}}{x}, \ x > \epsilon$$

Then for any $x > \epsilon$

$$F_+(x) = \frac{C}{\lambda_+^\epsilon}\int_\epsilon^x \frac{e^{-My}}{y}dy$$

Hence

$$F_+(x_k) - F_+(x_{k-1}) = \frac{C}{\lambda_+^\epsilon}\int_{x_{k-1}}^{x_k} \frac{e^{-My}}{y}dy$$

We approximate this integrale by

$$\frac{C}{\lambda_+^\epsilon}e^{-Mx_{k-1}}\int_{x_{k-1}}^{x_k}\frac{dy}{y}dy = \frac{C}{\lambda_+^\epsilon}e^{-Mx_{k-1}}\log\left(\frac{x_k}{x_{k-1}}\right)$$

Then the function G_+ satisfy

$$yF_+(A) = y_{k-1} + \frac{C}{\lambda_+^\epsilon}\int_{x_{k-1}}^{G_+(y)}\frac{e^{-My}}{y}dy$$

As previously the above integrale is approximated by

$$\frac{C}{\lambda_+^\epsilon}e^{-Mx_{k-1}}\log\left(\frac{G_+(y)}{x_{k-1}}\right)$$

Hence $G_+(y)$ can be approximated by

$$x_{k-1}\exp\left[\frac{\lambda_+^\epsilon}{C}\left(yF_+(A) - y_{k-1}\right)e^{-Mx_{k-1}}\right] \tag{3.6}$$

In the VG model M_T is approximated by M_T^ϵ. In the table 3.1, we observe the convergence of our method with respect to ϵ. Note that the errors are relative, and we mean by "true" price that obtained by *[Becker(2008)]*.

ϵ	price	Monte Carlo error	total error
10^{-1}	7.076	0.05%	24.7%
10^{-2}	9.347	0.08%	0.50%
10^{-3}	9.401	0.08%	0.04%

Table 3.1: Approximation of the continuous call lookback price in VG model. Les parameters are : $S_0 = 100$, $r = 0.0548$, $\delta = 0$, $T = 0.40504$, $S_+ = 100$, $\theta = -0.2859$, $\kappa = 0.2505$, $\sigma = 0.1927$ and $n = 1000000$. The "true" call price is 9.39827.

3.2 The CGMY case

It is a pure jump Lévy process (see [5]), with Lévy measure

$$\nu(dx) = C\frac{e^{-Mx}}{x^{1+Y}}\mathbb{1}_{x>0}dx + C\frac{e^{-G|x|}}{|x|^{1+Y}}\mathbb{1}_{x<0}dx$$

Where C, G et M are positive, and $Y \in (0,2)$. When $Y = 0$, we get the Variance-Gamma model. Its characteristic exponent is given by

$$\varphi(u) = \begin{cases} C\left((M-iu)\log\left(1-\frac{iu}{M}\right) + (G+iu)\log\left(1+\frac{iu}{G}\right)\right), & si\ Y=1 \\ C\Gamma(-Y)\left[M^Y\left(\left(1-\frac{iu}{M}\right)^Y - 1 + \frac{iuY}{M}\right) + G^Y\left(\left(1+\frac{iu}{G}\right)^Y - 1 - \frac{iuY}{G}\right)\right], & sinon \end{cases}$$

In the CGMY model, the pdf of Y_1^+ is

$$f_+(x) = \frac{C}{\lambda_+^\epsilon}\frac{e^{-Mx}}{x^{1+x}},\ x > \epsilon$$

Then its cdf is

$$F_+(x) = \frac{C}{\lambda_+^\epsilon}\int_\epsilon^x \frac{e^{-My}}{y^{1+Y}}dy$$

Hence

$$F_+(x_k) - F_+(x_{k-1}) = \frac{C}{\lambda_+^\epsilon}\int_{x_{k-1}}^{x_k}\frac{e^{-My}}{y}dy$$

Then we approximate $F_+(x_k) - F_+(x_{k-1})$ by

$$\frac{C}{\lambda_+^\epsilon}e^{-Mx_{k-1}}\int_{x_{k-1}}^{x_k} y^{1+Y}dy = \frac{C}{\lambda_+^\epsilon Y}e^{-Mx_{k-1}}\left(\frac{1}{x_{k-1}^Y} - \frac{1}{x_k^Y}\right)$$

So G_+ is solution of the equation

$$yF_+(A) = y_{k-1} + \frac{C}{\lambda_+^\epsilon}\int_{x_{k-1}}^{G_+(y)}\frac{e^{-My}}{y^{1+Y}}dy$$

ϵ	prix	erreur statistique	erreur totale
10^{-1}	14.212	0.07%	2.54%
10^{-2}	13.903	0.07%	0.30%
10^{-3}	13.868	0.07%	0.07%

Table 3.2: Approximation of the discrete put lookback price (where the number of discretization points is $N = 252$) in CGMY model. The parameters are : $S_0 = 100$, $r = 0.05$, $\delta = 0.02$, $T = 1$, $S_+ = 100$, $C = 4$, $G = 50$, $M = 60$, $Y = 0.7$ and $n = 1000000$. The "true" price is 13.8600.

We approximate the above integrale by

$$\frac{C}{\lambda_+^\epsilon Y} e^{-Mx_{k-1}} \left(\frac{1}{x_{k-1}^Y} - \frac{1}{(G_+(y))^Y} \right)$$

Hence $G_+(y)$ can be approximated by

$$\left[\frac{1}{x_{k-1}^Y} - \frac{\lambda_+^\epsilon Y}{C} e^{Mx_{k-1}} \left(yF_+(A) - y_{k-1} \right) \right]^{-\frac{1}{Y}} \tag{3.7}$$

The r.v. M_T is approximated by $\hat{M}_T^\epsilon$. In the table 3.2, we observe the convergence of our method with respect to ϵ. The errors are relative, and we mean by "true" price that obtained by *[Feng-Linetsky(2009)]*.

3.3 The NIG case

Like the VG model, the NIG (Normal Inverse Gaussian) model (see [7]) is a particular case of the hyperbolic models. It is charterized by four parameters : α, β, δ and μ. Where $0 \leq |\beta| \leq \alpha$, $\delta > 0$ and $\mu \in \mathbb{R}$. Its generating triplet are $(\gamma, 0, \nu)$, where

$$\begin{aligned} \gamma &= \mu + 2\frac{\alpha\delta}{\pi} \int_0^1 \sinh(\beta x) K_1(\alpha x) \\ \nu(dx) &= \frac{\alpha\delta}{\pi|x|} K_1(\alpha|x|) e^{\beta x} dx \end{aligned}$$

with

$$K_\lambda(z) = \frac{1}{2} \int_{\mathbb{R}^+} y^{\lambda-1} \exp\left(-\frac{1}{2} z \left(y + \frac{1}{y} \right) \right) dy$$

In financial applications we set $\mu = 0$. Then the NIG is represented by three parameters : (α, β, δ). The cdf of Y_1^+) is

$$f_+(x) = \frac{\alpha\delta}{\pi x} K_1(\alpha x) e^{\beta x}, \ x > \epsilon$$

And then its cdf is given by

$$F_+(x) = \frac{\alpha\delta}{\pi}\int_\epsilon^x \frac{K_1(\alpha y)}{y} e^{\beta y} dy$$

Therefore

$$F_+(x_k) - F_+(x_{k-1}) = \frac{\alpha\delta}{\pi}\int_{x_{k-1}}^{x_k} \frac{K_1(\alpha y)}{y} e^{\beta y} dy$$

To approximate the above integrale, we need to study the asymptotic behaviour of K_1. We have (see [1], formulas 9.7.2 et 9.8.7)

$$K_1(x) \underset{x\downarrow 0}{\sim} \frac{C}{x}, \text{ for a given C>0}$$

$$K_1(x) \underset{x\to+\infty}{\sim} \sqrt{\frac{\pi}{2x}} e^{-x}$$

Hence the following approximation

$$\frac{\alpha\delta}{\pi} x_{k-1} K_1(\alpha x_{k-1}) e^{\beta x_{k-1}} \int_{x_{k-1}}^{x_k} \frac{dy}{y^2} = \frac{\alpha\delta}{\pi} x_{k-1} K_1(\alpha x_{k-1}) e^{\beta x_{k-1}} \left(\frac{1}{x_{k-1}} - \frac{1}{x_k}\right)$$

In NIG case G_+ satisfy

$$yF_+(A) = y_{k-1} + \frac{\alpha\delta}{\pi}\int_{x_{k-1}}^{G_+(y)} \frac{K_1(\alpha y)}{y} e^{\beta y} dy$$

So we approximate $G_+(y)$ by

$$\left(\frac{1}{x_{k-1}} - \frac{\pi}{\alpha\delta}\frac{yF_+(A) - y_{k-1}}{x_{k-1}K_1(\alpha x_{k-1})} e^{-\beta x_{k-1}}\right)^{-1} \tag{3.8}$$

The Y_1^- case is treated is the same way, we only need to substitute β by $-\beta$. In this model M_T is approximated by $\hat{M}_T^\epsilon$. In the table 3.3, we observe the convergence of our method with respect to ϵ. The errors are relative, and we

ϵ	prix	erreur statistique	erreur totale
10^{-1}	13.48	0.0%	10.33%
10^{-2}	12.43	0.08%	1.74%
10^{-3}	12.25	0.08%	0.31%

Table 3.3: Approximation of the discrete put lookback price (where the number of discretization points is $N = 252$) in NIG model. The parameters are : $S_0 = 100$, $r = 0.05$, $\delta = 0.02$, $T = 1$, $S_+ = 100$, $\alpha = 15$, $\beta = -5$,$\tilde{\delta} = 0.5$ and $n = 1000000$. The “true” price is 12.2224.

mean by “true” price that obtained by *[Feng-Linetsky(2009)]*.

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