Montecarlo for Barrier Option:Algorithm

Lucia Caramellino

March 1, 2012

Contents

Premia 14

1 European Barrier Options

Suppose the underlying asset price evolves according to the Black and Scholes model with continuous yield δ, i.e.

$$dS_u = S_u(bdu + \sigma dB_u), \quad S_t = x$$

where $b = r - \delta$, r being the spot rate. As usual, we set T as the maturity, $\theta = T - t$ as the time to maturity and $f(S_T(x))$ as a suitable function which will be used to determine the payoff: since we are going to study european barrier call or put options, f turns out to be

$$f(S_T(x)) = (S_T(x) - K)_+ \quad \text{or} \quad f(S_T(x)) = (K - S_T(x))_+$$

respectively, where K stands for the exercise price.

We need also to introduce the barriers:

$$L, U \,:\, [t, +\infty) \longrightarrow [0, +\infty), \quad \text{with} \quad L(u) < U(u),\, \forall u.$$

Here, L denotes the *lower* barrier and U the *upper* one.

The payoff of a knock-out single or double barrier is given by $f(S_T(x))$ provided that the underlying asset price S does not hit on the barrier(s) during the time interval $[t, T]$; if it does, a pre-specified cash rebate R is paid out. Similarly, in a knock-in framework, the payoff is equal to $f(S_T(x))$ if S reaches the boundary and a cash rebate R is paid out if S stays beyond the barrier(s) until the maturity T. In formulas, the price of a barrier option is given by

$$F(t, x) = \mathbb{E}\Big(G(S(x))\Big)$$

being G the following functional of all the path $(S_s, t \le s \le T)$:

$$G(S(x)) = \begin{cases} e^{-r\theta} f(S_T(x)) \mathbf{1}_{\tau(x)>T} + e^{-r(\tau(x)-t)} R \mathbf{1}_{\tau(x)\le T} & \text{knock}-\text{out case} \\ e^{-r\theta} f(S_T(x)) \mathbf{1}_{\tau(x)\le T} + e^{-r\theta} R \mathbf{1}_{\tau(x)>T} & \text{knock}-\text{in case} \end{cases}$$

where τ denotes the hitting time on the barrier(s)[1].

Concernig the associated delta, as usual it is given by

$$\Delta(t, x) = \frac{\partial}{\partial x} F(t, x).$$

We present here a Monte Carlo procedure allowing to numerically compute the price and the associated delta of barrier options. As usual, one generates a large number, say M, of independent paths which approximate the underlying asset price S on the time interval $[t, T]$ and for each simulated path one computes the associated value of G.

The price F and the delta Δ of the option will be numerically evaluated by averaging over the M samples:

$$F(t, x) \sim \frac{1}{M} \sum_{i=1}^{M} G(S^{(i)}(x))$$

[1] See next sections for a more precise definition of τ.

and

$$\Delta(t,x) \sim \frac{F(t,x+\varepsilon)-F(t,x)}{\varepsilon} \sim \frac{1}{M\varepsilon}\sum_{i=1}^{M}\Big[G(S^{(i)}(x+\varepsilon)-G(S^{(i)}(x)\Big]$$

where ε stands for a constant close to zero[2].

The paths $S^{(i)}(x)$'s are generated as follows.

For a fixed positive integer N, set $h=\theta/N$ as the step size. The underlying stock price is then simulated at the times $t_k=t+hk$, $k=1,2,\dots,N$:

$$\begin{cases} S_t^{(i)}=x \\ S_{t+hk}^{(i)}=S_{t+h(k-1)}^{(i)}\exp\Big\{\Big(b-\frac{\sigma^2}{2}\Big)h+\sigma\sqrt{h}X_k^{(i)}\Big\} \quad k=1,2,\dots,N \end{cases}$$

where $(X_k^{(i)})_{1\le k\le N,1\le i\le M}$ is a set of independent standard gaussian random variables.

Thus, in the Monte Carlo-Baldi algorithm, at step k, $S_{t+hk}^{(i)}$ is generated, by means of $S_{t+h(k-1)}^{(i)}$. Now, one checks if $S_{t+hk}^{(i)}$ has or has not reached the boundary. In the first case, the simulation is stopped and the value of the functional G can be then computed. Otherwise, one checks if $S^{(i)}$ has crossed the barrier(s) during the time interval $(t+h(k-1),t+hk)$ by means of an approximation p_k^h of the conditional exit probability[3], given the observations $S_{t+h(k-1)}^{(i)}$ and $S_{t+hk}^{(i)}$, as follows. A Bernoulli random variable $Y_k^{(i)}$, with parameter p_k^h, is generated. If $Y_k^{(i)}=1$ then the boundary has been reached by $S^{(i)}$, so that the simulation of the i^{th} path is stopped and the associated value of the functional G is then computed; otherwise, the simulation is carried on and the step $(k+1)$ is considered, unless obviously $k=N$.

In the same way, the paths $\{S^{(i)}(x+\varepsilon)\}_{1\le i\le M}$ are generated, $\{G(S^{(i)}(x+\varepsilon)\}_{1\le i\le M}$ computed and the delta can be approximated. It worth to remark that in order to numerically compute the delta, the paths $S^{(i)}(x)$ and $S^{(i)}(x+\varepsilon)$ have to be generated by means of the same sample of gaussian r.v.'s $\{\hat{X}_k^{(i)}\}_k$ and bernoulli r.v.'s $\{Y_k^{(i)}\}_k$. Indeed, in the Monte Carlo approximation of the derivative of a function, such as the delta, it has been shown[4] that the error

[2] It is worth to observe that a too small value for ε does not give good results since it is not hard to show that in the barrier option framework the variance of the estimator diverges as $\varepsilon\to 0$. Anyway, numerical tests have shown that $\varepsilon=0.01$ provides good outcomes for the delta, as it follows by comparing the obtained results with the values exactly computed by closed formulas.

[3] The value of p_k^h is actually exact in the single barrier framework studied by Ikeda and Kunitomo.

[4] Such a property is clear and has been already remarked by several authors.

decreases when the correlation of the samples involved in the simulation goes to 1.

In summary, the algorithm can be splitted in two cicles:

- the "i-cicle", for $i = 1, 2, \ldots, M$, giving the i^{th} simulated path $S^{(i)}$, the value $G(S^{(i)})$ and thus the partial sum of the samples of the functional G up to the i^{th} simulation;
- the "k-cicle", for $k = 1, 2, \ldots, N$, inside the "i-cicle", through which the i^{th} path is simulated at times $t_k = t + hk$, giving the eventual exit and the value of $S_T^{(i)}$, if it is useful for determining the functional G.

At the end of the "i-cicle", one can average over the M obtained values $G(S^{(i)})$.

In the following, we give the details of the simulation procedure for the i^{th} path, i.e. what we have called the "k-cicle", specializing on the cases of single or double barrier options and on the knock-out or knock-in framework.

1.1 Single Barrier Options

Let L denote the lower barrier and let

$$\tau_L = \inf\{u > t \ ; \ S_u \leq L(u)\}$$

be the hitting time on the barrier. Similarly, if U denotes the upper barrier,

$$\tau_U = \inf\{u > t \ ; \ S_u \geq U(u)\}$$

stands for the hitting time on U. The price of a single barrier call or put option is written in terms of τ_L and τ_U.

We recall that $f(S_T(x)) = (S_T(x) - K)_+$ or $f(S_T(x)) = (K - S_T(x))_+$ according to the case of a call or a put option, respectively.

1.1.1 Down-Out Call/Put Options

The price of the option is $F(t, x) = \mathbb{E}\Big(G(S(x))\Big)$, where

$$G(S(x)) = e^{-r\theta} f(S_T(x)) \mathbf{1}_{\tau_L(x) > T} + e^{-r(\tau_L(x) - t)} R \mathbf{1}_{\tau_L(x) \leq T}.$$

For $i = 1, 2, \ldots, M$, the i^{th} path $S^{(i)}$ is simulated and $G(S^{(i)})$ is evaluated as follows.

One sets $S_t^{(i)} = x$ and computes $L(t)$. For $k = 1, 2, \ldots N$, $S_{t+hk}^{(i)}$ is simulated and $L(t + hk)$ is computed, where $h = \theta/N$.

♠ If $S_{t+hk}^{(i)} \leq L(t + hk)$ then

- one sets $G(S^{(i)}) = e^{-rhk}R$;
- the $(i+1)^{\text{th}}$ path is considered, unless $i = M$.

♠ If $S^{(i)}_{t+hk} > L(t+hk)$ then

- the conditional exit probability p^h_k, given the simulated points at times $t + h(k-1)$ and $t + hk$, is computed:

$$p^h_k = \exp\Big\{-\frac{2}{\sigma^2 h}\Big[(\ln S^{(i)}_{t+h(k-1)} - L(t+h(k-1)))(\ln S^{(i)}_{t+hk} - L(t+h(k-1))) - (\ln S^{(i)}_{t+h(k-1)} - L(t+h(k-1)))(L(t+hk) - L(t+h(k-1)))\Big]\Big\};$$

- a Bernoulli random variable $Y^{(i)}_k$, with parameter p^h_k, is generated and:
  - if $Y^{(i)}_k = 1$ then
    - one sets $G(S^{(i)}) = e^{-rhk}R$;
    - the $(i+1)^{\text{th}}$ path is considered, unless $i = M$;
  - if $Y^{(i)}_k = 0$ then
    - if $k < N$ then the step $(k+1)$ is considered;
    - if $k = N$ then ($T = t + hN$ so that) one sets $G(S^{(i)}) = e^{-r\theta} f(S^{(i)}_T)$.

1.1.2 Up-Out Call/Put Options

The price of the option is $F(t,x) = \mathbb{E}\Big(G(S(x))\Big)$, where

$$G(S(x)) = e^{-r\theta} f(S_T(x)) \mathbf{1}_{\tau_U(x) > T} + e^{-r(\tau_U(x) - t)} R \mathbf{1}_{\tau_U(x) \le T},$$

For $i = 1, 2, \ldots, M$, the i^{th} path $S^{(i)}$ is simulated and $G(S^{(i)})$ is evaluated as follows.

One sets $S^{(i)}_t = x$ and computes $U(t)$. For $k = 1, 2, \ldots N$, $S^{(i)}_{t+hk}$ is simulated and $U(t+hk)$ is computed, where $h = \theta/N$.

♠ If $S^{(i)}_{t+hk} \ge U(t+hk)$ then

- one sets $G(S^{(i)}) = e^{-rhk}R$;
- the $(i+1)^{\text{th}}$ path is considered, unless $i = M$.

♠ If $S^{(i)}_{t+hk} < U(t+hk)$ then

- the conditional exit probability p_k^h, given the simulated points at times $t+h(k-1)$ and $t+hk$, is computed:

$$p_k^h = \exp\Big\{-\frac{2}{\sigma^2 h}\Big[(U(t+h(k-1))-\ln S^{(i)}_{t+h(k-1)})(U(t+h(k-1))-\ln S^{(i)}_{t+hk}) \\ +(U(t+h(k-1))-\ln S^{(i)}_{t+h(k-1)})(U(t+hk)-U(t+h(k-1)))\Big]\Big\};$$

- a Bernoulli random variable $Y_k^{(i)}$, with parameter p_k^h, is generated and:
  - if $Y_k^{(i)}=1$ then
    - one sets $G(S^{(i)}) = e^{-rhk}R$;
    - the $(i+1)^{\text{th}}$ path is considered, unless $i=M$;
  - if $Y_k^{(i)}=0$ then
    - if $k<N$ then the step $(k+1)$ is considered;
    - if $k=N$ then ($T=t+hN$ so that) one sets $G(S^{(i)}) = e^{-r\theta}f(S_T^{(i)})$.

1.1.3 Down-In Call/Put Options

The price of the option is $F(t,x) = \mathbb{E}\Big(G(S(x))\Big)$, where

$$G(S(x)) = e^{-r\theta}f(S_T(x))\mathbf{1}_{\tau_L(x)\le T} + e^{-r\theta}R\mathbf{1}_{\tau_L(x)>T}.$$

For $i=1,2,\dots,M$, the i^{th} path $S^{(i)}$ is simulated and $G(S^{(i)})$ is evaluated as follows.

One sets $S_t^{(i)}=x$ and computes $L(t)$. For $k=1,2,\dots N$, $S^{(i)}_{t+hk}$ is simulated and $L(t+hk)$ is computed, where $h=\theta/N$.

♠ If $S^{(i)}_{t+hk}\le L(t+hk)$ then

- $S^{(i)}$ is generated at time T:

$$S_T^{(i)} = S^{(i)}_{t+hk}\exp\Big\{\Big(b-\frac{\sigma^2}{2}\Big)h(N-k)+\sigma\sqrt{h(N-k)}\hat{X}_k^{(i)}\Big\}$$

  where $\hat{X}_k^{(i)}$ is a further standard gaussian random variable;

- one sets $G(S^{(i)}) = e^{-rhk}f(S_T^{(i)})$;

- the $(i+1)^{\text{th}}$ path is considered, unless $i=M$.

♠ If $S^{(i)}_{t+hk} > L(t+hk)$ then

- the conditional exit probability p^h_k, given the simulated points at times $t+h(k-1)$ and $t+hk$, is computed:

$$p^h_k = \exp\Big\{-\frac{2}{\sigma^2 h}\Big[(\ln S^{(i)}_{t+h(k-1)} - L(t+h(k-1)))(\ln S^{(i)}_{t+hk} - L(t+h(k-1))) \\ -(\ln S^{(i)}_{t+h(k-1)} - L(t+h(k-1)))(L(t+hk) - L(t+h(k-1)))\Big]\Big\};$$

- a Bernoulli random variable $Y^{(i)}_k$, with parameter p^h_k, is generated and:
  - if $Y^{(i)}_k = 1$ then
    * $S^{(i)}$ is generated at time T:

    $$S^{(i)}_T = S^{(i)}_{t+hk} \exp\Big\{\Big(b - \frac{\sigma^2}{2}\Big)h(N-k) + \sigma\sqrt{h(N-k)}\hat{X}_k\Big\}$$

      where $\hat{X}^{(i)}_k$ is a further standard gaussian random variable;
    * one sets $G(S^{(i)}) = e^{-rhk} f(S^{(i)}_T)$;
    * the $(i+1)^{\text{th}}$ path is considered, unless $i = M$.
  - if $Y^{(i)}_k = 0$ then
    * if $k < N$ then the step $(k+1)$ is considered;
    * if $k = N$ then one sets $G(S^{(i)}) = e^{-r\theta} R$.

1.1.4 Up-In Call/put Options

The price of the option is $F(t,x) = \mathbb{E}\Big(G(S(x))\Big)$, where

$$G(S(x)) = e^{-r\theta} f(S_T(x))\mathbf{1}_{\tau_U(x)\le T} + e^{-r\theta} R\mathbf{1}_{\tau_U(x)>T}.$$

For $i = 1, 2, \dots, M$, the i^{th} path $S^{(i)}$ is simulated and $G(S^{(i)})$ is evaluated as follows.

One sets $S^{(i)}_t = x$ and computes $U(t)$. For $k = 1, 2, \dots N$, $S^{(i)}_{t+hk}$ is simulated and $U(t+hk)$ is computed, where $h = \theta/N$.

♠ If $S^{(i)}_{t+hk} \ge U(t+hk)$ then

- $S^{(i)}$ is generated at time T:

$$S^{(i)}_T = S^{(i)}_{t+hk} \exp\Big\{\Big(b - \frac{\sigma^2}{2}\Big)h(N-k) + \sigma\sqrt{h(N-k)}\hat{X}^{(i)}_k\Big\}$$

  where $\hat{X}^{(i)}_k$ is a further standard gaussian random variable;

- one sets $G(S^{(i)}) = e^{-rhk} f(S_T^{(i)})$;
- the $(i+1)^{\text{th}}$ path is considered, unless $i = M$.

♠ If $S_{t+hk}^{(i)} < U(t+hk)$ then

- the conditional exit probability p_k^h, given the simulated points at times $t + h(k-1)$ and $t + hk$, is computed:

$$p_k^h = \exp\Big\{-\frac{2}{\sigma^2 h}\Big[(U(t+h(k-1)) - \ln S_{t+h(k-1)}^{(i)})(U(t+h(k-1)) - \ln S_{t+hk}^{(i)}) + (U(t+h(k-1)) - \ln S_{t+h(k-1)}^{(i)})(U(t+hk) - U(t+h(k-1)))\Big]\Big\};$$

- a Bernoulli random variable $Y_k^{(i)}$, with parameter p_k^h, is generated and:
  - if $Y_k^{(i)} = 1$ then
    - $S^{(i)}$ is generated at time T:

$$S_T^{(i)} = S_{t+hk}^{(i)} \exp\Big\{\Big(b - \frac{\sigma^2}{2}\Big)h(N-k) + \sigma\sqrt{h(N-k)}\hat{X}_k^{(i)}\Big\}$$

      where $\hat{X}_k^{(i)}$ is a further standard gaussian random variable;
    - one sets $G(S^{(i)}) = e^{-rhk} f(S_T^{(i)})$;
    - the $(i+1)^{\text{th}}$ path is considered, unless $i = M$.
  - if $Y_k^{(i)} = 0$ then
    - if $k < N$ then the step $(k+1)$ is considered;
    - if $k = N$ then ($T = t + hN$ so that) one sets $G(S^{(i)}) = e^{-r\theta} R$.

1.2 Double Barrier Options

The price of a double barrier call or put option is written in terms of first time the underlying asset price hits on the barriers:

$$\tau = \inf\{u > t \ ; \ S_u \leq L(u) \text{ or } S_u \geq U(u)\} \equiv \tau_L \wedge \tau_U.$$

Again we recall that $f(S_T(x)) = (S_T(x) - K)_+$ or $f(S_T(x)) = (K - S_T(x))_+$ according to the case of a call or a put option, respectively.

1.2.1 Knock-Out Call/Put Options

The price of the option is $F(t,x) = \mathbb{E}\Big(G(S(x))\Big)$, where

$$G(S(x)) = e^{-r\theta} f(S_T(x)) \mathbf{1}_{\tau(x)>T} + e^{-r(\tau(x)-t)} R \mathbf{1}_{\tau(x)\leq T}.$$

For $i = 1, 2, \ldots, M$, the i^{th} path $S^{(i)}$ is simulated and $G(S^{(i)})$ is evaluated as follows.

One sets $S^{(i)}_t = x$ and computes $L(t)$ and $U(t)$. For $k = 1, 2, \ldots N$, $S^{(i)}_{t+hk}$ is simulated and $L(t+hk)$, $U(t+hk)$ are computed, where $h = \theta/N$.

♠ If $S^{(i)}_{t+hk} \leq L(t+hk)$ or $S^{(i)}_{t+hk} \geq U(t+hk)$ then

- one sets $G(S^{(i)}) = e^{-rhk} R$;
- the $(i+1)^{\text{th}}$ path is considered, unless $i = M$.

♠ If $S^{(i)}_{t+hk} > L(t+hk)$ and $S^{(i)}_{t+hk} < U(t+hk)$ then

- the conditional exit probability p^h_k, given the simulated points at times $t + h(k-1)$ and $t + hk$, is computed:
  if $S^{(i)}_{t+h(k-1)} + S^{(i)}_{t+hk} > L(t + h(k-1)) + U(t + h(k-1))$ then
  $$p^h_k = \exp\Big\{-\frac{2}{\sigma^2 h}\Big[(U(t+h(k-1)) - \ln S^{(i)}_{t+h(k-1)})(U(t+h(k-1)) - \ln S^{(i)}_{t+hk}) + (U(t+h(k-1)) - \ln S^{(i)}_{t+h(k-1)})(U(t+hk) - U(t+h(k-1)))\Big]\Big\};$$
  if $S^{(i)}_{t+h(k-1)} + S^{(i)}_{t+hk} < L(t + h(k-1)) + U(t + h(k-1))$ then
  $$p^h_k = \exp\Big\{-\frac{2}{\sigma^2 h}\Big[(\ln S^{(i)}_{t+h(k-1)} - L(t+h(k-1)))(\ln S^{(i)}_{t+hk} - L(t+h(k-1))) - (\ln S^{(i)}_{t+h(k-1)} - L(t+h(k-1)))(L(t+hk) - L(t+h(k-1)))\Big]\Big\};$$
- a Bernoulli random variable $Y^{(i)}_k$, with parameter p^h_k, is generated and:
  - if $Y^{(i)}_k = 1$ then
    * one sets $G(S^{(i)}) = e^{-rhk} R$;
    * the $(i+1)^{\text{th}}$ path is considered, unless $i = M$;
  - if $Y^{(i)}_k = 0$ then
    * if $k < N$ then the step $(k+1)$ is considered;
    * if $k = N$ then ($T = t + hN$ so that) one sets $G(S^{(i)}) = e^{-r\theta} f(S^{(i)}_T)$.

1.2.2 Knock-In Call/Put Options

The price of the option is $F(t,x) = \mathbb{E}\Big(G(S(x))\Big)$, where

$$G(S(x)) = e^{-r\theta} f(S_T(x))\mathbf{1}_{\tau(x)\le T} + e^{-r\theta} R\mathbf{1}_{\tau(x)>T}.$$

For $i = 1, 2, \dots, M$, the i^{th} path $S^{(i)}$ is simulated and $G(S^{(i)})$ is evaluated as follows.

One sets $S^{(i)}_t = x$ and computes $L(t)$ and $U(t)$. For $k = 1, 2, \dots N$, $S^{(i)}_{t+hk}$ is simulated and $L(t+hk)$, $U(t+hk)$ are computed, where $h = \theta/N$.

♠ If $S^{(i)}_{t+hk} \le L(t+hk)$ or $S^{(i)}_{t+hk} \ge U(t+hk)$ then

- $S^{(i)}$ is generated at time T:

  $$S^{(i)}_T = S^{(i)}_{t+hk} \exp\Big\{\Big(b - \frac{\sigma^2}{2}\Big)h(N-k) + \sigma\sqrt{h(N-k)}\hat{X}^{(i)}_k\Big\}$$

  where $\hat{X}^{(i)}_k$ is a further standard gaussian random variable;
- one sets $G(S^{(i)}) = e^{-rhk} f(S^{(i)}_T)$;
- the simulation of the $(i+1)^{\text{th}}$ path is considered, unless $i = M$.

♠ If $S^{(i)}_{t+hk} > L(t+hk)$ and $S^{(i)}_{t+hk} < U(t+hk)$ then

- the conditional exit probability p^h_k, given the simulated points at times $t + h(k-1)$ and $t + hk$, is computed:
  if $S^{(i)}_{t+h(k-1)} + S^{(i)}_{t+hk} > L(t+h(k-1)) + U(t+h(k-1))$ then

  $$p^h_k = \exp\Big\{-\frac{2}{\sigma^2 h}\Big[(U(t+h(k-1)) - \ln S^{(i)}_{t+h(k-1)})(U(t+h(k-1)) - \ln S^{(i)}_{t+hk}) + (U(t+h(k-1)) - \ln S^{(i)}_{t+h(k-1)})(U(t+hk) - U(t+h(k-1)))\Big]\Big\};$$

  if $S^{(i)}_{t+h(k-1)} + S^{(i)}_{t+hk} < L(t+h(k-1)) + U(t+h(k-1))$ then

  $$p^h_k = \exp\Big\{-\frac{2}{\sigma^2 h}\Big[(\ln S^{(i)}_{t+h(k-1)} - L(t+h(k-1)))(\ln S^{(i)}_{t+hk} - L(t+h(k-1))) - (\ln S^{(i)}_{t+h(k-1)} - L(t+h(k-1)))(L(t+hk) - L(t+h(k-1)))\Big]\Big\};$$

- a Bernoulli random variable $Y^{(i)}_k$, with parameter p^h_k, is generated and:

- if $Y_k^{(i)} = 1$ then
  * $S^{(i)}$ is generated at time T:

$$S_T^{(i)} = S_{t+hk}^{(i)} \exp\Big\{\Big(b - \frac{\sigma^2}{2}\Big)h(N-k) + \sigma\sqrt{h(N-k)}\hat{X}_k\Big\}$$

    where $\hat{X}_k$ is a further standard gaussian random variable;
  * one sets $G(S^{(i)}) = e^{-rhk} f(S_T^{(i)})$;
  * the $(i+1)^{\text{th}}$ path is considered, unless $i = M$.
- if $Y_k^{(i)} = 0$ then
  * if $k < N$ then the step $(k+1)$ is considered;
  * if $k = N$ then ($T = t + hN$ so that) one sets $G(S^{(i)}) = e^{-r\theta} R$.

2 Parisian Barrier Options

Suppose the underlying asset price evolves according to the Black and Scholes model with continuous yield δ, i.e.

$$dS_u = S_u(bdu + \sigma dB_u), \quad S_t = x$$

where $b = r - \delta$, r being the spot rate. As usual, we set T as the maturity, $\theta = T - t$ as the time to maturity and $f(S_T(x))$ as a suitable function which will be used to determine the payoff: since we are going to study parisian call or put barrier options, f turns out to be

$$f(S_T(x)) = (S_T(x) - K)_+ \quad \text{or} \quad f(S_T(x)) = (K - S_T(x))_+$$

respectively, where K stands for the exercise price.

We need to introduce the *lower* barrier L and the *upper* barrier U:

$$L, U \,:\, [t, +\infty) \longrightarrow [0, +\infty), \quad \text{with} \quad 0 \le L(u) < U(u),\, \forall u.$$

and a positive constant D which will play the role of a *delay*.

The payoff of a knock-out single or double parisian barrier option is given by $f(S_T(x))$ provided that the underlying asset price S does not stay above the upper barrier or below the lower one uninterruptedly for longer than a pre-specified time length D; if it does, it is nullified. Similarly, in the knock-in framework the payoff is equal to $f(S_T(x))$ if S stays beyond the barrier(s) uninterruptedly for longer than D; otherwise, the value is set equal to 0. In formulas, the price of a parisian barrier option is given by

$$F(t, x) = \mathbb{E}\Big(G(S(x))\Big)$$

being G the following functional of the path $(S_u, t \leq u \leq T)$:

$$G(S(x)) = \begin{cases} e^{-r\theta} f(S_T(x)) \mathbf{1}_{H_D(x)>T} & \text{knock} - \text{out case} \\ e^{-r\theta} f(S_T(x)) \mathbf{1}_{H_D(x)\leq T} & \text{knock} - \text{in case} \end{cases}$$

where H_D denotes the first time at which the underlying asset price has been observed to stay beyond the barrier(s) uninterruptedly for longer than D:

$$H_D = \inf\{u > t \ ; \ (u - g_u)\mathbf{1}_{S_u(x) \text{ is outside the barrier(s)}} \geq D\}$$

g_u being the final time up to u when the underlying asset price S hits on the boundary, $g_u = u$ otherwise[5].

We present here a Monte Carlo procedure allowing to numerically compute the price $F(t, x)$ of parisian barrier options. As usual, one generates a large number, say M, of independent paths which approximate the underlying asset price S on the time interval $[t, T]$ and for each simulated path one computes the associated value of G. The price of the option will be numerically evaluated by averaging over the M samples:

$$F(t, x) \sim \frac{1}{M} \sum_{i=1}^{M} G(S^{(i)}(x)).$$

The paths $S^{(i)}$'s are generated as follows.

For a fixed positive integer N, set $h = \theta/N$ as the step size. The underlying stock price is then simulated at the times $t_k = t + hk$, $k = 1, 2, \ldots, N$:

$$\begin{cases} S_t^{(i)} = x \\ S_{t+hk}^{(i)} = S_{t+h(k-1)}^{(i)} \exp\left\{\left(b - \frac{\sigma^2}{2}\right)h + \sigma\sqrt{h}X_k^{(i)}\right\} & k = 1, 2, \ldots, N \end{cases}$$

where $(X_k^{(i)})_{1\leq k\leq N, 1\leq i\leq M}$ is a set of independent standard gaussian random variables.

Thus, in the Monte Carlo-Parisian algorithm, at step k, $S_{t+hk}^{(i)}$ is generated, by means of $S_{t+h(k-1)}^{(i)}$.

If $k = 0$, one sets $\hat{g}_t^{(i)} = t$ and $\hat{H}_D^{(i)} = 0$, where $\hat{g}^{(i)}$ and $\hat{H}_D^{(i)}$ stand for the approximation of g and H_D respectively.

When $k \geq 1$, one checks if $S_{t+hk}^{(i)}$ has or has not reached the boundary. In the latter case, one sets $\hat{g}_{t+hk}^{(i)} = t + hk$, $\hat{H}_D^{(i)} = 0$ and the simulation is carried on, unless $k = N$. Otherwise, the algorithm behaves in two different ways according to the position of the path $S^{(i)}$ at time $t + h(k-1)$:

[5] See next sections for a more precise definition of H_D and g_u.

– if $S^{(i)}_{t+h(k-1)}$ has not breached the barrier(s), $\hat{g}^{(i)}_{t+hk}$ is updated by a suitable instant between[6] $t+h(k-1)$ and $t+hk$;

– if also $S^{(i)}_{t+h(k-1)}$ has breached the barrier(s), one checks if $S^{(i)}$ has crossed the boundary during the time interval $(t+h(k-1), t+hk)$ by means of an approximation p^h_k of the conditional exit probability[7], given the observations $S^{(i)}_{t+h(k-1)}$ and $S^{(i)}_{t+hk}$, as follows. A Bernoulli random variable $Y^{(i)}_k$, with parameter p^h_k, is generated. If $Y^{(i)}_k = 1$ then the boundary has been reached by $S^{(i)}$, so that $\hat{g}^{(i)}_{t+hk} = t+hk$ and the simulation is carried on; otherwise, the value of $g^{(i)}_{t+hk}$ is not changed.

Thus, $\hat{H}^{(i)}_D = t + hk - g^{(i)}_{t+hk}$ is computed. As soon as $\hat{H}^{(i)}_D > D$ the simulation of the i^{th} path is stopped and the associated value of the functional G is then computed; otherwise, the simulation is carried on and the step $(k+1)$ is considered, unless obviously $k = N$.

In summary, the algorithm can be splitted in two cicles:

- the "i-cicle", for $i = 1, 2, \dots, M$, giving the i^{th} simulated path $S^{(i)}$, the value $G(S^{(i)})$ and thus the partial sum of the samples of the functional G up to the i^{th} simulation;

- the "k-cicle", for $k = 1, 2, \dots, N$, inside the "i-cicle", through which the i^{th} path is simulated at times $t_k = t + hk$, giving an approximation $\hat{H}^{(i)}_D$ of $H^{(i)}_D$ and the value of $S^{(i)}_T$, if it is useful for determining the functional G.

At the end of the "i-cicle", one can average over the M obtained values $G(S^{(i)})$.

In the following, we give the details of the simulation procedure for the i^{th} path, i.e. what we have called the "k-cicle", specializing on the cases of single or double parisian barrier options and on the knock-out or knock-in framework.

2.1 Single Parisian Barrier Options

Let L denote the lower barrier and U the upper one. Let g_u be the final time up to u when the barrier is crossed. More in details, in the down parisian barrier option framework, set

$$g_u = \sup\{s \le u \; ; \; S_s = L(s)\};$$

[6] See next sections for the formula giving $\hat{g}^{(i)}_{t+hk}$.

[7] The value of p^h_k is actually exact in the single barrier framework when the barrier is exponential w.r.t. the time variable (this is the case studied by Ikeda and Kunitomo for standard barrier options).

as for the up parisian barrier option case,

$$g_u = \sup\{s \le u \ ; \ S_s = U(s)\}.$$

If $S_s > L(s)$ or $S_s < U(s)$ for any $s \le u$, put $g_u = u$. We can now define the first time H_D at which the underlying asset price S has been observed to stay beyond the barrier uninterruptedly for longer than D: for down parisian barrier options,

$$H_D = \inf\{u > t \ ; \ (u - g_u)\mathbf{1}_{S_u(x) \le L(u)} \ge D\} \tag{1}$$

and for up parisian barrier options,

$$H_D = \inf\{u > t \ ; \ (u - g_u)\mathbf{1}_{S_u(x) \ge U(u)} \ge D\} \tag{2}$$

The price of a single parisian barrier call or put option is written in terms of H_D.

We recall that $f(S_T(x)) = (S_T(x) - K)_+$ or $f(S_T(x)) = (K - S_T(x))_+$ according to the case of a call or a put option, respectively.

In the following, the maps $\Phi^h_L, \Phi^h_U \ : \ \{1, \ldots, N\} \times \{(\xi, \eta) \in \mathbb{R}^2 \ ; \ \xi \neq \eta\} \longrightarrow \mathbb{R}$, are defined as[8]:

$$\begin{aligned} \Phi^h_L(k, \xi, \eta) &= \frac{\ln L(t + hk) - \xi}{\eta - \xi}; \\ \Phi^h_U(k, \xi, \eta) &= \frac{\ln U(t + hk) - \xi}{\eta - \xi}. \end{aligned}$$

2.1.1 Down Call/Put Parisian Options

The price of the option is $F(t, x) = \mathbb{E}\Big(G(S(x))\Big)$, where

$$G(S(x)) = \begin{cases} e^{-r\theta} f(S_T(x))\mathbf{1}_{H_D > T} & \text{Down-Out Case;} \\ e^{-r\theta} f(S_T(x))\mathbf{1}_{H_D \le T} & \text{Down-In Case,} \end{cases}$$

[8] In the sequel, the maps Φ^h_L, Φ^h_U will be evaluated on the triple $(k, \ln S^{(i)}_{t+hk}, \ln S^{(i)}_{t+h(k+1)})$ subject to the constraints

$$\ln S^{(i)}_{t+hk} > L(t + hk), \quad \ln S^{(i)}_{t+h(k+1)} \le L(t + h(k + 1))$$

and

$$\ln S^{(i)}_{t+hk} < U(t + hk), \quad \ln S^{(i)}_{t+h(k+1)} \ge U(t + h(k + 1))$$

respectively, so that $\Phi^h_L(k, \ln S^{(i)}_{t+hk}, \ln S^{(i)}_{t+h(k+1)}) > 0$ and $\Phi^h_U(k, \ln S^{(i)}_{t+hk}, \ln S^{(i)}_{t+h(k+1)}) > 0$, at least for small values of h.

H_D being defined through (1).

For $i = 1, 2, \dots, M$, the i^{th} path $S^{(i)}$ is simulated and $G(S^{(i)})$ is evaluated as follows.

One sets $S_t^{(i)} = x$, computes $L(t)$ and puts

$$\hat{g}_t^{(i)} = t, \qquad \hat{H}_D^{(i)} = 0.$$

For $k = 1, 2, \dots N$, $S_{t+hk}^{(i)}$ is simulated and $L(t+hk)$ is computed, where $h = \theta/N$.

♠ If $S_{t+hk}^{(i)} > L(t+hk)$, one sets

$$\hat{g}_{t+hk}^{(i)} = t + hk.$$

♠ If $S_{t+hk}^{(i)} \leq L(t+hk)$ then

- if $S_{t+h(k-1)}^{(i)} > L(t+h(k-1))$, one sets

$$\hat{g}_{t+hk}^{(i)} = t + hk + \Phi_L^h(k-1, \ln S_{t+h(k-1)}^{(i)}, \ln S_{t+hk}^{(i)});$$

- if $S_{t+h(k-1)}^{(i)} \leq L(t+h(k-1))$ then
  - the conditional exit probability p_k^h, given the simulated points at times $t+h(k-1)$ and $t+hk$, is computed:

$$p_k^h = \exp\Big\{-\frac{2}{\sigma^2 h}\Big[(\ln S_{t+h(k-1)}^{(i)} - L(t+h(k-1)))(\ln S_{t+hk}^{(i)} - L(t+h(k-1))) \\ -(\ln S_{t+h(k-1)}^{(i)} - L(t+h(k-1)))(L(t+hk) - L(t+h(k-1)))\Big]\Big\};$$

  - a Bernoulli random variable $Y_k^{(i)}$, with parameter p_k^h, is generated and:
    * if $Y_k^{(i)} = 1$ one sets

$$\hat{g}_{t+hk}^{(i)} = t + hk;$$

    * if $Y_k^{(i)} = 0$, the value of $\hat{g}_{t+hk}^{(i)}$ is not changed.

Once $\hat{g}_{t+hk}^{(i)}$ has been updated, $\hat{H}_D^{(i)}$ is computed as

$$\hat{H}_D^{(i)} = t + hk - \hat{g}_{t+hk}^{(i)}.$$

Now, the algorithm differs according to the down-out or down-in framework as follows.

- **Down-Out Call/Put Parisian Options**
  - If $\hat{H}_D^{(i)} \geq D$ then
    - one sets $G(S^{(i)}) = 0$;
    - the $(i+1)^{\text{th}}$ step is considered, unless $i = M$;
  - if $\hat{H}_D^{(i)} < D$ then
    - if $k < N$, the $(k+1)^{\text{th}}$ step is considered;
    - if $k = N$, $(t + hk \equiv T$ so that$)$ one sets $G(S^{(i)}) = e^{-r\theta} f(S_T^{(i)})$.
- **Down-In Call/Put Parisian Options**
  - If $\hat{H}_D^{(i)} \geq D$ then
    - $S^{(i)}$ is generated at time T:

$$S_T^{(i)} = S_{t+hk}^{(i)} \exp\Big\{\Big(b - \frac{\sigma^2}{2}\Big)h(N-k) + \sigma\sqrt{h(N-k)}\hat{X}_k^{(i)}\Big\}$$

      where $\hat{X}_k^{(i)}$ is a further standard gaussian random variable;
    - one sets $G(S^{(i)}) = e^{-r\theta} f(S_T^{(i)})$;
    - the $(i+1)^{\text{th}}$ path is considered, unless $i = M$.
  - if $\hat{H}_D^{(i)} < D$ then
    - if $k < N$, the $(k+1)^{\text{th}}$ step is considered;
    - if $k = N$, one sets $G(S^{(i)}) = 0$.

2.1.2 Up Call/Put Parisian Options

The price of the option is $F(t,x) = \mathbb{E}\Big(G(S(x))\Big)$, where

$$G(S(x)) = \begin{cases} e^{-r\theta} f(S_T(x))\mathbf{1}_{H_D > T} & \text{Down-Out Case;} \\ e^{-r\theta} f(S_T(x))\mathbf{1}_{H_D \leq T} & \text{Down-In Case,} \end{cases}$$

H_D being defined through (2).

For $i = 1, 2, \ldots, M$, the i^{th} path $S^{(i)}$ is simulated and $G(S^{(i)})$ is evaluated as follows.

One sets $S_t^{(i)} = x$, computes $U(t)$ and puts

$$\hat{g}_t^{(i)} = t, \qquad \hat{H}_D^{(i)} = 0.$$

For $k=1,2,\dots N$, $S^{(i)}_{t+hk}$ is simulated and $U(t+hk)$ is computed, where $h=\theta/N$.

♠ If $S^{(i)}_{t+hk}<U(t+hk)$, one sets

$$\hat{g}^{(i)}_{t+hk}=t+hk.$$

♠ If $S^{(i)}_{t+hk}\geq U(t+hk)$ then

- if $S^{(i)}_{t+h(k-1)}<U(t+h(k-1))$, one sets

$$\hat{g}^{(i)}_{t+hk}=t+hk+\Phi^h_U(k-1,\ln S^{(i)}_{t+h(k-1)},\ln S^{(i)}_{t+hk});$$

- if $S^{(i)}_{t+h(k-1)}\geq U(t+h(k-1))$ then
  - the conditional exit probability p^h_k, given the simulated points at times $t+h(k-1)$ and $t+hk$, is computed:

$$p^h_k=\exp\Big\{-\frac{2}{\sigma^2 h}\Big[(\ln S^{(i)}_{t+h(k-1)}-U(t+h(k-1)))(\ln S^{(i)}_{t+hk}-U(t+h(k-1))) \\ -(\ln S^{(i)}_{t+h(k-1)}-U(t+h(k-1)))(U(t+hk)-U(t+h(k-1)))\Big]\Big\};$$

  - a Bernoulli random variable $Y^{(i)}_k$, with parameter p^h_k, is generated and:
    - if $Y^{(i)}_k=1$ one sets

$$\hat{g}^{(i)}_{t+hk}=t+hk;$$

    - if $Y^{(i)}_k=0$, the value of $\hat{g}^{(i)}_{t+hk}$ is not changed.

Once $\hat{g}^{(i)}_{t+hk}$ has been updated, $\hat{H}^{(i)}_D$ is computed as

$$\hat{H}^{(i)}_D=t+hk-\hat{g}^{(i)}_{t+hk}$$

Now, the algorithm differs according to the down-out or down-in framework as follows.

- **Up-Out Call/Put Parisian Options**
  - If $\hat{H}^{(i)}_D\geq D$ then
    - one sets $G(S^{(i)})=0$;

    - the $(i+1)^{\text{th}}$ step is considered, unless $i = M$;
  - if $\hat{H}_D^{(i)} < D$ then
    - if $k < N$, the $(k+1)^{\text{th}}$ step is considered;
    - if $k = N$, $(t + hk \equiv T$ so that) one sets $G(S^{(i)}) = e^{-r\theta} f(S_T^{(i)})$.

- **Up-In Call/Put Parisian Options**
  - If $\hat{H}_D^{(i)} \geq D$ then
    - $S^{(i)}$ is generated at time T:

$$S_T^{(i)} = S_{t+hk}^{(i)} \exp\Big\{\Big(b - \frac{\sigma^2}{2}\Big)h(N-k) + \sigma\sqrt{h(N-k)}\hat{X}_k^{(i)}\Big\}$$

      where $\hat{X}_k^{(i)}$ is a further standard gaussian random variable;
    - one sets $G(S^{(i)}) = e^{-r\theta} f(S_T^{(i)})$;
    - the $(i+1)^{\text{th}}$ path is considered, unless $i = M$.
  - if $\hat{H}_D^{(i)} < D$ then
    - if $k < N$, the $(k+1)^{\text{th}}$ step is considered;
    - if $k = N$, one sets $G(S^{(i)}) = 0$.

2.2 Double Parisian Options

Let L denote the lower barrier and U the upper one. Let g_u be the final time up to u when the barriers are crossed:

$$g_u = \sup\{s \leq u \ ; \ S_s = L(s) \text{ or } S_s = U(s)\};$$

if $L(s) < S_s < U(s)$ for any $s \leq u$, put $g_u = u$. We can now define the first time H_D at which the underlying asset price S has been observed to stay beyond the barrier uninterruptedly for longer than D:

$$H_D = \inf\{u > t \ ; \ (u - g_u)\mathbf{1}_{S_u(x) \leq L(u) \text{ or } S_u(x) \geq L(u)} \geq D\}.$$

The price of a double parisian barrier call or put option is written in terms of H_D.

We recall that $f(S_T(x)) = (S_T(x) - K)_+$ or $f(S_T(x)) = (K - S_T(x))_+$ according to the case of a call or a put option, respectively.

In the following, the maps $\Phi_{L,U}^h \ : \ A_{L,U} \subset \{1, \ldots, N\} \times \{(\xi, \eta) \in \mathbb{R}^2 \ ; \ \xi \neq \eta\} \longrightarrow \mathbb{R}$, is defined as:

$$\Phi_{L,U}^h(k, \xi, \eta) = \begin{cases} \Phi_L^h(k, \xi, \eta), & \text{if } \eta < \ln L(t + hk); \\ \Phi_U^h(k, \xi, \eta), & \text{if } \eta > \ln U(t + hk), \end{cases}$$

where Φ^h_L and Φ^h_U has been defined in the previous section, i.e.[9]

$$\Phi^h_{L,U}(k,\xi,\eta) = \begin{cases} \dfrac{\ln L(t+hk)-\xi}{\eta-\xi}, & \text{if } \eta < \ln L(t+hk); \\ \dfrac{\ln U(t+hk)-\xi}{\eta-\xi}, & \text{if } \eta > \ln U(t+hk). \end{cases}$$

The price of the option is $F(t,x) = \mathbb{E}\Big(G(S(x))\Big)$, where

$$G(S(x)) = \begin{cases} e^{-r\theta} f(S_T(x)) \mathbf{1}_{H_D>T}, & \text{Knock-Out Case} \\ e^{-r\theta} f(S_T(x)) \mathbf{1}_{H_D\leq T}, & \text{Knock-In Case} \end{cases}$$

For $i = 1, 2, \ldots, M$, the i^{th} path $S^{(i)}$ is simulated and $G(S^{(i)})$ is evaluated as follows.

One sets $S^{(i)}_t = x$, computes $L(t)$, $U(t)$, and puts

$$\hat{g}^{(i)}_t = t, \qquad \hat{H}^{(i)}_D = 0.$$

For $k = 1, 2, \ldots N$, $S^{(i)}_{t+hk}$ is simulated and $L(t+hk)$, $U(t+hk)$ are computed, where $h = \theta/N$.

♠ If $S^{(i)}_{t+hk} > L(t+hk)$ and $S^{(i)}_{t+hk} <\geq U(t+hk)$, one sets

$$\hat{g}^{(i)}_{t+hk} = t + hk.$$

♠ If $S^{(i)}_{t+hk} \leq L(t+hk)$ then

- if $S^{(i)}_{t+h(k-1)} > L(t+h(k-1))$, one sets

$$\hat{g}^{(i)}_{t+hk} = t + hk + \Phi^h_{L,U}(k-1, \ln S^{(i)}_{t+h(k-1)}, \ln S^{(i)}_{t+hk});$$

- if $S^{(i)}_{t+h(k-1)} \leq L(t+h(k-1))$ then

[9] In the sequel, the map $\Phi^h_{L,U}$ will be evaluated on the triple $(k, \ln S^{(i)}_{t+hk}, \ln S^{(i)}_{t+h(k+1)})$ subject to the constraints

$$\ln S^{(i)}_{t+hk} > L(t+hk), \quad \ln S^{(i)}_{t+h(k+1)} \leq L(t+h(k+1))$$

or

$$\ln S^{(i)}_{t+hk} < U(t+hk), \quad \ln S^{(i)}_{t+h(k+1)} \geq U(t+h(k+1)),$$

so that $\Phi^h_{L,U}(k, \ln S^{(i)}_{t+hk}, \ln S^{(i)}_{t+h(k+1)}) > 0$, at least for small values of h.

- the conditional exit probability p_k^h, given the simulated points at times $t+h(k-1)$ and $t+hk$, is computed:

$$p_k^h = \exp\Big\{-\frac{2}{\sigma^2 h}\Big[(\ln S^{(i)}_{t+h(k-1)} - L(t+h(k-1)))(\ln S^{(i)}_{t+hk} - L(t+h(k-1))) - (\ln S^{(i)}_{t+h(k-1)} - L(t+h(k-1)))(L(t+hk) - L(t+h(k-1)))\Big]\Big\};$$

- a Bernoulli random variable $Y_k^{(i)}$, with parameter p_k^h, is generated and:
  - if $Y_k^{(i)} = 1$ one sets

$$\hat{g}^{(i)}_{t+hk} = t + hk;$$

  - if $Y_k^{(i)} = 0$, the value of $\hat{g}^{(i)}_{t+hk}$ is not changed.

♠ If $S^{(i)}_{t+hk} \geq U(t+hk)$ then

- if $S^{(i)}_{t+h(k-1)} < U(t+h(k-1))$, one sets

$$\hat{g}^{(i)}_{t+hk} = t + hk + \Phi^h_{L,U}(k-1, \ln S^{(i)}_{t+h(k-1)}, \ln S^{(i)}_{t+hk});$$

- if $S^{(i)}_{t+h(k-1)} \geq U(t+h(k-1))$ then
  - the conditional exit probability p_k^h, given the simulated points at times $t+h(k-1)$ and $t+hk$, is computed:

$$p_k^h = \exp\Big\{-\frac{2}{\sigma^2 h}\Big[(\ln S^{(i)}_{t+h(k-1)} - U(t+h(k-1)))(\ln S^{(i)}_{t+hk} - U(t+h(k-1))) - (\ln S^{(i)}_{t+h(k-1)} - U(t+h(k-1)))(U(t+hk) - U(t+h(k-1)))\Big]\Big\};$$

  - a Bernoulli random variable $Y_k^{(i)}$, with parameter p_k^h, is generated and:
    - if $Y_k^{(i)} = 1$ one sets

$$\hat{g}^{(i)}_{t+hk} = t + hk;$$

    - if $Y_k^{(i)} = 0$, the value of $\hat{g}^{(i)}_{t+hk}$ is not changed.

Once $\hat{g}^{(i)}_{t+hk}$ has been updated, $\hat{H}^{(i)}_D$ is computed as

$$\hat{H}^{(i)}_D = t + hk - \hat{g}^{(i)}_{t+hk}.$$

Now, the algorithm differs according to the knock-out or knock-in framework as follows.

- **Knock-Out Double Call/Put Parisian Options**
  - If $\hat{H}^{(i)}_D \geq D$ then
    - one sets $G(S^{(i)}) = 0$;
    - the $(i+1)^{\text{th}}$ step is considered, unless $i = M$;
  - if $\hat{H}^{(i)}_D < D$ then
    - if $k < N$, the $(k+1)^{\text{th}}$ step is considered;
    - if $k = N$, ($t + hk \equiv T$ so that) one sets $G(S^{(i)}) = e^{-r\theta} f(S^{(i)}_T)$.
- **Knock-In Double Call/Put Parisian Options**
  - If $\hat{H}^{(i)}_D \geq D$ then
    - $S^{(i)}$ is generated at time T:

      $$S^{(i)}_T = S^{(i)}_{t+hk} \exp\left\{\left(b - \frac{\sigma^2}{2}\right)h(N-k) + \sigma\sqrt{h(N-k)}\hat{X}^{(i)}_k\right\}$$

      where $\hat{X}^{(i)}_k$ is a further standard gaussian random variable;
    - one sets $G(S^{(i)}) = e^{-r\theta} f(S^{(i)}_T)$;
    - the $(i+1)^{\text{th}}$ path is considered, unless $i = M$.
  - if $\hat{H}^{(i)}_D < D$ then
    - if $k < N$, the $(k+1)^{\text{th}}$ step is considered;
    - if $k = N$, one sets $G(S^{(i)}) = 0$.

References