COMPUTING VALUE-AT-RISK AND CONDITIONAL TAIL EXPECTATION IN LÉVY MODELS

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Abstract. We describe FFT-based method for computing standard risk measures in infinitely divisible distributions. The method efficiently recovers the cumulative distribution function from the characteristic function using the inversion theorem by means of the Fast Fourier Transform algorithm. The method for computing VaR and CTE implemented into Premia 14 is closely related to the papers Kim et al. (2010) and Kelani and Quittard-Pinon (2011).

Premia 14

1. Introduction

In recent years more and more attention has been given to stochastic models of financial markets which depart from the traditional Gaussian model. At this moment a wide range of models is available. One of the tractable empirical models are jump diffusions or, more generally, Lévy processes. We concentrate on the one-dimensional case. For an introduction on these models applied to finance, we refer to [5, 7].

In insurance and in the financial industry, pricing contracts at fair price is an important subject as well as hedging and assessing risk of portfolios or positions. Among the risk management tools promoted by the Basel committee, the most popular is the Value-at-Risk (VaR) which measures the potential loss in value of a risky asset or portfolio over a defined period for a given confidence interval. It was introduced by JP Morgan, and has been intensively used in the financial and insurance sector since then.

Key words and phrases. Lévy processes, VaR, CTE, CVaR, CGMY, FFT.

Nevertheless, VaR has a number of well-known limitations as a risk measure, see e.g. [2]. It has led to another risk measure, namely the conditional Value-at-Risk (CVaR) also known as Conditional Tail Expectation (CTE), see [10, 16, 19, 20]. By the definition, CTE is the average of VaRs larger than the VaR for a given tail probability. It should be noted that CVaR is a superior alternative to VaR because it satisfies all axioms of coherient risk measures and it is consistent with preference relations of risk-averse investors (see details in Rachev et al. [21]).

In recent years, many generalization of risk measures have been suggested, see e.g. [1, 22]. However, VaR and CTE still remain the most applicable risk measures. In general Lévy models special numerical procedures are needed for computing VaR and CTE, in contrast to the Gaussian case where explicit formulas are known. See details in [12, 11].

2. Infinitely divisible distributions: a short reminder

An infinitely divisible distribution (i.i.d.) is defined as a distribution which can be written – for every positive integer n - as the n-fold convolution of some distribution function (for details, see e.g. [21]). An i.i.d may have a Gaussian component and/or pure jump component. The latter is characterized by the density of jumps, which is called the Lévy density. We denote it by $F(dy)$.

An i.i.d. X can be completely specified by its characteristic exponent, ψ, definable from the equality (we confine ourselves to the one-dimensional case):

$$\phi_X(\xi)(= E[e^{i\xi X}]) = e^{-\psi(\xi)}. \tag{2.1}$$

The characteristic exponent is given by the Lévy-Khintchine formula:

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty}(1 - e^{i\xi y} + i\xi y\mathbf{1}_{|y|\leq 1})F(dy), \tag{2.2}$$

where σ^2 is the variance of the Gaussian component, and $F(dy)$ satisfies

$$\int_{\mathbf{R}\setminus\{0\}} \min\{1, y^2\}F(dy) < +\infty. \tag{2.3}$$

Example 2.1. [**KoBoL(CGMY) model**] The characteristic exponent of a pure jump KoBoL (CGMY) model of order $\nu \in (0,2), \nu \neq 1$, is given by

$$\psi(\xi) = -i\mu\xi + c\Gamma(-\nu)[\lambda_+^\nu - (\lambda_+ + i\xi)^\nu + (-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu], \tag{2.4}$$

where $c > 0$, $\mu \in \mathbf{R}$, and $\lambda_- < -1 < 0 < \lambda_+$, see [5]. The paper [6] uses different parameter's labels C, G, M, Y:

$$\psi(\xi) = -i\mu\xi + C\Gamma(-Y)[G^Y - (G + i\xi)^Y + M^Y - (M - i\xi)^Y]. \tag{2.5}$$

The relation between two parameterizations is quite easy to obtain:

$$c = C,\ \lambda_+ = G,\ \lambda_- = -M,\ \nu = Y. \tag{2.6}$$

Example 2.2. [**Normal Inverse Gaussian model**] A normal inverse Gaussian process (NIG) can be described by the characteristic exponent of the form (see [3])

$$\psi(\xi) = -i\mu\xi + \delta[(\alpha^2 - (\beta + i\xi)^2)^{1/2} - (\alpha^2 - \beta^2)^{1/2}], \tag{2.7}$$

where $\alpha > |\beta| > 0$, $\delta > 0$ and $\mu \in \mathbf{R}$.

Example 2.3. [**Variance Gamma model** The characteristic exponent of a Variance Gamma model is given by (see [15])

$$\psi(\xi) = -i\mu\xi + c[\ln(\lambda_+ + i\xi) - \ln\lambda_+ + \ln(-\lambda_- - i\xi) - \ln(-\lambda_-)], \tag{2.8}$$

where $c > 0$, $\mu \in \mathbf{R}$, and $\lambda_- < -1 < 0 < \lambda_+$.

Example 2.4. [**Kou model**] If $F_\mp(dy)$ are given by exponential functions on negative and positive axis, respectively:

$$F_\mp(dy) = c_\pm(\pm\lambda_\pm)e^{\lambda_\pm y},$$

where $c_\pm \geq 0$ and $\lambda_- < 0 < \lambda_+$, then we obtain Kou model. The characteristic exponent of the process is of the form

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \frac{ic_+\xi}{\lambda_+ + i\xi} + \frac{ic_-\xi}{\lambda_- + i\xi}. \tag{2.9}$$

The version with one-sided jumps is due to [8], the two-sided version was introduced in [9], see also [13].

3. Computing VaR and CTE

Let the random infintely divisible variable X represents the loss of a portfolio, and $F_X(x) = \mathbf{P}(X < x)$, $p_X = \frac{d}{dx}F_X(x)$, $\phi(\xi) = E[e^{i\xi X}]$ stand for the cumulative distribution function (cdf), the probability distribution function (pdf), and the characteristic function (chf) of X, respectively.

It is well known (see e.g. [12]), that the VaR of X at tail probability α is defined as follows.

$$\mathrm{VaR}_\alpha(X) = \inf\{y \in \mathbf{R} | F_X(y) \geq \alpha\}. \tag{3.1}$$

Further, the CTE at tail probability α is defined as the average of VaRs which are larger than the $\mathrm{VaR}_\alpha(X)$, that is

$$\mathrm{CTE}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_\epsilon(X) d\epsilon. \tag{3.2}$$

If $F_X(x)$ is continuous, then

$$F_X(x) = \int_{-\infty}^x p_X(y)dy, \tag{3.3}$$

and the following formulas are valid (see [12]):

$$\mathrm{VaR}_\alpha(X) = F_X^{-1}(\alpha), \tag{3.4}$$

and

$$\mathrm{CTE}_\alpha(X) = \frac{1}{1-\alpha}\int_{\mathrm{VaR}_\alpha(X)}^{+\infty} x p_X(x)dx = \frac{1}{1-\alpha} E\Big[X \mathbf{1}_{X \geq \mathrm{VaR}_\alpha(X)}\Big]. \tag{3.5}$$

If the probability density p_X is known, one can apply a quadrature rule to (3.3) and (3.5) for computing numerically VaR and CTE, respectively. However, in the case of infinitely divisible distributions as a rule explicit analytical formulas for pdf are not available. In order to recover the pdf p_X one can use the characteristic function ϕ_X which is typically known in the closed form. In the general case, p_X can be expressed in terms of the characteristic function $\phi_X(\xi)$, by using the Fourier transform

$$p_X(x) = (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{-ix\xi}\phi_X(\xi)d\xi, \tag{3.6}$$

where the chf $\phi_X(\xi)$ can be written via the characteristic exponent, see (2.1). The formula (3.6) can be efficiently realized by means of the Fast Fourier Transform algorithm, see the next section.

One can also substitute the formula (3.6) into (3.3) and express the cumulative distribution function F_X in terms of the Fourier integral (see [12, 11].

$$F_X(x) = \frac{e^{x\rho}}{\pi} \operatorname{Re} \int_0^{\infty} e^{-ix\xi} \frac{\phi_X(\xi + i\rho)}{\rho - i\xi} d\xi, x \in \mathbf{R}, \tag{3.7}$$

where $\rho > 0$. Note that the correspondent Fourier integral for CTE is more involved. The difference between results obtained by these two approaches is unsignificant. However, the method which uses quadrature rule and (3.6) is more simpe for a numerical implementation. Thus we have chosen to implement the first approach.

Further, we consider computing VaR and CTE for geometrical Lévy models. Let the stock price $S_t = S_0 e^{X_t}$ is an exponential Lévy process, then the chf of X_t is given by the formula $\phi_{X_t}(\xi) = e^{-t\psi(\xi)}$, where ψ is the characteristic exponent of the form (2.2). As well as [11], we consider a more general quantity $L_t = S_t - K$, where $K \geq 0$. The computation of the VaR for L_t is straightforward because it is related to the Value at Risk of X_t, which has already been obtained, see (3.4). From [11] we have

$$\mathrm{VaR}_\alpha(L_T) = S_0 e^{\mathrm{VaR}_\alpha(X_T)} - K. \tag{3.8}$$

Due to (3.5), we obtain

$$\mathrm{CTE}_\alpha(L_T) = \frac{1}{1-\alpha} \int_{\mathrm{VaR}_\alpha(X_T)}^{+\infty} (S_0 e^x - K) p_{X_t}(x) dx = \frac{S_0}{1-\alpha} \int_{\mathrm{VaR}_\alpha(X_T)}^{+\infty} e^x p_{X_t}(x) dx - K. \tag{3.9}$$

We remark that (3.9) includes the requirement that $E[e^{X_t}] < \infty$.

3.1. **Computing the pdf of an infinitely divisible distribution by using Fast Fourier Transform.** Let d be the step in x-space, ζ–the step in ξ-space, and $M = 2^m$ the number of the points on the grid; decreasing d and increasing (even faster) M, we obtain a sequence of approximations to the option price. An approximation for the pdf can be efficiently computed by using the Fast Fourier Transform (FFT). Consider the algorithm (the discrete Fourier transform (DFT)) defined by

$$G_l = DFT[g](l) = \sum_{k=0}^{M-1} g_k e^{2\pi ikl/M}, \quad l = 0, ..., M-1. \tag{3.10}$$

(It differs in sign in front of i from the algorithm fft in MATLAB). The DFT maps m complex numbers (the g_k's) into m complex numbers (the G_l's). The formula for the inverse DFT which recovers the set of g_k's exactly from G_l's is:

$$g_k = iDFT[G](k) = \frac{1}{M} \sum_{l=0}^{M-1} G_l e^{-2\pi ikl/M}, \quad k = 0, ..., M-1. \tag{3.11}$$

In our case, the data (pdf) consist of a real-valued array $\{g_k\}_{k=0}^{M}$. The resulting transform satisfies $G_{M-l} = \bar{G}_l$. Since this complex-valued array has real values G_0 and $G_{M/2}$, and $M/2-1$ other independent complex values $G_1, ..., G_{M/2-1}$, then it has the same "degrees of freedom" as the original real data set. In this case, it is inefficient to use full complex FFT algorithm. The main idea of FFT of real functions is to pack the real input array cleverly, without extra zeros, into a complex array of half of length. Then a complex FFT can be applied to this shorter length; the trick is then to get the required values from this result (see [17] for technical details). To distinguish DFT of real functions we will use notation RDFT.

Fix the space step $d > 0$ and number of the space points $M = 2^m$. Define the partitions of normalized log-price domain $[-\frac{Md}{2}; \frac{Md}{2})$ by points $x_k = -\frac{Md}{2} + kd$, $k = 0, ..., M-1$, and frequency domain $[-\frac{\pi}{d}; \frac{\pi}{d}]$ by points $\xi_l = \frac{2\pi l}{dM}$, $l = -M/2, ..., M/2$.

Using the formula (3.6) we can approximate the pdf p_X as follows.

$$\begin{aligned} p_X(x_k) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix_k\xi} \phi_X(\xi) d\xi \\ &\approx \frac{1}{2\pi} \int_{-\pi/d}^{\pi/d} e^{ix_k\xi} \phi_X(\xi) d\xi \approx \frac{1}{2\pi} \sum_{l=-M/2+1}^{M/2} e^{ix_k\xi_l} \phi_X(\xi_l) \frac{2\pi}{dM} \\ &\approx \Big(\frac{2}{Md} \operatorname{Re} \sum_{l=1}^{M/2-1} e^{2\pi ikl/M} p(\xi_l)(-1)^l + \frac{1}{Md}(1 + \operatorname{Re} \phi_X(\xi_{M/2})) \Big). \end{aligned}$$

Finally,

$$p_X(x_k)(x_k) \approx \frac{1}{d} iRDFT[\tilde{\phi_X}](k), \quad k = 0, ..., M-1, \tag{3.12}$$

where $(\tilde{\phi_X})_l = \phi_X(\xi_l) \cdot (-1)^l$. Note that real-FFT is two times faster than FFT.

Table 4.1. VaR and CTE in the CGMY model

A

Quantile risk Measure	Premia, d=0.0001	Premia, d=0.00005	FFT	MC	C.I.99%
$\mathrm{VaR}_{0.9}(L)$	0.162997	0.163055	0.16303	0.162332	[0.14748 , 0.17719]
$\mathrm{VaR}_{0.95}(L)$	0.287111	0.287111	0.28711	0.287141	[0.26756 , 0.30300]
$\mathrm{VaR}_{0.975}(L)$	0.410720	0.410720	0.41069	0.405142	[0.38026 , 0.43264]
$\mathrm{VaR}_{0.99}(L)$	0.578697	0.578618	0.57863	0.598236	[0.54706 , 0.65849]

B

Quantile risk Measure	Premia, d=0.0001	Premia, d=0.00005	FFT	MC
$\mathrm{CTE}_{0.9}(L)$	0.345468	0.344910	0.344812	0.344813
$\mathrm{CTE}_{0.95}(L)$	0.472005	0.471772	0.471422	0.472922
$\mathrm{CTE}_{0.975}(L)$	0.601702	0.601425	0.601138	0.606909
$\mathrm{CTE}_{0.99}(L)$	0.781253	0.781585	0.780711	0.790691

CGMY parameters: $C = 1$, $G = 5$, $M = 10$, $Y = 0.5$, $\mu = 0$.
VaR and CTE parameters: $K = 1$, $S = 1$, $T = 1$, α.
Panel A: VaR; Panel B: CTE.

4. Numerical examples

In this section, we assume a loss of the type $L_T = S_0 e^{X_T} - K$, where X_t follows the exponential CGMY (KoBoL) process (see Example 2.1). The parameters C, G, M, Y play an important role in capturing some properties of the stochastic process under study. In particular, the parameters M and G, respectively, control the rate of exponential decay in far parts of the right and the left tails of the probability density. We will use zero drift with the parameters $C = 1$, $G = 5$, $M = 10$, $Y = 0.5$ which were obtained in [15] by calibrating the CGMY model to the options prices on the S&P 500 index.

We compare the results obtained by the aprroach based on (3.3) and a quadrature rule (implemented into Premia), and the method which uses (3.7) (FFT-method). The Table 4.1 shows the Premia and FFT values of the standard risk measures compared to the ones simulated, the 99% confidence interval for the VaR(L) is also provided (see [11]).

5. The implementation into Premia 14

We implemented computing VaR and CTE under the exponential CGMY (KoBoL) model (see Example 2.1). One can use the routine for the other types of Lévy processes by replacing the corresponding part with the computation of the characteristic exponent. Notice that due to (3.9), the parameter of the CGMY model labeled M should satisfy the following inequality: $M > 1$. The method converges well if $G > 1$ as well.

The input parameters of the problem are Maturity T, Strike K, the Tail Probability α. Note that in the program implemented to Premia 14 one can manage by two parameters of the algorithm: the space step d, the scale of logprice range L. Parameter L controls the size of the truncated region in x-space; it corresponds to the region $(-L\ln(4)/d; L\ln(4)/d)$. The typical values of the parameter are $L = 1$, $L = 2$ and $L = 4$. By default we set $L = 2$. To improve the results one should decrease d, when L is fixed.

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