Functional quantization for numerics with an application to option pricing

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Functional Quantization with applications to option pricing.

Functional quantization (FQ) means quantization of stochastic processes viewed as infinite dimensional space valued random variables, e.g. the standart Brownian $\{W_t\}_{t\in[0,T]}$ motion on $[0,T]$ viewed as an $L^2_T = L^2([0,T])$-valued random variable. Its aim is to provide tools for the numerical integration of stochastic processes, e.g. the expectation of a mean value in time of a process. In a Gaussian framework, it relies mainly on the Karhunen-Loève (K-L) expansion which can be written in the case of $\{W_t\}$ as

$$W(\omega,t) \overset{L^2_T}{=} \sum_{k\geq 1} \sqrt{\lambda_k}\, \xi_k(\omega) e_k^W(t) \qquad \mathbb{P}(d\omega)\text{-}a.s., \tag{0.1}$$

where

$$e_k^W(t) := \sqrt{\frac{2}{T}} \sin\left(\pi(k-1/2)\frac{t}{T}\right), \qquad \lambda_k := \left(\frac{T}{\pi(k-1/2)}\right)^2, \qquad k\geq 1, \tag{0.2}$$

and where $\{\xi_k\}_{k\geq 1}$ is a sequence of iid normal valued random variables. Spatial discretization is then achieved thanks to vectorial quantization of the normal valued random variables ξ_k. Once an integer $N \geq 1$ have been given, two types of spatial discretization are proposed in this current version:

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- We have to choose first an integer d_N in order to truncate the sum in (0.1). Then W_t is replaced by

$$\widehat{W}^N_{prod}(t) = \sum_{k=1}^{d_N} \sqrt{\lambda_k}\widehat{\xi}_k e_k^W(t), \tag{0.3}$$

where $\widehat{\xi}_k$ is the N_k-optimal quantizer of the one-dimensional Normal distribution where $N_1 \times \cdots \times N_{d_N} = N$. Such a quantizer is called a *product-quantizer*. This method and in particular the way the integers d_N and N_k, $1 \le k \le d_N$ are chosen is described in the paper which follows (see subsection 5.1). From a practical point of view, (0.3) can be written as the collection of the trajectories

$$\varphi^N_{i_1,\ldots,i_n}(t) = \sqrt{\frac{2}{T}} \sum_{k=1}^{d_N} \frac{T}{\pi(k-\frac{1}{2})} \sin\left(\pi\left(k-\frac{1}{2}\right)\frac{t}{T}\right) x_{i_k}^{(N_k)}, \quad 1 \le i_k \le N_k, \quad 1 \le k \le d_N,$$

weighted by

$$\alpha^N_{i_1,\ldots,i_n} = \prod_{k=1}^{d_N} \alpha_{i_k}^{(N_k)},$$

where $(x_{i_k}^{(N_k)}, \alpha_{i_k}^{(N_k)})_{1\le i_k\le N_k}$ denotes the optimal (weighted) N_k-quantizer of the one-dimensional Normal distribution. Two grids are provided as examples with the C routine: $N = 966 = 23\times 7\times 3\times 2$ and $N = 96 = 12\times 4\times 2$. They have been constructed using the one dimensional quantizers of the Normal distributions (see `http://quantize.maths-fi.com/` for download).

- W_t is replaced by

$$\widehat{W}^N_{opti}(t) = \sum_{k=1}^{d_N^{opti}} (\widehat{X}^N)_k e_k^W(t), \tag{0.4}$$

where $(\widehat{X}^N)_k$ denotes the coordinates of $\widehat{X}^N$ which is the d_N^{opti}-dimensional N-quantization of $\mathcal{N}(0, Id)$ with respect to the norm:

$$y \mapsto \|y\|_W = \left(\sum_{k=1}^{d_N^{opti}} \lambda_k\, y_k^2\right)^{1/2}.$$

>From a practical point of view, (0.4) can be replaced by the collection of the trajectories

$$\varphi_i(t) = \sqrt{\frac{2}{T}} \sum_{k=1}^{d_N^{opti}} \sin\left(\pi\left(k-\frac{1}{2}\right)\frac{t}{T}\right) x^N_{i,k}, \quad 1 \le i \le N, \quad 1 \le k \le d_N^{opti},$$

weighted by α_i, where $\{\alpha_i, \{x_{i,k}\}_{1\leq k\leq d_N^{opti}}\}_{1\leq i\leq N}$ denotes the optimal (weighted) N-quantizer which defines $\widehat{X}^N$ and its ditribution. Two examples of grids are provided here, namely for $N = 400$ with $d_N^{opti} = 6$ and $N = 100$ with $d_N^{opti} = 4$. They have been obatined using optimization techniques.

Optimal quantization needs in the applications less points than product quantization for the same level of accuracy. This is why it has been set by default in Premia. Although this way of discretization is not explained in the following paper, some explanations can be found in `http://quantize.maths-fi.com/NMF06.pdf`.

Abstract

We investigate in this paper the numerical performances of quadratic functional quantization with some applications to Finance. We emphasize the rôle played by the so-called product quantizers and the Karhunen-Loève expansion of Gaussian processes, in particular the Brownian motion. We show how to build some efficient functional quantizers for Brownian diffusions. We propose a quadrature formula based on a Romberg log-extrapolation of "crude" functional quantization which speeds up significantly the method. Numerical experiments are carried out on two European option pricing problems: vanilla and Asian Call options in a Heston stochastic volatility model. It suggests that functional quantization is a very efficient integration method for various path-dependent functionals of a diffusion processes: it produces deterministic results which outperforms Monte Carlo simulation for usual accuracy levels.

Key words: Functional quantization, Product quantizers, Romberg extrapolation, Karhunen-Loève expansion, Brownian motion, SDE, Asian option, stochastic volatility, Heston model.

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1 Introduction

This paper is an attempt to investigate the numerical aspects of functional quantization of stochastic processes and their applications to the pricing of derivatives through numerical integration on path-spaces; we will mainly focus on the Brownian motion and the Brownian diffusions viewed as square integrable random vectors defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ taking their values in the Hilbert space $L^2_T := L^2_{\mathbb{R}}([0,T], dt)$ endowed with the usual norm defined by $|g|_{L^2_T} = (\int_0^T g^2(t)dt)^{1/2}$.

Abstract *Quadratic quantization* theory consists in studying the best approximation of X in $(L^2_H(\Omega, \mathbb{P}), \|\,.\,\|_2)$ by H-valued random vectors taking at most N values and all the induced questions: optimization of the values, asymptotic rate of the quantization error bounds, explicit construction of nearly optimal quantizers. The historical framework is the Euclidean one ($H = \mathbb{R}^d$) comes from Information Theory and Signal processing and was introduced in the late 1940's. Its aim is to provide an optimal spatial discretization of a random vector-valued signal X with distribution $\mathbb{P}_X$ by a random vector taking N values in the codebook $\{x_1, \ldots, x_N\}$ (the N-tuple $(x_1, \ldots, x_N)$ is called a N-quantizer). Then, instead of transmitting the complete signal $X(\omega)$ itself, one first selects the nearest codebook x_i in the codebook and transmits its (binary coded) *label* i. After reception, a proxy $\widehat{X}(\omega)$ of $X(\omega)$ is reconstructed using the codebook correspondence $i \mapsto x_i$ (called the codebook *bible*). For a given N, there is (at least) one N-quantizer which

minimizes over $(\mathbb{R}^d)^N$ the quadratic quantization error $\|X - \widehat{X}\|_2$ induced by replacing X by $\widehat{X}$. In d-dimension, this lowest quantization error goes to zero at a $N^{-\frac{1}{d}}$-rate as $N \to +\infty$. Stochastic optimization procedure based on simulation have been devised to compute these optimal quantizers. For an expository of mathematical aspects of quantization in finite dimension we refer to [6] and the references therein. For Signal processing and algorithmic aspects, we refer to [5], [4] and [19].

In the early 1990', optimal quantization has been introduced in Numerical Probability to devise some quadrature integration formulæ with respect to the distribution $\mathbb{P}_X$ on $\mathbb{R}^d$ using that $\mathbb{E}\, F(X) \approx \mathbb{E}\, F(\widehat{X})$ if N is large enough. This approach is efficient in medium dimensions (see [15], [16] and [19]) especially when many integrals need to be computed with respect to the same distribution $\mathbb{P}_X$: tables of the optimal weighted N-tuples can be computed and kept off-line like for Gauss points on the unit interval. Later, optimal quantization has been used to design some tree methods in order to solve multi-dimensional non-linear problems involving the computation of many conditional expectations: American option pricing, non-linear filtering for stochastic volatility models, portfolio optimization (see [18] for a review of applications to computational Finance).

More recently the infinite dimensional setting has been extensively investigated from a theoretical viewpoint with a special attention paid to *functional quantization* (FQ) *i.e.* the quantization of stochastic processes viewed as random vectors taking values in their path spaces such as $L^2_T := L^2([0,T], dt)$ (see [3],[11],[12], etc).

In this paper we aim to develop some first numerical applications of FQ. As concerns theoretical background we partially rely on [13]. We also provide some new numerically-oriented ingredients. We will focus on a financial framework: the pricing of path-dependent derivatives in a Heston stochastic volatility model.

More generally what our approach can be applied to the computation of the expectation $\mathbb{E}(F(X))$ where X is a Brownian diffusion (with explicit coefficients) and F is an additive (integral) functional defined on L^2_T by $\xi \mapsto F(\xi) := \int_0^T f(t, \xi(t))\, dt$.

In practice, true *optimal quantizers* of a process X are out of reach for numerical use, but some "*rate optimal*" sequences of quantizers do have some semi-closed form. So the starting point for numerics is to compute these "efficient" quantizers as well as the distribution of both their induced quantizations $\widehat{X}$ and quantization errors $\|X - \widehat{X}\|_2$ (another property – *stationarity* – will be needed, see sections 3.2 and 4.1.2 further on). Then, the quadrature formulæ involving these N-quantizers make up an efficient deterministic alternative to Monte Carlo simulation for the computation of $\mathbb{E}\, F(X)$.

As concerns Gaussian processes, this can be done by using an expansion

of X on its Karhunen-Loève (K-L) orthonormal basis. For Brownian diffusions, one maps the Brownian quantizers by solving an integral equation system (see section 4.3 for a presentation or [13]).

Let us give an example of such a rate optimal sequence of stationary quantizers in the simpler case where $X = B$ is the standard Brownian motion on $[0,T]$. Given N, we produce some optimal values d_N and $(N_k)_{1\le k\le d_N}$ – in a sense to be specified in subsection 4.1.1 – such that $N_1 \times \cdots N_n \le N$. Then, the quantizer used for B (at level N) is

$$\varphi^N_{i_1,\dots,i_n}(t) = \sqrt{\frac{2}{T}} \sum_{k=1}^{d_N} \frac{T}{\pi(k-\frac{1}{2})} \sin\left(\pi\left(k-\frac{1}{2}\right)\frac{t}{T}\right) x^{(N_k)}_{i_k}, \quad 1 \le i_k \le N_k, \quad 1 \le k \le d_N,$$

and its weight is

$$\alpha^N_{i_1,\dots,i_n} = \prod_{k=1}^{d_N} \alpha^{(N_k)}_{i_k},$$

where $(x^{(N_k)}_{i_k}, \alpha^{(N_k)}_{i_k})_{1\le i_k\le N_k}$ denotes the optimal (weighted) N_k-quantizer of the one-dimensional Normal distribution (see some examples at `http://perso-math.univ-mlv.fr/users/pr`
Then $\mathbb{E}\,F(X)$ is approximated by the weighted sum by

$$\sum_{i_1,\dots,i_n} \alpha^N_{i_1,\dots,i_n} \int_0^T f(t, \varphi_{i_1,\dots,i_n}(t))\,dt.$$

However, "crude" FQ theoretically converges at a rather poor rate, usually $(\log N)^{-\theta}$ for some θ depending on the pathwise regularity of the process X (*e.g.* $\theta = 1/2$ for the Brownian motion). So, hoping to compete successfully with Monte Carlo simulations needs to bet on its performances for "reasonably low" values of N (say $N \le 10\,000$). In fact to help winning this bet, we will introduce two speeding up procedures based on specific properties of FQ: one is stationarity, the other is a Romberg like extrapolation which is introduced in section 5.2.

The paper is organized as follows: in Section 2 we provide some background on functional quantization of (Gaussian) processes X viewed as L^2_T-valued random vectors. Section 3.1 is devoted to stationarity and its first computational applications (one-dimensional optimal quantizers, etc). In Section 3.2 some new weighted quadrature formulæ are established for $\mathbb{E}\,F(X)$ when F is a $|\,.\,|_{L^2_T}$ smooth functional. In Section 4 we give several examples of efficient quantizers: first for Gaussian processes the *Karhunen-Loève product quantizers* and some of their "non-Voronoi" variants (sections 4.1.1 and 4.2); then some explicit rate optimal sequences of quantizers are proposed for Brownian diffusions. A procedure is described to tabulate the optimal product quantizers. In Section 5.1, the computation procedure of K-L product quantizers for the Brownian motion are described (as well as

an extract of the tables available on the web). In Section 5.2, the speeding up Romberg log-extrapolation method is made explicit. In Section 6 the results of two numerical experiments are presented: the pricing of vanilla calls in a Heston stochastic volatility model (as a benchmark since an FFT-semi-closed form is available) and the pricing of an Asian option in the same model. The results are quite promising (although we decided not to implemented any of the usual "variance reducer"). In particular, we point out the efficiency of the Romberg log-extrapolation (sometimes combined with a linear interpolation method) which numerically outperforms Monte Carlo simulation in both examples (within the range of tested values). In Section 7, we outline an FQ-MC method to integrate irregular functionals $F(X)$: then FQ becomes a control variate random variable.

2 Preliminaries on quadratic functional quantization

Let $(H, (. \mid .)_H)$ be a separable Hilbert space and $X : (\Omega, \mathcal{A}, \mathbb{P}) \to H$ be a square integrable H-valued random vector with distribution $\mathbb{P}_X$ defined on H endowed with its Borel σ-field $\mathcal{B}or(H)$. One denotes by $\|.\|_2$ the usual quadratic norm on $L^2_H(\Omega, \mathbb{P})$ defined by $\|X\|_2 = \sqrt{\mathbb{E}(|X|^2_H)}$.

Let $x := (x_1, \ldots, x_N) \in H^N$ be an N-quantizer and let $\mathrm{Proj}_x : H \to \{x_1, \ldots, x_N\}$ be a projection following the *nearest neighbour rule*. It means that the Borel partition made of the so-called *Voronoi cells* $C_i(x) := \mathrm{Proj}_x^{-1}(\{x_i\})$, $i = 1, \ldots, N$, of H satisfies

$$\mathrm{Proj}_x^{-1}(\{x_i\}) \subset \{\xi \in H \mid |x_i - \xi|_H = \min_{1 \le j \le N} |x_j - \xi|_H\},\ 1 \le i \le N.$$

The partition$(C_i(x))_{i=1,\ldots,N}$ is called a *Voronoi tessellation* of H induced by x. One defines the *Voronoi quantization* of X induced by x by

$$\widehat{X}^x := \mathrm{Proj}_x(X).$$

(the exponent ^{x} will often be dropped or replaced by its size ^{N}). It is the best $L^2(\mathbb{P})$-approximation of X by $\{x_1, \ldots, x_N\}$-valued random vectors since, for any random vector $X' : \Omega \to \{x_1, \ldots, x_N\}$,

$$\|X - X'\|_2^2 = \int_\Omega |X(\omega) - X'(\omega)|^2_H \, \mathbb{P}(d\omega) \ge \int \min_{1 \le i \le N} |X(\omega) - x_i|^2_H \, \mathbb{P}(d\omega) = \|X - \widehat{X}^x\|_2^2$$

There are infinitely many Voronoi tessellations, all producing the same quadratic *quantization error* $\|X - \widehat{X}^x\|_2$. In fact the boundaries of any Voronoi tessellation are contained in the union of finitely many median hyperplanes $H_{ij} \equiv (x_i - x_j \mid \frac{x_i + x_j}{2} - .)_H = 0$ $(x_i \ne x_j)$. Hence, if the distribution $\mathbb{P}_X$ weights no hyperplane, then $\widehat{X}^x$ is $\mathbb{P}$-$a.s.$ uniquely defined.

The second step of the optimization procedure is to find an N-tuple $x \in H^N$, if any, which minimizes the quantization error over H^N. In fact one checks by the triangular inequality that the function

$$Q_N^X : (x_1, \ldots, x_N) \mapsto \|X - \widehat{X}^x\|_2 = \| \min_{1 \le i \le N} |X - x_i|_H \|_2$$

is Lipschitz continuous on H^N. When $N = 1$, $Q_1^2(x) = \mathbb{E}|X - x|_H^2$ is a strictly convex function which reaches its minimum $\mathrm{Var}(|X|_H)$ at $x^* := \mathbb{E}\, X$. Then, one shows by induction on N (see [11] for details), that Q_N^X always reaches a minimum at some optimal N-quantizer $x^* := (x_1^*, \ldots, x_N^*)$. As soon as $|\mathrm{supp}\mathbb{P}_X| \ge N$, any such optimal N-quantizer has pairwise distinct components. The key argument is that the function Q_N^X is weakly lower semi-continuous on H^N. (If $H = \mathbb{R}$ and $\mathbb{P}_X$ has a log-concave density, the optimal N-quantizer is unique, up to a permutation of its components). One shows using an everywhere dense sequence in H that $\min_{H^N}(Q_N^X)^2$ goes to 0 as N goes to ∞. Elucidating the rate of this convergence is a much more demanding problem, even in finite dimension. It is elucidated for non-singular $\mathbb{R}^d$-valued random vectors by the so-called Zador Theorem (see [6]).

Theorem 1. *(Zador, Bucklew & Wise, Graf & Luschgy) Assume that* $X \in L^{2+\eta}_{\mathbb{R}^d}(\Omega, \mathbb{P})$ *for some* $\eta > 0$. *Let* f *denote the density of the absolutely continuous part of* $\mathbb{P}_X$ *(which can be possibly* 0*). Then*

$$\min_{(\mathbb{R}^d)^N} (Q_N^X)^2 = \min_{x \in (\mathbb{R}^d)^N} \|X - \widehat{X}^x\|_2^2 = \frac{J_{2,d}}{N^{2/d}} \left(\int_{\mathbb{R}^d} f^{\frac{d}{d+2}}(\xi) d\xi \right)^{1+2/d} + o\left(\frac{1}{N^{\frac{2}{d}}} \right) \quad \text{as} \quad N \to +\infty.$$

When $f \not\equiv 0$, this yields a sharp rate for the quadratic quantization error since the integral in the right hand side is always finite under the assumption of the theorem. When $f \equiv 0$, this no longer provides a sharp rate, although such sharp rates can be established for some special distributions (self-similar distributions on fractal sets, etc). The true value of $J_{2,d}$ – which corresponds to the uniform distribution over $[0,1]^d$ – is unknown although one knows that $J_{2,d} = d/(2\pi e) + o(d)$.

3 Numerical integration using (functional) quantization

In this section, we first recall what stationarity of an N-quantizer is, then we provide some quadrature formulæ (some of them are new) and finally we describe in details the log-Romberg speeding up procedure (which remains partially heuristic given the present state of the art.

3.1 Stationarity quantizers

Definition 1. *An N-quantizer $x := (x_1, \dots, x_N) \in H^N$ is* stationary *if it satisfies*

$$\forall\, i \neq j, \quad x_i \neq x_j \qquad \textit{and} \qquad \mathbb{P}(X \in \cup_i \partial C_i(x)) = 0 \tag{3.1}$$

($\mathbb{P}_X$-negligible boundary of the Voronoi cells) and

$$\mathbb{E}(X \mid \widehat{X}^x) = \widehat{X}^x. \tag{3.2}$$

The random vector $\widehat{X}^x$ is called a stationary N-quantization of X.

In particular, any stationary quantizer satisfies $\mathbb{E}(X) = \mathbb{E}(\widehat{X}^x)$. Since the σ-fields generated by $\widehat{X}^x$ and $\{\{X \in C_i(x)\},\, i = 1, \dots, N\}$ coincide, Equation (3.2) also reads

$$x_i = \frac{\mathbb{E}(\mathbf{1}_{C_i(x)}(X)X)}{\mathbb{P}(X \in C_i(x))} = \mathbb{E}(X \mid \{X \in C_i(x)\}), \qquad i = 1, \dots, N. \tag{3.3}$$

provided $\mathbb{P}(X \in C_i(x)) > 0,\ i = 1, \dots, N$.

In fact stationary quantizers are the critical points of (the square of) quadratic quantization error: The function $x \mapsto D_N^X(x) := \|X - \widehat{X}^x\|_2^2$ is continuously differentiable at any N-quantizer x satisfying (3.1) and

$$\frac{\partial D_N^X}{\partial x_i}(x) := 2\,\mathbb{E}(\mathbf{1}_{C_i(x)}(X)(x_i - X)) = 2\int_{C_i(x)} (x_i - \xi)\,\mathbb{P}_X(d\xi), \quad 1 \le i \le N. \tag{3.4}$$

Consequently, any (local) minimum of the quantization error function is stationary. Optimal N-quantizer(s) are usually not the only stationary quantizers (see Proposition 4 below about Karhunen-Loève product quantizers). However, in 1-dimension for log-concave one-dimensional p.d.f., there is a unique stationary N-quantizer (the optimal one).

Note that owing to (3.2) the quantization error has then a simpler expressions

$$\mathbb{E}|X - \widehat{X}^x|_H^2 = \mathbb{E}|X|_H^2 - \mathbb{E}|\widehat{X}^x|_H^2 = \mathbb{E}|X|_H^2 - \sum_{i=1}^N |x_i|_H^2\,\mathbb{P}(X \in C_i(x)). \tag{3.5}$$

Similar equalities hold with the variances $\sigma^2(|X|_H)$ and $\sigma^2(|\widehat{X}^x|_H)$ of X and $\widehat{X}^x$ since their first moments coincide. We will see further on (Sections 3.2, 4 and 5.2) that stationary quantizers are an important class of quantizers for numerics.

In finite dimension, several numerical methods to compute (locally) optimal quantizers are based on the stationary equation. In 1-dimension the stationary quantizers are obtained by a Newton-Raphson procedure. In higher

dimension, we turn to stochastic gradient procedure. We refer to ([19]) for detailed explanations. Thus, a tabulation of optimal N-quantizers of the $\mathcal{N}(0;1)$ distribution has been carried out and kept off-line. Files can be can be downloaded at any of the following the URL's

`www.proba.jussieu.fr/pageperso/pages.html` or `perso-math.univ-mlv.fr/users/printems.jacques/n`

It contains, for every $N \in \{1,\dots,400\}$,

– the (unique) optimal N-quantizer x^N,

– the $\mathbb{P}_\xi$-masses $\mathbb{P}_\xi(C_i(x^N)), i = 1,\dots,N$, of its Voronoi cells (*i.e.* the distribution of $\widehat{\xi}^{x^N}$, $\xi \sim \mathcal{N}(0;1)$),

– the induced quadratic quantization error $\|\xi - \widehat{\xi}^{x^N}\|_2$ (using (3.5)),

When X is a bi-measurable process and $H = L^2_T$, the stationarity condition in its form (3.3) has consequences on the pathwise regularity of the elementary quantizers x_i : they have (at least) the regularity of $t \mapsto X_t$ from $[0,T]$ into $L^2(\Omega,\mathcal{A},\mathbb{P})$ (see [11, 12] for details).Furthermore, if X is a centered Gaussian process, one shows that stationary quantizers lie in the self-reproducing space of X (see [11]), like the Cameron-Martin space $H^1 := \{h \in L^2_T \,/\, h(t) = \int_0^t \dot{h}(s)ds,\ \dot{h} \in L^2_T\}$ for the Brownian motion.

3.2 Quadrature formulæ for numerical integration

The basic idea is that, on the one hand, a good quantization $\widehat{X}^x$ is close to X *in distribution* and, on the other hand, for every Borel functional $F : H \to \mathbb{R}$ and every $x = (x_1,\dots,x_N) \in H^N$,

$$\mathbb{E}\,F(\widehat{X}^x) = \sum_{1\le i\le N} \mathbb{P}_X(C_i(x))\,F(x_i). \tag{3.6}$$

So if one has a numerical access to both the N-quantizer x and its "companion" distribution $(\mathbb{P}_X(C_i(x)))_{1\le i\le N}$, the computation in (3.6) is straightforward. In the proposition below are established some error bounds for $\mathbb{E}\,F(X) - \mathbb{E}\,F(\widehat{X}^x)$ based on L^p-quantization errors $\|X - \widehat{X}^x\|_p$ (with $p = 2$ or 4).

Item (a) devoted to Lipschitz continuous functionals is classical, item (b) extends a second order quadrature formula involving stationary quantizers coming from [15] (see also [19]). Other quadrature formulæ based on L^p-quantization, $p \neq 2$, can be derived.

Proposition 1. *Let $X \in L^2_H(\Omega,\mathbb{P})$ and let $F : H \to \mathbb{R}$ be a Borel functional defined on H*

(a) First order quadrature formula: *If F is Lipschitz continuous, then*

$$|\mathbb{E}\,F(X) - \mathbb{E}\,F(\widehat{X}^x)| \le [F]_{\mathrm{Lip}}\|X - \widehat{X}^x\|_2$$

for every N-quantizer $x \in H^N$. In particular, if $(x^N)_{N\geq 1}$ denotes a sequence of quantizers such that $\lim_N \|X - \widehat{X}^{x^N}\|_2 = 0$, then the distribution $\sum_{i=1}^{N} \mathbb{P}_X(C_i(x^N))\delta_{x_i^N}$ of $\widehat{X}^{x^N}$ weakly converges to the distribution $\mathbb{P}_X$ of X as $N \to +\infty$.

(*b*) SECOND ORDER QUADRATURE FORMULÆ: *Assume that x is a stationary quantizer for X.*

– *Let $\theta : H \to \mathbb{R}_+$ be a nonnegative convex function. If $\theta(X) \in L^2(\mathbb{P})$ and if F is locally Lipschitz with at most θ-growth, i.e. $|F(x) - F(y)| \leq [F]_{\text{Liploc}}|x-y|\,(\theta(x)+\theta(y))$, then $F(X) \in L^1(\mathbb{P})$ and*

$$|\mathbb{E}\,F(X) - \mathbb{E}\,F(\widehat{X}^x)| \leq 2[F]_{\text{Liploc}}\|X - \widehat{X}^x\|_2\|\theta(X)\|_2. \tag{3.7}$$

– *If F is differentiable on H with an α-Hölder differential $\mathrm{D}F$ ($\alpha \in (0,1]$), then*

$$|\mathbb{E}\,F(X) - \mathbb{E}\,F(\widehat{X}^x)| \leq [\mathrm{D}F]_\alpha\|X - \widehat{X}^x\|_2^{1+\alpha}. \tag{3.8}$$

When F is twice differentiable and D^2F is bounded then, one may replace $[\mathrm{D}F]_1 = [\mathrm{D}F]_{\text{Lip}}$ by $\frac{1}{2}\|\mathrm{D}^2F\|_\infty$ in (3.8).

– *If $\mathrm{D}F$ is is locally Lipschitz with at most θ-growth, θ convex, $\theta(X) \in L^4(\mathbb{P})$, then*

$$|\mathbb{E}\,F(X) - \mathbb{E}\,F(\widehat{X}^x)| \leq 3[\mathrm{D}F]_{\text{Liploc}}\|X - \widehat{X}^x\|_4^2\|\theta(X)\|_4. \tag{3.9}$$

(*c*) AN INEQUALITY FOR CONVEX FUNCTIONALS: *Assume that x is a stationary quantizer. Then for any convex functional $F : H \to \mathbb{R}$*

$$\mathbb{E}\,F(\widehat{X}^x) \leq \mathbb{E}\,F(X). \tag{3.10}$$

The proofs of these quadrature formulæ are postponed to an annex.

Remark: The error bound (3.9) involves $\|X - \widehat{X}^x\|_4$ about which very little is known when x is a stationary (or even optimal) quadratic quantizer of X: its rate of convergence as N goes to infinity is not elucidated. So one often uses a less elegant (and probably less sharp) bound: assume that $\theta(X) \in L^p(\mathbb{P})$ *for every* $p \geq 1$, then, for every $\varepsilon \in (0,1]$,

$$|\mathbb{E}\,F(X) - \mathbb{E}\,F(\widehat{X}^x)| \leq [\mathrm{D}F]_{\text{Liploc}}\|X - \widehat{X}^x\|_2^{2-\varepsilon}\|X - \widehat{X}^x\|_4^{\varepsilon}(1 + 3\|\theta(X)\|_{\frac{1}{\varepsilon}}). \tag{3.11}$$

Examples: • The typical regular functionals defined on $(L^2_T, |\,.\,|_{L^2_T})$ (most important example for stochastic processes) are the integral functionals F defined by

$$\forall\, \xi \in L^2_T, \qquad F(\xi) = \int_0^T f(t, \xi(t))\,dt$$

where $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a Borel function with at most linear growth in x uniformly in t. In particular, F is Lipschitz continuous as soon as $f(t,.)$ is (uniformly in t), convex if $f(t,.)$ is for every t, etc; in particular F is differentiable with an α-Hölder differential as soon as $f(t,.)$ is differentiable for every $t \in [0,T]$ with an α-Hölder partial differential $\frac{\partial f}{\partial x}(t,.)$ (uniformly in t). Then

$$\forall\, \xi \in L^2_{_T}, \qquad \mathrm{D}F(\xi) = \int_0^T \frac{\partial f}{\partial x}(t, \xi(t))dt.$$

• The functional F defined for every $\xi \in L^2_{_T}$ by

$$F(\xi) := \int_0^T e^{\sigma\xi(t)+\rho t}dt \qquad (\rho \in \mathbb{R})$$

is convex, locally Lipschitz with θ-linear growth, infinitely differentiable. Furthermore, using that $|e^u - e^v| \leq |u-v|(e^u + e^v)$ and Schwarz inequality, one derives that

$$[F]_{\mathrm{Liploc}} := \sigma e^{\rho_+ {}^T} \qquad \text{and} \qquad \theta(\xi) = |e^{\sigma\xi}|_{_{L^2_T}}. \tag{3.12}$$

4 Functional quantization revisited from the numerical viewpoint

In this section several results on functional quantization are re-visited to emphasize all the combinatorial and computational aspects that make possible numerical applications. Of course, among all gaussian processes, the Brownian motion plays a central rôle. So, from now on, we will assume that X is a bi-measurable process defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ satisfying

$$\mathbb{E}\,|X|^2_{L^2_T} = \int_0^T \mathbb{E}(X_s^2)ds < +\infty$$

so that it can be viewed as an $L^2_{_T}$-valued random vector (up to a $\mathbb{P}$-negligible set). However many results below remain true in an abstract Hilbert framework.

4.1 Gaussian processes

For convenience in this section we will assume from now on that all random processes X are centered *i.e.*

$$\mathbb{E}\,X = 0_{_H}.$$

4.1.1 Karhunen-Loève product quantizers

Let $(e_n)_{n\geq 1}$ be an ortho-normal basis of $L^2_{\mathcal{T}}$. One may expand the paths of $(X_t)_{t\in[0,T]}$ on this basis *i.e.*

$$X(\omega) \overset{L^2_{\mathcal{T}}}{=} \sum_{n\geq 1} (X(\omega)|e_n)_{L^2_{\mathcal{T}}}\, e_n \qquad \mathbb{P}(d\omega)\text{-}a.s. \tag{4.13}$$

Since X is a Gaussian process, the sequence $((X|e_n))_{n\geq 1}$ is a Gaussian sequence of random variables so that (4.13) can be written

$$X(\omega) \overset{L^2_{\mathcal{T}}}{=} \sum_{n\geq 1} \sqrt{c_n}\,\xi_n(\omega) e_n \qquad \mathbb{P}(d\omega)\text{-}a.s. \tag{4.14}$$

where $c_n = \mathrm{Var}((X|e_n))$ and$(\xi_n)_{n\geq 1}$ is a Gaussian sequence of $\mathcal{N}(0;1)$-distributed random variables, usually not mutually independent. However there is a basis which plays a special r$\widehat{\text{o}}$le with respect to the process X: its Karhunen-Loève (denoted K-L) basis $(e^X_n)_{n\geq 1}$ which achieves the infinite dimensional PCA of its covariance operator Γ_X defined by

$$\forall\, f\in L^2_{\mathcal{T}}, \qquad \Gamma_X(f) := \left(t \mapsto \int_0^T f(s)\,\mathbb{E}(X_t X_s) ds\right).$$

The operator Γ_X is a non-negative self-adjoint compact operator, so it can be diagonalized in an orthonormal basis – the K-L basis – $(e^X_n)_{n\geq 1}$ of $L^2_{\mathcal{T}}$:

$$\Gamma_X(e^X_n) = \lambda_n e^X_n, \qquad n\geq 1,$$

where the eigenvalues λ_n make up a nonincreasing sequence of nonnegative real numbers satisfying

$$\sum_{n\geq 1} \lambda_n = \mathbb{E}\,|X|^2_{L^2_{\mathcal{T}}} < +\infty.$$

Without loss of generality one may assume that

$$\forall\, n\geq 1, \qquad \lambda_n > 0 \tag{4.15}$$

since otherwise supp$X \neq H$. Then, the K-L eigenbasis is unique. (In case X is a finite dimensional Gaussian vector, most of what follows remains true by setting $d := \max\{n\geq 1 : \lambda_n > 0\}$ and considering $\{1,\ldots,d\}$ instead of $\{1,\ldots,n,\ldots\}$ as an index set.)

Then, it follows from the so-called "reproducing property" that

$$\forall\, f,\, g\in L^2_{\mathcal{T}},\ \mathrm{Cov}\left((f|X)_{L^2_{\mathcal{T}}}, (g|X)_{L^2_{\mathcal{T}}}\right) = \int_{[0,T]^2} f(t)g(s)\mathbb{E}(X_t X_s) ds\, dt = (f|\Gamma_X(g))_{L^2_{\mathcal{T}}} \tag{4.16}$$

so that

$$c_n^2 = \mathbb{E}\,(X|e_n^X)_{L_T^2}^2) \;=\; \Big(e_n^X|\Gamma_X(e_n^X)\Big)_{L_T^2} = \lambda_n \tag{4.17}$$

$$\mathrm{Cov}((X|e_n^X)_{L_T^2},(X|e_m^X)_{L_T^2}) \;=\; (e_n^X|\Gamma_X(e_m^X))_{L_T^2} = \delta_{n,m}\lambda_n \tag{4.18}$$

where $\delta_{n,m}$ is for Kronecker symbol. Consequently

$$X(\omega) \overset{L_T^2}{=} \sum_{n\geq 1} \sqrt{\lambda_n}\; \xi_n(\omega) e_n^X \qquad \mathbb{P}(d\omega)\text{-}a.s. \tag{4.19}$$

where the sequence

$$\xi_n := \frac{(X|e_n^X)_{L_T^2}}{\sqrt{\mathrm{Var}((X|e_n^X)_{L_T^2})}},\; n\geq 1$$

is now i.i.d. and $\mathcal{N}(0;1)$-distributed. Expansion (4.19) is known as the K-L expansion of X. It combines both orthonormality of the K-L basis $(e_n^X)_{n\geq 1}$ and the mutual independence of its coordinates ξ_n. Furthermore, the equality (4.19) holds in $L^2(d\mathbb{P}\otimes dt)$. In particular it holds $\mathbb{P}(d\omega)$-$a.s.$ at dt-almost every time $t\in[0,T]$. It is the PCA of the process X in that all d dimensional truncations of (4.19) produce the best d-dimensional approximation of X in the least square sense.

A very natural way to produce a functional quantization for Gaussian processes in L_T^2 using at most N elementary quantizers is to use a *product quantizer* of the form

$$\widehat{X}_t = \sum_{n\geq 1} \sqrt{\lambda_n}\, \widehat{\xi}_n\, e_n^X(t) \tag{4.20}$$

where $\widehat{\xi}_n := \widehat{\xi}_n^{x^{(N_n)}} = \mathrm{Proj}_{x^{(N_n)}}(\xi_n)$ is an optimal N_n-quantization of ξ_n and $N_1\times\cdots\times N_n\leq N$, $N_1,\ldots,N_n\geq 1$. Note that for large enough n, $N_n=1$ so that $\widehat{\xi}_n = 0$ which makes the above series a finite sum. Also keep in mind that the p.d.f. of normal distribution being log-concave the optimal N_n quantizer $x^{(N_n)} := (x_1^{(N_n)},\ldots,x_{N_n}^{(N_n)})$ is unique (and already tabulated as mentioned above).

The $N_1\times\cdots\times N_n$-quantizer χ that produces the above Voronoi quantization (4.20) is of the form

$$\chi_{\underline{i}}(t) = \sum_{n\geq 1} \sqrt{\lambda_n}\, x_{i_n}^{(N_n)}\, e_n^X(t), \qquad \underline{i} = (i_1,\ldots,i_n,\ldots)\in \prod_{n\geq 1}\{1,\ldots,N_n\}. \tag{4.21}$$

Definition 2. *A quantizer χ defined by (4.21) is called a K-L* product quantizer. *For convenience and when there is no ambiguity concerning the reference basis we will often denote χ by*

$$\chi = \sqrt{\lambda}\otimes x \qquad \textit{with} \qquad x = \prod_{n\geq 1} x^{(N_n)}.$$

Furthermore, one denote by $\mathcal{O}_{pq}(X,N)$ *the set*

$$\mathcal{O}_{pq}(X,N) := \{\chi \,/\, \chi \text{ } K\text{-}L \text{ product quantizer of size at most } N \text{ as defined by (4.21)}\}$$

The proposition below describes the geometric structure

Proposition 2. *Let* $\chi = \sqrt{\lambda} \otimes x$ *be a* K*-*L *product quantizer as defined by (4.21). Let* $d_x := \max\{k : N_k > 1\} \in \mathbb{N}$ *denote the "quantization dimension" (highest non-trivially quantized dimension).*
(a) Then, the quadratic quantization error induced by χ *satisfies*

$$\|X - \widehat{X}^{\chi}\|_2^2 = \sum_{n\geq 1} \lambda_n \|\xi_n - \widehat{\xi}_n^{N_n}\|_2^2 = \mathbb{E}|X|_{L^2_T}^2 + \sum_{n=1}^{d_x} \lambda_n \left(\|\xi_k - \widehat{\xi}_k^{N_n}\|_2^2 - \mathbb{E}|\xi_n|^2\right) \tag{4.22}$$

(b) For every multi-index $\underline{i} \in \prod_{n\geq 1}\{1,\dots,N_n\}$*, the associated Voronoi cell of* χ *is*

$$C_{\underline{i}}(\chi) = \prod_{n\geq 1} (\sqrt{\lambda_n} C_{i_n}(x^{(N_n)})). \tag{4.23}$$

Remark. In fact, both claims only rely on the orthonormality of the basis $(e_n^X)_{n\geq 1}$ and do not make use of the specificity of the K-L basis.

Proof. *(a)* One notes that $\|X - \widehat{X}^{\chi}\|_2^2 = \mathbb{E}\min_{\underline{i}} |X - \chi_{\underline{i}}|^2$ where $\chi_{\underline{i}}$ is given by (4.21). Then

$$\begin{aligned}
\mathbb{E}\min_{\underline{i}} |X - \chi_{\underline{i}}|^2 &= \mathbb{E}\left(\min_{1\leq i_1\leq N_1,\cdots,1\leq i_{d_x}\leq N_{d_x}} \left|\sum_{n\geq 1}\sqrt{\lambda_n}\,\xi_n e_n - \sum_{n=1}^{d_x}\sqrt{\lambda_n}\,x_{i_n}^{(N_n)} e_n\right|^2\right) \\
&= \mathbb{E}\left(\min_{1\leq i_1\leq N_1,\cdots,1\leq i_{d_x}\leq N_{d_x}} \sum_{n=1}^{d_x}\lambda_n\,|\xi_n - x_{i_n}^{(N_n)}|^2 + \sum_{n\geq d_x+1}\lambda_n\,\xi_n^2\right) \\
&= \sum_{n=1}^{d_x}\lambda_n\,\mathbb{E}\left(\min_{1\leq i_n\leq N_n}|\xi_n - x_{i_n}^{(N_n)}|^2\right) + \sum_{n\geq d_x+1}\lambda_n \\
&= \sum_{n=1}^{d_x}\lambda_n\,\mathbb{E}\left(\min_{1\leq i_n\leq N_n}|\xi_n - x_{i_n}^{(N_n)}|^2\right) + \mathbb{E}|X|_{L^2_T} - \sum_{n=1}^{d_x}\lambda_n
\end{aligned}$$

The first equality follows from the fact that, for every $n > d_x$, $x^{(N_n)} = \mathbb{E}(\xi_n) = 0$.
(b) One may assume without loss of generality that, for every $n \geq 1$, the components of $x^{(N_n)}$ are in an ascending order *i.e.* $i \mapsto x_i^{(N_n)}$ is nondecreasing. Let $\underline{i} := (i_1,\dots,i_{d_x},1,\dots)$ and $\underline{j} := (j_1,\dots,j_{d_x},1,\dots)$. Then, if $\zeta = \sum_n \zeta_n e_n \in H$, $|\zeta - \chi_{\underline{i}}|^2 < |\zeta - \chi_{\underline{j}}|^2$ if and only if

$$\sum_{n=1}^{d_x}\sqrt{\lambda_n}^2\left(x_{i_n}^{(N_n)} - x_{j_n}^{(N_n)}\right)\left(\frac{\zeta_n}{\sqrt{\lambda_n}} - \frac{x_{i_n}^{(N_n)} + x_{j_n}^{(N_n)}}{2}\right) < 0.$$

Then, for every fixed n, setting $j_n = i_{n\pm1}$ and $j_{n'} = i_{n'}$ if $n' \neq n$ implies that

$$\frac{x_{i_n}^{(N_n)} + x_{i_n-1}^{(N_n)}}{2} < \frac{\zeta_n}{\sqrt{\lambda_n}} < \frac{x_{i_n+1}^{(N_n)} + x_{i_n}^{(N_n)}}{2} \quad i.e. \quad \frac{\zeta_n}{\sqrt{\lambda_n}} \in C_{i_n}(x^{(N_n)}).$$

One checks that this condition is sufficient. $\diamond$

Then the lowest quadratic quantization error induced by K-L product quantizers having at most N codebooks is obtained as the solution of the following optimization problem

$$\min\left\{\sum_{n=1}^{d}\lambda_n \ \min_{\mathbb{R}^{N_n}}\|\xi - \widehat{\xi}^{N_n}\|_2^2 + \sum_{n\geq d+1}\lambda_n,\ N_1\times\cdots\times N_n \leq N,\ N_1,\ldots,N_m \geq 2,\ d\geq 1\right\}. \tag{4.24}$$

This provides an upper-bound for the lowest quantization error over all quantizers with at most N codebooks. This approach is the starting point for theoretical estimation of the rate of convergence of the quantization error in [11] (and in [12] with d-dimensional marginal blocks instead of 1-dimensional ones; this holds for any orthonormal basis of L_T^2 which is extensively exploited in these references).

Solving numerically the optimization problem (4.24) for a wide range of values of N when it is possible is a first step to use functional quantization for numerics (see Section 5.1 for the Brownian motion).

Now, let us come to the specific feature of the K-L expansion which makes possible numerical implementation of the quadrature formulæ established in Section 3.2. *A closed formula is available for the distribution of* $\widehat{X}^{\chi}$ *when* χ *is a* K-L *product quantizer*, namely

$$\forall\, \underline{i} \in \prod_{n\geq1}\{1,\ldots,N_n\},\quad \mathbb{P}(\widehat{X}^{\chi} = \chi_{\underline{i}}) = \prod_{n\geq1}\mathbb{P}(\xi\in C_{i_n}(x^{(N_n)})),\qquad \xi\sim\mathcal{N}(0;1). \tag{4.25}$$

This follows from the combination of parallelepipedic shape (4.23) of the Voronoi cells $C_{\underline{i}}$ and the independence of the normal random variables $\xi_n, n\geq 1$ in the expansion (4.19) since

$$\mathbb{P}(\widehat{X}^{\chi} = \chi_{\underline{i}}) = \mathbb{P}\left(\cap_{n\geq1}\{\sqrt{\lambda_n}\xi_n \in \sqrt{\lambda_n}C_{i_n}(x^{(N_n)}\}\right) = \prod_{n\geq1}\mathbb{P}(\xi_n\in C_{i_n}(x^{(N_n)}).$$

The weight vector $(\mathbb{P}(\xi\in C_i(x^{(N_n)})))_{i=1,\ldots,N_n}$ is simply the distribution of the optimal N_n-quantization $\widehat{\xi}^{N_n} := \xi^{x^{(N_n)}}$, $\xi\sim\mathcal{N}(0;1))$. hence, if one denotes by erf the distribution function of $\mathcal{N}(0;1)$, one has

$$\mathbb{P}(\xi\in C_i(x^{(N_n)})) = \mathrm{erf}\left(x_{i+1/2}^{(N_n)}\right) - \mathrm{erf}\left(x_{i-1/2}^{(N_n)}\right),\ i=1,\ldots,N_n$$

with $x_{i+1/2}^{(N_n)} := \frac{x_{i+1}^{(N_n)} + x_i^{(N_n)}}{2}$, $i = 1, \ldots, N_n - 1$, $x_{1/2}^{(N_n)} = -\infty$, $x_{N_n+1/2}^{(N_n)} = +\infty$ (They are available at the formerly given URL's). For our purposes here, no higher values than 100 are necessary for N_n.

Practical rule for numerical implementation: *Numerical implementation of the functional quantization of a Gaussian process X is possible as soon as closed form is available for the eigensystem $(\overset{X}{e}_n, \lambda_n)_{n\geq 1}$.*

The most important example of an explicit K-L system is of course the Brownian motion $(W_t)_{t\in[0,T]}$ whose K-L eigensystem is given by

$$\overset{W}{e}_n(t) := \sqrt{\frac{2}{T}} \sin\left(\pi(n-1/2)\frac{t}{T}\right), \qquad \lambda_n := \left(\frac{T}{\pi(n-1/2)}\right)^2, \qquad n \geq 1. \tag{4.26}$$

Other common Gaussian processes have explicit K-L expansions like the Brownian bridge ($\overset{X}{e}_n(t) := 2/T \sin(\pi n \frac{t}{T})$ and $\lambda_n := (T/(\pi\, n))^2$). The stationary Ornstein-Uhlenbeck process admits a semi-closed form for its K-L system (see *e.g.* [8], p.195).

As a conclusion to this chapter let us cite an upper-bound obtained in [11] by solving the optimization problem (4.24). This yields the following theoretical rate of convergence for the quantization error of a Gaussian process X.

Proposition 3. *Let X be a Gaussian process with a K-L eigensystem $(\overset{X}{e}_n, \lambda_n)_{n\geq 1}$. Assume that $\lambda_n \leq c^*\, n^{-b}$, $b > 1$, $c^* > 0$. Then*

$$\min\left\{\|X - \widehat{X}^{\chi}\|_2,\ \chi \in \mathcal{O}_{pq}(X, N)\right\} \leq \left(c^*\left(\frac{b}{2}\right)^{b-1}\left(\frac{1}{b-1} + 4\, C_{\mathcal{N}(0;1)}\right)\right)^{1/2} \frac{1}{(\log N)^{\frac{b-1}{2}}} \tag{4.27}$$

where $C_{\mathcal{N}(0;1)} := \sup_{N\geq 1}\left(N^2 \min_{x\in\mathbb{R}^N} \|\xi - \widehat{\xi}^x\|_2\right)$. In particular, for the standard Brownian motion W,

$$\min\left\{\|W - \widehat{W}^{\chi}\|_2,\ \chi \in \mathcal{O}_{pq}(W, N)\right\} \leq \frac{2T}{\pi}\left(1 + 4C_{\mathcal{N}(0;1)}\right)^{1/2} \frac{1}{(\log N)^{\frac{1}{2}}}.$$

Furthermore, it is also established in [11] using entropy methods that the above rate is the true one: if $\lambda_n \geq c_* n^{-b}$ ($c_* > 0$) for every $n \geq 1$, then there is some real $c'_* > 0$ such that

$$\min_{x\in H^N} \|X - \widehat{X}^x\|_2 \geq c'_*(\log N)^{-\frac{b-1}{2}}, \qquad N \geq 1. \tag{4.28}$$

Consequently the $O(\log N)^{-\frac{1}{2}}$)-rate is optimal for the Brownian motion and

There exists some rate optimal sequences $(\chi_N)_{N\geq 1}$ of K-L product quantizers for W.

Remarks. • A sharp rate based on a product quantization of X by d-dimensional marginal blocks instead of 1-dimensional ones is established in [12] when $\lambda_n = c_\lambda n^{-b} + o(n^{-b})$ $(c_* > 0)$:

$$\min_{x\in H^N} \|X - \widehat{X}^x\|_2 = \frac{c_\lambda^{\frac{1}{2}} b^{\frac{b}{2}}}{2^{\frac{b-1}{2}}(b-1)^{\frac{1}{2}}} (\log N)^{-\frac{b-1}{2}} + o\left((\log N)^{-\frac{b-1}{2}}\right). \quad (4.29)$$

• A conjecture confirmed by numerical experiments is that $C_{\mathcal{N}(0;1)} = \lim_N \left(N^2 \min_{x\in\mathbb{R}^N} \|\xi - \widehat{\xi}^x\|_2\right) = \frac{\pi}{2}\sqrt{3}$ (the second equality follows from Zador's Theorem, see [12] for more details).

Extensions. • The upper-bound (4.27) remains true if one replaces *mutatis mutandis* the K-L eigensystem by any system $(e_n, c_n^2)_{n\geq 1}$ where $(e_n)_{n\geq 1}$ is an orthonormal basis of L^2_T and $c_n^2 = \mathrm{Var}((X \,|\, e_n)_{L^2_T})$ (but there no explicit formula for the distribution of these product quantizations). This is a way to get the quantization error rates for many Gaussian processes like the fractional Brownian motion using *e.g.* the Haar basis (see [11] for some examples).

• The upper-bound (4.27) also remains true if one considers a system $(e_n, c_n^2)_{n\geq 1}$ such that $\sum_{n\geq 1} c_n \xi_n e_n$ converges to X in which e_n is no longer orthogonal (but still normed) provided that the random variables ξ_n in (4.19) remain independent. This follows from the independence of $\xi_n - \widehat{\xi}_n^{N_n}, n \geq 1$ and the stationarity property satisfied by $\widehat{\xi}_n^{N_n}, n \geq 1$ (see Section 4.2 below).

4.1.2 Stationarity of K-L product quantizer

The following proposition emphasizes a new important and typical feature of the K-L basis: that K-L product quantizers are stationary. It is an important asset for numerical purposes since one may then apply the second order quadrature formulæ established in Proposition 1(b).

Proposition 4. *Let $\chi \in \mathcal{O}_{pq}(X, N)$ be a K-L product quantizer of X. Then χ is stationary for X.*

Proof (See also [6], Lemma 4.8). Let $\chi = \sqrt{\lambda} \otimes x$. The ξ_n being independent in the K-L expansion (4.19), then the $\widehat{\xi}_n$ are independent in (4.20) as well. Furthermore, it is obvious from (4.20) and the identity $\widehat{\xi}_n =$

$(\widehat{X}^\chi|e_n^X)_{L^2_T}/\sqrt{\lambda_n},\ n\geq 1$, that $\sigma(\widehat{X}^\chi)=\sigma(\widehat{\xi}_n,\ n\geq 1)$. Consequently

$$\begin{aligned}\mathbb{E}(X\,|\,\widehat{X}^\chi) &= \sum_n \sqrt{\lambda_n}\,\mathbb{E}(\xi_n\,|\,\widehat{\xi}_m,\ m\geq 1)\,e_n^X = \sum_n \sqrt{\lambda_n}\,\mathbb{E}(\xi_n\,|\,\widehat{\xi}_n,\ \widehat{\xi}_m,\ m\geq 1,\ m\neq n)\,e_n^X\\ &= \sum_n \sqrt{\lambda_n}\,\mathbb{E}(\xi_n|\,\widehat{\xi}_n)e_n^X = \sum_n \sqrt{\lambda_n}\,\widehat{\xi}_n e_n^X \qquad \text{(stationarity of } \widehat{\xi}_n),\\ &= \widehat{X}^\chi. \qquad \diamond\end{aligned}$$

4.2 An example of computable non-Voronoi rate optimal quantizers: the antiderivative of the Brownian motion

We will illustrate in this short paragraph how rate optimal product quantizers of the Brownian motion can produce (non Voronoi) rate optimal quantizers of its antiderivative (nevertheless with an explicit distribution).

First note that one can integrate a Karhunen-Loève expansion of the Brownian motion. In fact, $h\mapsto \int_0^\cdot h(s)ds$ being a Lipschitz continuous function from L^2_T into $(C([0,T]),\|\,.\,\|_{\sup})$, one has, in $L^2_{(C([0,T]),\|.\|_{\sup})}(\mathbb{P})$ (and $\mathbb{P}$-*a.s.* in L^2_T):

$$\int_0^t W_s ds \overset{L^2_T}{=} \sum_{n\geq 1}\lambda_n\,\xi_n\sqrt{\frac{2}{T}}\left(1-\cos\left(\frac{t}{\sqrt{\lambda_n}}\right)\right) \quad \text{with}\ \ \lambda_n:=\left(\frac{T}{\pi(n-1/2)}\right)^2,\ n\geq 1 \tag{4.30}$$

$$= 2\sqrt{\frac{2}{T}}\sum_{n\geq 1}\lambda_n\,\xi_n\sin^2\left(\frac{t}{2\sqrt{\lambda_n}}\right) \tag{4.31}$$

where: – $(\xi_n)_{n\geq 1}$ is i.i.d., normally distributed (and comes from the K-L expansion of W),

– the sequence $\left(t\mapsto\sqrt{\frac{2}{T}}\left(1-\cos\left(\frac{t}{\sqrt{\lambda_n}}\right)\right)\right)_{n\geq 1}$ is *not* orthonormal in L^2_T.

In fact, the expansion (4.30) *converges* $\mathbb{P}$-*a.s. and in* $L^1(\mathbb{P})$, *uniformly in* $t\in[0,T]$, since

$$\sup_{t\in[0,T]}\left|\sum_{n\geq 1}\lambda_n\,\xi_n\sin^2\left(\frac{t}{2\sqrt{\lambda_n}}\right)\right|\leq\sum_{n\geq 1}\lambda_n|\xi_n|.$$

The series on the right hand of the inequality lies in $L^1(\mathbb{P})$ since $\sum_{n\geq 1}\lambda_n<+\infty$ and $\xi_n\sim\xi^1\in L^1(\mathbb{P})$. The same uniform $L^1(\mathbb{P})$- convergence holds for the integrated product quantizer expansion, that is

$$\widetilde{\int_0^\cdot W_s ds}:=\int_0^\cdot \widehat{W}_s^\chi ds = 2\sqrt{\frac{2}{T}}\sum_{n\geq 1}\lambda_n\,\widehat{\xi}_n\sin^2\left(\frac{t}{2\sqrt{\lambda_n}}\right) \tag{4.32}$$

since, by stationarity of the quantizer $x^{(N_n)}$ of ξ_n, $\mathbb{E}|\widehat{\xi}_n|\leq\mathbb{E}|\xi_n|$ for every $n\geq 1$ (the $\mathbb{P}$-*a.s.* convergence is trivial since $\widehat{\xi}_n=0$ for large enough n).

One has to be aware that $\widetilde{\int_0^\cdot W_s ds}^{\chi}$ *is not a Voronoi quantization* since it is defined on the Voronoi tessellation of the Brownian motion. For this very reason it is easy to compute and furthermore it satisfies a kind of stationary equation: one checks that $\sigma(\widetilde{\int_0^\cdot W_s ds}) = \sigma(\widehat{W}) = \sigma(\xi_n,\, n \geq 1)$ so that, $h \mapsto \int_0^\cdot h(s)ds$ being continuous and linear on L^2_T,

$$\mathbb{E}\left(\int_0^\cdot W_s ds \,|\, \widetilde{\int_0^\cdot W_s ds}\right) = \mathbb{E}\left(\int_0^\cdot W_s ds \,|\, \widehat{W}^{\chi}\right) = \widetilde{\int_0^\cdot W_s ds}.$$

Proposition 5. *Let $\chi^N \in \mathcal{O}_{pq}(W, N)$, $N \geq 1$, let λ_n be defined by (4.30) and let $\widetilde{\int_0^\cdot W_s ds}^{N} := \int_0^\cdot \widehat{W}_s^{\chi^N} ds$ be defined by (4.32).*
(a) The quadratic quantization error is given by

$$\left\| \int_0^\cdot W_s ds - \widetilde{\int_0^\cdot W_s ds}^{\chi^N} \right\|_2^2 = 3 \sum_{n\geq 1} \lambda_n^2 \left(1 - (-1)^{n-1} \frac{4\sqrt{\lambda_n}}{3T}\right) \min_{x\in\mathbb{R}^{N_n}} \|\xi - \widehat{\xi}^x\|_2^2. \tag{4.33}$$

If $(\chi^N)_{N\geq 1}$ is rate optimal for W then $\left\| \int_0^\cdot W_s ds - \widetilde{\int_0^\cdot W_s ds}^{\chi^N} \right\|_2 = O((\log N)^{-1})$ which is not rate optimal. There is a rate optimal sequence $\zeta^N \in \mathcal{O}_{pq}(W, N)$ for $\int_0^\cdot W_s ds$ i.e. such that

$$\left\| \int_0^\cdot W_s ds - \widetilde{\int_0^\cdot W_s ds}^{\zeta^N} \right\|_2 = O((\log N)^{-\frac{3}{2}}).$$

(b) Furthermore, the $L^1(\mathbb{P})$-mean $\|\,.\,\|_{\sup}$-quantization error satisfies

$$\mathbb{E}\left(\sup_{t\in[0,T]} \left| \int_0^t W_s ds - \widetilde{\int_0^t W_s ds}^{\chi^N} \right| \right) \leq 2\sqrt{\frac{2}{T}} \sum_{n\geq 1} \lambda_n \min_{x\in\mathbb{R}^{N_n}} \|\xi - \widehat{\xi}^x\|_2. \tag{4.34}$$

Proof: *(a)* Temporarily set $E_n(t) = 1 - \cos\left(\frac{t}{\sqrt{\lambda_n}}\right)$. Then $|E_n|^2_{L^2_T} = T\left(\frac{3}{2} - 2(-1)^{n-1}\frac{\sqrt{\lambda_n}}{T}\right)$ and

$$\left| \int_0^\cdot W_s ds - \widetilde{\int_0^\cdot W_s ds}^{\chi^N} \right|^2_{L^2_T} = \frac{2}{T} \sum_{n,m\geq 1} \lambda_n \lambda_m \mathbb{E}(\xi_n - \widehat{\xi}_n)(\xi^m - \widehat{\xi}^m)\, (E^n \,|\, E^m)_{L^2_T}$$

so that

$$\left\| \int_0^\cdot W_s ds - \widetilde{\int_0^\cdot W_s ds}^{\chi^N} \right\|_2^2 = \frac{2}{T} \sum_{n\geq 1} \lambda_n^2\, \mathbb{E}(\xi_n - \widehat{\xi}_n)^2 |E^n|^2_{L^2_T}.$$

The above equality follows from the fact that the random variables $\xi_n - \widehat{\xi}_n, n \geq 1$, are independent and centered since $\mathbb{E}(\xi_n - \widehat{\xi}_n) = \mathbb{E}(\mathbb{E}(\xi_n|\widehat{\xi}_n) -$

$\widehat{\xi}_n) = 0$. The first rate follows from the optimal size allocation for W in (4.24) then plugged in the right hand side of (4.33). The second follows from the optimal size allocation directly in (4.33). We refer to [11] for details.

(b) easily follows from

$$\sup_{t\in[0,T]}\left|\int_0^t W_s ds - \widetilde{\int_0^t W_s ds}^{\chi^N}\right| = 2\sqrt{\frac{2}{T}}\sup_{t\in[0,T]}\left|\sum_{n\geq 1}\lambda_n(\xi_n-\widehat{\xi}_n)\sin^2\left(\frac{t}{2\sqrt{\lambda_n}}\right)\right| \leq 2\sqrt{\frac{2}{T}}\sum_{n\geq 1}\lambda_n\,|\xi_n-\widehat{\xi}_n|.\ \diamond$$

Remarks. • One derives similarly from (4.34) in claim (b) that the lowest $L^1(\mathbb{P})$-mean $L^\infty(dt)$-quantization error goes to zero at (least at) a $O\left((\log(N))^{-1}\right)$-rate (this is not rate optimal, see [14]).

• Some rates can be obtained for higher iterated integrals (and the Brownian bridge too).

4.3 Explicit non-Voronoi rate optimal quantization of Brownian diffusions

In [13] the exact quantization error rate for a class of Brownian diffusions (including most 1-dimensional ones) is established (see also [2]). It is a constructive approach based on the Lamperti transform and stochastic calculus techniques. This rate is $O((\log N)^{-\frac{1}{2}})$ like for the Brownian motion as soon as the diffusion coefficient is not too degenerate. We shortly describe below how to construct an explicit (non-Voronoi) quantizer sequence that yields the rate. Let

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x_0 \tag{4.35}$$

be a Brownian diffusion where σ is a nonvanishing function. Let $(\chi^N)_{N\geq 1}$ be a sequence of rate optimal K-L product quantizers of the Brownian motion. The components of χ^N are explicit C^∞ functions. Some quantizers for X can be designed from the sequence $(\chi^N)_{N\geq 1}$ as follows: let L denote the Lamperti transform defined by

$$L(y) := \int_0^y \frac{d\xi}{\sigma(\xi)}$$

(assumed to be real-valued and increasing). Then, $Y_t := L(X_t)$ satisfies an *SDE*

$$dY_t = \beta(Y_t)dt + dW_t, \quad Y_0 = L(x_0)$$

with a linear Brownian perturbation term (and an explicit function β specified in [13]). Then, one defines, an N-quantizer of X by setting

$$x_i^N(t) = L^{-1}(u_i^N(t)) \quad \text{where} \quad u_i^N(t) = L(v_0) + \int_0^t \beta(u_i^N(t))dt + \chi_i^N(t), \quad i = 1,\ldots,N$$

(for notational convenience, we temporarily switch to the simpler notation i for the index rather than $\underline{i}$: here, the multi-index feature plays no rôle). Elementary computations show that $x^N = (x_i^N)_{1\le i\le N}$ is solution of the system of integral equations

$$x_i^N(t) = v_0 + \int_0^t [b(x_i^N(s)) - \frac{1}{2}\sigma\sigma'(x_i^N(s))]ds + \int_0^t \sigma(x_i^N(s))d\chi_i^N(s),\ i = 1,\dots,N. \tag{4.36}$$

Let us note that the Ito correction term $-\frac{1}{2}\sigma\sigma'$ would disappear if (4.35) was written in the Stratonovich sense.

When b and σ are both Lipschitz continuous the sequence $(x^N)_{N\ge1}$ is rate optimal in $L^p_{L^2_T}(\Omega,\mathbb{P})$ for every $p\in[1,2)$ (see Theorems 1 and 2 in [13]). More precisely, the sequence of non-Voronoi N-quantizations

$$\widetilde{X}_t^{x^N} := \sum_{1\le i\le N} x_i^N(t)\mathbf{1}_{C_i(\chi^N)}(W),\ N\ge 1,$$

satisfies

$$\| \, |X - \widetilde{X}^{x^N}|_{L^2_T} \, \|_p = O((\log N)^{-1/2}), \qquad p\in[1,2).$$

When $p=2$ an easy adaptation of the proof of Theorem 1 in [13] yields a $O((\log N)^{-\frac{1}{2}+\varepsilon})$-rate in $L^p_{L^2_T}(\Omega,\mathbb{P})$ for every $\varepsilon>0$.

The quantization $\widetilde{X}^{x^N}$ is not Voronoi since it is defined on the Voronoi tessellation of W, but its distribution is simply the $\mathbb{P}_W$-weights of the cells $C_i(\chi^N)$ which are known by (4.25). Numerical implementation of these quantizers needs to use a discretization scheme of the integral system (4.36). This is done in Section 6.1 to price options in a Heston stochastic volatility model.

This approach to functional quantization of diffusion heavily relies on the Lamperti transform which is structurally 1-dimensional. However an extension to multi-dimensional diffusions is possible *e.g.* if

$$\sigma(x) = (DS(x))^{-1}$$

where S is $\mathcal{C}^2$-diffeomorphism on $\mathbb{R}^d$ (see Section 5 in [13] for details). Other specific multi-dimensional settings can be dealt with in a *constructive* way as it will be seen in Section 6.2 with the stochastic volatility Heston model: some stochastic integrals $\int_0^t f(Z_s)dW_s$ will be approximated by functional quantization when Z is a 1-dimensional diffusion process independent with the Brownian motion W.

5 Toward numerical implementation

5.1 Optimal product quantizers: the "blind" optimization procedure

Assume that *closed forms* are available for both components of the K-L eigensystem $(e_n^X, \lambda_n)_{n\geq 1}$ of a Gaussian process X as it is the case for the Brownian motion (or the Brownian bridge).

Then, we are in the position to solve numerically the optimization problem (4.24) which yields for every $N \in \{1, \ldots, N_{\max}\}$ the best product quantizer $\chi_{\rm rec}^N$ and its "companion parameters" (distribution of $\widehat{X}^{\chi_{\rm rec}^N}$, quantization error $\|X - \widehat{X}^{\chi_{\rm rec}^N}\|_2$). The reason for implementing such a *blind* optimization procedure is that the theoretical values which produce the exact asymptotic rate are not efficient within the range of values of of numerical interest. Furthermore, this "blind" optimization procedure is reasonably fast and its results can be kept off line. It is carried out in two steps.

Phase 1 (Optimization phase at fixed N): Producing the K-L product N-quantizer $\chi_{\rm opt}^N = \sqrt{\lambda} \otimes x_{\rm opt}$ with minimal quadratic quantization error *among all product quantizers of size exactly* N.

This phase is carried out by using the "library" storing the optimal N_n-quantizers $x^{(N_n)}$ and their own companion parameters of the $\mathcal{N}(0;1)$ distribution.

In practice, $N_n \leq 100$ is enough for values of N as high as 10^6 since decompositions (d_x too small) involving not enough factors will clearly be far from optimality.

Phase 2 (Record Selection phase): Storing for every $N \in \{1, \ldots, N_{\max}\}$,

– the size $N_{\rm rec} := N_{\rm rec}(N) \in \{1, \ldots, N\}$ which produces the lowest quadratic quantization error,

– the optimal decomposition $N_{\rm rec} = N_1^{\rm rec} \times \cdots \times N_n^{\rm rec} \times \cdots \times N_{d_{\rm rec}}^{\rm rec}$, (with $n \mapsto N_n^{\rm rec}$ non-increasing and $N_{d_{\rm rec}}^{\rm rec} \geq 2$, $N_{d_{\rm rec}+1}^{\rm rec} = 1$),

from which one retrieves instantly

– the K-L product quantizer $\chi_{\rm rec}^N = \chi_{\rm opt}^{N_{\rm rec}}$ which solves the optimization problem (4.24) at level N.

– the distribution of its related quantization $\widehat{X}^{\chi_{\rm rec}^N}$ (using formula (4.25)),

– the corresponding quantization error by (4.22).

Table 1 below provides $N_{\rm rec}$, the quantization error and the record decomposition first for several typical values of N, namely $N = 1$, 10, 100, 1 000, 10 000 (the full record table, the record quantizer list including the distributions are available at the same URL). Figure 8 show the K-L product quantizers of the Brownian motion on $[0,1]$ for $N = 10, 48$ and for the "record value" of $N = 100$ that is $N_{\rm rec} = 96$.

N	$N_{\rm rec}$	Quantiz. Error	$N_{\rm rec}$ Decomposition
1	1	0.7071	1
10	10	0.3138	5 – 2
100	96	0.2264	12 – 4 – 2
1 000	966	0.1881	23 – 7 – 3 – 2
10 000	9 984	0.1626	26 – 8 – 4 – 3 – 2 – 2
100 000	97 920	0.1461	34 – 10 – 6 – 4 – 3 – 2 – 2

Table 1. *Brownian motion: Some typical "record" values for numerical implementations*

Figure 2 shows the graphs of both $N \mapsto \|W - \widehat{W}^{\chi^N_{\rm rec}}\|_2^2$ and $N \mapsto \|W - \widehat{W}^{\chi^N_{\rm opt}}\|_2^2$ for $N \in \{1, \ldots, 1\,000\}$. Figure 3 depicts $N \mapsto \|W - \widehat{W}^{\chi^N_{\rm rec}}\|_2^{-2}$ in a log-log scale which emphasizes the $\log N$ behaviour of the distortion. The coefficients obtained by a linear regression yield

$$\|W-\widehat{W}^{\chi^N_{\rm opt}}\|_2^{-2} \approx 4\log N+2 \quad i.e. \quad \|W-\widehat{W}^{\chi^N_{\rm rec}}\|_2^2 \approx \frac{0.25}{\log N + 0.5},\ 1 \le N \le 10\,000.$$

The lower and upper bounds provided by (4.28) and (4.27) respectively are on $[0, 1]$,

$$\frac{1}{\pi^2}\frac{2^2}{2^{2-1}(2-1)} = \frac{2}{\pi^2} \approx 0.2026 < \ \mathbf{0.25} \ < 1.2040 \approx \frac{1}{\pi^2}(1 + 2\pi\sqrt{3}). \quad (5.37)$$

confirm the above bound (5.37).

Firstly, we shortly describe the operating optimization procedure to obtain optimal product quantizers of size at most N for every $N \le N_{\max}$. Then, we propose a Romberg like extrapolation method which speeds up the convergence of the functional quantization method so that it produces very accurate results for moderate values of N, say less than 10 000. We include it in this section since it still relies on some conjectures concerning F-K product quantizers.

5.2 The Romberg log-extrapolation

For convenience we will only consider the case of the Brownian motion since we have a sharp rate for its quadratic quantization error rate, but the method works with any other process for which such a result holds. Let $\Psi : (L^2_T, |\,.\,|_{L^2_T}) \to \mathbb{R}$ be a three times differentiable functional such that $\mathrm{D}^2\Psi$ is bounded and Lipschitz. Let $(\chi^N)_{N\ge1}$ denote a sequence of rate optimal K-L product quantizers of the Brownian motion W and let $\widehat{W}^N := \widehat{W}^{\chi^N}$ denote the related Voronoi quantizations. First note that by Proposition 4 (stationarity)

$$\mathbb{E}(\mathrm{D}\Psi(\widehat{W}^N) \mid W - \widehat{W}^N)_{L^2_T} = \mathbb{E}(\mathrm{D}\Psi(\widehat{W}^N) \mid \mathbb{E}(W \mid \widehat{W}^N) - \widehat{W}^N)_{L^2_T} = 0.$$

Then, it follows from the Taylor formula that there is some bounded random vector ζ such that

$$\mathbb{E}(\Psi(W)) = \mathbb{E}(\Psi(\widehat{W}^N)) + \frac{1}{2}\mathbb{E}(\mathrm{D}^2\Psi(\widehat{W}^N).(W-\widehat{W}^N)^{\otimes 2}) + \mathbb{E}(\zeta.(W-\widehat{W}^N)^{\otimes 3}). \tag{5.38}$$

The sequence $(\chi^N)_N$ being rate optimal $\mathbb{E}(\mathrm{D}^2\Psi(\widehat{W}^N).(W-\widehat{W}^N)^{\otimes 2}) = O\left((\log N)^{-1}\right)$. However, recent finite dimensional results (see [1], Theorem 6) as well as several numerical experiments suggest a true expansion: this means conjecturing the existence of a real constant $\kappa_\Psi > 0$ such that

$$\mathbb{E}(\mathrm{D}^2\Psi(\widehat{W}^N).(W-\widehat{W}^N)^{\otimes 2}) = 2\,\kappa_\Psi(\log N)^{-1} + o((\log N)^{-1}) \ \text{ as } \ N \to \infty. \tag{5.39}$$

If one also assumes that $\mathbb{E}|W-\widehat{W}^N|^3_{L^2_T} = o((\log N)^{-1})$ (which also holds as a conjecture), then a speeding up Romberg log-extrapolation can be implemented as follows: one computes $\mathbb{E}(\Psi(\widehat{W}^M))$ and $\mathbb{E}(\Psi(\widehat{W}^N))$ for some $M = M(N) < N$, $M(N) \asymp N^r$, $r \in (0,1)$. Then solving the linear system

$$\mathbb{E}(\Psi(W)) = \mathbb{E}(\Psi(\widehat{W}^M)) + \frac{\kappa_\Psi}{\log M} + o((\log M)^{-1}),\ \mathbb{E}(\Psi(W)) = \mathbb{E}(\Psi(\widehat{W}^N)) + \frac{\kappa_\Psi}{\log N} + o((\log N)^{-1})$$

yields the *Romberg* log-*extrapolation* formula

$$\mathbb{E}(\Psi(W)) = \frac{\log N \times \mathbb{E}(\Psi(\widehat{W}^N)) - \log M \times \mathbb{E}(\Psi(\widehat{W}^M))}{\log N - \log M} + o\left((\log N)^{-1}\right). \tag{5.40}$$

So we passed from a $O((\log N)^{-1})$-rate to an $o\left((\log N)^{-1}\right)$-rate. The conjecture (supported by numerical simulations not reproduced here) concerning $\mathbb{E}|W-\widehat{W}^N|^3_{L^2_T}$ is that a $o\left((\log N)^{-(\frac{3}{2}-\varepsilon)}\right)$, $\varepsilon > 0$, rate holds. In fact very little is known on the $L^{p'}$-quantization error induced by L^p-optimal quantizers when $p' > p$ even in 1-dimension. If the same rate holds in (5.39), then a $o\left((\log N)^{-(\frac{3}{2}-\varepsilon)}\right)$-rate holds in (5.40).

An alternative to this approach can be to replace $\log N$ by $\|W-\widehat{W}^{\chi^N_{\mathrm{opt}}}\|_2^{-2}$ in (5.40) (and *idem* for M), as suggested by B. Wilbertz [23]. Some tests in [23] show that it often improves the accuracy of the extrapolation and has a stabilizing effect on the choice of the couples (N, M).

6 Numerical experiments using a Heston stochastic volatility model

In this section we use the functional quantization based quadrature formulæ to price vanilla Calls and Asian Calls in a Heston model. This is a stochastic volatility model introduced by Heston in 1993 (see [7]) in which the squared

volatility process is driven by a CIR process. Namely, the dynamics of the "risky" asset price process is given by

$$\left\{\begin{array}{lcl} dS_t & = & S_t(r\,dt+\sqrt{v_t})\,dW_t^1, \qquad S_0=s_0>0,\\ *[.4em]dv_t & = & k(a-v_t)dt+\vartheta\sqrt{v_t}\,dW_t^2, \quad v_0>0, \quad \text{with} \quad <W^1,W^2>_t=\rho\,t,\ \rho\in[-1,1], \end{array}\right.$$

where r denotes the (constant) interest rate and (v_t) denotes the square stochastic volatility process and a, k, ϑ are non-negative real parameters. The equation for the (v_t) has a unique (strong) pathwise continuous solution living in $\mathbb{R}_+$ (see *e.g.* [10] and [9], p.235). Note that in this section, we emphasize the numerical aspects and that in many situations we have no proof yet to support rigorously the results that we observe. This is due to the fact that the volatility process has a non Lipschitz diffusion coefficient. In particular the assumptions required in [13] to get some error rates for functional quantization of diffusion are not satisfied (except in some special cases as pointed out below).

6.1 A benchmark: pricing vanilla options in a Heston model

The pricing of vanilla calls and puts is simply a benchmark to evaluate the efficiency of the method since a quasi-closed form for their premium is available (based on an FFT). It involves some integrals of the characteristic function the couple $(v_t, \int_0^t v_s ds)$ for which a true closed form is available (see *e.g.* [10] or [7]). We use it to compute the reference premia in our experiments (its approximate accuracy is 10^{-2}, see [7]). Our aim is to price by functional quantization (at time 0) European Calls on the underlying asset (S_t) with strike price K and maturity $T>0$, *i.e.*

$$\text{Call}^{Hest}(S_0,K,r)=e^{-rT}\mathbb{E}((S_T-K)_+).$$

As a first step, we follow an approach which works for more general dynamics of the stochastic volatility. First we project W^1 onto W^2 so that

$$W_t^1=\rho W_t^2+\sqrt{1-\rho^2}\,\widetilde{W}_t^1,$$

with $\widetilde{W}^1$ a standard Brownian motion independent of W^2. Then, Itô calculus shows that

$$S_t=s_0\exp\left(-\frac{\rho^2}{2}\bar{v}_t\,t+\rho\int_0^t\sqrt{v_s}dW_s^2\right)\exp\left((r-\frac{1-\rho^2}{2}\bar{v}_t)t+\sqrt{1-\rho^2}\int_0^t\sqrt{v_s}d\widetilde{W}_s^1\right)$$

with $\bar{v}_t=\frac{1}{t}\int_0^t v_s ds$. Consequently, using the independence of $\widetilde{W}^1$ and W^2, one derives that

$$\begin{array}{rcl} \text{Call}^{Hest}(S_0,K,r,v_0,T) & = & \mathbb{E}\left(e^{-rT}\mathbb{E}\left((S_T-K)_+\,|\,\mathcal{F}_T^{W^2}\right)\right)=\mathbb{E}\left(\text{Call}_{BS}\left(S_0^{(v)},K,r,\left((1-\rho^2\right.\right.\right.\\ *[.4em]\text{with} \qquad S_0^{(v)} & = & s_0\exp\left(-\frac{\rho^2}{2}\bar{v}_T T+\rho\int_0^T\sqrt{v_s}dW_s^2\right) \end{array}$$

where $\mathrm{Call}_{BS}(s_0, K, r, \sigma, T)$ denotes the regular (r, σ, T)-Black-Scholes model premium function (for *vanilla* calls). Then the equation satisfied by (v_t) yields([1])

$$\int_0^t \sqrt{v_s} dW_s^2 = \frac{v_t - v_0 - kt(a - \bar{v}_t)}{\vartheta} \tag{6.41}$$

so that finally

$$\mathrm{Call}^{Hest}(S_0, K, r, v_0, T) \quad = \quad \mathbb{E}\left(\Phi_c(v_{_T} - v_0, \bar{v}_{_T})\right) \tag{6.42}$$

with $$\Phi_c(v, \bar{v}) = \mathrm{Call}_{BS}\left(s_0 \exp\left(\rho T\left[\left(\frac{k}{\vartheta} - \frac{\rho}{2}\right)\bar{v} + \frac{v}{T\vartheta} - \frac{ka}{\vartheta}\right]\right), K, r, \left((1-\rho^2)\bar{v}\right)^{\frac{1}{2}}, T\right).$$

An analogous formula holds for $\mathrm{Put}_{Hest}(S_0, K, r, v_0, T)$ by replacing *mutatis mutandis* Call_{BS} by Put_{BS} in (6.42). Then function Φ_c is $\mathcal{C}^\infty$ on $(0, +\infty)^2$. Note that when $\rho = 0$, (6.42) only depends on the L^2-continuous linear functional $\bar{v}_{_T}$.

Following the quantization procedure described in section 4.3, we set $b(v) = -k(v - a)$ and $\sigma(v) = \vartheta\sqrt{v}$. Unfortunately, since the function σ is non-Lipschitz at 0, we cannot rigorously claim from [13] that solutions of the the integral system (4.36) produce a rate optimal sequence (y^N) for (v_t). However, we will see (when $\frac{\vartheta^2}{4k} < a$) that it produces satisfactory numerical results.

A Setting: $a = \frac{\vartheta^2}{4k}$. This special setting will make possible to investigate the efficiency of functional quantization for smooth functionals since, in this setting, the solutions of the integral equation (4.36) can be made explicit. Hence, there is no error due to the time discretization scheme of (4.36).

So, in some way it is more illustrative of the numerical performances of functional quantization. This follows from a fact pointed out by Rogers in [21]: one may assume without loss of generality that the process (v_t) is the square of a scalar Ornstein-Uhlenbeck process

$$dX_t = -\frac{k}{2}X_t dt + \frac{\vartheta}{2}dW_t^2, \qquad X_0 = \sqrt{v_0}. \tag{6.43}$$

Having in mind that the N-quantizers $\chi^N = \sqrt{\lambda} \otimes x \in \mathcal{O}_{pq}(W, N)$ given by (4.21) read

$$\chi_{\underline{i}}^N(t) = \sqrt{\frac{2}{T}} \sum_{n \geq 1} x_{i_n}^{(N_n)} \frac{T}{\pi(n - 1/2)} \sin\left(\pi(n - 1/2)\frac{t}{T}\right), \qquad \underline{i} = (i_1, \ldots, i_n, \ldots) \in \prod_{n \geq 1} \{1, \ldots, N_n\},$$

[1] The key point in what follows is to express the stochastic integral $\int_0^t \sqrt{v_s} dW_s^2$ as a functional of v_t, v_0 and an integral functional of (v_s). If the variance process follows a general diffusion process $dv_t = b(v_t)dt + \vartheta(v_t)dW_t^2$ then one may apply under appropriate regularity assumption, Itô's formula to the function $\varphi(v) := \sqrt{v}/\vartheta(v)$ to get such an expression.

the integral system (4.36) associated to X, namely

$$x_{\underline{i}}(t) = \sqrt{v_0} - \frac{k}{2}\int_0^t x_{\underline{i}}(s)\,ds + \frac{\vartheta}{2}\chi^N_{\underline{i}}(t), \qquad \underline{i} = (i_1,\ldots,i_n,\ldots) \in \prod_{n\geq 1}\{1,\ldots,N_n\}, \tag{6.44}$$

has a closed form given by

$$x^N_{\underline{i}}(t) \;=\; e^{-kt/2}\sqrt{v_0} + \frac{\vartheta}{2}\sum_{n\geq 1} x^{(n)}_{i_n}\,\widetilde{c}_n\,\varphi_n(t) \qquad \text{with} \qquad \widetilde{c}_n := \frac{T^2}{(\pi(n-1/2))^2 + (kT/2)^2}$$

$$\text{and} \qquad \varphi_n(t) \;:=\; \sqrt{\frac{2}{T}}\left(\frac{\pi}{T}(n-1/2)\sin\left(\pi(n-1/2)\frac{t}{T}\right) + \frac{k}{2}\left(\cos\left(\pi(n-1/2)\frac{t}{T}\right) - e^{-kt/2}\right)\right).$$

Then, following [13], we have for every $p\in[1,2)$,

$$\|\widetilde{X}^N - X\|_p \leq C_{p,k,\vartheta,T}\|\widehat{W^2}^{\chi^N} - W^2\|_2 = O\left((\log N)^{-\frac{1}{2}}\right) \tag{6.45}$$

where $\widetilde{X}^N$ is the non-Voronoi quantization defined by

$$\widetilde{X}^N_t \;=\; \sum_{i=1}^N x^N_{\underline{i}}(t)\mathbf{1}_{C_{\underline{i}}(\chi^N)}(W^2) = e^{-kt/2}\sqrt{v_0} + \frac{\vartheta}{2}\sum_{n\geq 1}\widehat{\xi}^{x^{(n)}}_n\,\widetilde{c}_n\,\varphi_n(t), \quad t\in[0,T].$$

One designs a (non-Voronoi) N-quantization for the process (v_t) by setting

$$\widetilde{v}^N_t = (\widetilde{X}^N_t)^2 = \sum_{\underline{i}}(x^N_{\underline{i}}(t))^2\mathbf{1}_{C_{\underline{i}}(\chi^N)}(W^2). \tag{6.46}$$

Then, one derives from (6.45) that, for every $p\in[1,2]$,

$$\|\,|\widetilde{v}^N - v|_{L^2_T}\,\|_p = O\left((\log N)^{-(\frac{1}{2}-\varepsilon)}\right) \quad \text{for every } \varepsilon>0. \tag{6.47}$$

Finally, a first approximation of $\mathrm{Call}^{Hest}(S_0,K,r,v_0,T)$ is based on (6.42)

$$\begin{aligned}\widehat{\mathrm{CrCall}}^{Hest}(s_0,K,T,v_0,r) \;&:=\; \mathbb{E}(\Phi_c(\widetilde{v}_{_T} - v_0, \overline{\widetilde{v}}_{_T})) \qquad (6.48)\\ &=\; \sum_{\underline{i}}\Phi_c\left((x^N_{\underline{i}})^2(T) - v_0, \overline{(x^N_{\underline{i}})^2}(T)\right)\mathbb{P}(\widehat{W}^2 = \chi^N_{\underline{i}})\end{aligned}$$

where the probability distribution $(\mathbb{P}(\widehat{W}^2 = \chi^N_{\underline{i}}))_{\underline{i}}$ is given by (4.25). The notation "Cr" is for "Crude".

The Call-Put parity equation provides a second proxy of the Call by setting

$$\widehat{\mathrm{PrCall}}^{Hest}(s_0,K,T,v_0,r) := s_0 - e^{-rT}K + \mathbb{E}(\Phi_p(\widetilde{v}_{_T} - v_0, \overline{\widetilde{v}}_{_T})). \tag{6.49}$$

When $\rho\neq 0$, no error bound is available for these proxies since we do not know the rate of pointwise quantization of $v_{_T}$ by the quadratic functional quantizer $\widetilde{v}_{_T}$.

When $\rho = 0$, $\bar{\Phi}_c(v_{_T} - v_0, \bar{v}_{_T}) = \bar{\Phi}_c(0, \bar{v}_{_T})$ does not depend on $v_{_T}$ and is clearly Lipschitz in $\bar{v}_{_T}$, so the theoretical rate of convergence is $O\left((\log N)^{-(\frac{1}{2}-\varepsilon)}\right)$ (without any acceleration techniques).

As concerns the Romberg log-extrapolation, one notices that both functions $\sigma \mapsto \mathrm{Call}_{BS}(s_0, K, r, \sigma, T)$ and its Put counterpart are infinitely differentiable on $(0, +\infty)$ and $u \mapsto \bar{u}_{_T} := \frac{1}{T}\int_0^T u(s)ds$ is an $L^2_{_T}$-continuous linear functional. On the other hand, the solution of the integral equation $x(t) = x(0) - \frac{k}{2}\int_0^t x(s)ds + \frac{\vartheta}{2}\,\xi(t)$ is also an $L^2_{_T}$-continuous linear functional functional of ξ. Consequently, one may write

$$\mathrm{Call}^{Hest}(s_0, K, T, r) = \mathbb{E}(\Psi_c(W^2)) = s_0 - e^{-rT}K + \mathbb{E}(\Psi_p(W^2))$$

where Ψ_p and Ψ_c are infinitely differentiable and Ψ_p is bounded with all its differentials which suggests a favourable framework to implement the Romberg log-extrapolation.

Following the results of former experiments carried out with the Asian option in a Black & Scholes model (see [20]) we compute time integrals by the midpoint method with $n = 20$, *i.e.*

$$\overline{(x_{\underline{i}}^N)^2} = \frac{1}{T}\int_0^T (x_{\underline{i}}^N(s))^2 ds \approx \frac{1}{n}\sum_{k=1}^{n}(x_{\underline{i}}^N(t_k))^2 \quad \text{with} \quad t_k = \frac{(2k-1)T}{2n}.$$

The set of parameters of the Heston model is specified as follows

$$s_0 = 50, \quad r = 0.05, \quad T = 1, \quad \rho = 0.5, \quad v_0 = a = 0.01, \quad \vartheta = 0.1, \quad k = 0.25.$$

Note that $\mathbb{E}\, v_t = a$, $t \in [0, T]$ (and $a = \vartheta^2/(4k)$).

We carried out our numerical experiments on a whole vector of strike prices $K \in \{44, 45, \ldots, 55, 56\}$ (with step 1) to evaluate the performances of the method for in-, at- and out-of-the-money options. The Heston Call premium vector were computed using:

– a "crude" FQ integration using $\widehat{\mathrm{CrCall}}^{Hest}(s_0, K)$ given by formula (6.48) with optimal product N-quantizer of sizes $N = N_{\mathrm{rec}} = 96, 966, 9\,984, 97\,920$.

– a Romberg log-extrapolation $\widehat{\mathrm{RbgCrCall}}^{Hest}(s_0, K)$ based on (5.40) for the three couples $(N, M) = (96, 966)$, $(966, 9984)$ and $(9\,984, 97\,920)$. (Since $\rho \neq 0$, we have no theoretical evidence that it speeds up the convergence so this is just a numerical experiment).

– a K-linear interpolation method (which is purely numerical at this stage): the principle is to interpolate $\widehat{\mathrm{RbgCrCall}}^{Hest}(s_0, K)$ and its counterpart $\widehat{\mathrm{RbgPrCall}}^{Hest}(s_0, K)$ (obtained using the model-free Call-Put parity equation) by setting for every $K \in \{K_{\min}, \ldots, K_{\max}\}$

$$\widehat{\mathrm{IRCall}}^{Hest}(s_0, K) = \frac{(K - K_{\min})\widehat{\mathrm{RbgCrCall}}^{Hest}(s_0, K) + (K_{\max} - K)\widehat{\mathrm{RbgPrCall}}^{Hest}(s_0, K)}{K_{\max} - K_{\min}}.$$

Doing so we put proportionally more weight on the "less random" variable which tends to make the global error smaller. It means we put more weight on the Put when K is small, on the Call, when K is large. This is a purely heuristic approach ([2])

– Finally the variance of the Heston Calls were computed (by a Monte Carlo simulation) in order to compare the accuracy of the FQ approaches with respect to MC confidence intervals.

The implementation was achieved on a G5 (2.5 Ghz) Apple computer using MATLAB. The results are reported in Tables 2 below (relative errors) and in Figure 8 (absolute errors).

Our comments on these first set of results are the following: the rate of the "Crude FQ" approach is $O(1/\log(N))$ in accordance with the theoretical rate (*e.g.* passing from $N = 96 \approx 100$ to $N = 9\,998 \approx 100^2$ divides the error by two). But its absolute accuracy is not sufficient for financial applications (see Figure 8(a)).

On the other hand, the results obtained by the Romberg log-extrapolation method and the (induced) K-interpolation method are outstanding: *for both tested couples the error lies within* 0.5 *cent* (see Figure 8(b)-(c)). For the couple $(M, N) = (966, 9\,984)$ the error induced is lower than 0.2 cent by Romberg log-extrapolation and lower than 0.1 cent by K-interpolation for all strike prices (note that the differences with the reference price in Table 2 below are only due to rounding effect to the nearest cent). This can be considered as a good indication on the conjecture (5.39). On this very example the K-interpolation only yields a slight improvement. Its main asset is in fact its robustness with respect to time discretization (as emphasized in further simulations). From a theoretical viewpoint, these results plead in favour of the existence of higher order term at rate $O((\log N)^{-3/2})$ in the expansion (5.40). Furthermore, a comparison between Figures 8(b) and 8(c) shows the expected effect of the K-linear interpolation of Romberg log-extrapolations on the values of the *absolute* errors of the Heston Call for small values of the strike K (deep in-the-money options), especially in a rough spatial discretization setting ($(M, N) = (96, 966)$).

As concerns the comparison with a "*crude*" Monte Carlo method, we reported in Table 2 below (third row) $2 \times Std_{_N}$, where $Std_{_N}$ denotes the (relative) standard deviation of a MC estimator (for $N = 10\,000$). This quantity defines its 95.5%-confidence interval. Note that by "crude" MC, we simply mean a standard MC estimator without any variance reduction techniques.

One verifies that $2 \times Std_{10\,000}$ is slightly higher (say 10 to 30%) than the

[2] In a standard Monte Carlo method if two r.v. $X \neq X'$ have the same expectation m there is an optimal way to to compute m by considering independent copies of $\lambda X + (1-\lambda)X'$, with $\lambda = \mathbb{E}((X'-X)X')/\mathbb{E}(X-X')^2$. We have no reason to do so in a FQ approach so we adopted the linear interpolation.

relative error induced by "crude" FQ-integration with $N = 9\,984$ (within brackets, fourth row).

In terms of velocity, computing the whole premium vector (13 strike prices) by functional quantization for $(M, N) = (966, 9\,984)$ including the Romberg log-extrapolation and the K-interpolation takes (with $n = 20$) less than 0.5 second (this outperforms any Monte Carlo simulation but not the "reference" Inverse Fourier Transform method (second row) but this is not our aim at this stage since we use this setting as a benchmark).

K	44	45	46	47	48	49	
Heston Call(Ref)	**8.18**	**7.26**	**6.36**	**5.49**	**4.68**	**3.93**	
"Crude" Monte Carlo ($2 \times Std_{10\,000}$)	(0.64%)	(0.72%)	(0.82%)	(0.94%)	(1.08%)	(1.26%)	(1
"crude" FQ ($\widehat{\text{CrCall}}^{Hest}(s_0, K)$) $N = 9\,984$	8.14 (0.50%)	7.21 (0.57%)	6.31 (0.67%)	5.45 (0.79%)	4.64 (0.94%)	3.89 (1.11%)	(1
Romberg on "crude" FQ ($\widehat{\text{RbgCrCall}}^{Hest}(s_0, K)$) (M,N)=(966-9984)	**8.18** (0.07%)	**7.25** (0.07%)	**6.36** (0.11%)	**5.49** (0.13%)	**4.68** (0.16%)	**3.93** (0.20%)	(0
K**-interpol. of Romberg** FQ ($\widehat{\text{IRCall}}^{Hest}(s_0, K)$) (M,N)=(966-9984)	**8.18** (0.00%)	**7.26** (0.00%)	**6.36** (0.02%)	**5.49** (0.05%)	**4.68** (0.08%)	**3.93** (0.11%)	(0

K	51	52	53	54	55	56
Heston Call(Ref)	**2.68**	**2.18**	**1.765**	**1.42**	**1.14**	**0.91**
"Crude" Monte Carlo ($2 \times Std_{10\,000}$)	(1.66%)	(1.90%)	(2.14%)	(2.42%)	(2.70%)	(3.02%)
"crude" FQ ($\widehat{\text{CrCall}}^{Hest}(s_0, K)$) $N = 9\,984$	2.64 (1.53%)	2.14 (1.77%)	1.73 (2.01%)	1.39 (2.21%)	1.11 (2.38%)	0.89 (2.61%)
Romberg on "crude" FQ ($\widehat{\text{RbgCrCall}}^{Hest}(s_0, K)$) (M,N)=(966-9984)	**2.68** (0.28%)	**2.18** (0.32%)	**1.765** (0.33%)	**1.42** (0.29%)	**1.14** (0.18%)	**0.91** (0.10%)
K**-interpol. of Romberg** FQ ($\widehat{\text{IRCall}}^{Hest}(s_0, K)$) (M,N)=(966-9984)	**2.68** (0.19%)	**2.18** (0.23%)	**1.765** (0.23%)	**1.42** (0.22%)	**1.14** (0.14%)	**0.91** (0.10%)

Table 2. *A setting: Relative Standard deviation of the* 10 000 *Monte Carlo estimator, Heston Call by "Crude" Functional Quantization ($N = 9\,984$), Romberg* log*-extrapolation ($(M, N) = (966, 9\,984)$) and K-linear interpolation.*

Concerning the behaviour of the method with other sets of parameters, numerical experiments not reproduced here (see [20]) show that the smaller the correlation ρ is (in absolute value), the more efficient functional quantization is (the impact of the non L^2_T-continuous pointwise functional v_T decreases). Other experiments not reproduced here either show that, as expected, the error increases (for both FQ and MC) as the volatility ϑ of the volatility process grows, but remains quite satisfactory until $\vartheta = 30\,\%$ (when $a = 0.01$).

B SETTING: $\vartheta^2/(4k) < a$. Any solution (v_t) of (6.41) is positive and, once again an adaptation of the proof of Theorem 1 in [13] would show that $\| \, |v - \widetilde{v}^N|_{L^2_T} \, \|_2 = O((\log N)^{-\frac{1}{2}+\varepsilon})$. This time any numerical implementation of this functional quantization method requires to discretize the integral

system (4.36) for the process $(v_t)_t$ defined by the second equation in (6.41). This means setting $b(v) = k(a - v)$ and $\sigma(v) = \vartheta\sqrt{v}$. We implemented a (slightly non-homogeneous) Euler scheme defined by: $y_{\underline{i}}^{n,N}(0) = v_0$,

$$y_{\underline{i}}^{n,N}(t_{k+1}) = y_{\underline{i}}^{n,N}(t_k) + k\left(a - \frac{\vartheta^2}{4k} - y_{\underline{i}}^{n,N}(t_k)\right)(\Delta t)_k + \vartheta\sqrt{y_{\underline{i}}^{n,N}(t_k)}(\chi_{\underline{i}}^{N}(t_{k+1}) - \chi_{\underline{i}}^{N}(t_k)), \tag{6.50}$$

$k = 0, \ldots, n$, with $t_0 = 0$, $t_k = (2k-1)T/(2n)$, $k = 1, \ldots, n$, $t_{n+1} = T$ and $(\Delta t)_k = t_k - t_{k-1}$. Then, one sets

$$y_{\underline{i}}^{n,N}(t) = y_{\underline{i}}^{n,N}(t_k), \quad t \in ((k-1)T/n, kT/n),\ k = 1, \ldots, n.$$

This scheme is designed to optimize the computation of the integral below by a midpoint method. Then, one designs a (non-Voronoi) N-quantization for the process (v_t) by setting

$$\widetilde{v}_t^{n,N} = \sum_{\underline{i}} y_{\underline{i}}^{n,N}(t)\mathbf{1}_{C_{\underline{i}}(\chi^N)}(W^2). \tag{6.51}$$

The integral $\frac{1}{T}\int_0^T \widetilde{v}_t dt$ is then approximated by

$$\frac{1}{T}\int_0^T \widetilde{v}_t^{n,N} dt = \frac{1}{n}\sum_{k=1}^{n} \widetilde{v}^{n,N}(t_k).$$

Since the functions $y_{\underline{i}}^N$ are positive and the functions b, σ and χ^N are smooth on $(0, +\infty)$, it is classical background in numerical analysis of ODE that

$$y_{\underline{i}}^N - y_{\underline{i}}^{n,N} = \frac{\zeta_{\underline{i}}^N}{n} + O_{\underline{i}}(1/n^2)$$

where $\zeta_{\underline{i}}^N$ satisfies an ODE involving b, σ and their higher order derivatives. This expansion holds for the uniform convergence on compact sets.

This suggests to implement a Romberg time extrapolation to speed up the convergence of Euler scheme. More precisely, some standard computations based on a Taylor expansion of Φ_c show that

$$2\,\Phi_c(y_{\underline{i}}^{2n,N} - v_0, \overline{y}_{\underline{i}}^{2n,N}) - \Phi_c(y_{\underline{i}}^{n,N} - v_0, \overline{y}_{\underline{i}}^{n,N}) = \Phi_c(y_{\underline{i}}^{N} - v_0, \overline{y}_{\underline{i}}^{N}) + \frac{C_n}{n^2} \tag{6.52}$$

with $\sup_n \mathbb{E}(C_n) < +\infty$.

The first tested method is to estimate the premium of the Heston Call by a Romberg log-extrapolation (5.40) of the expectation of the left hand side of the above equation (6.52), still denoted $\widehat{\text{RbgCrCall}}^{Hest}(s_0, K)$. We used the same size couples (M, N) and introduced the same K-interpolated estimator as in A setting (but the "crude" FQ approach was no longer tested given the results of A setting). For numerical tests we set

$$s_0 = 50, \quad k = 2, \quad r = 0.05, \quad T = 1, \quad \rho = 0.5, \quad v_0 = a = 0.01, \quad \vartheta = 0.1.$$

The time discretization of the integral system was processed with $2n = 64$. This is finer than in A setting but this time we do not simply evaluate some functions at the discretization times, we discretize an integral system.

K	44	45	46	47	48	49
Call^{Hest}(FFT Ref. premium)	**8.18**	**7.26**	**6.38**	**5.53**	**4.73**	**3.99**
"Crude" Monte Carlo ($2 \times Std_{10\,000}$)	(0.35%)	(0.40%)	(0.51%)	(0.57%)	(0.63%)	(0.68%)
Romberg on "crude" FQ ($\widehat{\text{RbgCrCall}}^{Hest}(s_0,K)$) (M,N)=(966-9984)	**8.18** (0.00%)	**7.25** (0.00%)	**6.38** (0.00%)	**5.53** (0.00%)	**4.73** (0.00%)	**3.99** (0.00%)
K**-interpol. of Romberg** FQ ($\widehat{\text{IRCall}}^{Hest}(s_0,K)$) (966-9984)	**8.18** (0.00%)	**7.26** (0.00%)	**6.38** (0.00%)	**5.53** (0.00%)	**4.73** (0.00%)	**3.99** (0.00%)

K	51	52	53	54	55	56
Call^{Hest}(FFT Ref. premium)	**2.74**	**2.23**	**1.795**	**1.43**	**1.13**	**0.89**
"Crude" Monte Carlo ($2 \times Std_{10\,000}$)	(0.75%)	(0.75%)	(0.77%)	(0.77%)	(0.77%)	(0.77%)
Romberg on "crude" FQ ($\widehat{\text{RbgCall}}^{Hest}(s_0,K)$) (966-9984)	**2.74** (0.00%)	**2.23** (0.00%)	**1.795** (0.00%)	**1.43** (0.01%)	**1.13** (0.04%)	**0.89** (0.11%)
K**-interpol. of Romberg** FQ ($\widehat{\text{IRCall}}^{Hest}(s_0,K)$) (966-9984)	**2.74** (0.01%)	**2.23** (0.04%)	**1.795** (0.11%)	**1.43** (0.17%)	**1.13** (0.27%)	**0.89** (0.41%)

Table 3. *B setting: Relative Standard deviation of the "crude"* 10 000 *Monte Carlo estimator, Heston Call by Romberg* log*-extrapolation and* K*-linear interpolation with* $(M,N) = (966, 9\,984)$.

Additional information of interest are the following:

$$\text{CPU Time} < 0.8\,s, \qquad \text{mean error (over } K) = 5.10^{-4}.$$

The results reported in Table 3 (relative errors) and Figure 5 (absolute errors) confirm those obtained in the special A setting. Again the K-linear interpolation gives better results for lower values of K (deep in-the-money options) as expected (see Figure 5(b)).

6.2 Pricing "Heston" Asian Call by functional quantization

The Asian Call premium is defined by

$$\text{AsCall}^{Hest} = e^{-rT}\mathbb{E}\left(\left(\frac{1}{T}\int_0^T S_s ds - K\right)_+\right).$$

Note that here the functional is not smoother than Lipschitz and that no closed form is available for this option. We only consider the more general case $\vartheta^2/(4k) < a$. We adopt the same notations as for the vanilla Call.

First, we approximate the temporal mean by a midpoint quadrature formula *i.e.*

$$\frac{1}{T}\int_0^T S_s ds \approx \frac{1}{n}\sum_{k=1}^{n} S_{t_k}$$

where $t_k = (2k-1)T/(2n)$, $k = 1, \ldots, n$. Following (6.41), we recall that for every $t \in [0,T]$,

$$\begin{aligned} S_t &= s_0 \exp\left((r - \frac{1}{2}\bar{v}_t)t + \rho \int_0^t \sqrt{v_s} dW_s^2\right) \exp\left(\sqrt{1-\rho^2} \int_0^t \sqrt{v_s} d\widetilde{W}_s^1\right) \\ &= s_0 \exp\left(t\left((r - \frac{\rho a k}{\vartheta}) + \bar{v}_t(\frac{\rho k}{\vartheta} - \frac{1}{2})\right) + \frac{\rho}{\vartheta}(v_t - v_0)\right) \exp\left(\sqrt{1-\rho^2} \int_0^t \sqrt{v_s} d\widetilde{W}_s^1\right). \end{aligned}$$

We start from the chaining rule for conditional expectations to compute this premium *i.e.*

$$\mathrm{AsCall}^{Hest}(s_0, K) = e^{-rT}\mathbb{E}\left(\mathbb{E}\left(\left(\frac{1}{T}\int_0^T S_s ds - K\right)_+ | \sigma((v_t)_{0\le t\le T})\right)\right). \tag{6.53}$$

Then one sets

$$\widehat{S}_t^{n,N} = \sum_{\underline{i},\underline{j}} s_{\underline{i},\underline{j}}^{n,N}(t) \mathbf{1}_{\chi_{\underline{i}}^N}(\widetilde{W}^1) \mathbf{1}_{\chi_{\underline{j}}^N}(W^2) \tag{6.54}$$

where the multi-indices $\underline{i}$ and $\underline{j}$ run over $\prod_{k\ge 1}\{1, \ldots, N_k\}$ and

$$s_{\underline{i},\underline{j}}^{n,N}(t) = s_0 \exp\left(t\left((r - \frac{\rho a k}{\vartheta}) + \bar{y}_{\underline{j}}^{n,N}(t)(\frac{\rho k}{\vartheta} - \frac{1}{2})\right) + \frac{\rho}{\vartheta}(y_{\underline{j}}^{n,N}(t) - v_0)\right) \exp\left(\sqrt{1-\rho^2} \int_0^t \sqrt{y_{\underline{j}}^{n,N}} d\chi_{\underline{i}}^N\right)$$

where $y_{\underline{j}}^{n,N}$ are obtained as in (6.50). The rest of the procedure is quite similar to that implemented for the Heston vanilla Call: "crude" functional quantization approach directly based on (6.53) and (6.54) (and a time Romberg extrapolation). Indeed we set for a given time discretization size n

$$\begin{aligned} \widehat{\mathrm{CrAsCall}}^{Hest}(s_0, K) &= 2\,\mathbb{E}(\Phi_{as}(s_0, K, 2n, \chi^N)) - \mathbb{E}(\Phi_{as}(s_0, K, n, \chi^N)), \\ \text{where} \quad \Phi_{as}(s_0, K, n, \chi^N) &= e^{-rT}\mathbb{E}\left(\left(\frac{1}{T}\int_0^T \widehat{S}_t^{n,N}\, dt - K\right)_+ | \mathcal{F}_T^{\widehat{W}^2}\right) \\ &= e^{-rT} \sum_{\underline{i},\underline{j}} \left(\frac{1}{T}\int_0^T s_{\underline{i},\underline{j}}^{n,N}(t) dt - K\right)_+ \mathbb{P}(\widetilde{W}^1 \in C_{\underline{i}}(\chi^N)) \mathbf{1}_{\chi_{\underline{j}}^N}(W^2) \\ \text{so that } \mathbb{E}(\Phi_{as}(s_0, K, n, \chi^N)) &= e^{-rT} \sum_{\underline{i},\underline{j}} \left(\frac{1}{T}\int_0^T s_{\underline{i},\underline{j}}^{n,N}(t) dt - K\right)_+ \mathbb{P}(\widetilde{W}^1 \in C_{\underline{i}}(\chi^N)) \mathbb{P}(W^2 \in C_{\underline{j}}(\chi^N)). \end{aligned}$$

Like for Heston vanilla Calls, its poor (but expected) rate of convergence leads to introduce a space Romberg log-extrapolation $\widehat{\mathrm{RCrAsCall}}^{Hest}(s_0, K)$ and, finally, a K-linear interpolation $\widehat{\mathrm{IRAsCall}}^{Hest}(s_0, K)$ between $\widehat{\mathrm{RCrAsCall}}^{Hest}(s_0, K)$ and its counterpart $\widehat{\mathrm{RPrAsCall}}^{Hest}(s_0, K)$ resulting from the Asian Call-Put parity equation

$$\mathrm{AsCall}^{Hest}(s_0, K) - \mathrm{AsPut}(s_0, K) = s_0 \frac{1 - e^{-rT}}{rT} - K e^{-rT}.$$

The reference price was computed using a Monte Carlo simulation of size 10^8 (including a time Romberg extrapolation with $2n = 32$ and some variance reduction techniques).

In this framework the quantization procedure has a greater complexity since we have to sum over a bivariate functional product quantizer: namely the complexity of the quantizing procedure is approximately N^2 (since $M \ll N$), but this is partially balanced by the lower volatility of the Asian pseudo-asset $\frac{1}{t}\int_0^t S_s ds$ compared to the original traded asset (S_t). Nevertheless, the comparison with a crude MC simulation (through the computation of the standard deviation parameter) is made in this framework with respect to a 10^6-*MC estimator* (using the same time Romberg extrapolation but no variance reduction).

We evaluated the method with both couples $(M, N) = (96, 966)$ and $(M, N) = (966, 9\,984)$. The time discretization size is $2n = 32$. The results are reported in Table 4 below (relative errors for the first couple only) and are depicted in Figure 6 (absolute errors).

K	44	45	46	47	48	49
Heston Asian Call(10^8-MC Reference)	**6.92**	**5.97**	**5.03**	**4.11**	**3.245**	**2.46**
"Crude" Monte Carlo ($2 \times Std_{10^6}$)	(0.08%)	(0.10%)	(0.11%)	(0.14%)	(0.16%)	(0.20%)
Romberg on "crude" FQ ($\widehat{\text{RCrAsCall}}^{Hest}(s_0, K)$) (M,N)=(96-966)	6.92 (0.01%)	5.97 (0.04%)	5.03 (0.05%)	4.12 (0.09%)	3.25 (0.17%)	2.47 (0.32%)
K-interpol. of Romberg FQ ($\widehat{\text{IRAsCall}}^{Hest}(s_0, K)$) (M,N)=(96-966)	**6.92** (0.01%)	**5.97** (0.02%)	**5.03** (0.02%)	**4.11** (0.04%)	**3.24** (0.05%)	**2.46** (0.04%)

K	51	52	53	54	55	56
Heston Asian Call(MC Ref)	**1.25**	**0.84**	**0.54**	**0.34**	**0.21**	**0.125**
"Crude" Monte Carlo ($2 \times Std_{10^6}$)	(0.31%)	(0.39%)	(0.50%)	(0.63%)	(0.81%)	(1.04%)
Romberg on "crude" FQ ($\widehat{\text{RCrAsCall}}^{Hest}(s_0, K)$) (M,N)=(96-966)	1.26 (1.16%)	0.85 (2.06%)	0.56 (3.73%)	0.36 (6.58%)	0.23 (11.53%)	0.15 (19.96%
K-interpol. of Romberg FQ ($\widehat{\text{IRAsCall}}^{Hest}(s_0, K)$) (M,N)=(96-966)	**1.25** (0.17%)	**0.84** (0.37%)	**0.545** (0.78%)	**0.34** (1.37%)	**0.21** (2.15%)	**0.125** (2.84%)

Table 4. *B setting: Relative Standard deviation of the* 10^6*-Monte Carlo estimator, Heston Asian Call by Romberg* log*-extrapolation and the induced* K*-linear interpolation with* $(M, N) = (96, 966)$.

Additional information of interest are the following:

$$(M, N) = (96, 966), \qquad \text{CPU Time} < 4.4\,s, \qquad \text{mean quadratic error (over } K) = 7.51\,10^{-4}.$$

The main comments are the following: when $(M, N) = (96, 966)$ the Romberg log-extrapolated premia are not satisfactory (although the errors remain within 2 cents and decrease when the options goes out-of-the money so that the relative error remains bounded). On the other hand, the K-interpolation method produces premia with an error lying within 0.5 *cent* with respect to the Monte Carlo reference price. This is quite satisfactory

although this may induce non vanishing relative errors for deep out-of-the-money options (but in practice Asian options are dealt closer to the money than plain vanilla options). The functional quantization approach is fifteen times faster than a 10^6-MC simulation having exactly the same characteristics (in particular a time Romberg extrapolation with $2n = 32$): CPU time is 4.4 seconds with FQ with 13 strike prices *versus* 66 seconds for MC (on our device).

When $(M, N) = (966, 9\,984)$, the error induced by the Romberg log-extrapolation lies now within 0.5 cent. As concerns the K-interpolated premia one observes a little improvement (maximal error within 0.4 cent on Figure 6(b)) but the computation time is of no interest for application (close to 430 seconds $\approx 7\ mn$) but the mean accuracy remains similar ($6.35\ 10^{-4}$). We can see again the efficiency of the K-linear interpolation in the low discretization setting $(96, 966)$. It gives the same kind of precision as for $(966, 9\,984)$ (compare Figure 6(a)-(b)).

As a conclusion to this section, we would like to insist on the following fact: the quantizing sizes have been selected *a priori*, namely $N_{\rm rec}$ for $N = 100,\ 1\,000,\ 10\,000,\ 100\,000$. For every problem some parameter dependent couples (M, N) produce significantly more accurate results. We decided not to report these results which are not significant in view of operating applications. However, the next step is to carry on a systematic search for some possibly globally more performing couples (M, N). This requires to have more insight on the interaction between M and N in the log-Romberg extrapolation. Finally, we would like to emphasize that *we decided not to implement the FQ counterpart of any usual "control variate random variable"*. When implemented they do improve the results (see [20] for the case of the Asian option in Black-Scholes model). Note that what have been proposed here straightforwardly applies to the C.I.R. interest rate model.

7 FQ as a control variate variable in a FQ-MC method for non smooth functionals.

Numerical integration on the L^2_T-space by functional quantization turns out to performs surprisingly well as emphasized on the two formerly investigated option pricing problems. It provides very accurate deterministic proxies for medium values of N, say $N \approx 10\,000$. However, in both cases the underlying functionals had some regularity on L^2_T. For less regular functionals (like indicator functions, etc) it can be interesting is to use numerical FQ for small values of N – say $N \approx 100$ – as a control variate random variable in a Monte Carlo simulation.

We will briefly outline this approach now. Let us consider the case of a functional $F(W)$ of the Brownian motion W (but what follows formally

applies too any Gaussian process with an explicit K-L expansion). In order to compute $\mathbb{E}(F(W))$, one writes

$$\begin{aligned}\mathbb{E}(F(W)) &= \mathbb{E}(F(\widehat{W}^N)) + \mathbb{E}\left(F(W) - F(\widehat{W}^N)\right) \\ &= \underbrace{\mathbb{E}(F(\widehat{W}^N))}_{(a)} + \underbrace{\frac{1}{M}\sum_{m=1}^{M} F(W^{(m)}) - F(\widehat{W^{(m)}}^N)}_{(b)} + R_{N,M} \qquad (7.55)\end{aligned}$$

where $(W^{(m)})_{m=1,\dots,M}$ are M independent copies of the standard Brownian motion and $R_{N,M}$ is a remainder term defined by (7.55). Term (a) is computed by quantization and Term (b) is computed by a Monte Carlo simulation of the K-L expansion of the Brownian motion. Then,

$$\mathbb{E}|R_{N,M}|^2_{L^2_T} \le \frac{\mathbb{E}|F(W) - F(\widehat{W}^N)|^2}{M} \quad \text{and} \quad \sqrt{M}\, R_{N,M} \xrightarrow{\mathcal{L}} \mathcal{N}(0; \|F(W) - F(\widehat{W}^N)\|_2)$$

as $M \to +\infty$ so that if F is simply a Lipschitz functional (*e.g.* like the payoff of the Asian Call in a Black-Scholes model) and if $(\widehat{W}^N)_{N\ge1}$ is a rate optimal sequence of product quantization, then

$$\|F(W) - F(\widehat{W}^N)\|_2 \le \frac{[F]_{\mathrm{Lip}} C_W}{(\log N)^{\frac12}} \quad \text{and} \quad \| \, |R_{N,M}|_{L^2_T} \, \|_2 \le \frac{[F]_{\mathrm{Lip}} C_W}{(M \log N)^{\frac12}}.$$

The simulation of $\widehat{W}^N$ from $W = \sum_{n\ge1} \sqrt{\lambda_n}\, \xi_n e_n^W$ amounts to solving for every $n = 1, \dots, m_N$, the nearest neighbour problem for the simulated Gaussian variable ξ_n into the N_n-quantizer set $\{x_1^{(N_1)}, \dots, x_{N_n}^{(N_n)}\}$.

8 Provisional remarks

First let us mention that several speeding up procedures (especially the Romberg log-extrapolation, etc) remains partially heuristic and subsequently would need some theoretical support: this means that deeper investigations on these specific theoretical aspects of functional quantization should be carried out.

On the other hand, further numerical developments to still improve the efficiency of quadrature formulæ based on functional quantization could be:

– to search for some (reasonably) "universal" good couples (M, N) that would improve the performances of our selected couples (some numerical work is in progress in that direction, see [23]). This needs to have more insight on the interaction between M and N in the Romberg log-extrapolation.

– to investigate the quantization using higher dimensional marginals (see [23]).

– to replace the Romberg log-extrapolation by a "three step" extrapolation method to cancel two terms instead of one in the expansion of the error

$\mathbb{E}(\Psi(W) - \psi(\widehat{W}))$. This requires some insight on the rate of convergence of quantities like $\mathbb{E}(\Theta(\widehat{W}).(W - \widehat{W})^{\otimes 3})$.

– the implementation of an L^2_T-valued extension of the $CLVQ$ procedure used in finite dimension to get some (locally) optimal quantizers (see [19]). The $CLVQ$ procedure is the stochastic gradient descent derived in d-dimension from the integral representation of the distortion gradient function (see (3.4) and [16] and [19]). However, the bounds obtained in (5.37) show that the gain to be expected from such a stochastic optimization remains limited.

Finally, let us mention that the K-L expansion of the Brownian motion W is in fact $a.s.$ converging in $(C([0,T]), \|\,.\,\|_{\sup})$. This follows from the Kolmogorov criterion and the Lévy-Ito-Nisio Theorem (see *e.g.* [22] p.104 and p.431 respectively). $A.s.$ uniform convergence holds for the Schauder basis as well. This suggests to evaluate the performances of K-L product quantizers for the $\|\,.\,\|_{\sup}$-norm (theoretically, see [14], but also numerically): the family of $\mathbb{P}_W$-$a.s.$ $\|\,.\,\|_{\sup}$-continuous functional is much wider than for the $\|\,.\,\|_2$-norm and contains most natural functionals (supremum, Brownian hitting times, stopped functionals, etc) involved in path-dependent options (lookback, barriers, down-and-out, etc).

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Annex: proof of the quadrature formulæ

(a) This error bounds readily follows from $|F(X) - F(\widehat{X}^x)| \leq [F]_{\mathrm{Lip}}|X - \widehat{X}^x|$.

(b) Formula (3.7) can be derived as follows:

$$|F(X) - F(\widehat{X}^x)| \leq [F]_{\mathrm{Liploc}}|X - \widehat{X}^x|(\theta(X) + \theta(\widehat{X}^x)).$$

Hence by the Schwarz inquality

$$\mathbb{E}|F(X) - F(\widehat{X}^x)| \leq [F]_{\mathrm{Liploc}}\|X - \widehat{X}^x\|_2(\|\theta(X)\|_2 + \|\theta(\widehat{X}^x)\|_2).$$

Now θ^2 is convex since θ is and $u \mapsto u^2$ is increasing and convex on $\mathbb{R}_+$. Consequently

$$\mathbb{E}\,\theta^2(\widehat{X}^x) = \mathbb{E}\,\theta^2(\mathbb{E}(X|\widehat{X}^x)) \leq \mathbb{E}(\mathbb{E}(\theta^2(X)|\widehat{X}^x)) = \mathbb{E}(\theta^2(X))$$

which completes the proof. Concerning (3.8), one starts from a Taylor expansion, where DF denotes the differential of F and $\|\,.\,\|$ the operator norm on $L(H)$,

$$\begin{aligned} |F(X) - F(\widehat{X}^x) - (DF(\widehat{X}^x), X - \widehat{X}^x)| &\leq \sup_{z\in(X,\widehat{X}^x)} \|DF(z) - DF(\widehat{X}^x)\||X - \widehat{X}^x| \\ &\leq [DF]_\alpha|X - \widehat{X}^x|^{1+\alpha}. \end{aligned}$$

Consequently $$\left|\mathbb{E}\,F(X) - \mathbb{E}\,F(\widehat{X}^x) - \mathbb{E}\left((DF(\widehat{X}^x)\,|\,X - \widehat{X}^x)\right)\right| \leq [DF]_\alpha\mathbb{E}\,|X - \widehat{X}^x|^{1+\alpha}.$$

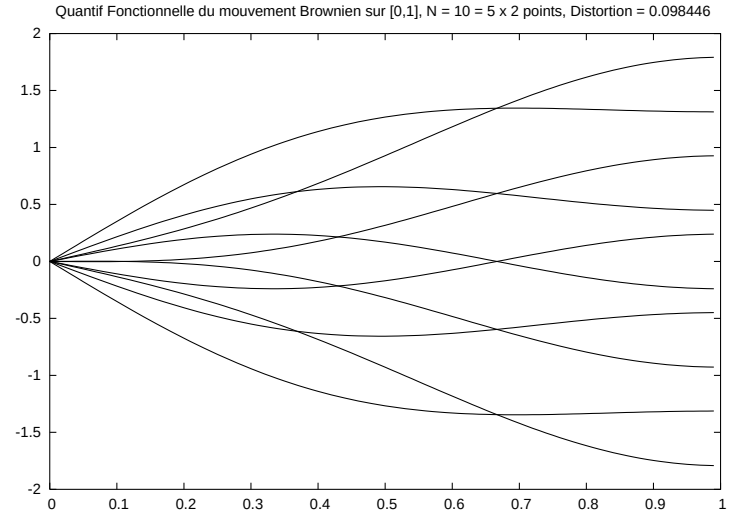
Now

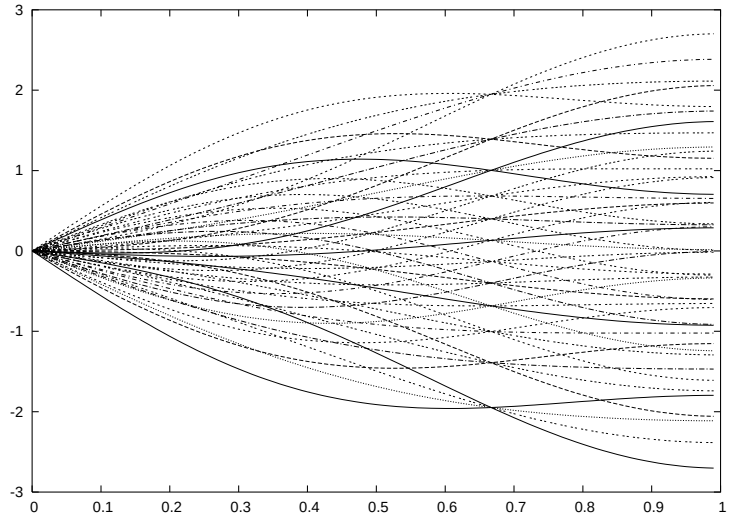
$$\mathbb{E}\left((DF(\widehat{X}^x)\,|\,X - \widehat{X}^x)\right) = \mathbb{E}\left(\left(DF(\widehat{X}^x)\,|\,\mathbb{E}(X - \widehat{X}^x|\widehat{X}^x)\right)\right) = \mathbb{E}\left(\left(DF(\widehat{X}^x)\,|\,0_H\right)\right) = 0.$$

To establish the last quadrature formula, one notes, using the convexity of θ that

$$\begin{aligned} \sup_{z\in(X,\widehat{X}^x)} \|DF(z) - DF(\widehat{X}^x)\| &\leq [DF]_{\mathrm{Liploc}}|X - \widehat{X}^x|(\theta(\widehat{X}^x) + \sup_{z\in(X,\widehat{X}^x)} \theta(z)) \\ &\leq [DF]_{\mathrm{Liploc}}|X - \widehat{X}^x|(\theta(\widehat{X}^x) + \max(\theta(X), \theta(\widehat{X}^x))) \\ &\leq [DF]_{\mathrm{Liploc}}|X - \widehat{X}^x|(2\theta(\widehat{X}^x) + \theta(X)). \end{aligned}$$

and one concludes as above by combining Jensen and Schwarz Inequalities. $\diamond$





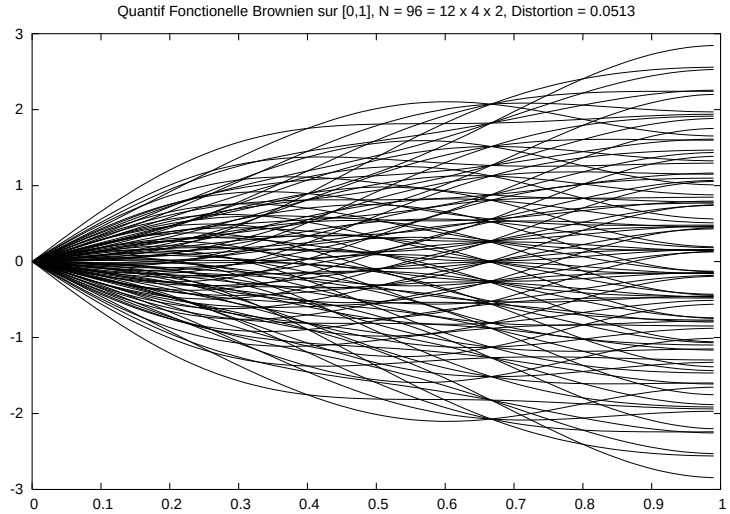


Figure 1: *The* $N_{\rm rec}$*-quantizer* $\chi^N_{\rm rec} = \sqrt{\lambda} \otimes x$ *for* $N = 10$ *(*$N_{\rm rec} = 2 \times 5 = 10$*),* $N = 50$ *(*$N_{\rm rec} = 12 \times 4 = 48$*) and* $N = 100$ *(*$N_{\rm rec} = 12 \times 4 \times 2 = 96$*).*

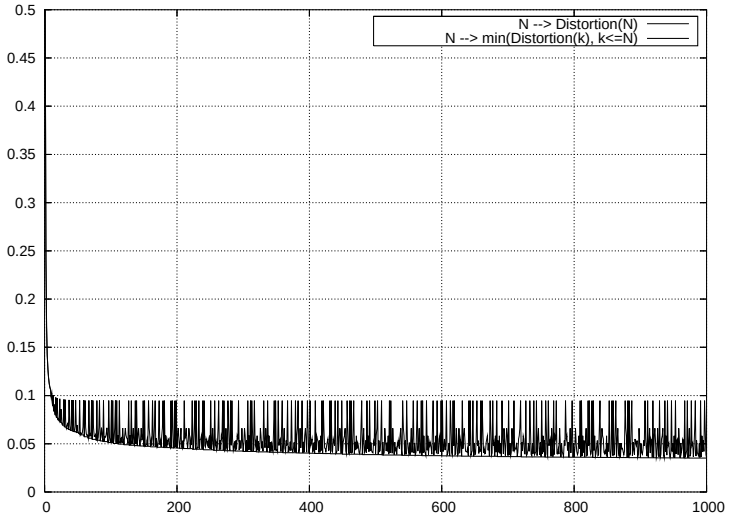


Figure 2: $N \mapsto \|W - \widehat{W}\chi_{\mathrm{opt}}^N\|_2^2$ *and* $N \mapsto \|W - \widehat{W}\chi_{\mathrm{rec}}^N\|_2^2 = \min_{\chi \in \mathcal{O}_{pq}(W,N)} \|W - \widehat{W}\chi\|_2^2$, $N = 1, \ldots, 1\,000$.

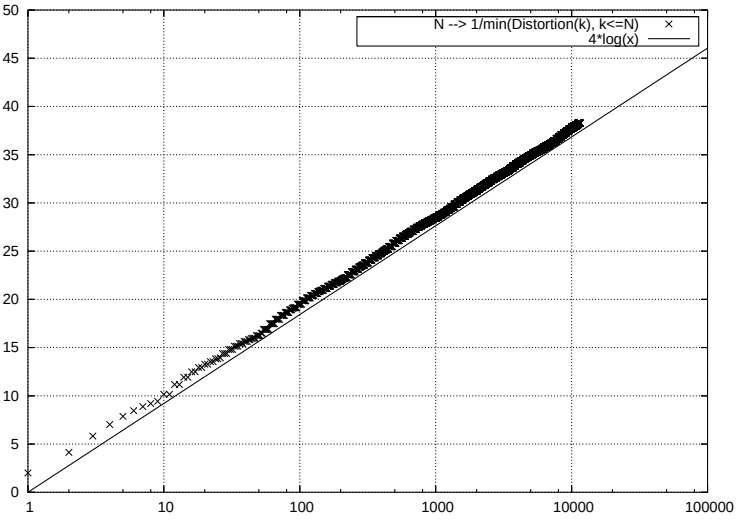


Figure 3: $\log N \mapsto \dfrac{1}{\|W - \widehat{W}\chi_{\mathrm{rec}}^N\|_2^2}$, $N = 1, \ldots, 10\,000$.

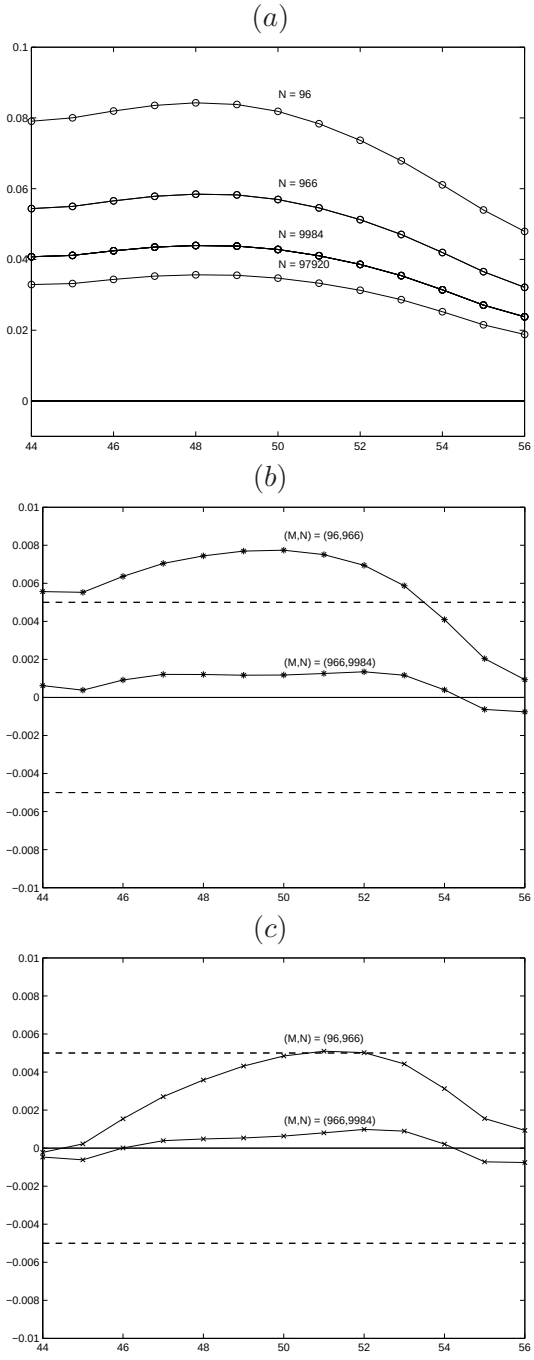


Figure 4: *Heston vanilla Call, A setting (absolute errors):* $T = 1$, $s_0 = 50$, $k = 0.250$, $a = 0.01$, $\rho = 0.5$, $\vartheta = 0.1$. (a) $K \longmapsto \mathrm{Call}^{Hest}(s_0, K) - \widehat{\mathrm{CrCall}}^{Hest}(s_0, K)$, $K \in \{44, \ldots, 56\}$. *Pricing by "crude" Functional Quantization (−−∘−−),* $N = 96,\ 966,\ 9\,984,\ 97\,920$. (b) $K \longmapsto \mathrm{Call}^{Hest}(s_0, K) - \widehat{\mathrm{RbgCrCall}}^{Hest}(s_0, K)$, $K \in \{44, \ldots, 56\}$. *Pricing by a Romberg* log*-extrapolation (—∗—) with* (M, N)=(96-966), (966-9984). (c) $K \longmapsto \mathrm{Call}^{Hest}(s_0, K) - \widehat{\mathrm{IRCAll}}^{Hest}(s_0, K)$, $K \in \{44, \ldots, 56\}$. *Pricing by K-linear interpolation of Romberg* log*-extrapolations (→←) with* $(M, N) = (96, 966), (966, 9984)$.

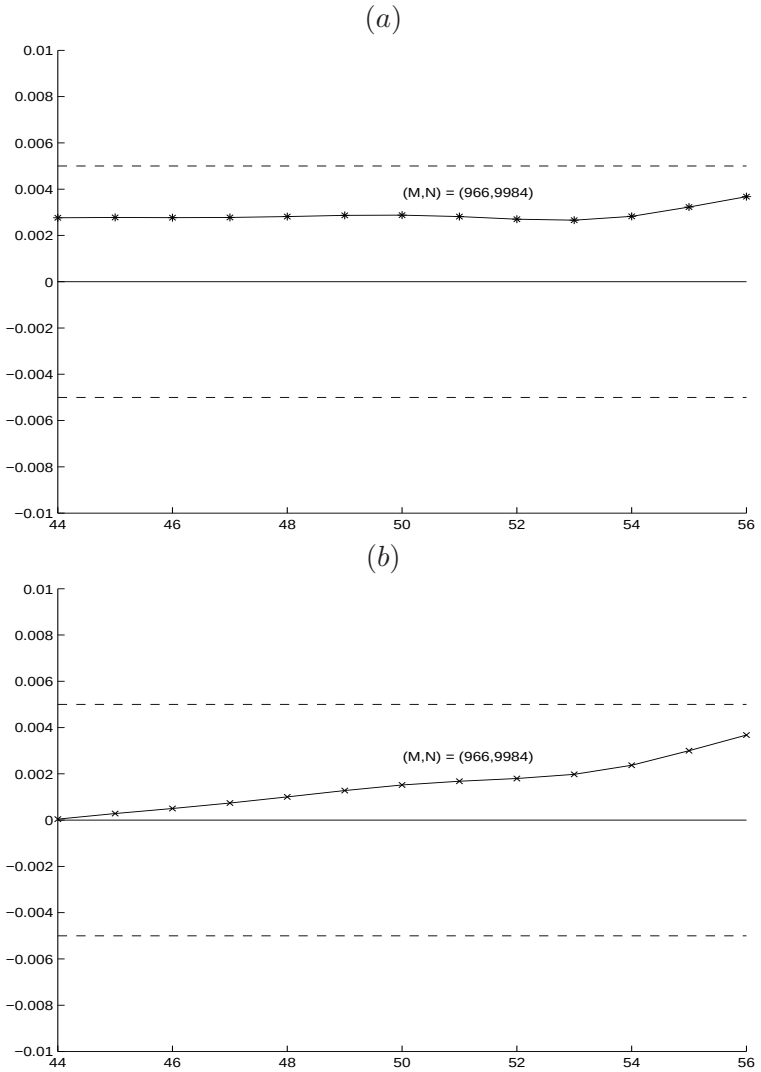


Figure 5: *Heston vanilla Call, B setting (absolute errors):* $T = 1$, $s_0 = 50$, $k = 2$, $a = 0.01$, $\rho = 0.5$, $\vartheta = 0.1$ (a) $K \longmapsto \mathrm{Call}^{Hest}(s_0, K) - \widehat{\mathrm{RbgCrCall}}^{Hest}(s_0, K)$, $K \in \{44, \ldots, 56\}$. *Pricing by Romberg* log*-extrapolation (−−∗−−),* $(M, N) = (966, 9984)$. (b) $K \longmapsto \mathrm{Call}^{Hest}(s_0, K) - \widehat{\mathrm{IRCall}}^{Hest}(s_0, K)$, $K \in \{44, \ldots, 56\}$. *Pricing by* K*-linear interpolation of Romberg* log*-extrapolations (⇀↽) with* $(M, N) = (966, 9984)$.

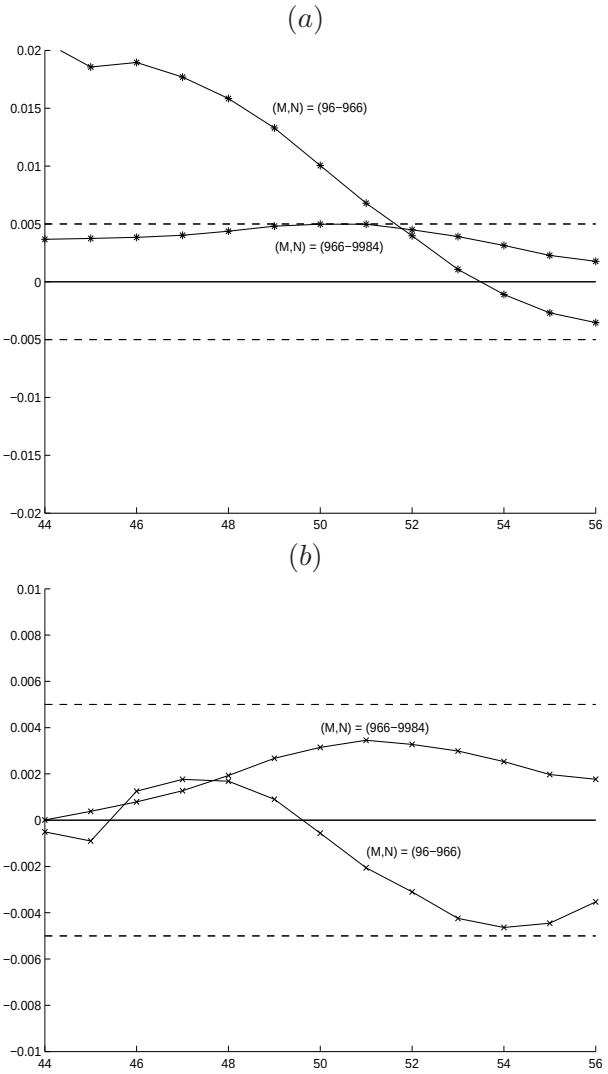


Figure 6: *Heston Asian Call, B setting (absolute errors):* $T = 1$, $s_0 = 50$, $k = 2$, $a = 0.01$, $\rho = 0.5$, $\vartheta = 0.1$ (a) $K \longmapsto \mathrm{AsCall}^{Hest}(s_0, K) - \widehat{\mathrm{RbgAsCall}}^{Hest}(s_0, K)$, $K \in \{44, \ldots, 56\}$. *Pricing by Romberg* log*-extrapolation (—∗—),* (M, N)=$(96, 966)$, $(966, 9984)$. (b) $K \longmapsto \mathrm{AsCall}^{Hest}(s_0, K) - \widehat{\mathrm{IRAsCall}}^{Hest}(s_0, K)$, $K \in \{44, \ldots, 56\}$. *Pricing by* K*-linear interpolation of Romberg* log*-extrapolations (—×—) with* (M, N)= $(96, 966)$, $(966, 9984)$.