Princing Swaptions Within an Affine Framework

Xin WANG

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2 Notations

2.1 The instantaneous short rate model

We use the following conventions:

- t denotes today,
- $x(t) = \{x_j(t),\ j = 1, \ldots, J\}$ denotes a vector of Markov processes whose risk-neutral dynamics are such that the instantaneous drifts and covariances are linear in the state variables, with which we characterize a general J-factor affine model,
- $r(t)$ denotes the instantaneous short rate which is defined as a linear combination of the state variables: $r(t) = \delta + \sum_{j=1}^{J} x_j(t)$, with δ a constant,
- $P^T(t, x(t))$ denotes the bond price on date t for an euro on date T in the future,
  If we use $\mathbb{E}_t^Q$ to denote the conditional expectation at time t under the risk-neutral mesure, for a T-maturity zero-coupon bond at time t, we will get :
$$P^T(t, x(t)) = \mathbb{E}_t^Q \left[e^{-\int_t^T r(s)ds} \right].$$

2.2 The swaption

We use the following conventions:

- T_0 denotes the exercice date of the swaption,

- $(T_1, ..., T_N)$ denote the dates that the coupon payments are made,
- $(C_1, ..., C_N)$ denote the payments at dates T_i,$i = 1, ..., N$,
- $CB(T_0)$ denotes the price of this underlying coupon bond at the date T_0, and we have:

$$CB(T_0) = \sum_{i=1}^{N} C_i P^{T_i}(T_0, x(T_0)),$$

- K denotes the strike price of a swaption,
- $Swn(t, x_j(t))$ denotes the price of a swaption which is defined as:

$$Swn(t, x_j(t)) = \mathbb{E}_t^Q \left[e^{-\int_t^{T_0} r(s)ds} \max(CB(T_0) - K, 0) \right].$$

3 Decription of the short rate models

3.1 Three-Factor Gaussian Model

We consider a three-dimensional Gaussian model with the three state variables dynamics as follows:

$$\begin{cases} dx_i(t) = -\kappa_i x_i(t)dt + \sigma_i dz_i^Q(t) \qquad i \in \{1, 2, 3\}, \\ r(t) = \delta + \sum_{i=1}^{3} x_i(t), \end{cases}$$

where

- δ, κ_i and σ_i, $i \in \{1, 2, 3\}$ are constants for all the three factors,
- $dz_i^Q(t)$, $i \in \{1, 2, 3\}$ are three Brownian motions which are dependent with each other: $d < z_i^Q, z_j^Q >_t = \rho_{ij}dt$ where $\rho_{ij} = 1$, $i = j$; $\rho_{ij} \neq 0$, $i \neq j$.

Thus, the model has κ_i, σ_i, δ and $x_i(0)$, $i \in \{1, 2, 3\}$ as its inputs.
For this model, we have the following formula for the bond price:

$$P^T(t, x(t)) = \mathbb{E}_t^Q \left[e^{-\int_t^T r(s)ds} \right] = e^{B_0(T-t) - \sum_{i=1}^{3} B_{\kappa_i}(T-t)x_i(t)},$$

where the deterministic functions $B_0(\tau)$ and $B_{\kappa_i}(\tau)$ satisfy a system of ordinary differential equations known as Ricatti equations, which we describe in the Appendix **(1)**,**(2)**.

3.2 Two-Factor CIR Model

We consider a two-dimensional Cox-Ingersoll-Ross model (CIR for short) with the two state variables following independant square root processes:

$$\begin{cases} dx_i(t) = -\kappa_i(\theta_i - x_i(t))dt + \sigma_i\sqrt{x_i(t)}dz_i^Q(t) & i \in \{1,2\}, \\ r(t) = \delta + \sum_{i=1}^2 x_i(t), \end{cases}$$

where

- κ_i, θ_i and σ_i are constants for the two factors,
- $dz_i^Q(t)$, $i \in \{1,2\}$ are two Brownian motions which are independent with each other.

Thus, the model has κ_i, θ_i, σ_i, δ and $x_i(0)$, $i \in \{1,2\}$ as its inputs.
For this model, we are the bond price in the form as:

$$P^T(t, x(t)) = \mathbb{E}_t^Q \left[e^{-\int_t^T r(s)ds} \right] = e^{B_0(T-t) - \sum_{i=1}^2 B_i(T-t)x_i(t)}$$

where the deterministic functions $B_0(\tau)$ and $B_i(\tau)$ are given in the Appendix **(9)**,**(10)**.

4 The W-forward risk measure

The price of a swaption in today is given by the expected discounted cash flow, where the expectation is under the risk-neutral measure $\mathbb{E}^Q$. However, we notice that sometimes it is more convenient to price swaptions by calculating expectations under the so-called forward measures $\mathbb{E}^W$ rather than the risk-neutral measure. The W-forward risk measure is defined by:

$$\mathbb{E}_t^W [Z(W)] = \mathbb{E}_t^Q \left[e^{-\int_t^W r(s)ds} Z(W) \right] / P^W(t, x(t))$$

for all non-negative Itô process Z. Therefore, we get the price of a swaption as:

$$
\begin{aligned}
Swn(t,x(t)) &= \mathbb{E}_t^Q\left[e^{-\int_t^{T_0} r(s)ds}\max(CB(T_0-K,0))\right]\\
&= \sum_{i=1}^{N} C_i\mathbb{E}_t^Q\left[e^{-\int_t^{T_i} r(s)ds}\mathbf{1}_{CB(T_0)>K}\right]-K\mathbb{E}_t^Q\left[e^{-\int_t^{T_0} r(s)ds}\mathbf{1}_{CB(T_0)>K}\right]\\
&= \sum_{i=1}^{N} C_iP^{T_i}(t,x(t))\mathbb{E}_t^Q\left[\frac{e^{-\int_t^{T_i} r(s)ds}}{P^{T_i}(t,x(t))}\mathbf{1}_{CB(T_0)>K}\right]\\
&\quad -KP^{T_0}(t,x(t))\mathbb{E}_t^Q\left[\frac{e^{-\int_t^{T_0} r(s)ds}}{P^{T_0}(t,x(t))}\mathbf{1}_{CB(T_0)>K}\right]\\
&= \sum_{i=1}^{N} C_iP^{T_i}(t,x(t))\mathbb{E}_t^{T_i}\left[\mathbf{1}_{CB(T_0)>K}\right]-KP^{T_0}(t,x(t))\mathbb{E}_t^{T_i}\left[\mathbf{1}_{CB(T_0)>K}\right]
\end{aligned}
$$

Thus, we need to estimate $\mathbb{E}_t^T\left[\mathbf{1}_{CB(T_0)>K}\right]$.

5 The estimation of $\mathbb{P}(Y>K)$ using the cumulants of Y

We define firstly the cumulants as the coefficients of a Taylor series expansion of the logarithm of the characteristic function. So if we define $G(k)=\mathbb{E}[e^{ikY}]$ as the characteristic function of Y, then, the cumulant c_j are given via:

$$\log\left[G(k)\right]=\sum_{j=1}^{\infty} c_j\frac{(ik)^j}{j!},$$

So the m-th order cumulant is uniquely defined by the first m moments $\mu_m=\mathbb{E}[Y^m]$ of the distribution (see Appendix **(18)-(24)**). Then we use the inverse Fourier transform to approximate $\mathbb{P}(Y>K)$ by performing a cumulant expansion, and we get:

$$\mathbb{P}(Y>K)=\sum_{j=0}^{M}\gamma_j\lambda_j,$$

where γ_j and λ_j, $j\in\{1,\dots,M\}$ are given in the Appendix **(25)-(32)** and **(34)-(41)**. This expansion uses the first M moments of Y.

6 Computation of the moments of $CB(T_0)$

From the section 4, to give an approximation of $\mathbb{E}_t^T[\mathbf{1}_{CB(T_0)>K}]$, it is enough for us to compute the first M moments of $CB(T_0)$. We choosed $M = 7$ because it offers an excellent balance between speed and accuracy, and, because it is computationally expensive to determine the higher-order cumulants c_6 and c_7), so we find it convient to set these both to zero. So we are:

$$E_t^T\left[(CB(T_0)^m)\right] = E_t^T\left[\sum_{i_1,\dots,i_m=1}^{N} (C_{i_1}\dots C_{i_m}) \times \left(e^{F_0 - \int_{j=1}^{J} x_j(T_0)F_j}\right)\right],$$

$$m \in \{1,\dots,5\}.$$

- For three-factor gaussian model,

$$E_t^{T_i}\left[(CB(T_0)^m)\right] = E_t^{T_i}\left[\sum_{i_1,\dots,i_m=1}^{N} \left(C_{i_1}\dots C_{i_m} \times e^{F_0} \times L(t,T_0,T_i)\right)\right],$$

(see Appendix **9.1.1**,**(3)**,**(4)**,**(6)**,**(7)**,**(8)**).

- For two-factor CIR model,

$$E_t^{T_i}\left[(CB(T_0)^m)\right] = E_t^{T_i}\left[\sum_{i_1,\dots,i_m=1}^{N} (C_{i_1}\dots C_{i_m} \times L(t,T_0,T_i))\right],$$

(see section **9.1.2**,**(11)**-**(17)**).

7 Monte Carlo methods

7.1 Three-Factor Gaussian Model

For this model, we know that $x_i(t), i \in \{1,2,3\}$ are Gaussian processes both under the risk-neutral measure and under the W-forward risk measure. Thus, under the W-forward risk measure we have for $i \in \{1,2,3\}$:

$$x_i(T_0) \sim \mathcal{N}(m_i(T_0), \sigma_i^2(T_0)),$$

where

$$m_i(T_0) = x_i(t)e^{-\kappa_i(T_0)} - \sum_{j=1}^{3} \frac{\sigma_i\sigma_j\rho_{ij}}{\kappa_j}\left[B_{\kappa_i}(T_0) - e^{-\kappa_j(W-T_0)}B_{\kappa_i+\kappa_j}(T_0)\right],$$

$$\sigma_i^2(T_0) = \sum_{j=1}^{3} \sigma_i\sigma_j\rho_{ij}B_{\kappa_i+\kappa_j}(T_0),$$

and $B_{\kappa_i}(\tau)$ is given in the Appendix **(1)**. Then, the Monte Carlo price are obtained using the exact Gaussian distribution of the state variable at maturity to avoid any time discretization bias.

7.2 Two-Factor CIR Model

We cannot calculate the exact distribution of $x_i(t), i \in \{1,2\}$ for this model, instead, the Monte Carlo prices are obtained this time by using an improved Euler discretization scheme of the stochastic differential equation (which could be called as scheme of E(0), see References [**2**]). We consider the regular grid $t_j = \frac{jT_0}{n}$, $j \in \{0,\dots,n\}$. We try to simulate the $x_i(T_0)$ on this grid and we got the approximate value $\hat{x}^n_{i,t_n}, i \in \{1,2\}$ via:

$$\begin{cases} \hat{x}^n_{i,t_0} = x_i(0), \\ \hat{x}^n_{i,t_{j+1}} = \left(\left(1 - \frac{\kappa_i T_0}{2n}\right)\sqrt{\hat{x}^n_{i,t_j}} + \frac{\sigma_i\left(Z^Q_{i,t_{j+1}} - Z^Q_{i,t_j}\right)}{2\left(1-\frac{\kappa_i T_0}{2n}\right)}\right)^2 + \left(\kappa_i\theta_i - \frac{\sigma_i^2}{4}\right)\frac{T_0}{n}. \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j \in \{0,\dots,n-1\} \end{cases}$$

8 Program algorithm

The functions below are common to all the programs:

- `int write_time(double dif)`
  This function computes the time which the whole program uses to calculate the price of a swaption and it writes the result in the screen in the format as: "`Computation time: ? h ? m ? s`".

- `double B_ki(double ki,double t)/double B_i(int i,double t)`
  `double B_0(double t)`
  These two functions compute the coefficients B_{k_i} and B_0 for the three-factor Gaussian model or B_i and B_0 for the two factor CIR model which are served to calculate the bond price given in the Appendix **(1)**,**(2)**,**(9)**,**(10)**.

- `double P(double t,double Ti)`
  This function returns the bond price in the date `t` for an euro in the date `Ti` (see section **2.1** and **2.2**).

The following functions are used in Cumulant Approximation method:

- `double erf(double x)`
  This function is used to calculate $N(d)$ given in the Appendix **(33)**.

- `double Ci(int *I,int m)`
  This function returns $\sum_{i_1,i_2,\ldots,i_m=1}^{N}(C_{i_1}\ldots C_{i_m})$, which is served to calculate the m-th moment, $m \in \{1,\ldots,M\}$.

- `double F_ki(int i,int *I,int m)/double F_i(int i,int *I,int m)`
  `double F_0(int *I,int m)`
  These two functions are to compute the coefficients F_{κ_j} and F_0 for the three-factor Gaussian model (see Appendix **(3)**,**(4)**) or F_j and F_0 for the two-factor CIR model (see Appendix **(11)**,**(12)**) which are used to compute the m-th moment, $m \in \{1,\ldots,M\}$.

- `double N_i(double t,int i,int *I,int m)/double N_i(double t,int i,int *I,double W,int m)`
  `double M_W(double t,double W,int *I,int m)`
  These two functions are used to compute the coefficients which are given in the Appendix **(7)**,**(8)**,**(14)**,**(15)** for the Laplace transform.

- `double L(double t,double W,int *I,int m)`
  These functions are used to compute the Laplace transform $L(t,T_0,T_i)$ (see Appendix **(6)**,**(13)**) of the variable $x_i(T_0), i \in \{1,2,3\}$ under the forward-neutral measure to express the expectation of products of bond prices at some future date.

- `double moment(double t,double W)`
  There are five functions like this which are to compute the first five moments (see section **5**).

- `double coeff(double t,double W)`
  We used the first $M(M=7)$ moments and the formulae in the Appendix **(18)-(24)** to calculate the cumulants and the coefficients $\gamma_j^{T_i}$ and $\lambda_j^{T_i}$, $i=1,\ldots,N; j=0,\ldots,M$.

- `double price(double t)`
  This simple function is using the formula

$$Swn(t,X_j(t)) = \sum_{i=1}^{N} C_i P^{T_i}(t)\left(\sum_{j=0}^{M}\gamma_j^{T_i}\lambda_j^{T_i}\right) - KP^{T_0}(t)\left(\sum_{j=0}^{M}\gamma_j^{T_0}\lambda_j^{T_0}\right)$$

  to compute the price of the swaption which we seeked.

The following functions are used in Monte Carlo method:

- `double SHUFL()`
  This function is used to generate a variable which follows an uniform distribution $\mathcal{U}(0,1)$.

- `double BoxMuller()`
  This function uses the Box-Muller theorem to transform from a two-dimensional continuous uniform distribution $\mathcal{U}(0,1)$ to a two-dimension bivariate normal distribution $\mathcal{N}(0,1)$(see Appendix **9.3**).

- `int prix_MC(double t)`
  This function gives the price of a T_0-maturity swaption with the strike K on N coupon payments dates. The number of Monte Carlo simulations is stored in `MC`. We compute also the confidence interval which is stored in `variance`.

- To simulate the $x(T_0)$:
  - for three-factor Gaussian model:
    * `int experance(double t,double W)`
      We compute the experances of $x_i(T_0), i \in \{1,\ldots,3\}$ which are stored in a table `ex[3]`.
    * `int covariance(double t)`
      We compute the covariance matrix of $x_i(T_0), i \in \{1,\ldots,3\}$ which is stored in a matrix `cov[3][3]`.
    * `int cholesky(double a[d][d])`
      We simulate the variable $x_i(T_0), i \in \{1,\ldots,3\}$ which follow the normal distributions $ex + \sqrt{cov}\mathcal{N}(0,I_3)$, where we get the root of $cov[3][3]$ by using the cholesky method.
  - for two-factor CIR model:
    * `double approximate(int j,double t)`
    * `double CB_T0(double t)`
      We use an ameliorated Euler approximation to simulate the variables $x_i(T_0)$.

9 Results and conclusions

9.1 Three-Factor Gaussian Model

We have experimented some tests to better check the accuracy of the computed prices.

1. We tested firstly with $N = 1$:
   - We supposed that the three factors are independent, then we can calculate the price in closed formula and we compared it to the results came from the cumulant expansion approximation method and Monte Carlo method when $N = 1$ and $\rho_{ij} = 0 (i \neq j)$.
   - We then checked the two methods with $N = 1$, but with the dependant state variables, which is the normal case. We noticed that the approximation is excellent and the cumulant expansion approximation method expends much less computation time than Monte Carlo.
2. We experimented afterwards with different payment dates $T_i, i = 1, ..., N$ and $N = 10$, $N = 20$:
   - For the cumulant expansion approximation method, we launched the programs for $M = 4$ et $M = 7$.
   - For Monte Carlo method, we launched it for 500 000 times and we calculated the confidence of intervals.
     We saw clearly that along with the augment of N, our programe ran more and more slowly, even so, the cumulant expansion approximation method expends still less time than Monte Carlo, and the absolute error relative to the true solution is less than a few parts in 10^{-4}. We stored the results of the two methods in two files *Gaussian.dat* et *MC_Gaussian.dat* and then we give the graphes (only when N=20) as follows:

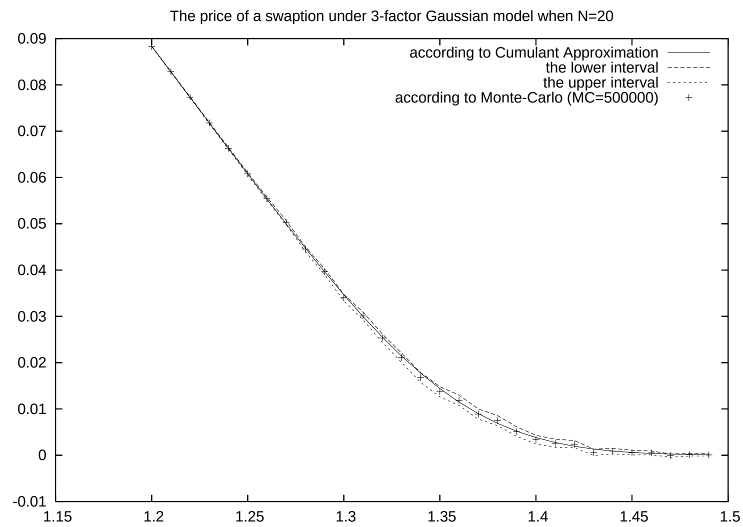

Figure 1: The price of a swaption under three-factor Gaussian model when $N = 20,\ K \in \{1.15, 1.5\}$

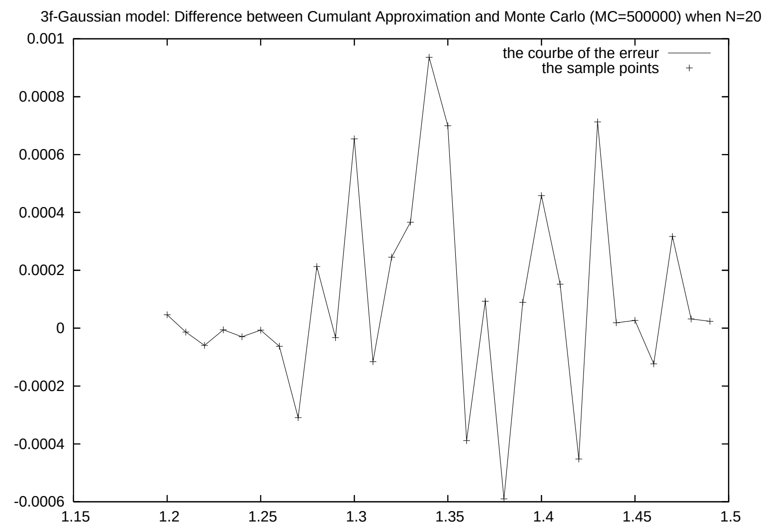

Figure 2: Three-factor Gaussian model: Difference between Cumulant Approximation and Monte Carlo ($MC = 500000$) when $N = 20$, $K \in \{1.15, 1.5\}$

9.2 Two-Factor CIR Model

1. To prove that the Laplace transform is correct, we wrote two sub-program to compare the following two values
   - $E_t^Q\left[e^{-\int_t^{T_0} r_v dv} e^{F_0^* - x_1(T_0)F_1^* - x_2(T_0)F_2^*}\right]$
   - $e^{M(T_0-t)-x_1(t)N_1(T_0-t)-x_2(t)N_2(T_0-t)}$

   where $M(\tau)$ and $N_i(\tau)$ can be referenced in the Appendix **(14)**, **(15)**. In the same time, we saw that the improved Euler approximation avoid the situation when $\hat{x}_{t_i}^n$ is equal to 0 during the recurrence.

2. We experimented the two methods with $N = 20$, the approximation this time is still excellent. The absolute error relative to the true solution is less than a few parts in 10^{-4}. We give the graphes as follows:

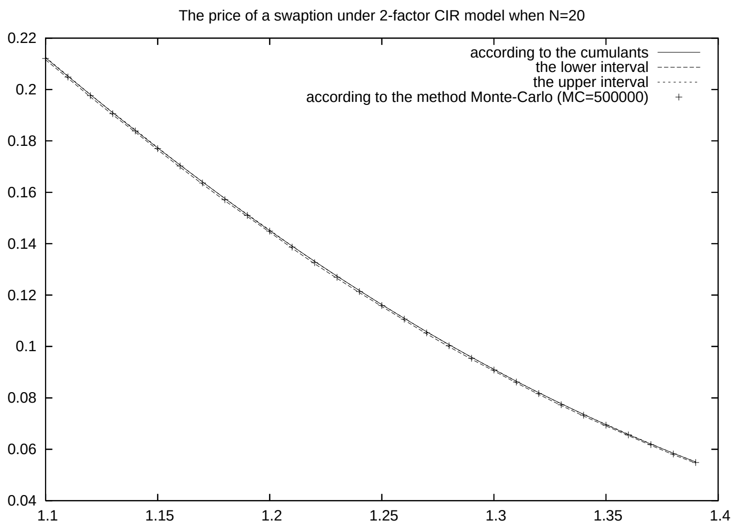

Figure 3: The price of a swaption under two-factor CIR model when $N = 20$, $K \in \{1.1, 1.4\}$

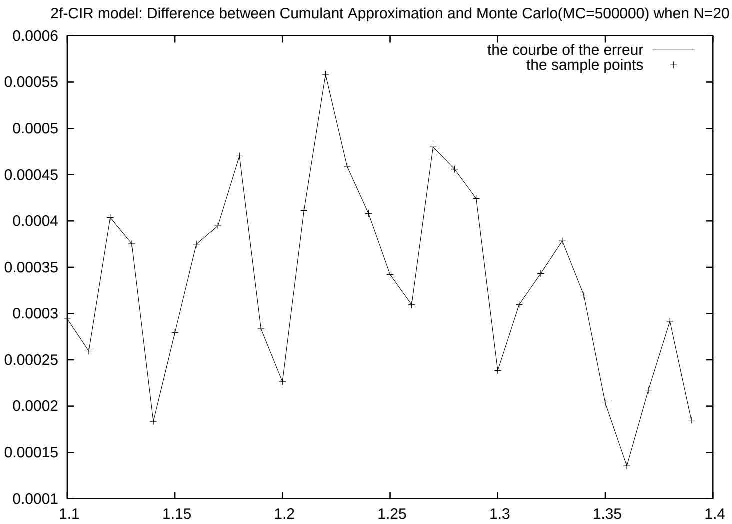

Figure 4: Two-factor CIR model: Difference between Cumulant Approximation and Monte Carlo ($MC = 500000$) when $N = 1$, $K \in \{1.1, 1.4\}$

10 Appendix

10.1 The coefficients for the model

10.1.1 Three-Factor Gaussian Model

For all τ, we have:

$$B_{\kappa_i}(\tau) = \frac{1 - e^{-\kappa_i \tau}}{\kappa_i}, \tag{1}$$

$$B_0(\tau) = -\delta\tau + \frac{1}{2}\sum_{i,j}^{3} \frac{\sigma_i\sigma_j\rho_{ij}}{\kappa_i\kappa_j}\left[\tau - B_{\kappa_i}(\tau) - B_{\kappa_j}(\tau) + B_{\kappa_i+\kappa_j}(\tau)\right], \tag{2}$$

$$F_{\kappa_j} = \sum_{k=1}^{m} B_{\kappa_j}(T_{i_k} - T_0) \qquad m \in \{1, \ldots, M\}, \tag{3}$$

$$F_0 = \sum_{k=1}^{m} B_0(T_{i_k} - T_0) \qquad m \in \{1, \ldots, M\}. \tag{4}$$

When we change the probability under the W forward mesure, the state variables have the dynamics:

$$dx_i(t) = \left(-\kappa_i x_i(t) - \sum_{j=1}^{3} \sigma_i\sigma_j\rho_{ij}B_{\kappa_j}(W - t)\right)dt + \sigma_i dz_i^W(t) \tag{5}$$

To compute all the moments of the coupon bond price at the maturity date T_0, we have used the Laplace transform of the state variable under the forward-neutral measure:

$$\begin{aligned} L(t,T,W) &\equiv E_t^W\left[e^{-\sum_{i=1}^3 F_i x_i(T)}\right] \\ &= e^{M(T-t)-\sum_{i=1}^3 N_i(T-t)x_i(t)} \end{aligned} \tag{6}$$

where

$$N_i(\tau) = F_i e^{-\kappa_i \tau} \tag{7}$$

$$\begin{aligned} M(\tau) &= \sum_{i,j} \frac{\sigma_i\sigma_j\rho_{ij}}{\kappa_j} F_i \left[B_{\kappa_i}(\tau) - e^{-\kappa_j(W-\tau)B_{\kappa_i+\kappa_j}(\tau)}\right] \\ &+ \frac{1}{2}\sum_{i\geq j} \sigma_i\sigma_j\rho_{ij}F_iF_jB_{\kappa_i+\kappa_j(\tau)} \end{aligned} \tag{8}$$

10.1.2 Two-Factor CIR Model

For all τ, we have:

$$\begin{aligned} B_0(\tau) &= -\delta\tau \\ &+\sum_{i=1}^2\left(\frac{2\kappa_i\theta_i}{\gamma_i-\kappa_i}\tau - \frac{2\kappa_i\theta_i}{\sigma_{i^2}}\log\left[\frac{(\kappa_i+\gamma_i)(e^{\gamma_i\tau}-1)+2\gamma_i}{2\gamma_i}\right]\right), \end{aligned} \tag{9}$$

$$B_i(\tau) = \frac{2(e^{\gamma_i\tau}-1)}{(\kappa_i+\gamma_i)(e^{\gamma_i\tau}-1)+2\gamma_i}, \tag{10}$$

$$F_0 = \sum_{k=1}^m B_0(T_{i_k}-T_0) \qquad m\in\{1,\ldots,M\}, \tag{11}$$

$$F_j = \sum_{k=1}^m B_j(T_{i_k}-T_0) \qquad m\in\{1,\ldots,M\}, \tag{12}$$

where we define $\gamma_i \equiv \sqrt{\kappa_i^2+2\sigma_i^2}$. Then we give the moments of the distribution of a coupon bond by noting:

$$\begin{aligned} L(t,T_0,W) &\equiv E_t^W\left[e^{F_0-x_1(T_0)F_1-x_2(T_0)F_2}\right] \\ &= \frac{1}{P^W(t)}e^{M(T_0-t)-x_1(t)N_1(T_0-t)-x_2(t)N_2(T_0-t)} \end{aligned} \tag{13}$$

where

$$N_i(\tau) = \frac{F_i^*(\lambda_{i,+}e^{\gamma_i\tau} - \lambda_{i,-}) + \frac{2}{\sigma_i^2}(e^{\gamma_i\tau} - 1)}{F_i^*(e^{\gamma_i\tau} - 1) - (\lambda_{i,-}e^{\gamma_i\tau} - \lambda_{i,+})} \tag{14}$$

$$\begin{aligned} M(\tau) &= F_0^* - \delta\tau + \sum_{i=1}^{2} \times \\ &\left(\frac{2\kappa_i\theta_i}{\gamma_i - \kappa_i}\tau - \frac{2\kappa_i\theta_i}{\sigma_{i^2}} log\left[\frac{(F_i^* - \lambda_{i,-})e^{\gamma_i\tau} - (F_i^* - \lambda_{i,+})}{\frac{2\gamma_i}{\sigma_i^2}}\right]\right) \end{aligned} \tag{15}$$

and

$$F_i^* = F_i + B_i(W - T_0) \tag{16}$$

$$\lambda_{i,\pm} \equiv \frac{-\kappa_i \pm \gamma_i}{\sigma_i^2} \tag{17}$$

10.2 The relevant cumulants and other parameters

We calculate the first seven moments by the definition: $\mu_m = \mathbb{E}[CB(T_0)^m]$, $m \in \{1, \ldots, M\}$, $M = 7$. We provide the first seven cumulants c_i in terms of the moments μ_i as follows:

$$c_1 = \mu_1, \tag{18}$$

$$c_2 = \mu_2 - \mu_1^2, \tag{19}$$

$$c_3 = \mu_3 - 3\mu_1\mu_2 + 2\mu_1^3, \tag{20}$$

$$c_4 = \mu_4 - 4\mu_1\mu 3 - 3\mu_2^2 + 12\mu_1^2\mu_2 - 6\mu_1^4, \tag{21}$$

$$c_5 = \mu_5 - 5\mu_1\mu 4 - 10\mu_2\mu_3 + 20\mu_1^2\mu_3 + 30\mu_1\mu_2^2 - 60\mu_1^3\mu_2 + 24\mu_1^5, \tag{22}$$

$$\begin{aligned} c_6 &= \mu_6 - 6\mu_1\mu 5 - 15\mu_2\mu_4 + 30\mu_1^2\mu_4 - 10\mu_3^2 + 120\mu_1\mu_2\mu_3 - 120\mu_1^3\mu_3 + 30\mu_2^3 \\ &\quad -270\mu_1^2\mu_2^2 + 360\mu_1^4\mu_2 - 120\mu_1^6, \end{aligned} \tag{23}$$

$$\begin{aligned} c_7 &= \mu_7 - 7\mu_1\mu 6 - 21\mu_2\mu_5 - 30\mu_3\mu_4 + 140\mu_1\mu_3^2 - 630\mu_1\mu_2^3 + 210\mu_1\mu_2\mu_4 - 1260\mu_1^2\mu_2\mu_3 \\ &\quad +42\mu_1^2\mu_5 + 2520\mu_1^3\mu_2^2 - 210\mu_1^3\mu_4 + 210\mu_2^2\mu_3 + 840\mu_1^4\mu_3 - 2520\mu_1^5\mu_2 + 720\mu_1^7. \end{aligned} \tag{24}$$

We define also:

$$\lambda_0 = N\left[\frac{c_1 - K}{\sqrt{c_2}}\right], \tag{25}$$

$$\lambda_1 = \frac{1}{\sqrt{2\pi c_2}} e^{-\frac{(K-c_1)^2}{2c_2}} \left[c_2\right], \tag{26}$$

$$\lambda_2 = c_2 N\left[\frac{c_1 - K}{\sqrt{c_2}}\right] + \frac{1}{\sqrt{2\pi c_2}} e^{-\frac{(K-c_1)^2}{2c_2}} \left[c_2(K - c_1)\right], \tag{27}$$

$$\lambda_3 = \frac{1}{\sqrt{2\pi c_2}} e^{-\frac{(K-c_1)^2}{2c_2}} \left[c_2(K - c_1)^2 + 2c_2^2\right], \tag{28}$$

$$\lambda_4 = 3c_2^2 N\left[\frac{c_1 - K}{\sqrt{c_2}}\right] + \frac{1}{\sqrt{2\pi c_2}} e^{-\frac{(K-c_1)^2}{2c_2}} \left[c_2(K - c_1)^3 + 3c_2^2(K - c_1)\right], \tag{29}$$

$$\lambda_5 = \frac{1}{\sqrt{2\pi c_2}} e^{-\frac{(K-c_1)^2}{2c_2}} \left[c_2(K - c_1)^4 + 4c_2^2(K - c_1)^2 + 8c_2^3\right], \tag{30}$$

$$\lambda_6 = 15c_2^3 N\left[\frac{c_1 - K}{\sqrt{c_2}}\right] + \frac{1}{\sqrt{2\pi c_2}} e^{-\frac{(K-c_1)^2}{2c_2}} \left[c_2(K - c_1)^5 + 5c_2^2(K - c_1)^3 + 15c_2^3(K - c_1)\right], \tag{31}$$

$$\lambda_7 = \frac{1}{\sqrt{2\pi c_2}} e^{-\frac{(K-c_1)^2}{2c_2}} \left[c_2(K - c_1)^6 + 6c_2^2(K - c_1)^4 + 24c_2^3(K - c_1)^2 + 48c_2^4\right], \tag{32}$$

where

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{x^2}{2}} dx, \tag{33}$$

and

$$\gamma_0 = 1 + \frac{3}{c_2^2}\left(\frac{c_4}{4!}\right) - \frac{15}{c_2^3}\left(\frac{c_6}{6!} + \frac{1}{2}\frac{c_3^2}{(3!)^2}\right), \tag{34}$$

$$\gamma_1 = -\frac{3}{c_2^2}\left(\frac{c_3}{3!}\right) + \frac{15}{c_2^3}\frac{c_5}{5!} - \frac{105}{c_{2^4}}\left(\frac{c_7}{7!} + \frac{c_3 c_4}{(3!)(4!)}\right), \tag{35}$$

$$\gamma_2 = -\frac{6}{c_2^3}\left(\frac{c_4}{4!}\right) + \frac{45}{c_2^4}\left(\frac{c_6}{6!} + \frac{1}{2}\frac{c_3^2}{(3!)^2}\right), \tag{36}$$

$$\gamma_3 = \frac{1}{c_2^3}\left(\frac{c_3}{3!}\right) - \frac{10}{c_2^4}\frac{c_5}{5!} + \frac{105}{c_{2^5}}\left(\frac{c_7}{7!} + \frac{c_3 c_4}{(3!)(4!)}\right), \tag{37}$$

$$\gamma_4 = \frac{1}{c_2^4}\left(\frac{c_4}{4!}\right) - \frac{15}{c_2^5}\left(\frac{c_6}{6!} + \frac{1}{2}\frac{c_3^2}{(3!)^2}\right), \tag{38}$$

$$\gamma_5 = \frac{1}{c_2^5}\left(\frac{c_5}{5!}\right) - \frac{21}{c_{2^6}}\left(\frac{c_7}{7!} + \frac{c_3 c_4}{(3!)(4!)}\right), \tag{39}$$

$$\gamma_6 = \frac{1}{c_2^6}\left(\frac{c_6}{6!} + \frac{1}{2}\frac{c_3^2}{(3!)^2}\right), \tag{40}$$

$$\gamma_7 = \frac{1}{c_{2^7}}\left(\frac{c_7}{7!} + \frac{c_3 c_4}{(3!)(4!)}\right). \tag{41}$$

10.3 Box-Muller transformation

If x_1 and x_2 are uniformly and independently distributed between 0 and 1, then z_1 and z_2 as defined below have a normal distribution with mean $\mu = 0$ and variance $\sigma^2 = 1$.

$$z_1 = \sqrt{-2\ln x_1}\cos(2\pi x_2), \tag{42}$$

$$z_2 = \sqrt{-2\ln x_1}\sin(2\pi x_2). \tag{43}$$

References

[1] Pierre COLLIN-DUFRESNE, Robert S.GOLDSTEIN, *Pricing swaption within an affine framework, Journal of Derivatives Fall 2002*

[2] Aurélien ALFONSI, *On the discretization schemes for the CIR (and Bessel squared) processes, may 2, 2005*