Time Dependent Heston Model

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Premia 14

1 Formulation of the problem

We consider the solution of the stochastic differential equation (SDE):

$$dX_t = \sqrt{v_t}dWt - \frac{v_t}{2}dt,\ X_0 = x_0, \tag{1}$$
$$dv_t = \kappa(\theta_t - v_t)dt + \xi_t\sqrt{v_t}dBt,\ v_0, \tag{2}$$

$$d < W, B >_t = \rho_t dt,$$

where $(B_t, W_t)_{0\leq t\leq T}$ is a two-dimensional correlated Brownian motion on a given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$. In our setting, $(X_t)_{0\leq t\leq T}$ is the log of the forward price and $(v_t)_{0\leq t\leq T}$ is the square of the volatility which follows a CIR process with an initial value $v_0 > 0$, a positive mean reversion κ, a positive long-term level $(\theta_t)_{0\leq t\leq T}$, a positive volatility of volatility $(\xi_t)_{0\leq t\leq T}$ and a correlation $(\rho_t)_{0\leq t\leq T}$. These time dependent parameters are assumed to be measurable and bounded on $[0, T]$.

To develop the approximation method, we set the following perturbed process w.r.t. $\varepsilon \in [0, 1]$,

$$dX_t^\varepsilon = \sqrt{v_t^\varepsilon}dWt - \frac{v_t^\varepsilon}{2}dt,\ X_0^\varepsilon = x_0, \tag{3}$$
$$dv_t^\varepsilon = \kappa(\theta_t - v_t^\varepsilon)dt + \varepsilon\xi_t\sqrt{v_t^\varepsilon}dBt,\ v_0^\varepsilon = V_0, \tag{4}$$

so that the above perturbed process coincides with the initial one for $\varepsilon = 1$ and we have

$$X_t^1 = X_t,\ \ v_t^1 = v_t.$$

The main purpose is to give an accurate analytic approximation, in a certain sense, of the expected payoff of a put option:

$$g(\varepsilon) = e^{-\int_0^T r_t dt}\mathbb{E}\left[\left(K - e^{\int_0^T (r_t - q_t)dt + X_T^\varepsilon}\right)_+\right] \tag{5}$$

where r (resp. q) is the risk-free rate (resp. the dividend yield), T is the maturity and $\varepsilon = 1$.

2 Pricing formula

In the following, let the function $(x, y) \mapsto P_{BS}(x, y)$ denotes the put function price in a Black-Scholes model with spot e^x, strike K, total variance y, risk-free rate $r_{eq} = \frac{\int_0^T r_t}{T}$, dividend yield $q_{eq} = \frac{\int_0^T q_t}{T}$ and maturity T. We recall that $P_{BS}(x, y)$ has the following explicit expression:

$$P_{BS}(x, y) = Ke^{-r_{eq}T}\mathcal{N}\left(\frac{1}{\sqrt{y}}\log(\frac{Ke^{-r_{eq}T}}{e^x e^{-q_{eq}T}}) + \frac{1}{2}\sqrt{y}\right) - e^x e^{-r_{eq}T}\mathcal{N}\left(\frac{1}{\sqrt{y}}\log(\frac{Ke^{-r_{eq}T}}{e^x e^{-q_{eq}T}}) - \frac{1}{2}\sqrt{y}\right).$$

The main result is given by the following accurate expansion for the **put price in the time dependent Heston model**

$$g(1) = e^{-\int_0^T r_t dt}\mathbb{E}\left[\left(K - e^{\int_0^T (r_t - q_t)dt + X_T^1}\right)_+\right] = P_{BS}(x_0, var_T) + \sum_{i=1}^{2} a_{i,T} \frac{\partial^{i+1} P_{BS}}{\partial x^i y}(x_0, var_T)$$
$$+ \sum_{i=0}^{1} b_{2i,T} \frac{\partial^{2i+2} P_{BS}}{\partial x^{2i} y^2}(x_0, var_T) + O\left(\left[\xi_{sup}\sqrt{T}\right]^3 \sqrt{T}\right) \quad (6)$$

The computation of coefficients var_T, $a_{i,T}$ and $b_{i,T}$ can be achieved (using iterations) only when functions θ_t, ξ_t, ρ_t are **piecewise** or **constant**. These formulas are given explicitly in [1].

For the Premia code, we implement this method in both cases **piecewise** and **constant** in order to compute the **put**, **call** and the **delta** prices . For the **piecewise** case the Premia user can change and implement his own data for functions θ_t, ξ_t, ρ_t and this only by changing the initial file data in the Premia software. Functions $\frac{\partial^{i+j} P_{BS}}{\partial x^i y^j}$, for $i, j \in \{0, 1, 2\}$, and $(var_T, a_{i,T}, b_{i,T})$ are implemented in the Premia code, respectively in functions **int** greeksBS and **int** expansion_terms.

References

[1] Benhamou, E., Gobet, E., Miri, M. Time-dependent Heston model. SSRN preprint (2009). 2