Calibration by a weighted Monte Carlo method .

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This calibration method introduced in [1] consists in generating ν sample paths according to some a priori model and in assigning non uniform weights to these paths according to the market prices of some Put options.

The a priori model.

The sample paths are generated according to the following stochastic volatility model:

$$\begin{cases} dS_t = S_t((r-q)dt + \sigma_t dB_t^1) \\ d\sigma_t = -\beta(\sigma_t - \sigma_0)dt + \alpha dB_t^2 \\ d < B^1, B^2 >_t = \rho dt, -1 \leq \rho \leq 1 \end{cases}$$

where

- r is the risk free interest rate.
- q is the dividend rate
- σ_t is the stochastic volatility.
- B_t^1, B_t^2 are brownian motions.
- ρ is the correlation between the brownian motions.
- σ_0 is the mean volatility.
- α is the perturbation .
- $\beta > 0$ is the rate of return to mean.

More precisely,we generate discretized sample-paths by the standard Euler scheme with timestep Δt:

$$\begin{cases}\overline{S}_{(n+1)\Delta t} = \overline{S}_{n\Delta t}(1+(r-q)\Delta t + \overline{\sigma}_{n\Delta t}\sqrt{\Delta t} g^1_{n+1}), n = 1,\dots,M \\ \overline{\sigma}_{(n+1)\Delta t} = \sigma_0 + (\overline{\sigma}_{n\Delta t} - \sigma_0)(1-\beta\Delta t) + \alpha\sqrt{\Delta t}(\rho g^1_{n+1} + \sqrt{1-\rho^2} g^2_{n+1})\end{cases} \tag{1}$$

where $(g^1_n, g^2_n)_{1\leq n\leq M}$ are i.i.d according to $\mathcal{N}_2(0, Id)$.

We construct a set of ν sample paths of (1) which we denote by

$$\omega^{(i)} = (\overline{S}_{\Delta t}(\omega^{(i)}), \dots, \overline{S}_{M\Delta t}(\omega^{(i)})), i = 1, \dots, \nu$$

Computation of the weights.

The choice of the weights $(p_i)_{1\leq i\leq \nu}$ is done by minimizing the Kullback-Leibler relative entropy $D(p|u) = \ln(\nu) + \sum_{i=1}^{\nu} p_i \ln(p_i)$ with respect to the uniform distribution under the constraint that the corresponding price of the benchmark puts is equal to their market prices.
More precisely, let $P_1, ..., P_N$ denote the market prices of N european puts, and $g_{1j}, g_{2j}, \dots, g_{\nu j}, j = 1, \dots, N$ represent the present values of the cashflows of the jth benchmark along the different paths, i.e $g_{ij} = e^{-rk_j\Delta t}(\overline{S}_{k_j\Delta t}(\omega^{(i)}) - K_j)^+$, where K_j is the strike of the jth benchmark and $k_j\Delta t$ is its maturity. The price relations for the benchmark instruments can be written in the form

$$\sum_{i=1}^{\nu} p_i g_{ij} = P_j, j = 1, \dots, N \tag{2}$$

The mathematical problem is to minimize the convex function of p

$$D(p|u) = \ln(\nu) + \sum_{i=1}^{\nu} p_i \ln(p_i)$$

under the linear constraints (2).
Let's introduce Langrange multipliers $(\lambda_1, ..., \lambda_N)$. The problem is equivalent to find p which maximizes

$$\min_{\lambda\in\mathbb{R}^N} \Big(- D(p|u) + \sum_{j=1}^{N} \lambda_j \Big(\sum_{i=1}^{\nu} p_i g_{ij} - P_j \Big) \Big)$$

and can be solved by considering the dual formulation

$$\min_{\lambda}\left(\max_{p}\left(-D(p|u)+\sum_{j=1}^{N}\lambda_j\left(\sum_{i=1}^{\nu}p_i g_{ij}-P_j\right)\right)\right)$$

A straightforward argument shows that the probability vector that realizes the supremum for each λ has the Boltzmann-Gibbs form

$$p_i = p(\omega^{(i)}) = \frac{1}{Z(\lambda)} e^{\sum_{j=1}^{N} g_{ij}\lambda_j}$$

where

$$Z(\lambda) = \sum_{i=1}^{\nu} e^{\sum_{j=1}^{N} g_{ij}\lambda_j}$$

Then the optimal λ minimizes

$$W(\lambda) = \ln\left(\sum_{i=1}^{\nu} e^{\sum_{j=1}^{N} g_{ij}\lambda_j}\right) - \sum_{j=1}^{N}\lambda_j P_j$$

It is also interesting not to satisfy exactly the constraint by choosing λ which minimizes the penalized function:

$$\ln\left(\sum_{i=1}^{\nu} e^{\sum_{j=1}^{N} g_{ij}\lambda_j}\right) - \sum_{j=1}^{N}\lambda_j P_j + \frac{1}{2}\sum_{j=1}^{N} w_j\lambda_j^2$$

We recover the previous case by choosing $w_j = 0, 1 \le j \le N$.

Numerical resolution

We first renormalize the payoff by introducing $\alpha_j = \sup_{1\le i\le\nu}|g_{ij}-P_j|$ and setting $\tilde{g}_{ij} = \frac{g_{ij}-P_j}{\alpha_j}$.

We find $(\tilde{\lambda}_1, ..., \tilde{\lambda}_N)$ which minimizes

$$\ln\left(\sum_{i=1}^{\nu} e^{\sum_{j=1}^{N}\tilde{\lambda}_j\tilde{g}_{ij}}\right) + \frac{1}{2}\sum_{j=1}^{N}\frac{w_j}{\alpha_j^2}\tilde{\lambda}_j^2$$

by a conjugate gradient algorithm and deduce the corresponding weights

$$p_i = \frac{1}{\tilde{Z}(\tilde{\lambda})} e^{\sum_{j=1}^{N}\tilde{g}_{ij}\tilde{\lambda}_j}$$

where

$$\tilde{Z}(\tilde{\lambda}) = \sum_{i=1}^{\nu} e^{\sum_{j=1}^{N}\tilde{g}_{ij}\tilde{\lambda}_j}$$

References

[1] M.Avellaneda, R.Buff, C.Friedman, N.Grandchamp, L.Kruk, J.Newman. *Weighted Monte Carlo : A New Technique for Calibrating Asset-Pricing Models*. International Journal of Theoretical and Applied Finance Vol. 4, No. 1 (2001) 91-119