Introduction

Unlike European options, American options cannot be valued by closed-form formulae and require the use of numerical methods. There are two main numerical approaches: one is the probabilistic approach, based on the approximation of diffusions by Markov chains, the other is the analytic approach related to the discretization of variational inequalities. In this survey, we will concerned with the pricing of American options on one or two stocks in the Black-Scholes setting for which various numerical methods have been developed. We recall here the connections between the modern theory of American options with the optimal stopping and variational inequalities theories.

1 The value function of American options

1.1 The model

We will consider American options written on two dividend-paying stocks. Let S_t^i $(i = 1, 2)$ be the stock-price at time t of the stock i which satisfies the following stochastic differential equation:

$$\frac{dS_t^i}{S_t^i} = (r - \delta_i)dt + \sum_{j=1}^{2} \sigma_{ij} dW_t^j \quad i = 1, 2 \tag{1}$$

where, under the so-called risk neutral probability measure which will be denoted by P, W is a standard 2-dimensional Brownian motion. The nonnegative constant r is the interest rate and the nonnegative constants δ_i are the dividend rate of the stock i.
The matrix $\Sigma = (\sigma_{ij})_{1\leq i,j\leq n}$ is assumed to be invertible.
An American option with maturity T is defined by a continuous nonnegative adapted process $(Z_t)_{0\leq t\leq T}$ satisfying $E(\sup_{0\leq t\leq T} Z_t) < \infty$, where Z_t stands for the payoff of the option when exercise occurs at time t. We will be concerned with payoff processes given by $Z_t = \psi(S_t)$ where ψ is a continuous nonnegative function satisfying

$$\exists M > 0 \quad \forall x \in]0, \infty[^2 \quad \psi(x) + ||\nabla\psi(x)|| \leq M(1 + ||x||) \tag{2}$$

We recall the proposition which gives the value function of an American option (see e.g. Jaillet et al. Proposition 2.2) and highlights the connection with Optimal Stopping Theory.

Proposition 1.1. *The value at time t of an American option with maturity T and payoff function ψ is given by $U(t, S_t)$ where*

$$\begin{aligned} U(t, x) &= \sup_{\tau\in\mathcal{T}_{0,T-t}} E(e^{-r\tau}\psi(S_\tau^{0,x})) \\ &= E(e^{-r\tau^*}\psi(S_{\tau^*}^{0,x})) \end{aligned}$$

where $\mathcal{T}_{0,T-t}$ is the set of all stopping times with values in $[0, T-t]$. and

$$\tau^* = \inf\{u \in [0, T-t] \ | \ C(t+u, S_u^{0,x}) = \psi(S_u^{0,x})\}.$$

Recall that τ^* is the smallest optimal stopping time.

Remark 1.1. *The process $\left(e^{-rt}U(t, S_t^x)\right)_{0\leq t\leq T}$ is the Snell envelope of the process $\left(e^{-rt}\psi(S_t^x)\right)_{0\leq t\leq T}$ for any $x \in R^2$.*

For Probabilistic methods, see Tree Methods

1.2 Exercise region

We introduce the following set:

$$\mathcal{E} = \{(t,x) \in [0, T[\times R^n \ | \ U(t,x) = \psi(x)\}.$$

Clearly, it is never optimal to exercise prior to maturity out of $\mathcal{E}$ where the payoff due to the option's sale is greater than the one due to the exercise. Moreover, the smallest optimal stopping times τ^* satisfies

$$\tau^* = \inf\{u \geq 0 \ | \ (t+u, S_u) \in \mathcal{E}\} \wedge (T-t).$$

Definition 1.1. *The coincidence set $\mathcal{E}$ is called the exercise region of the American option.*

1.3 Variational inequalities and American options

The purpose of this subsection is to derive the variational inequality satisfied by the value function U. Actually, it is convenient to make the change of variable $X_t = \log(S_t)$ in equation 1. We have

$$dX_t^i = (r - \delta_i - \frac{1}{2}\sum_{j=1}^{2}\sigma_{ij}^2)dt + \sum_{j=1}^{2}\sigma_{ij}dW_t^j \quad i = 1, 2 \tag{3}$$

Jaillet-Lamberton-Lapeyre have studied in the Black-Scholes diffusion case the American option value by the method of variational inequalities based on the work of Bensoussan-Lions. The viscosity solutions theory introduced by Crandall and P.L Lions is also a means to characterize the option value.
Introduce the parabolic operator $\mathcal{L}$ by:

$$\mathcal{L}U = \frac{\partial U}{\partial t} + \mathcal{A}U$$

where

$$\mathcal{A}F = \frac{1}{2}\sum_{i,j=1}^{2} a_{ij}\frac{\partial^2 F}{\partial x_i \partial x_j} + \sum_{i=1}^{2}(r - \delta_i - \frac{1}{2}\sum_{j=1}^{2}\sigma_{ij}^2)\frac{\partial F}{\partial x_i} - rF$$

where the matrix $A = (a_{ij})_{1\leq i,j\leq 2}$ equals $\Sigma^*\Sigma$.
Formally, the function V is the solution in a certain weak sense of the following obstacle problem on $[0,T[\times R^2$

$$\left\{\begin{array}{c}\max(\mathcal{L}U, \psi - U) = 0\\ U(T,.) = \psi\end{array}\right. \tag{4}$$

Let us recall what we understand by solution of problem (4).

1.3.1 Variational Inequality

In order to write the variational form of problem (4), we introduce some weighted Sobolev spaces. Let m be a nonnegative integer and let $1 \leq p \leq \infty$ and $0 < \gamma < \infty$. $W^{m,p,\gamma}(R^2)$ will denote the space of all functions u in $L^p(R^2, e^{-\gamma|x|}dx)$ whose weak derivatives of all orders $\leq m$ exist and belong to $L^p(R^2, e^{-\gamma|x|}dx)$. We will write H_γ (resp. V_γ) instead of $W^{0,2,\gamma}(R^2)$ (resp. $W^{1,2,\gamma}(R^2)$) and the inner product on H_γ will be denoted by $(.,.)_\gamma$. Moreover, define

$$|u|_\gamma = \left(\int_{R^2} |u(x)|^2 e^{-\gamma|x|}\, dx\right)^{\frac{1}{2}} = (u,u)_\gamma^{\frac{1}{2}}$$

and

$$||u||_\gamma = \left(|u|_\gamma^2 + \sum_{i=1}^{2} |\frac{\partial u}{\partial x_i}|_\gamma^2\right)^{\frac{1}{2}}.$$

We define a bilinear form on V_γ in the following way:

$$a_\gamma(u,v) = \sum_{i=1}^{2}\left(\int \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} e^{-\gamma|x|}\, dx - \gamma \int \frac{\partial u}{\partial x_i} v \frac{x_i}{|x|} e^{-\gamma|x|}\, dx\right) + \rho \int u v e^{-\gamma|x|}\, dx$$

With these notations, a function u defined on $[0,T[\times R^2$ is in $L^2([0,T];V_\gamma)$ if

$$\int_0^T ||u(t,.)||_\gamma^2\, dt < \infty.$$

Theorem 1.1 below states that the function U equals the unique solution of the variational inequality (5) (see Bensoussan-Lions and Jaillet-Lamberton-Lapeyre).

Theorem 1.1. *Assume $\psi \in W^{1,p,\gamma}(R^2)$ with $p > 2$.*

1- There is one and only one function u satisfying $u \in L^2([0,T];V_\gamma)$ and $\frac{\partial u}{\partial t} \in L^2([0,T];H_\gamma)$ such that:

$$\left\{\begin{array}{l} u(T,x) = \phi(x)\\ u \geq \phi \ a.e \ in \ [0,T[\times R^2\\ \forall v \in V_\gamma, (\ v \geq \psi \rightarrow -(\frac{\partial u}{\partial t}, v-u)_\gamma + a_\gamma(u, v-u) \geq 0\end{array}\right. \tag{5}$$

2- The function U is equal to the solution of (5).

Remark 1.2. *If ψ satisfies (2) then we can find $\gamma > 0$ such that ψ belongs to $W^{1,p,\gamma}(R^2)$ with $p > 2$. Hence, theorem 1.1 is valid in our context.*

1.3.2 Viscosity solution

First, we present the notation used in this section and recall the definition of viscosity solution suitable for our obstacle problem. We denote by $\mathcal{C}^{1,2}([0,T] \times R^n)$ the set of functions once continuously differentiable in t and twice continuously differentiable in x.

Definition 1.2. *A function $u : [0,T] \times R^2 \to R$ is a viscosity subsolution (resp. supersolution) of (4) if u is upper (resp. lower) semicontinuous such that*
1) $u(T,.) \leq$ (resp. $\geq$) ψ.
2) for every $w \in \mathcal{C}^{1,2}(]0,T[\times R^2)$, if $(t_0,x_0) \in]0,T[\times R^2$ is a local maximum (resp. minimum) of $u - w$ then

$$\max(\frac{\partial w}{\partial t}(t_0,x_0) + \mathcal{A}w(t_0,x_0), \psi(x_0) - u(t_0,x_0)) \leq \text{ (resp. } \geq) \, 0$$

A viscosity solution is both a viscosity subsolution and supersolution of (4).

We have the following result:

Proposition 1.2. *The function V is the unique viscosity solution of (4).*

For the numerical analysis of our equation, see Finite Difference Methods