CDO Pricing : Copula

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Premia 14

1 Link between CDOs and Copulas

We will thereafter consider a synthetic CDO with some given maturity T. This is based upon n CDS with nominals $N_j, j = 1, \dots, n$ and maturity also equal to T. We denote by δ_j the *recovery rate* for credit j and by $M_j = (1-\delta_j)N_j$ the corresponding *loss given default*.

For the n names in the collateral pool, we consider the associated default times $\tau_1, \dots, \tau_n$ defined on a common probability space $(\Omega, \mathcal{G}, \mathbb{P})$. In the following, we will consider only reduced-form models of default times defined by

$$\tau_i = \inf\Big\{u \in \mathbb{R}^+, \int_0^u h_i(v)dv \geq -\log(U_i)\Big\}, \qquad (H_\tau)$$

where the h_i are deterministic and continuous positive functions, the U_i are some uniform random variables.

In order to compute (by a semi-analytic approach) the price of one CDO tranche, all we need is the portfolio loss distribution *i.e.* the portfolio aggregate loss on the credit portfolio at time t:

$$L(t) = \sum_{j=1}^{n} M_j \mathbf{1}_{\{\tau_j \leq t\}}$$

which is a pure jump process. This distribution depend on the joint distribution of the default times $\tau_1, \dots, \tau_n$ that we modelling using a classical factor approach and Copula functions.

We denote by F and S respectively the *joint distribution* and survival functions such that for all $(t_1, \dots t_n) \in [0,T]^n$, $F(t_1, \dots, t_n) = \mathbb{P}(\tau_1 \leq t_1, \dots, \tau_n \leq t_n)$ and $S(t_1, \dots, t_n) = \mathbb{P}(\tau_1 > t_1, \dots, \tau_n > t_n)$. $F_1, \dots, F_n$ represent the *marginal* distribution functions and $S_1, \dots, S_n$ the corresponding survival functions. By the assumption (H_τ), we have

$$S_i(t) = \mathbb{P}(\tau_i > t) = \exp(-\int_0^t h_i(v)dv). \qquad (1)$$

We refer to Appendix for the proof of (1).

We will consider now a latent factor V such that conditionally on V, the default times are independent. We will denote by $p_t^{i|V} = \mathbb{P}(\tau_i \leq t|V)$ and $q_t^{i|V} = 1 - p^{i|V}(t)$ the conditional default and survival probabilities. It is easy to check that

$$S(t_1, \dots, t_n) = \int \prod_{i=1}^{n} q_t^{i|v} f(v)dv,$$

$$F(t_1, \dots, t_n) = \int \prod_{i=1}^{n} p_t^{i|v} f(v)dv.$$

So, if we can easily compute the conditional default probabilities and integrate along the density of the factor V, we are able to compute the joint distribution of the default times.

Définition 1. A copula C is a multivariate joint distribution on the m-dimensional unit cube $[0,1]^m$ such that every marginal distribution is uniform on the interval $[0,1]$.

$$C : (u_1, \cdots, u_m) \in [0,1]^m \longmapsto \mathbb{P}(U_1 \leq u_1, \cdots, U_m \leq u_m)$$

$U_1, \cdots, U_m$ is a random vector whose marginals are uniform on $[0,1]$.

2 Gaussian Copula

We consider a standard Gaussian random variable V, and we define the Gaussian vector $(X_1, \ldots, X_n)$ by

$$X_i = \rho V + \sqrt{1-\rho^2} V_i$$

where V_i are *independent* ($\forall i, j, V_i \perp V_j$ and $\forall i, V_i \perp V$) standard Gaussian random variables. We define the uniform random variable $U_i = 1 - \mathcal{N}(X_i)$ where $\mathcal{N}$ is the cumulative distribution function of a standard Gaussian variable. The joint distribution of $(U_1, \ldots, U_n)$ is known as the Gaussian copula. Then, we get

$$p_t^{i|V} = \mathcal{N}\Big(\frac{\mathcal{N}^{-1}(F_i(t)) - \rho V}{\sqrt{1-\rho^2}}\Big). \tag{2}$$

with $F_i(t) = 1 - \exp(-\int_0^t h_i(v)dv)$.

We refer to Appendix for the proof of (2).

3 Clayton Copula

We consider a positive random variable V following a standard Gamma distribution $\Gamma(\lambda, \alpha)$ with parameters $\lambda = 1, \alpha = 1/\theta$ where $\theta > 0$. Its probability density is given by $f(x) = \frac{1}{\Gamma(\theta^{-1})} \exp(-x) x^{(1-\theta)/\theta}$ for $x > 0$. We define the uniform random variables $U_1, \cdots, U_n$

$$U_i = 1 - \Psi\Big(-\frac{\log(\bar{U}_i)}{V}\Big),$$

where $\bar{U}_1, \ldots, \bar{U}_n$ are independent uniform random variables also independent from V, and Ψ is the Laplace transform of f. The joint distribution of $(U_1, \ldots, U_n)$ is known as the Clayton copula.

The conditional default probabilities can be expressed as

$$p_t^{i|V} = \exp\Big(V(1 - F_i(t)^{-\theta})\Big),$$

with $F_i(t) = 1 - \exp(-\int_0^t h_i(v)dv)$.

4 NIG Copula

For more details on the NIG copula, we refer to [1]. The Normal Inverse Gaussian distribution (NIG) is a mixture of normal and inverse Gaussian distributions. A non-negative r.v.

Y has inverse Gaussian distribution with parameters $\alpha > 0$ and $\beta > 0$ if its density function is of the form:

$$f_{\mathcal{IG}}(y;\alpha,\beta) = \begin{cases} \frac{\alpha}{\sqrt{2\pi\beta}} y^{-3/2} \exp\left(-\frac{(\alpha-\beta y)^2}{2\beta y}\right) & \text{if } y > 0 \\ 0 & y \leq 0. \end{cases}$$

A r.v. X follows a Normal Inverse Gaussian (NIG) distribution with parameters α, β, μ and δ if:

$$\begin{aligned} &X|Y = y \sim \mathcal{N}(\mu + \beta y, y) \\ &Y \sim \mathcal{IG}(\delta\gamma, \gamma^2) \text{ with } \gamma := \sqrt{\alpha^2 - \beta^2}, \end{aligned}$$

with parameters satisfying the following conditions: $0 \leq |\beta| < \alpha$ and $\delta > 0$. We write $X \sim \mathcal{NIG}(\alpha, \beta, \mu, \delta)$ and the density function is given by:

$$f_{\mathcal{NIG}}(x;\alpha,\beta,\mu,\delta) = \frac{\delta\alpha \exp(\delta\gamma + \beta(x-\mu))}{\pi\sqrt{\delta^2 + (x-\mu)^2}} K_1(\alpha\sqrt{\delta^2 + (x-\mu)^2}),$$

where $K_1(w) = \frac{1}{2}\int_0^\infty \exp(\frac{-1}{2}w(t+t^{-1}))dt$ is the modified Bessel function of the third kind. The probability function is given by

$$\begin{aligned} F_{\mathcal{NIG}}(x) := \int_{-\infty}^{x} f_{\mathcal{NIG}}(t)dt &= \int_0^\infty \mathcal{N}\left(\frac{x-(\mu+\beta y)}{\sqrt{y}}\right) f_{\mathcal{IG}}(y;\delta\gamma,\gamma^2)dy \\ &= \int_0^1 \mathcal{N}\left(\frac{x-(\mu-\beta\log(t))}{\sqrt{-\log(t)}}\right) f_{\mathcal{IG}}(-\log(t);\delta\gamma,\gamma^2)\frac{1}{t}dt \end{aligned}$$

The first equality is due to the fact that the NIG distribution stems from a convolution of the normal and the inverse Gaussian distribution. The second one follows from the change of variable $t = \exp(-y)$.

The main properties of the NIG distribution class are the scaling property

$$X \sim \mathcal{NIG}(\alpha,\beta,\mu,\delta) \Longrightarrow cX \sim \mathcal{NIG}(\frac{\alpha}{c},\frac{\beta}{c},c\mu,c\delta)$$

and the closure under convolution for independent r.v. X and Y

$$\begin{aligned} &X \sim \mathcal{NIG}(\alpha,\beta,\mu_1,\delta_1), Y \sim \mathcal{NIG}(\alpha,\beta,\mu_2,\delta_2) \\ &\Longrightarrow X + Y \sim \mathcal{NIG}(\alpha,\beta,\mu_1+\mu_2,\delta_1+\delta_2) \end{aligned}$$

In our implementation, we consider a random variable V following a NIG distribution with parameters

$$V \sim \mathcal{NIG}(\alpha,\beta,-\frac{\alpha\beta}{\gamma},\alpha)$$

where $\gamma = \sqrt{\alpha^2-\beta^2}$, and we define the vector $(X_1,\cdots,X_n)$

$$X_i = \rho V + \sqrt{1-\rho^2}V_i,$$

where V_i are independent (and independent from V) NIG random variables with parameters

$$V_i \sim \mathcal{NIG}(\frac{\sqrt{1-\rho^2}}{\rho}\alpha, \frac{\sqrt{1-\rho^2}}{\rho}\beta, -\frac{\sqrt{1-\rho^2}}{\rho}\frac{\alpha\beta}{\gamma}, \frac{\sqrt{1-\rho^2}}{\rho}\alpha).$$

To simplify notations we denote $F_{\mathcal{NIG}(s)}(x)$ the cumulative distribution of a NIG random variable with parameters $\mathcal{NIG}(s\alpha, s\beta, -s\frac{\alpha\beta}{\gamma}, s\alpha)$. Using the scaling property and stability under convolution of NIG distribution we get $X_i \sim \mathcal{NIG}(1/\rho)$. We define the uniform random variable $U_i = 1 - F_{\mathcal{NIG}(1/\rho)}(X_i)$. The joint distribution of $(U_1, \ldots, U_n)$ is known as the NIG copula. We get

$$p_t^{i|V} = F_{\mathcal{NIG}(\sqrt{1-\rho^2}/\rho)}\Big(\frac{F^{-1}_{\mathcal{NIG}(1/\rho)}(F_i(t)) - \rho V}{\sqrt{1-\rho^2}}\Big).$$

5 Student Copula

Définition 2. Let G and Y be two independent r.v. s.t. $G \sim \mathcal{N}(0,1)$ and $Y \sim \chi^2(\nu)$. Then, $X := \sqrt{\frac{\nu}{Y}}G$ is a Student r.v. with parameter ν. The density of the Student law is given by

$$t_\nu(t) := \frac{\Gamma((\nu+1)/2)}{\sqrt{\nu\pi}} \frac{1}{\left(1+\frac{t^2}{\nu}\right)^{(\nu+1)/2}}, t \in \mathbb{R}.$$

In our implementation, we define for $i = 1, \cdots, n$

$$X_i = \sqrt{\frac{\nu}{Y}}\left(\rho V + \sqrt{1-\rho^2}V_i\right),$$

where V, Y and $V_1, \cdots, V_n$ are independent r.v. s.t. $V \sim \mathcal{N}(0,1)$, $V_i \sim \mathcal{N}(0,1)$ and $Y \sim \chi^2(\nu)$. X_i is a Student r.v. with parameter ν. Let us define $U_i := 1 - T_\nu(X_i)$, where T_ν is the cumulative distribution function of a Student r.v. with parameter ν. The joint distribution of $(U_1, \ldots, U_n)$ is known as the Student copula. We get

$$p_t^{i|V} = \mathcal{N}\Big(\frac{T_\nu^{-1}(F_i(t))\sqrt{\frac{Y}{\nu}} - \rho V}{\sqrt{1-\rho^2}}\Big). \tag{3}$$

The proof of (3) is postponed to the Appendix.

6 Double T Copula

Let M be a Student r.v. s.t. $M \sim S(\nu)$. We define the vector $(X_1, \cdots, X_n)$ by

$$X_i := \rho\sqrt{\frac{\nu-2}{\nu}}M + \sqrt{1-\rho^2}\sqrt{\frac{\bar{\nu}-2}{\bar{\nu}}}Z_i,$$

where $(Z_i)_{i=1\cdots n}$ are independent Student variables with parameter $\bar{\nu}$. Let $T_{\nu,\bar{\nu}}$ denote the cumulative distribution of X_i. The definition of $T_{\nu,\bar{\nu}}$ can be found by using the law of M and Z_i:

$$\mathbb{P}(X_i \le z) = \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy \left(\rho\sqrt{\frac{\nu-2}{\nu}}x + \sqrt{1-\rho^2}\sqrt{\frac{\bar{\nu}-2}{\bar{\nu}}}y\right) \mathbf{1}_{x+y\le z} t_\nu(x) t_{\bar{\nu}}(y).$$

Then, we define $U_i := 1 - T_{\nu,\bar{\nu}}(X_i)$. The joint distribution of $(U_1, \dots, U_n)$ is known as the Double-t copula. We get

$$p_t^{i|V} = T_{\bar{\nu}}\Big(\frac{T_{\nu,\bar{\nu}}^{-1}(F_i(t)) - \rho\sqrt{\frac{\nu-2}{\nu}}M}{\sqrt{1-\rho^2}\sqrt{\frac{\bar{\nu}-2}{\bar{\nu}}}}\Big). \tag{4}$$

The proof of (4) is similar to the proof of (3).

7 Numerical values for the Copula parameters

Type	Parameters (example value)
Gaussian	Correlation ρ (0.03)
Clayton	θ (0.2)
NIG	Correlation ρ (0.06), α (1.2), β (-0.2)
Student	Correlation ρ (0.02), Degree of freedom $t1$ (5)
Double t	Correlation ρ (0.03), Degree of freedom $t1$ (5), Degree of freedom $t2$ (7)

8 Appendix

8.1 Proof of (1)

We have

$$\begin{aligned}\tau_i &= \inf\{u \in \mathbb{R}^+ : -\int_0^u h_i(v)dv \geq -\log(U_i)\}, \\ &= \inf\{u \in \mathbb{R}^+ : \exp(-\int_0^u h_i(v)dv) \leq U_i\}.\end{aligned}$$

Since $u \longmapsto \exp(-\int_0^u h_i(v)dv)$ is adecreasing function (denoted f) with values in $[0,1]$, we get

$$\mathbb{P}(\tau_i > t) = \mathbb{P}(\inf\{u \in \mathbb{R}^+ : f(u) \leq U_i\} > t) = \mathbb{P}(f(t) > U_i) = f(t) = \exp(-\int_0^t h_i(v)dv).$$

8.2 Proof of (2)

U_i is a uniform r.v.: $1 - U_i = \Phi(X_i)$ and Φ is the cumulative density function of X_i, then $\Phi(X_i) \sim \mathcal{U}[0,1]$. $(\mathbb{P}(U_i \leq k) = \mathbb{P}(\Phi(X_i) \leq k) = \mathbb{P}(X_i \leq \Phi^{-1}(k)) = \Phi(\Phi^{-1}(k)) = k)$.
From the definition of τ_i, we get $\tau_i = \inf\{t : S_i(t) \leq U_i\} = \inf\{t : 1 - F_i(t) \leq U_i\} = \inf\{t : F_i(t) \geq 1 - U_i\} = \inf\{t : F_i(t) \geq \Phi(X_i)\} = \inf\{t : X_i \leq \Phi^{-1}(F_i(t))\}$. Then

$$\begin{aligned}\mathbb{P}(\tau_i \leq t|V) &= \mathbb{P}(\inf\{u \in \mathbb{R}^+ : X_i \leq \Phi^{-1}(F_i(u))\} \leq t|V) \\ &= \mathbb{P}(\Phi^{-1}(F_i(t)) > X_i|V) \text{ since } u \longmapsto \Phi^{-1}(F_i(u)) \text{ is increasing} \\ &= \mathbb{P}(\rho V + \sqrt{1-\rho^2}\bar{V}_i < \Phi^{-1}(F_i(t))|V) \\ &= \mathbb{P}\left(\bar{V}_i < \frac{\Phi^{-1}(F_i(t)) - \rho V}{\sqrt{1-\rho^2}}|V\right) \\ &= \Phi\left(\frac{\Phi^{-1}(F_i(t)) - \rho V}{\sqrt{1-\rho^2}}\right).\end{aligned}$$

8.3 Clayton Copula

The Laplace transform of f is $\Psi(s) = \frac{1}{(1+s)^{\theta^{-1}}}$. Indeed,

$$\begin{aligned}\Psi(s) &= \frac{1}{\Gamma(\theta^{-1})}\int_0^\infty e^{-sx}e^{-x}x^{\frac{1}{\theta}-1}dx\\ &= \frac{1}{\Gamma(\theta^{-1})}\int_0^\infty e^{-(s+1)x}x^{\frac{1}{\theta}-1}dx\\ &= \frac{1}{\Gamma(\theta^{-1})}\int_0^\infty e^{-t}t^{\frac{1}{\theta}-1}dt\frac{1}{(s+1)^{\theta^{-1}}}.\end{aligned}$$

Since $\Gamma(\theta^{-1}) = \int_0^\infty e^{-t}t^{\frac{1}{\theta}-1}dt$, we get the result The inverse function is $\Psi^{-1}(s) = s^{-\theta} - 1$. The r.v. U_i are uniform:

$$\begin{aligned}\mathbb{P}(U_i \le k) = \mathbb{P}(1-U_i \le k) &= \mathbb{P}\left(\Psi\Big(-\frac{\log(\bar{U}_i)}{V}\Big) \le k\right) = \mathbb{P}\left(-\frac{\log(\bar{U}_i)}{V} \ge \Psi^{-1}(k)\right)\\ &= \mathbb{P}(-\log(\bar{U}_i) \ge V(k^{-\theta}-1)) = \mathbb{P}(\bar{U}_i \le \exp(V(1-k^{-\theta})))\\ &= \mathbb{E}[\mathbb{E}[\mathbf{1}_{\bar{U}_i \le \exp(V(1-k^{-\theta}))}|V]] = \mathbb{E}[\mathbb{P}(\bar{U}_i \le \exp(V(1-k^{-\theta}))|V)]\\ &= \mathbb{E}[\exp(V(1-k^{-\theta}))] = \frac{1}{\Gamma(\theta^{-1})}\int_0^\infty \exp(v(1-k^{-\theta}))e^{-v}v^{\frac{1}{\theta}-1}dv\\ &= \frac{1}{\Gamma(\theta^{-1})}\int_0^\infty e^{-vk^{-\theta}}v^{\frac{1}{\theta}-1}dv = \frac{1}{\Gamma(\theta^{-1})}\int_0^\infty e^{-x}x^{\frac{1}{\theta}-1}dx\frac{1}{k^{-\theta}}\left(\frac{1}{k^{-\theta}}\right)^{\frac{1}{\theta}-1} = k.\end{aligned}$$

Let us prove that the conditional default probability is $p_t^{i|V} = \exp\Big(V(1-F_i(t)^{-\theta})\Big)$. To do so, we write $\tau_i = \inf\{t : S_i(t) \le U_i\} = \inf\{t : 1-F_i(t) \le U_i\} = \inf\{t : F_i(t) \ge 1-U_i\} = \inf\{t : F_i(t) \ge \Psi\Big(-\frac{\log(\bar{U}_i)}{V}\Big)\} = \inf\{t : \bar{U}_i \le \exp(-V\Psi^{-1}(F_i(t)))\}$. Then, since $u \longmapsto \exp(-V\Psi^{-1}(F_i(u)))$ is an increasing function, we get

$$\begin{aligned}\mathbb{P}(\tau_i \le t|V) &= \mathbb{P}(\inf\{u \in \mathbb{R}^+ : \bar{U}_i \le \exp(-V\Psi^{-1}(F_i(u)))\} \le t|V)\\ &= \mathbb{P}(\bar{U}_i \le \exp(-V\Psi^{-1}(F_i(t)))|V) = \exp\Big(V(1-F_i(t)^{-\theta})\Big).\end{aligned}$$

8.4 Proof of (3)

From the definition of τ_i, we get $\tau_i = \inf\{t : S_i(t) \le U_i\} = \inf\{t : 1-F_i(t) \le U_i\} = \inf\{t : F_i(t) \ge 1-U_i\} = \inf\{t : F_i(t) \ge T_\nu(X_i)\} = \inf\{t : X_i \le T_\nu^{-1}(F_i(t))\}$. Then

$$\begin{aligned}\mathbb{P}(\tau_i \le t|V) &= \mathbb{P}(\inf\{u \in \mathbb{R}^+ : X_i \le T_\nu^{-1}(F_i(u))\} \le t|V)\\ &= \mathbb{P}(T_\nu^{-1}(F_i(t)) > X_i|V) \text{ since } u \longmapsto T_\nu^{-1}(F_i(u)) \text{ is increasing}\\ &= \mathbb{P}\left(\rho V + \sqrt{1-\rho^2}V_i < \sqrt{\frac{Y}{\nu}}T_\nu^{-1}(F_i(t))|V\right)\\ &= \mathbb{P}\left(V_i < \frac{\sqrt{\frac{Y}{\nu}}T_\nu^{-1}(F_i(t)) - \rho V}{\sqrt{1-\rho^2}}|V\right)\\ &= \mathcal{N}\left(\frac{\sqrt{\frac{Y}{\nu}}T_\nu^{-1}(F_i(t)) - \rho V}{\sqrt{1-\rho^2}}\right).\end{aligned}$$

References

[1] Kalemanova, Schmid, Werner (2005) *The Normal Inverse Gaussian distribution for synthetic CDO pricing.* Preprint. 2