Exact retrospective Monte Carlo computation of arithmetic average Asian options

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Abstract

Taking advantage of the recent litterature on exact simulation algorithms (Beskos, Papaspiliopoulos and Roberts [1]) and unbiased estimation of the expectation of certain fonctional integrals (Wagner [27], Beskos et al. [2] and Fearnhead et al. [6]), we apply an exact simulation based technique for pricing continuous arithmetic average Asian options in the Black & Scholes framework. Unlike existing Monte Carlo methods, we are no longer prone to the discretization bias resulting from the approximation of continuous time processes through discrete sampling. Numerical results of simulation studies are presented and variance reduction problems are considered.

Introduction

Although the Black & Scholes framework is very simple, it is still a challenging task to efficiently price Asian options. Since we do not know explicitly the distribution of the arithmetic sum of log-normal variables, there is no closed form solution for the price of an Asian option. By the early nineties, many researchers attempted to address this problem and hence different approaches were studied including analytic approximations (see Turnbull and Wakeman [24], Vorst [26], Levy [17] and more recently Lord [18]), PDE methods (see Vecer [25], Rogers and Shi [21], Ingersoll [11], Lelievre and Dubois [5]), Laplace transform inversion methods (see Geman and Yor [10], Geman and Eydeland [8]) and, of course, Monte Carlo simulation methods (see Kemna and Vorst [15], Broadie and Glasserman [3], Fu, Madan and Wang [7]).

Monte Carlo simulation can be computationally expensive because of the usual statistical error. Variance reduction techniques are then essential to accelerate the convergence (one of the most efficient technique is the Kemna&Vorst control variate based on the geometric average). One must also account for the inherent discretization bias resulting from approximating the continuous average of the stock price with a discrete one. It is crucial to choose with care the discretization scheme in order to have an accurate solution (see Lapeyre and Temam [16]). The main contribution of our work is to fully address this last feature by the use, after a suitable change of variables, of an exact simulation method inspired from the recent work of Beskos et al. ([1] and [2]) and Fearnhead et al. [6].

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In the first part of this note, we construct an unbiaised estimator of the price of an European option when the underlying solves a one-dimensional stochastic differential equation. By a suitable change of variables, one may suppose that the diffusion coefficient is equal to one. Then, according to the Girsanov theorem, one may deal with the drift coefficient by introducing an exponential martingale weight. Because of the one-dimensional setting, the stochastic integral in this exponential weight is equal to a standard integral with respect to the time variable up to the addition of a function of the terminal value of the path. The entire series expansion of the exponential function permits to replace this exponential weight by a computable weight with the same conditional expectation given the Brownian path. This idea was first introduced by Wagner [27],[28],[29] and [30] in a statistical physics context and it was very recently revisited by Beskos et al. [2] and Fearnhead et al. [6] for the estimation of partially observed diffusions.

The second part is devoted to the application of this method to continuous Asian option pricing within the Black & Scholes framework. Throughout the paper, $S_t = S_0 \exp\left(\sigma W_t + (r - \delta - \frac{\sigma^2}{2})t\right)$ represents the stock price at time t, T the maturity of the option, r the short interest rate, σ the volatility parameter, δ the dividend rate and $(W)_{t\in[0,T]}$ denotes a standard Brownian motion on the risk-neutral probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We are interested in computing the price $C_0 = \mathbb{E}\left(e^{-rT} f\left(\frac{1}{T}\int_0^T S_t dt\right)\right)$ of a European option with pay-off $f\left(\frac{1}{T}\int_0^T S_t dt\right)$ assumed to be square integrable under the risk neutral measure $\mathbb{P}$. We propose a new change of variables which is singular at initial time and we show how to apply the method based on the unbiased estimator of Wagner [27] presented in the first section.

1 The unbiased estimator (U.E)

Consider the stochastic process $(\xi_t)_{0\le t\le T}$ determined as the solution of a general stochastic differential equation of the form :

$$\begin{cases} d\xi_t &= b(\xi_t)dt + \sigma(\xi_t)dW_t \\ \xi_0 &= \xi \in \mathbb{R} \end{cases} \tag{1}$$

where b and σ are scalar functions satisfying the usual Lipschitz and growth conditions with σ non vanishing. To simplify this equation, [1] suggests to use the following change of variables : $X_t = \eta(\xi_t)$ where η is a primitive of $\frac{1}{\sigma}$ $(\eta(x) = \int_.^x \frac{1}{\sigma(u)}du)$.

Under the additional assumption that $\frac{1}{\sigma}$ is continuously differentiable, one can apply ItÃ´'s lemma to get

$$\begin{aligned} dX_t &= \eta'(\xi_t)d\xi_t + \frac{1}{2}\eta''(\xi_t)\, d\!<\xi,\xi>_t \\ &= \frac{b(\xi_t)}{\sigma(\xi_t)}dt + dW_t - \frac{\sigma'(\xi_t)}{2}dt \\ &= \underbrace{\left(\frac{b(\eta^{-1}(X_t))}{\sigma(\eta^{-1}(X_t))} - \frac{\sigma'(\eta^{-1}(X_t))}{2}\right)}_{a(X_t)} dt + dW_t \end{aligned}$$

So $\xi_t = \eta^{-1}(X_t)$ where $(X_t)_t$ is a solution of the stochastic differential equation

$$\begin{cases} dX_t &= a(X_t)dt + dW_t \\ X_0 &= x. \end{cases} \tag{2}$$

Thus, without loss of generality, one can start from equation (2) instead of (1).

In finance, the pricing of contingent claims often comes down to the problem of computing an expectation of the form

$$C_0 = \mathbb{E}\left(f(X_T)\right) \tag{3}$$

where X is a solution of the SDE (2) and f is a scalar function such that $f(X_T)$ is square integrable. In a simulation based approach, one is usually unable to exhibit an explicit solution of this SDE and will therefore resort to numerical discretization schemes, such as the Euler or Milstein schemes, which introduce a bias. Here, we are going to present a technique which permits to compute exactly the expectation (3) under mild assomptions.

Let us denote by $(W_t^x)_{t\in[0,T]}$ the process $(W_t + x)_{t\in[0,T]}$, by $\mathbb{Q}_{W^x}$ its law and by $\mathbb{Q}_X$ the law of the process $(X_t)_{t\in[0,T]}$. From now on, we will denote by $(Y_t)_{t\in[0,T]}$ the canonical process, that is the coordinate mapping on the set $C([0,T],\mathbb{R})$ of real continuous maps on $[0,T]$ (see [20] or [14]).

One needs the following assumption to be true

Assumption 1 : *Under $\mathbb{Q}_{W^x}$, the process*

$$L_t = \exp\left[\int_0^t a(Y_u)dY_u - \frac{1}{2}\int_0^t a^2(Y_u)du\right]$$

is a martingale.

According to Rydberg [22], a sufficient condition for this assumption to hold is

-Existence and uniqueness in law of a solution to the SDE (2).

-$\forall t\in[0,T]$, $\int_0^t a^2(Y_u)du < \infty$, $\mathbb{Q}_X$ and $\mathbb{Q}_{W^x}$ almost surely on $C([0,T],\mathbb{R})$.

Thanks to this assumption, one can apply the Girsanov theorem to get that $\mathbb{Q}_X$ is absolutely continuous with respect to $\mathbb{Q}_{W^x}$ and its Radon-Nikodym derivative is equal to

$$\frac{d\mathbb{Q}_X}{d\mathbb{Q}_{W^x}} = \exp\left[\int_0^T a(Y_t)dY_t - \frac{1}{2}\int_0^T a^2(Y_t)dt\right].$$

Consider A the primitive of the drift a, and assume that

Assumption 2 : *a is continuously differentiable.*

Since, by ItÃ´'s lemma, $A(W_T^x) = A(x) + \int_0^T a(W_t^x)dW_t^x + \frac{1}{2}\int_0^T a'(W_t^x)dt$, we have

$$\frac{d\mathbb{Q}_X}{d\mathbb{Q}_{W^x}} = \exp\left[A(Y_T) - A(x) - \frac{1}{2}\int_0^T a^2(Y_t) + a'(Y_t)dt\right].$$

So, the expectation (3) writes

$$C_0 = \mathbb{E}\left(f(W_T^x)\exp\left[A(W_T^x) - A(x) - \frac{1}{2}\int_0^T a^2(W_t^x) + a'(W_t^x)dt\right]\right). \tag{4}$$

In order to implement an importance sampling method, let us introduce a positive density ρ on the real line and a process $(Z_t)_{t\in[0,T]}$ distributed according to the following law $\mathbb{Q}_Z$

$$\mathbb{Q}_Z = \int_{\mathbb{R}} \mathcal{L}\Big((W_t^x)_{t\in[0,T]}|W_T^x = y\Big)\rho(y)dy.$$

By (4), one has

$$C_0 = \mathbb{E}\left(\psi(Z_T)\exp\left[-\int_0^T \phi(Z_t)dt\right]\right) \tag{5}$$

where $\psi : z \mapsto f(z)\frac{e^{A(z)-A(x)-\frac{(z-x)^2}{2T}}}{\sqrt{2\pi}\rho(z)}$ and $\phi : z \mapsto \frac{a^2(z)+a'(z)}{2}$. The free parameter ρ is chosen in such a way that it reduces the variance of the simulation.

In his first paper [27], Wagner constructs an unbiased estimator of the expectation (5) when ψ is a constant, $(Z_t)_{t\in[0,T]}$ is an $\mathbb{R}^d-$valued Markov process with known transition function and ϕ is a measurable function such that $\mathbb{E}\left(e^{\int_0^T |\phi(Z_t)|dt}\right) < +\infty$. His main idea is to expand the exponential term in a power series, then, using the transition function of the underlying Markov process and symmetry arguments, he constructs a signed measure ν on the space $\mathcal{Y} = \bigcup_{n=0}^{+\infty}([0,T] \times \mathbb{R}^d)^{n+1}$ such that the expectation at hand is equal to $\nu(\mathcal{Y})$. Consequently, any probability measure μ on Y that is absolutely continuous with respect to ν gives rise to an unbiased estimator ζ defined on $(\mathcal{Y}, \mu)$ via $\zeta(y) = \frac{d\nu}{d\mu}(y)$. In practice, a suitable way to construct such an estimator is to use a Markov chain with an absorbing state. Wagner also discusses variance reduction techniques, specially importance sampling and a shift procedure consisting on adding a constant c to the integrand ϕ and then multiplying by the factor e^{-cT} in order to get the right expectation. Wagner [29] extends the class of unbiased estimators by perturbating the integrand ϕ by a suitably chosen function ϕ_0 and then using mixed integration formulas representation. Very recently, Beskos et al. [2] obtained a simplified unbiased estimator for (5), termed Poisson estimator, using Wagner's idea of expanding the exponential in a power series and his shift procedure. To be specific, the Poisson estimator writes

$$\psi(Z_T)e^{c_p T - cT}\prod_{i=1}^{N}\frac{c - \phi(Z_{V_i})}{c_P} \tag{6}$$

where N is a Poisson random variable with parameter c_P and $(V_i)_i$ is a sequence of independent random variables uniformly distributed on $[0,T]$. Fearnhead et al. [6] generalized this estimator allowing c and c_P to depend on Z and N to be distributed according to any positive probability distribution on $\mathbb{N}$. They termed the new estimator the generalized Poisson estimator. We introduce a new degree of freedom by allowing the sequence $(V_i)_i$ to be distributed according to any positive density on $[0,T]$. This gives rise to following unbiased estimator for (5) :

Lemma 1. — *Let p_Z and q_Z denote respectively a positive probability measure on $\mathbb{N}$ and a positive probability density on $[0,T]$. Let N be distributed according to p_Z and $(V_i)_{i\in\mathbb{N}^*}$ be a sequence of independent random variables identically distributed according to the density q_Z, both independent from each other conditionally on the process $(Z_t)_{t\in[0,T]}$. Let c_Z be a real number which may depend on Z. Assume that*

$$\mathbb{E}\left(|\psi(Z_T)|e^{-c_Z T}\exp\left[\int_0^T |c_Z - \phi(Z_t)|dt\right]\right) < \infty.$$

Then

$$\psi(Z_T)e^{-c_Z T}\frac{1}{p_Z(N)\,N!}\prod_{i=1}^{N}\frac{c_Z - \phi(Z_{V_i})}{q_Z(V_i)} \tag{7}$$

is an unbiased estimator of C_0.

Proof. The result follows from

$$\begin{aligned}\mathbb{E}\left(\psi(Z_T)e^{-c_Z T}\frac{1}{p_Z(N)\,N!}\prod_{i=1}^{N}\frac{c_Z - \phi(Z_{V_i})}{q_Z(V_i)}\Big|(Z_t)_{t\in[0,T]}\right) &= \psi(Z_T)e^{-c_Z T}\sum_{n=0}^{+\infty}\frac{\left(\int_0^T c_Z - \phi(Z_t)dt\right)^n}{p_Z(n)\,n!}p_Z(n)\\ &= \psi(Z_T)\exp\left(-\int_0^T \phi(Z_t)dt\right).\end{aligned}$$

□

Using (7), one is now able to compute the expectation at hand by a simple Monte Carlo simulation. The practical choice of p_Z and q_Z conditionnaly on Z is studied in the appendix A.

Remark 2. — *One can derive two estimators of C_0 from the result of Lemma 1 :*

$$\delta_1 = \frac{1}{n}\sum_{i=1}^{n} f(Z_T^i)\frac{e^{A(Z_T^i)-A(x)-\frac{(Z_T^i-x)^2}{2T}}}{\sqrt{2\pi}\rho(Z_T^i)}e^{-c_Z T}\frac{1}{p_Z(N^i)\,N^i!}\prod_{j=1}^{N^i}\frac{c_Z-\phi(Z_{V_j^i}^i)}{q_Z(V_j^i)}$$

$$\delta_2 = \frac{\displaystyle\sum_{i=1}^{n} f(Z_T^i)\frac{e^{A(Z_T^i)-A(x)-\frac{(Z_T^i-x)^2}{2T}}}{\sqrt{2\pi}\rho(Z_T^i)}\frac{1}{p_Z(N^i)\,N^i!}\prod_{j=1}^{N^i}\frac{c_Z-\phi(Z_{V_j^i}^i)}{q_Z(V_j^i)}}{\displaystyle\sum_{i=1}^{n}\frac{e^{A(Z_T^i)-A(x)-\frac{(Z_T^i-x)^2}{2T}}}{\sqrt{2\pi}\rho(Z_T^i)}\frac{1}{p_Z(N^i)\,N^i!}\prod_{j=1}^{N^i}\frac{c_Z-\phi(Z_{V_j^i}^i)}{q_Z(V_j^i)}}.$$

2 Application : the pricing of continuous Asian options

In the Black & Scholes model, the stock price is the solution of the following SDE under the risk-neutral measure $\mathbb{P}$

$$\frac{dS_t}{S_t} = (r-\delta)dt + \sigma dW_t \tag{8}$$

where all the parameters are constant : r is the short interest rate, δ is the dividend rate and σ is the volatility.
Throughout, we denote $\gamma = r - \delta - \frac{\sigma^2}{2}$. The path-wise unique solution of (8) is

$$S_t = S_0\,\exp(\sigma W_t + \gamma t)\,.$$

We consider an option with pay-off of the form

$$f\left(\frac{1}{T}\int_0^T S_t dt\right) \tag{9}$$

where f is a given function such that $\mathbb{E}\left(f^2\left(\frac{1}{T}\int_0^T S_t dt\right)\right) < \infty$, T is the maturity of the option. Note that an Asian call corresponds to the special case $f(x) = (x-K)_+$ and that an Asian put corresponds to $f(x) = (K-x)_+$.

The fundamental theorem of arbitrage-free pricing ensures that the price of the option under consideration is

$$C_0 = \mathbb{E}\left(e^{-rT} f\left(\frac{1}{T}\int_0^T S_u du\right)\right).$$

At first sight, the problem seems to involve two variables : the stock price and the integral of the stock price with respect to time. Dealing with the PDE associated with Asian option pricing, Rogers and Shi [21] used a suitable change of variables to reduce the spatial dimension of the problem to one. We are going to use a similar idea.

Let

$$\left\{\begin{array}{lcl} \xi_t & = & \dfrac{S_0}{t}\displaystyle\int_0^t e^{\sigma(W_t-W_u)+\gamma(t-u)}du \\ \xi_0 & = & S_0. \end{array}\right. \tag{10}$$

Obviously, the two variables ξ_T and $\frac{1}{T}\int_0^T S_u du$ have the same law. Hence, the price of the Asian option becomes

$$C_0 = \mathbb{E}\left(e^{-rT} f\left(\frac{1}{T}\int_0^T S_u du\right)\right) = \mathbb{E}\left(e^{-rT} f(\xi_T)\right).$$

Remark 3. — *The pricing of floating strike Asian options is also straightforward using this method. It is even more natural to consider these options since it unveils the appropriate change of variables as we shall see below.*

Let us consider a floating strike Asian call for example. We have to compute

$$C_0 = \mathbb{E}\left(e^{-rT}\big(\frac{1}{T}\int_0^T S_u du - S_T\big)_+\right).$$

Using $\widetilde{S}_t = S_t e^{\delta t}$ as a numÃ©raire (see the seminal paper of Geman et al. [9]), we immediately obtain that

$$C_0 = \mathbb{E}_{\mathbb{P}_{\widetilde{S}}}\left(S_0 e^{-\delta T}\big(\frac{1}{T}\int_0^T \frac{S_u}{S_T}du - 1\big)_+\right)$$

where $\mathbb{P}_{\widetilde{S}}$ is the probability measure associated to the numÃ©raire $\widetilde{S}_t$. It is defined by its Radon-Nikodym derivative $\frac{d\mathbb{P}_{\widetilde{S}}}{d\mathbb{P}} = e^{\sigma W_T - \frac{\sigma^2}{2}T}$.

Under $\mathbb{P}_{\widetilde{S}}$, the process $B_t = W_t - \sigma t$ is a Brownian motion and we can write that

$$\begin{aligned} C_0 &= \mathbb{E}_{\mathbb{P}_{\widetilde{S}}}\left(S_0 e^{-\delta T}\big(\tfrac{1}{T}\int_0^T e^{\sigma(B_u - B_T)+(r-\delta+\frac{\sigma^2}{2})(u-T)}du - 1\big)_+\right) \\ &= \mathbb{E}\left(S_0 e^{-\delta T}\big(\tfrac{1}{T}\int_0^T e^{\sigma(W_u - W_T)+(r-\delta+\frac{\sigma^2}{2})(u-T)}du - 1\big)_+\right) \\ &= \mathbb{E}\left(e^{-\delta T}\big(\xi_T - S_0\big)_+\right) \end{aligned}$$

where ξ_t is the process defined by (10) but with $\gamma = r - \delta + \frac{\sigma^2}{2}$. We see therefore that the problem simplifies to the fixed strike Asian pricing problem.

Let us write down the stochastic differential equation that rules the process $(\xi_t)_{t\in[0,T]}$. Using ItÃ´'s lemma, we get

$$\begin{cases} d\xi_t &= \frac{\xi_0 - \xi_t}{t}dt + \xi_t\left(\sigma dW_t + (\gamma + \frac{\sigma^2}{2})dt\right) \\ \xi_0 &= S_0. \end{cases}$$

Note that we are faced with a singularity problem near 0 because of the term $\frac{\xi_0 - \xi_t}{t}$. We are going to reduce its effect using another change of variables.

Using ItÃ´'s lemma, we show that

$$C_0 = \mathbb{E}\left(e^{-rT} f\left(S_0 e^{X_T}\right)\right) \tag{11}$$

where $X_t = \log(\xi_t/\xi_0)$ solves the following SDE

$$\begin{cases} dX_t &= \sigma dW_t + \gamma dt + \frac{e^{-X_t} - 1}{t}dt \\ X_0 &= 0. \end{cases} \tag{12}$$

Lemma 4. — *Existence and strong uniqueness hold for the stochastic differential equation (12).*

Proof. Existence is obvious since we have a particular solution X_t. The diffusion coefficient being constant and the drift coefficient being a decreasing function in the spatial variable, we have also strong uniqueness for the SDE. $\square$

Because of the singularity of the term $\frac{e^{-X_t}-1}{t}$ in the drift coefficient, the law of $(X_t)_{t\geq 0}$ is not absolutely continuous with respect to the law of $(\sigma W_t)_{t\geq 0}$. That is why we now define $(Z_t)_{t\geq 0}$ by the following SDE with an affine non-hommogenous drift coefficient :

$$\left\{\begin{array}{rcl} dZ_t &=& \sigma dW_t + \gamma dt - \dfrac{Z_t}{t}dt \\ Z_0 &=& X_0 = 0. \end{array}\right. \tag{13}$$

The drift coefficient exhibits the same behavior as the one in (12) in the limit $t \to 0$ in order to ensure the desired absolute continuity property. It is affine in the spatial variable so that $(Z_t)_{t\geq 0}$ is a Gaussian process and as such is easy to simulate recursively.

Lemma 5. — *The process*

$$Z_t = \frac{\sigma}{t}\int_0^t s dW_s + \frac{\gamma}{2}t \tag{14}$$

is the unique solution of the stochastic differential equation (13).

Proof. Using ItÃ´'s Lemma, we easily check that Z_t given by (14) is a solution of (13). Again, constant diffusion coefficient and decreasing drift coefficient ensures strong uniqueness. □

Remark 6. — *For the computation of the price $C_0 = \mathbb{E}\left(e^{-rT}(S_0e^{X_T} - K)_+\right)$ of a standard Asian call option, the random variable $e^{-rT}(S_0e^{Z_T} - K)_+$ provides a natural control variate. Indeed, since Z_T is a Gaussian random variable with mean $\frac{\gamma}{2}T$ and variance $\frac{\sigma^2 T}{3}$, one has*

$$\mathbb{E}\left(e^{-rT}(S_0e^{Z_T} - K)_+\right) = S_0e^{(\frac{\gamma}{2}+\frac{\sigma^2}{6}-r)T}\mathcal{N}\left(d + \sigma\sqrt{\frac{1}{3}T}\right) - Ke^{-rT}\mathcal{N}(d)$$

where $\mathcal{N}$ is the cumulative standard normal distribution function and $d = \frac{\log(S_0/K)+\frac{\gamma}{2}T}{\sigma\sqrt{\frac{1}{3}T}}$.

Notice that in [15], Kemna and Vorst suggest the use of the control variate $e^{-rT}\left(S_0\exp\left(\frac{1}{T}\int_0^T \sigma W_t + \gamma t\, dt\right) - K\right)_+$ which has the same law than $e^{-rT}\left(S_0e^{Z_T} - K\right)_+$ as $\dfrac{1}{T}\displaystyle\int_0^T \sigma W_t + \gamma t\, dt$ is also a Gaussian variable with mean $\frac{\gamma}{2}T$ and variance $\frac{\sigma^2 T}{3}$.

In order to define a new probability measure under which $(Z_t)_{t\geq 0}$ solves the SDE (12), one introduces

$$L_t = \exp\left[\int_0^t \frac{e^{-Z_s} - 1 + Z_s}{\sigma s}dW_s - \frac{1}{2}\int_0^t \left(\frac{e^{-Z_s} - 1 + Z_s}{\sigma s}\right)^2 ds\right].$$

Because of the singularity of the coefficients in the neighborhood of $s = 0$, one has to check that the integrals in L_t are well defined. This relies on the following lemma

Lemma 7. — *Let $\epsilon > 0$. In a random neighborhood of $s = 0$, we have*

$$|Z_s| \leq cs^{\frac{1}{2}-\epsilon} \text{ and } |X_s| \leq cs^{\frac{1}{2}-\epsilon}$$

where c is a constant depending on σ,γ and ϵ.

Since $\forall\epsilon > 0$,

$$\forall z \leq cs^{\frac{1}{2}-\epsilon}, \left(\frac{e^{-z} - 1 + z}{\sigma s}\right)^2 \leq Cs^{-4\epsilon},$$

we can choose $\epsilon < \frac{1}{4}$ to deduce that L_t is well defined.

Proof. We easily check that the Gaussian process $(B_t)_{t\in[0,T]}$ defined by $B_t = \displaystyle\int_0^{(3t)^{\frac{1}{3}}} s dW_s$ is a standard Brownian motion. Thanks to the law of iterated logarithm for the Brownian motion (see for example [14] p. 112), there exists $t_1(\omega)$ such that[2],

$$\forall t \leq t_1(\omega), |B_t(\omega)| \leq t^{\frac{1}{2}-\frac{\epsilon}{3}}.$$

Therefore,

$$\forall t \leq (3t_1(\omega))^{\frac{1}{3}}, \quad |Z_t(\omega)| = \Big|\frac{\sigma}{t}B_{\frac{t^3}{3}}(\omega) + \frac{\gamma}{2}t\Big| \leq \frac{\sigma}{3^{\frac{1}{2}-\frac{\epsilon}{3}}}t^{\frac{1}{2}-\epsilon} + \frac{\gamma}{2}t.$$

Taking $c = \max(\frac{\sigma}{3^{\frac{1}{2}-\frac{\epsilon}{3}}}, \frac{\gamma}{2})$ yields

$$\forall t \leq (3t_1(\omega))^{\frac{1}{3}} \wedge 1, \quad |Z_t(\omega)| \leq ct^{\frac{1}{2}-\epsilon}.$$

On the other hand, recall that $X_t = \log(\xi_t/\xi_0) = \log\left(\frac{1}{t}e^{\sigma W_t + \gamma t}\displaystyle\int_0^t e^{-\sigma W_u - \gamma u}du\right)$. So, using the law of iterated logarithm for the Brownian motion, we deduce that there exists $t_2(\omega)$ such that

$$\forall t \leq t_2(\omega), \quad 0 \leq \frac{1}{t}e^{\sigma W_t(\omega)+\gamma t}\int_0^t e^{-\sigma W_u(\omega) - \gamma u}du \leq \frac{1}{t}e^{\sigma t^{\frac{1}{2}-\epsilon}+\gamma t}\int_0^t e^{\sigma u^{\frac{1}{2}-\epsilon} - \gamma u}du.$$

Denote $g(t) = \frac{1}{t}e^{\sigma t^{\frac{1}{2}-\epsilon}+\gamma t}\int_0^t e^{\sigma u^{\frac{1}{2}-\epsilon}-\gamma u}du$ and let us investigate the order in time near zero of this function. We have that

$$\begin{aligned} e^{\sigma t^{\frac{1}{2}-\epsilon}+\gamma t} &= 1 + \sigma t^{\frac{1}{2}-\epsilon} + \mathcal{O}(t^{1-2\epsilon}) \\ \int_0^t e^{\sigma u^{\frac{1}{2}-\epsilon}-\gamma u}du &= t + \frac{\sigma}{\frac{3}{2}-\epsilon}t^{\frac{3}{2}-\epsilon} + \mathcal{O}(t^{2-2\epsilon}) \end{aligned}$$

hence

$$g(t) = 1 + (\sigma + \frac{\sigma}{\frac{3}{2}-\epsilon})t^{\frac{1}{2}-\epsilon} + \mathcal{O}(t^{1-2\epsilon}),$$

so $X_t(\omega) \leq \log(g(t)) \underset{t\to 0}{\sim} (\sigma + \frac{\sigma}{\frac{3}{2}-\epsilon})t^{\frac{1}{2}-\epsilon}$, which ends the proof for X_t. $\square$

Proposition 8. — *$(L_t)_{t\in[0,T]}$ is a martingale and, consequently, for all $g : \mathcal{C}([0,T]) \to \mathbb{R}$ measurable, the random variables $g((X_t)_{0\leq t\leq T})$ and $g((Z_t)_{0\leq t\leq T})L_T$ are simultaneously integrable and then*

$$\mathbb{E}\Big(g((X_t)_{0\leq t\leq T})\Big) = \mathbb{E}\Big(g((Z_t)_{0\leq t\leq T})L_T\Big).$$

Proof. We have already shown existence and strong uniqueness for both SDE (12) and (13). Showing that the stopping time

$$\tau_n(Y) = \inf\left\{t \in \mathbb{R}^+ \text{ such that } \int_0^t \left(\frac{e^{-Y_s} - 1 + Y_s}{\sigma s}\right)^2 ds \geq n\right\}, \text{ with the convention } \inf\{\emptyset\} = +\infty,$$

have infinite limits when n tends to $+\infty$, $\mathbb{Q}_X$ and $\mathbb{Q}_Z$ almost surely, follows from the previous lemma. $\square$

[2] ω is an element of the underlying probability space Ω.

One has

$$L_T = \exp\left[\int_0^T \frac{e^{-Z_t}-1+Z_t}{\sigma^2 t}dZ_t - \int_0^T \frac{e^{-Z_t}-1+Z_t}{\sigma^2 t}\left(\frac{e^{-Z_t}-1+Z_t}{2t}+\gamma-\frac{Z_t}{t}\right)dt\right].$$

Set $A(t,z) = \dfrac{1-z+\frac{z^2}{2}-e^{-z}}{\sigma^2 t}$. The function $A :]0,T]\times\mathbb{R}\to\mathbb{R}$ is continuously differentiable in time and twice continuously differentiable in space. So, we can apply ItÃ´'s Lemma on the interval $[\epsilon, T]$ for $\epsilon > 0$:

$$A(T,Z_T) = A(\epsilon, Z_\epsilon) + \int_\epsilon^T \frac{e^{-Z_t}-1+Z_t}{\sigma^2 t}dZ_t - \int_\epsilon^T \frac{1-Z_t+\frac{Z_t^2}{2}-e^{-Z_t}}{\sigma^2 t^2}dt + \int_\epsilon^T \frac{1-e^{-Z_t}}{2t}dt$$

Using the lemma 5, we let $\epsilon\to 0$ to obtain

$$A(T,Z_T) = \int_0^T \frac{e^{-Z_t}-1+Z_t}{\sigma^2 t}dZ_t - \int_0^T \frac{1-Z_t+\frac{Z_t^2}{2}-e^{-Z_t}}{\sigma^2 t^2}dt + \int_0^T \frac{1-e^{-Z_t}}{2t}dt.$$

Then

$$L_T = \exp\left[A(T,Z_T) - \int_0^T \phi(t,Z_t)dt\right]$$

where ϕ is the mapping

$$\phi(t,z) = \frac{e^{-z}-1+z-\frac{z^2}{2}}{\sigma^2 t^2} + \frac{1-e^{-z}}{2t} + \frac{e^{-z}-1+z}{\sigma^2 t}\left(\frac{e^{-z}-1+z}{2t}+\gamma-\frac{z}{t}\right). \tag{15}$$

By (11) and Proposition 8, we get

$$C_0 = \mathbb{E}\left(e^{-rT}f(S_0e^{Z_T})\exp\left[A(T,Z_T)-\int_0^T\phi(t,Z_t)dt\right]\right).$$

We can use the unbiased estimator if, for a real number c_Z possibly dependent on Z, we have

$$\mathbb{E}\left(e^{A(T,Z_T)-rT-c_ZT}|f(S_0e^{Z_T})|e^{\int_0^T|c_Z-\phi(t,Z_t)|dt}\right) < \infty. \tag{16}$$

In order to be able to deal with both call and put options, a sufficient condition for (16) to be true when $f(x) = (x-K)_+$ or $f(x) = (K-x)_+$ is the following conjecture

Conjecture 9. —

$$\mathbb{E}\left(e^{A(T,Z_T)-rT-c_ZT}(e^{Z_T}+1)e^{\int_0^T|\phi(t,Z_t)|dt}\right) < \infty.$$

Given the complexity of the function ϕ, it is difficult to give a theoretical proof of this result. Nevertheless, numerical tests are very satisfactory.

Let p_Z and q_Z denote respectively a positive probability measure on $\mathbb{N}$ and a positive probability density on $[0,T]$. Let N be distributed according to p_Z and $(U_i)_{i\in\mathbb{N}^*}$ be a sequence of independent random variables identically distributed according to the density q_Z, both independent conditionnaly on the process $(Z_t)_{t\in[0,T]}$. Assuming the conjecture 9, we can write that

$$C_0 = \mathbb{E}\left(e^{A(T,Z_T)-rT-c_ZT}f(S_0e^{Z_T})\frac{1}{p_Z(N)\,N!}\prod_{i=1}^N\frac{c_Z-\phi(U_i,Z_{U_i})}{q_Z(U_i)}\right) \text{ for } f(x)=(x-K)_+ \text{ or } f(x)=(K-x)_+. \tag{17}$$

The above expectation can be computed by the Monte Carlo method. It is very important then that the random variable of interest is not only integrable but also square integrable in order that the central limit theorem holds and so it becomes possible to build confidence intervals. The square integrability condition writes

$$\mathbb{E}\left(e^{2A(T,Z_T)-2rT-2c_ZT}f^2(S_0e^{Z_T})\frac{\left(\int_0^T\frac{(c_Z-\phi(t,Z_t))^2}{q_Z(t)}dt\right)^N}{p_Z(N)^2\,(N!)^2}\right)<\infty, \tag{18}$$

which is again very difficult to check whatever the choice of p_Z and q_Z. But, at least, we may choose the probability distribution q such that the integral $\int_0^T\frac{\phi^2(t,Z_t)}{q_Z(t)}dt$ is well defined. To do so, we need the following lemma

Lemma 10. — *Let $\epsilon>0$. In a random neighborhood of zero, we have that*

$$\phi(t,Z_t)+\frac{2Z_t^3}{3\sigma^2t^2}-\frac{Z_t}{2t}=\mathcal{O}(t^{-\epsilon}) \tag{19}$$

and consequently, for distributions q of the form $q(t)=Ct^a$ with $a>-1$ and C a normalizing constant, we have that

$$\int_0^T\frac{\phi^2(t,Z_t)}{q(t)}dt<\infty\ \text{a.s}\quad\text{if and only if } a<0.$$

Proof. We rewrite (15) this way

$$\phi(t,z)=\left(\frac{1-e^{-z}}{2}+\gamma\frac{e^{-z}-1+z}{\sigma^2}\right)\frac{1}{t}-\left(\frac{1-z+\frac{z^2}{2}-e^{-z}-\frac{1}{2}(e^{-z}-1+z)(e^{-z}-1-z)}{\sigma^2}\right)\frac{1}{t^2}$$

and make the following Taylor expansions

$$\frac{1-z+\frac{z^2}{2}-e^{-z}-\frac{1}{2}(e^{-z}-1+z)(e^{-z}-1-z)}{\sigma^2}=\frac{2}{3\sigma^2}z^3+\mathcal{O}(z^4)$$

$$\text{and }\frac{1-e^{-z}}{2}+\gamma\frac{e^{-z}-1+z}{\sigma^2}=\frac{1}{2}z+\mathcal{O}(z^2).$$

Using lemma 7, we deduce that, in a random neighborhood of zero,

$$\phi(t,Z_t)+\frac{2Z_t^3}{3\sigma^2t^2}-\frac{Z_t}{2t}=\mathcal{O}(t^{-\epsilon}).$$

We then have

$$\phi(t,Z_t)=-\frac{2Z_t^3}{3\sigma^2t^2}+\frac{Z_t}{2t}+R(t,Z_t),$$

where the remainder term $R(t,Z_t)$ is such that there exists $t_1(\omega)$ for which $\forall t\le t_1(\omega),\ |R(t,Z_t(\omega))|\le t^{-\epsilon}$.

Hence, for $a\in(-1,0)$, taking ϵ such that $2\epsilon+a<1$ ensures that

$$\int_0^T\frac{|R(t,Z_t(\omega))|^2}{t^a}dt<\infty.$$

We have seen in the proof of lemma 7 that $Z_t = \frac{\sigma}{t} B_{\frac{t^3}{3}} + \frac{\gamma}{2} t$ where B_t is a standard Brownian motion. So, clearly,

$$\int_0^T \frac{1}{t^a}\left(\frac{2Z_t^3}{3\sigma^2 t^2} - \frac{Z_t}{2t}\right)^2 dt < \infty \text{ a.s.} \quad \text{if and only if } \int_0^T \frac{1}{t^a}\left(\frac{2}{3\sigma^2 t^2}(\frac{\sigma}{t}B_{\frac{t^3}{3}})^3 - \frac{1}{2t}(\frac{\sigma}{t}B_{\frac{t^3}{3}})\right)^2 dt < \infty \text{ a.s.}$$

Using the change of variables $u = \frac{t^3}{3}$, we write that

$$\begin{aligned}\int_0^T \frac{1}{t^a}\left(\frac{2}{3\sigma^2 t^2}(\frac{\sigma}{t}B_{\frac{t^3}{3}})^3 - \frac{1}{2t}(\frac{\sigma}{t}B_{\frac{t^3}{3}})\right)^2 dt &= \int_0^{\frac{T^3}{3}} \frac{1}{(3u)^{\frac{a}{3}}}\left(\frac{2\sigma}{3(3u)^{\frac{5}{3}}}B_u^3 - \frac{\sigma}{2(3u)^{\frac{2}{3}}}B_u\right)^2 du \\ &= \int_0^{\frac{T^3}{3}} \frac{1}{(3u)^{\frac{a+1}{3}}}\widetilde{B}_u du\end{aligned}$$

where the law of $\widetilde{B}_u := u^{\frac{1}{3}}\left(\frac{2\sigma}{3(3u)^{\frac{5}{3}}}B_u^3 - \frac{\sigma}{2(3u)^{\frac{2}{3}}}B_u\right)^2$ is independent of u by the scaling property of the Brownian motion. We can now apply the so-called *Jeulin's lemma* (see [12] Lemma 3.22, p. 44 or [13]) which can be written this way (see [19]) :

Jeulin's Lemma 11. — *Let $(H_t)_{t\in[0,T]}$ be a measurable and non negative process such that, for fixed t, the law of H_t does not depend on t, and*

$$\mathbb{E}(H_t) < +\infty \text{ and } \mathbb{P}(H_t > 0) = 1.$$

Consider a deterministic, positive and $\sigma-$finite measure $\nu(dt)$ on $[0,T]$. Then, the event $\{\int_0^T H_t \nu(dt) < +\infty\}$ has probability zero or one according to $\nu([0,T]) = +\infty$ or $\nu([0,T]) < +\infty$.

Consequently,

$$\int_0^T \frac{1}{t^a}\left(\frac{2Z_t^3}{3\sigma^2 t^2} - \frac{Z_t}{2t}\right)^2 dt < \infty \text{ a.s.} \quad \text{if and only if } \int_0^{\frac{T^3}{3}} \frac{1}{(3u)^{\frac{a+1}{3}}} dt,$$

which is true if and only if $a \in (-1, 0)$. □

Remark 12. — *According to this lemma, when using a uniform variable for q_Z, which corresponds to the use of the generalized Poisson estimator of Fearnhead et al. [6], the square integrability condition (18) is not satisfied and it is not legitimate to build confidence intervals. Yet, we were unable to illustrate this result by numerical computations.*

2.0.1 Numerical implementation

We first discuss the practical choice of the probability distributions p_Z and q_Z in order to compute (17) by the Monte Carlo method. As suggested in the appendix A, we choose a Poisson distribution for p. Its mean is set to $c_p T$ where c_p is a free parameter. The choice of q_Z is more intricate. Lemma 10 leads us to consider probability distributions of the form $q_Z(t) = Ct^a$ with $a \in (-1, 0)$. Using lemma 7 and the expansion (19), we see that $|\phi|$ is approximately of order $\frac{1}{\sqrt{t}}$. So, as suggested in the appendix A, we choose the following distribution for q_Z : $q_Z(t) = \frac{1}{2\sqrt{t}\sqrt{T}} \mathbb{1}_{[0,T]}(t)$.

To fix the ideas, let us consider a call option. The price C_0 simplifies then to

$$C_0 = \mathbb{E}\left(e^{A(T,Z_T)-rT}(S_0 e^{Z_T} - K)_+ e^{c_p T - c_Z T}\prod_{i=1}^N \frac{2\sqrt{U_i}\,(c_Z - \phi(U_i, Z_{U_i}))}{c_p\sqrt{T}}\right).$$

Remark 13. — *Simulating from the probability distribution q_Z is straightforward using the inverse of the cumulative distribution function. But we frequently simulate very small values (of order 10^{-9}) which pose over-floating problems with the computation of ϕ. The solution that we propose is to use the equivalent of ϕ given in lemma 10 instead of its exact expression (15) for U_i smaller than 10^{-7}.*

Variance reduction : Subsequently, we investigate different ways to reduce the variance. We already have two levers for reducing the variance which are the parameters c_p and c_Z.

Also, as pointed out in Remark 6, we can use a control variate technique. In order to get the best out of it and, at the same time, to smoothen integrability problems, we compute a conditional expectation on the trajectory of Z_t. That is, for every simulated path $(Z_t^j)_{t\in[0,T]}$, we compute

$$\frac{1}{n}\sum_{k=1}^{n}\prod_{i=1}^{N_k}\frac{2\sqrt{U_i^k}\left(c_Z-\phi(U_i^k,Z^j_{U_i^k})\right)}{c_p\sqrt{T}} \tag{20}$$

instead of

$$\prod_{i=1}^{N}\frac{2\sqrt{U_i}\left(c_Z-\phi(U_i,Z^j_{U_i})\right)}{c_p\sqrt{T}}.$$

This method of computation is more time consuming and we have to choose very carefully the parameter n so that the variance reduction we obtain is sufficient to gain on the exchange. An empirical study suggested the choice of $c_p = c_Z = \frac{T}{2}$ and $n = 5$.

To summarize, we approximate C_0 by the following optimized unbiased estimation :

$$\begin{aligned} C_0 \approx \frac{1}{M}\sum_{j=1}^{M} e^{-rT}(S_0e^{Z_T^j}-K)_+ &\left(e^{A(T,Z_T^j)}e^{c_pT-c_ZT}\left(\frac{1}{n}\sum_{k=1}^{n}\prod_{i=1}^{N_k}\frac{2\sqrt{U_i^k}\left(c_Z-\phi(U_i^k,Z^j_{U_i^k})\right)}{c_p\sqrt{T}}\right)-1\right)\\ &+S_0e^{(\frac{\gamma}{2}+\frac{\sigma^2}{6}-r)T}\mathcal{N}\left(d+\sigma\sqrt{\frac{1}{3}T}\right)-Ke^{-rT}\mathcal{N}(d). \end{aligned}$$

Remark 14. — *Using the call-put parity, one may compute the put option instead of the call when the variance is lower for the put and vice versa. In the implementation of the method, we choose to compute the put whenever $S_0e^{rT} < K$ and to compute the call otherwise.*

3 Conclusion

Clearly, the U.E method is not yet competitive regarding computation time. Nevertheless, unlike the usual discretization methods which are prone to discretization errors, it gives an exact price within a Monte Carlo confidence interval. We also point out the limit of the U.E method for long maturities and high volatilities.

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A The practical choice of p and q in the U.E method

The best choice for the probability law p of N and the common density q of the variables $(V_i)_{i\geq 1}$ is obviously the one for which the variance of the simulation is minimum. In a very general setting, it is difficult to tackle this issue. In order to have a first idea, we are going to restrict ourselves to the computation of $\mathbb{E}\left(\frac{1}{p(N)\,N!}\prod_{i=1}^{N}\frac{g(V_i)}{q(V_i)}\right)$ where $g:[0,T]\to\mathbb{R}$.

Lemma 15. — *When g is a measurable function on $[0,T]$ such that $0<\int_0^T |g(t)|dt<+\infty$, the variance of $\frac{1}{p(N)\,N!}\prod_{i=1}^{N}\frac{g(V_i)}{q(V_i)}$ is minimal for*

$$q_{opt}(t)=\frac{|g(t)|}{\int_0^T|g(t)|dt}\mathbb{1}_{[0,T]}(t)\ \text{ and }\ p_{opt}(n)=\frac{\left(\int_0^T|g(t)|dt\right)^n}{n!}\exp\left(-\int_0^T|g(t)|dt\right).$$

Proof. Minimizing the variance in (7) comes down to minimizing the expectation of the square of $\frac{1}{p(N)\,N!}\prod_{i=1}^{N}\frac{g(V_i)}{q(V_i)}$. Set

$$F(p,q)=\mathbb{E}\left(\frac{1}{(p(N)\,N!)^2}\prod_{i=1}^{N}\frac{g^2(V_i)}{q^2(V_i)}\right)=\sum_{n=0}^{+\infty}\frac{\left(\int_0^T\frac{g^2(t)}{q(t)}dt\right)^n}{p(n)\,(n!)^2}.$$

Using Cauchy-Schwartz inequality we obtain a lower bound for $F(p,q)$

$$\begin{aligned}F(p,q)=\sum_{n=0}^{+\infty}\left(\frac{\left(\int_0^T\frac{g^2(t)}{q(t)}dt\right)^{\frac{n}{2}}}{p(n)\,n!}\right)^2 p(n) &\geq \left(\sum_{n=0}^{+\infty}\frac{\left(\int_0^T\frac{g^2(t)}{q(t)}dt\right)^{\frac{n}{2}}}{n!}\right)^2\\ &=\left(\sum_{n=0}^{+\infty}\frac{\left(\int_0^T\left(\frac{g(t)}{q(t)}\right)^2q(t)dt\right)^{\frac{n}{2}}}{n!}\right)^2 \geq \left(\sum_{n=0}^{+\infty}\frac{\left(\int_0^T|g(t)|dt\right)^n}{n!}\right)^2\\ &=\exp\left(2\int_0^T|g(t)|dt\right).\end{aligned}$$

We easily check that this lower bound is attained for q_{opt} and p_{opt}. ∎

The optimal probability distribution p_{opt} is the Poisson law with parameter $\int_0^T|g(t)|dt$. This justifies our use of a Poisson distribution for p.