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## 1 Introduction

A barrier option is activated or extinguished when a specified asset price, index, or rates reaches a specified level. Some models of barrier option assume continuous monitoring of the barrier, others specify fixed times for monitoring of the barrier (typically daily closing). In this last case, the price can not be computed in closed form. Indeed, the price is equal to a n-dimensional integral on a multivariate normal distribution (where n is the number of monitoring instants), and the classical analytical methods become rapidly inefficient. Even Monte Carlo methods imply times of computing too important to be a real tradind tool.

In their paper "A continuity correction for discrete barrier options" (Mathematical Finance, vol 7, N°4 (october 1997)) Broadie, Glasserman and kou [1] (henceforth BGK) propose to lie the "discrete price" to the "continuous price" for which there exists a closed formula. In a first part we recall this result and from this, we obtain an operational closed formula. In a second part we look at the profils of the price and delta of the option at some date before a monitoring instant, and then display the real hedging problem that raise such options.

## 2 Integration of the BGK's formula

We are under the usual Black-Scholes market assumptions. In particular, the asset price follows the stochastic differential equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

where $(W_t)_{t\geq 0}$ is a standard brownian motion, and we denote r the constant instantaneous risk-free interest rate. In their article (BGK) introduce a continuity correction for approximate pricing of discrete barrier options.

**Theorem 1.** *Let $V_m(H)$ be the price of a discretely monitored knock-in or knock-out down call or up put with barrier H. Let $V(H)$ be the price of the price of the corresponding continuously monitored barrier option. Then*

$$V_m(H) = V(H e^{\pm\beta\sigma\sqrt{\frac{T}{m}}}) \; + \; o(\frac{1}{\sqrt{m}})$$

*where + applies if* $Htext>S_0$, *- applies if* $Htext<S_0$, *and* $\beta = \frac{\zeta(\frac{1}{2})}{\sqrt{2\Pi}} \approx 0.5826$, *with* $\zeta$ *the riemann zeta function.*

However the above formula is not operational for all that to the extent that the price and delta can only be computed at monitoring instant.Starting from this result, our aim is to obtain a value for each of these two quantities at every instant of the life of the option.
We denote: $V_T^x(L,t)$ ≡price at time t of a “down and out” call option with maturity T, barrier L, and n monitoring instant $t_{1,...,}t_n$ uniformly distributed on [0,T], the underlying asset starting from x. $V_{c,T}^x(L,t)$ ≡price at time t of a “down and out” call option with maturity T, continuous barrier L, the underlying asset starting from x. $BS_T(x)$ ≡price at time 0 of a standard call option with maturity T, the underlying asset starting from x. $FF_{t,T}^x(L)$ ≡price of a “down and out” call option with maturity T, barrier L and one monitoring instant t, the underlying asset starting from x.
In the sequel, the strike K is fixed and $E^*$ [] denote the expectation under the risk neutral probability. Let t $\in [0,T]$ , for instance $ttext<t_1$

$$V_T^{S_0}(L,t) = e^{-r(t_1-t)}E^*\left[V_T^{S_0}(L,t_1)|F_t\right]$$

but

$$V_T^{S_0}(L,t_1) = V_{T-t_1}^{S_{t_1}}(L,0)1(S_{t_1} > L)$$

and from the theorem

$$V_{T-t_1}^{S_{t_1}}(L,0) = V_{c,T-t_1}^{S_{t_1}}(L^{'},t)$$

then

$$\begin{aligned} V_T^{S_0}(L,t) &= e^{-r(t_1-t)}E^*\left[V_{c,T-t_1}^{S_{t_1}}(L^{'},t)1(S_{t_1} > L)|F_t\right] \; (0) \\ &= e^{-r(t_1-t)}E^*\left[BS_{T-t_1}(S_{t_1})1(S_{t_1} > L)|F_t\right] - e^{-r(t_1-t)}E^*\left[(\frac{L^{'}}{S_{t_1}})^{\lambda}BS_{T-t_1}(\frac{(L^{'})^2}{S_{t_1}})1(S_{t_1} > L\right. \end{aligned}$$

where $\lambda = (\frac{2r}{\sigma^2}) - 1$ we look at the firts term of the (RHS) of (0)

$$E^*\left[BS_{T-t_1}(S_{t_1})1(S_{t_1} > L)|F_t\right] = E^*\left[BS_{T-t_1}(S_t\frac{S_{t_1}}{S_t})1(S_t\frac{S_{t_1}}{S_t} > L)|F_t\right] \; (1)$$

Define

$$\begin{array}{rcl} S_0^{\nabla} & = & S_t \\ \mathrm{S}_u^{\nabla} & = & S_0^{\nabla}\exp((r-\sigma^2)u+\sigma B_u) \end{array}$$

then

$$\begin{array}{rcl} (1) & = & E^*\left[BS_{T-t_1}(S_{t_1-t}^{\nabla})1(S_{t_1-t}^{\nabla}>L)\right] \\ & = & E^*\left[BS_{T-t-(t_1-t)}(S_{t_1-t}^{\nabla})1(S_{t_1-t}^{\nabla}>L)\right] \\ & = & e^{-r(T-t_1)}E^*\left[1(S_{t_1-t}^{\nabla}>L)(S_{T-t}^{\nabla}-K)_+\right] \end{array}$$

hence

$$(1)=FF_{t_1-t,T-t}^{S_0^{\nabla}}(L)$$

Now we look at the second term of the (RHS) of (0)

$$\begin{array}{rcl} E^*\left[(\frac{L^{'}}{S_{t_1}})^{\lambda}BS_{T-t_1}(\frac{(L^{'})^2}{S_{t_1}})1(S_{t_1}>L)|F_t\right] & = & E^*\left[(\frac{L^{'}}{S_t}\frac{S_t}{S_{t_1}})^{\lambda}BS_{T-t_1}(\frac{(L^{'})^2}{S_t}\frac{S_t}{S_{t_1}})1(\frac{S_{t_1}}{S_t}>\frac{L}{S_t})|F_t\right] \quad (2) \\ & = & (\frac{L^{'}}{S_t})^{\lambda}E^*\left[(\frac{S_t}{S_{t_1}})^{\lambda}BS_{T-t_1}(\frac{(L^{'})^2}{S_t}\frac{S_t}{S_{t_1}})1(\frac{S_{t_1}}{S_t}>\frac{L}{S_t})|F_t\right] \end{array}$$

we recall theorems of Girsanov and Levy

**Theorem 2.** *(Girsanov)Let $L$ and $M$ be two $P$-local martingales such that $\varepsilon(L)$ is a real martingale, then $M$-$\langle M,L\rangle$ is a $Q$-locale martingale where $\frac{dQ}{dP}=\varepsilon(L)$ .*

**Theorem 3.** *(Lévy) Let $X$ be a$Q$-continuous local martingale with $\langle X,X\rangle_t=t$ „ then $X$ is a brownian motion.*

So by Girsanov's theorem applied with

$$M=B\ ,\ L=-\lambda\sigma B$$

we know that

$$(\widetilde{B}_u=B_u+\lambda\sigma u)_{0\leqslant u\leqslant t_1-t}$$

is Q-local martingale where

$$\frac{dQ}{dP^*}|F_{t_1-t}=\varepsilon(-\lambda\sigma B_{t_1-t})$$

but

$$\left\langle\widetilde{B_u},\widetilde{B_u}\right\rangle=u$$

so by Lévy's theorem

$$(\widetilde{B}_u)_{0\leqslant u\leqslant t_1-t}$$

is a Q-brownian motion. Hence with

$$c=(\frac{L'}{S_t})^{\lambda}\exp((-\lambda r+\lambda\frac{\sigma^2}{2})+\lambda^2\frac{\sigma^2}{2})(t_1-t))$$

we can rewrite (2) as

$$\begin{aligned}(2)=cE^Q[\quad & BS_{T-t_1}(\frac{(L^{'})^2}{S_t}\exp(-(r-\frac{\sigma^2}{2}-\lambda\frac{\sigma^2}{})(t_1-t)-\sigma\widetilde{B}_{t_1-t}))\\ & 1(S_t\exp((r-\frac{\sigma^2}{2}-\lambda\frac{\sigma^2}{})(t_1-t)+\sigma\widetilde{B}_{t_1-t}>L)|F_t]\\ =cE^Q[\quad & BS_{T-t_1}(\frac{(L^{'})^2}{S_t}\exp(-(2r-(\lambda+1)\frac{\sigma^2}{})(t_1-t))\exp((r-\frac{\sigma^2}{2})(t_1-t)-\sigma\widetilde{B}_{t_1-t}))\\ & 1(\frac{(L^{'})^2}{S_t}\exp(-(2r-(\lambda+1)\frac{\sigma^2}{})(t_1-t))\exp((r-\frac{\sigma^2}{2})(t_1-t)-\sigma\widetilde{B}_{t_1-t})<\frac{(L^{'})^2}{L}|F_t]\end{aligned}$$

Define

$$\begin{aligned}S_0^{\nabla\nabla}&=\frac{(L^{'})^2}{S_t}\exp(-(2r-(\lambda+1)\frac{\sigma^2}{})(t_1-t))\\ \mathrm{S}_u^{\nabla\nabla}&=S_0^{\nabla\nabla}\exp((r-\frac{\sigma^2}{2})u+\sigma\widetilde{B}_u)\end{aligned}$$

since -$\widetilde{B}_u$ and $\widetilde{B}_u$ have the same distribution we have

$$\begin{aligned}(2)&=cE^Q[BS_{T-t_1}(\mathrm{S}_{t_1-t}^{\nabla\nabla})1(\mathrm{S}_{t_1-t}^{\nabla\nabla}<\frac{(L^{'})^2}{L})]\\ &=cE^Q[BS_{T-t_1}(\mathrm{S}_{t_1-t}^{\nabla\nabla})]-cE^Q[BS_{T-t_1}(\mathrm{S}_{t_1-t}^{\nabla\nabla})1(\mathrm{S}_{t_1-t}^{\nabla\nabla}>\frac{(L^{'})^2}{L})]\end{aligned}$$

on one hand

$$E^Q[BS_{T-t_1}(\mathrm{S}_{t_1-t}^{\nabla\nabla})]=e^{r(t_1-t)}BS_{T-t}(\mathrm{S}_0^{\nabla\nabla})$$

on the other hand

$$\begin{aligned}&E^Q[BS_{T-t_1}(\mathrm{S}_{t_1-t}^{\nabla\nabla})1(\mathrm{S}_{t_1-t}^{\nabla\nabla}>\frac{(L^{'})^2}{L})]\\ =\;&e^{-r(t-t_1)}E^Q[e^{-r(T-t)}(\mathrm{S}_{T-t}^{\nabla\nabla}-K)_+1(\mathrm{S}_{t_1-t}^{\nabla\nabla}>\frac{(L^{'})^2}{L})]\\ =\;&e^{-r(t-t_1)}FF_{t_1-t,T-t}^{S_0^{\nabla\nabla}}(\frac{(L^{'})^2}{L})\end{aligned}$$

Finaly we can write

$$V_T^{S_0}(L,t) = FF^{S_0^{\nabla}}_{t_1-t,T-t}(L) - cBS_{T-t}(\mathrm{S}_0^{\nabla\nabla}) + cFF^{S_0^{\nabla\nabla}}_{t_1-t,T-t}(\frac{(L^{'})^2}{L})$$

From this formula we can obtain a value for the delta of the option: define

$$\begin{aligned}
\delta_1 &= \frac{\partial FF^{S_0^{\nabla}}_{t_1-t,T-t}(L)}{\partial S_t} \\
\delta_2 &= \frac{\partial FF^{S_0^{\nabla\nabla}}_{t_1-t,T-t}(\frac{(L^{'})^2}{L})}{\partial S_t} \\
\delta_3 &= \frac{\partial BS_{T-t}(\mathrm{S}_0^{\nabla\nabla})}{\partial S_t} \\
p_2 &= FF^{S_0^{\nabla\nabla}}_{t_1-t,T-t}(\frac{(L^{'})^2}{L}) \\
p_3 &= BS_{T-t}(\mathrm{S}_0^{\nabla\nabla})
\end{aligned}$$

hence we have

$$\frac{\partial V_T^{S_0}(L,t)}{\partial S_t} = \delta_1 - \frac{\partial c}{\partial S_t}p_3 - c\delta_3\frac{\partial(\mathrm{S}_0^{\nabla\nabla})}{\partial S_t} + \frac{\partial c}{\partial S_t}P_2 - c\delta_2\frac{\partial(\mathrm{S}_0^{\nabla\nabla})}{\partial S_t}$$

## 3 Profils of the price

Let's now look at the profils of the price (and corresponding delta) of discrete barrier option through the example of a "down and out" call option with maturity T, strike K, barrier L and one monitoring instant t. We consider its price at time t as a function of the spot $\mathrm{S}_t$. The map

$$V : S_t \to V(S_t) = e^{-r(T-t)}1(S_t > L)E^*\left[(S_T - K)_+|F_t\right]$$

is obviously not continuous on L, that distinguish stongly continuous barrier option (for wich the price goes to 0 on $\mathrm{L}^+$) from discrete barrier option. Here the price is equal to 0 on L and its value on $\mathrm{L}^+$ is the price of a standard call option with maturity T-t and initial asset price $\mathrm{L}^+$.

On the other hand, the price at time t-$\varepsilon$ can be expressed as follows

$$\begin{aligned}
V\left(S_{t-\varepsilon}\right) &= e^{-r\varepsilon}E^*\left[V(S_t)|F_{t-\varepsilon}\right] \\
&= e^{-r\varepsilon}E^*\left[1(S_t > L)BS_{T-t}(S_t)|F_{t-\varepsilon}\right]
\end{aligned}$$

We can see this price as these of a digital call option with maturity that pays $BS_{T-t}(S_t)$ if $\mathrm{S}_t > L$ and 0 otherwise. This is a function of the spot $\mathrm{S}_{t-\varepsilon}$ wich

is $C^\infty$ in spite of the discontinuity of the price at time t. However the effects of this discontinuity are all the more visible because $\varepsilon$ is small. To illustrate this phenomenon, we give 5 pairs of graphics illustrations of the price and delta varying the spot, with 5 differents monitoring instants going from 0.002 to 0.01.
Let us place at time 0: -Both a "time area" on the left of the monitoring instant and a "spot area" appear for wich the price is unstable in so far as the slope takes bigger and bigger values as the pricing date is close to the monitoring instant. -as a consequence the corresponding values of delta are typically too large for the real trading life. This last fact brings the limits of the Black Scholes models to light, and induce us to consider others models, especially with constraints on the delta.

## References

[1] M.BROADIE P.GLASSERMANN S.KOU. A continuity correction for discrete barrier options. *Mathematical Finance*, 7, 1997. 1