

# GARCH pricing for PREMIA

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March 1, 2012

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### 1. GARCH MODELS AND OPTION PRICING

The GARCH(p,q) [E82, B86] family of time series models provides a parsimonious description of the returns associated to the quoted prices in stock markets that captures the most important stylized facts that have been empirically described, namely, leptokurtosis and volatility clustering.

When this family is used in the modeling of the underlying asset of a contingent product, the incompleteness of the associated market appears immediately as a problem. Several directions have been proposed to tackle this issue. In these pages we review the one presented by Duan in [D95] whose legitimacy is based on a utility maximization argument.

**The model.** Duan [D95] considers a discrete time economy in which the price of the asset at time  $t$  is given by  $S_t$ . The one-period log-returns associated to the price process  $\{S_t\}_{t \in \mathbb{N}}$  are conditionally normally distributed under the physical probability  $P$ . More specifically:

$$\log \left( \frac{S_t}{S_{t-1}} \right) = r + \lambda \sigma_t - \frac{1}{2} \sigma_t^2 + \sigma_t \epsilon_t, \quad (1)$$

where  $r$  is the one-period risk-free interest rate,  $\{\epsilon_t\} \sim \text{IIDN}(0, 1)$ ,  $\lambda \in \mathbb{R}$  can be interpreted as the unit risk premium, and the conditional variance  $\sigma_t^2$  is uniquely determined by the autoregressive equation:

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^q \alpha_i \sigma_{t-i}^2 \epsilon_{t-i}^2 + \sum_{i=1}^p \beta_i \sigma_{t-i}^2, \quad (2)$$

where  $\alpha_0 > 0$ ,  $\alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_p \geq 0$  and they are additionally subjected to the constraint

$$\sum_{i=1}^q \alpha_i + \sum_{i=1}^p \beta_i < 1 \quad (3)$$

so that (1) and (2) admit a unique stationary solution.

**The pricing measure.** Duan [D95] formulates a valuation principle that he calls locally risk-neutral valuation relationship (LRNVR) that he proves is satisfied whenever the market agents fulfill standard risk aversion conditions. The LRNVR materializes in the existence of a pricing measure  $Q$  under which the GARCH process given by (1) and (2) takes the form

$$\log \left( \frac{S_t}{S_{t-1}} \right) = r - \frac{1}{2} \sigma_t^2 + \sigma_t \epsilon_t, \quad (4)$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^q \alpha_i (\sigma_{t-i}^2 \epsilon_{t-i}^2 - \lambda \sigma_{t-i}) + \sum_{i=1}^p \beta_i \sigma_{t-i}^2. \quad (5)$$

**Prices and hedging strategies.** Using the risk-neutral probability  $Q$ , the value  $V_t$  of the derivative product with measurable payoff function  $h$  at time  $t$  is given by the conditional expectation

$$V_t = e^{-(T-t)r} E_Q [h \mid \mathcal{F}_t], \quad (6)$$

where  $\{\mathcal{F}_t\}_{t \in \mathbb{N}}$  is the filtration generated by the process (1) and (2). In particular, the selling price of the option at the beginning of the contract is given by:

$$V_0 = e^{-Tr} E_Q [h]. \quad (7)$$

In this pricing framework, the hedging delta is defined as the first partial derivative of the option price with respect to the underlying asset price. In the particular case of a European call option with payoff function  $h = (S_T - K)_+$  delta is given by

$$\Delta_t^C = e^{-(T-t)r} E_Q \left[ \frac{S_T}{S_t} \mathbf{1}_{S_T \geq K} \mid \mathcal{F}_t \right]. \quad (8)$$

Regarding the European put, the delta is obtained out of the call-put parity relation, that is,  $\Delta_t^P = \Delta_t^C - 1$ .

## 2. HISTORICAL MODEL CALIBRATION

In this section we consider a GARCH(1,1) model of the form

$$\log \left( \frac{S_t}{S_{t-1}} \right) = \mu_t + \sigma_t \epsilon_t, \quad (9)$$

where  $\mu_t$  is a predictable prescription for the conditional mean and  $\sigma_t$  is determined by the autoregressive equation (2), with  $p = q = 1$ . More specifically, we will examine two different prescriptions for the conditional mean  $\mu_t$ , namely:

- (i): Standard GARCH model [B86]:  $\mu_t = \mu$  for all  $t$ , with  $\mu \in \mathbb{R}$  a constant.
- (ii): Duan model [D95]:  $\mu_t := r + \lambda \sigma_t - \frac{1}{2} \sigma_t^2$ . This is the model in equation (1).

The goal of this section is explaining how to estimate the parameters  $\theta := \{\alpha_0, \alpha_1, \beta, \lambda\}$  given a historical prices sample  $\{S_0, \dots, S_T\}$  and the risk-free interest rate  $r$  for that period. The chosen calibration method consists of taking as estimated parameters those that maximize the Gaussian quasi-likelihood function at the given sample.

We start by constructing a sample of log-returns  $\mathbf{r} := \{r_1, \dots, r_T\}$  out of the prices sample, that is  $r_t := \log(S_t/S_{t-1})$ ,  $t \in \{1, \dots, T\}$ . The Gaussian hypothesis on the innovations and the predictability of the conditional mean  $\{\mu_t\}$  and variance  $\{\sigma_t\}$  processes imply that

$$r_t \mid \mathcal{F}_{t-1} \sim N(\mu_t, \sigma_t^2),$$

with  $\mathcal{F}_t$  the filtration generated by the GARCH process. Consequently, the associated conditional likelihood is given by

$$l_t(r_t; \theta) := \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp \left( -\frac{(r_t - \mu_t)^2}{2\sigma_t^2} \right),$$

and hence the quasi log-likelihood is:

$$\log L(r_1, \dots, r_T; \theta) = -\frac{T}{2} \log 2\pi - \frac{1}{2} \sum_{t=1}^T \log \sigma_t^2 - \frac{1}{2} \sum_{t=1}^T \frac{(r_t - \mu_t)^2}{\sigma_t^2}, \quad (10)$$

conditional on the initial values  $r_0$  and  $\sigma_0$ . We will set  $r_0 = r$  and  $\sigma_0 = \alpha_0/(1 - \alpha_1 - \beta_1)$  which is the stationary variance of the process (1)-(2) with  $p = q = 1$ .

It can be shown that the estimator  $\hat{\theta}$  of  $\theta$  obtained out of the constrained optimization of (10) is consistent under standard regularity conditions even if the underlying distribution is not conditionally normal [W81, GMT84]. Besides, this estimator is asymptotically normal; more specifically:

$$\sqrt{T}(\hat{\theta} - \theta) \sim N(0, J^{-1} I J^{-1}), \quad \text{with}$$

$$J = E_0 \left[ -\frac{\partial^2 \log l_t(\mathbf{r}; \theta)}{\partial \theta \partial \theta'} \right] \quad \text{and} \quad I = E_0 \left[ \frac{\partial \log l_t(\mathbf{r}; \theta)}{\partial \theta} \frac{\partial \log l_t(\mathbf{r}; \theta)}{\partial \theta'} \right].$$

The symbol  $E_0$  stands for the expectation with respect to the true distribution. In practice  $I$  and  $J$  are estimated by replacing  $E_0$  with the empirical mean and the unknown parameter  $\theta$  with its estimator  $\hat{\theta}$  [G97, page 49].

The constrained optimization of the quasi-likelihood is carried out using the PNL function

`pnl_optim_intpoints_bfgs_solve` that requires as an input the gradient of this function with respect to the parameters that need to be calibrated. We now describe how to compute this gradient for the two different models that we contemplate in this section.

### 2.1. Gradient of the quasi-likelihood for the standard GARCH model.

Given a sample of log-returns  $\mathbf{r} := \{r_1, \dots, r_T\}$  we want to calibrate to it a model of the form

$$\begin{cases} r_t &= \mu + \sigma_t \epsilon_t, \\ \sigma_t^2 &= \alpha_0 + \alpha_1 \sigma_{t-1}^2 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2. \end{cases} \quad (11)$$

Given that  $E[r_t] = \mu$  we can estimate the parameter  $\mu$  by using the sample mean of  $\mathbf{r}$  and then we proceed by calibrating the model

$$\begin{cases} r_t &= \sigma_t \epsilon_t, \\ \sigma_t^2 &= \alpha_0 + \alpha_1 \sigma_{t-1}^2 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2. \end{cases} \quad (12)$$

to the mean adjusted sample  $\hat{\mathbf{r}} := \mathbf{r} - \mu$ . In view of this, we can focus without loss of generality on the calibration of a model of the type (12) to a sample  $\mathbf{r} := \{r_1, \dots, r_T\}$  with zero mean. In this case, the quasi log-likelihood is:

$$\log L(\mathbf{r}; \theta) = -\frac{T}{2} \log 2\pi - \frac{1}{2} \sum_{t=1}^T \log \sigma_t^2 - \frac{1}{2} \sum_{t=1}^T \frac{r_t^2}{\sigma_t^2}, \quad \theta := (\alpha_0, \alpha_1, \beta_1) \quad (13)$$

conditional on the initial values  $r_0$  and  $\sigma_0^2$ . We set  $r_0 = 0$  and  $\sigma_0^2 = \text{var}(\mathbf{r})$ , that is, the sample variance of  $\mathbf{r}$ . The different components of the gradient of  $\log L$  can be computed by using the chain rule:

$$\frac{\partial \log L(\mathbf{r}; \theta)}{\partial \theta} = \sum_{t=1}^T \frac{\partial \log l_t(\mathbf{r}; \theta)}{\partial \sigma_t^2} \frac{\partial \sigma_t^2}{\partial \theta} = \sum_{t=1}^T \frac{1}{2} \left[ \frac{r_t^2}{\sigma_t^4} - \frac{1}{\sigma_t^2} \right] \frac{\partial \sigma_t^2}{\partial \theta}. \quad (14)$$

In order to fully characterize (14), we use (12) to explicitly write down the process  $\{\sigma_t^2\}$  as a function of the sample  $\mathbf{r}$  and the parameters  $\theta := (\alpha_0, \alpha_1, \beta_1)$ . Indeed, it can be shown by induction that

$$\sigma_t^2 = \alpha_0 \sum_{j=0}^{t-1} \beta_1^j + \alpha_1 \sum_{j=1}^{t-1} \beta_1^{t-j-1} r_j^2 + \sigma_0^2 \beta_1^t. \quad (15)$$

Consequently,

$$\frac{\partial \sigma_t^2}{\partial \alpha_0} = \sum_{j=0}^{t-1} \beta_1^j, \quad (16)$$

$$\frac{\partial \sigma_t^2}{\partial \alpha_1} = \sum_{j=1}^{t-1} \beta_1^{t-j-1} r_j^2, \quad (17)$$

$$\frac{\partial \sigma_t^2}{\partial \beta_1} = \alpha_0 \sum_{j=0}^{t-1} j \beta_1^{j-1} + \alpha_1 \sum_{j=1}^{t-2} (t-j-1) \beta_1^{t-j-2} r_j^2 + t \sigma_0^2 \beta_1^{t-1}. \quad (18)$$

The last four equalities substituted in (14) yield the three components of the gradient  $\partial \log L(\mathbf{r}; \theta) / \partial \theta$ .

**2.2. Gradient of the quasi-likelihood for the Duan GARCH model.** In this case a sample of log-returns  $\mathbf{r} := \{r_1, \dots, r_T\}$  and a risk-free interest rate  $r$  are given to which we want to calibrate a model of the form

$$\begin{cases} r_t &= \mu_t + \sigma_t \epsilon_t, \\ \sigma_t^2 &= \alpha_0 + \alpha_1 \sigma_{t-1}^2 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2. \end{cases} \quad (19)$$

with  $\mu_t := r + \lambda \sigma_t - \frac{1}{2} \sigma_t^2$ . In this case:

$$\log L(\mathbf{r}; \theta, \lambda) = -\frac{T}{2} \log 2\pi - \frac{1}{2} \sum_{t=1}^T \log \sigma_t^2 - \frac{1}{2} \sum_{t=1}^T \frac{(r_t - \mu_t)^2}{\sigma_t^2}, \quad \theta := (\alpha_0, \alpha_1, \beta_1). \quad (20)$$

As we saw before, the different components of the gradient of  $\log L$  can be computed by using the chain rule:

$$\begin{aligned} \frac{\partial \log L(\mathbf{r}; \theta)}{\partial \theta} &= \sum_{t=1}^T \frac{\partial \log l_t(\mathbf{r}; \theta)}{\partial \sigma_t^2} \frac{\partial \sigma_t^2}{\partial \theta} \\ &= \sum_{t=1}^T \frac{1}{2} \left[ \frac{(r_t - \mu_t)}{\sigma_t^4} \left( 2 \left( \frac{\partial \mu_t}{\partial \sigma_t^2} \right) \sigma_t^2 + (r_t - \mu_t) \right) - \frac{1}{\sigma_t^2} \right] \frac{\partial \sigma_t^2}{\partial \theta}, \\ \frac{\partial \log L(\mathbf{r}; \theta)}{\partial \lambda} &= \sum_{t=1}^T \frac{\partial \log l_t(\mathbf{r}; \theta)}{\partial \sigma_t^2} \frac{\partial \sigma_t^2}{\partial \lambda} + \frac{\partial \log l_t(\mathbf{r}; \theta)}{\partial \mu_t} \frac{\partial \mu_t}{\partial \lambda} \\ &= \sum_{t=1}^T \frac{1}{2} \left[ \frac{(r_t - \mu_t)}{\sigma_t^4} \left( 2 \left( \frac{\partial \mu_t}{\partial \sigma_t^2} \right) \sigma_t^2 + (r_t - \mu_t) \right) - \frac{1}{\sigma_t^2} \right] \frac{\partial \sigma_t^2}{\partial \lambda} + \frac{(r_t - \mu_t)}{\sigma_t^2} \frac{\partial \mu_t}{\partial \lambda} \end{aligned} \quad (21)$$

In this expression the variables  $\{\sigma_t^2\}$  are determined recursively using (19) and setting  $r_0 = 0$  and  $\sigma_0^2 = \text{var}(\mathbf{r})$ . Additionally:

$$\frac{\partial \mu_t}{\partial \sigma_t^2} = \frac{\lambda}{2\sqrt{\sigma_t^2}} - \frac{1}{2}, \quad \frac{\partial \mu_t}{\partial \lambda} = \sqrt{\sigma_t^2}.$$

Finally, in order to fully characterize (21) and (22), we need to compute the derivatives  $\partial \sigma_t^2 / \partial \theta$ . Given the complexity of the conditional mean term, in this situation it is very difficult to write down an explicit expression like (15) and use it to compute  $\partial \sigma_t^2 / \partial \theta$  and  $\partial \sigma_t^2 / \partial \lambda$ . We will therefore compute these derivatives recursively

by differentiating with respect to the parameters  $\theta$  and  $\lambda$  on both sides of the second equality in (19) where the variables  $\{\sigma_{t-1}^2 \epsilon_{t-1}^2\}$  are replaced by  $\{(r_t - \mu_t)^2\}$ . The following recursions are initialized by setting  $\partial \sigma_0^2 / \partial \theta = \partial \sigma_0^2 / \partial \lambda = 0$ :

$$\frac{\partial \sigma_t^2}{\partial \alpha_0} = 1 - 2\alpha_1(r_{t-1} - \mu_{t-1}) \left( \frac{\partial \mu_{t-1}}{\partial \sigma_{t-1}^2} \right) \left( \frac{\partial \sigma_{t-1}^2}{\partial \alpha_0} \right) + \beta_1 \frac{\partial \sigma_{t-1}^2}{\partial \alpha_0}, \quad (23)$$

$$\frac{\partial \sigma_t^2}{\partial \alpha_1} = (r_{t-1} - \mu_{t-1})^2 - 2\alpha_1(r_{t-1} - \mu_{t-1}) \left( \frac{\partial \mu_{t-1}}{\partial \sigma_{t-1}^2} \right) \left( \frac{\partial \sigma_{t-1}^2}{\partial \alpha_1} \right) + \beta_1 \frac{\partial \sigma_{t-1}^2}{\partial \alpha_1} \quad (24)$$

$$\frac{\partial \sigma_t^2}{\partial \beta_1} = -2\alpha_1(r_{t-1} - \mu_{t-1}) \left( \frac{\partial \mu_{t-1}}{\partial \sigma_{t-1}^2} \right) \left( \frac{\partial \sigma_{t-1}^2}{\partial \beta_1} \right) + \beta_1 \frac{\partial \sigma_{t-1}^2}{\partial \beta_1} + \sigma_{t-1}^2, \quad (25)$$

$$\frac{\partial \sigma_t^2}{\partial \lambda} = -2\alpha_1(r_{t-1} - \mu_{t-1}) \left[ \left( \frac{\partial \mu_{t-1}}{\partial \sigma_{t-1}^2} \right) \left( \frac{\partial \sigma_{t-1}^2}{\partial \lambda} \right) + \frac{\partial \mu_{t-1}}{\partial \lambda} \right] + \beta_1 \frac{\partial \sigma_{t-1}^2}{\partial \lambda}. \quad (26)$$

The last six equalities substituted in (21) and (22) yield the four components of the gradient  $\partial \log L(\mathbf{r}; \theta) / \partial \theta$  and  $\partial \log L(\mathbf{r}; \theta) / \partial \lambda$ .

### 3. VARIANCE REDUCTION VIA EMPIRICAL MARTINGALE SIMULATION (EMS)

In view of (7), option prices are computed in this context using Monte Carlo simulations. Given that the standard estimator of the mean does not respect the martingale condition Duan *et al* [DS98, DGS01] have proposed a correction that fixes this problem and additionally reduces its variance. We briefly review the construction in [DS98, DGS01].

Let  $\hat{S}_0(i), \dots, \hat{S}_T(i)$ ,  $i \in \{1, \dots, n\}$ , be a family of  $n$  simulated asset prices paths using (4) and (5). The goal is constructing a new family of paths  $S_0^*(i), \dots, S_T^*(i)$ ,  $i \in \{1, \dots, n\}$  that can be used as Monte Carlo sample paths to compute option prices using (7); the construction requires the use of a family of  $n + 1$  auxiliary paths  $Z_1^*(i), \dots, Z_T^*(i)$ ,  $i \in \{0, \dots, n\}$  recursively defined together with the  $S_j^*(i)$  as:

- (i): For  $t = 0$ : for any  $i \in \{1, \dots, n\}$  we have  $\hat{S}_0(i) = S_0$  and  $S_0^*(i) := S_0$ .
- (ii): For  $t = 1$ : define

$$Z_1(i) := \hat{S}_1(i), \text{ for any } i \in \{1, \dots, n\}, \quad (27)$$

$$Z_1(0) := \frac{1}{ne^r} \sum_{i=1}^n Z_1(i), \quad (28)$$

$$S_1^*(i) := S_0 \frac{Z_1(i)}{Z_1(0)}, \text{ for any } i \in \{1, \dots, n\}. \quad (29)$$

- (iii): For  $t \in \{2, \dots, T\}$ : define

$$Z_j(i) := S_{j-1}^*(i) \frac{\hat{S}_j(i)}{\hat{S}_{j-1}(i)}, \text{ for any } i \in \{1, \dots, n\}, \quad (30)$$

$$Z_j(0) := \frac{1}{ne^{rj}} \sum_{i=1}^n Z_j(i), \quad (31)$$

$$S_j^*(i) := S_0 \frac{Z_j(i)}{Z_j(0)}, \text{ for any } i \in \{1, \dots, n\}. \quad (32)$$

#### 4. PREMIA FUNCTIONS

All the functions below use the library Pnl which is a scientific library originally designed to provide a unified framework for contributors to Premia.

##### 4.1. Function to carry out the EMS correction.

- **int** `ems(const PnlMat* path, double interest, int frequency, PnlMat* ems_path)`

**Description:** this function carries out the EMS correction method on a family of simulated paths that are given in the columns of `PnlMat* path`. `interest` contains the annualized interest and `frequency` the rate at which the quotes contained in `PnlMat* path` are given (1 for daily, 5 for weekly, etc).

##### 4.2. Functions for GARCH pricing and hedging.

- **int** `pnl_garch_price(double today_price, double alpha_zero, double alpha_one, double beta_one, double lambda, double interest, int frequency, double K, int T, int N, emsopt choice, int type_generator, option_price* garch_price)`

**Description:** computes the price of a European vanilla option (call and put) using a Duan-type GARCH(1,1) model for the underlying asset. `today_price` denotes the price of the underlying asset at the beginning of the contract, `garch_price` `alpha_zero`, `alpha_one`, `beta_one`, and `lambda` are the parameters of the GARCH model, `interest` is the annualized interest, `frequency` is the frequency associated to the model used (1 for daily, 5 for weekly, etc), `K` and `T` are the option strike and time to maturity, `N` is the number of Monte Carlo paths, `emsopt` (`ems_on` or `ems_off`) selects if the EMS correction is used or not, and `type_generator` declares the pnl random generator used in the creation of Monte Carlo paths.

- **int** `pnl_garch_delta(const PnlVect* h, double alpha_zero, double alpha_one, double beta_one, double lambda, double interest, int frequency, double K, int T, int N, emsopt choice, int type_generator, option_price* delta)`

**Description:** computes the hedging delta for a vanilla option (call and put) whose underlying asset is modeled using a GARCH(1,1) model. The price of the underlying asset between the beginning of the contract and the moment at which delta is computed has followed the path specified by the column vector `h`.

The rest of the parameters use the same syntax as in the preceding function.

##### 4.3. Functions for historical GARCH model calibration.

- **int** `garch_historical_fit(param* input, double tolerance, PnlVect* start-point, option opt, int iter_max, int print_algo_steps, PnlVect* x_output)`

**Description:** fits either a standard GARCH(1,1) or a Duan model to a historical sample via quasi log-likelihood maximization. The optimization is carried out using the BFGS interior point algorithm presented in [AGJ00]. `input` contains the historical prices and in the Duan model case the annualized interest rate and the frequency associated to the

historical quotes (1 for daily, 5 for weekly, etc). The chosen model is selected in `opt`.

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