Pricing american option under stochastic volatility and stochastic interest rate: implementation in PREMIA

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Abstract

Using the analytical approach provided in Medvedev and Scaillet (2009), We price the American option in a three factor model with stochastic volatility and stochastic interest rate.

1 Introduction

We consider the pricing problem of American option in a three factor model with stochastic volatility and stochastic interest rate. The implementation of this problem is based on the approach given by Medvedev and Scaillet. By introducing an explicit and intuitive proxy for the exercise rule to the American option, they derived an asymptotic expansion for the price of American option price with short maturity.

The rest of this file is as follows: in Section 2 we introduce the model and the modified problem, then we present the solution of the modified problem in Section 3, the program manual of the implementation in PREMIA is given in Section 4.

2 Model description and its PDE

The risk-neutral dynamic of the three factor model is given as follows:

$$\begin{aligned} dS_t &= (r_t - \delta)S_t dt + \sigma_t S_t dW_t^{(1)} \\ d\sigma_t &= a(\sigma_t)dt + b(\sigma_t)dW_t^{(2)} \\ dr_t &= \alpha(r_t, t)dt + \beta(r_t)dW_t^{(3)}, \end{aligned}$$

where $dW_t^{(i)} dW_t^{(j)} = \rho_{ij} dt, i, j = 1, 2, 3$. From now on we assume that $\rho_{23} = 0$.

Then the put option price $\mathbf{P}(S,\sigma,r,t)$ satisfies the partial differential equation (PDE):

$$\begin{aligned} 0 = \quad & \partial_t \mathbf{P} + \partial_S \mathbf{P} S(r-\delta) + \partial_\sigma \mathbf{P} a(\sigma) + \partial_r \mathbf{P} \alpha(r,t) + \frac{1}{2}\partial^2_{SS}\mathbf{P} S^2\sigma^2 \\ & + \frac{1}{2}\partial^2_{\sigma\sigma}\mathbf{P} b^2(\sigma) + \frac{1}{2}\partial^2_{rr}\mathbf{P}\beta^2(r) + \partial^2_{S\sigma}\mathbf{P}\sigma S b(\sigma)\rho_{12} \\ & + \partial^2_{Sr}\mathbf{P}\sigma S\beta(\sigma)\rho_{13} + \partial^2_{\sigma r}\mathbf{P} b(\sigma)\beta(\sigma)\rho_{23} - r\mathbf{P}. \end{aligned} \tag{2.1}$$

with boundary conditions:

$$\begin{aligned} & \mathbf{P}(\infty,\sigma,r,t) = 0, \quad \mathbf{P}(S,\sigma,r,T) = \max\{K-S,0\}, \qquad (2.2) \\ & \mathbf{P}(\overline{S}(T-t),\sigma,r,t) = \max\{K-\overline{S}(T-t),0\}, \quad \partial_S\mathbf{P}(\overline{S}(T-t),\sigma,r,t) = -1, \end{aligned}$$

where we use notation $\mathbf{P} := \mathbf{P}(S,\sigma,r,t)$ for convenience in (2.1).

Change variables of (S,σ,r,t) to (θ,σ,r,τ) with $\theta = \frac{\log(K/S)}{\sigma\sqrt{T-t}}$ and $\tau = T-t$, then denote the put option price $\mathbf{P}(S,\sigma,r,t)$ as $P(\theta,\sigma,r,\tau)$, make use of the following relationships:

$$\partial_t\mathbf{P} = \frac{1}{2}\frac{\theta}{\tau}\partial_\theta P - \partial_\tau P, \qquad \partial_S\mathbf{P} = -\frac{\partial_\theta P}{\sigma S\sqrt{\tau}}$$

$$\partial_\sigma\mathbf{P} = -\frac{\theta}{\sigma}\partial_\theta P + \partial_\sigma P, \qquad \partial_r\mathbf{P} = \partial_r P,$$

$$\partial^2_{SS}\mathbf{P} = \frac{\partial^2_{\theta\theta}\mathbf{P}}{\sigma^2 S^2\tau} + \frac{\partial_\theta P}{\sigma S^2\sqrt{\tau}},$$

$$\partial^2_{\sigma\sigma}\mathbf{P} = \partial^2_{\sigma\sigma}P - \frac{2\theta}{\sigma}\partial^2_{\theta\sigma}P + \frac{2\theta}{\sigma^2}\partial_\theta P + \frac{\theta^2}{\sigma^2}\partial^2_{\theta\theta}P,$$

$$\partial^2_{rr}\mathbf{P} = \partial^2_{rr}P, \qquad \partial^2_{S\sigma}\mathbf{P} = \frac{\theta\partial^2_{\theta\theta}P}{S\sigma^2\sqrt{\tau}} - \frac{\partial^2_{\theta\sigma}P}{S\sigma\sqrt{\tau}} + \frac{\partial_\theta P}{S\sigma^2\sqrt{\tau}},$$

$$\partial^2_{Sr}\mathbf{P} = -\frac{\partial^2_{\theta r}P}{S\sigma\sqrt{\tau}}, \qquad \partial^2_{\sigma r} = -\frac{\theta}{\sigma}\partial^2_{\theta r}P + \partial^2_{\sigma r}P,$$

we can tranform PDE (2.1) as

$$\begin{aligned} & \frac{\theta}{2}\partial_\theta P + \frac{\partial^2_{\theta\theta}P}{2} - \tau\partial_\tau P + \sqrt{\tau}\left[\frac{\partial_\theta P}{\sigma}\left(\frac{\sigma^2}{2} - r + \delta\right) + b\rho_{12}\left(\frac{\theta\partial^2_{\theta\theta}P}{\sigma} - \partial^2_{\theta\sigma}P + \frac{\partial_\theta P}{\sigma}\right) - \beta\rho_{13}\partial^2_{\theta r}P\right] \\ & + \tau\left[a\left(\partial_\sigma P - \frac{\theta}{\sigma}\partial_\theta P\right) + \frac{b^2}{2}\left(\partial^2_{\sigma\sigma}P - \frac{2\theta}{\sigma}\partial^2_{\theta\sigma}P + \frac{2\theta}{\sigma^2}\partial_\theta P + \frac{\theta^2}{\sigma^2}\partial^2_{\theta\theta}P\right)\right] \\ & + \tau\left[\frac{\beta^2\partial^2_{rr}P}{2} + b\beta\rho_{23}\left(\partial^2_{\sigma r}P - \frac{\theta}{\sigma}\partial^2_{\theta r}P\right) - rP + \alpha_0(r)\partial_r P\right] + \sum_{i=1}^{\infty}\tau^{i+1}\alpha_i(r)\partial_r P = 0 \end{aligned} \tag{2.3}$$

with boundary condition

$$P(\infty,\sigma,r,\tau) = 0, \tag{2.4}$$

$$P(y,\sigma,r,\tau) = K\max\{1-e^{-\sigma y\sqrt{\tau}},0\} = K(1-e^{-\sigma y\sqrt{\tau}}), \tag{2.5}$$

where $\alpha(r,t) = \alpha(r, T-\tau) = \sum_{i=0}^{\infty} \alpha_i(r)\tau^i$.

Denote the solution of (2.3) under the boundary conditions (2.4) and (2.5) by $P(\theta, \sigma, r, \tau; y)$, then the American put price $P^\star(\theta, \sigma, r, \tau)$ can be approximated from below by:

$$P^\star(\theta, \sigma, r, \tau) \simeq \max_{y \geq \theta} P(\theta, \sigma, r, \tau; y) = P(\theta, \sigma, r, \tau; \overline{y}(\theta, \sigma, r, \tau)). \tag{2.6}$$

3 Asymptotic expansion of American option price

Now we present the asymptotic expansion of American option price given by Medvedev and Scaillet (2009).

Assume that $P(\theta, \sigma, r, \tau)$ as in PDE (2.3) has regular asymptotic expansion near maturity of the form:

$$P(\theta, \sigma, r, \tau) = \sum_{n=1}^{\infty} P_n(\theta, \sigma, r)\tau^{n/2}, \tag{3.1}$$

where $P_n(\theta, \sigma, r), n = 1, 2, \cdots$ are the coefficients of the short-maturity asymptotic expansion in τ. Next proposition from Medvedev and Scaillet (2009) gives the characteristic of the coefficients in (3.1).

Proposition 1 Consider the PDE (2.3) with boundary condition (2.4) and (2.5) with the regular asymptotic expansion (3.1) with (θ, τ)in the vicinity of $(0, 0)$. For any solution to this problem there exists functions $C_1(\sigma, r), C_2(\sigma, r), ...$ such that for each n:

$$P_n(\theta, \sigma, r) = C_n(\sigma, r)[p_n^0(\theta, \sigma, r)\Phi(\theta) + q_n^0(\theta, \sigma, r)\phi(\theta)] + p_n^1(\theta, \sigma, r)\Phi(\theta) + q_n^1(\theta, \sigma, r)\phi(\theta),$$

where $p_n^0(\theta, \sigma, r) \in \prod^0(n, \theta, \sigma, r), p_n^1(\theta, \sigma, r) \in \prod^1(n-2, \theta, \sigma, r), q_n^0(\theta, \sigma, r) \in \prod^0(n-1, \theta, \sigma, r), q_n^1(\theta, \sigma, r) \in \prod^1(n-3, \theta, \sigma, r)$ with the coefficients depending on model parameters and $C_1(\sigma, r), C_2(\sigma, r), \cdots, C_{n-1}(\sigma, r)$, $\Phi(\theta)$ and $\phi(\theta)$ is the cumulative distribution function and probability distribution function of normal distribution respectively.

$p_n^0(\theta, \sigma, r), q_n^0(\theta, \sigma, r), p_n^1(\theta, \sigma, r), q_n^1(\theta, \sigma, r)$ can be solved by substituting the expansion (3.1) into PDE (2.3). To determine functions $C_n(\sigma, r), n = 1, 2, ...$, we impose an explicit early exercise rule by requiring that the put option is exercised as soon as it hits the barrier $\theta = y$. This condition allows us to uniquely identify the coefficients $C_n(\sigma, r), n = 1, 2, \cdots$. Note that the put option payoff is $P(y, \sigma, r, \tau) = K[1 - \exp(-\sigma y\sqrt{\tau})]$, by Tylor's expansion we have

$$P(y, \sigma, r, \tau) = K[1 - \exp(-\sigma y\sqrt{\tau})] = \sigma y K\sqrt{\tau} - \frac{\sigma^2 y^2 K}{2}\tau + ... \tag{3.2}$$

By equating $P_1(\theta, \sigma, r)$ in (3.1) evaluated at $\theta = y$ to $\sigma y K$ in (3.2), we determine $C_1(\sigma, r)$. Then we find $p_2^0, p_2^1, q_2^0, q_2^1$, and equating $P_1(\theta, \sigma, r)$ to $\frac{\sigma^2 y^2 K}{2}$, we obtain $C_2(\sigma, r)$. All the other functions can be determined recursively by the same way.

4 Program Manual

We implement the American option pricing of the affine model

$$\begin{aligned} dS_t &= r_t S_t dt + \sqrt{v_t} S_t dW_t^{(1)}, \\ dv_t &= k_v(\overline{v} - v_t)dt + \sigma_v \sqrt{v_t} dW_t^{(2)}, \\ dr_t &= k_r(\overline{r} - r_t)dt + \sigma_r \sqrt{r_t} dW_t^{(3)}. \end{aligned}$$

For the affine model above, the coefficients and functions of the 4th order expansion for the American option price is given in the appendix.

Included files:
The program directory contains 3 files:
this documentation file “docu.pdf"
“ame_put.cpp"
“ame_put.out"

Compile and run program:
Compilation commande under Linux:
g++ ame_put.cpp -o ame_put.out

Commande to execute the algorithm:
./ame_put.out

Parameters of the product:
spot: stock price at the initial time
strike: strike of the American option
maturity: maturity of the American option, the expansion asymptotic works well for small maturity.
volatility: stock volatility at the initial time
interest: interest rate at the initial time
dividend:divident of the underlying stock

Model Parameters:
kv: k_v in model (4.1)
vbar: $\overline{v}$ in model (4.1)
sigmav: σ_v in model (4.1)
kr: k_r in model (4.1)
rbar: $\overline{r}$ in model (4.1)
sigmar: σ_r in model (4.1)
rho12: ρ_{12} in model (4.1)
rho13: ρ_{13} in model (4.1)
rho23: ρ_{23} in model (4.1), note that our method works well only when $\rho_{23} = 0$.

5 Appendix

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\begin{aligned}
C_1(v,r) &= \frac{Ky\sqrt{v_t}}{\Phi(y)y+\phi(y)}, \\
C_2(v,r) &= \frac{-\Phi(y)C_1\sqrt{v_t}^2+2\Phi(y)C_1r_t-\phi(y)yb_1\rho_{12}C_1-Ky^2v_t^3}{2v_t[\phi(y)(y^2+y)+\Phi(y)]}, \\
C_3(v,r) &= \frac{1}{24v_t^2[\Phi(y)(y^3+3y)+\phi(y)(y^2+2)]} \\
&\quad \times \Big[-3C_1v_t^4+\phi(y)+48\Phi(y)yb_1\rho_{12}\frac{\partial C_2}{\partial v}v_t^2+48\phi(y)b_1\frac{\partial C_2}{\partial v}v_t^2+24\phi(y)r_tC_1v_t^2 \\
&\quad -4\phi(y)b_1^2y^2C_1-24\Phi(y)yC_2v_t^3+24\Phi(y)yr_tC_1v_t^2-3\phi(y)b_1^2\rho_{12}^2C_1y^4 \\
&\quad +6\phi(y)b_1y^2\rho_{12}C_1v_t^2-24\phi(y)C_2v_t^3-96\Phi(y)yb_1\rho_{12}C_2v_t \\
&\quad -72\phi(y)b_1\rho_{12}C_2v_t-12\phi(y)a_vv_tC_1-12\phi(y)b_1y^2\rho_{12}C_1r_t+48\Phi(y)yC_2v_tr_t \\
&\quad +12\phi(y)b_1\rho_{12}C_1r_t+48\Phi(y)yb_r\rho_{13}v_t^2\frac{\partial C_2}{\partial r}+48\phi(y)b_r\rho_{13}v_t^2\frac{\partial C_2}{\partial r}-12\phi(y)C_1r_t^2 \\
&\quad -12\phi(y)b_r\rho_{13}v_tC_1+12\phi(y)C_1v_t^2r_t+48\phi(y)C_2v_tr_t-2\phi(y)b_1^2C_1 \\
&\quad +6\phi(y)b_1\rho_{12}C_1v_t^2-\phi(y)b_1^2\rho_{12}^2C_1+10\phi(y)b_1^2y^2\rho_{12}^2C_1+4Ky^3v_t^5\Big],
\end{aligned}
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\begin{aligned}
C_4(v,r) \quad = \quad & -\frac{1}{48(v_t^3(3\Phi(y)+\Phi(y)y^4+6\Phi(y)y^2+5\phi(y)y+\phi(y)y^3))} \\
& \times \big[6\phi(y)y^5b_1^2\rho_{12}^2C_1r_t+12\Phi(y)C_2v_t^5+2Ky^4v_t^7+12\phi(y)yb_1\rho_{12}C_1v_t^2r_t \\
& -36\phi(y)yb_1\rho_{12}C_1r_t^2+24\phi(y)yb_r\rho_{13}v_tC_1r_t+96\Phi(y)b_r\rho_{13}v_t^2\frac{\partial C_2}{\partial r}r_t-144\Phi(y)y^2v_t^2C_3r_t \\
& +42\phi(y)yb_1^2\rho_{12}^2C_1r_t-10\phi(y)yb_1^2C_1v_t^2-13\phi(y)y^5b_1^3\rho_{12}^3C_1-12\phi(y)yC_1v_t^2r_t^2 \\
& +6\phi(y)yC_1v_t^4r_t-44\phi(y)y^3b_1^2\rho_{12}^2C_1r_t+24\phi(y)ya_vv_tC_1r_t+48\Phi(y)a_rv_t^3\frac{\partial C_2}{\partial r} \\
& -24\Phi(y)a_rv_t^2C_1+8\phi(y)yC_1r_t^3+72\Phi(y)C_3v_t^4-28\phi(y)yb_1\rho_{12}b_r\rho_{13}v_tC_1 \\
& +24\Phi(y)b_r\rho_{13}v_t^3C_1+12\phi(y)y^3b_1b_r\rho_{13}v_tC_1-12\phi(y)C_1-12\phi(y)yb_r\rho_{13}v_t^3C_1 \\
& +48\Phi(y)b_r\rho_{13}v_t^2C_2+48\Phi(y)b_r\rho_{13}^2v_t^3b_{r_1}\frac{\partial C_2}{\partial r}-48\phi(y)yb_1\rho_{12}b_r\rho_{13}v_t^2\frac{\partial C_2}{\partial r} \\
& -48\Phi(y)b_r\rho_{13}v_t^4r-144\Phi(y)b_1\rho_{12}b_r\rho_{13}v_t^2\frac{\partial C_2}{\partial r}-144\Phi(y)b_r\rho_{13}v_t^3\frac{\partial C_3}{\partial r} \\
& -144\Phi(y)y^2v_t^3b_r\rho_{13}\frac{\partial C_3}{\partial r}-144\phi(y)yb_r\rho_{13}v_t^3\frac{\partial r}{+}8\phi(y)yb_r\rho_{13}^2v_t^2b_{r_1}C_1 \\
& -144\Phi(y)b_1\rho_{12}\frac{\partial C_3}{\partial v}v_t^3-144\Phi(y)y^2v_t^3b_1\rho_{12}\frac{\partial C_3}{\partial v}-144\phi(y)yb_1\rho_{12}\frac{\partial C_3}{\partial v}v_t^3 \\
& +48\Phi(y)C_2v_tr_t^2-48\Phi(y)C_2v_t^3r_t-144\Phi(y)C_3v_t^2r_t+2\phi(y)yb_1^3\rho_{12}C_1 \\
& -18\phi(y)y^3b_1^3\rho_{12}C_1-12\phi(y)y^3b_1\rho_{12}C_1v_t^2r_t+48\Phi(y)r_tC_1v_t^2r_t-12\phi(y)yb_1^2C_1r_t \\
& +8\phi(y)yb_1\rho_{12}v_t^2a_{v_1}C_1+432\Phi(y)y^2v_t^2b_1\rho_{12}C_3+288\Phi(y)b_1\rho_{12}C_3v_t^2 \\
& +432\phi(y)yb_1\rho_{12}C_3v_t^2+48\phi(y)ya_rv_t^3\frac{\partial C_2}{\partial r}+48\Phi(y\phi(y)ya_rv_t^3\frac{\partial C_2}{\partial r}+48\Phi(y)y^2v_t^3a_r\frac{\partial C_2}{\partial r} \\
& +12\phi(y)y^3b_1\rho_{12}C_1r_t^2+96\Phi(y)b_1\rho_{12}\frac{\partial C_2}{\partial v}v_t^2r_t+8\phi(y)y^3b_1^2C_1r_t-24\Phi(y)r_tC_1v_t^4 \\
& +12\Phi(y)b_1^2C_1v_t^2-\phi(y)yC_1v_t^6-48\Phi(y)C_2v_t^3r_t+72\Phi(y)b_1^2C_2v_t \\
& -48\Phi(y)a_vv_t^2C_2+24\Phi(y)a_vv_t^3C_1+48\Phi(y)a_vv_t^3\frac{\partial C_2}{\partial v}+24\phi(y)yb_1^2\frac{\partial^2 C_2}{\partial v^2}v_t^3 \\
& -96\phi(y)yb_1^2\frac{\partial C_2}{\partial v}v_t^2+33\phi(y)y^3b_1^3\rho_{12}^3C_1+10\phi(y)y^3b_1^2\rho_{12}^2C_1v_t^2+48\Phi(y)b_1^2\rho_{12}^2v_t^3\frac{\partial^2 C_2}{\partial v^2} \\
& -96\phi(y)ya_vv_t^2C_2-12\phi(y)ya_vv_t^3C_1-12\phi(y)yb_1\rho_{12}a_vv_tC_1+48\phi(y)ya_vv_t^3\frac{\partial C_2}{\partial v} \\
& +12\phi(y)y^3b_1\rho_{12}a_vv_tC_1+48\Phi(y)y^2v_t^3a_v\frac{\partial C_2}{\partial v}-96\Phi(y)y^2v_t^2a_vC_2-12\phi(y)y^3b_1^2\rho_{12}^2C_2v_t \\
& -48\phi(y)yC_2v_t^3r_t-9\phi(y)yb_1^3\rho_{12}^3C_1-\phi(y)yb_1^2\rho_{12}^2C_1v_t^2+3\phi(y)yb_1\rho_{12}C_1v_t^4 \\
& -24\phi(y)yb_1\rho_{12}v_t^2r_tC_1-192\Phi(y)b_1\rho_{12}C_2v_tr_t-48\phi(y)yb_1\rho_{12}C_2v_tr_t-144\phi(y)yC_3v_t^2r_t \\
& -144\Phi(y)b_1^2\rho_{12}^2\frac{\partial C_2}{\partial v}v_t^2+24\Phi(y)b_1^2\frac{\partial^2 C_2}{\partial v^2}v_t^3-48\Phi(y)b_1^2\frac{\partial C_2}{\partial v}v_t^2-3\phi(y)y^5b_1^2\rho_{12}^2C_1v_t^2 \\
& +\phi(y)y^7b_1^3\rho_{12}^3C_1-4\phi(y)y^3b_1^2C_1v_t^2+4\phi(y)y^5b_1^3\rho_{12}C_1+3\phi(y)y^3b_1\rho_{12}C_1v_t^4 \\
& +24\Phi(y)y^2v_t^3b_1^2\frac{\partial^2 C_2}{\partial v^2}-96\Phi(y)y^2v_t^2b_1^2\frac{\partial C_2}{\partial v}-48\Phi(y)b_1\rho_{12}\frac{\partial C_2}{\partial v}v_t^4-48\phi(y)yb_1^2\rho_{12}^2\frac{\partial C_2}{\partial v}v_t^2 \\
& +128\phi(y)yb_1^2C_2v_t-48\Phi(y)y^2v_t^3C_2r_t+144\Phi(y)y^2v_tb_1^2C_2+24\phi(y)yb_1\rho_{12}C_2v_t^3 \\
& +100\phi(y)yb_1^2\rho_{12}^2C_2v_t+72\phi(y)yC_3v_t^4+48\Phi(y)b_1\rho_{12}C_2v_t^3+192\Phi(y)b_1^2\rho_{12}^2C_2v_t \\
& +72\Phi(y)y^2v_t^4C_3\big],
\end{aligned}
$$

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\begin{aligned}
C_5(v,r) \quad = \quad & \frac{1}{5760v_t^4[\Phi(y)(15y+y^5+10y^3)+\phi(y)(y^4+9y^2+8)]} \\
& \times \Bigg[\Phi(y)\Bigg(-5760y^3v_t^4\rho_{23}b_rb_1\frac{\partial^2C_3}{\partial v\partial r}+86400yb_1\rho_{12}v_t^3b_r\rho_{13}\frac{\partial C_3}{\partial r} \\
& -11520yb_r\rho_{13}v_t^4\frac{\partial C_2}{\partial r}r_t+5760y\rho_{23}b_rb_1v_t^3\frac{\partial C_2}{\partial v}+23040yb_r\rho_{13}v_t^2b_1^2\frac{\partial C_2}{\partial r} \\
& +23040y^3v_t^4b_r\rho_{13}\frac{\partial C_4}{\partial r}+69120yb_r\rho_{13}v_t^4\frac{\partial C_4}{\partial r}-17280yv_t^2C_3r_t^2 \\
& +5760yv_t^3b_1^2\frac{\partial^2C_2}{\partial r^2}r_t-23040yv_t^2a_vC_2r_t-34560yv_t^3b_r\rho_{13}\frac{\partial C_3}{\partial r}r_t \\
& -23040yb_1\rho_{12}v_t^3a_r\frac{\partial C_2}{\partial r}-11520yv_t^3C_2r_t^2-17280yb_r\rho_{13}v_t^3C_3 \\
& -207360yb_1\rho_{12}C_4v_t^3-5760y^3v_t^4a_v\frac{\partial C_3}{\partial v}-92160y^3v_t^3b_1\rho_{12}C_4 \\
& -23040yv_t^2b_1^2\frac{\partial C_2}{\partial v}r_t+69120yC_4v_t^3r_t+11520yb_1\rho_{12}v_t^4a_v\frac{\partial^2C_2}{\partial r^2}-2880y^3v_t^4b_1^2\frac{\partial^2C_3}{\partial r^2} \\
& -17280yb_r\rho_{13}v_t^3a_v\frac{\partial C_2}{\partial r}-17280y\rho_{23}b_rb_1\frac{\partial C_3^2}{\partial v\partial r}v_t^4+5760yb_r^2v_t^3\frac{\partial C_2}{\partial r} \\
& -17280yb_r^2\rho_{13}^2v_t^4\frac{\partial^2C_3}{\partial r^2}+5760y^3v_t^4C_3r_t+11520yv_t^3a_r\frac{\partial C_2}{\partial r}r_t \\
& +5760yb_r\rho_{13}v_t^4a_{r_1}\frac{\partial C_2}{\partial r}-2880y\rho_{23}b_rb_1v_t^4\frac{\partial C_2}{\partial r}+11520y\rho_{23}b_rb_1^2v_t^2\rho_{12}\frac{\partial C_2}{\partial r} \\
& -2880y^3v_t^4b_r^2\frac{\partial C_3^2}{\partial^2 r}+34560yb_1^2\frac{\partial C_3}{\partial v}v_t^3-34560y^3v_t^2b_1^2C_3-4320yv_t^6C_3 \\
& -34560yv_t^3b_1\rho_{12}\frac{\partial C_3}{\partial v}r_t+34560yv_tb_1^2C_2r_t-28800yb_1^3\rho_{12}v_t^3\frac{\partial^2C_2}{\partial r^2} \\
& +23040y^3v_t^3C_4r_t+17280yv_t^4C_3r_t-155520yb_1^2\rho_{12}^2v_t^2C_3-8640yb_1^2\frac{\partial^2C_3}{\partial r^2}v_t^4 \\
& +11520yv_t^3a_v\frac{\partial C_2}{\partial v}r_t-34560yv_t^4b_1\rho_{12}C_3-60480yb_1^2v_t^2C_3 \\
& -5760yv_t^5a_r\frac{\partial C_2}{\partial r}-5760yv_t^5a_v\frac{\partial C_2}{\partial v}-17280yb_1^2\rho_{12}^2v_t^4\frac{\partial^2C_3}{\partial r^2}-17280ya_vv_t^4\frac{\partial C_3}{\partial v} \\
& +34560ya_vv_t^3C_3+5760yv_t^4b_1^2\frac{\partial C_2}{\partial v}+5760yv_t^5C_2r_t-5760y\rho_{23}b_rb_1v_t^2\frac{\partial C_2}{\partial r}r_t \\
& +86400yb_1^2\rho_{12}^2v_t^3\frac{\partial C_3}{\partial v}-2880yv_t^5b_1^2\frac{\partial^2C_2}{\partial r^2}+103680yv_t^2b_1\rho_{12}C_3r_t+17280yv_t^4C_3r_t \\
& +23040y^3v_t^4b_1\rho_{12}\frac{\partial C_4}{\partial v}+5760yb_1\rho_{12}v_t^4a_{v_1}\frac{\partial C_2}{\partial v}+69120yb_1\rho_{12}\frac{\partial C_4}{\partial v}v_t^4 \\
& -103680yb_1^3\rho_{12}v_tC_2+23040yb_1\rho_{12}v_t^3C_2r_t-11520yb_1\rho_{12}v_t^4\frac{\partial C_2}{\partial v}r_t+80640yb_1^3\rho_{12}v_t^2\frac{\partial C_2}{\partial v} \\
& +5760ya_rv_t^3C_2+2880yb_r^2v_t^4b_{r_2}\rho_{13}\frac{\partial C_2}{\partial r}-5760yb_r\rho_{13}v_t^4C_2-17280yb_r\rho_{13}^2v_t^4b_{r_1}\frac{\partial C_3}{\partial r} \\
& -8640yb_r^2v_t^4\frac{\partial^2C_3}{\partial r^2}-5760y\rho_{23}b_rb_1v_t^2C_2-40320yb_1\rho_{12}v_t^3a_v\frac{\partial C_2}{\partial v}
\end{aligned}
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\begin{aligned}
&+2880y\rho_{23}b_rb_1v_t^3C_1-8640yv_t^3b_1^2C_2-34560yb_1\rho_{12}v_t^4b_r\rho_{13}\frac{\partial^2C_3}{\partial v\partial r}+5760yv_t^4a_vC_2\\
&+17280yv_t^5b_r\rho_{13}\frac{\partial C_3}{\partial r}-2880yr_t^2v_t^4C_1+17280y^3v_t^3a_vC_3\\
&-11520yb_1\rho_{12}v_t^3a_{v_1}C_2+57600yb_1\rho_{12}v_t^2a_vC_2-34560yv_t^5C_4-11520y^3v_t^5C_4\\
&-5760y^3v_t^4a_r\frac{\partial C_3}{\partial r}-17280ya_rv_t^4\frac{\partial C_3}{\partial r}+2880ya_rv_t^4C_1\\
&+17280y^3v_t^3b_1^2\frac{\partial C_3}{\partial v}+17280yv_t^5b_1\rho_{12}\frac{\partial C_3}{\partial v}+5760ya_rv_t^4b_{r_1}\rho_{13}\frac{\partial C_2}{\partial r}\Bigg)\\
&+\phi(y)\Bigg(-17280b_r\rho_{13}v_t^3C_3-360y^2v_t^5b_r\rho_{13}C_1+16320b_r\rho_{13}v_t^2b_1^2\frac{\partial C_2}{\partial r}\\
&-2880v^5b_r\rho_{13}^2b_{r_1}\frac{\partial C_2}{\partial r}-17280b_r^2\rho_{13}^2v_t^4\frac{\partial^2C_3}{\partial r^2}-960b_r\rho_{13}v_t^3a_{r_1}C_1\\
&+1440y^2b_r\rho_{13}v_tb_1^2C_1-13440y^2b_1^2\rho_{12}^2b_r\rho_{13}v_t^2\frac{\partial C_2}{\partial r}+720y^2b_1^2\rho_{12}\rho_{23}b_rv_tC_1\\
&-11520b_r\rho_{13}v_t^4\frac{\partial C_2}{\partial r}r_t+1920y^2b_r\rho_{13}v_t^2b_1^2\frac{\partial C_2}{\partial r}-240y^2b_r^2\rho_{13}^3v_t^3b_{r_2}C_1+1920b_r^2\rho_{13}^3v_t^4b_{r_2}\frac{\partial C_2}{\partial r}\\
&-1080v_t^5b_r\rho_{13}C_1+1440b_1\rho_{12}a_rv_t^2C_1-120v_t^2b_1^2\rho_{12}^2C_1r_t+240b_1^3\rho_{12}C_1r_t\\
&+2160y^2b_1^2C_1r_t^2+5760b_1\rho_{12}\frac{\partial C_2}{\partial v}v_t^2r_t^2-120b_1^4\rho_{12}^2y^8C_1-15y^2v_t^8C_1-840b_1^3y^4v_t^2\rho_{12}C_1\\
&+390b_1^2y^4v_t^4\rho_{12}^2C_1-3771y^2b_1^4\rho_{12}^4C_1-1440y^2v_t^4b_1\rho_{12}r_tC_1+2400b_1^3y^4v_t^2\rho_{12}^3C_1\\
&-177b_1^4\rho_{12}^4C_1-6720y^2b_1^3\rho_{12}C_1r_t-11760b_1^3y^4\rho_{12}^3C_1r_t\\
&+12960y^2b_1^3\rho_{12}^3C_1r_t+240C_1r_t^4+150v_t^4b_1^2\rho_{12}^2C_1+60b_1^3\rho_{12}^3y^8v_t^2C_1\\
&+17280b_1^2\frac{\partial C_3}{\partial v}v_t^3-480b_1y^4\rho_{12}C_1r_t^3+5760v_t^3b_1^2\frac{\partial^2C_2}{\partial r^2}r_t-34560v_t^3b_r\rho_{13}\frac{\partial C_3}{\partial r}r_t\\
&-960a_vv_t^3a_{v_1}C_1+2880y^2b_1\rho_{12}v_t^2a_vC_2-720v_t^5b_1\rho_{12}C_2-11520a_vv_t^4\frac{\partial C_3}{\partial v}\\
&-5760v_t^5a_v\frac{\partial C_2}{\partial v}+2880v_t^4a_vC_2-1080v_t^5a_vC_1+17280a_vv_t^3C_3-240y^2C_1r_t^4+1920C_2v_tr_t^3\\
&+17280v_t^4C_3r_t-17280v_t^2C_3r_t^2+2880y^2b_1\rho_{12}C_1r_t^3+360v_t^4b_1\rho_{12}C_1r_t\\
&+360b_1^2y^6v_t^2\rho_{12}^2C_1r_t-240y^2b_1^2C_1v_t^2r_t+1440v_t^3b_r\rho_{13}C_1r_t-1440y^2a_vv_tC_1r_t^2\\
&-1440b_1y^4\rho_{12}a_vv_tC_1r_t-23040b_1\rho_{12}b_r\rho_{13}v_t^2\frac{\partial C_2}{\partial r}r_t+5760y^2b_1\rho_{12}b_r\rho_{13}v_t^2\frac{\partial C_2}{\partial r}r_t\\
&+7680y^2b_1\rho_{12}b_r\rho_{13}v_tC_1r_t+24000v_tb_1^2C_2r_t-120b_1^3\rho_{12}^3y^8C_1r_t-7200v_t^3b_1^2C_2\\
&-14400b_1\rho_{12}C_2v_tr_t^2-3360b_1\rho_{12}b_r\rho_{13}v_tC_1r_t-960y^2v_t^3b_1\rho_{12}b_r\rho_{13}C_1\\
&-3530b_1^4y^6\rho_{12}^4C_1-840b_1^3y^6v_t^2\rho_{12}^3C_1+435b_1^4\rho_{12}^4y^8C_1+960b_1\rho_{12}v_t^2a_{v_1}C_1r_t\\
&-11520\rho_{23}b_rb_1\frac{\partial^2C_3}{\partial v\partial r}v_t^4+5760b_1\rho_{12}b_r\rho_{13}v_t^3\frac{\partial C_2}{\partial v}+1900b_1^4y^6\rho_{12}^2C_1\\
&-960y^2b_1\rho_{12}v_t^2a_{v_1}C_1r_t+120y^2v_t^6C_1r_t-360y^2v_t^4C_1r_t^2-960a_rv_t^3b_{r_1}\rho_{13}C_1\\
&+720b_1y^4v_t^3\rho_{12}b_r\rho_{13}C_1+2880v_t^3b_r\rho_{13}C_1r_t-480b_1^2y^4b_r\rho_{13}v_tC_1+480y^2v_t^4b_r\rho_{13}^2b_{r_1}C_1
\end{aligned}
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\begin{aligned}
&+240b_r\rho_{13}^3v_t^3b_{r_1}^2C_1+1920b_r\rho_{13}^2v_t^3b_{r_1}C_2-360b_1^2y^6\rho_{12}^2b_r\rho_{13}v_tC_1+240b_r^2\rho_{13}^3v_t^3b_{r_2}C_1\\
&-5760y^2v_t^4\rho_{23}b_rb_1\frac{\partial C_3^2}{\partial v\partial r}-360b_1y^4v_t^4\rho_{12}C_1r_t+5760y^2b_1\rho_{12}a_vv_tC_1r_t-2880a_rv_t^2C_1r_t\\
&-1440b_1y^4\rho_{12}b_r\rho_{13}v_tC_1r_t+15v_t^8C_1-5760v_t^4b_r\rho_{13}\frac{\partial C_2}{\partial r}r_t-1440y^2b_r\rho_{13}v_tC_1r_t^2\\
&+5760y^2v_t^4C_3r_t+720v_t^2b_1\rho_{12}C_1r_t^2-17280v_t^2a_vC_2r_t-1440b_1\rho_{12}a_vv_tC_1r_t\\
&+11520v_t^3a_v\frac{\partial C_2}{\partial v}r_t+1440y^2a_vv_t^3C_1r_t-240v_t^7C_2+480a_vv_tb_1^2C_1-360b_1^2y^6\rho_{12}^2a_vv_tC_1\\
&+2880a_vv_t^3r_tC_1+720v_t^3b_1\rho_{12}a_vC_1+480b_1^2y^4v_t^2C_1r_t-480b_1^3y^6\rho_{12}C_1r_t-120v_t^6C_1r_t\\
&-23040v_t^5C_4-4320v_t^6C_3+60b_1y^4v_t^6\rho_{12}C_1+240b_1^3y^6v_t^2\rho_{12}C_1-90b_1^2y^6v_t^4\rho_{12}^2C_1\\
&-15b_1^4\rho_{12}^4C_1y^10-2880r_t^2v_t^4C_1+720v_t^6r_tC_1+1440b_r\rho_{13}v_tC_1r_t^2-5760\rho_{23}b_rb_1v_t^2\frac{\partial C_2}{\partial r}r_t\\
&+5760b_r\rho_{13}v_t^2\frac{\partial C_2}{\partial r}r_t^2+720b_1y^4\rho_{12}C_1v_t^2r_t^2+23040y^2v_t^3C_4r_t-11520v_t^3C_2r_t^2\\
&-16320y^2b_1^2\rho_{12}^2C_2v_tr_t-720b_1^2C_1r_t^2+1440C_2v_t^5r_t-2880v_t^3C_2r_t^2+360v_t^4C_1r_t^2\\
&-480C_1v_t^2r_t^3+46080C_4v_t^3r_t-480b_1^2y^4C_1r_t^2+2520b_1^2\rho_{12}^2C_1r_t^2\\
&+240b_1^2\rho_{12}^2v_t^3a_{v_2}C_1-4320y^2b_1^2\rho_{12}^2v_t^2C_3-720b_1^2\rho_{12}\rho_{23}b_rv_tC_1-2880\rho_{23}b_rb_1v_t^4\frac{\partial C_2}{\partial r}\\
&+5760\rho_{23}b_rb_1v_t^3\frac{\partial C_2}{\partial v}+2880y^2b_1\rho_{12}b_r\rho_{13}v_t^2C_2-12480b_1\rho_{12}b_r\rho_{13}v_t^2C_2\\
&+2880y^2b_1\rho_{12}b_r\rho_{13}^2v_t^3b_{r_1}\frac{\partial C_2}{\partial r}+1320b_1^2\rho_{12}^2b_r\rho_{13}v_tC_1+48Ky^5v_t^9-120b_1^2y^4v_t^4C_1\\
&+17280v_t^5b_1\rho_{12}\frac{\partial C_3}{\partial v}+4440b_1^2y^4\rho_{12}^2C_1r_t^2-1080b_1^3\rho_{12}^3C_1r_t-8640b_1^3\rho_{12}^3v_t^3\frac{\partial^2C_2}{\partial r^2}\\
&+5760y^2b_1^2\rho_{12}^2\frac{\partial C_2}{\partial v}v_t^2r_t+86400v_t^2b_1\rho_{12}C_3r_t+1440y^2v_t^3b_r\rho_{13}C_1r_t\\
&-2400y^2v_t^2b_1^2\rho_{12}^2C_1r_t-720y^2a_v^2v_t^2C_1-120b_1^2\rho_{12}^2a_vv_tC_1+720y^2v_t^5b_1\rho_{12}C_2\\
&+240b_1^3y^6\rho_{12}^3C_2v_t+23040y^2v_t^4b_1\rho_{12}\frac{\partial C_4}{\partial v}+720y^2v_t^4b_1\rho_{12}C_1r_t+2880y^2v_t^2b_1\rho_{12}C_1r_t^2\\
&-2880y^2v_t^2b_1\rho_{12}C_1r_t^2+3720y^2v_t^2b_1^2\rho_{12}^2C_1r_t-420y^2v_t^4b_1^2C_1+2880y^2b_1^3\rho_{12}^3v_t^3\frac{\partial^2C_2}{\partial r^2}\\
&-360b_1^2y^6\rho_{12}^2C_1r_t^2+1440b_1^2y^4\rho_{12}^2C_2v_tr_t+8640v_t^3b_1\rho_{12}C_2r_t-25920v_t^4b_1\rho_{12}C_3\\
&+1440v_t^6b_1\rho_{12}\frac{\partial C_2}{\partial v}-2880y^2v_t^3b_1\rho_{12}C_2r_t+5760b_r\rho_{13}v_t^2C_2r_t+1920y^2v_tb_1^2C_2r_t\\
&+5760a_rv_t^3C_2+480b_1^2v_t^2r_tC_1-900v_t^4b_1^2C_1-13440y^2b_1^3\rho_{12}^3\frac{\partial C_2}{\partial v}v_t^2\\
&+5760b_1^2\rho_{12}^2v_t^3\frac{\partial^2C_2}{\partial r^2}r_t-2880v_t^4r_tC_1r_t-180v_t^6b_1\rho_{12}C_1-840y^2v_t^2b_1^3\rho_{12}^3C_1\\
&+960y^2a_vv_tb_1^2C_1-480b_1^2y^4a_vv_tC_1+11520b_1\rho_{12}v_t^4a_v\frac{\partial^2C_2}{\partial r^2}-11520b_1\rho_{12}v_t^4\frac{\partial C_2}{\partial v}r_t\\
&-17280b_1^2\rho_{12}^2v_t^4\frac{\partial^2C_3}{\partial r^2}-92160y^2v_t^3b_1\rho_{12}C_4-115200b_1\rho_{12}C_4v_t^3+2880C_1v_t^2r_tr_t^2
\end{aligned}
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$$+480y^2C_1v_t^2r_t^3+5760b_r\rho_{13}^2v_t^3b_{r_1}\frac{\partial C_2}{\partial r}r_t-34560v_t^3b_1\rho_{12}\frac{\partial C_3}{\partial v}r_t-10440y^2b_1^2\rho_{12}^2C_1r_t^2$$
$$+1440v_t^3a_vC_1r_t-17280b_1\rho_{12}v_t^3a_r\frac{\partial C_2}{\partial r}-2880v_t^5b_1^2\rho_{12}^2\frac{\partial^2 C_2}{\partial r^2}-2880b_1\rho_{12}v_t^2r_tC_1r_t$$
$$+2880y^2b_1\rho_{12}C_2v_tr_t^2+35040b_1^2\rho_{12}^2C_2v_tr_t-5760v_t^4b_1\rho_{12}\frac{\partial C_2}{\partial v}r_t-23040b_1^2\rho_{12}^2\frac{\partial C_2}{\partial v}v_t^2r_t$$
$$+1920\rho_{23}b_rb_1v_tC_1r_t+30y^2v_t^4b_1^2\rho_{12}^2C_1-120v_t^2b_1^3\rho_{12}C_1+17280y^2v_t^3b_1^2\frac{\partial C_3}{\partial v}$$
$$-1440v_t^4b_1\rho_{12}r_tC_1-252b_1^4C_1-276y^2b_1^4C_1-480b_1^2y^4\rho_{12}^2v_t^2a_{v_1}C_1-240b_1^2\rho_{12}^2v_t^2a_{v_1}C_1$$
$$+46080b_1\rho_{12}\frac{\partial C_4}{\partial v}v_t^4+5760b_r\rho_{13}v_t^4a_{r_1}\frac{\partial C_2}{\partial r}+46080b_r\rho_{13}v_t^4\frac{\partial C_4}{\partial r}-80b_1^4y^6C_1$$
$$+5760v_t^5C_2r_t+240v_t^2b_1^2\rho_{12}^2C_1r_t+5280y^2v_t^3b_1^2\rho_{12}^2C_2+960b_1^3y^4\rho_{12}v_tC_2$$
$$-27600b_1^3\rho_{12}^3C_2v_t+17280b_1\rho_{12}v_t^3C_2r_t+22320y^2b_1^3\rho_{12}^3C_2v_t+4800y^2b_1^3\rho_{12}v_t^2\frac{\partial C_2}{\partial v}$$
$$-1440b_1\rho_{12}C_1r_t^3+240b_1^2C_1v_t^2r_t-3000b_1^2y^4v_t^2\rho_{12}^2C_1r_t+2400b_1^3y^6\rho_{12}^3C_1r_t+564b_1^4\rho_{12}^2C_1$$
$$+2652y^2b_1^4\rho_{12}^2C_1+2880v_t^4b_1^2\frac{\partial C_2}{\partial v}-5760b_1^2\frac{\partial^2 C_3}{\partial r^2}v_t^4-34560y^2v_t^2b_1^2C_3$$
$$+576b_1^4y^4C_1+1440b_1^3y^4\rho_{12}^3\frac{\partial C_2}{\partial v}v_t^2-60v_t^2b_1^3\rho_{12}^3C_1+960y^2b_1^2v_t^2r_tC_1-6372b_1^4y^4\rho_{12}^2C_1$$
$$+5760v_t^4b_1^2\rho_{12}^2\frac{\partial C_2}{\partial v}-2880y^2v_t^4b_1^2\rho_{12}^2\frac{\partial C_2}{\partial v}+53760b_1^3\rho_{12}v_t^2\frac{\partial C_2}{\partial v}+22560b_1^3\rho_{12}^3\frac{\partial C_2}{\partial v}v_t^2$$
$$-360y^2v_t^5a_vC_1+720b_1y^4v_t^3\rho_{12}a_vC_1-5760y^2v_t^4a_r\frac{\partial C_3}{\partial r}-1440y^2b_1\rho_{12}a_rv_t^2C_1$$
$$-11520a_rv_t^4\frac{\partial C_3}{\partial r}+4320a_rv_t^4C_1-5760v_t^5a_r\frac{\partial C_2}{\partial r}-7800y^2b_1^2\rho_{12}^2b_r\rho_{13}v_tC_1$$
$$+2880b_r^2v_t^4b_{r_2}\rho_{13}\frac{\partial C_2}{\partial r}-480b_r^2v_t^3b_{r_2}\rho_{13}C_1-480b_1y^4\rho_{12}b_r\rho_{13}^2v_t^2b_{r_1}C_1+3960b_1^2y^4\rho_{12}^2b_r\rho_{13}v_tC_1$$
$$+1440v_t^6b_r\rho_{13}\frac{\partial C_2}{\partial r}+22560b_1^2\rho_{12}^2b_r\rho_{13}v_t^2\frac{\partial C_2}{\partial r}+480v_t^4b_r\rho_{13}^2b_{r_1}C_1-240y^2b_r\rho_{13}^3v_t^3b_{r_1}^2C_1$$
$$+2160y^2b_1\rho_{12}b_r\rho_{13}^2v_t^2b_{r_1}C_1+1920b_r\rho_{13}^3v_t^4b_{r_1}^2\frac{\partial C_2}{\partial r}-17280b_r\rho_{13}^2v_t^4b_{r_1}\frac{\partial C_3}{\partial r}$$
$$-5760\rho_{23}b_rb_1v_t^2C_2-240a_v^2v_t^2C_1+8466b_1^4y^4\rho_{12}^4C_1-110880b_1^2\rho_{12}^2v_t^2C_3+5760a_rv_t^4b_{r_1}\rho_{13}\frac{\partial C_2}{\partial r}$$
$$-11520b_r\rho_{13}v_t^3a_v\frac{\partial C_2}{\partial r}+480b_r\rho_{13}v_t^2a_vC_1-720y^2b_r^2\rho_{13}^2v_t^2C_1+720b_r^2\rho_{13}^2v_t^2C_1$$
$$-240v_t^3b_1\rho_{12}b_r\rho_{13}C_1-34560b_1\rho_{12}v_t^4b_r\rho_{13}\frac{\partial^2 C_3}{\partial v\partial r}-5760b_r^2v_t^4\frac{\partial^2 C_3}{\partial r^2}+5760b_r^2v_t^3\frac{\partial C_2}{\partial r}$$
$$-960b_r^2v_t^2C_1-720y^2b_1^2v_t^2a_{v_1}C_1+480v_t^4b_1\rho_{12}a_{v_1}C_1-9600b_1\rho_{12}v_t^3a_{v_1}C_2$$
$$+37440b_1\rho_{12}v_t^2a_vC_2+17280y^2v_t^3a_vC_3-720b_1\rho_{12}b_r\rho_{13}^2v_t^2b_{r_1}C_1-2880y^2v_t^4b_r^2\frac{\partial^2 C_3}{\partial r^2}$$
$$-8640b_1\rho_{12}b_r\rho_{13}^2v_t^3b_{r_1}\frac{\partial C_2}{\partial r}-1440y^2b_r\rho_{13}v_t^2a_vC_1+23040y^2v_t^4b_r\rho_{13}\frac{\partial C_4}{\partial r}$$

$$
\begin{aligned}
&+69120 b_1 \rho_{12} v_t^3 b_r \rho_{13} \frac{\partial C_3}{\partial r} + 5760 \rho_{13}^2 v_t^3 \frac{\partial C_2}{\partial r} - 8640 b_r \rho_{13} v_t^4 C_2 \\
&+5760 v_t^4 b_1 \rho_{12} b_r \rho_{13} \frac{\partial C_2}{\partial r} + 9600 b_1^2 v_t^2 \rho_{12} \frac{\partial C_2}{\partial r} + 1440 b_1^2 y^4 \rho_{12}^2 b_r \rho_{13} v_t^2 \frac{\partial C_2}{\partial r} \\
&+17280 v_t^5 b_r \rho_{13} \frac{\partial C_3}{\partial r} + 3840 \rho_{23} b_r b_1 v_t^3 C_1 - 2880 y^2 v_t^4 b_1 \rho_{12} b_r \rho_{13} \frac{\partial C_2}{\partial r} \\
&-67680 \rho_{12} v_t C_2 - 6960 v_t^3 b_1^2 \rho_{12}^2 C_2 - 10080 y^2 b_1^3 \rho_{12} v_t C_2 - 4560 b_1^3 y^4 C_2 v_t \\
&+5760 b_1 \rho_{12} v_t^4 a_{v_1} \frac{\partial C_2}{\partial v} - 960 y^2 v_t^3 b_1^2 C_2 - 2880 y^2 v_t^4 b_1^2 \frac{\partial^2 C_3}{\partial r^2} + 960 b_r \rho_{13}^2 v_t^2 r_t \\
&-960 y^2 b_r \rho_{13}^2 v_t^2 b_{r_1} C_1 r_t + 1440 a_v v_t C_1 r_t^2 + 4560 b_1^3 y^4 \rho_{12} C_1 r_t + 480 y^2 v_t^4 b_1 \rho_{12} a_{v_1} C_1 \\
&-240 b_1^2 v_t^2 a_{v_1} C_1 + 1680 y^2 b_1^2 \rho_{12}^2 v_t^2 a_{v_1} C_1 + 11520 r_t - 31680 b_1^2 v_t^2 C_3 \\
&-240 y^2 b_1^2 \rho_{12}^2 v_t^3 a_{v_2} C_1 + 3000 b_1^2 y^4 \rho_{12}^2 a_v v_t C_1 - 5760 y^2 v_t^4 a_v \frac{\partial C_3}{\partial v} \\
&-28800 \rho_{12} v_t^3 a_v \frac{\partial C_2}{\partial v} - 3480 y^2 b_1^2 \rho_{12}^2 a_v v_t C_1 - 11520 y^2 v_t^5 C_4 - 17280 v_t^2 b_1^2 \frac{\partial C_2}{\partial v} r_t \\
&+11520 v_t^3 a_r \frac{\partial C_2}{\partial r} r_t - 20160 b_1^3 \rho_{12} v_t^3 \frac{\partial^2 C_2}{\partial r^2} + 69120 b_1^2 \rho_{12}^2 v_t^3 \frac{\partial C_3}{\partial v} + 720 b_1^2 y^4 v_t^2 \rho_{12}^2 C_1 r_t \\
&-2880 v_t^5 b_1^2 \frac{\partial^2 C_2}{\partial r^2} - 480 b_1^2 v_t^3 a_{v_2} C_1 - 720 b_1^2 y^4 v_t^3 \rho_{12}^2 C_2 \Bigg) \Bigg],
\end{aligned}
$$

where

$$
\begin{aligned}
a_r &= k_r(\overline{r} - r_t), \\
a_{r_1} &= -k_r \\
a_v &= (k_v(\overline{v} - v_t^2) - b_1^2)/2/v_t \\
a_{v_1} &= -(k_v\overline{v} - b_1^2)/2/v_t^2 - k_v/2 \\
a_{v_2} &= (k_v\overline{v} - b_1^2)/v_t^3 \\
b_1 &= \sigma_v/2 \\
b_r &= \sigma_r\sqrt{r_t} \\
b_{r_1} &= \sigma_r/\sqrt{r_t}/2 \\
b_{r_2} &= -(1/4)\sigma_r r_t^{-3/2}
\end{aligned}
$$

References

[1] A., Medvedev, O., Scaillet, 2009, Pricing American options under stochastic volatility and stochastic interest rates.