# Interest Rate Derivatives

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## Contents



Premia 14

## 1 Options on Zero Coupon Bond

Let $r_t$ the dynamics for the instantaneous spot rate process. We note $B(t,T)$ the value of a unit-principal zero coupon bond at time $t$ with maturity $T$. The price of a zero-coupon bond (ZCB) at time t for the maturity T is characterized by a a unit amount of currency available at time T:

$$B(t,T) = E_Q \left( e^{-\int_t^T r(s)ds} \middle| F_t \right).$$

The price of a European call option with maturity T, strike K and written on a unit-principal zero-coupon bond with maturity $S > T$

$$ZBC(t,T,S,K) = E_Q \left( e^{-\int_t^T r(s)ds} (B(T,S) - K)_+ \middle| F_t \right).$$

The price of a European Put option with maturity T, strike K and written on a unit-principal zero-coupon bond with maturity $S > T$

$$ZBP(t,T,S,K) = E_Q \left( e^{-\int_t^T r(s)ds} (K - B(T,S))_+ \middle| F_t \right).$$

with Q risk-neutral measure. The put-call parity for bond options gives

$$ZBC(t,T,S,K) - ZBP(t,T,S,K) = P(T,S) - KP(t,T)$$

## 2 Cap and floor

We note $B(t,T)$ the value of a zero coupon bond at time $t$ with maturity $T$.We note $L(t,T_i,\tau)$ the forward rate which set at time $T_i$ the cash flow received at time $T_i+\tau$. Arbitrage leads to

$$1 + \tau L(t,T_i,\tau) = \frac{B(t,T_i)}{B(t,T_{i+1})} \tag{1}$$

spot libor rate is given by

$$1 + \tau L(T_i,T_i,\tau) = \frac{1}{B(T_i,T_{i+1})} \tag{2}$$

suppose that $t < T_M < T_{M+1}$ **Caplet and floorlet**

- we note $Cplt(t,T_M,K,\tau,N)$ the european caplet with maturity $T_M$, strike $K$ on the spot libor rate $L(t,t,\tau)$ with nominal value $N$ then at time $T_{M+1} = T_M + \tau$ the payoff is given by

$$N\tau(L(T_M,T_M,\tau) - K)_+$$

- we note $Flt(t,T_M,K,\tau,N)$ the european floorlet with maturity $T_M$, strike $K$ on the spot libor rate $L(t,t,\tau)$ with nominal value $N$ then at time $T_{M+1} = T_M + \tau$ the payoff is given by

$$N\tau(K - L(T_M,T_M,\tau))_+$$

the cash flow at time $T_{M+1} = T_M + \tau$ is fixed at time $T_M$
**Cap and floor** suppose that $t = T_0 < T_1 < .. < T_M$

- we note $Cap(t,T_s,T_M,K,\tau,N)$ the european cap with maturity $T_M$, strike $K$ on the spot rate $L(t,t,\tau)$ then at times $T_{s+1},...,T_M$ the option leads the cash flows $N\tau(L(T_s,T_s,\tau)-K)_+, N\tau(L(T_{s+1},T_{s+1},\tau)-K)_+,..,N\tau(L(T_{M-1},T_{M-1},\tau)-K)_+$ ie

$$\text{at } T_i \text{ cash flow } N\tau(L(T_{i-1},T_{i-1},\tau) - K)_+$$

- we note $Floor(t, T_s, T_M, K, \tau, N)$ the european floor with maturity $T_M$, strike $K$ on the spot libor rate $L(t, t, \tau)$ then at times $T_{s+1}, ..., T_M$ the option leads the cash flows $N\tau(K - L(T_s, T_s, \tau))_+, N\tau(K - L(T_{s+1}1, T_{s+1}, \tau))_+, .., N\tau(K - L(T_{M-1}, T_{M-1}, \tau))_+$

$$\text{at } T_i \text{ cash flow } N\tau(K - L(T_{i-1}, T_{i-1}, \tau))_+$$

1. a cap is a portfolio of caplets

$$Cap(t, T_s, T_M, K, \tau, N) = \sum_{i=s}^{M-1} Cplt(t, T_i, K, \tau, N)$$

2. a floor is a portfolio of floorlets

$$Floor(t, T_s, T_M, K, \tau, N) = \sum_{i=s}^{M-1} Flt(t, T_i, K, \tau, N)$$

Is possible to derive explicit formualas for cap/floor prices under the analitytically tractable short rate models given in Premia :

1. a cap is a portfolio of European Put options on zero-coupon bond

$$Cap(t, T_s, T_M, K, \tau, N) = \sum_{i=s}^{M-1} ZBP(t, T_i, K', \tau)$$

2. a floor is a portfolio of European Call options on zero-coupon

$$Floor(t, T_s, T_M, K, \tau, N) = \sum_{i=s}^{M-1} ZBC(t, T_i, K', \tau)$$

where

$$K' = \frac{1}{1 + K\tau}$$

# 3 Swaption

A swaption is an option on swap rate between time $T_\alpha$ and $T_\beta$ which is given by :

$$S_{\alpha,\beta}(t) = \frac{B(t, T_\alpha) - B(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i B(t, T_i)}$$

where $B(t,T_i)$ is the zero coupon bond price, and $T_i \quad _{(i=\alpha,..,\beta)}$ the different maturities of the swap rate. A payer swaption is an option over a swap rate maturing at time $T_\alpha$ given by :

$$PSwpt_{\alpha,\beta}(t) = E_t\left[e^{-\int_t^{T_\alpha} r(s)ds} \sum_{i=\alpha+1}^{\beta} \tau_i B(T_\alpha,T_i)\left(S_{\alpha,\beta}(T\alpha) - K\right)_+\right].$$

A receiver swaption is an option over a swap rate given by :

$$RSwpt_{\alpha,\beta}(t) = E_t\left[e^{-\int_t^{T_\alpha} r(s)ds} \sum_{i=\alpha+1}^{\beta} \tau_i B(T_\alpha,T_i)\left(K - S_{\alpha,\beta}(T\alpha)\right)_+\right].$$

A swaption can also be seen as an option of strick 1 over a certain coupon bearing:

$$RSwpt_{\alpha,\beta}(t) = E_t\left[e^{-\int_t^{T_\alpha} r(s)ds}\left(CB_{\alpha,\beta}(T_\alpha) - 1\right)_+\right], \tag{3}$$

where

$$CB_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} c_i B(t,T_i)$$

whith $c_i = K\tau_i$ for $i = \alpha+1,..,\beta-1$ and $c_\beta = 1 + K\tau_\beta$.
Supposing now that there is an analitical formula for the zero coupon bonds of the form :

$$B(t,T) = A_1 e^{-A_2\, r(t)},$$

then there exist an $r^*$ such that analitical value of coupon bearing at time $t = T_\alpha$ with $r(T_\alpha) = r^*$ is 1:

$$\sum_{i=\alpha+1}^{\beta} c_i A_1(T_\alpha,T_i) e^{-A_2(T_\alpha,T_i)r^*} = 1$$

Replacing the strick 1 in (3) it appears that parenthesis is a sum : $(\sum)_+$ which is positive if and only if each term of the sum is positive. Thus :

$$\begin{aligned} RSwpt_{\alpha,\beta}(t) &= \sum_{i=\alpha+1}^{\beta} c_i E_t\left[e^{-\int_t^{T_\alpha} r(s)ds}\left(B(T_\alpha,T_i) - K_i\right)_+\right] \\ &= \sum_{i=\alpha+1}^{\beta} c_i CALL(t,T_\alpha,T_i,K_i) \end{aligned}$$

Where $K_i = A_1(T_\alpha,T_i)e^{-A_2(T_\alpha,T_i)r^*}$ and $CALL(t,T_\alpha,T_i,K_i)$ is a call option at time $t$ on a zero coupon bond $B(T_\alpha,T_i)$ maturing at time $T_\alpha$ with a strick $K_i$. Equally for a payer swaption :

$$PSwpt_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} c_i PUT(t,T_\alpha,T_i,K_i)$$

Where $K_i = A_1(T_\alpha, T_i)e^{-A_2(T_\alpha, T_i)r^*}$ and $PUT(t, T_\alpha, T_i, K_i)$ is a put option at time $t$ on a zero coupon bond $B(T_\alpha, T_i)$ maturing at time $T_\alpha$ with a strick $K_i$.
Thus an analytical formula for call and put option leads to an analytical formula for a swaption.

# References