Default contagion in large homogeneous portfolios

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Premia 14
Credit Derivatives

1 Model and assumptions

We consider the problem of computing the price of a Collateralized Debt Obligation (CDO) and Credit Default Swap (CDS) using the Hebertsson approach [1]. In this setting, we give the main steps for the algorithm that we use in Premia to find these prices.

Let us assume that the investor has bought a derivative with a given return depending on the defaults of d entities. The main task is to find the good price of the credit derivatives provided that these entities are exchangeable. For some given random variables $\tau_1, \cdots, \tau_d$, that represent the default times of the d entities, we introduce the point process $N_{t,i} = 1_{\{\tau_i \leq t\}}$, for $1 \leq i \leq d$. Let us introduce now the following filtrations:

$$\mathcal{F}_{t,i} = \sigma(N_{s,i}, s \leq t), \qquad \mathcal{F}_t = \cup_{i=1}^d \mathcal{F}_{t,i}$$

and the associated $\mathcal{F}_t$-intensity: $\lambda_{t,i}$ of the point process $N_{t,i}$. In the following, we Assume the exchangeability of the entities: $\lambda_t = \lambda_{t,i}$ for any $i = 1, \cdots, d$ and $t \leq T$. More precisely, we will assume that

$$\lambda_t = a + \sum_{i=1}^{d-1} b_i 1_{\{T_i \leq t\}}, \qquad t \leq T \tag{1}$$

where the sequence $(T_i)_{1 \leq i \leq d}$ denotes the ordering default times $(\tau_i)_{1 \leq i \leq d}$. The constants $a, b_1, \cdots b_{d-1}$ are defined such that the process λ is positive.

Proposition 1. *(Herbertsson [1]) There exists a Markov jump process $(Y_t)_{t \geq 0}$ on a finite state $E = \{0, 1, \cdots, d\}$ such that the stopping time*

$$T_k = \inf\{t > 0, Y_t = k\}$$

are the ordering of d default times $\tau_1, \tau_2, \cdots, \tau_d$ with the intensity given in (1). The generator Q of Y is given by

$$Q_{k,k+1} = (d-k)\left(a + \sum_{j=1}^{k} b_j\right), \qquad Q_{k,k} = -Q_{k,k+1} \text{ for all } k = 0, 1 \cdots, d-1,$$

where the others entries of Q are zero. The Markov process always starts in $\{0\}$.

Proposition 2. *Let us consider d entities with default intensity given by (1) and let $q \in \mathbb{N}$, $1 \leq q \leq d$. We have*

$$\mathbb{P}(\tau_1 \geq t, \cdots \tau_q \geq t) = \alpha e^{Qt} s^{(q)}, \qquad \mathbb{P}(\tau_1 \geq t, \cdots \tau_q \geq t | Y_t = j) = \frac{C_{d-j}^q}{C_d^q} \text{ pour } j \leq d - q$$

where $\alpha = (1, 0, \cdots 0)$ denotes the initial distribution on E and $s_j^q = \frac{C_{d-j}^q}{C_d^q}, 1 \leq j \leq d$.

Now, knowing the joint conditional and unconditional law of the default times, we can compute explicitly the price of the CDO and the index CDS.

2 Index CDS Spread

The buyer of this credit derivative pays $\kappa \times dt$ on the period dt and in counterpart he receives a recovery $(1-R)$ at each default time. Our goal is to find the annual spread κ such that the payment leg and default leg are equals.

Payment leg: On the period $[0,T]$, the payment leg is given by $PL = \int_0^T e^{-rs}\kappa(1-L_s)ds$ where L denotes the loss process given by $L_s = \frac{(1-R)}{d}\sum_{i=1}^d N_{s,i}$. Then, using the grill of payment $0 \le t_1 \le \cdots \le t_n = T$ we get:

$$\mathbb{E}(PL) = \kappa \sum_{i=1}^{n} e^{-rt_i}(1-\mathbb{E}(L_{t_i}))(t_i - t_{i-1}).$$

Default leg: The default leg denotes the sum of loss due to defaults. In the period $[0,T]$, the default leg $DL = \int_0^T e^{-rs}dL_s$. Using an integration by parts formula one gets $DL = e^{-rT}L_T + \int_0^T re^{-rt}L_t dt$. The default leg on the grill of default $0 \le t_1 \le \cdots \le t_n$ is given by:

$$\mathbb{E}(DL) = e^{-rT}L_T + r\sum_{i=1}^{n}\mathbb{E}(L_{t_i})(t_i - t_{i-1}).$$

3 CDO spread

As in index Credit Default Swap, we should find the payment leg and the default leg on the CDO tranche $[a,b]$ to find a fair price of the CDO spread.

Payment leg: Let $M_{a,b}$ denotes the process of impact loss on the tranche CDO $[a,b]$, we recall $M_{a,b}(t) = (L_t - a)1_{\{L_t \ge a\}} + (b-a)1_{\{L_t \ge b\}}$, hence the payment leg on the period $[0,T]$,$PL = \kappa\int_0^T e^{-rs}(b-a-M_{a,b}(t))dt$. On the grill of payment we get:

$$\mathbb{E}(PL) = \kappa \sum_{i=1}^{n} e^{-rt_i}(b-a-\mathbb{E}(M_{a,b}(t_i)))(t_i - t_{i-1}).$$

Default leg: The default leg of the CDO tranche $[a,b]$ on $[0,T]$, $DL = \int_0^T e^{-rs}dM_{a,b}(t)$. Using integration by parts formula one gets: $DL = e^{-rT}M_{a,b}(T) + r\int_0^T e^{-rs}M_{a,b}(s)ds$. On the grill of default leg we get:

$$\mathbb{E}(DL) = e^{-rT}M_{a,b}(T) + r\sum_{i=1}^{n}\mathbb{E}(M_{a,b}(t_i))(t_i - t_{i-1}).$$

Hence to get the price of the index CDS and the tranche of CDO $[a,b]$, we should find the expectation of the loss on the grill of the payment and default leg.

Lemma 1. *Let us consider a portfolio of d entities and a given generator Q, the expectation of the loss on the index CDS and the expectation of the loss on a CDO tranche $[a,b]$ at time t are given by:*

$$\mathbb{E}[L_t] = \alpha e^{Qt} l, \qquad \mathbb{E}[M_{a,b}(t)] = \alpha e^{Qt} m_{a,b}.$$

where $\alpha = (1,0,\cdots,0)$ denotes the initial distribution of the Markov process Y, l is the vector of loss, at the j-th default $l_j = \frac{(1-R)}{d}j$, $1 \le j \le d$ and $m_{a,b}$ is the vector of loss on $[a,b]$, at the j-th default $m_{a,b}(j) = \left[\frac{(1-R)}{d}j - a\right]1_{\{j\in[n_l,n_u]\}} + (b-a)1_{\{j>n_u\}}$ $\Big(n_l$ is the smallest number of defaults affecting the CDO tranche , $n_l = \left[\frac{a}{1-R}d\right]$ and n_u is the smallest number such that the return of the CDO tranche is null $n_u = \left[\frac{b}{1-R}d\right]\Big)$.

Using the previous lemma, we get an explicit formula on the default leg of an index CDS on $[0,T]$.

$$\begin{aligned}\mathbb{E}[DL] = e^{-rT}\mathbb{E}(L_T) + r\int_0^T e^{-rt}\mathbb{E}(L_t)dt &= e^{-rT}\alpha e^{QT} l + r\int_0^T \alpha e^{(Q-rI)t} l dt \\ &= \alpha\left(e^{-rT}e^{QT} + r(Q-rI)^{-1}(e^{(Q-rI)T} - I)\right) l\end{aligned}$$

As previously, we deduce an explicit form of the default leg on CDO tranche $[a, b]$:

$$\mathbb{E}(DL) = \alpha \left(e^{-rT} e^{QT} + r(Q - rI)^{-1}(e^{(Q-rI)T} - I) \right) m_{a,b}$$

We get the annual spread κ of the CDO tranche $[a, b]$ such that $\mathbb{E}(PL) = \mathbb{E}(DL)$:

$$\kappa = \frac{\alpha \left(e^{-rT} e^{QT} + r(Q - rI)^{-1}(e^{(Q-rI)T} - I) \right) m_{a,b}}{\sum_{i=1}^{n} e^{-rt_i} \left(b - a - \alpha e^{Qt_i} m_{a,b} \right) (t_i - t_{i-1})}.$$

and the spread of an index CDS:

$$\kappa = \frac{\alpha \left(e^{-rT} e^{QT} + r(Q - rI)^{-1}(e^{(Q-rI)T} - I) \right) l}{\sum_{i=1}^{n} e^{-rt_i} \left(1 - \alpha e^{Qt_i} l \right) (t_i - t_{i-1})}.$$

4 Computation of the index CDS spread and CDO spread in PREMIA

In this part, we give the different steps to compute the spread of index CDS and a tranche of CDO:

Step 1: Given the number of firms d, the maturity of the contract T, the interest rate r, the recovery R, the default intensity depending on the vector $\hat{a} = (a, b_1, \cdots, b_{d-1})$ and given the matrix Q

Step 2: Give the grill payments $(0 \leq t_1 \leq t_n = T)$ and find for any point t_i on the grill $exp(Qt_i)$, hence you can write a cycle to compute the denominator of the spread.

Step 3: From step 2, you can deduce $\exp(QT)$ hence you compute easily the numerator of the spread, compute the spread.

The price of any CDO tranche with detachment point less than 0.03 is given in upfront and the others in bp. Since the intensity process has many parameters (d, a,b_1,$\cdots$, b_d), we will assume that it exists a constant piecewise function b such that $b_i = b(i)$, $1 \leq i \leq d$. In the Premia code, we implemented the following constant piecewise function b as follows: for $1 \leq i < 7 \wedge d$, $b_i = c_1$, for $7 \leq i < 13 \wedge d$ $b_i = c_2$, for $13 \leq i < 19 \wedge d$, $b_i = c_3$, for $19 \leq i \leq 26 \wedge d$,$b_i = c_4$, for $26 \leq i \leq 46 \wedge d$, $b_i = c_5$ and for $46 \leq i \leq d$, $b_i = c_6$. The user can choose another piecewise function, but should be careful since it is impossible to calibrate the jump of the intensity process if one doesn't consider the standard CDO tranches from Itraxx where the upper number($n_u = \frac{det*125}{(1-0.4)}$) of these tranches are 7, 13, 19, 26, 46.

References

[1] Herbertsson, A.: Default contagion in a large homogeneous portfolios. SSRN preprint (2007). 1