Fourier method on Lévy process, Affine process and Lévy time change process

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Abstract

We study some standard Fourier methods [7, 1] and apply them to the pricing of European options in model class on which we known the characteristic exponent. We will consider advanced equity option models incorporating stochastic volatility. More specifically, we focus on the Heston Stochastic Volatility model (with and without jumps in the stock price process), the Barndorff-Nielsen-Shephard model and Lévy models with stochastic time.

Our goal is to give a general framework to test these models as is done in [11].

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1 Pricing methods

1.1 Carr & Madan method

Carr and Madan [7] developed pricing methods for the classical vanilla options which can be applied in general when the characteristic function of the risk-neutral stock price process is known. Let α be a positive constant such that the αth moment of the stock price exists. For all stock price models encountered here, typically a value of $\alpha = 0.75$ will do fine. Carr and Madan then showed that the price $C(K,T)$ of a European call option with strike K and time to maturity T is given by:

$$C(K,T) = \frac{\exp\left(\pounds\alpha\log\left(K\right)\right)}{\pi}\int_{\mathbb{R}^+}\exp\left(\pounds\imath\xi\log\left(K\right)\right)\varphi(\xi)d\xi, \tag{1}$$

where

$$\varphi(\xi) = \mathrm{e}^{\pounds rT}\,\frac{\Phi\left(\xi\pounds(\alpha+1)\imath, T\right)}{\alpha^2+\alpha\pounds\xi^2+\imath(2\alpha+1)\xi}. \tag{2}$$

Then $C(K,T)$ is computed using discrete Fourier transform.

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1.2 Modified version of Carr & Madan method

The previous algorithm is slightly modified in our implementation. We use a Black-Scholes price for a fixed implied volatility σ_{BS} and substract this price in the previous formulation :

$$C(K,T) = C_{\sigma_{BS}}(K,T) + \frac{\exp\left(£\alpha \log\left(K\right)\right)}{\pi} \int_{\mathbb{R}^+} \exp\left(£\imath\xi \log\left(K\right)\right) \varphi^\star(\xi) d\xi, \tag{3}$$

where

$$\varphi^\star(\xi) = \mathrm{e}^{£rT} \frac{\Phi\left(\xi £(\alpha+1)\imath, T\right) - \Phi^{BS}_{\sigma_{BS}}\left(\xi £(\alpha+1)\imath, T\right)}{\alpha^2 + \alpha £\xi^2 + \imath(2\alpha+1)\xi}. \tag{4}$$

This modification cancel some effect due to the non-integrability condition of the payoff function.

1.3 Attari method

Attari [1] proposed the following methods for the classical vanilla options which can be applied in general when the characteristic function of the risk-neutral stock price process is known. Consider the simple case where the current stock price is given by $(S_t) = S_0\,\mathrm{e}^{(r-q)t+x_t}$ with respect to the risk-neutral density $\nu_{t\to T}(x)$, associated with x_t.

We can write the value of a call option with strike price K, $C(t,S_t;T,K)$, as

$$\begin{aligned} C(t,S_t;T,K) &= \mathrm{e}^{£r(T-t)}\,\mathbb{E}\left[\left(S_T £ K\right)^+ \middle| S_t\right] \qquad (5)\\ &= \mathrm{e}^{£r(T-t)}\left[\mathbb{E}\left[S_T 1_{S_T>K} \middle| S_t\right] - K\mathbb{E}\left[1_{S_T>K}\middle| S_t\right]\right] \\ &= \mathrm{e}^{q(T-t)}\,S_t \int_l^\infty \mathrm{e}^x\,\nu_{t\to T}(x)dx - \mathrm{e}^{£r(T-t)}\,K\int_l^\infty \nu_{t\to T}(x)dx \\ &= \mathrm{e}^x\,q(T-t)S_t\Pi_1 - \mathrm{e}^{£r(T-t)}\,K\Pi_2 \end{aligned}$$

where,$l = \ln \frac{K}{S_t}\,\mathrm{e}^{£(r-q)(T-t)}$, $\Pi_1 = \int_0^\infty \mathrm{e}^x\,q(x)dx$ and $\Pi_2 = \int_0^\infty q(x)dx$. Following the definition of the characteristic function, we have

$$\nu_{t\to T}(x) = \int_{\mathbb{R}} \mathrm{e}^{-\imath\xi}\,\Phi(\xi, T-t).$$

Then,

$$\Pi_j = \frac{1}{2} + \frac{1}{\pi}\int_{\mathbb{R}^+} \frac{\Phi_i(\xi,T-t)\cos\left(\xi l\right) + \Phi_i(\xi,T-t)\sin\left(\xi l\right)}{\xi} d\xi, \tag{6}$$

where $\Phi_r(\cdot,\cdot) = \mathrm{Re}\left(\Phi(\cdot,\cdot)\right)$, $\Phi_i(\cdot,\cdot) = \mathrm{Im}\left(\Phi(\cdot,\cdot)\right)$. Expression (5) should be rearranging as proposed in [1] then we obtain, with $\tau = T-t$:

$$C(t,S_t;T,K) = \mathrm{e}^{q(\tau)}\,S_t - \mathrm{e}^{-r(\tau)}\,K \tag{7}$$
$$\left(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty\left(\Phi_r(\xi,\tau)+\frac{\Phi_i(\xi,\tau)}{\xi}\right)\cos\left(\xi l\right)+\left(\Phi_i(\xi,\tau)-\frac{\Phi_r(\xi,\tau)}{\xi}\right)\sin\left(\xi l\right)\right)\frac{d\xi}{1+\xi^2}$$

This involves a single numerical integration instead of the two needed to evaluate expression (5). Also, the integrand has a ξ^2 term in the denominator giving a faster rate of decay.

2 Equities Models

2.1 Exponential Lévy models

2.1.1 Meixner process

The Meixner process is a special type of Lévy process which originates from the theory of orthogonal polynomials. It was introduce in financial application by Schoutens [10]. A natural method is by replacing the Brownian motion in the BS-model, by a more sophisticated Lévy process. The model which produces exactly Meixner(a,b,d,m) daily log-returns for the stock is given by

$$S_t = S_0\exp\left(M_t\right), \tag{8}$$

where the density of the Meixner distribution (Meixner(a, b, d, m)) is given by

$$f\left(x;a,b,m,d\right)=\frac{\left(2\cos\left(b/2\right)\right)^{2d}}{2a\pi\Gamma(2d)}\exp\left(\frac{b(x\pounds m)}{a}\right)\left|\Gamma\left(d+\frac{\imath(x\pounds m)}{a}\right)\right|^2$$

where $a>0$, $\pounds\pi<b<\pi, d>0$, and $m\in\mathbb{R}$.

2.2 Affine models

2.2.1 Bates Model

See [3] for a complet description of the model. We use the following notations:

$$\begin{aligned} dS_t &= (r-q-\lambda_y\mu)S_t dt+\sqrt{V_t}S_t dW_t^1 + J_y S_t dq_t^y \\ dV_t &= \kappa_\nu\left(\eta_\nu+V_t\right)+\theta_\nu\sqrt{V_t}dW_t^2 \\ d\left\langle W^1,W^2\right\rangle &= \rho dt \end{aligned} \tag{9}$$

where :

- $(1+J_y)$ is a log-normally distributed with mean μ_y and variance σ_y^2,
- q^y is an independent Poisson process with arrival rate λ_y,
- $\mu=\left(\exp\left(\mu_y+\sigma_y^2/2\right)-1\right)$, is the jump drift correction

2.2.2 Duffie Pan Singleton model

This class of models were introduced in [8]. Authors extended previous Bates models adding: jumps on the volatility process and correlated jumps between spot and volatility processes. We use the following notations:

$$\frac{dS_t}{S_t} = (r - q - \lambda_y \mu^y - \lambda_\rho \mu_\rho)dt + \sqrt{V_t} dW_t^1 + J_y dq_t^y + J_\rho^y S_t dq_t^\rho$$
$$d\nu_t = \kappa_\nu \left(\eta_\nu + \nu_t\right) + \theta_\nu \sqrt{\nu_t} dW_t^2 + J_\nu dq_t^\nu + J_\rho^\nu S_t dq_t^\rho$$
$$d\left\langle W^1, W^2 \right\rangle = \rho dt \tag{10}$$

where :

- $(1 + J_y)$ is a log-normally distributed with mean μ_y and variance σ_y^2, then the jump drift correction μ is equal to $\theta(1) - 1$ with $\theta^y(c) = e^{\mu_y c + \sigma_y^2 / 2c}$.
- J_ν has an exponential distribution with mean μ_ν, with $\theta^\nu(c) = \frac{1}{1 - \mu_\nu c}$.
- J_ρ has a 2dimensional random variable. The marginal distribution of the jump size in ν is exponential with mean μ_ρ. Conditional on a realization, say z_ν, of the jump size in ν, the jump size in x is normally distributed with mean $\mu_\rho - z_\nu \rho_j$, and variance σ_ρ^2. $\theta^\rho(c_1, c_2) = \frac{e^{\mu_\rho c_1 + \sigma_\rho^2 / 2c_2}}{1 - \mu_\rho (c_2 - \rho_j c_1)}$.
- q^y, q^ν, q^ρ are independent Poisson process with arrivals rates λ_y, λ_ν, λ_ρ.

2.2.3 Barndorff-Nielsen Shephard Model

This class of models were introduced in [2] and have a comparable structure to Heston class of model. The volatility is now modeled by an Ornstein Uhlenbeck process driven by a subordinator. We use the classical and tractable example of the Gamma-OU process. The marginal law of the volatility is Gamma-distributed. Volatility can only jump upwards and then it will decay exponentially. A co-movement effect between up-jumps in volatility and (down)-jumps in the stock price is also incorporated. The price of the asset will jump downwards when an up-jump in volatility takes place. In the absence of a jump, the asset price process moves continuously and the volatility decays also continuously. Other choices for OU-processes can be made. The squared volatility now follows a SDE of the form:

$$d\sigma_t^2 = -\lambda \sigma_t^2 dt + dz_{\lambda t}$$

where $\lambda > 0$ and $(z_t)_{t \leq 0}$ is a subordinator. The risk neutral dynamic of the log price $x_t = \log(S_t)$ are given by

$$dx_t = (r - q - \lambda k(-\rho) - \sigma^2/2)dt + \sigma_t dW_t + \rho dz_t, \quad x_0 = log(S_0).$$

where $k(u) = \log\left(\mathbb{E}\left[\exp\left(-uz_1\right)\right]\right)$. Choice z_t as a compound Poisson process,

$$z_t = \sum_{n=1}^{N_t} x_n$$

where N_t is a Poisson process with intensity parameter α and each x_n follows an exponential law with mean $\frac{1}{\beta}$. One can show that the process σ_t^2 is a stationary process with a marginal law that follows a Gamma distribution with mean α and variance $\frac{\alpha}{\beta}$. In this case,

$$k(u) = \frac{-\alpha u}{\beta + u}.$$

2.3 Lévy Models with Stochastic Time

Another way to build in stochastic volatility effects is by making time stochastic. Periods with high volatility can be looked at as if time runs faster than in periods with low volatility. In other words, time as to re-scale in term of volume of trade. The above mentioned Lévy process will be subordinated (or time-changed) by this stochastic clock. By definition of a subordinator, the time needs to increase and the process modeling the rate of time change $(y_t)_{t\geq 0}$ needs also to be positive. The volume trade-time elapsed in units of calendar time is then given by the integrated process $(Y_t)_{t\geq 0}$ where

$$Y_t = \int_0^t y_s ds. \tag{11}$$

2.3.1 Characteristic function of a Lévy stochastic clock

Let $(Y_t)_{t\geq 0}$ be the process we choose to model our volume-trade time. Let us denote by $\varphi(u; t, y_0)$ the characteristic function of $(Y_t)_{t\geq 0}$ given y_0. The price process $(S_t)_{t\geq 0}$ is now modeled as follows:

$$S_t = S_0 \frac{\mathrm{e}^{(r-q)t}}{\mathbb{E}\left[\mathrm{e}^{X_{Y_t}} \middle| y_0\right]} \mathrm{e}^{X_{Y_t}}, \tag{12}$$

where $(Xt)_{t\geq 0}$ is a Lévy process. The factor $\frac{\mathrm{e}^{(r-q)t}}{\mathbb{E}\left[\mathrm{e}^{X_{Y_t}} \middle| y_0\right]}$ puts us immediately into the risk-neutral world by a mean-correcting argument. Basically, we model the stock price process as the ordinary exponential of a time-changed Lévy process. The process incorporates jumps (through the Lévy process X_t) and stochastic volatility (through the time change Y_t). he characteristic function $\phi(u, t)$ for the log of our stock price is given by:

$$\Phi(\xi, t) = \mathbb{E}\left[\mathrm{e}^{\imath\xi \log(S_t)} \middle| S_0, y_0\right] = \mathrm{e}^{\imath\xi((r-q)t+\log(S_0))} \frac{\varphi\left(-\imath\psi(\xi); t, y_0\right)}{\varphi\left(-\imath\psi(-\imath); t, y_0\right)^{\imath\xi}}, \tag{13}$$

where ψ is the characteristic exponent of the Lévy process. We present some example of process $(Y_t)_{t\geq 0}$ using a time change process.

2.3.2 CIR stochastic clock

Carr, Geman, Madan and Yor [6] use as the rate of time change the CIR process that solves the SDE:

$$dy_t = \kappa\,(\eta - y_t)\,dt + \lambda\sqrt{y_t}dW_t, \tag{14}$$

where $(W_t)_{t\geq 0}$ is a standard brownian motion. The characteristic function of Y_t (given y_0) is explicitely given by:

$$\varphi_Y\,(\xi, t) = \mathrm{e}^{\kappa^2\eta t/\lambda^2}\,\frac{\exp\left(2y_0\imath\xi/\left(\kappa + \gamma(\xi)\coth\left(\gamma(\xi)t/2\right)\right)\right)}{\left(\cosh\left(\gamma(\xi)t/2\right) + \kappa/\gamma(\xi)\sinh\left(\gamma(\xi)t/2\right)\right)^{2\kappa\eta/\lambda^2}}, \tag{15}$$

where $\gamma\,(\xi) = \left(\kappa^2 - 2\lambda^2\imath\xi\right)^{\frac{1}{2}}$.

2.3.3 Gamma-OU stochastic clock

The rate of time change is now solution of the SDE

$$dy_t = -\lambda y_t dt + dz_{\lambda t}. \tag{16}$$

Choice z_t as a compound Poisson process,

$$z_t = \sum_{n=1}^{N_t} x_n$$

where N_t is a Poisson process with intensity parameter α and each x_n follows an exponential law with mean $\frac{1}{\beta}$. The characteristic function of Y_t (given y_0) is explicitely given by:

$$\varphi_Y\,(\xi, t) = \exp\left(y_0\imath\xi\lambda^{-1}\left(1 - \mathrm{e}^{-\lambda t}\right) + \frac{\lambda\alpha}{\imath\xi - \lambda\beta}\left(\beta\log\left(\frac{\beta}{\beta - \imath\xi\lambda^{-1}\left(1 - \mathrm{e}^{-\lambda t}\right)}\right) - \imath\xi t\right)\right). \tag{17}$$

3 Calibration

Using Fast Fourier Transforms, one can compute within a second the complete option surface on an ordinary computer. We apply the above calculation method in our calibration procedure and estimate the model parameters by minimizing the difference between market prices and model prices in a least-squares sense.

4 Extensions

- **Model:** We can extend this work to more Equities models which can be construct form existing method like Meixner stochastic volatility model [10] using a Meixner process and a CIR process as stochastic time.
- **Options class:** Pricing forward start options with Fourier method [4], to compute the dynamic of volatility smile. Options on the quadratic variation should also be computed using results on characteristic function of the quadratic process as is done in[5].
- **Pricing method:** As it is explain in [12], we can use Monte-Carlo for exotic options like Cliquet options and Wiener-Hopf factorization [9] for American and barrier options.

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