A Mean-Reverting SDE on Correlation matrices

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We introduce in Premia a mean-reverting SDE whose solution is naturally defined on the space of correlation matrices. This SDE can be seen as an extension of the well-known Wright-Fisher diffusion. Since in the original paper, we developed several discretization scheme, we implemented in Premia only the one related to Wishart process.

Let us now introduce the process. We first advise the reader to have a look at our notations for matrices located at the end of the first part of [**?**], even though they are rather standard. We consider $(W_t, t \geq 0)$, a d-by-d square matrix process whose elements are independent real standard Brownian motions, and focus on the following SDE on the correlation matrices $\mathfrak{C}_d(\mathbb{R})$:

$$\begin{aligned} X_t =& x + \int_0^t (\kappa(c - X_s) + (c - X_s)\kappa)\, ds \\ &+ \sum_{n=1}^{d} a_n \int_0^t \left(\sqrt{X_s - X_s e_d^n X_s} dW_s e_d^n + e_d^n dW_s^T \sqrt{X_s - X_s e_d^n X_s} \right), \end{aligned} \tag{1}$$

where $x, c \in \mathfrak{C}_d(\mathbb{R})$ and $\kappa = diag(\kappa_1, \ldots, \kappa_d)$ and $a = diag(a_1, \ldots, a_d)$ are nonnegative diagonal matrices such that

$$\kappa c + c\kappa - (d-2)a^2 \in \mathcal{S}_d^+(\mathbb{R}) \text{ or } d = 2. \tag{2}$$

Under these assumptions, we have already shown in [**?**] that this SDE has a unique weak solution which is well-defined on correlation matrices, i.e. $\forall t \geq 0, X_t \in \mathfrak{C}_d(\mathbb{R})$. We will have also shown that strong uniqueness holds if we assume moreover that $x \in \mathfrak{C}_d^*(\mathbb{R})$ and

$$\kappa c + c\kappa - da^2 \in \mathcal{S}_d^+(\mathbb{R}). \tag{3}$$

Looking at the diagonal coefficients, conditions (2) and (3) imply respectively $\kappa_i \geq (d-2)a_i^2/2$ and $\kappa_i \geq da_i^2/2$. This heuristically means that the speed of the mean-reversion has to be high enough with respect to the noise in order to stay in $\mathfrak{C}_d(\mathbb{R})$. Throughout the paper, we will denote $MRC_d(x, \kappa, c, a)$ the law of the process $(X_t)_{t\geq 0}$ and $MRC_d(x, \kappa, c, a; t)$ the law of X_t. Here, MRC stands for Mean-Reverting Correlation process. When using these notations, we

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implicitly assume that (2) holds. In the rest of this note, all notations used are well defined in [**?**].

First, we split the infinitesimal generator of $MRC_d(x, \kappa, c, a)$ as the sum

$$L = L^{\xi} + \tilde{L},$$

where $\tilde{L}$ is the infinitesimal generator of $MRC_d(x, \frac{d-2}{2}a^2, I_d, a)$ and L^{ξ} is the operator associated to $\xi(t, x)$ given by

$$\xi'(t, x) = \kappa(c - x) + (c - x)\kappa - \frac{d-2}{2}[a^2(I_d - x) + (I_d - x)a^2], \; \xi(0, x) = x \in \mathfrak{C}_d(\mathbb{R}). \quad (4)$$

By the splitting technique methods, if we suppose that $\hat{\xi}_t^x$ and $\hat{X}_t^x$ denote respectively two potential weak second order schemes on $\mathfrak{C}_d(\mathbb{R})$ for L^{ξ} and $\tilde{L}$. If B is an independent Bernoulli variable of parameter $1/2$. Then

$$(a)\; B\hat{X}_t^{\hat{\xi}_t^x} + (1 - B)\hat{\xi}_t^{\hat{X}_t^x} \quad \text{and} \quad (b)\; \hat{\xi}_{t/2}^{\hat{X}_t^{\hat{\xi}_{t/2}^x}}$$

are potential weak second order schemes for L.

It is then sufficient to derive a second potential order scheme associated to $\tilde{L}$. By a straightforward calculus, one can observe that the operator itself $\tilde{L}$ can be splitted into an elementary operator that commute and have the same law by index permutation. In other words, we have $\tilde{L} = \sum_{i=1}^{d} a_i^2 L_i$, where $L_i L_j = L_j L_i$ for every $1 \leq i, j \leq d$, and each operator L_i is associated to the process $MRC_d(x, \frac{d-2}{2}e_d^i, I_d, e_d^i)$. If we suppose that $\hat{X}_t^{1,x}, \ldots \hat{X}_t^{d,x}$ are second order discretization associated respectively to $L_1, \ldots, L_d$. then

$$\underline{\xi(t/2, \hat{X}_{a_d t}^{d, \ldots^{\hat{X}_{a_1 t}^{1, \xi(t/2, x)}}}) \text{ is a potential second order scheme for } MRC_d(x, \kappa, c, a).} \quad (5)$$

The last point is how to derive a suitable scheme for $(\hat{X}^{i,x})_{1 \leq i \leq d}$. In this case we use the Wishart process discretization. Since all operation $(L_i)_{2 \leq i \leq d}$ are given by the law of L_1 under index permutation between the first index and i^{th} one, we shall present the result only for the operator L_1. If we suppose that $\hat{Y}_t^{1,x}$ is a potential second order scheme for $WIS_d(x, d - 1, 0, e_d^1)$, then

$$\mathbf{p}(\tilde{Y}_{\phi(t)}^x) \text{ is a potential second order scheme for } L_1, \quad (6)$$

where $\forall x \in \mathcal{S}_d^+(\mathbb{R}), \; \mathbf{p}(x)_{i,j} = \frac{x_{i,j}}{\sqrt{x_{i,i}x_{j,j}}}$ and

$$\phi(t) = \begin{cases} t - (5 - d)\frac{t^2}{2} \text{ if } d \geq 5 \\ \frac{-1+\sqrt{1+2(5-d)t}}{5-d} \text{ otherwise,} \end{cases}$$

Finally, we combine (5) and (6) in order to give a second high discretization of the MRC process.

References