Calibration of Libor market model with stochastic volatility: Implementation in PREMIA

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Premia 14

1 Introduction

We consider an extension of Libor market model with a high-dimensional Heston type stochastic volatility processes, which matches cap and swaption volatility smiles and skews observed in the markets and allows for stable calibration to the cap-strike matirx as well. Moreover the extension of the model is made by preseving the local covariance structure of the market model.

Here we use the road map proposed by Belomestny, Mathew and Schoenmakers (2009) for calibration of this extension Libor market model, which is FFT based, fast and easy to implement.

The content of this file is as follows. In section 2, we introduce the extended Libor market model we considered in this paper and the pricing formula for caplet-based on this model is derived in section 3. Then we discribe the calibration road map to the caps and pricing of swaptions in section 4. Section 5 is about the program manual of the implementaion in PREMIA.

2 Model Discription

2.1 The general Libor model

For the tenor structure with a fixed sequence of tenor dates $0 := T_0 < T_1 < \cdots < T_n$, the Libor forward rate is defined as:

$$L_i(t) = \frac{1}{\delta_i}\left(\frac{B_i(t)}{B_{i+1}(t)} - 1\right), \quad 0 \le t \le T_i, 1 \le i < n$$

where the day-count fractions $\delta_i := T_{i+1} - T_i, i = 1, 2, \cdots, n-1$, and $B_i(t), i = 1, 2, \cdots, n$, is the zero-conpon bond price at time t.

The dynamics of the Libor model is as follows:

$$\frac{dL_i}{L_i} = (\cdots)dt + \gamma_i^\top dW, \quad i = 1, 2, \cdots, n-1, \tag{2.1}$$

where the pre-speciafied volatility process $\gamma_i \in \mathbb{R}^m$ is adapted to the filtration generated by the standard Brownian motion $W \in \mathbb{R}^m$, and the drift term, adumbrated by the dots, is known under differenct numeraire measures, such as the risk-neutral, spot, terminal and all measures induced by individual bonds taken as numeraire. We name the model (2.1) as Libor *market* model, when the volatility processes $t \to \gamma_i(t) := \gamma_i$ in (2.1) are deterministic.

2.2 The extended Libor market model

The extension of Libor market model is proposed by adding a term of Heston type volatility on the extended Brownian motions $\widetilde{W}$, then the dynamics of the extended Libor market model is given as

$$\frac{dL_i}{L_i} = (\cdots)dt + \sqrt{1-r_i^2}\gamma_i^\top dW + r_i\beta_i^\top\sqrt{v}d\widetilde{W}, \quad 1 \le i < n \tag{2.2}$$

$$d\nu_k = \kappa_k(\theta_k - \nu_k)dt + \sigma_k\sqrt{\nu_k}\left(\rho_k d\widetilde{W}_k + \sqrt{1-\rho_k^2}d\overline{W}_k\right), \quad 1 \le k \le d, \tag{2.3}$$

where $\widetilde{W}$ and $\overline{W}$ are mutually independent d-dimensional standard Brownian motions, both independent of W. The coefficients r_i are constants that may be considered “allotment" or "proportion" factor quantifying how much the original input market model should be in play. For $r_i = 0$ for all i, the extended model restored to the standard Libor market model. As such, at nozero value of the r_i, the extended model may be regarded as a perturbation of the former. In this paper, we set for $i = 1, 2, \cdots, n$, $r_i \equiv r$ and $\nu_k(0) \equiv \theta_k \equiv 1, 1 \le k < n$.

In model (2.2) the coefficients $\beta_i \in \mathbb{R}^d$ are choosen to be deterministic vector functions. To preseved the covariance of the Libor market model, it requires that

$$\int_0^t \beta_i^\top \beta_j ds = \int_0^t \gamma_i^\top \gamma_j ds, \quad 1 \le i, j < n. \tag{2.4}$$

In order to obtain closed-form expressions for characteristic functions of (log-)Libors later on, we need $\beta(t)$ to be piecewise constant in time. But here for the progmatic sake, we choose of a constant vectors β_i according to

$$\beta_i = \sigma_i^{\text{Black}} e_i \quad \text{where} \quad \left(\sigma_i^{\text{Black}}\right)^2 := \frac{1}{T_i}\int_0^{T_i} |\gamma_i(s)|^2 ds, \tag{2.5}$$

in order to match the covariance constraint (2.4) roughly. Note that $\gamma_i \in \mathbb{R}^m$, even when $m < n-1$, matching (2.4) may require $d = n-1$. In the calibration procedure, we assume the input market Libor volatility structure $\gamma \in \mathbb{R}^{(n-1)\times m}$ to be of full rank, that is $m = n-1$. So thoughout all our work we take $d = m = n-1$ and we also assume that $\beta \in \mathbb{R}^{(n-1)\times d}$ to be a squared upper triangular matrix.

2.3 Extended model dynamics under terminal measure

Donote the (time independent) solution of (2.5) by $\overline{\gamma} \in \mathbb{R}^{(n-1)\times d}$. Let $r_i \equiv r$, under the terminal measure $\mathbb{P}_n$, the dynamics of the extended Libor market model (2.2) is

$$\begin{aligned}\frac{dL_i}{L_i} = & -\sum_{j=i+1}^{n-1} \frac{\delta_j L_j}{1+\delta_j L_j}\left[(1-r^2)\gamma_i^\top \gamma_j + r^2 \sum_{k=1}^{d} \overline{\gamma}_{ik}\overline{\gamma}_{jk}\nu_k\right] dt \\ & + \sqrt{1-r^2}\gamma_i^\top dW^{(n)} + r\sum_{k=1}^{d}\sqrt{\nu_k}\overline{\gamma}_{ik} d\widetilde{W}_k^{(n)} \end{aligned} \tag{2.6}$$

wtih the volativility as given by

$$d\nu_k = \kappa_k(\theta_k - \nu_k)dt + \sigma_k\sqrt{\nu_k}\left(\rho_k d\widetilde{W}_k^{(n)} + \sqrt{1-\rho_k^2}d\overline{W}_k^{(n)}\right), \quad 1 \le k \le d, \tag{2.7}$$

2.4 Extended model dynamics under forward measure

For pricing caplet, we need to know the forward Libor dynamics in forward measure P_{i+1}. Then by rearraging terms in (2.6) we have

$$\begin{aligned}\frac{dL_i}{L_i} = & \sqrt{1-r^2}\gamma_i^\top\left(dW^{(n)} - \sqrt{1-r^2}\sum_{j=i+1}^{n-1}\frac{1+\delta_j L_j}{\delta_j L_j}\gamma_j dt\right) \\ & + r\sum_{k=1}^{d}\overline{\gamma}_{ik}\sqrt{\nu_k}\left(d\widetilde{W}_k^{(n)} - \sum_{j=i+1}^{n-1}\frac{1+\delta_j L_j}{\delta_j L_j}\overline{\gamma}_{jk}\sqrt{\nu_k}dt\right) \\ =: & \sqrt{1-r^2}\overline{\gamma_i}^\top dW^{(i+1)} + r\sum_{k=1}^{d}\overline{\gamma_{ik}}\sqrt{\nu_k}d\widetilde{W}_k^{(i+1)}. \end{aligned} \tag{2.8}$$

Since L_i is a martingale under P_{i+1}, then both of $W^{(i+1)}$ and $\widetilde{W}^{(i+1)}$ in (**??**) are standard Brownian motions under forward measure P_{i+1}. In term of these new Brownian motions the volatility dynamics are

$$d\nu_k = \quad \kappa_k(1-\nu_k)dt + r\sigma_k\rho_k \sum_{j=i+1}^{n-1} \frac{\delta_j L_j}{1+\delta_j L_j}\gamma_{jk}\nu_k dt$$
$$+\rho_k\sigma_k\sqrt{\nu_k}d\widetilde{W}_k^{(i+1)} + \sqrt{1-\rho_k^2}\sigma_k\sqrt{\nu_k}d\overline{W}_k^{(n,i+1)}, \tag{2.9}$$

where $\overline{W}^{(n,i+1)}$ in (2.9) is a standard Brownian motion under both measures P_{i+1} and P_n. By freezing the Libors at their initial values in (2.9), we obtain approximative CIR dynamic

$$d\nu_k \approx \kappa_k^{i+1}\left(\theta_k^{(i+1)} - \nu_k\right)dt + \sigma_k\sqrt{\nu_k}\left(\rho_k d\widetilde{W}_k^{(i+1)} + \sqrt{1-\rho_k^2}d\overline{W}_k^{(i+1)}\right), \tag{2.10}$$

with reversion speed parameter

$$\kappa_k^{(i+1)} := \kappa_k - r\sigma_k\rho_k \sum_{j=i+1}^{n-1} \frac{\delta_j L_j(0)}{1+\delta_j L_j(0)}\overline{\gamma}_{jk}, \tag{2.11}$$

and mean reversion level

$$\theta_k^{(i+1)} := \frac{\kappa_k}{\kappa_k^{(i+1)}}. \tag{2.12}$$

We will use the above freezing volatility dynamics in the calibration routines.

3 Pricing Caplet

The arbitrage-free price at time zero $C_j(K)$ of a caplet at time $T_j, 1 \leq j < n$ with strike K paying $\delta_k(F_j(T_j) - X)^+$ at time T_{j+1} is given by

$$C_j(K) = \delta_j B_{j+1}(0)E_{j+1}\left[(L_j(T_j) - K)^+\right], \tag{3.1}$$

where E_{j+1} is the expectation taking under the forward measure P_{i+1}.

To calculate the caplet price (3.1), we use the method of FFT-method of Carr and Madan (1999). The main result is as follows.

In term of the log-moneyness variable

$$y := \ln\left(\frac{K}{L_j(0)}\right), \tag{3.2}$$

the j-th caplet price can be expressed as

$$\mathbf{C}_j(y) := C_j(e^y L_j(0)) = \delta_j B_{j+1}(0)L_j(0)E_{j+1}\left(e^{X_j(T_j)-e^y}\right)^+,$$

where $X_j(t) = \ln L_j(t) - \ln L_j(0)$ and we define the cahracteristic function of the process $X_j(t)$ under P_{j+1} as $\varphi_{j+1}(\cdot;t)$. Then for the auxiliary function

$$\mathbf{O}_j(y) := \delta_j^{-1}B_{j+1}^{-1}(0)L_j^{-1}(0)\,C_j(y) - (1-e^y)^+. \tag{3.3}$$

from Proposition 1 of Belomestny (2009), we have its Fourier trandform of the function $\mathbf{O}_j$ in term of $\varphi(\cdot;t)$ as

$$\mathbf{F}\{\mathbf{O}_j\}(z) = \int_{-\infty}^{\infty} \mathbf{O}_j(y)e^{iyz}dy = \frac{1-\varphi_{j+1}(z-i;T_j)}{z(z-i)}. \tag{3.4}$$

Combining (3.2), (3.3) and (3.4), we have the caplet price as

$$\begin{aligned} C_j(K) = \quad & \delta_j B_{j+1}(0)(L_j(0)-K)^+ \\ & + \frac{\delta_j B_{j+1}(0)L_j(0)}{2\pi}\int_{-\infty}^{\infty} \frac{1-\varphi_{j+1}(z-i;T_j)}{z(z-i)} e^{-iz\ln\left(\frac{K}{L_j(0)}\right)}dz. \end{aligned} \tag{3.5}$$

The explicit formula of the conditional characteristic funcion $\varphi_{j+1}(\cdot;t)$ is given asd in Appendix 8.0.1 of Belomestny et al (2009).

4 Calibration algorithm

The calibration algorithm is based on the pre-calibration Libor market model, which is calibrate to ATM caps and ATM swaptions using Schoenmakers (2005). The pre-calibration processure is not essential in this algorithm, we just use the pre-calibration result directly in our model. Here is pre-calibration volatility structure for Libor market model

$$\gamma_i(t) = c_i g_i(T_i - t)e_{i+1-m(t)}, \qquad 0 < t \le T_i, 1 \le i \le n,$$

where function

$$g(s) = g_\infty + (1 - g_\infty + as)e^{-bs},$$

and $e_i \in \mathbb{R}^{n-1}$ are unit vectors such that $(e_{i,k})$ is upper triangular. For the pre-calibrated Libor market model the loading factors c_i are readily computed from

$$\left(\sigma_{T_i}^{ATM}\right)^2 T_i = c_i \int_0^{T_i} g^2(s)ds, \qquad i = 1, \cdots, n-1. \tag{4.1}$$

In this calibration routine, the loading factors c_i will be calibrated as newly for flexibility. We will use the pre-calibration result $g_\infty = 2.578, a = 5.001, b = 2.000$ for the Libor market model in the our calibration algorithm.

We will calibrate the model parameters sets $\{\kappa_i, \sigma_i, \rho_i, c_i; 1 \le i < n\}$ and r to the market cap-strike volatility matrix by an iteration step, where $\kappa_i, \sigma_i, \rho_i, 1 \le i < n$ are parameters in the volatility dynamics (2.9) and $c_i, 1 \le i < n$ are the loading factors in Libor market model. At each step we minimizing the weight sum of the difference between the Black-scholes caplet price and the computed caplet price using (3.5) for different strike, the weight are taken to be proportional to Black-Scholes vegas.

The iteration step is as follows:

- We star from step $i = n-1$. Calibrate r and the parameter set $(\kappa_{n-1}, \sigma_{n-1}, \rho_{n-1}, c_{n-1})$ to the T_{n-1} column of the cap-strike matirx using teh explicitly known characteristic function φ_n of the $\ln[L_{n-1}(T_{n-1}/L_{n-1}(0))]$.

- For the steps from $i = n-2$ down to 1 we carry out the next iteration step:

- For the k-th step $i = n-k$, transform the yet know parameter set $(\kappa_j, \sigma_j, \rho_j, c_j), i < j < n$ via (2.11) and (2.12) into the corresponding set $(\kappa_j^{(i+1)}, \sigma_j^{(i+1)}, \rho_j^{(i+1)}, c_j), i < j < n$. By teh upper triangular structure of the square matrix $\overline{\gamma}$ we obviously have $\kappa_i^{(i+1)} = \kappa_i$ and hense by (2.12) $\theta_i^{(i+1)} = 1$. Then calibrate the unknown parameter set $(\kappa_j, \sigma_j, \rho_j, c_j), i < j < n$ to the T_i column of the cap-strike matrix using (3.5).

5 Program manual

Including files:
The program directory contains 22 files:

- this documentation file.
- the source program file: “calibrate_lmm_vola_stoc.cpp" written in C++
- a binary executable file “calibrate.out" generated by the compilation of “calibrate_lmm_vola_stoc.cpp"
- an output file of calibration results: “cali_result"
- an example of input market data: “initial_curve.dat" “cap_strike_matrix.dat"
- header and link files (altogether 16 files):“intg.h" “intg.C" “normal_df.h" “normal_df.C" “error_msg.h" “optype.h" “realfft.h" “realfft.C" “min.h" “min.cpp" “routines.c" “routines.cpp" “routines.o" “f2c.h" “libf2c.a" “iterate"

Compilation:
Compilation command under Linux: g++ intg.C normal_df.C realfft.C min.cpp routines.cpp libf2c.a cali_lmm_vola_stoc.cpp -o calibrate.out, generates the executable binary file “calibrate.out".

Execution:
Type ./calibrate.out to run the calibration program.

Five parameters can be calibrated to each line of the cap-strike volatility matrix in this program:

- r: $0 \leq r \leq 1$
- ke: $0 \leq ke \leq 2$
- sigma: $0.0006 \leq sigma \leq 2$
- rho: $-0.9999 \leq rho \leq 0.9999$
- lf:$0.1 \leq lf \leq 4$

Results are pressented at the end of program and in the result file "cali_result"