The Libor Market Model with Jumps

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Premia 14

Abstract

The aim of this note is to use a Lévy-driven model to describes the joint arbitrage-free dynamics of a set of forward Libor rates. Such model is called a Libor market model. This note is based on the paper of Tankov and Kohatsu-Higa (so for more details see [4]).

1 Preliminaries

We consider a d-dimensional Lévy process Z without diffusion componet. Thus $(\gamma, \sigma) \in \mathbb{R} \times \mathbb{R}^+$, and ν is a Radon measure on $\mathbb{R}\backslash\{0\}$ satisfying

$$\int_{\mathbb{R}} \left(1 \wedge x^2\right) \nu(dx) < \infty.$$

By the Lévy-Itô decomposition, X can be written in the form

$$Z_t = \gamma t + \int_{|x|>1, s\in[0,t]} xJ(dx \times ds) + \lim_{\delta \downarrow 0} \int_{\delta \leq |x| \leq 1, s\in[0,t]} x\widetilde{J}(dx \times ds) \tag{1.1}$$

Here $\gamma \in \mathbb{R}^d$, J is a Poisson measure on $\mathbb{R} \times [0, \infty)$ with intensity $\nu(dx)dt$, $\widetilde{J}(dx \times ds) = J(dx \times ds) - \nu(dx)ds$ and ν is a Radon measure on $\mathbb{R}\backslash\{0\}$ satisfying $\int_{\mathbb{R}} (1 \wedge x^2)\, \nu(dx) < \infty$. Given $\epsilon > 0$, we define the process R^ϵ by

$$R_t^\epsilon = \int_{0\leq|x|\leq\epsilon, s\in[0,t]} x\widetilde{J}(dx \times ds), \;\; t \geq 0. \tag{1.2}$$

Note that we have

$$\mathbb{E}R_t^\epsilon = 0.$$

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On the other hand we denote by Σ^ϵ the covariance matrix of R_1^ϵ, and thus for any $i,j \in \{1,\dots,d\}$

$$\Sigma_{i,j}^\epsilon = \int_{|x|\le\epsilon} x_i x_j \nu(dx).$$

Define the process Z^ϵ by

$$Z_t^\epsilon = \int_{|x|>\epsilon, s\in[0,t]} xJ(dx\times ds),\ t\ge 0.$$

Then we have

$$Z_t = \gamma_\epsilon t + Z_t^\epsilon + R_t^\epsilon,\ t\ge 0, \tag{1.3}$$

where

$$\gamma_\epsilon = \gamma - \int_{\epsilon<|x|\le 1} x\nu(dx). \tag{1.4}$$

We will call $(T_i^\epsilon)_{i\ge 1}$ the jump times of the process Z^ϵ.

2 Approximation of multidimensional SDE

Let X be a n-dimensional stochastic process, and the unique solution of the stochastic differential equation

$$dX_t = h\left(X_{t^-}\right) dZ_t, \quad t\in[0,1], \tag{2.5}$$

where h is a $n\times d$ matrix. A suitable approximation of X is $\bar X$ defined by

$$d\bar X_t = h\left(\bar X_{t^-}\right)\left(\gamma_\epsilon dt + dW_t^\epsilon + dZ_t^\epsilon\right), \tag{2.6}$$

where W^ϵ is a d-dimensional Brownian motion with covariance matrix Σ^ϵ. The choice of this approximation is explain in [4]. The process $\bar X$ can be also written in this form

$$\begin{aligned}\bar X_t &= \bar X_{\eta_t} + \int_{\eta_t}^t h\left(\bar X_s\right) dW_s^\epsilon + \int_{\eta_t}^t h\left(\bar X_s\right)\gamma_\epsilon ds\\ \bar X_{T_i^\epsilon} &= \bar X_{T_i^\epsilon -} + h\left(\bar X_{T_i^\epsilon -}\right)\Delta Z_{T_i^\epsilon},\end{aligned}$$

where $\eta_t = \sup T_i^\epsilon,\ T_i^\epsilon \le t$. The idea of [4] is to approximate $\bar X$ by

$$Y^0 + \frac{\partial}{\partial\alpha} Y^\alpha\bigg|_{\alpha=0},$$

where the family of processes $(Y^\alpha)_{0\le\alpha\le 1}$ is defined by

$$Y_t^\alpha = \bar X_{\eta_t} + \int_{\eta_t}^t h\left(Y_s^\alpha\right) dW_s^\epsilon + \int_{\eta_t}^t h\left(Y_s^\alpha\right)\gamma_\epsilon ds$$

Hence a new approximation of X, called $\tilde{X}$, is defined by

$$\begin{aligned}
\tilde{X}_t &= Y_{0,t} + Y_{t,1}, \quad t > \eta_t \\
\tilde{X}_{T_i^\epsilon} &= \tilde{X}_{T_i^\epsilon -} + h\left(\tilde{X}_{T_i^\epsilon -}\right) \Delta Z_{T_i^\epsilon} \\
Y_{0,t} &= \tilde{X}_{\eta_t} + \int_{\eta_t}^t h\left(Y_{0,s}\right) \gamma_\epsilon ds \\
Y_{1,t} &= \int_{\eta_t}^t h\left(Y_{0,s}\right) dW_s^\epsilon + \sum_{i=1}^n \int_{\eta_t}^t \frac{\partial h}{\partial x_i}\left(Y_{0,s}\right) Y_{1,s}^i \gamma_\epsilon ds.
\end{aligned}$$

The random vector $Y_{1,t}$ is Gaussian with mean zero and covariance matrix Ω_t satisfying

$$\Omega_t = \int_{\eta_t}^t \left(\Omega_s M_s + M_s^\perp \Omega_s^\perp + N_s\right) ds,$$

where $M^\perp$ is the transpose of the matrix M and

$$M_t^{ij} = \frac{\partial h^{ij}\left(Y_{0,t}\right)}{\partial x_j} \gamma_\epsilon^j, \quad N_t = h\left(Y_{0,t}\right) \Sigma^\epsilon h^\perp \left(Y_{0,t}\right).$$

3 Libor market model

Let $T_i = T_1 + (i-1)\delta,\ i = 1, \ldots, n+1$ be a set dates, called tenor dates. The Libor rate L_t^i is the forward interest rate, defined at date t for the period $[T_i, T_{i+1}]$. The Libor rate can be expressed with respect to prices of zero-coupon bonds.

$$L_t^i = \frac{1}{\delta}\left(\frac{B_t(T_i)}{B_t(T_{i+1})} - 1\right),$$

where $B_t(T)$ is the price at time t of a zero-coupon bond with maturity T. A arbitrage-free dynamics of $L_t^1, \ldots, L_t^n$ (see [3]) is

$$\frac{dL_t^i}{dL_{t-}^i} = \sigma_{i,t} dZ_t - \int_{\mathbb{R}^d} \sigma_{i,t} z \left[\prod_{j=i+1}^{n+1}\left(1 + \frac{\delta L_t^j \sigma_t^j z}{1 + \delta L_t^j}\right) - 1\right] \nu(dz) dt, \tag{3.7}$$

where Z is a d-dimensional martingale pure jump Lévy process, with Lévy measure ν, and $\sigma_{i,t}$ are d-dimensional deterministic volatility functions. The dynamics are given under the so-called terminal measure. This means the last zero-coupon bond, $B_t(T_{n+1})$, is used as the numéraire. So the price at time t of an option with payoff $H = f\left(L_{T_1}^1, \ldots, L_{T_1}^n\right)$ at time T_1 is given by

$$\pi_t(H) = \frac{B_t(T_1)}{\prod_{i=1}^n \left(1 + \delta L_t^i\right)} \mathbb{E}\left[f\left(L_{T_1}^1, \ldots, L_{T_1}^n\right) \prod_{i=1}^n \left(1 + \delta L_{T_1}^i\right) / \mathcal{F}_t\right].$$

We introduce the process $(n+1)$-dimensional X with $X_t^0 = t$ and $X_t^i = L_t^i$ (for $i = 1, \dots, n$), a $(d+1)$-dimensional process $\tilde{Z} = (t, Z_t)^{\perp}$, and a $(n+1) \times (d+1)$-dimensional function h with $h^{11} = 1$, $h^{1j} = 0$ for $2 \leq j \leq d+1$, $h^{i1} = f^i(x)$ and $h^{ij} = \sigma_{i,x_0}^{j-1}$ (for $2 \leq j \leq d+1$) with

$$f^i(x) = -\int_{\mathbb{R}^d} \sigma_{i,x_0} z \left[\prod_{j=i+1}^{n+1} \left(1 + \frac{\delta x_j \sigma_t^j z}{1 + \delta x_j} \right) - 1 \right] \nu(dz) dt,$$

so that the equation (3.7) takes the form

$$dX_t = h\left(X_{t^-}\right) d\tilde{Z}_t.$$

For details about this model, see [4].

References

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References