

# Pricing caps on Libor rates in Lévy models

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## Contents

<b>1</b>	<b>Model</b>	<b>1</b>
<b>2</b>	<b>Pricing caps</b>	<b>3</b>
<b>3</b>	<b>Program manual</b>	<b>4</b>
	<b>References</b>	<b>7</b>

## Abstract

Here, we give an exact formula to price caps on libor rates in Lévy models. For this, we use the paper of Ernst Eberlein and Fehmi Özban[1].

## 1 Model

Let  $(\Omega, F_{T^*}, \mathbb{P}_{T^*}, (F_t)_{0 \leq t \leq T^*})$  be a complete stochastic basis, and  $T_0 < T_1 < \dots < T_n < T_{n+1} = T^*$  be a discrete-tenor structure with  $\delta = T_{i+1} - T_i$ ,  $i=0, \dots, n$  where  $T_i^* = T^* - i\delta$ , the libor rates is then defined by:

- For any maturity  $T_i$  there is a bounded deterministic function  $\lambda(., T_i)$  which represents the volatility of the forward Libor rate process  $L(., T_i)$ .
- We assume a strictly decreasing and positive initial term structure  $B(0, T)$  ( $T \in ]0, T^*]$ ). Consequently the initial term structure  $L(0, T)$  of forward Libor rates is given for  $T \in ]0, T^* - \delta]$  by

$$L(0, T) = \frac{1}{\delta} \left( \frac{B(0, T)}{B(0, T + \delta)} - 1 \right).$$

Under  $\mathbb{P}_{T^*}$ , we suppose that

$$1 + \delta L(t, T_1^*) = (1 + \delta L(0, T_1^*)) \exp\left(\int_0^{T_1^*} \lambda(s, T_1^*) dL_s^{T^*}\right)$$

where

$$L_t^{T^*} = \int_0^t b_s ds + \int_0^t c_s^{\frac{1}{2}} dW_s^{T^*} + \int_0^t \int_{\mathbb{R}} x(\mu^L - \nu^{T^*, L})(ds, dx)$$

and

$$\int_0^t \lambda(s, T_1^*) b_s ds = -\frac{1}{2} \int_0^t c_s \lambda^2(s, T_1^*) ds - \int_0^t \int_{\mathbb{R}} (e^{\lambda(s, T_1^*)x} - 1 - \lambda(s, T_1^*)x) \nu^{T^*, L}(ds, dx).$$

$(L_t^{T^*})_{0 \leq t \leq T^*}$  is a non-homogeneous Lévy process, i.e.  $W^{T^*}$  is a brownian motion,  $c_s$  is positive such that  $\int_0^{T^*} (|b_s| + |c_s|) ds < \infty$ ,  $\mu^L$  is the random measure of jumps of the process, and  $\nu^{T^*, L}(ds, dx) = F_s(dx)ds$  is the  $\mathbb{P}_{T^*}$ -compensator of  $\mu^L$ . We assume that the Lévy measures  $F_s$ , which are measures on  $\mathbb{R}$  with  $F_s(0) = 0$  and  $\int_0^{T^*} \int_{\mathbb{R}} (x^2 \wedge 1) F_s(dx) ds < \infty$ , satisfy the following additionnal integrability assumption

$$\int_0^{T^*} \int_{|x|>1} \exp(ux) F_s(dx) ds < \infty$$

for  $|u| \leq (1+\epsilon)M$ , where  $M, \epsilon > 0$  are constants such that  $\sum_{i=1}^n |\lambda(\cdot, T_i^*)| \leq M$ .

Thus we postulate that  $\forall j \in [0; n-1]$  the libor rate process can be written under  $\mathbb{P}_{T_j^*}$

$$1 + \delta L(t, T_{j+1}^*) = (1 + \delta L(0, T_{j+1}^*)) \exp\left(\int_0^t \lambda(s, T_{j+1}^*) dL_s^{T_j^*}\right)$$

where

$$L_t^{T_j} = \int_0^t b_s^{T_j} ds + \int_0^t c_s^{\frac{1}{2}} dW_s^{T_j^*} + \int_0^t \int_{\mathbb{R}} x(\mu^L - \nu^{T_j^*})(ds, dx)$$

and

$$\int_0^t \lambda(s, T_{j+1}^*) b_s^{T_j^*} ds = -\frac{1}{2} \int_0^t c_s \lambda^2(s, T_{j+1}^*) ds - \int_0^t \int_{\mathbb{R}} (e^{\lambda(s, T_{j+1}^*)x} - 1 - \lambda(s, T_{j+1}^*)x) \nu^{T_j^*, L}(ds, dx).$$

and

$$\frac{dP_{T_{j+1}^*}}{dP_{T_j^*}} = \varepsilon_{T_{j+1}^*} \left( \int_0^t \lambda(s, T_{j+1}^*) c_s^{\frac{1}{2}} dW_s^{T_j^*} + \int_0^t \int_{\mathbb{R}} (e^{\lambda(s, T_{j+1}^*)x} - 1) (\mu^L - \nu^{T_j^*})(ds, dx) \right)$$

and

$$W_t^{j+1} = W_t^j - \int_0^t \lambda(s, T_{j+1}^*) c_s^{\frac{1}{2}} ds$$

is a brownian motion under  $\mathbb{P}_{T_{j+1}^*}$

and

$$\nu^{T_{j+1}, L} = e^{\lambda(t, T_{j+1}^*) x} \nu^{T_j, L}$$

is a  $\mathbb{P}_{T_{j+1}^*}$ -compensator of  $\mu^L$

To implement in using this model, we must use the following assumption:

- the deterministic function  $L(., T_i)$  is constant for  $0 \leq i \leq n-1$ .

## 2 Pricing caps

Now, we can get an exact formula for caps price. Indeed, let  $0 \leq T_0 \leq T_1 \leq \dots \leq T_n$  be a tenor structure where  $T_{i+1} - T_i = \delta$ , the price of j-th caplet at time t is given by

$$\mathbb{E}_{\mathbb{P}^*} \left[ \frac{B_t}{B_{T_j}} \delta (L(T_{j-1}, T_{j-1}) - \mathcal{K})^+ \middle| F_t \right].$$

Thus, the value of cap at time  $t \leq T_0$  is given by

$$FC_t = \sum_{j=1}^n \mathbb{E}_{\mathbb{P}^*} \left[ \frac{B_t}{B_{T_j}} \delta (L(T_{j-1}, T_{j-1}) - \mathcal{K})^+ \middle| F_t \right].$$

For  $1 \leq j \leq n$ , let's consider the measure  $\mathbb{P}_{T_j}$  defined by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}_{T_j}} = B_{T_j} B(0, T_j).$$

Thus, we obtain that

$$\sum_{j=1}^n B(t, T_j) \mathbb{E}_{T_j} \left[ \delta (L(T_{j-1}, T_{j-1}) - \mathcal{K})^+ \middle| F_t \right].$$

Then we suppose that dynamics of libor rates  $L(t, T_{j-1})$  under the forward measure associated with the date  $T_{j-1}$  is given by

$$1 + \delta L(T_{j-1}, T_{j-1}) = (1 + \delta L(0, T_{j-1})) \exp(X_{T_{j-1}})$$

where  $(X_t)$  is defined by

$$X_t = \int_0^t \lambda(s, T_{j-1}) dL_s^{T_j}$$

and the deterministic function  $L(., T_i)$  is constant for  $0 \leq i \leq n-1$ .

To use the following theorem makes possible to find a formula to price caps. For this, we need the following result

$$\mathcal{X}(u) = \mathbb{E}_{\mathbb{P}_{T_j}} \left[ e^{iuX_{T_{j-1}}} \right] = \exp \left( \int_0^{T_{j-1}} \int_{\mathbb{R}} \left( e^{iu\lambda(s, T_{j-1})x} - iue^{\lambda(s, T_{j-1})x} - (1 - iu) \right) \nu^{T_j, L}(ds, dx) \right).$$

**Theorem 2.1** *Let  $\zeta_j = -\ln(1 + \delta L(0, T_{j-1}))$  and  $\mathcal{K}' = \delta\mathcal{K} + 1$ . Then  $L(T_{j-1}, T_{j-1}) = e^{-\zeta_j + X_{T_{j-1}}}$ . Assume that  $\text{mgf}(-R) < \infty$  where  $\text{mgf}(u) = \mathbb{E}_{T_{j-1}} \left[ e^{X_{T_{j-1}}} \right]$ . Let  $V_j(\zeta_j, \mathcal{K}')$  be the time-0 price of the  $j$ -th caplet and let  $L[v_{\mathcal{K}'}]$  be the bilateral Laplace transform of  $v_{\mathcal{K}'}$ , i.e.*

$$L[v_{\mathcal{K}'}] = \int_{-\infty}^{\infty} e^{-zx} v_{\mathcal{K}'}(x) dx, \quad z = R + iu \in \mathbb{C}, u \in \mathbb{R}.$$

Then

$$V_j(\zeta_j, \mathcal{K}') = B(0, T_j) \frac{e^{\zeta_j R}}{2\pi} \lim_{M \rightarrow \infty} \int_M^M e^{iu\zeta_j} L[v_{\mathcal{K}'}](R + iu) \mathcal{X}(iR - u) du$$

whenever the right-hand side exists.

Then, we must give the Lévy process cases that we want to use to price caps.

### 3 Program manual

In this program, we can choose between six different Lévy measure cases:

1. Lévy process where the Lévy measure is defined by

$$\nu = a \frac{e^{-\lambda x}}{x^2} 1_{\{x>0\}} \quad , \lambda, a > 0.$$

2. Lévy process where the Lévy measure is defined by

$$\nu = a \frac{e^{-\lambda x}}{x} 1_{\{x>0\}} \quad , \lambda, a > 0.$$

3. Lévy process where the Lévy measure is defined by

$$\nu = a \frac{e^{-\lambda x}}{x^{\alpha+1}} 1_{\{x>0\}} \quad , \lambda, a > 0 \text{ and } 0 < \alpha < 1.$$

4. Lévy process where the Lévy measure is defined by

$$\nu(x) = \frac{a_-}{|x|^{1+\alpha_-}} e^{-\lambda_- |x|} 1_{x<0} + \frac{a_+}{x^{1+\alpha_+}} e^{-\lambda_+ x} 1_{x>0}$$

with  $a_+, a_-, \lambda_+, \lambda_- > 0$ ,  $\alpha_-, \alpha_+ \neq 0, 1$  and  $\alpha_-, \alpha_+ < 2$ .

5. Lévy process where the Lévy measure is defined by

$$\nu(x) = \frac{a_-}{|x|} e^{-\lambda_- |x|} 1_{x<0} + \frac{a_+}{x} e^{-\lambda_+ x} 1_{x>0}$$

with  $a_+, a_-, \lambda_+, \lambda_- > 0$ .

6. Lévy process where the Lévy measure is defined by

$$\nu(x) = \frac{a_-}{|x|^2} e^{-\lambda_- |x|} 1_{x<0} + \frac{a_+}{x^2} e^{-\lambda_+ x} 1_{x>0}$$

with  $a_+, a_-, \lambda_+, \lambda_- > 0$ .

When we run the program, we have to choose between this six cases and give parameters of Lévy density that we choose for this program.

- For the first case, we have to enter 0 and after we have to enter the parameters  $a, \lambda$ .
- For the second case, we have to enter 1 and after we have to enter the parameters  $a, \lambda$ .
- For the third case, we have to enter 2 and after we have to enter the parameters  $a, \lambda, \alpha$ .
- For the fourth case, we have to enter 3 and after we have to enter the parameters  $a_-, a_+, \lambda_-, \lambda_+, \alpha_-, \alpha_+$ .

- For the fifth case, we have to enter 4 and after we have to enter the parameters  $a_-, a_+, \lambda_-, \lambda_+$ .
- For the sixth case, we have to enter 5 and after we have to enter the parameters  $a_-, a_+, \lambda_-, \lambda_+$ .

After to choose the Lévy density and enter this parameters, the program ask us to enter the strike  $\mathcal{K}$ , the number of dates  $n$ , the date  $T^*$ , the volatility of the forward Libor rate process  $L(., T_i)$  for  $0 \leq i \leq n - 1$ .

Now, we have to calculate bond prices. For this, we have two methods:

- in the first method, we use this formula  $B(0, t) = \exp(-rt)$ .
- in the second method, the previous formula is wrong. Thus, we calculate bond prices by interpolation in using the values and dates of bond prices already known.

Thus, we have to do:

- if we choose the first method to calculate bond prices, the program ask us to enter the constant interest rate  $r$ .
- if we choose the second method to calculate bond prices, the program ask us to enter the number of bond prices that we know, their dates and their values.

## References

- [1] Ernst Eberlein and Fehmi Özban. The lévy libor model. *Finance and Stochastics*, 9(3), july 2005.