Libor Market Model

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Derivative products

Date structure $\{T_n = T_0 + n\tau; n = 1..M\}$

assets and rates:

- $B(t, T)$ is the zero coupon bond at time t with maturity T.

- the foward swap rate starting at date T_s and ending at T_M is

$$S(t, T_s, T_M) = \frac{B(t, T_s) - B(t, T_M)}{\sum_{j=s+1}^{M} \tau B(t, T_j)}$$

- the spot swap rate is $S(T_s, T_s, T_n) = S(T_s, T_M)$.

- $L(t, T_i, \tau)$ the forward rate with maturity T_i and length τ

Arbitrage leads to

$$1 + \tau L(t, T_i, \tau) = \frac{B(t, T_i)}{B(t, T_{i+1})} \tag{1}$$

spot libor rate is given by

$$1 + \tau L(T_i, T_i, \tau) = \frac{1}{B(T_i, T_{i+1})} \tag{2}$$

Main products of interest for the libor Market Model

- Caplet and floorlet

- Cap floor

- swaption

Libor Market Model

Recall that

$$1 + \tau L(t, T_i, \tau) = \frac{B(t, T_i)}{B(t, T_{i+1})}$$

and

$$S(t, T_s, T_M) = \frac{B(t, T_s) - B(t, T_M)}{\sum_{j=s+1}^{M} \tau B(t, T_j)}$$

$$\frac{B(t,T_i)}{B(t,T_s)} = \prod_{j=s}^{i-1} \frac{1}{1+\tau L(t,T_j,\tau)}$$

as such the forward swap rate as a function of the forward rates is given by the relation

$$S(t,T_s,T_M) = \frac{1-\prod_{j=s}^{M-1}\frac{1}{1+\tau L(t,T_j,\tau)}}{\sum_{j=s+1}^{M}\tau\prod_{k=s}^{j-1}\frac{1}{1+\tau L(t,T_k,\tau)}} \qquad (3)$$

In the libor market model we suppose the following dynamic for the forward Libor rates

$$dL(t, T_i, \tau) = L(t, T_i, \tau)\gamma(t, T_i, \tau)dW^{Q^{T_{i+1}}}$$

where

- $\{W_t^{Q^{T_{i+1}}}; t \geq 0\}$ is a d dimensional brownian motion
- is the forward probability $Q^{T_{i+1}}$ associated with the numeraire $B(t, T_{i+1})$
- $\gamma(t, T_i, \tau)$ is a deterministic function.

- $\sigma_B(t,T)$ is the volatility of the zero coupon bond

we have the relation

$$\frac{\gamma(t,T_i,\tau)\tau L(t,T_i,\tau)}{1+\tau L(t,T_i,\tau)} = \sigma_B(t,T_i) - \sigma_B(t,T_{i+1})$$

and we deduce that

for $i > j+1$

$$
\begin{aligned}
\frac{dL(t,T_j,\tau)}{L(t,T_j,\tau)} &= \gamma(t,T_j,\tau)dW_t^{Q^{T_i}} \\
&- \sum_{k=j+1}^{i-1} \frac{\tau L(t,T_k,\tau)\gamma(t,T_k,\tau)\gamma(t,T_j,\tau)}{1+\tau L(t,T_k,\tau)} dt
\end{aligned}
$$

for $j \geq i$

$$
\begin{aligned}
\frac{dL(t,T_j,\tau)}{L(t,T_j,\tau)} &= \gamma(t,T_j,\tau)dW_t^{Q^{T_i}} \\
&+ \sum_{k=i}^{j} \frac{\tau L(t,T_k,\tau)\gamma(t,T_k,\tau)\gamma(t,T_j,\tau)}{1+\tau L(t,T_k,\tau)} dt
\end{aligned}
$$

Libor market model: a stochastic volatility extension

the standart LMM is unable to fit the smile

$$\Downarrow$$

a stochastic volatility extension

The model

Under the risk neutral measure Q the zero coupon bond follows the dynamic

$$
\begin{aligned}
\frac{dB(t,T)}{B(t,T)} &= r(t)dt + \sqrt{V_t}\sigma_B(t,T)'dW_t \\
dV_t &= \kappa(\theta - V_t)dt + \epsilon\sqrt{V_t}dZ_t
\end{aligned}
$$

where

- $(W_t; t \geq 0)$ is a d dimensional brownian motion under Q

- $(Z_t; t \geq 0)$ is a 1 dimensional brownian motion under Q,

- $\sigma_B(t,T)$ is a $1 * d$ vector

we have

$$\begin{aligned}
\frac{dL(t,T_j,\tau)}{L(t,T_j,\tau)} &= \sqrt{V_t}\gamma(t,T_j,\tau)'[dW_t^Q - \sqrt{V_t}\sigma_B(t,T_{j+1})dt] \\
dV_t &= \kappa(\theta - V_t)dt + \epsilon\sqrt{V_t}dZ_t
\end{aligned}$$

with

$$\gamma(t,T_j,\tau) = \frac{1+\tau L(t,T_j,\tau)}{\tau L(t,T_j,\tau)}[\sigma_B(t,T_j) - \sigma_B(t,T_{j+1})] \tag{4}$$

we make the hypothesis

$$\boxed{\{\gamma(t,T_j,\tau);\ t \geq 0;\ j = 1..M\} \text{ are deterministic functions.}}$$

We note $\gamma(t,T_j,\tau) = (\gamma^1(t,T_j,\tau), \gamma^2(t,T_j,\tau), .., \gamma^d(t,T_j,\tau))$

From (4) and under the hypothesis $\sigma_B(t,T_1) = 0$ we obtain

$$\sigma_B(t,T_{j+1}) = -\sum_{k=1}^{j} \frac{1 + \tau L(t,T_j,\tau)}{\tau L(t,T_j,\tau)} \gamma(t,T_k,\tau)$$

correlation between the forward rate factors and the volatility factor:

$$\frac{\gamma(t,T_j,\tau)'dW_t}{||\gamma(t,T_j,\tau)||}dZ_t = \rho_j(t)dt$$

If we note $W_t = (W_t^1, .., W_t^d)$ and $dW_t^i dZ_t = \rho^i dt$ we have

$$\begin{aligned} ||\gamma(t,T_j,\tau)||\rho_j(t)dt &= \gamma(t,T_j,\tau)'dW_t dZ_t && (5) \\ &= \sum_{i=1}^{d} \rho^i \gamma^i(t,T_j,\tau)dt && (6) \end{aligned}$$

under $Q^{T_{j+1}}$ the probability measure associated with $B(t, T_{j+1})$ as numeraire we have

$$\begin{cases} \frac{dL(t,T_j,\tau)}{L(t,T_j,\tau)} = \sqrt{V_t}\gamma(t, T_j, \tau)' dW_t^{Q^{T_{j+1}}} \\ dV_t = \kappa(\theta - (1 + \frac{\epsilon}{\kappa}\xi_j(t))V_t)dt + \epsilon\sqrt{V_t} dZ_t^{Q^{T_{j+1}}} \end{cases}$$

where $W_t^{Q^{T_{j+1}}}$ resp. $Z_t^{Q^{T_{j+1}}}$ is a $1 * d$ resp. 1 dimensional brownian motion under $Q^{T_{j+1}}$ and

$$\xi_j(t) = \sum_{k=1}^{j} \frac{\tau L(t, T_k, \tau)}{1 + \tau L(t, T_k, \tau)} \rho_k(t) ||\gamma(t, T_k, \tau)||$$

the authors propose to freeze this stochastic process and define

$$\xi_j^0(t) = \sum_{k=1}^{j} \frac{\tau L(0, T_k, \tau)}{1 + \tau L(0, T_k, \tau)} \rho_k(t) ||\gamma(t, T_k, \tau)||$$

$$\begin{aligned} \tilde{\xi}_j(t) &= 1 + \frac{\epsilon}{\kappa} \xi_j(t) \\ \tilde{\xi}_j^0(t) &= 1 + \frac{\epsilon}{\kappa} \xi_j^0(t) \end{aligned}$$

thus the dynamic is given by

$$\begin{cases} \frac{dL(t,T_j,\tau)}{L(t,T_j,\tau)} = \sqrt{V_t}\gamma(t,T_j,\tau)' dW_t^{Q^{T_{j+1}}} \\ dV_t = \kappa(\theta - \tilde{\xi}_j^0(t)V_t)dt + \epsilon\sqrt{V_t}dZ_t^{Q^{T_{j+1}}} \end{cases}$$

Moment generating function for the caplet

Computing the moment generating function for $X_u = ln\frac{L(u,T_j,\tau)}{L(t,T_j,\tau)}$, define

$$\phi(t, X_t, V_t, z) = E^{Q^{T_{j+1}}}\left[e^{zX_{T_j}}|\mathcal{F}_t\right]$$

The function $\phi(t, x, V, z)$ satisfies the pde

$$\begin{cases} \partial_t \phi + \kappa(\theta - \tilde{\xi}_j^0(t)V)\partial_V \phi - \frac{1}{2}||\gamma(t, T_j, \tau)||^2 V \partial_x \phi \\ +\frac{1}{2}\epsilon^2 V \partial^2_{VV}\phi + \epsilon\rho_j(t)V||\gamma(t, T_j, \tau)||\partial^2_{Vx}\phi + \frac{1}{2}||\gamma(t, T_j, \tau)||^2 V \partial^2_{xx}\phi = 0 \\ \phi(T, x, V, z) = e^{zx} \end{cases}$$

we define the function

$$\phi_T(z) = \phi(t, 0, V_t, z) \tag{7}$$

Moment generating function for the swaption

For the swaption pricing: recall

$$
\begin{aligned}
S(t, T_s, T_M) &= \frac{B(t, T_s) - B(t, T_M)}{\sum_{j=s+1}^{M} \tau B(t, T_j)} \\
&= \frac{1 - \prod_{j=s}^{M-1} \frac{1}{1+\tau L(t, T_j, \tau)}}{\sum_{j=s+1}^{M} \tau \prod_{k=0}^{j-1} \frac{1}{1+\tau L(t, T_k, \tau)}}
\end{aligned}
$$

using Ito's lemma we deduce that

$$
\begin{aligned}
dS(t,T_s,T_M) &= \sum_{j=s}^{M-1} \frac{\partial S(t,T_s,T_M)}{\partial L(t,T_j,\tau)} L(t,T_j,\tau)\sqrt{V_t}\gamma(t,T_j,\tau)' \\
&\quad [dW_t - \sqrt{V_t}\sigma_S(t)dt] \\
dV_t &= \kappa(\theta - \tilde{\xi}_S(t)V_t)dt + \epsilon\sqrt{V_t}[dZ_t + \xi_S(t)dt]
\end{aligned}
$$

with

$$
\begin{aligned}
\sigma_S(t) &= \sum_{j=s}^{M-1} \alpha_j(t)\sigma_B(t, T_{j+1}) \\
\tilde{\xi}_S(t) &= 1 + \frac{\epsilon}{\kappa}\sum_{j=s}^{M-1} \alpha_j(t)\xi_j(t) \\
\alpha_j(t) &= \frac{\tau B(t, T_{j+1})}{\sum_{j=s}^{M-1} \tau B(t, T_{j+1})} \\
\frac{\partial S(t, T_s, T_M)}{\partial L(t, T_j, \tau)} &= \frac{\tau S(t, T_s, T_M)}{(1 + \tau L(t, T_j, \tau))} \\
&\left(\frac{B(t, T_M)}{B(t, T_s) - B(t, T_M)} + \frac{\sum_{k=j+1}^{M} \tau B(t, T_k)}{\sum_{j=s+1}^{M} \tau B(t, T_j)}\right)
\end{aligned}
$$

the dynamic of the forward swap rate is given by

$$\left\{ \begin{array}{l} dS(t,T_s,T_M) = \sum_{j=s}^{M-1} \frac{\partial S(t,T_s,T_M)}{\partial L(t,T_j,\tau)} L(t,T_j,\tau)\sqrt{V_t}\gamma(t,T_j,\tau)' dW_t^{Q^S} \\ dV_t = \kappa(\theta - \tilde{\xi}_S(t)V_t)dt + \epsilon\sqrt{V_t}dZ_t^{Q^S} \end{array} \right.$$

with

$$\begin{aligned} dW_t^{Q^S} &= dW_t - \sqrt{V_t}\sigma_S(t)dt \\ dZ_t^{Q^S} &= dZ_t - \sqrt{V_t}\xi_S(t)dt \end{aligned}$$

where $W_t^{Q^S}$ resp. $Z_t^{Q^S}$ is a $1 * d$ dimensional resp 1 dimensionnal brownian motion under Q^S.

freezing the volatility for the forward swap rate and the drift of the volatility we get

$$\begin{cases} \frac{dS(t,T_s,T_M)}{S(t,T_s,T_M)} = \sum_{j=s}^{M-1} \omega_j(0)\sqrt{V_t}\gamma(t,T_j,\tau)' dW_t^{Q^S} \\ dV_t = \kappa(\theta - \tilde{\xi}_S^0(t)V_t)dt + \epsilon\sqrt{V_t}dZ_t^{Q^S} \end{cases}$$

$$\omega_j(0) = \frac{\partial S(0,T_s,T_M)}{\partial L(0,T_j,\tau)} \frac{L(0,T_j,\tau)}{S(0,T_s,T_M)}$$

$$\tilde{\xi}_S^0(t) = 1 + \frac{\epsilon}{\kappa} \sum_{j=s}^{M-1} \alpha_j(0) \xi_j^0(t)$$

Computing the moment generating function for $X_u = ln\frac{S(u,T_s,T_M)}{S(t,T_s,T_M)}$, define

$$\phi(t, X_t, V_t, z) = E^{Q^S}\left[e^{zX_T}|\mathcal{F}_t\right]$$

The function $\phi(t,x,V,z)$ satisfies the pde

$$\left\{ \begin{array}{l} \partial_t\phi + \kappa(\theta - \tilde{\xi}_S^0(t)V)\partial_V\phi - \frac{1}{2}||\gamma_{s,M}(t)||^2 V\partial_x\phi \\ +\frac{1}{2}\epsilon^2 V\partial_{VV}^2\phi + \epsilon\rho^S(t)V||\gamma_{s,M}(t)||\partial_{Vx}^2\phi + \frac{1}{2}||\gamma_{s,M}(t)||^2 V\partial_{xx}^2\phi = 0 \\ \phi(T,x,V,z) = e^{zx} \end{array} \right.$$

with

$$\begin{aligned} \gamma_{s,M}(t) &= \sum_{j=s}^{M-1}\omega_j(0)\gamma(t,T_j,\tau) \\ \rho^S(t) &= \frac{\sum_{j=s}^{M-1}\omega_j(0)||\gamma(t,T_j,\tau)||\rho_j(t)}{||\gamma_{s,M}(t)||} \end{aligned}$$

furthermore the authors suggest, arguing a calibration objective not presented in the paper, to approximate

$$\rho^S(t) \sim \sum_{j=s}^{M-1} \omega_j(0)\rho_j(t)$$

In fact, this approximation is useless because only $\rho^S(t)||\gamma_{s,M}(t)||$ is needed and (6) is used.

We define the function $\phi_T(z)$ by

$$\phi_T(z) = \phi(t, 0, V_t, z) \tag{8}$$

Computing the moment generating function

The pdes are identical as such we write both in a compact form

$$\begin{cases} \partial_t\phi + \kappa(\theta - \beta(t)V)\partial_V\phi - \frac{1}{2}\lambda(t)^2 V\partial_x\phi \\ +\frac{1}{2}\epsilon^2 V\partial^2_{VV}\phi + \epsilon\rho(t)V\lambda(t)\partial^2_{Vx}\phi + \frac{1}{2}\lambda(t)^2 V\partial^2_{xx}\phi = 0 \\ \phi(T, x, V, z) = e^{zx} \end{cases}$$

for the caplet

$$
\begin{aligned}
\beta(t) &= \tilde{\xi}_j^0(t) \\
\lambda(t) &= ||\gamma(t, T_j, \tau)|| \\
\rho(t) &= \rho_j(t) \\
\zeta(t) &= ||\gamma(t, T_j, \tau)||\rho_j(t)
\end{aligned}
$$

for the swaption

$$
\begin{aligned}
\beta(t) &= \tilde{\xi}_S^0(t) \\
\lambda(t) &= ||\gamma_{s,M}(t)|| \\
\rho(t) &= \rho^S(t) \\
\zeta(t) &= \rho^S(t)||\gamma_{s,M}(t)||
\end{aligned}
$$

we emphazis the time dependence of the parameters. Looking for a solution of the form $\phi(t,x,V,z) = e^{A(t,z)+B(t,z)V+zx}$ we obtain the Riccati's equations

$$
\begin{aligned}
-\partial_t A(t,z) &= \kappa\theta B(t,z) && (9)\\
-\partial_t B(t,z) &= \frac{1}{2}\epsilon^2 B(t,z)^2 + (\rho(t)\epsilon\lambda(t)z - \kappa\beta(t))B(t,z) + \frac{1}{2}\lambda(t)^2(z^2 - z) && (10)\\
&= b_2(t)B(t,z)^2 + b_1(t)B(t,z) + b_0(t) && (11)
\end{aligned}
$$

with terminal conditions $A(T,z) = 0$ and $B(T,z) = 0$

Under the hypothesis that the volatility is piecewise constant and the maturity of the option is T_N the solution of the above system is given by

$$
\begin{cases}
B(t,z) = B(T_{i+1},z) + \frac{-b_1+d-2B(T_{i+1},z)b_2}{2b_2(1-ge^{d(T_{i+1}-t)})}(1-e^{d(T_{i+1}-t)}) \\
A(t,z) = A(T_{i+1},z) + \frac{a_0}{2b_2}\left((-b_1+d)(T_{i+1}-t) - 2ln\left(\frac{1-ge^{d(T_{i+1}-t)}}{1-g}\right)\right)
\end{cases}
$$

for $t \in [T_i \; T_{i+1}]$ and $i \in \{0..N-1\}$ with

$$
\begin{aligned}
A(T_N, z) &= 0 \\
B(T_N, z) &= 0 \\
a_0 &= \kappa\theta \\
b_1 &= \rho(T_i)\epsilon\lambda(T_i)z - \kappa\beta(T_i) \\
b_0 &= \frac{\lambda(T_i)^2}{2}(z^2 - z) \\
b_2 &= \frac{\epsilon^2}{2} \\
d &= \sqrt{\Delta} \\
\Delta &= b_1^2 - 4b_0b_2 \\
g &= \frac{-b_1 + d - 2B(T_{i+1}, z)b_2}{-b_1 - d - 2B(T_{i+1}, z)b_2}
\end{aligned}
$$

Remark: For computational prupose we embed the caplet/floorlet structure in the swpation structure. In fact we have

$$L(t, T_i, \tau) = S(t, T_i, T_{i+1})$$

as such for pricing a caplet or a swaption we will use the same algorithm.

Derivatives pricing

For the caplet $Cplt(t, T_M, K, \tau, N)$ we have

$$
\begin{aligned}
Cplt(t, T_M, K, \tau, N) &= B(t, T_M + \tau)\tau N E_t^{Q^{T_M+\tau}}[(L(T_M, T_M, \tau) - K)_+] \\
&= B(t, T_M + \tau)\tau N L(t, T_M, \tau)\left(I_1 - \frac{K}{L(t, T_M, \tau)} I_2\right)
\end{aligned}
$$

with

$$
\begin{aligned}
I_1 &= E_t^{Q^{T_M+\tau}}\left[e^{ln\frac{L(T_M,T_M,\tau)}{L(t,T_M,\tau)}}\mathbf{1}_{\{\frac{L(T_M,T_M,\tau)}{L(t,T_M,\tau)} > \frac{K}{L(t,T_M,\tau)}\}}\right] \\
I_2 &= E_t^{Q^{T_M+\tau}}\left[\mathbf{1}_{\{\frac{L(T_M,T_M,\tau)}{L(t,T_M,\tau)} > \frac{K}{L(t,T_M,\tau)}\}}\right]
\end{aligned}
$$

For the floorlet $Flt(t, T_M, K, \tau, N)$ we have

$$\begin{aligned} Flt(t, T_M, K, \tau, N) &= B(t, T_M + \tau)\tau N L(t, T_M, \tau) \\ &\quad \left((1 - I_2) \frac{K}{L(t, T_M, \tau)} - (1 - I_1) \right) \end{aligned}$$

For the european payer swaption $Swpt(t, T_s, T_M, K, \tau, N)$

$$\begin{aligned} Swpt(t, T_s, T_M, K, \tau, N) &= \sum_{i=s}^{M-1} B(t, T_{i+1})\tau N S(t, T_s, T_M) \\ &\quad \left(I_1 - \frac{K}{S(t, T_s, T_M)} I_2 \right) \end{aligned}$$

$$
\begin{aligned}
I_1 &= E_t^{Q^S}\left[e^{ln\frac{S(T_s,T_s,T_M)}{S(t,T_s,T_M)}}\mathbf{1}_{\{\frac{S(T_s,T_s,T_M)}{S(t,T_s,T_M)}>\frac{K}{S(t,T_s,T_M)}\}}\right] \\
I_2 &= E_t^{Q^S}\left[\mathbf{1}_{\{\frac{S(T_s,T_s,T_M)}{S(t,T_s,T_M)}>\frac{K}{S(t,T_s,T_M)}\}}\right]
\end{aligned}
$$

For the european receiver swaption $Swpt(t, T_s, T_M, K, \tau, N)$

$$
\begin{aligned}
Swpt(t, T_s, T_M, K, \tau, N) &= \sum_{i=s}^{M-1} \tau B(t, T_{i+1}) N S(t, T_s, T_M) \\
&\quad \left(\frac{K}{S(t, T_s, T_M)}(1 - I_2) - (1 - I_2)\right)
\end{aligned}
$$

Computing the integrals

We have the following expressions for I_1 and I_2

$$
\begin{aligned}
I_1 &= \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \frac{Im\{e^{-iuln\left(\frac{K}{X(t)}\right)} \phi_T(1+iu)\}}{u} du \\
I_2 &= \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \frac{Im\{e^{-iuln\left(\frac{K}{X(t)}\right)} \phi_T(iu)\}}{u} du
\end{aligned}
$$

where $\phi_T(u)$ is given by (7) or (8) depending on whether a swaption or a caplet is priced and $X(t) = L(t, T_M, \tau)$ resp. $X(t) = S(t, T_s, T_M)$

for the caplet/floorlet resp. the swaption (receiver or payer).
FFT method is also possible

Numerical examples

For our numerical experiments we choose a two factors model with the following piecewise volatility structure:

$$\gamma(t, T_k, \tau) = (\gamma^1(t, T_k, \tau), \gamma^2(t, T_k, \tau)).$$

if $t \in [T_j \; T_{j+1}[$

$$\begin{aligned}
\gamma^1(t, T_k, \tau) &= 0.2 \\
\gamma^2(t, T_k, \tau) &= \frac{0.01 - 0.05e^{-0.1(j-k)}}{\sqrt{0.04 + 0.00075j}}
\end{aligned}$$

and

$$\begin{aligned}
dW_t^1 dZ_t &= \rho^1 dt = 0.5dt \\
dW_t^2 dZ_t &= \rho^2 dt = 0.2dt
\end{aligned}$$

the yield curve is flat at 5%, $V_0 = 1$, $\epsilon = 0.6$, $\kappa = 1$ and $\theta = 1$.

Swaption payer prices in bps

swaption maturity	Tenor	strike	price
1	1	ATM	64.519
1	5	ATM	405.221
1	10	ATM	1179.612
3	1	ATM	116.830
3	5	ATM	739.835
3	10	ATM	2057.297
5	1	ATM	161.735
5	5	ATM	1009.870
5	10	ATM	1904.210
1	1	0.8 ATM	114.683
1	5	0.8 ATM	609.080
1	10	0.8 ATM	1472.062
3	1	0.8 ATM	151.380
3	5	0.8 ATM	869.485
3	10	0.8 ATM	2201.807
5	1	0.8 ATM	185.766
5	5	0.8 ATM	1087.164
5	10	0.8 ATM	2257.460

Swaption payer prices in bps

swaption maturity	Tenor	strike	price
1	1	1.2 ATM	34.655
1	5	1.2 ATM	267.585
1	10	1.2 ATM	954.980
3	1	1.2 ATM	91.083
3	5	1.2 ATM	636.496
3	10	1.2 ATM	1934.698
5	1	1.2 ATM	142.306
5	5	1.2 ATM	944.592
5	10	1.2 ATM	1623.445

Arbitrage free discretization of the Libor Market Model

Definition of *arbirtrage-free* discretization

In the BGM models it is supposed that all the libor $L(t,T_i,\tau)$ under their own forward measure $Q^{T_{i+1}}$ has no drift and deterministic log-volatility :

$$\forall i=1,..,M: \quad dL(t,T_i,\tau)=L(t,T_i,\tau)\gamma_i(t).dW_t^{Q^{T_{i+1}}}.$$

Considering a numeraire $N(t)$, we denote by D_i the deflated bonds :

$$\forall i=1,..,M+1: D_i(t)=\frac{B(t,T_i)}{N(t)}.$$

By definition of a numeraire the deflated bonds are martingale under their corresponding measure Q^N associated to the numeraire N. This

martingale property is of course for the continuous filtration.
The deflated bonds price can be defined by the libors :

$$\forall t < T_i : \quad D_i(t) = \frac{B(t, T_{i_t})}{N(t)} \prod_{j=i_t}^{i-1} \frac{1}{1 + \tau L(t, T_j, \tau)}, \quad \text{for } i = i_t, .., M+1$$

where i_t is the unique integer such that $T_{i_t - 1} \leq t < T_{i_t}$.

Definition : A discretisation $0 = t_0 < t_1 < ... < t_n = T_{M+1}$ is said to be *arbitrage-free* if all the discrete deflated bonds are discrete martingale. In other words, if we denote $\widehat{D}_i(t_j)$ the computed deflated bond D_i in time t_j, we must have :

$$\forall i = 1, .., M+1, \quad j = 0, .., n-1 : \widehat{D}_i(t_j) = E\left[\widehat{D}_i(t_{j+1})_{/F_j}\right] \qquad (12)$$

where F_j is the filtration associed to the discrete brownian process over $t_0, t_1, ..., t_n$.

Remark : Thus the condition to an *arbitrage-free* discretisation can be resumed to these backward discrete relations.

Two usefull numeraires for *arbitrage-free* There are two numeraires that will be usefull to seek *arbitrage-free* dicretization.
the terminal numeraire :

$$N_T(t) = B(t, T_{M+1})$$

and the spot numraire (i_t such that: $T_{i_t - 1} \leq t < T_{i_t}$):

$$N_S(t) = \frac{B(t, T_{i_t})}{B(0, T_1) \prod_{j=1}^{i_t - 1} B(T_j, T_{j+1})}$$

Both numeraires have the great advantage, for a libor model, to give the expression of the deflated bonds only with respect to the libors :

$$D_i(t) = \prod_{j=i}^{M} \Big(1 + \tau L(t, T_j, \tau)\Big) \quad \text{for the terminal numeraire} \tag{13}$$

$$D_i(t) = B(0, T_1) \prod_{j=1}^{i-1} \frac{1}{1 + \tau L(t, T_j, \tau)} \quad \text{for the spot numeraire} \tag{14}$$

for all $i = 1, ..., M+1$.

Thus in the *libors discrete world*, denoting for all $i = 0, .., n$ and $j = 1, .., M$ $\widehat{L}(t_i, T_j, \tau)$ the numerical computed value of the libors, the discrete deflated bonds price are:

$$\widehat{D}_i(t) = \prod_{j=i}^{M} \Big(1 + \tau \widehat{L}(t, T_j, \tau)\Big) \quad \text{for the terminal numeraire} \tag{15}$$

$$\widehat{D}_i(t) = B(0, T_1) \prod_{j=1}^{i-1} \frac{1}{1 + \tau \widehat{L}(t, T_j, \tau)} \quad \text{for the spot numeraire} \tag{16}$$

Continuous martingales versus discrete Martingale

- Martingale property of the continuous deflated bonds does not imply the martingale property of the discrete deflated bonds

- in a BGM model the libors $\widehat{L}_i$ or log-libors $log(\widehat{L}_i)$ are computed through a standart Euler scheme then the $\widehat{D}_i$ have no (discrete) martingale

Main strategy to solve this problem:

Other assets associated to the libors by a bijective relation will be discretized to make the *arbitrage-free* discretisation true

$$\Downarrow$$

that is to say to make the discrete deflated bonds martingale.

Two assets that can be considered: X and Y given by :

$$X_i(t) = L(t,T_i,\tau)\prod_{j=i+1}^{M}\Big(1+\tau L(t,T_j,\tau)\Big) \quad \forall i = 1,..,M. \tag{17}$$

$$Y_i(t) = \tau L(t,T_i,\tau)\prod_{j=1}^{i}\frac{1}{1+\tau L(t,T_j,\tau)} \quad \forall i = 1,..,M. \tag{18}$$

Taking $Y_{M+1}(t) = \prod_{j=1}^{M}\frac{1}{1+\tau\hat{L}(t,T_j,\tau)}$ we have the following equalities:

$$\sum_{j=1}^{M+1} Y_j(t) = 1. \tag{19}$$

The libors can aslo be written with respect to this assets :

$$
\begin{aligned}
L_i(t, T_i, \tau) &= \frac{X_i(t)}{1 + \tau X_{i+1}(t) + .. + \tau X_M(t)} \quad \forall i = 1, 2, .., M. \qquad (20) \\
L_i(t, T_i, \tau) &= \frac{Y_i(t)}{\tau(Y_{i+1}(t) + .. + Y_{M+1}(t))} \quad \forall i = 1, 2, .., M. \qquad (21)
\end{aligned}
$$

The deflated bonds can also be written with respect to these assets :

$$D_i(t) = 1 + \tau \sum_{j=i}^{M} X_j(t) \quad \text{for terminal numeraire} \tag{22}$$

$$D_i(t) = B(0, T_1) \sum_{j=i}^{M+1} Y_j(t) \quad \text{for spot numeraire} \tag{23}$$

and vice versa :

$$X_i(t) = \frac{1}{\tau}(D_i(t) - D_{i+1}(t)) \quad \text{for terminal numeraire} \tag{24}$$

$$Y_i(t) = \frac{D_i(t) - D_{i+1}(t)}{B(0, T_1)} \quad \text{for spot numeraire} \tag{25}$$

Theorem: The assets X and Y are martingale respectively under the terminal and spot measure.

Theorem: Under their measure the EDS verified by X and Y are the following :

$$\begin{aligned} \frac{dX_i(t)}{X_i(t)} &= \left(\gamma_i(t) + \sum_{j=i+1}^{M} \frac{\tau X_j(t) * \gamma_j}{1 + \tau X_j(t) + ... + \tau X_M(t)}\right) .dW^{Q^{N_T}} \quad \forall i = 1, . \\ \frac{dY_i(t)}{Y_i} &= \left(\gamma_i + \sum_{i_t}^{i} \frac{Y_j * \gamma_j}{Y_{j-1} + .. + Y_1 - 1}\right) .dW^{Q^{N_S}} \quad \forall i = 1, .., M+1. \end{aligned}$$

Implementation of caps and swaptions with X and Y

Theorem With a standart log Euler scheme, the discrete assets $\widehat{X}$ and $\widehat{Y}$ are discrete martingales.

Considering a receiver swaption of a swap rate between T_α and T_β ($\alpha < \beta < M+1$), under the numeraire measure Q^N we have for its price at time $t = 0$:

$$\frac{RS_{\alpha,\beta}}{N(0)} = E\left[\frac{1}{N(T_\alpha)}\left(1 - B(T_\alpha, T_\beta) - K\sum_{j=\alpha+1}^{\beta}\tau B(T_\alpha, T_j)\right)\right] \quad (28)$$

$$\frac{RS_{\alpha,\beta}}{N(0)} = E\left[D_\alpha(T_\alpha) - D_\beta(T_\alpha) - K\tau\sum_{j=\alpha+1}^{\beta} D_j(T_\alpha)\right] \quad (29)$$

Swaption price for asset X : With equality (22) in (29) we get under terminal measure :

$$\frac{RS_{\alpha,\beta}}{B(0,T_{M+1})} = \tau E\left[\sum_{j=\alpha}^{\beta-1} X_j(T_\alpha) - K\tau \sum_{j=\alpha+1}^{\beta}\left(1+\tau\sum_{k=j}^{M} X_k(T_\alpha)\right)\right] \quad (30)$$

Swaption price for asset Y : With equality (23) in (29) we get under spot measure:

$$\frac{RS_{\alpha,\beta}}{N_S(0)} = B(0,T_1)E\left[\sum_{j=\alpha}^{\beta-1} Y_j(T_\alpha) - K\tau \sum_{j=\alpha+1}^{\beta}\sum_{k=j}^{M+1} Y_k(T_\alpha)\right] \quad (31)$$

Caplet price for asset X : Using (20) we have for a caplet price over $L(T_i, T_i, \tau)$ under terminal measure :

$$\frac{Caplet_i}{N_T(0)} = E\left[X_i(T_\alpha)\frac{1+\tau\sum_{j=i}^{M} X_j(T_\alpha)}{1+\tau\sum_{j=i+1}^{M} X_j(T_\alpha)} - K\left(1+\tau\sum_{j=i}^{M} X_j(T_\alpha)\right)\right] \tag{32}$$

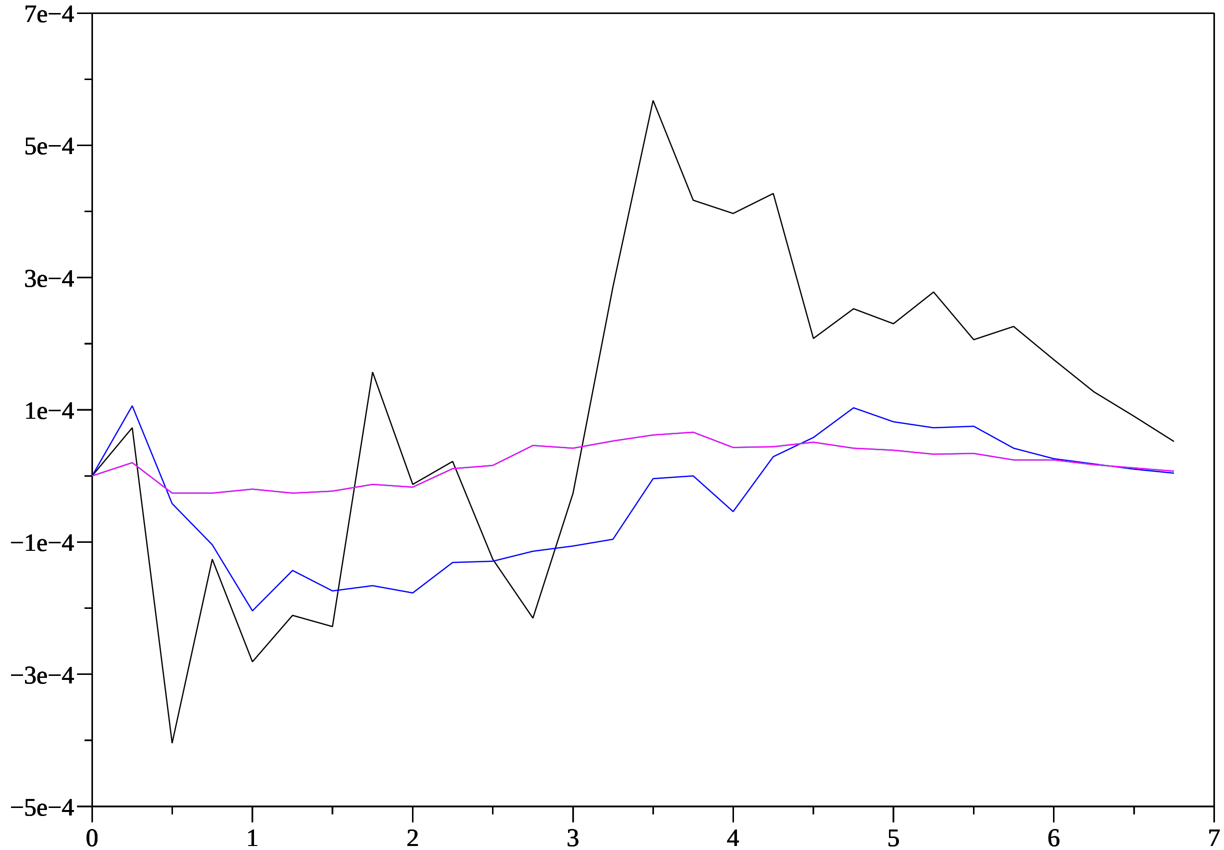
Caplet price for asset Y : Using (21) we have for a caplet price over $L(T_i, T_i, \tau)$ under spot measure :

$$\frac{Caplet_i}{N_S(0)} = B(0,T_1)E\left[Y_i(T_\alpha)\frac{\sum_{j=i}^{M+1} Y_j(T_\alpha)}{\sum_{j=i+1}^{M+1} Y_j(T_\alpha)} - K\left(1+\tau\sum_{j=i}^{M+1} Y_j(T_\alpha)\right)\right] \tag{33}$$

Simulations results The zero coupon bonds can be expressed with an expectation. We have

$$B(0,T_i) = N(0)\frac{B(0,T_i)}{N(0)} = N(0)E(D_i(T_i)).$$

Thanks to the exact martingale property of the discrete deflated bonds $\widehat{D}_i$ we can say that if we compute the expectation of the previous formula, the error only comes from noises due to the number of Monte-Carlo draws and will tend to vanish when this number goes to infinity.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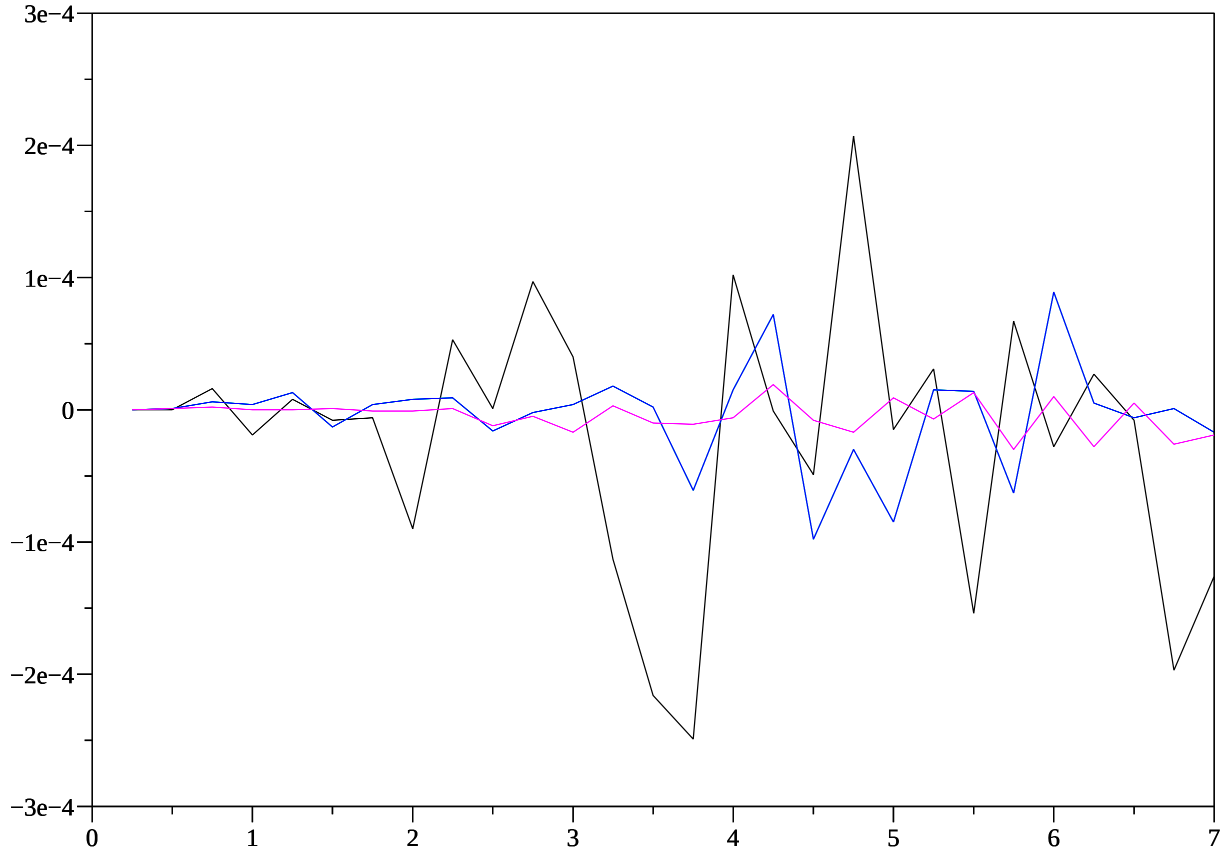
7

Computation bonds error with martingale asset X for 10000, 100000 and 1000000 Monte-carlo draws, for $\tau = 0.25$ and $M = 28$ (Number of factor=1, $\gamma_i = 0.15$ and $L(0, T_i, T_i) = 0.05$).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Computation bonds error with martingale asset Y for 10000, 100000 and 1000000 Monte-carlo draws, for $\tau = 0.25$ and $M = 28$ (Number of factor=1, $\gamma_i = 0.15$ and $L(0, T_i, T_i) = 0.05$)

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