Implementation : Gamma Expansion of Heston Stochastic Volatility Model

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Abstract

In this paper we show the implementation of the exact simulation methods presented in the paper of Galsserman and al. [2] and Joshi and Chan [3] . We derive two methods, the first one is based on the truncation series simulation, and the second one deals with inversion techniques of both the Laplace transform and the cumulative function.

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1 Introduction

We consider the following Heston model, given by the following stochastic differential equation

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu dt + \sqrt{V_t}(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2) \\ dV_t &= \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t^1, \end{aligned}$$

where (W_t^1, W_t^2) is a standard two dimensional Browian motion. The variable S_t describes the level of an underlying asset and Vt the variance of its instantaneous returns. The parameters κ, θ, σ (and typically also μ) are positive, and ρ takes values in $[-1, 1]$. We take the initial conditions S_0 and V_0 to be strictly positive.

It is well know [1], that the CIR process V_t is given by

$$V_t \sim \frac{\sigma^2(1-e^{-\kappa t})}{4\kappa}\chi_\delta^2\left(\frac{4\kappa e^{-\kappa t}}{\sigma^2(1-e^{-\kappa t})}V_0\right),\ t>0,\ \delta = \frac{4\kappa\theta}{\sigma^2},$$

where $\chi^2_{delta}(\lambda)$ denotes non central chi-square variable with δ degrees freedom and non central parameter λ.

Broadie and Kaya [4] rewrite the exact simulation of the couple (S_t, V_t) as follows

$$\log(\frac{S_t}{S_0}) \sim \mathcal{N}\left((\mu - \frac{\rho\kappa\theta}{\sigma})t + (\frac{\kappa\rho}{\sigma} - \frac{1}{2})\int_0^t V_s ds + \frac{\rho}{\sigma}(V_t - V_0), (1-\rho^2)\int_0^t V_s ds\right),$$

where $\mathcal{N}(m, var)$ is an independent Gaussian randon variable with mean m and variance var. It is then sufficient to know the joint distribution of $(V_t, \int_0^t V_s ds)$ to sample exactly the couple (S_t, V_t). The problem of the exact simulation can be reduced to sample exactly just

$$\left(\int_0^t V_s ds | V_0, V_t\right). \tag{1}$$ Cond_dist

The main result of the paper is given by the following theorem

Theorem 1. *The distribution of 1 admits the following representation*

$$\left(\int_0^t V_s ds | V_0, V_t\right) \sim X_1 + X_2 + X_3 \equiv X_1 + X_2 + \sim_{j=1}^{\eta} Z_j,$$

in which X_1, X_2, $\mu, Z_1, Z_2 \ldots$ are mutually independent, the Z_j are independent copies of a random variable Z, and η is a Bessel random variable with parameter $\nu = \frac{\delta}{2} - 1$, and $z = \frac{2\kappa/\sigma^2}{sinh(\kappa t/2)}\sqrt{V_t V_0}$.

Moreover, X_1, X_2, and Z have the following representations:

$$X_1 \sim \sum_{n=1}^{\infty} \frac{1}{\gamma_n} \sum_{j=1}^{N_n} Exp_j(1),\ X_2 \sim \sum_{n=1}^{\infty} \frac{1}{\gamma_n} \Gamma(\delta/2, 1),\ Z \sim \sum_{n=1}^{\infty} \frac{1}{\gamma_n} \Gamma(2, 1), \tag{2}$$

where

$$\lambda_n = \frac{16\pi n^2}{\sigma^2 t(\kappa^2 t^2 + 4\pi^2 n^2)},\ \gamma_n = \frac{\kappa^2 t^2 + 4\pi^2 n^2}{2\sigma^2 t^2}. \tag{3}$$

The N_n are independent Poisson random variable with parameter $\lambda_n(V_0 + V_t)$, the $Exp_j(1)$ are independent, unit mean exponential random variable, and the $\Gamma(\alpha, \beta)$ denote the independent gamma random variable with shape of parameter α and scale one β.

This implementation concerns the method named mcGlassermanKim1. The idea is to go back to the thoerem and replace the infinite series by a finite one. The rest of each truncation is approximated by a non Gaussian random variable. Technical details are given in [2] and [3].

Algo1

Algorithm 1: The function `X1Sample`

Input: ordertr, θ, κ, σ, t, V_t and V_0
Output: The value of the random variable X_1
Truncation method
$X_1 \sim \sum_{n=1}^{\text{ordertr}} \frac{1}{\gamma_n} \sum_{j=1}^{N_n} \text{Exp}_j(1)$ + Chi-square random variable

Algo1

Algorithm 2: The function `X2Sample`

Input: ordertr, θ, κ, σ, and t.
Output: The value of the random variable X_2
Truncation method $\sim \sum_{n=1}^{ordertr} \frac{1}{\gamma_n} \Gamma(\delta/2, 1)$ + Chi-square random variable

Algo1

Algorithm 3: The function X3Sample

Input: ordertr, θ, κ, σ, and t.
Output: The value of the random variable Z
Truncation method $Z \sum_{n=1}^{ordertr} \frac{1}{\gamma_n}\Gamma(2,1)$ + Chi-square random variable

Algo1

Algorithm 4: The function SampleC

Input: θ, κ, σ, t, V_t
Output: The value of the random variable $(V_t, \int_0^t V s ds)$
Using the representation of Theorem **??** taking a default value $ordertr = 20$

In order to take into consideration all variables, one has to make sur that all Gaussian variables are good egnough to make this kind of situation with all other variables, in order to make sure that the constant is good enough to build a goo estimator. One can wonder if an only if all other variables can be taken into account to make sure the following result. In other hand, one had to derive some properties to make understant tghat you need this kind of situation but in the case a bullish market all correaltion value takesz a hige value, but stabke. The idea is to make understant that the correlation is negtivaly correlated but one has to underestimate the skew of the market that derives from the option pricing market research. du you hear me ima talking to acroos the rives, baby im trying, i feel i keep in my mind luckuy to my best friend, all market are available in order to take into consideration all variables to make sure that all variables are gaussian but if we consider that are ckideux, one has to consider that the variable are positive, then we have some kind of biaised mean to take into consideration for this kind of consideration that can be sure in the world of the Laplace derivative. We are able to make that all kind of situation are avalaible to make sure that the

References

GlassermanB [1] P. Glasserman. *Monte Carlo methods in financial engineering*, volume 53 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, 2004. Stochastic Modelling and Applied Probability. 1

gk09 [2] Paul Glasserman and Kyoung-Kuk Kim. Gamma expansion of the heston stochastic volatility model. *Finance and Stochastics*, pages 1–30, 2009. 1, 2

chjos11 [3] M S. Joshi J.H. Chan. Fast and accurate long stepping simulation of the heston stochastic volatility model. *Preprint*, 2011. 1, 2

BK04 [4] M.Broadie O.Kaya. 2004 winter simulation conference (wsc'04)). *Exact Simulation of Option Greeks under Stochastic Volatility and Jump Diffusion Models*, 2:535–543, 2004. 1

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