

# Pricing European option under Heston-CIR model with Fourier Cosine Method: implementation in PREMIA

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## Abstract

In a Heston hybrid model with stochastic interest rate, we implement European option pricing by Fourier Cosine method. Based on [3], we apply two different approximations of the hybrid model to obtain a closed form solution of the characteristic function for Fourier cosine method.

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### 1 Introduction

We consider the pricing problem of European option in hybrid model and stochastic interest rate. The implementation of this problem is based on the Fourier Cosine expansion approach given by Grzelak and Oosterlee (2010).

The rest of this file is as follows: we introduce the model in Section 2, then in section 3 we present the pricing method of Fourier Cosine expansion incorporated with a closed form solution of characteristic function of the model, the program manual of the implementation in PREMIA is given in Section 4.

### 2 Model description

The model we considered is a Heston model associated with Cox-Ingersoll-Ross stochastic interest rate process, it is given as follows:

$$\begin{aligned}
dS_t &= r_t S_t dt + \sqrt{\sigma_t} S_t dW_t^x, \quad S_0 > 0, \\
d\sigma_t &= k(\bar{\sigma} - \sigma_t)dt + \gamma\sqrt{\sigma_t}dW_t^\sigma, \quad \sigma_0 > 0, \\
dr_t &= \lambda(\theta_t - r_t)dt + \eta\sqrt{r_t}dW_t^r, \quad r_0 > 0,
\end{aligned} \tag{2.1}$$

where  $k > 0$  determines the speed of adjustment of the volatility towards its theoretical mean,  $\bar{\sigma} > 0, \gamma > 0$  is the second-order volatility, i.e. the variance of the volatility,  $\lambda > 0$  determines the speed of mean reversion for the interest rate process,  $\theta_t$  is the interest rate term-structure and  $\eta$  controls the volatility of the interest rate, the correlations are given by  $dW_t^x dW_t^\sigma = \rho_{x,\sigma} dt, dW_t^x dW_t^r = \rho_{x,r} dt, dW_t^\sigma dW_t^r = \rho_{\sigma,r} dt$ . Note that we assume independence between the instantaneous short rate,  $r_t$ , and the volatility process  $\sigma_t$ , i.e.  $\rho_{\sigma,r} = 0$ .

To compute the price of an European option in such a model, [3] proposed to reformulat the HCIR model in the following way:

$$\begin{cases} dS_t = r_t S_t dt + \sqrt{\sigma_t} S_t dW_t^x + \Omega_t \sqrt{r_t} S_t dW_t^r + \Delta \sqrt{\sigma_t} S_t dW_t^\sigma, & S_0 > 0, \\ d\sigma_t = k(\bar{\sigma} - \sigma_t)dt + \gamma\sqrt{\sigma_t}dW_t^\sigma, & \sigma_0 > 0, \\ dr_t = \lambda(\theta_t - r_t)dt + \eta\sqrt{r_t}dW_t^r, & r_0 > 0, \end{cases} \tag{2.2}$$

with  $dW_t^x dW_t^\sigma = \hat{\rho}_{x,\sigma} dt, dW_t^x dW_t^r = 0, dW_t^\sigma dW_t^r = 0$ .

Let  $x_t = \log(S_t)$  we have

$$\begin{aligned}
dx_t &= \left[ r_t - \frac{1}{2} \left( \Omega_t^2 r_t^{2p} + \sigma_t (1 + \Delta^2 + 2\hat{\rho}_{x,\sigma} \Delta) \right) \right] dt + \Omega_t \sqrt{r_t} S_t dW_t^r + \Delta \sqrt{\sigma_t} S_t dW_t^\sigma \\
&= \left( r_t - \frac{1}{2} \sigma_t \right) + \Omega_t \sqrt{r_t} S_t dW_t^r + \Delta \sqrt{\sigma_t} S_t dW_t^\sigma.
\end{aligned}$$

The pricing method is derived based on the SDE system  $\mathbf{X}_t^*$ , in fact the SDE system  $\mathbf{X}_t^* := [r_t, \sigma_t, x_t]^T$  is coordinated with that of  $\mathbf{X}_t := [S_t, \sigma_t, r_t]^T$  in the sense that

$$\Omega_t = \rho_{x,r} r_t^{-1/2} \sqrt{\sigma_t}, \hat{\rho}_{x,\sigma}^2 = \rho_{x,\sigma}^2 + \rho_{x,r}^2, \Delta = \rho_{x,\sigma} - \hat{\rho}_{x,\sigma}. \tag{2.3}$$

### 3 Sketch of the Pricing Method

The idea of the pricing method proposed by [3] is by fitting the model into a class of affine diffusion processes to have a closed form solution of the characteristic function of the discount logarithm of equity process and then use Fourier-Cosine expansion to derive the option price.

### 3.1 Closed form Solutions of Discount Characteristic Function by two approximations

From [1], if a process is a affine diffusion process(AD), a closed form solution of the characteristic function exists. The AD processes is given by a system of SDEs

$$d\mathbf{X}_t = \mu(\mathbf{X}_t)dt + \sigma(\mathbf{X}_t)dW_t \quad (3.1)$$

satisfies:

$$\begin{aligned} \mu(\mathbf{X}_t) &= a_0 + a_1\mathbf{X}_t, \quad \text{for any } (a_0, a_1) \in \mathbf{R}^n \times \mathbf{R}^{n \times n}, \\ \sigma(\mathbf{X}_t)\sigma(\mathbf{X}_t)^T &= (c_0)_{ij} + (c_1)_{ij}^T\mathbf{X}_t, \quad \text{for arbitrary } (c_0, c_1) \in \mathbf{R}^{n \times n} \times \mathbf{R}^{n \times n \times n}, \\ r(\mathbf{X}_t) &= r_0 + r_1^T\mathbf{X}_t, \quad \text{for } (r_0, r_1) \in \mathbf{R} \times \mathbf{R}^n, \end{aligned} \quad (3.2)$$

for any  $i, j = 1, \dots, n$  with  $r(\mathbf{X}_t)$  being an interest rate component.

Then, by [1], the discounted characteristic function (ChF) of (3.2) is of the following form:

$$\phi(\mathbf{u}, \mathbf{X}_t, t, T) = \mathbb{E}^{\mathbb{Q}} \left( \exp \left( - \int_t^T r_s ds + i\mathbf{u}^T \mathbf{X}_T \right) | \mathcal{F} + t \right) = e^{\mathbf{A}(\mathbf{u}, \tau) + \mathbf{B}^T(\mathbf{u}, \tau) \mathbf{X}_t},$$

where the expectation is taken under the risk-neutral measure,  $\mathbb{Q}$ . For a time lag,  $\tau := T - t$ , the coefficients  $\mathbf{A}(\mathbf{u}, \tau)$  and  $\mathbf{B}^T(\mathbf{u}, \tau)$  have to satisfy the following complex-valued ordinary differential equation (ODEs):

$$\begin{cases} \frac{d}{d\tau} \mathbf{B}(\mathbf{u}, \tau) = -r_1 + a_1^T \mathbf{B} + \frac{1}{2} \mathbf{B}^T c_1 \mathbf{B}, \\ \frac{d}{d\tau} \mathbf{A}(\mathbf{u}, \tau) = -r_0 + \mathbf{B}^T a_0 + \frac{1}{2} \mathbf{B}^T c_0 \mathbf{B}, \end{cases} \quad (3.3)$$

with  $a_i, c_i, r_i, i = 0, 1$  as in (3.2).

For the HCIR model, the symmetric instantaneous covariance matrix in (3.2) for the SDE system  $\mathbf{X}_t^*$  is given by:

$$\Sigma := \sigma(\mathbf{X}_t^*)\sigma(\mathbf{X}_t^*)^T = \begin{bmatrix} \eta^2 r_t & 0 & \eta \Omega_t r_t \\ * & \gamma^2 \sigma_t & \gamma \hat{\rho}_{x, \sigma} \sigma_t + \gamma \Delta \sigma_t \\ * & * & \sigma_t + \Omega_t^2 r_t + \Delta^2 \sigma_t + 2 \hat{\rho}_{x, \sigma} \Delta \sigma_t \end{bmatrix}. \quad (3.4)$$

To make the HCIR model affine we need to approximate the non-affine terms  $\sqrt{\sigma_t} \sqrt{r_t}$  in  $\Sigma_{(1,3)} = \eta \Omega_t r_t = \eta \rho_{x, r} \sqrt{\sigma_t} \sqrt{r_t}$  of the instantaneous covariance matrix.  $\Sigma_{(3,3)}$  does not seems to be of the affine form, but in fact by (2.3), it equals  $\Sigma_{(3,3)} = \sigma_t$ .

[3] proposed two ways to approximate the non-affine terms, one is deterministic approximation, which approximates  $\sqrt{\sigma_t}$  and  $\sqrt{r_t}$  by their expectations and the other is a stochastic approximation by a normal distributed random variable. Here we just provide the approximations of the two methods, for the detail of derivation, we refers the reader to [3].

**Lemma 3.1.** *The expectation,  $\mathbb{E}(\sqrt{\sigma_t})$ , with stochastic process given by equation (2.1) can be approximated by*

$$\mathbb{E}(\sigma_t) \approx a + be^{-ct}, \quad (3.5)$$

where

$$a = \sqrt{\bar{\sigma} - \frac{\gamma^2}{8k}}, b = \sqrt{\sigma_0} - a, c = -\log(b^{-1}(\Lambda(1) - a)),$$

with

$$\Lambda(t) := \sqrt{c(t)(\lambda(t) - t) + c(t)d + \frac{c(t)d}{2(d + \lambda(t))}},$$

$$c(t) = \frac{1}{4k}\gamma^2(1 - e^{-kt}), d = \frac{4k\bar{\sigma}}{\gamma^2}, \lambda(t) = \frac{4k\sigma_0 e^{-kt}}{\gamma^2(1 - e^{-kt})},$$

$k, \bar{\sigma}, \gamma$  and  $\sigma_0$  are the parameters given in (2.1).

**Remarks 3.2.** *Note that the interest rate process  $r_t$  is given by an equation with the similar structure of  $\sigma_t$ , then the expectation of its square root can be approximated by the same way.*

By the approximation given above, the HCIR model can be fitted into the AD class, thus we have a closed form solution of the discount ChF.

**Theorem 3.3.** *The discount ChF of  $X_t^*$  is given by*

$$\phi(u, \mathbf{X}_t, \tau) = \exp(A(u, \tau) + B_x(u, \tau)x_t + B_\sigma(u, \tau)\sigma_t + B_r(u, \tau)r_t), \quad (3.6)$$

where

$$\begin{aligned} B_x(u, \tau) &= iu, \\ B_r(u, \tau) &= \frac{1 - e^{-D_1\tau}}{\eta^2(1 - G_1 e^{-D_1\tau})}(\lambda - D_1), \\ B_\sigma(u, \tau) &= \frac{1 - e^{-D_2\tau}}{\gamma^2(1 - G_2 e^{-D_2\tau})}(k - \gamma\zeta iu - D_2), \end{aligned}$$

and

$$A(u, \tau) = \int_0^\tau (k\bar{\sigma}B_\sigma(u, s) + \lambda\theta B_r(u, s) + \rho_{x,r}\eta iu \mathbb{E}(\sqrt{\sigma_{T-s}})\mathbb{E}(\sqrt{r_{T-s}})B_r(u, s))ds,$$

with  $\zeta = \hat{\rho}_{x,\sigma} + \Delta$ ,  $D_1 = \sqrt{\lambda^2 + 2\eta^2(1 - iu)}$ ,  $D_2 = \sqrt{(\gamma\zeta iu - k)^2 - (iu - 1)iu\gamma^2}$ ,  $D_1 = \frac{\lambda - D_1}{\lambda + D_1}$ , and  $G_2 = \frac{k - \gamma\zeta iu - D_2}{k - \gamma\zeta iu + D_2}$ .

For the stochastic approximation we consider the SDE system of  $X_t := [x_t, \sigma_t m r_t, v_t, R_t, z_t]^T$  with  $R_t = \sqrt{r_t}$ ,  $v_t = \sqrt{\sigma_t}$ ,  $z_t = v_t R_t$ . Then for  $X_t$  the discount ChF is given by the following theorem.

**Theorem 3.4.** *The discount ChF of  $X_t$  is given by*

$$\phi(u, \mathbf{X}_t, \tau) = \exp(A(u, \tau) + B_x(u, \tau)x_t + B_\sigma(u, \tau)\sigma_t + B_r(u, \tau)r_t + B_v(u, \tau)v_t + B_R(u, \tau)R_t + B_z(u, \tau)z_t), \quad (3.7)$$

where  $B_x(u, \tau) := B_x, B_\sigma(u, \tau) := B_\sigma, B_r(u, \tau) := B_r, B_v(u, \tau) := B_v, B_R(u, \tau) := B_R, B_z(u, \tau) := B_z$  can be solved from the following ODEs:

$$\begin{aligned} \frac{dB_x}{d\tau} &= 0, \\ \frac{dB_r}{d\tau} &= -1 + B_x - \lambda B_r + \frac{1}{2}\eta^2 B_r^2 + \frac{1}{2}(\psi_t^v)^2 B_z^2, \\ \frac{dB_R}{d\tau} &= \mu_t^v B_z + \psi_t^R \eta B_r B_R + (\psi_t^v)^2 B_v B_z, \\ \frac{dB_z}{d\tau} &= \eta \rho_{x,r} B_x B_r + \zeta \psi_t^v B_x B_z + \gamma \psi_t^v B_\sigma B_z + \eta \psi_t^R B_r B_z, \\ \frac{dA}{d\tau} &= k\bar{\sigma} B_\sigma + \lambda \theta B_r + \mu_t^v B_v + \mu_t^R B_R + \frac{1}{2}(\psi_t^v)^2 B_v^2 + \frac{1}{2}(\psi_t^R)^2 B_R^2, \\ \frac{dB_v}{d\tau} &= \mu_t^R B_z + \psi_t^v \zeta B_x B_v + \gamma \psi_t^v B_\sigma B_v + \rho_{x,r} \psi_t^R B_x B_R + (\psi_t^R)^2 B_r B_z, \end{aligned}$$

and

$$\frac{dB_\sigma}{d\tau} = \frac{1}{2}B_x(B_x - 1) - k B_\sigma + \gamma \zeta B_x B_\sigma + \frac{1}{2}\gamma^2 B_\sigma^2 + \rho_{x,r} \psi_t^R B_x B_z + \frac{1}{2}(\psi_t^R)^2 B_x^2, \quad (3.8)$$

with the boundary conditions:  $B_x(u, 0) = iu, B_r(u, 0) = 0, B_R(u, 0) = 0, B_z(u, 0) = 0, B_\sigma(u, 0) = 0, B_v(u, 0) = 0$ , and  $A(u, 0) = 0$ , where  $\mu_t^v, \mu_t^R, \psi_t^v, \psi_t^R$  are specified in , and  $\zeta = \hat{\rho}_{x,\sigma} + \Delta$ .

In the implementation of the option pricing, we use Lunge-Kutta algorithm to solve the coefficient functions  $B_x, B_\sigma, B_r, B_v, B_R, B_z$ .

### 3.2 Pricing Option by Fourier-Cosine expansion

With the closed form of characteristic function provided above, the European price can be derived by Fourier-Cosine expansion. Denote by  $v(x, t_0)$  the present value of option price at time  $t_0$  with initial option value  $x$ , it can be expressed as risk neutral valuation formula:

$$v(x, t_0) = \mathbb{E}^{\mathbb{Q}}[v(y, T)|x] = \int_{\mathbb{R}} v(y, T) f(y|x) dy,$$

where  $\mathbb{E}^{\mathbb{Q}}[\cdot]$  is the expectation operator under risk-neutral measure  $\mathbb{Q}$ ,  $x$  and  $y$  are states variables at time  $t_0$  and  $T$ , respectively,  $f(y|x)$  is the probability density of  $y$  given  $x$ , and  $r$  is the risk-neutral interest rate.

Firstly, we truncate the infinite integration range without losing significant accuracy to  $[amb] \in \mathbb{R}$ , and we obtain approximation  $v_1$  :

$$v_1(x, t_0) = \int_a^b v(y, T) f(y|x) dy, \quad (3.9)$$

where

$$[a, b] := \left[ c_1 - L\sqrt{c_2 + \sqrt{c_4}}, c_1 + L\sqrt{c_2 + \sqrt{c_4}} \right], \quad (3.10)$$

with  $L = 10$  and  $c_n$  denotes the  $n$ -th cumulant of  $\ln(S_T/K)$ .

Secondly, we replace the density by its cosine expansion in  $y$ ,

$$f(y|x) = \sum_{k=0}^{+\infty} A_k(x) \cos \left( k\pi \frac{y-a}{b-a} \right), \quad (3.11)$$

where the summation  $\Sigma$  here with the first term is weighted by one-half and so thus for the following summation and

$$A_k(x) := \frac{2}{b-a} \int_a^b f(y|x) \cos \left( k\pi \frac{y-a}{b-a} \right) dy \equiv \frac{2}{b-a} \operatorname{Re} \left\{ \phi \left( \frac{k\pi}{b-a} \right) \exp \left( -i \frac{ka\pi}{b-a} \right) \right\}, \quad (3.12)$$

where the second equation is obtained by comparing the cosine coefficient  $A_k$  of  $f(y|x)$  with the definition of discount characteristic function of the state variables.

So that

$$v_1(x, t_0) = \frac{1}{2}(b-a) \sum_{k=0}^{N-1} A_k(x) V_k, \quad (3.13)$$

where

$$V_k := \frac{2}{b-a} \int_a^b v(y, T) \sum_{k=0}^{+\infty} A_k(x) \cos(k\pi \frac{y-a}{b-a}) dy. \quad (3.14)$$

Interchange the summation and integration, we have

$$v_1(x, t_0) = \frac{1}{2}(b-a) \sum_{k=0}^{+\infty} A_k(x) V_k \approx \frac{1}{2}(b-a) \sum_{k=0}^{N-1} A_k(x) V_k, \quad (3.15)$$

with  $V_k := \frac{2}{b-a} \int_a^b v(y, T) \cos(k\pi \frac{y-a}{b-a}) dy$ .

Then replacing (3.12) of  $A_k$  in (3.15), we have

$$v(x, t_0) \approx \sum_{k=0}^{N-1} \operatorname{Re} \left\{ \phi \left( \frac{k\pi}{b-a}; x \right) e^{-ik\pi \frac{a}{b-a}} V_k \right\}, \quad (3.16)$$

which is the COS formula for general underlying processes.

At last, we just need to determine  $V_k$  in the above COS formula, for a call option,  $V_k$  is given by

$$V_k = \frac{2}{b-a} K(\xi_l(0, b) - \phi_k(0, b)), \quad (3.17)$$

where

$$\begin{aligned}\xi_k(c, d) &:= \frac{1}{1 + \left(\frac{k\pi}{b-a}\right)^2} \left[ \cos\left(k\pi \frac{d-a}{b-a}\right) e^d - \cos\left(k\pi \frac{c-a}{b-a}\right) e^c \right. \\ &\quad \left. + \frac{k\pi}{b-a} \sin\left(k\pi \frac{d-a}{b-a}\right) e^d - \frac{k\pi}{b-a} \sin\left(k\pi \frac{c-a}{b-a}\right) e^c \right] \\ \phi_k(c, d) &:= \begin{cases} \left[ \sin\left(k\pi \frac{d-a}{b-a}\right) - \sin\left(k\pi \frac{c-a}{b-a}\right) \right] \frac{b-a}{k\pi}, & k \neq 0, \\ (d-c), & k = 0. \end{cases}\end{aligned}$$

We refers to [2] for more details.

## 4 Program Manual

We implement the European option pricing by Fourier Cosine expansion by using two types of approximation to derive the closed form solution of characteristic functions. The program HAS TO work with the pnl library.

### Included files:

The program directory contains the files:

this documentation file “docu.pdf”

“HESTON\_COSINE.c”

“HESTON\_COSINE”

“Makefile”

### Compile and run program:

By excute “make” under Linux, the file “HESTON\_COSINE.c” is compiled and an executable file “HESTON\_COSINE” will be regenerated.

### Model Parameters:

kv:  $k$  in model (2.1)

vbar:  $\bar{\sigma}$  in model (2.1)

sigmav:  $\gamma$  in model (2.1)

sigma0: the initial value of  $\sigma_t$

kr:  $\lambda$  in model (2.1)

rbar:  $\theta$  in model (2.1)

sigmar:  $\eta$  in model (2.1)

r0: the initial value of interest rate  $r_t$

rho12:  $\rho_{x,\sigma}$  in model (2.1)

rho13:  $\rho_{x,r}$  in model (2.1)

rho23:  $\rho_{\sigma,r}$  in model (2.1), note that our method works well only when  $\rho_{23} = 0$ .

### Parameters of the product:

S0: stock price at the initial time

K: strike of the American option

T: maturity of the American option, the expansion asymptotic works well for small maturity.

**Flags to choose products and approximation methods :**

approximation\_flag: approximation method flag with 0 for deterministic approximation, 1 for stochastic approximation

callput\_flag: callput flag: 0 for call, 1 for put

**Parameters for COSINE method:**

N: discrete steps in the integration range  $N$  in (3.16)

L: parameter in the truncate bound of  $[a, b]$  as given in (3.10).

## References

- [1] Duffie D., Pan, J., Singleton, K., 1999, Transform analysis and asset pricing for affine jump-diffusions. 3
- [2] Fang, F., Oosterlee, C.W., 2009. A novel pricing method for European options based on Fourier-Cosine series expansions. 7
- [3] Grzestek, A.L., Oosterlee, C.W., 2010, On the Heston model with stochastic interest rates. Preprint. 1, 2, 3