Premia 14
DYNAMIC TESTS

Contents

1 About Dynamic Hedging

Firstable let's talk about what dynamic hedging consists in.

It's a technique of portfolio insurance or position risk management in which an option-like return pattern is created by increasing or reducing the position in the underlying to simulate the delta change in value of an option position. Dynamic hedging relies on liquid and reasonably continuous markets.

Ideally a continuous market is a market characterized by great depth and liquididity and by small price changes between successive transactions. In other words : the spot's trajectory is assumed to be continuous. Many financial models assume markets are continuous, but care must be taken that discontinuities do not lead to misapplication of a model.

By elaboring the Dynamic Tests we address the question of hedging options discretely in time. The Black-Scholes analysis requires continuous hedging, which is possible in theory but impossible-even undesirable-in practice. The simpliest model for discrete hedging is to rehedge at fixed intervals of time h ; a strategy commonly used with h ranging from one day to one week. In the tests we will consider in detail the errors in following a pure Black-Scholes hedging strategy in discrete time.

To fix ideas, we shall take the point of view of the seller of the option. Consider for example the case of a european call option with maturity T. The picture is the following : we sell the option at time 0 at price C_0, that is we receive at time 0 this amount of money, but in turn it is mandatory for us to pay at time T the pay-off $(S_T - K)^+$, which may be very high depending on the movements of the underlying.

The idea is to go to the underlying market to buy some shares in order to hedge the possibility of a high increase of the underlying. So the very possibility to trade options, at least in a safe or quite dafe manner, is closely related to the access to the underlying market, or more generally to hedging instruments. The hedge may be performed dynamically to balance in a required way the option pay-off.

2 Tools needed in Dynamic Tests

2.1 Generating the value of the spot

We assume the value of the spot is the solution of the S.D.E. :

$dS_t = S_t\left(\mu dt + \sigma dB_t\right)$, $S_0 = s_0$.

This solution is : $S_t = s_0 \times e^{\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right)}$, so we deduce the formula we use in the routines : $S_{t+h} = S_t \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)h + \sigma\left(B_{t+h} - B_t\right)\right)$ where $B_{t+h} - B_t = \sqrt{h}X$ and $X \rightsquigarrow N(0,1)$.

With this method, we simulate the new value of the spot, step by step, multiplying the older spot by $\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)h + \sigma\sqrt{h}X\right)$ with $X \rightsquigarrow N(0,1)$.

2.2 Using a Brownian Bridge

We found interesting to be able to filter some noteworthy paths of the spot, selecting for example paths passing by a wanted spot's value at a wanted moment. The aim of this effort is to observe the behaviour of the pricing routines in extreme situations of the spot's trajectory.

A Brownian Bridge is a centered Gaussian process defined on $T = [0,1]$ and with covariance $\Gamma(s,t) = s(1-t)$ on $(s \leq t)$. The easiest way to prove that Γ is a covariance is to observe that for the process $X_t = B_t - tB_1$ where B is a Brownian Motion, $E\left[X_s X_t\right] = s(1-t)$ for $s \leq t$. This gives us also immediately a continuous version of the Brownian Bridge. We observe that $X_1 = 0$ a.s. hence all the paths go a.s. from 0 at time 0 to 0 at time 1 ; this is the reason for the name given to this process. Naturally, the notion of Bridge may be extended to higher dimensions and to intervals over than $[0,1]$.

In our case, we want the path of the spot to pass at S_{T1} at time T_1. For this we would like to replace the Brownian motion in the last formula by a Brownian Bridge in aim to the path passes by the wanted target point. So we set $X_{x,y}^{0,T1}(t) = \frac{t}{T1}(y-x) + x + B_t - \frac{t}{T1}B_{T1}$ with $T1$: time target, y: brownian bridge's target value, x: brownian bridge's starting value and B a brownian motion ; we've also defined a Brownian Bridge which path goes from x at time 0 to y at time $T1$.

We set $X_{x,y}^{0,T1}(0) = x = 0$ and $X_{x,y}^{0,T1}(T1) = y = \frac{1}{\sigma}\left(\ln\left(\frac{S_{T1}}{S_0}\right) - \left(\mu - \frac{\sigma^2}{2}\right)T1\right)$ because we want : $S_{T1} = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)T1 + \sigma X_{x,y}^{0,T1}(T1)\right)$.

We obtain finally $S_{t+h} = S_t \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)h + \sigma\left(X_{x,y}^{0,T1}(t+h) - X_{x,y}^{0,T1}(t)\right)\right)$. Now, we would like to simulate the quantity : $\Gamma = X_{x,y}^{0,T1}(t+h) - X_{x,y}^{0,T1}(t)$. After all caculs we found that $\Gamma \rightsquigarrow N\left(\left(y - X_{x,y}^{0,T1}(t)\right)\frac{h}{T1-t}; h\left(1 - \frac{h}{T1-t}\right)\right)$.

It gives us the formula dealing the new value of the spot :

$$S_{t+h} = S_t \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)h\right)\exp\left(\sigma\left(y - X_{x,y}^{0,T1}(t)\frac{h}{T1-t} + \sqrt{h\left(1 - \frac{h}{T1-t}\right)}Z\right)\right)$$

where $Z \rightsquigarrow N(0,1)$.

We note that this last formula is right only for $(t \leq T1)$, for $(t > T1)$ we use the formula given in section 2.1.

3 What Dynamic Tests consist in

If we still take the point of view of the seller of the option, we have at our disposal the following hedging formula giving us the expression of our Profit&Loss : $P\&L = C_0 + \sum_{n=0}^{N-1}\Delta_n(S_{nh})\left(\widetilde{S}_{(n+1)h} - \widetilde{S}_{nh}\right) - \widetilde{\varphi}(S_T)$ where C_0 is the selling price, Δ_n is the hedge ratio, $\widetilde{S}_{nh}$ is the actualized value of the spot at time nh, $\widetilde{\varphi}(S_T)$ the pay-off and N the number of hedges before exercise. It means that we sell the option at time 0 at price C_0, we hedge N times at regular time step h and we deal the pay-off at time $T = Nh$.

Indeed the market maker is not able to hedge continuously : it's impossible in practice. If he could he would perfectly replicate the option and the P&L would equal 0. That's why we elaborated Dynamic Tests which simulate the market maker's behaviour hedging at discrete time (for example every day at closing price).

By doing this, he produces an hedging error, also the P&L never equals 0. The Dynamic Tests simulate several spot's trajectories with continuous formulas given before at discrete time, and calculate for each path the corresponding P&L with the discrete hedging formula. After this, we calculate some statistics on these P&L like the mean, the square deviation, and we determine the maximum and the minimum of these found P&L. We keep also some relevant spot's trajectories generating extremal P&L positions in aim to output some graphics. With these graphic outputs, we can observe the behaviour of the hedging error done by the market maker hedging at discrete time with continuous models.

Using the brownian bridge, we can be interested in some critical situations : we can decide to observe the behaviour of the pricing routines when the spot's trajectory passes at a critical point ; for example we can impose the spot to reach the limit, in the case of a barrier option, at an interesting date like the maturity, because we know that can cause numerical problems due to the expression of the Δ at this position.

A Dynamic Test gives us two kinds of informations : it evaluates the hedging error made by the market maker by hedging at discrete time on continuous models (like it is in reality), and it detects some problems of the pricing routines behaviour. That's why elaborating these Dynamic Tests was very interesting and relevant.