# mc_lookbackmin_andersen

Input parameters:

- Number of iterations $N$
- Generator_Type
- Confidence Value

Output parameters:

- Price $P$
- Error Price $\sigma_P$
- Delta $\delta$
- Error delta $\sigma_\delta$
- Confidence Interval: [Inf Price, Sup Price]
- Confidence Interval: [Inf Delta, Sup Delta]

**Description** : Computation for a Lookback Option on Minimum of its Price (see [1]) and its Delta with the Standard Monte Carlo Simulation or Quasi-Monte Carlo simulation. For MC simulation, this method also provides estimations for the integration error and confidence intervals.

The underlying asset price evolves according to the Black and Scholes model, that is:

$$dS_u = S_u((r-d)du + \sigma dB_u), \quad S_{T-t} = s$$

then

$$S_T = s \exp\left((r - d - \frac{\sigma^2}{2})t\right) \exp(\sigma B_t)$$

$S_T$ denotes the spot at maturity $T$, $s$ is the initial spot, $t$ the time to maturity.

We note $m_T = \min_{[T-t,T]}(S_u)$ the minimum reached before maturity.
The Price of an option is:

$$P = E\left[\exp(-rt)f(K, S_T, m_T)\right]$$

where $f$ denotes the payoff of the option and $K$ the strike.
The Delta is given by:

$$\delta = \frac{\partial}{\partial s}E[\exp(-rt)f(K, S_T, m_T)]$$

Estimators are expressed as:

$$\widetilde{P} = \frac{1}{N}\exp(-rt)\sum_{i=1}^{N} P(i)$$

$$\widetilde{\delta} = \frac{1}{N}\exp(-rt)\sum_{i=1}^{N}\frac{\partial}{\partial s}P(i) = \frac{1}{N}\exp(-rt)\sum_{i=1}^{N}\delta(i)$$

The values for $P(i)$ and $\delta(i)$ are detailed for each option.

- **Put Fixed Euro**: The payoff is $(K - \min_{[T-t,T]}(S_u))^+$

$$P(i) = (K - m_T(i))^+$$

$$\delta_i = \begin{cases} -\frac{\partial m_T(i)}{\partial s} = -\frac{m_T(i)}{s} & \text{if} \quad P(i) > 0 \\ 0 & \text{otherwise} \end{cases}$$

- **Call Floating Euro**: The payoff is $(S_T - \min_{[T-t,T]}(S_u))$

$$P(i) = S_T(i) - m_T(i)$$

$$\delta_i = \frac{\partial S_T(i)}{\partial s} - \frac{\partial m_T(i)}{\partial s} = \frac{S_T(i) - m_T(i)}{s} = \frac{P(i)}{s}$$

## Simulation of the minimum

The conditional law induced by $m_T$, given the observation $S_T$, has a very simple form, so that we can easily simulate the minimum of $S$ over the time interval $[T-t, T]$.
Setting $W_u = \ln S_u$, then obviously $m_T$ is the exponential of the minimum

of $W$ and the conditional probability distribution function of the minimum of $W$, given $W_{T-t} = w_1$ and $W_T = w_2$, is:

$$G_{min}(w; w_1, w_2) = \begin{cases} \exp\left[-\frac{2}{\sigma^2 t}(w - w_1)(w - w_2)\right] & \text{if } w \leq \min(w_1, w_2) \\ 1 & \text{otherwise} \end{cases}$$

In the Monte Carlo algorithm, the associated inverse function $G^{-1}_{min}$ is used in order to simulate values for $m_T$. We have the following expression:

$$G^{-1}_{min}(y; w_1, w_2) = \frac{1}{2}\left(w_1 + w_2 - \sqrt{(w_1 - w_2)^2 - 2\sigma^2 t \ln y}\right)$$

where $y$ is uniform on $[0, 1]$.

Then, at step $i$, $m_T(i)$ is simulated as follows:
- $S_T(i)$ is generated as $s \exp\left((r - d - \frac{\sigma^2}{2})t\right) \exp(\sigma B_t(i))$ with $B_t(i) = \sqrt{t} g_i$ and $g_i$ is a standard gaussian variable;
- $y(i)$ is generated as a uniform variable on $[0, 1]$, independent on $g_i$;
- $w_{T-t} = \ln s$ and $w_T(i) = \ln S_T(i)$;
- $W_T(i)$ is computed as $G^{-1}_{min}(y(i); w_{T-t}, w_T(i))$;
- Finally, $m_T(i) = \exp(W_T(i))$.

**Remark:** Algorithm to simulate independent $g_i$ and $y_i$ is not the same depending on whether we realize a MC simulation or a QMC one. In this second case, we have to use a two-dimensional low-discrepancy sequence. Details about this point are given in the presentation of simulation method, especially in the section about simulation of random variables. The both implementations are described in the next point.

## Algorithm

/* Value to construct the confidence interval */
For example if the confidence value is equal to 95% then the value $z_\alpha$ used to construct the confidence interval is 1.96.
/* Initialisation */
/* Size of the random vector we need in the simulation */
Value for the dimension is 2 because we will need a two-dimensional low-discrepancy sequence to generate independent variables.
/*Median forward stock and delta values*/
Computation of intermediate values we use several times in the program.

- /* Monte Carlo sampling */

Initialization of the simulation: generator type, dimension, size $N$ of the sample.
/* Test after initialization for the generator */
Test if the dimension of the simulation is compatible with the selected generator. (See remarks on QMC simulation, especially on dimension of low-discrepancy sequences). For Quasi Monte Carlo in this model, we never have any problem with the dimension, fixed to 2 at the beginning of the programm.

Definition of a parameter which exprimes if we realize a MC or QMC simulation. Some differences then appear in the algorithm for simulation of random vectors and in results of the simulation.

/* We test if simulation is MC or QMC.
This involves two parts in the program because simulation for random vector must be called from different functions */
It is more efficient to test only once if we do a MC or QMC simulation and then to have two parts in the program, instead of testing it at each iteration inside the algorithm.

/* **MC simulation case** */
/* Begin N iterations */

/* For MC simulation, generation of two independent variables, a gaussian one and a uniform one, can be realized with the same preudo random number generator without problem of independence*/
- /* Simulation of a gaussian variable according to the generator type (MC) */
We recall that for a MC simulation, we use the Gauss-Abramovitz algorithm.
- /* Second variable: uniform to generate the minimum */
This variable is independent on the first gaussian one.
Computation of $S_T(s)$ and $\min_{[T-t,T]} S_u(s)$ from the initial spot value $s$.
The minimum value is simulated according to the conditional law induced by $s$ and $S_T$. The different steps for the simulation are described previously.

- /* Price and Delta */
/* PutFixedEuro */
/* CallFloatingEuro */
At the iteration $i$, computation of $P(i)$ and $\delta(i)$ for the option with formula previously expressed.

/*Sum*/
Computation of the sums $\sum P_i$ and $\sum \delta_i$ for the mean price and the mean delta.

/*Sum of squares*/
Computation of the sums $\sum P_i^2$ and $\sum \delta_i^2$ necessary for the variance price and the variance delta estimations.

/* End N iterations */

- /* Price */

The price estimator is:

$$P = \frac{1}{N} \exp(-rt) \sum_{i=1}^{N} P_i$$

The error estimator is $\sigma_P$ with :

$$\sigma_P^2 = \frac{1}{N-1} \left( \frac{1}{N} \exp(-2rt) \sum_{i=1}^{N} P_i^2 - P^2 \right)$$

- /* Price Confidence Interval */

The confidence interval is given as:

$$IC_P = [P - z_\alpha \sigma_P; P + z_\alpha \sigma_P]$$

with $z_\alpha$ computed from the confidence value.

- /* Delta */

$$\delta = \frac{1}{N} \exp(-rt) \sum_{i=1}^{N} \delta_i$$

The error estimator is $\sigma_\delta$ with:

$$\sigma_\delta^2 = \frac{1}{N-1} \left( \frac{1}{N} \exp(-2rt) \sum_{i=1}^{N} \delta_i^2 - \delta^2 \right)$$

- /* Delta Confidence Interval */

The confidence interval is given as:

$$IC_\delta = [\delta - z_\alpha \sigma_\delta; \delta + z_\alpha \sigma_\delta]$$

with $z_\alpha$ computed from the confidence value.

/* **QMC simulation** */
We just give a description of the points which differ from MC simulation.

Otherwise algorithm follows the same steps.
/* Begin N iterations */

/* In QMC simulation, to generate two independent variables, a gaussian one and a uniform one, we have to use a two-dimensional low-discrepancy sequence. Then we call the function 'D-uniform' with parameter simulation-dim=2.*/
- /* First variable transformed into a gaussian */
We recall that we use an inverse function for QMC simulation. We use the first coordinate of the selected low-discrepancy sequence.
- /* Second variable, uniform to generate the minimum */
This variable comes from the second dimension of the selected low-discrepancy sequence.

The algorithm goes on as for a MC simulation, except that we don't compute variance and confidence interval because they don't work for QMC simulation.

# References

[1] L.ANDERSON R.BROTHERTON-RATCLIFFE. Exact exotics. *Risk*, 9:85–89, Oct 1996. 1