Pricing of Inflation Indexed Derivatives *

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Abstract: This paper is based on the C++ implementation in PREMIA of pricing of inflation indexed derivatives. Especially we will consider pricing of inflation indexed caplets and swaps in the JY model and of caplets in a stochastic volatility model for forward consumer price indices.

Key words: Inflation indexed derivatives; JY model; Libor market model; caplets; swaps.

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1 Introduction and notations

As an indicator that reflects the change of purchase power, inflation is defined in terms of the percentage increments of a reference index, the Consumer Price Index (CPI), which is a representative basket of goods and services. Note that according to the set of goods and services, inflation is not uniformly.

To control the risk of the variations in the purchasing power of currency, European governments have been issuing inflation-indexed (II) bonds since the beginning of the 80's, nowadays II derivatives such as II zero-coupon swaps and II year-on-year swaps have been more and more popular. The pricing of the II derivatives based on a foreign-currency analogy, where the nominal and real rates and the CPI is interpreted as the "exchange rate" between the nominal and real economies, then the valuation of an inflation-indexed payoff becomes equivalent to that of a cross-currency interest rate derivative.

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Denoting by $I(t)$ the value of the CPI at time t, the reference basket of good and services can be bought with $I(0)$ units of currency at time $t = 0$, whereas at time $t = T$ one needs to spend $I(T)$. In real economy, all the payoff is scaled to the price of a set of reference goods and services. The real bond $P_r(t, T)$ is defined as the price at time t and in CPI units of a contract paying one unit of goods and services at T, then we have

$$P_r(T, T) = I(T).$$

While in nominal economy, the nominal bond $P_n(t, T)$ is the t-time price (say, in euros) of one euro in T. Based on the real and nominal bond, we can define the corresponding instantaneous forward rates, spot rate, forward interest rate and bank money account, respectively.

Given the future times T_{i-1} and T_i, the forward interest rates, at time t, are defined as

$$F_x(t; T_{i-1}, T_i) = \frac{P_x(t, T_{i-1}) - P_x(t, T_i)}{\tau_i P_x(t, T_i)}, \quad x \in \{n, r\}.$$

The nominal and real instantaneous forward rates at time t for maturity T are defined as

$$f_x(t, T) = \frac{\partial \ln P_x(t, T)}{\partial T}, \quad x \in \{n, r\}.$$

We then denote the nominal and real instantaneous short rates, respectively, by

$$n(t) = f_n(t, t) \quad \text{and} \quad r(t) = f_r(t, t).$$

And the two bank money account

$$B_x(t) = e^{\int_0^t x(s) ds}, \quad x \in \{n, r\}.$$

We denote by Q_n and Q_r the nominal and real risk-neutral measures, respectively, and by E_x the expectation associated with $Q_x, x \in \{n, r\}$. Finally, the forward CPI at time t for maturity T_i is denoted by $\mathcal{I}_i(t)$ and defined by

$$\mathcal{I}_i(t) := I(t) \frac{P_r(t, T_i)}{P_n(t, T_i)}.$$

2 Inflation indexed swaps and caplets(floorlets)

Now we introduce the three main traded II derivatives: zero-coupon swaps(ZCIIS), year-on-year swaps(YYIIS), and caplets/floorlets(IIC/IIF) here.

Given the tenor structure as $\{T_0, T_1, \cdots, T_n\}$ with $\tau_i := T_i - T_{i-1}$.

A ZCIIS is, at final time T_M, Party B pays Party A the fixed amount

$$N((K+1)^M - 1)$$

during time interval $[0, T_M]$, where K and N are, respectively, the contract fixed rate and nominal value, in particular, if $T_M = M$ years, K_M is assumed to be a yearly compounded rate, that is

$$K_M = (1+K)^M - 1.$$

In exchange for this fixed payment, Party A pays Party B, at the final time T_M, the floating amount

$$N\left(\frac{I(T_M)}{I_0} - 1\right),$$

i.e. the payoff of the ZCIIS at time T_M, denoted by $\mathrm{ZCIIS}(T_M; T_M, K)$, is

$$\mathrm{ZCIIS}(T_M; T_M, K) = N\left(\left(\frac{I(T_M)}{I(0)} - 1\right) - K_M\right).$$

A YYIIS, at each time T_i, is Party B pays Party A the fixed amount

$$N\varphi_i K,$$

where φ_i is the contract fixed-leg year fraction for the interval $[T_{i-1}, T_i]$, while Party A pays Party B the (floating) amount

$$N\psi_i\left(\frac{I(T_i)}{I(T_{i-1})} - 1\right),$$

where ψ_i is the floating-leg year fraction for the interval $[T_{i-1}, T_i]$, $T_0 := 0$ and N is again the contract nominal value, then the payoff of the YYIIS at time T_M, denote by $\mathrm{YYIIS}(T_i; T_{i-1}, T_i, K)$, is

$$\mathrm{YYIIS}(T_i; T_{i-1}, T_i, K) = N\left(\psi_i\left(\frac{I(T_i)}{I(T_{i-1})} - 1\right) - \varphi_i K\right).$$

For sake of simplicity, we set $\varphi_i = \psi = 1$ in the following.

An Inflation-Indexed Caplet (IIC) is a call option on the inflation rate implied by the CPI index. Analogously, an Inflation-Indexed Floorlet (IIF) is a put option on the same inflation rate. In formulas, at time T_i, the IICF payoff is

$$N\psi_i\left[\omega\left(\frac{I(T_i)}{I(T_{i-1})} - 1 - k\right)\right]^+$$

where k is the IICF strike, ψ_i is the contract year fraction for the interval $[T_{i-1}, T_i]$, N is the contract nominal value, and $\omega = 1$ for a caplet and $\omega = -1$ for a floorlet.

Now we derive the model-independent pricing formula of ZCIIS, which is not based on specific assumptions on the evolution of the interest rate market, but simply follow from the absence of arbitrage. Under assumption of absence of arbitrage, the price at $0 = T_0 < t < T_M$ of a ZCIIS maturing at $T_M = M$ years is,

$$\begin{aligned} \mathrm{ZCIIS}(t; T_M, K) &= NE^n \left[\frac{B_n(t)}{B_n(T_M)} \left(\frac{I(T_M)}{I(T_0)} - (1+K)^M \right) | \mathcal{F}_t \right] \\ &= N \left(\frac{1}{I(T_0)} E^n \left[\frac{B_n(t)}{B_n(T_M)} I(T_M) | \mathcal{F}_t \right] - P_n(t, T_M)(1+K)^M \right) \end{aligned} \tag{2.1}$$

where $\mathcal{F}$ denotes the ęÒ-algebra generated by the relevant underlying processes up to time t. Note that the term $E^n(B_n(t)/B_n(T_M) I(T_M)|\mathcal{F}_t)$ in the last equation is the nominal price at time t for a contract paying off one unit of the CPI index at bond maturity, which equals to the nominal price at time t of a real zero coupon bond, in fact, for $\forall t < T_M$, we have

$$E_n \left[\frac{B_n(t)}{B_n(T_M)} I(T_M) | \mathcal{F}_t \right] = I(t) E_r \left[\frac{B_r(t)}{B_r(T_M)} | \mathcal{F}_t \right] = I(t) P_r(t, T_M)$$

Therefore (2.1) becomes

$$\mathrm{ZCIIS}(t; T_M, K) = N \left(\frac{I(t)}{I(T_0)} P_r(t, T_M) - P_n(t, T_M)(1+K)^M \right)$$

which at time $t = 0$ simplifies to

$$\mathrm{ZCIIS}(0; T_M, K) = N \left(P_r(t, T_M) - P_n(t, T_M)(1+K)^M \right). \tag{2.2}$$

The market quotes values of the so-called Zero coupon swap rate $K = K(T_M)$ for some given maturities T_M, then with (2.2), we can get the real discount factor from the corresponding nominal one. The following table contains the market data for zero coupon swap rate and nominal discount factors in the second and third columns and get the real discount factors from (2.2) in the forth column.

$T_i(year)$	ZC swap rate (%)	$P_n(0,T)$	$P_r(0,T)$
1	2.1112	0.97701	0.99764
2	2.1875	0.94982	0.99183
3	2.2400	0.91835	0.98145
4	2.2775	0.88433	0.96769
5	2.2925	0.84862	0.95045
6	2.3000	0.81179	0.93046
7	2.3100	0.77460	0.90887
8	2.3200	0.73785	0.88644
9	2.32500	0.70218	0.86354
10	2.3350	0.66773	0.84109

Table 1-1 Market ZCIIS swap rate, market nominal discount factors and derived real discount factor via (2.2). Base on US CPI of date 03/11/2004.

3 The Jarrow and Yildrim model

Within the Heath-Jarrow-Morton framework, under the real-world probability space $(\omega, \mathcal{F}, P)$, with associated filtration $\mathcal{F}$, Jarrow and Yildrim proposed an evolution for the nominal and real instantaneous forward rates and for the CPIčž

$$\begin{aligned} df_n(t,T) &= \alpha_n(t,T)dt + \zeta_n(t,T)dZ_n(t) \\ df_r(t,T) &= \alpha_r(t,T)dt + \zeta_r(t,T)dZ_r(t) \\ dI(t) &= I(t)\mu^I(t)dt + \sigma_I I(t)dZ_I(t) \end{aligned} \tag{3.1}$$

where the three Brownian motion $Z_n(t), Z_r(t)$ and $Z_I(t)$ with correlations $\rho_{n,r}, \rho_{n,I}$ and $\rho_{r,I}$; α_n, α_r, μ are adapted process; ζ_n, ζ_r are deterministic functions; σ_I is a positive constant; Initial conditions $I(0) = I_0 > 0$ and $f_x(0,T), x \in \{n, r\}$ are given by the Market.

To ease the calculation of the derivatives' prices in the next sections, we choose to model the forward rate volatilities as

$$\zeta_n(t,T) = \sigma_n \exp\left(-a_n(T-t)\right), \quad \zeta_r(t,T) = \sigma_r \exp\left(-a_r(T-t)\right), \tag{3.2}$$

where $\sigma_n, \sigma_r, a_n, a_r$ are positive constants. Then integrate $f_x(t,T), x \in \{n, r\}$ in $[0,t]$, the Jorrow and Yildrim model can be rewritten as

$$\begin{aligned} dn(t) &= [\Theta_n(t) - a_n n(t)]dt + \sigma_n dZ'_n(t) \\ dr(t) &= [\Theta_r(t) - \rho_{r,I}\sigma^I\sigma_r - a_r r(t)]dt + \sigma_r dZ'_r(t) \\ dI(t) &= I(t)[n(t) - r(t)]dt + \sigma^I I(t)dZ'_I(t) \end{aligned} \tag{3.3}$$

where

$$\Theta_x(t) = \frac{\partial f_x(0,t)}{\partial T}|_{T=t} + a_x f_x(0,t) + \frac{\sigma_x^2}{2a_x}(1-e^{-2a_x t}), \quad x \in \{n, r\}.$$

Under assumption that both nominal and real spot rates are normally distributed under their respective risk-neutral measures, the real rate is proved to be an Ornstein-Uhlenbeck process under the nominal measure Q_n, and the inflation index $I(t), \forall t > 0$ is lognormally distributed under Q_n, then for $\forall t < T$,

$$I(T) = I(t)e^{\int_t^T [n(u)-r(u)]du - \frac{1}{2}(\sigma^I)^2(T-t) + \sigma^I(Z_I'(T)-Z_I'(t))}.$$

3.1 Pricing of YYIIS

Consider the floating leg of YYIIS in the Jarrow and Yildrim model,

$$\begin{aligned} YYIIS(t; T_{i-1}, T_i) &= E^n\left[\frac{B_n(t)}{B_n(T_{i-1})}P_r(T_{i-1}, T_i)|\mathcal{F}_t\right] \\ &= E^{i-1}\left[\frac{P_n(t, T_{i-1})}{P_n(T_{i-1}, T_{i-1})}P_r(T_{i-1}, T_i)|\mathcal{F}_t\right] \\ &= P_n(t, T_{i-1})A_r(T_{i-1}, T_i)E^{i-1}[e^{-\beta_r(T_{i-1},T_i)r(T_{i-1})}|\mathcal{F}_t] \end{aligned}$$

where

$$\beta_r(t,T) = \frac{1}{a_r}[1 - e^{a_r(T-t)}],$$

$$A_r(t,T) = \frac{P_r(0,T)}{P_r(0,t)}\exp\left\{\beta_r(t,T)f_r^M(0,t) - \frac{\sigma_r^2}{4a_r}(1-e^{-2a_r t})\beta_r(t,T)^2\right\}.$$

the second equation derived by switch to measure associated by $P_n(t, T_{i-1})$. As we note before, the real spot rate $r(T)$ conditionned to $t < T$ is an Ornstein Uhlenbeck process, his probability density function is Guassian. Hence, any change of measure will only affect his drift, that is his mean value, and not his gaussian character or its variance. It follows then that

$$\begin{aligned} YYIIS^{FL}(t; T_{i-1}, T_i) &= P_n(t, T_i)A_r(T_{i-1}, T_i)E^M\left[e^{-\beta_r(T_{i-1},T_i)r(T_{i-1})}|\mathcal{F}_t\right] \\ &= P_n(t, T_i)A_r(T_{i-1}, T_i)e^{-\beta_r(T_{i-1},T_i)m_r^t(T_{i-1}) + \frac{1}{2}\beta_r(T_{i-1},T_i)^2\nu_r^t(T_{i-1})} \end{aligned}$$

where $m_r^t(T_{i-1})$ and $(\nu_r)^t(T_{i-1})$ are the mean and variance of $r(T_{i-1})$ conditional on $\mathcal{F}_t$. To price YYIIS is then sufficient to compute the Q^{i-1} mean of $r(T_{i-1})|\mathcal{F}_t$. By change- of-measure theory and Girsanov theorem,

$$\frac{dQ^{i-1}}{dQ^n}|\mathcal{F}_t = P_n(0, T_{i-1})\exp\left\{-\sigma_n\int_0^t \beta_n(u, T_{i-1})dz_n'(u) + \frac{1}{2}\sigma_n^2\int_0^t \beta_n^2(u, T_{i-1})du\right\} \doteq \mathbb{Z}(t),$$

and the only thing which is left to do is to perform substitutions

$$\begin{aligned} dZ_x^i(t) &= dZ_x'(t) - < dZ_x'(t), d\ln \mathbb{Z}(t) > \\ &= dZ_x'(t) + \sigma_n \rho_{x,n} \beta_n(t, T_{i-1}) dt, \quad x \in \{n, r, I\}, \end{aligned}$$

to finally get

$$dr(t) = [-\rho_{n,r}\sigma_n\sigma_r\beta_n(t, T_{i-1}) + \theta_r(t) - \rho_{r,I}\sigma_I\sigma_r - a_r r(t)]dt + \sigma_r dZ_r^i(t)$$

where Z_r^{i-1} is now a Q^{i-1}- Brownian motion.
Integrating this equation (which still leads to an OU dynamics), leads to

$$\begin{aligned} & m_r^t(T_{i-1}) = E[r(T_{i-1})|\mathcal{F}_t] \\ = \ & r(t)e^{-a_r(T_{i-1}-t)} + \int_0^{T_{i-1}} e^{-a_r(T_{i-1}-u)[\theta_r(u,T_{i-1}) - \rho_{n,r}\sigma_n\sigma_r\beta_n(t,T_{i-1})]du} \end{aligned}$$

after tedious calculations, to the final result

$$\mathrm{YYIIS}^{FL}(t; T_{i-1}, T_i, K) = P_n(t, T_{i-1}) \frac{P_r(t, T_i)}{P_r(t, T_{i-1})} e^{\gamma_i(t)} \tag{3.4}$$

with

$$\begin{aligned} \gamma_i(t) &= \sigma_r\beta_r(T_{i-1}, T_i)\{\beta_r(t, T_{i-1})[\rho_{r,I}\sigma_I - \frac{1}{2}\sigma_r\beta_r(t, T_{i-1}) \\ &\quad + \frac{\rho_{n,r}\sigma_n}{a_n + a_r}(1 + a_r\beta_n(t, T_{i-1}))] - \frac{\rho_{n,r}\sigma_n}{a_n + a_r}\beta_n(t, T_{i-1})\}. \end{aligned}$$

The price of YYIIS on $[0, T_M]$ is thus given by

$$\mathrm{YYIIS}(0; T_M, K) = \sum_{i=1}^{M} P_n(t, T_i) \left[\frac{P_n(t, T_{i-1}) P_r(t, T_i)}{P_n(t, T_i) P_r(t, T_{i-1})} e^{\gamma(t, T_{i-1}, T_i)} - (K + 1) \right] \tag{3.5}$$

and as $P_x(0, 0) = 1$ (the t-time price of one euro at t is obviously 1), the first payment recovers the price of a one year ZCIIS.

3.2 Pricing of Caplets/Floorlets

In the Jarrow and Yildrim model, the caplets and floorlets are priced vis a very simple Black and Scholes formula.

Consider the t-time price of a i-th caplet

$$\mathrm{Cplt}_i(t; K) = P_n(t, T_i) E^i \left[\left(\omega \frac{I(T_i)}{I(T_{i-1})} - \omega(1 + K) \right)^+ |\mathcal{F}_t \right].$$

Note that $\ln[T_i(t)/T_{i-1}(t)]$ for $I(t)$ is lognormal, then the caplets prices are given by

$$\mathrm{Cplt}_i(t;K) = \omega P_n(t,T_i)[m_I\Phi(\omega d_+) - (K+1)\Phi(\omega d_-)], \tag{3.6}$$

where

$$d_\pm = \frac{\ln(m_I/(K+1)) \pm V_I^2/2}{V_I}$$

and m_I is the Q^i mean of $I(T_i)/I(T_{i-1})$ and V_I the variance of $\ln[I(T_i)/I(T_{i-1})]$, both conditional on $\mathcal{F}_t$. $\Phi(\cdot)$ is the standard Gaussian cumulative distribution function. As we discuss before, the change of measure would only affect the mean of $I(t)$ and not his variance, then under Q^i, we have

$$\mathrm{YYIIS}(t;T_{i-1},T_i,K) = P_n(t,T_i)\left\{E^i\left[\frac{I(T_i)}{I(T_{i-1})}|\mathcal{F}_t\right] - (K+1)\right\},$$

and compare it to (3.4), we have

$$m_I = E^i\left[\frac{I(T_i)}{T_{i-1}}|\mathcal{F}_t\right] = \frac{P_n(t,T_{i-1})P_r(t,T_i)}{P_n(t,T_i)P_r(t,T_{i-1})}e^{\gamma(t,T_{i-1},T_i)}.$$

Then the variance V_I^2 is not difficult to obtained, but more tricky.

4 The stochastic volatility model

4.1 Model

Hereafter, we will drop the subscript n when denoting nominal quantities, for ease of notation. To recover smile-consistent prices for inflation-indexed caps and floors, Mecurio and Moreni (2005) proposed a stochastic volatility model for forward CPI, with volatility dynamics has a heston-like evolution with a common volatility $V(t)$ that follows a mean-reverting square-root process, the and the nominal forward Libor rate fits into the Libor market model (BGM) model. Under the a given reference measure Q, the model is presented as

$$\begin{aligned} dF_i(t)/F_i(t) &= (\cdots)dt + \sigma_i^F dZ_i^{Q,F}(t) \\ d\mathcal{I}_i(t)/\mathcal{I}_i(t) &= (\cdots) + \sigma_i^I\sqrt{V(t)}dZ_i^{Q,I}(t) \\ dV(t) &= \alpha(\theta - V(t))dt + \epsilon\sqrt{V(t)}dW^Q(t) \end{aligned}$$

where $\sigma_i^F, \sigma_i^I, \alpha, \theta$ and ϵ are positive constants, $2\alpha\theta > \epsilon$ to ensure positiveness of V, and where we allow for correlations between Brownian motions $Z_i^{Q,F}, Z_i^{Q,I}, W^Q$.

As what we do with the Libor market model, using the changes of measure technique, under the nominal spot Libor measure $\mathcal{Q}_0$, which is related to the numeral

$$N(t) = P(t, \beta(t)) \prod_{i=1}^{\beta(t)} (1 + \tau F_i(t)), \quad \beta(t) = T_j, \quad \text{if} \quad T_{j-1} < t \leq T_j,$$

the model (4.1) can be rewritten as

$$\begin{aligned} dF_i(t)/F_i(t) &= \sigma_i^F \left[\sum_{l=\beta(t)+1}^{i} \sigma_i^F \rho_{i,l}^F \frac{\tau_l F_l(t)}{1 + \tau_l F_l(t)} dt + dZ_i^{0,F}(t) \right] \\ d\mathcal{I}_i(t)/\mathcal{I}_i(t) &= \sqrt{V(t)} \sigma_i^I \left[\sum_{l=\beta(t)+1}^{i} \sigma_i^F \rho_{l,j}^{F,I} \frac{\tau_l F_l(t)}{1 + \tau_l F_l(t)} dt + dZ_i^{0,I}(t) \right] \end{aligned} \tag{4.1}$$

and the two Brownian motions are correlated with correlation parameters

$$\rho_{i,l}^F dt = dZ_i^{0,F}(t) dZ_l^{0,F}$$

and

$$\rho_{i,l}^{F,I} dt = dZ_i^{0,I}(t) dZ_l^{0,F}$$

while $V(t)$ evolves as in (4.1).

The pricing of caplets depends on $\mathcal{I}_j$ and $\mathcal{I}_{j-1}$, then we derived the SDE of the relevant quantities $\mathcal{I}_j(\cdot), \mathcal{I}_{j-1}(\cdot)$ and $X_j(\cdot) := \ln(\mathcal{I}_j(\cdot)/\mathcal{I}_{j-1}(\cdot))$

$$\begin{aligned} d\mathcal{I}_j(t)/\mathcal{I}_j(t) &= \sqrt{V(t)} \sigma_j^I dZ_j^I(t) \qquad (4.2) \\ d\mathcal{I}_{j-1}(t)/\mathcal{I}_{j-1}(t) &= \sqrt{V(t)} \sigma_{j-1}^I \left[\frac{\tau_j F_j(t)}{1 + \tau_j F_j(t)} \sigma_j^F \rho_{j,j-1}^{F,I} dt + dZ_{j-1}^I(t) \right] \\ dX_j(t) &= \left[\frac{V(t)}{2} \left((\sigma_{j-1}^I)^2 - (\sigma_j^I)^2 \right) + \sqrt{V(t)} \sigma_{j-1}^I \sigma_j^F \rho_{j,j-1}^{F,I} \frac{\tau_j F_j(t)}{1 + \tau_j F_j(t)} \right] dt \\ &\quad + \sqrt{V(t)} [\sigma_j^I dZ_j^I(t) - \sigma_{j-1}^I dZ_{j-1}^I(t)] \end{aligned}$$

while the volatility evolves according to

$$\begin{aligned} dV(t) &= [\alpha\theta - \epsilon m_j(t) \sqrt{V(t)} - \alpha V(t)] dt + \epsilon \sqrt{V(t)} dW(t) \\ m_j(t) &= \sum_{l=\beta(t)+1}^{j} \frac{\tau_l F_l(t)}{1 + \tau_l F_l(t)} \sigma_l^F \rho_l^{F,V} \end{aligned} \tag{4.3}$$

where $dZ_i^F(t) dW(t) = \rho^{F,V} dt$, for each I.

Denote the correlation between forward CPI's as $\rho_{j,l}^I dt = dZ_j^I(t) dZ_l^I(t)$ and correlation between forward CPI and volatility as $\rho_i^{I,V} = dZ_i^I(t) dW(t)$.

4.2 Pricing of Caplets/Floorlets

The price at time $t \leq T_j$ of j-th caplet, is, under measure Q^j is

$$\begin{aligned}\text{Cplt}_j(t,K) &= P(t,T_j)E_t^J\left(\frac{\mathcal{I}_j(T_j)}{\mathcal{J}-\infty(T_{j-1})}-(K+1)\right)^+ \\ &= P(t,T_j)\int_{-\infty}^{+\infty}(e^s-e^k)^+ q_i^j(s)ds\end{aligned}$$

where $k = \ln(K+1), q_t^j(s)ds := \mathcal{Q}^j\{\ln(\mathcal{I}_j(T_j)/\mathcal{I}_{j-1}(T_{j-1})) \in [s, s+ds]|\mathcal{F}_t\}$, and E_t^j denotes the expetation under $\mathcal{Q}^j$ conditional on the σ-algebra $\mathcal{F}_t$ generated by the relevant processes up to time t.

The caplet price can be rewritten in term of the Fourier Transform as

$$\begin{aligned}\text{Cplt}_j(t)(t,e^k) &= \frac{e^{-\eta k}}{2\pi}\int_{-\infty}^{+\infty} e^{-iuk}\frac{\phi_t^j(u-(\eta+1)i)}{(\eta+iu)(\eta+1+iu)}du \\ &= P(t,T_j)\frac{e^{-\eta k}}{\pi}\text{Re}\int_0^{+\infty} e^{-iuk}\frac{\phi_t^j(u-(\eta+1)i)}{(\eta+iu)(\eta+1+iu)}du\end{aligned}$$

where $\phi_t^j(\cdot)$ is the conditional characteristic function of $\ln(\mathcal{I}_j(T_j))/ln(\mathcal{I}_{j-1}(T_{j-1}))$,

$$\phi_t^j(u) = E_t^j[e^{iu\ln(\mathcal{I}_j(T_j)/\mathcal{I}_{j-1}(T_{j-1}))}],$$

where $\eta \in \mathbb{R}^+$ is used to ensure L^2-integrability when $k \to -\infty$.

We will derive the explicit expression of caplet price in three cases: the uncorrelated case and two approximation of correlated cases.

Uncorrelated case: In this case, we assume that $\rho_{i,l}^{F,I} = 0$ and $\rho_i^{F,V} = 0$ for each $i, l = 1, \cdots, M$. Then the evolution of (4.2) and (4.3)

$$\begin{aligned}dY_j(t) &= -\frac{1}{2}V(t)(\sigma_j^I)^2dt + \sqrt{V(t)}\sigma_j^I(t)dZ_j^I(t) \\ dX_j(t) &= \frac{V(t)}{2}((\sigma_{j-1}^I)^2-(\sigma_j^I)^2)dt + \sqrt{V(t)}[\sigma_j^I dZ_j^I(t) - \sigma_{j-1}^I dZ_{j-1}^I(t)] \\ dV(t) &= [\alpha\theta - \alpha V(t)]dt + \epsilon\sqrt{V(t)}dW(t)\end{aligned}$$

where $Y_j(t) = \ln(\mathcal{I}_j(t))$.

To price caplets it's sufficient to make function $\phi_t^i(u)$ explicit,

$$\phi_t^j(t) = E_t^j[e^{iu(Y_j(T_j)-Y_{j-1}(T_{j-1}))}] = E_t^j[e^{-iuY_{j-1}(T_{j-1})}E_{T_{j-1}}^j(e^{iuY_j(T_j)})].$$

Note that $E^j_{T_{j-1}}(e^{iuY_j(T_j)})$ is the characteristic function of $\ln(\mathcal{I}_j(T_j))$ conditional on $\mathcal{F}_{T_{j-1}}$, then we have from (4.2)

$$E^j_{T_{j-1}}(e^{iuY_j(T_j)}) = exp\{A_Y(\tau_j, u) + B_Y(\tau_j, u)V(T_{j-1}) + iuY_j(T_{j-1})\}$$

where

$$B_Y(s, u) = \frac{\gamma - b}{2a}\left[\frac{1 - e^{\gamma s}}{1 - \frac{b-\gamma}{b+\gamma}e^{\gamma s}}\right]$$

$$A_yY(s, u) = \frac{\alpha\theta(\gamma - b)}{2a}s - frac\alpha\theta a \ln\left[\frac{1 - \frac{b-\gamma}{b+\gamma}e^{\gamma s}}{1 - \frac{b-\gamma}{b+\gamma}}\right]$$

and

$$a = \epsilon^2, \quad c = -iu(\sigma^I_j)^2/2 - (\sigma^I_j)^2 - (\sigma^I_j)^2u^2/2,$$
$$b = iu\sigma^I_j\epsilon\rho^-_j\alpha \quad \gamma = \sqrt{b^2 - 4ac}.$$

With $X_j(T_{j-1}) = Y_j(T_{j-1})$, we have

$$\phi^j_t(u) = e^{A_Y(\tau_j,u)}E^j_t[e^{iuX_j(T_{j-1})+B_Y(\tau_j,u)V(T_{j-1})}],$$

in fact, it's the characteristic function of the couple $(X_j(T_{j-1}), V(T_{j-1}))$ evaluated at point $(u, -iB_Y(\tau_j, u))$, then proceeding as before, we have

$$\phi^j_t(u) = \exp\{A_Y(\tau_i, u) + A_X(T_{j-1} - t, u)B_X(T_{j-1} - t, u)V(t) + iuX_j(t)\}$$

where

$$B_X(\tau, u) = B_Y(\tau_j, u) + \frac{\gamma' - b' - 2a'B_Y(\tau_j, u)}{2a'}\left[\frac{1 - e^{\gamma'\tau}}{1 - \frac{2a'B_Y(\tau_j,u)+b'-\gamma'}{2a'B_Y(\tau_j,u)+b'+\gamma' e^{\gamma'\tau}}}\right]$$

$$A_X(\tau, u) = \frac{\alpha\theta(\gamma' - b')}{2a'}\gamma - \frac{\alpha\theta}{a'}\ln\left[\frac{1 - \frac{2a'B_Y(\tau_i,u)+b'-\gamma'}{2a'B_Y(\tau_i,u)+b'+\gamma'}e^{\gamma'\tau}}{1 - \frac{2a'B_Y(\tau_i,u)+b'-\gamma'}{2a'B_Y(\tau_i,u)+b'+\gamma'}}\right]$$

where

$$a' = \epsilon^2/2 \quad b' = iu\epsilon(\sigma^I_j\rho^{I,V}_j - \sigma^I_{j-1}\rho^{I,V}_{j-1}) - \alpha$$
$$c' = iu((\sigma^I_{j-1})^2 - (\sigma^I_j)^2) - ((\sigma^I_{j-1})^2 + (\sigma^I_j)^2) - 2\sigma^I_j\sigma^I_{j-1}\rho^t_{j,j-1}u^2/2$$
$$\gamma' = \sqrt{b^2 - 4ac}.$$

Approximated Dynamics for non-zero Correlations The typical technique for the non-zero correlation dynamics is freezing the drift terms that involve forward rate

$$D_l(t) := \sqrt{V(t)}\frac{F_l(t)}{1 + \tau_l F_l(t)}.$$

A simple approximate is $D_l(t) \approx D_l(0)$, then changing the asymptotic volatility value from θ to

$$\theta' := \theta - \frac{\epsilon}{\alpha} \sum_{t=1}^{j} D_l(0) \tau_l \sigma_l^F \rho_l^{F,V},$$

which leads to the following SDEs for X_j and V

$$\begin{aligned} dX_j(t) &\approx \left[\frac{V(t)}{2} ((\sigma_{j-1}^I)^2 - (\sigma_j^I)^2 + D_j(0) \tau_j \sigma_{j-1}^I \sigma_j^F \rho_{j,j-1}^{F,I}) \right] dt \\ &\quad + \sqrt{V(t)} [\sigma_j^I dZ_j^I(t) - \sigma_{j-1}^I dZ_{j-1}^I(t)] \\ dV(t) &\approx \alpha(\theta' - V(t)) dt + \epsilon \sqrt{V(t)} dW(t) \end{aligned}$$

Another approximation is to set

$$D_l(t) \approx \frac{F_l(t)}{1 + \tau_l F_l(t)} \frac{V(t)}{\sqrt{V(t)}} \approx \frac{F_l(0)}{1 + \tau_l F_l(0)} \frac{V(t)}{\sqrt{V(0)}} = D_l(0) \frac{V(t)}{V(0)},$$

then the pricing procedure is similar to uncorrelated case, with the modified SDE for X_j and V

$$\begin{aligned} dX_j(t) &\approx V(t) \left[\frac{1}{2} ((\sigma_{j-1}^I)^2 - (\sigma_j^I)^2) + \frac{D_l(0)}{V(0)} \tau_j \sigma_{j-1}^I \sigma_j^F \rho_{j,j-1}^{F,I} \right] dt \\ &\quad + \sqrt{V(t)} [\sigma_j^I dZ_j^I(t) - \sigma_{j-1}^I dZ_{j-1}^I(t)] \\ dV(t) &\approx \alpha'(\theta' - V(t)) dt + \epsilon \sqrt{V(t)} dW(t) \end{aligned}$$

where

$$\alpha' = \alpha + \frac{\epsilon}{V(0)} \sum_{l=1}^{j} D_l(0) \tau_l \sigma) l^F \rho_l^{F,V}$$
$$\theta' = \alpha \theta / \alpha'.$$

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