Pricing caps and swaptions on zero coupon bond prices in Lévy models

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Abstract

Here, we are going to give an exact formula to price caps on zero coupon bond price rates in Lévy models. For this, we use the paper of Eberlein and kluge [1].

1 Model

In this section, we give an expression of the characteristic function of a Lévy process in terms of its triplet (c,ν,b).

Theorem 1.1 ***(Levy-Kinchin representation)***
Let $(L_t)_{0\leq}$ be a Lévy processes on $\mathbb{R}$ with characteristic triplet (c, ν, b). Then

$$\mathbb{E}[e^{izX_t}] = e^{t\psi(z)}, z \in \mathbb{R}^d$$

with $\psi(z) = -\frac{1}{2}zcz + ibz + \int_{\mathbb{R}}(e^{izx} - 1 - izx1_{(|x|\leq 1)})\nu(dx)$.

Now, we can look at the dynamics of the **instantaneous forward rates** for $T \in [0, T^*]$ which are given by

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds - \int_0^t \sigma(s,T)dL_s \ (0 \leq t \leq T).$$

(L_t) is a Lévy Process on $\mathbb{R}$ with characteristic triplet (c,ν,b) and the initial values f(0,T) are deterministic and bounded and measurable in T. Moreover, α and σ are stochastic processes with values in $\mathbb{R}$ and $\mathbb{R}^d$ respectively defined on $\Omega \times [0,T^*] \times [0,T^*]$ that satisfy the following conditions:

- (ω,s,T) $\rightarrow \alpha(\Omega, s, T)$ and (ω,s,T) $\rightarrow \sigma(\Omega, s, T)$ are measurable with respect to $\mathcal{P} \oplus \mathcal{B}([0,T^*])$.
- For s>T we have $\alpha(\Omega, s, T)$=0 and $\sigma(\Omega, s, T)$=0.
- $\sup_{s,T\leq T^*}(|\alpha(\Omega, s, T)| + (|\sigma(\Omega, s, T)|)$.

Thus, in using the following expression $B(0,T) = \exp\Big(-\int_t^T f(t,u)du\Big)$ with $t \leq T$, we can deduce an expression for zero coupon bond price rates:

$$B(t,T) = B(0,T)e^{\int_0^t (r(s)-A(s,T))ds + \int_0^t \Sigma(s,T)dL_s}$$

where

$$A(s,T) = \int_s^T \alpha(s,u)du$$

and

$$\Sigma(s,T) = \int_s^T \sigma(s,u)du, (s \leq t).$$

Now, we put a new condition on the Lévy measure. Indeed, in this model we suppose :

- Assumption($\mathbb{EM}$): There are constant M,$\varepsilon > 0$ such that for $\forall u \in [-(1+\varepsilon)M, (1+\varepsilon)M]^d \quad \int_{|x|>1} e^{<u,x>}\nu(ds) < \infty$.
- Assumption($\mathbb{DET}$): The volatility structure σ is deterministic and continious in the first argument. Moreover, for $0 \leq s, T \leq T^*$ we have $0 \leq \Sigma^i(s,T) \leq M \quad (i \in 1, ..., d)$ where M is the constant from assumption ($\mathbb{EM}$).

Or if the Lévy measure satisfy the additionnal condition $\int_{|x|>1}|x|\nu(dx) < \infty$, $\mathbb{E}[e^{izL_t}]$ can write as

$$e^{t(-\frac{1}{2}zcz+ibz+\int_{\mathbb{R}^d}(e^{izx}-1-izx)\nu(dx)}$$

where

$$b = b + \int_{\mathbb{R}^d} x 1_{|x|>1}\nu(dx).$$

Thus, the function θ characterized by

$$\theta(z) = z.b + \frac{1}{2}z.cz + \int_{\mathbb{R}}(e^{z.x} - 1 - z.x)\nu(dx))$$

verify the following relation:

$$\mathbb{E}[e^{iu.L_t}] = e^{t\theta(iu)}.$$

In this new model, we can derive **explicit formulas for caps and swaptions**.

2 Pricing caps

Here, we want to price a call with strike K and maturity t on a bond which matures at T is then given by

$$C(t,T,K) = \mathbb{E}[\frac{1}{B_t}(B(t,T) - K)^+].$$

For this, we consider the **forward martingale measure for the settlement day t** P_t defined by

$$\frac{dP_t}{dP} = \frac{1}{B_t B(0,t)} = exp\Big(-\int_0^t A(s,t)ds + \int_0^t \Sigma(s,t)dL_s\Big).$$

Under the measure P_t the price of this call is

$$B(0,t)\mathbb{E}_{P_t}[(Dexp(X) - K)^+]$$

where

$$D = \frac{B(0,T)}{B(0,t)}e^{\int_0^t(\theta(\Sigma(s,T))-\theta(\Sigma(s,t)))dL_s}$$

and

$$X = \int_0^t (\Sigma(s,T) - \Sigma(s,t))dL_s$$

is F_t-measurable. Suppose the distribution of X possesses a Lebesgue-density. Choose an r<-1 such that $M_t^X(-r)<\infty$. Then we have:

$$C_0(t,T,K) = \frac{1}{2\pi}KB(0,t)e^{r\zeta}\int_{-\infty}^{\infty} e^{iu\zeta}\frac{1}{(r+iu)(r+1+iu)}M_t^x(-r-iu)du$$

where M_t^X is the **moment generating function** of the random variable X with respect to the measure P_t and $\zeta = log(\frac{B(0,t)}{B(0,T)}) - \int_0^t \Big(\theta(\Sigma(s,T)) - \theta(\Sigma(s,T))\Big)ds + log(K)$.

To choose r, we can use the following lemma:

Lemma 2.1 *Choose M and ε in assumption ($\mathbb{EM}$) such that $\Sigma(s,T) \leq M'$ conponenwise for an M' < M and for all s,T$\in [0,T^*]$. Then, for each r $\in [-1 - \frac{M-M'}{M'}, -1]$ we have $M_t^X < \infty$. Moreover, for z$\in \mathbb{C}$ with $\mathcal{R}z = -r$*

$$M_t^X(z) = exp\int_0^t \bigg(\theta\Big(z\Sigma(s,T) + (1-z)\Sigma(s,t)\Big) - \theta\Big(\Sigma(s,T)\Big)\bigg)ds$$

where the function θ verify the following expression $\theta(iu) = \psi(u) \quad ,u \in \mathbb{R}$.

This result is very important. Indeed, it is difficult to know the distribution of X under the measure P_t.

3 Pricing swaptions

In this section, we propose an expression for pricing swaptions. For this, we need the following condition:

Assumption($\mathbb{VOL}$): For all T$\in$[0,T^*] we have $\sigma(.,T) \neq (0,...,0)$ and $\sigma(s,T) = \sigma_2(T)\sigma_1(s) \quad ,(0 \leq s \leq T)$ where

$$\sigma_1 : [0,T^*] \rightarrow \mathbb{R}^d$$

and

$$\sigma_1 : [0,T^*] \rightarrow \mathbb{R}^d$$

are continously differentiable.

The three cases of volatility stucture that we are going to study satisfy this assumption.

Let $B_C(t, T_1, ..., T_n)$ be the time t price 1 and maturity T_n paying to its owner an amount of $C_1, ..., C_n$ at the dates $T_1, ..., T_n$.
We want to price a call with strike 1 and maturity t on that bond is obtained by taking the expectation of the discounted payoff. Thus, the formula is:

$$C_0 = \mathbb{E}[\frac{1}{B_t}(\sum_{i=1}^{n} CiB(t, T_i) - 1)^+]$$

As before, we modify the probability and we get after some modifications:

$$C_0 = B(0, t)\mathbb{E}_{P_t}[(\sum_{i=1}^{n}(D_i e^{B_i X}) - 1)^+]$$

where

$$D_i = \frac{B(0, T_i)}{B(0, t)} C_i exp(\int_0^t (\theta(\Sigma(s, t)) - \theta(\Sigma(s, T_i)))ds$$

and

$$X = \int_0^t (\Sigma(s, T_n) - \Sigma(s, t))dL_s.$$

To obtain an explicit expression, we need the following result:

Theorem 3.1 *Suppose the distribution of X possesses a Lebesgue-density. Choose an r<-1 such that $M_t^X(-r)<\infty$ and let Z be the unique zero of the stictly increasing and continuous function*

$$g(x) = \sum_{i=1}^{n} D_i e^{B_i x} - 1$$

Thus, we have

$$C_0 = \frac{1}{2\pi} B(0, t) \lim_{Y\to\infty} \int_{-Y}^{Y} L[v](r + iu) M_t^X(-R - iu)du$$

where

$$L[v](r + iu) = e^{(r+iu)Z}(\frac{1}{r + iu} - \sum_{i=1}^{n}(D_i e^{B_i Z} \frac{1}{B_i + r + iu}))$$

and $z \in \mathbb{C}, \mathcal{R}z = -r$ with

$$M_t^X(z) = exp(\int_0^t (\theta(z\Sigma(s, T_n) + (1 - z)\Sigma(s, t)) - \theta(\Sigma(s, T)))ds).$$

4 Lévy process and volatility stucture cases used to price caps and swaptions

We are going to price caps and swaptions for differents Lévy processes and volatility stuctures.
We study five cases for the Lévy process which are:

1. Brownian motion

2. Brownian motion+Compound Poisson process with intensity λ and jump size distribution f where

$$f(x) = \frac{1}{var\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2var^2}}$$

and the Lévy measure is defined by $\nu = \lambda f(x)$

3. Lévy process where the Lévy measure is defined by

$$\nu = a\frac{e^{-\lambda x}}{x} 1_{\{x>0\}} \quad , \lambda, a > 0.$$

4. Lévy process where the Lévy measure is defined by

$$\nu = a\frac{e^{-\lambda x}}{x^2} 1_{\{x>0\}} \quad , \lambda, a > 0.$$

5. Lévy process where the Lévy measure is defined by

$$\nu = a\frac{e^{-\lambda x}}{x^{\alpha+1}} 1_{\{x>0\}} \quad , 0 < \alpha < 1 \; and \; \lambda, a > 0.$$

For the volatility structure, we use three cases given by Eberlein and Kluge which verify the condition ($\mathbb{VOL}$):

1. $\sigma(s,T) = \tilde{\sigma}$ (Ho-Lee volatility stucture).

2. $\sigma(s,T) = \tilde{\sigma}e^{-x(t-s)}$ (vasicek volatility stucture).

3. $\sigma(s,T) = \tilde{\sigma}e^{-x(t-s)}\frac{1+\gamma T}{1+\gamma s}$ (Moraleda-Vorst volatility stucture).

for $\tilde{\sigma}, \gamma > 0$ and $x \neq 0$.

5 Manual program

1. The program ask us to choose the volatility stucture and the Lévy process that you want to use.

2. The program ask us to enter the stike K, the drift b, the brownian coefficient c.

3. The program ask us to enter volatility structure parameters:

   - In the first case, the program ask us to enter $\tilde{\sigma}$.
   - In the second case, the program ask us to enter $\tilde{\sigma}$ and x.
   - In the third case, the program ask us to enter $\tilde{\sigma}$, x, γ.

4. The program ask us to enter the Lévy process parameters:

   - In the first case, we have nothing to enter.
   - In the second case, the program ask us to enter μ, var, λ.
   - In the third case, the program ask us to enter λ, a.
   - In the fourth case, the program ask us to enter λ, a.
   - In the last case, the program ask us to enter λ, a, α.

5. - In the case cap, the program ask us to enter t, T and the strike K.
   - In the case swaption, the program ask us to enter the number of dates n, the date t, the dates T_i and the amounts C_i.

6. Finally, we must calculate the zero coupon bond prices. For this, we have two methods:

   - In the first method, we use this formula $B(0,t) = \exp(-rt)$.
   - In the second method, the previous formula is wrong. Thus, we calculate bond prices by interpolation in using the values and dates of bond prices already known.

Thus, we have to do:

- If we choose the first method to calculate bond prices, the program ask us to enter the constant interest rate r.

- If we choose the second method to calculate bond prices, the program ask us to enter the number of bond prices that we know, their dates and their values.

References

[1] Ernst Eberlein and Wolfgang Kluge. Exact pricing formulae for caps and swaptions in a lévy term structure model.