A closed-form extension to the Black-Cox model

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Abstract

In the Black-Cox model, a firm defaults when its value hits an exponential barrier. Here, we propose an hybrid model that generalizes this framework. The default intensity can take two different values and switches when the firm value crosses a barrier. Of course, the intensity level is higher below the barrier. We get an analytic formula for the Laplace transform of the default time and present numerical methods to numerically recover its distribution. We explain how this model can be calibrated to Credit Default Swap prices and show its tractability on different kinds of data. Last, we discuss the extension to multiple barriers and intensity levels.

Keywords: Credit Risk, Intensity Model, Structural Model, Black-Cox Model, Parisian options.

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1 Introduction and model setup

Modelling firm defaults is one of the fundamental matter of interest in finance. It has stimulated research over the past decades. Clearly, the recent worldwide financial crisis and its bunch of resounding bankruptcies have underlined once again the need to better understand credit risk. In this paper, we focus on the modelling of a single default. Usually, these models are divided into two main categories: structural and reduced form (or intensity) models.

Structural models aim at explaining the default time with economic variables. In his pathbreaking work, Merton [14] connects the default of a firm with its ability to pay back its debt. The firm value is defined as the sum of the equity value and the debt value, and is supposed to be a geometric Brownian motion. At the bond maturity, default occurs if the debtholders cannot be reimbursed. In this framework, the equity value is seen as a call option on the firm value. Then, Black and Cox [4] have extended this framework by triggering the default as soon as the firm value goes below some critical barrier. Thus, the default can occur at any time and not only at the bond maturity. Many extensions of the Black Cox model, based on first passage time, have been proposed in the literature. We refer to the book of Bielecki and Rutkowski [3] for a nice presentation. Recently, attention has be paid to the calibration of these models to Credit Default Swap (CDS in short) data (Brigo and Morini [5], Dorfleitner and al. [11]). However, though economically sounded, these models can hardly be used intensively on markets to manage portfolios especially for hedging. Unless considering dynamics with jumps (see Zhou [18] for example), their major drawback is that the default time is predictable and no default can occur when the firm value is clearly above the barrier. In other words, they understimate default probabilities and credit spreads for short maturities.

The principle of reduced form models is to describe the dynamics of the instantaneous probability of default that is also called intensity. This intensity is described by some autonomous dynamics and the default event is thus not related to any criterion on the solvency of the firm. We refer to the book of Bielecki and Rutkowski [3] for an overview of these models. In general, they are designed for being easily calibrated to CDS market data and are in practice more tractable to manage portfolios.

However, none of these two kinds of model is fully satisfactory. In first passage time models, the default intensity is zero away from the barrier and the default event can be forecasted. Intensity models are in line with CDS market data, but remain disconnected to the rationales of the firm like its debt and equity values. Thus, they cannot exploit the information available on equity markets. To overcome this shortcoming, and to provide a unified framework for pricing equity and credit products, hybrid models have been introduced, assuming that the default intensity is a (decreasing) function of the stock. Here, We mention the works of Atlan and Leblanc [2], and Carr and Linetsky [6] who consider the case of a defaultable constant elasticity model.

In this paper, we propose an hybrid model, which extends the Black-Cox model and in which the default intensity depends on the firm value. Let us present it in detail. We consider the usual framework when dealing with credit risk and firm value models. Namely, we assume that we are under the risk-neutral probability measure and that the (riskless) short interest rate is constant and equal to $r > 0$. We denote by $(\mathcal{F}_t, t \geq 0)$ the default-free filtration and consider a $(\mathcal{F}_t)$-Brownian motion $(W_t, t \geq 0)$. We assume that the firm value $(V_t, t \geq 0)$ evolves according to the Black-Scholes model and therefore satisfies the following dynamics:

$$dV_t = rV_t dt + \sigma V_t dW_t, \quad t \geq 0, \tag{1}$$

where $\sigma > 0$ is the volatility coefficient. To model the default event, we assume that the default intensity has the following form:

$$\lambda_t = \mu_2 \mathbf{1}_{\{V_t \leq C\,\mathrm{e}^{\alpha t}\}} + \mu_1 \mathbf{1}_{\{V_t > C\,\mathrm{e}^{\alpha t}\}}, \tag{2}$$

where $C > 0$, $\alpha \in \mathbb{R}$, and $\mu_2 > \mu_1 \geq 0$. This means that the firm has an instantaneous probability of default equal to μ_2 or μ_1 depending on whether its value is below the time-varying barrier $C\,\mathrm{e}^{\alpha t}$ or above. More precisely, let ξ denote an exponential random variable of parameter 1 independent of the filtration $\mathcal{F}$. Then, we define the instant of default of the firm by:

$$\tau = \inf\{t \geq 0, \int_0^t \lambda_s ds \geq \xi\}. \tag{3}$$

As usual, we also introduce $(\mathcal{H}_t, t \geq 0)$ the filtration generated by the process $(\tau \wedge t, t \geq 0)$ and define $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, so that $(\mathcal{G}_t, t \geq 0)$ embeds both default-free and defaultable information.

This framework is a natural extension of the pioneering Black-Cox model introduced in [4], which can indeed be simply seen as the limiting case of our model when $\mu_1 = 0$ and $\mu_2 \to +\infty$. In the work of Black and Cox, bankruptcy can in addition happen at the maturity date of the bonds issued by the firm when the firm value is below some level. Here, we do not consider this possibility, even though it is technically feasible, because it would make the default predictable in some cases. In the Black-Cox model, the barrier $C\,\mathrm{e}^{\alpha t}$ is meant to be a safety covenant under which debtholders can ask for being reimbursed. Here, default can happen either above or below the barrier, which represents instead the border between two credit grades. Let us briefly explain what typical parameter configurations could be for this model. For a very safe firm, we expect that its value start above the barrier with μ_1 very close to 0. The parameter μ_2 should also be rather small since its cannot be downgraded too drastically. Instead, for firms that are close to bankruptcy, we expect to have $C < V_0$ and a high intensity of default μ_2. Then, the parameters should be such that the firm is progressively drifted to the less risky region (i.e. $r - \sigma^2/2 - \alpha > 0$). In fact, the CDS prices often reflect two possible outcomes in such critical situations. Either the firm makes bankruptcy in the next future, or it survives and is then strengthened (see Brigo and Morini [5] for the Parmalat crisis case).

Now, we present the main theoretical result of this paper which gives the explicit formula for the Laplace transform of the default time.

Theorem 1.1. *Let us set* $b = \frac{1}{\sigma}\log(C/V_0)$, $m = \frac{1}{\sigma}(r - \alpha - \sigma^2/2)$ *and* $\mu_b = \mu_2 \mathbf{1}_{\{b>0\}} + \mu_1 \mathbf{1}_{\{b \leq 0\}}$. *The default cumulative distribution function* $P(\tau \leq t)$ *is a function of* t, b, m,

μ_1 and μ_2 and is fully characterized by its Laplace transform defined for $z \in \mathbb{C}_+ := \{z \in \mathbb{C}, \mathcal{R}\mathrm{e}\,(z) > 0\}$,

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-zt}\, P(\tau \le t)dt \;&=\; \mathrm{e}^{mb-|b|\sqrt{2(z+\mu_b)+m^2}} \left(\frac{1}{z+\mu_1} - \frac{1}{z+\mu_2}\right) \times \Bigg\{ -\mathbf{1}_{\{b>0\}} \\ &\quad + \frac{-m+\sqrt{2(z+\mu_2)+m^2}}{\sqrt{2(z+\mu_1)+m^2}+\sqrt{2(z+\mu_2)+m^2}} \Bigg\} + \frac{1}{z} - \frac{1}{z+\mu_b}.\end{aligned} \tag{4}$$

Theorem 1.1 can actually fit in the framework of Theorem 4.1 in the particular case of one barrier ($n = 1$). Hence, we refer the reader to Section 4 for a proof of Theorem 1.1. Let us mention here that other Black-Cox extensions based on analytical formulas for Parisian type options have been proposed in the recent past. Namely, Chen and Suchanecki [8], Moraux [15] and Yu [17] have considered the case where the default is triggered when the stock has spent a certain amount of time in a row or not under the barrier. Nonetheless, both extensions present the drawback that the default is actually predictable and the default intensity is either 0 or non-finite. This does not hold in our framework.

The paper is structured as follows. First, we present in Section 2 two methods to numerically invert the Laplace transform of the default cumulative distribution function given by Theorem 1.1. For each method, we state in a precise way its accuracy which heavily relies on the regularity of the function to be recovered. The required regularity assumptions are actually proved to be satisfied by the default cumulative distribution function, in Appendix A. Then, our main purpose is to explain how this closed formula for the Laplace transform combined with an efficient inversion technique can be exploited in a fast calibration procedure to CDS market data. Section 3 is devoted to practical applications of the theoretical results obtained before. We present a calibration procedure for the model and show on different practical settings how the model can fit the market data. The results are rather encouraging even if the meaning of the calibrated parameters can be discussed in a few examples. Last, in Section 4 we extend Theorem 1.1 to the case of multiple barriers with one intensity of default per slice.

2 Numerical methods for Laplace inversion

From Theorem 1.1, we know that the default time distribution is tractable using the semi-analytical formula for its Laplace transform. In this section, we are investigating different ways of inverting this Laplace transform to recover the cumulative distribution function of the default time τ, and also its first order derivatives with respect to each parameter. Recovering these derivatives enables us to quickly compute the sensitivities with respect to the different parameters, which is of great importance for the calibration procedure, if one wants to use a gradient descent to minimize some distance between the real and theoretical prices.

In this section, $f : \mathbb{R} \to \mathbb{R}$ is a real valued function that vanishes on $\mathbb{R}_-$ and such that $f(t)\,\mathrm{e}^{-\gamma t}$ is integrable for some $\gamma > 0$. We will denote by $\hat{f}(z) = \int_0^\infty \mathrm{e}^{-zt} f(t)dt$ its Laplace transform for $z \in \mathbb{C}$ when the integral is well-defined, i.e at least when $\mathcal{R}\mathrm{e}\,(z) \geq \gamma$. The scope of this section is to present numerical methods to recover f from $\hat{f}$ and analyze their accuracies. Basically in our model, f will be either $\mathbb{P}(\tau \leq t)$ or its derivative w.r.t. one of the model parameters.

2.1 The Fourier series approximation

From the formulas obtained for the Laplace transform of the default time, it is clear that these Laplace transforms are analytical in the complex half-plane $\mathbb{C}_+$. Thanks to [16], we know how to recover a function from its Laplace transform.

Theorem 2.1. *Let f be a continuous function defined on $\mathbb{R}_+$ and γ a positive number. If the function $f(t)\,\mathrm{e}^{-\gamma t}$ is integrable, then its Laplace transform $\hat{f}(z) = \int_0^\infty \mathrm{e}^{-zt} f(t)dt$ is well defined on $\{z \in \mathbb{C}, \mathcal{R}\mathrm{e}\,(z) \geq \gamma\}$, and f can be recovered from the contour integral*

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \mathrm{e}^{st}\,\hat{f}(s)ds = \frac{\mathrm{e}^{\gamma t}}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-ist}\,\hat{f}(\gamma - is)ds, \quad t > 0. \tag{5}$$

For any real valued function satisfying the hypotheses of Theorem 2.1, we introduce the following discretisation of Equation (5) with step $h > 0$

$$f_h(t) = \frac{h\,\mathrm{e}^{\gamma t}}{2\pi}\sum_{k=-\infty}^{\infty} \mathrm{e}^{-ikht}\,\hat{f}\,(\gamma - ikh)\,. \tag{6}$$

From [1, Theorem 5], one can prove using the Poisson summation formula that

Proposition 2.2. *If f is a continuous bounded function satisfying $f(t) = 0$ for $t < 0$, we have*

$$\forall t < 2\pi/h, \;\; |f(t) - f_h(t)| \leq \|f\|_\infty\, \frac{\mathrm{e}^{-2\pi\gamma/h}}{1 - \mathrm{e}^{-2\pi\gamma/h}}. \tag{7}$$

2.2 The fast Fourier transform approach

In this section, we focus on the inversion using an FFT based algorithm. First, let us recall that for a given integer $N \in \mathbb{N}^*$, the forward discrete Fourier transform (DFT) of $(x_k, k = 0, \ldots, N-1)$ is defined by

$$\hat{x}_l = \sum_{k=0}^{N-1} \mathrm{e}^{-2i\pi kl/N}\, x_k, \quad \text{for } l = 0, \ldots, N-1.$$

It is well known that there are Fast Fourier Transform algorithms that enable to compute $(\hat{x}_l, l = 0, \ldots, N-1)$ with a time complexity proportional to $N\log(N)$. In their path-breaking paper, Cooley and Tukey [10] have given such an algorithm for the special case where N is a power of 2. Many improvements of this algorithm have been proposed in the literature relaxing this constraint on N. In finance, the use of the FFT for option pricing has been popularized by Carr and Madan [7]. Here, we use the FFT algorithm in a different manner to compute the cdf of τ and its derivatives with respect to each parameter up to some time $T > 0$.

Let us assume that we want to recover the function f on the interval $[0, T]$. Typically, T will represent the largest maturity of the CDS that one wishes to consider. We set $h < 2\pi/T$, so that $h < 2\pi/t$ for any $t \in (0, T]$ and we can therefore control the error between the Fourier series f_h and f thanks to Proposition 2.2:

$$\forall t \in (0, T], |f(t) - f_h(t)| \leq \|f\|_\infty \frac{\mathrm{e}^{-2\pi\gamma/h}}{1 - \mathrm{e}^{-2\pi\gamma/h}}.$$

Since f is real valued, $\hat{f}(\bar{z}) = \overline{\hat{f}(z)}$, and we obtain

$$f_h(t) = \frac{h\,\mathrm{e}^{\gamma t}}{2\pi}\hat{f}(\gamma) + \frac{h\,\mathrm{e}^{\gamma t}}{\pi}\mathcal{R}\mathrm{e}\left(\sum_{k=1}^{\infty} \mathrm{e}^{-ikht}\,\hat{f}\left(\gamma - ikh\right)\right), \tag{8}$$

that we want to approximate by the following finite sum

$$f_h^N(t) = \frac{h\,\mathrm{e}^{\gamma t}}{2\pi}\hat{f}(\gamma) + \frac{h\,\mathrm{e}^{\gamma t}}{\pi}\mathcal{R}\mathrm{e}\left(\sum_{k=1}^{N} \mathrm{e}^{-ikht}\,\hat{f}\left(\gamma - ikh\right)\right). \tag{9}$$

For $1 \leq l \leq N$, we set $t_l = 2\pi l/(Nh)$ to get

$$\begin{aligned} f_h^N(t_l) &= \frac{h\,\mathrm{e}^{\gamma t_l}}{2\pi}\hat{f}(\gamma) + \frac{h\,\mathrm{e}^{\gamma t_l}}{\pi}\mathcal{R}\mathrm{e}\left(\sum_{k=1}^{N} \mathrm{e}^{-2i\pi kl/N}\,\hat{f}\left(\gamma - ikh\right)\right) \\ &= \frac{h\,\mathrm{e}^{\gamma t_l}}{2\pi}\hat{f}(\gamma) + \frac{h\,\mathrm{e}^{\gamma t_l}}{\pi}\mathcal{R}\mathrm{e}\left(e^{-2i\pi(l-1)/N}\sum_{k=1}^{N} \mathrm{e}^{-2i\pi(k-1)(l-1)/N}\,e^{-2ik\pi/N}\hat{f}\left(\gamma - ikh\right)\right). \end{aligned}$$

Therefore, $(f_h^N(t_l), l = 1, \ldots, N)$ can be computed easily using the direct FFT algorithm on the vector $(e^{-2ik\pi/N}\hat{f}\left(\gamma - ikh\right), k = 1, \ldots, N)$.

Now, let us analyze the error induced by approximating $(f(t_l))_l$ by $(f_h^N(t_l))_l$. The following proposition gives an upper bound of the error involved in the truncation of the series appearing in f_h.

Proposition 2.3. *Let f be a function of class $\mathcal{C}^3$ on $\mathbb{R}_+$ such that there exists $\epsilon > 0$ satisfying $\forall k \leq 3$, $f^{(k)}(s) = \mathcal{O}(\mathrm{e}^{(\gamma-\epsilon)s})$. Let us assume moreover that $f(0) = 0$. Let $A \in (0, 2\pi)$. Then, there exists a constant $K > 0$ independent of t such that:*

$$\forall t \in (0, A/h],\ |f_h^N(t) - f_h(t)| \leq K(1 + 1/t)\frac{\mathrm{e}^{\gamma t}}{N^2}. \tag{10}$$

Proof. From three successive integrations by parts, we get:

$$\begin{aligned}\widehat{f}(\gamma - ikh) &= \int_0^\infty \mathrm{e}^{(ikh-\gamma)u} f(u)du \\ &= \frac{-f'(0)}{(ikh-\gamma)^2} + \frac{f''(0)}{(ikh-\gamma)^3} - \int_0^\infty \frac{f^{(3)}(u)}{(ikh-\gamma)^3} \mathrm{e}^{(ikh-\gamma)u} du.\end{aligned}$$

We set $E_k = \sum_{j=0}^{k-1} e^{-ijht} = (1-e^{-ikht})/(1-e^{-iht})$ and get by a summation by parts

$$\sum_{k=0}^{N} \frac{\mathrm{e}^{-ikht}}{(ikh-\gamma)^2} = \frac{E_{N+1}}{(iNh-\gamma)^2} + \sum_{k=1}^{N} E_k \frac{h(2\gamma + i(2k-1)h)}{(ikh-\gamma)^2(i(k-1)h-\gamma)^2} - \frac{1}{\gamma^2}.$$

Therefore, we deduce that:

$$\begin{aligned}\frac{2\pi}{he^{\gamma t}}(f_h^N(t) - f_h(t)) &= 2f'(0)\,\mathcal{R}\mathrm{e}\left(\frac{E_{N+1}}{(iNh-\gamma)^2} + \sum_{k=N+1}^{\infty} E_k \frac{h(2\gamma+i(2k-1)h)}{(ikh-\gamma)^2(i(k-1)h-\gamma)^2}\right) \\ &\quad +2\,\mathcal{R}\mathrm{e}\left(\sum_{k=N+1}^{\infty} \mathrm{e}^{-ikht} \frac{f''(0) - \int_0^\infty f^{(3)}(u)\,\mathrm{e}^{(-\gamma+ikh)u}\,du}{(ikh-\gamma)^3} du\right).\end{aligned}$$

Then, using that for any $k \in \mathbb{N}$ $|E_k| \le 2/|1-e^{-iht}|$, we get:

$$\begin{aligned}\left|\frac{2\pi}{he^{\gamma t}}(f_h^N(t) - f_h(t))\right| &\le \frac{4|f'(0)|}{|1-e^{-iht}|}\left(\frac{1}{\gamma^2 + (Nh)^2} + \sum_{k=N+1}^{\infty} h \frac{\sqrt{(2\gamma)^2 + ((2k-1)h)^2}}{(\gamma^2+(kh)^2)(\gamma^2+((k-1)h)^2)}\right) \\ &\quad +2(|f''(0)| + C/\epsilon) \sum_{k=N+1}^{\infty} \frac{1}{(\gamma^2+(kh)^2)^{3/2}},\end{aligned}$$

where $C = \sup_{t\ge 0} |f^{(3)}(t)\,\mathrm{e}^{(\epsilon-\gamma)t}|$. The result follows form noticing that $\sup_{y\in[0,A]} \frac{y}{|1-e^{-iy}|} < \infty$. $\square$

Remark 2.4. *When $b \neq 0$, the functions $P_{b,m,\mu_1,\mu_2}(t)$ and $\partial_p P_{b,m,\mu_1,\mu_2}(t)$ for $p \in \{b, m, \mu_1, \mu_2\}$ satisfy the above assumption thanks to Proposition A.3.*

When $b = 0$, we can check from (31) that there exist some constants c and c_p for $p \in \{m, \mu_1, \mu_2\}$ such that the following expansions hold when $k \to +\infty$:

$$L_{0,m,\mu_1,\mu_2}(\gamma - ikh) = \frac{c}{(\gamma-ikh)^2} + O(1/k^3), \text{ and } \partial_p L_{0,m,\mu_1,\mu_2}(\gamma - ikh) = \frac{c_p}{(\gamma-ikh)^2} + O(1/k^3).$$

Therefore, we can use the same proof as in Proposition 2.3 to bound the truncation error by $K(1+1/t)\frac{\mathrm{e}^{\gamma t}}{N^2}$. On the contrary, the derivative with respect to b satisfies $\partial_b L_{0,m,\mu_1,\mu_2}(\gamma - ikh) = (m + \sqrt{2(\gamma - ikh + \mu_1) + m^2}) L_{0,m,\mu_1,\mu_2}(\gamma - ikh) = \frac{c_b}{(\gamma-ikh)^{3/2}} + O(1/k^{5/2})$. Thus, the same proof only gives a truncation error bounded by $K(1+1/t)\frac{\mathrm{e}^{\gamma t}}{N^{3/2}}$ in this case, which however still goes to zero when N is large enough.

Corollary 2.5. *Let f be a bounded function of class $\mathcal{C}^3$ on $\mathbb{R}_+$ such that there exists $\epsilon > 0$ satisfying $\forall k \leq 3,\ f^{(k)}(s) = \mathcal{O}(\mathrm{e}^{(\gamma-\epsilon)s})$. Let $A \in (0, 2\pi)$ and $h \leq A/T$.*

Then, there exists a constant $K > 0$ such that

$$\forall l \geq 1, t_l \leq T,\ |f_h^N(t_l) - f(t_l)| \leq K \max\left(\frac{\mathrm{e}^{\gamma T}}{N^2}, \frac{h}{2\pi N}\right) + \|f\|_\infty \frac{\mathrm{e}^{-2\pi\gamma/h}}{1 - \mathrm{e}^{-2\pi\gamma/h}}.$$

Proof. It is sufficient to use Propositions 2.2 and 2.3, and to remark that $\max_{t \in [t_1, T]}(\mathrm{e}^{\gamma t}/t) \leq \max(\mathrm{e}^{\gamma t_1}/t_1, \mathrm{e}^{\gamma T}/T)$. □

Practical implementation in our model. Now, let us explain how to choose the parameters in our model in order to achieve a precision of order $\varepsilon > 0$. First, let us notice that we can take $\gamma > 0$ as close to 0 as we wish thanks to Proposition A.3. The following conditions

$$h < 2\pi/T,\ \frac{2\pi\gamma}{h} = \log(1 + 1/\varepsilon),\ N > \max\left(\frac{h}{2\pi\varepsilon}, \sqrt{\frac{\mathrm{e}^{\gamma T}}{\varepsilon}}\right) \tag{11}$$

ensure by Corollary 2.5 that $\sup_{l \geq 1, t_l \leq T} |f_h^N(t_l) - f(t_l)|$ is of order ε.

In practice, it is important to make the time grid $(t_l, l = 1 \dots N)$ on which we recover the cdf (and its derivatives) coincide with the payment dates of all the products considered. Typically, this grid should encompass the quarterly time grid to easily compute the CDS prices and their sensitivities. More precisely, we will compute the integrals defining default and payment leg prices in (14) and (15) using the Simpson rule, which is very efficient since the integrated functions are regular enough (namely $\mathcal{C}^4$) as stated by Proposition A.3. To do so, we need a time grid at least twice thinner than the payment grid, and therefore $1/8$ has to be a multiple of $t_1 = \frac{2\pi}{Nh}$. Since in this paper we consider CDS up to $T = 10$ years, we make the following choice:

$$T = 10,\ h = \frac{5\pi}{8T},\ \gamma = \frac{h}{2\pi}\log(1 + 1/\varepsilon), N = \max\left(2^{\left\lceil \log_2\left(\max\left(\frac{h}{2\pi\varepsilon}, \sqrt{\frac{\mathrm{e}^{\gamma T}}{\varepsilon}}\right)\right)\right\rceil}, 2^7\right), \tag{12}$$

which automatically guarantees the latter condition: $1/8$ is clearly a multiple of $t_1 = 16/N$.

2.3 The Euler summation

The Laplace inversion based on the FFT is very efficient and enables to very quickly compute the c.d.f. and its derivatives on the whole time interval. However, the time grid has to be regular, which may be a possible drawback when dealing with bespoke products that have unusual payment dates. Here, we present another method to recover the function f from its Laplace transform at a given time $t \geq 0$.

Unlike the FFT approach, we can here choose h as a function of t, and the trick consists in choosing $h = \pi/t$ to get an alternating series in (8):

$$f_{\pi/t}(t) = \frac{\mathrm{e}^{\gamma t}}{2t}\hat{f}(\gamma) + \frac{\mathrm{e}^{\gamma t}}{t}\sum_{k=1}^{\infty}(-1)^k \mathcal{R}\mathrm{e}\left(\hat{f}\left(\gamma + i\frac{k\pi}{t}\right)\right). \tag{13}$$

Rather than simply truncating the series like in the FFT algorithm, we use the Euler summation technique as described by [1], which consists in computing the binomial average of q terms from the N-th term of the series appearing in (13). The following proposition describes the convergence rate of the binomial average to the infinite series $f_{\pi/t}(t)$ when p goes to ∞. Its proof can be found in Labart and Lelong [13].

Proposition 2.6. *Let $q \in \mathbb{N}^*$ and f be a function of class $\mathcal{C}^{q+4}$ such that there exists $\epsilon > 0$ satisfying $\forall k \leq q+4,\ f^{(k)}(s) = \mathcal{O}(\mathrm{e}^{(\gamma-\epsilon)s})$. We consider the truncation of the series in (13)*

$$f^N_{\pi/t}(t) = \frac{\mathrm{e}^{\gamma t}}{2t}\widehat{f}(\gamma) + \frac{\mathrm{e}^{\gamma t}}{t}\sum_{k=1}^{N}(-1)^k\,\mathcal{R}\mathrm{e}\left(\widehat{f}\left(\gamma + i\frac{\pi k}{t}\right)\right),$$

and $E(q,N,t) = \sum_{k=0}^{q}\binom{q}{k}2^{-q}f^{N+k}_{\pi/t}(t)$. Then,

$$\left|f_{\pi/t}(t) - E(q,N,t)\right| \leq \frac{t\,\mathrm{e}^{\gamma t}\,|f'(0) - \gamma f(0)|}{\pi^2}\frac{N!\,(q+1)!}{2^q\,(N+q+2)!} + \mathcal{O}\left(\frac{1}{N^{q+3}}\right)$$

when N goes to infinity.

In practice, for $q = N = 15$ and $\gamma = 11.5/t$, we have $\mathrm{e}^{\gamma t}\frac{N!\,(q+1)!}{2^q\,(N+q+2)!} \approx 3.13 \times 10^{-10}$, and it is therefore sufficient to make the summation accurate up to the 9^{th} decimal place. On the other hand, we have $|f_{\pi/t}(t) - f(t)| \leq \|f\|_\infty \frac{e^{-2\gamma t}}{1-e^{-2\gamma t}}$ from (2.2), which is of order 10^{-10}. Finally, the overall error is of order 10^{-10}. Note that, for a fixed t, the computation cost of $E(q,N,t)$ is proportional to $N+q$.

3 Calibration to CDS data and numerical results

In this section, we want to illustrate how the model presented in this paper can be calibrated to the CDS market data. Here, our aim is not to provide the ultimate calibration procedure for the model. This task requires to have a market feedback, and we let it to practitioners. We have decided instead to make one of the simplest choice, and we minimize the Euclidean distance between the theoretical and market CDS prices. Thus, we want to illustrate on market data picked from the past in which cases the model seems to give a rather good fit.

First, we will recall briefly what is a Credit Default Swap and give its theoretical price under the model. Then, we will explain in detail our calibration procedure. Last, we will discuss the calibration results on several cases.

3.1 Pricing of CDS

Credit Default Swaps are products that provide a financial protection on a given period in exchange of regular payments if a firm goes bankrupt. Here, we describe a synthetic CDS on a unit notional value that starts at time 0, with a maturity T and a payment grid $T_0 = 0 < T_1 < \cdots < T_n = T$. Usually, payments occur quarterly. For $t \in [0,T)$, $\beta(t)$ denotes the index in $\{1,\ldots,n\}$ of the next payment date, i.e. such that $T_{\beta(t)-1} \leq t < T_{\beta(t)}$.

If the default happens before T, the default leg pays the fraction LGD of the notional that is not recovered (loss given default). For sake of simplicity, we assume that $LGD \in [0,1]$ is deterministic. Since we also consider a constant interest rate $r > 0$, the default leg price is then given by

$$DL(0,T) = \mathbb{E}[\mathrm{e}^{-r\tau}\, \mathbf{1}_{\{\tau \le T\}} LGD] = LGD\left[\mathrm{e}^{-rT}\, \mathbb{P}(\tau \le T) + \int_0^T r\,\mathrm{e}^{-ru}\, \mathbb{P}(\tau \le u)du\right]. \quad (14)$$

The payment leg consists in regular (time-proportional) payments up to time $\tau \wedge T$. This means that they occur until the maturity T as long as the firm has not defaulted yet. The rate R of these payments is decided at the beginning of the CDS contract, and the price at time 0 of the payment leg is given by:

$$PL(0,T) = R \times \mathbb{E}\left[\sum_{i=1}^{n} (T_i - T_{i-1})\,\mathrm{e}^{-rT_i}\, \mathbf{1}_{\{\tau > T_i\}} + (\tau - T_{\beta(\tau)-1})\,\mathrm{e}^{-r\tau}\, \mathbf{1}_{\{\tau \le T\}}\right].$$

By integrating by parts, we get that

$$\begin{aligned}
\mathbb{E}[(\tau - T_{\beta(\tau)-1})\,\mathrm{e}^{-r\tau}\, \mathbf{1}_{\{\tau \le T\}}] &= -\int_0^T \mathrm{e}^{-ru}(u - T_{\beta(u)-1})d\mathbb{P}(\tau > u) \\
&= -\sum_{i=1}^{n} \mathrm{e}^{-rT_i}(T_i - T_{i-1})\mathbb{P}(\tau > T_i) + \int_0^T \mathrm{e}^{-ru}\, \mathbb{P}(\tau > u)du \\
&\quad -\int_0^T r\,\mathrm{e}^{-ru}(u - T_{\beta(u)-1})\mathbb{P}(\tau > u)du,
\end{aligned}$$

and therefore, we obtain that

$$PL(0,T) = R\left[\int_0^T \mathrm{e}^{-ru}\, \mathbb{P}(\tau > u)du - \int_0^T r\,\mathrm{e}^{-ru}(u - T_{\beta(u)-1})\mathbb{P}(\tau > u)du\right]. \quad (15)$$

The second term in the bracket can often be neglected in practice, but we have not made this approximation for our numerical experiments. We remark also that this is the only term that depends on the time-grid structure. This is the reason why we do not recall this dependency in our notations for the payment leg that mainly depends on the starting and ending dates.

Up to now[1], the market practice has been to quote the fair CDS spread $R(0,T)$ that makes both legs equal:

$$R(0,T) = LGD\frac{\mathrm{e}^{-rT}\, \mathbb{P}(\tau \le T) + \int_0^T r\,\mathrm{e}^{-ru}\, \mathbb{P}(\tau \le u)du}{\int_0^T \mathrm{e}^{-ru}\, \mathbb{P}(\tau > u)du - \int_0^T r\,\mathrm{e}^{-ru}(u - T_{\beta(u)-1})\mathbb{P}(\tau > u)du}. \quad (16)$$

This rate depends on the default time only through its cumulative distribution function $(\mathbb{P}(\tau \le t), t \in [0,T])$. In our model, we get the following result.

[1]The ISDA has recommended in early 2009 to switch and to quote CDS through the upfront value $U(0,T)$ such that $U(0,T) + PL(0,T) = DL(0,T)$. The CDS spread R is then standardized to some specific values. (see `www.cdsmodel.com/information/cds-model`)

Proposition 3.1. *With a deterministic interest rate $r > 0$ and a deterministic recovery rate $1 - LGD \in [0,1]$, the CDS price with the intensity model (2) is given by:*

$$R^{\mathbf{model}}(0,T) = LGD \frac{\mathrm{e}^{-rT} P_{b,m,\mu_1,\mu_2}(T) + \int_0^T r\,\mathrm{e}^{-ru} P_{b,m,\mu_1,\mu_2}(u)du}{\int_0^T \mathrm{e}^{-ru} P^c_{b,m,\mu_1,\mu_2}(u)du - \int_0^T r\,\mathrm{e}^{-ru}(u - T_{\beta(u)-1}) P^c_{b,m,\mu_1,\mu_2}(u)du},$$

where $b = \frac{1}{\sigma}\log(C/V_0)$ and $m = \frac{1}{\sigma}(r - \alpha - \sigma^2/2)$. Moreover, if we neglect the second integral in the denominator this rate is nondecreasing with respect to C, α, μ_1 and μ_2, and we get the following bounds:

$$\mu_1 \lesssim \frac{R^{\mathbf{model}}(0,T)}{LGD} \lesssim \mu_2. \tag{17}$$

Proof. The monotonicity properties is a direct consequence of Proposition A.1. Let us prove (17). From (2), we clearly have $\mu_1 \le \lambda_t \le \mu_2$ for any $t \ge 0$. From (3), we have $P^c_{b,m,\mu_1,\mu_2}(t) = \mathbb{E}[\mathrm{e}^{-\int_0^t \lambda_s ds}]$ and then:

$$\mathrm{e}^{-\mu_2 t} \le P^c_{b,m,\mu_1,\mu_2}(t) \le \mathrm{e}^{-\mu_1 t},\ 1 - \mathrm{e}^{-\mu_1 t} \le P_{b,m,\mu_1,\mu_2}(t) \le 1 - \mathrm{e}^{-\mu_2 t}.$$

Plugging these inequalities in (14) and (15), we get:

$$\begin{aligned} \frac{\mu_1}{r+\mu_1}(1 - \mathrm{e}^{-(r+\mu_1)T}) &\le DL^{\mathbf{model}}(T) \le \frac{\mu_2}{r+\mu_2}(1 - \mathrm{e}^{-(r+\mu_2)T}), \\ \frac{1}{r+\mu_1}(1 - \mathrm{e}^{-(r+\mu_2)T}) &\lesssim PL^{\mathbf{model}}(T) \lesssim \frac{1}{r+\mu_2}(1 - \mathrm{e}^{-(r+\mu_1)T}), \end{aligned}$$

neglecting $\int_0^T r\,\mathrm{e}^{-ru}(u - T_{\beta(u)-1}) P^c_{b,m,\mu_1,\mu_2}(u)du$. Then, we easily get (17). □

3.2 The Calibration procedure

Now, we want to describe the calibration method we have used in our numerical experiments. We denote by $T^{(1)} < \cdots < T^{(\nu)}$ the maturities of the quoted CDS, and $R^{\mathbf{market}}(0,T^{(1)}), \dots, R^{\mathbf{market}}(0,T^{(\nu)})$ their market prices. In practice, we have $\nu = 8$ market data sets for

$$T^{(1)} = 0.5,\ T^{(2)} = 1,\ T^{(3)} = 2,\ T^{(4)} = 3,\ T^{(5)} = 4,\ T^{(6)} = 5,\ T^{(7)} = 7 \text{ and } T^{(8)} = 10 \text{ years}, \tag{18}$$

and quarterly payments. Our goal is to minimize the following distance between model and market prices:

$$\min_{b,m\in\mathbb{R},0<\mu_1<\mu_2} \sum_{i=1}^{\nu} (R^{\mathbf{model}}(0,T^{(i)}) - R^{\mathbf{market}}(0,T^{(i)}))^2. \tag{19}$$

As already mentioned, there are probably better criteria to be minimized according to the market data and the purpose of the calibration. Here, we do not wish to discuss this point, but we rather want to qualitatively show what kind of CDS rate curves $T \mapsto R^{\mathbf{market}}(0,T)$ the model can fit. That is why we have chosen a very simple criterion to minimize.

To minimize (19), we simply use a gradient algorithm, which is very fast and takes advantage of the closed formula (4) and the Laplace inversion methods presented in Section 2. To do so, we need to compute the CDS prices $R^{\mathbf{model}}(0,T^{(i)})$ and their derivatives with respect to each parameter $p \in \{b, m, \mu_1, \mu_2\}$. In Section 2.2, we have explained in detail how to recover $P_{b,m,\mu_1,\mu_2}(t)$ on a time-grid from its Laplace transform (4) using the FFT. More precisely, we have used the FFT parameters given by (12) with $\varepsilon = 10^{-5}$. Similarly, we obtain by FFT the derivatives $\partial_p P_{b,m,\mu_1,\mu_2}(t)$ on the same time-grid. Their Laplace transforms can be obtained by simply differentiating formula (4). However, we have noticed that finite differences can also be used as a good proxy of the derivatives. Then, it is easy to compute the default and payment legs and their sensitivities with respect to each parameter. Numerical integration is performed using Simpson's rule. This is very efficient thanks to the regularity of the cdf (Proposition A.3). Last, we compute CDS prices and their derivatives.

To test this calibration procedure, we have computed CDS prices in our model considering them as Market data, and then we have try to find the parameters back by minimizing (19). The minimization is really fast and takes very few seconds. Thanks to (17), we start the gradient algorithm from the point

$$b = 0,\ m = 0,\ \mu_1 = \min_{i=1,\dots,\nu} R^{\mathbf{market}}(0,T^{(i)})/LGD,\ \mu_2 = \max_{i=1,\dots,\nu} R^{\mathbf{market}}(0,T^{(i)})/LGD.$$

Unfortunately, it sometimes fails and the gradient algorithm is trapped in local minima. This is partly due to a rather sensitive dependency between the parameters b and m. Then, it can be worth starting the gradient algorithm from a point where these parameters are both non zero. However, it is difficult to have a guess on the values of b and m. We have used the following way to get a prior on (b, m).

- We take a finite set $\mathcal{S} \subset \mathbb{R}^2$, typically $\mathcal{S} = \{-B + 2iB/n, i = 0,\dots,n\} \times \{-M + 2iM/n, i = 0,\dots,n\}$ for some $B, M > 0, n \in \mathbb{N}^*$. For $(b,m) \in \mathcal{S}$, we minimize the criterion (19) with respect to μ_1 and μ_2, keeping b and m constant. In practice, we have mostly taken $B, M \in \{1, 2\}$ and $n = 8$.
- Then, we select the couple $(b,m) \in \mathcal{S}$ that achieves the smallest score and use it (with the optimized parameters μ_1 and μ_2) as the initial point of the gradient algorithm for (19).

This procedure generally improves the basic one. However, our minimization problem is ill-posed and significantly different parameters can lead to rather close CDS rates. Let us take the case of a constant intensity model $\lambda > 0$, which leads to a flat CDS rate curve from (17). This case corresponds to many different sets of parameters in our model, namely:

1. $\mu_1 = \mu_2 = \lambda$, with $b, \lambda \in \mathbb{R}$ arbitrarily chosen,
2. $\mu_1 = \lambda, b \to -\infty$, with $m \in \mathbb{R}$ and $\mu_2 > \mu_1$ arbitrarily chosen,
3. $\mu_2 = \lambda, b \to +\infty$, with $m \in \mathbb{R}$ and $\mu_2 > \mu_1$ arbitrarily chosen.

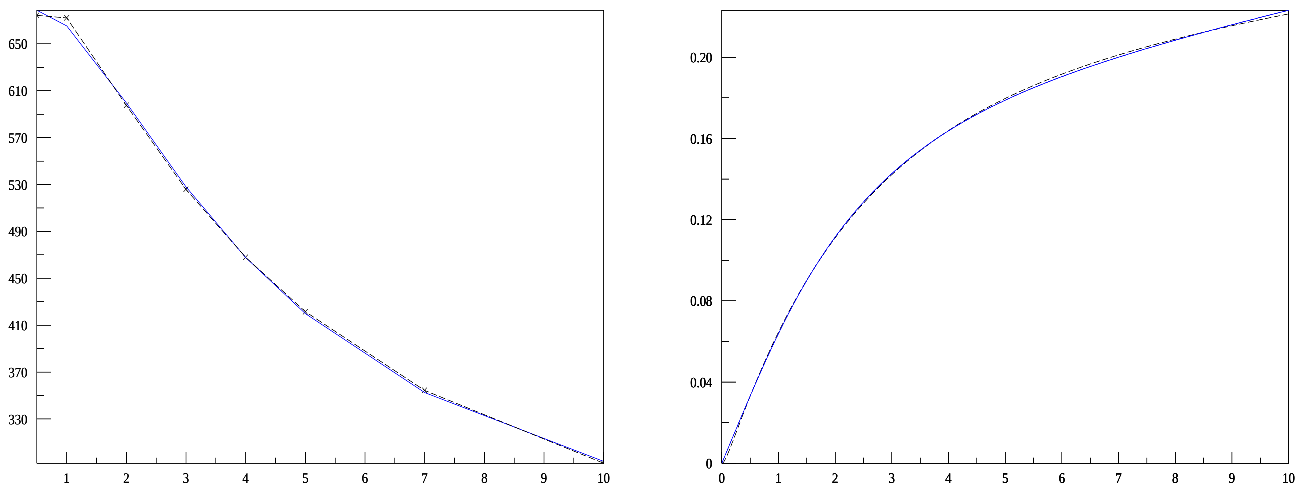

Figure 1: In the l.h.s. picture are plotted the CDS prices as function of the maturities (18). Prices are given in basis points (10^{-4}) with $LGD = 1$ and $r = 5\%$. The r.h.s picture shows the corresponding cumulative distribution functions. The dashed line is obtained with $b = -0.2$, $m = 0.6$, $\mu_1 = 0.005$ and $\mu_2 = 0.3$ and the solid line is with $b = 2.168849$, $m = 0.912237$, $\mu_1 = 0.008414$ and $\mu_2 = 0.067515$.

Thus, calibrating very flat CDS spreads can lead to many different satisfactory parameter configurations. We have found other less trivial examples when testing our calibration procedure. In Figure 1, are given two sets of parameters which give CDS prices which are close up to a 1% relative error but have very similar cdfs. This shows that simply calibrating the model to the only CDS prices (that only depend on the default cdf) can be insufficient to determine parameters in an univocal manner. Further information on the dependency between the firm value and the default event can be necessary in some cases for that.

3.3 Calibration on Market data

Now, we want to give calibration results under very different CDS rate data. Here, we chose to calibrate all the four parameters (b, m, μ_1, μ_2) to check if they are sufficient to fit the market data well. However, some of these parameters have an economic meaning. For example, the firm value can be related to its balance sheet and any other relevant information available in practice. In that case, one would like to fix some parameters or restrict them to lie in some interval. Here, for the sake of simplicity, we only consider the information given by the CDS prices and leave a more elaborated calibration for further research.

We have picked up very different examples from 2006 to 2009 on Crédit Agricole (bank, CA in short), PSA, Ford (car companies) and Saint-Gobain (glass maker, SG in short). In all our examples, we have set $LGD = 0.6$, except for Crédit Agricole for which we have taken $LGD = 0.8$ as it is commonly done for bank companies. We have also taken $r = 5\%$ for the sake of simplicity, since r has anyway a rather minor impact on the CDS spread values. The maturities observed on the market are the one listed in (18). In all the figures, we have plotted in dotted lines the CDS market data and in solid lines the CDS prices obtained with the calibrated model. Prices are given in basis points (10^{-4}). For each example, we give the calibrated parameters (b, m, μ_1, μ_2). To interpret them into the original firm value framework, we have also indicated the corresponding values of V_0/C and α using (26), taking the one-year at-the-money implied volatility as a proxy of the firm value volatility. However, as pointed in Section 3.2, significantly different parameters can lead to analogous CDS prices. The calibration to CDS prices only allows to fit the default cdf. This is why we have added in each case a subplot of the calibrated cdf, $(P_{b,m,\mu_1,\mu_2}(t), t \in [0, T^{(8)}])$.

We have split the results into three classes.

- The curve $T \mapsto R^{\mathbf{market}}(0, T)$ is mostly increasing. Roughly speaking, it happens when the firm's future is more unsure than its present.
- The curve $T \mapsto R^{\mathbf{market}}(0, T)$ is mostly decreasing. This usually means that the firm is in a critical period. If it overcomes this time, its future will be less risky.
- Most of the market data correspond to the two previous cases. However, when a firm switches from one regime to the other, the CDS curve tends to be flat, keeping often however a gentle slope.

3.3.1 Increasing CDS spreads

We start with data before the subprime crisis on companies that presented a low risk profile. Their calibration are plotted in Figure 2. Not surprisingly, in this case the model is able to fit the prices well, with a relative error of a few percents. As one could expect, the firm value starts in both cases above the threshold C in the "μ_2 region" and is drifted to the "μ_1 region" since the parameter m is negative (or equivalently, $\alpha > r - \sigma^2/2$).

We have also considered increasing patterns with a higher level of risk, and the calibrating results are drawn in Figure 3. The Ford curve (left) is really well fitted. The Saint-Gobain rates (left) are globally well captured, but some irregularities are smoothed by the calibrated curve. Once again, the firm value starts above the threshold in the safer side, which confirms the heuristic interpretation made above on increasing CDS curves.

3.3.2 Decreasing CDS spreads

Now, we want to test if the model is also able to fit decreasing CDS curves. As already mentioned, it happens when a firm goes through a difficult period. We give in Figure 4 two stressed examples on Ford company, taken at the climax of its crisis in November 2008 (left) and in February 2009 (right). Both curves are correctly fitted. The most significant

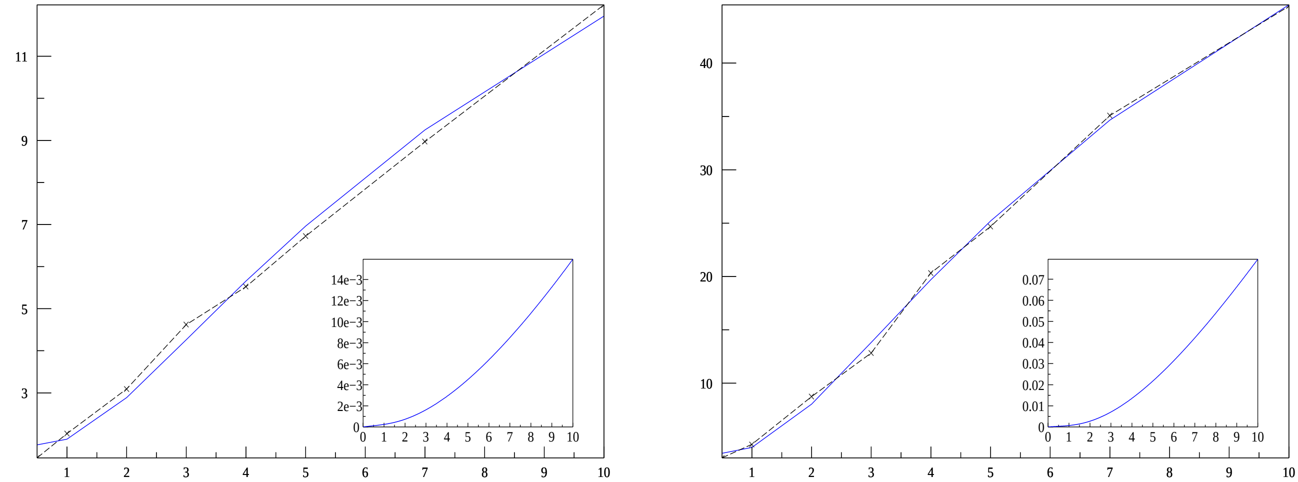

Figure 2: *Left, CA 08/31/06:* $b = -2.3415$, $m = -0.2172$, $\mu_1 = 2.164 \times 10^{-4}$, $\mu_2 = 5.597 \times 10^{-3}$, $V_0/C = 1.753$, $\alpha = -1.78 \times 10^{-2}$. *Right, PSA 05/03/06:* $b = -2.3878$, $m = -0.3745$, $\mu_1 = 5.581 \times 10^{-4}$, $\mu_2 = 2.214 \times 10^{-2}$, $V_0/C = 1.757$, $\alpha = 2.038 \times 10^{-2}$.

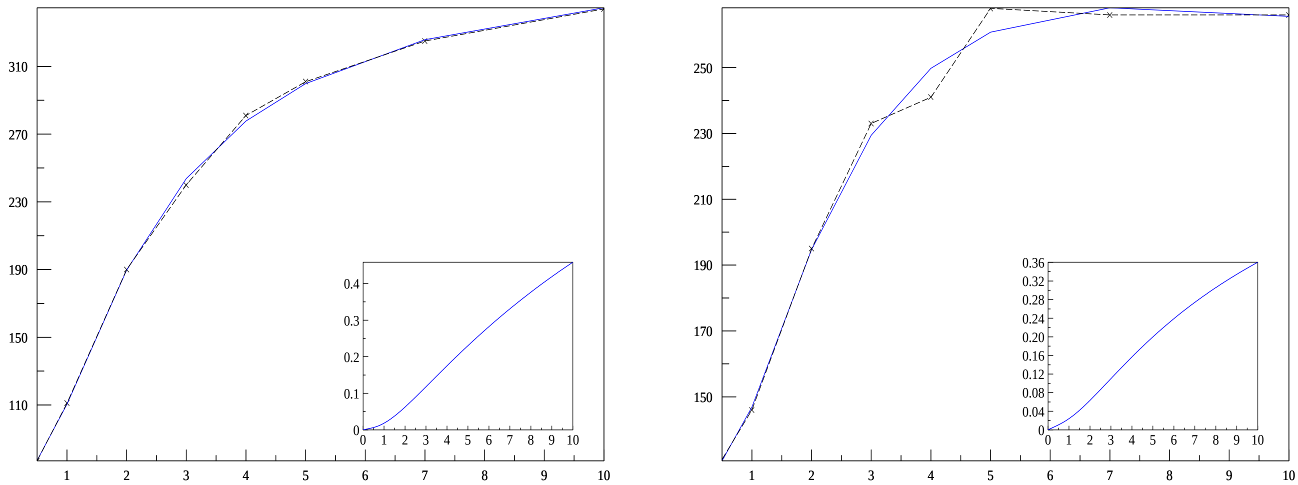

Figure 3: *Left, Ford 11/30/06:* $b = -1.734$, $m = -1.363$, $\mu_1 = 1.2 \times 10^{-2}$, $\mu_2 = 7.05 \times 10^{-2}$, $V_0/C = 2.173$, $\alpha = 0.436$. *Right, SG 10/08/08:* $b = -1.897$, $m = 0.1725$, $\mu_1 = 2.135 \times 10^{-2}$, $\mu_2 = 0.652$, $V_0/C = 2.8506$, $\alpha = -0.3213$.

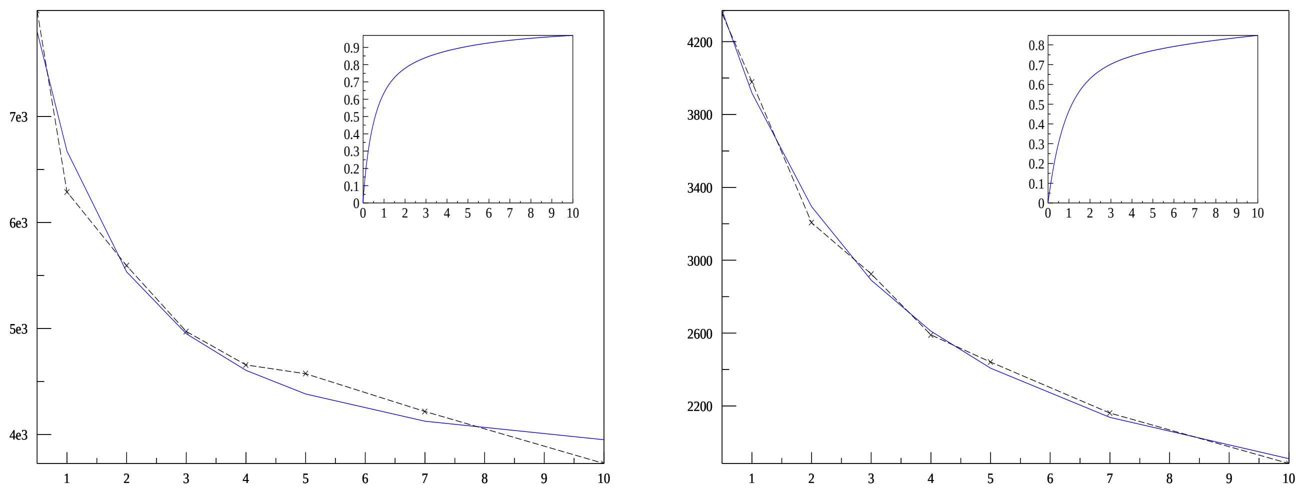

Figure 4: *Left, Ford 11/24/08:* $b = 0.209$, $m = 0.344$, $\mu_1 = 0.2014$, $\mu_2 = 1.986$, $V_0/C = 0.716$, $\alpha = -1.3$. *Right, Ford 02/25/09:* $b = 0.8517$, $m = 0.5277$, $\mu_1 = 6.85 \times 10^{-2}$, $\mu_2 = 0.7806$, $V_0/C = 0.3355$, $\alpha = -1.2676$

relative difference between market and model prices is equal to 6% on November data and 2% on February data. As expected, in both cases, the firm value starts below the threshold in the "μ_2 region" and goes gradually to the "μ_1 region" since $m > 0$ (or equivalently, $\alpha < r - \sigma^2/2$).

Now, we want to test the model on decreasing but less stressed patterns. We also want to see if it can in addition fit an initial bump. Indeed, it happens quite often on decreasing curves that the 6-month rate is however lower than the one-year rate. Roughly speaking, this means that the firm is in difficulty but the market however believes that it has some guarantee to live in the very short future. We have drawn in Figure 5 two examples on PSA (left) and Saint-Gobain (right). In the first case, the model does not seem able to replicate the initial bump, but the remaining part of the curve is well fitted. The bump is approximated by a flat curve in between. Doing this, the gradient algorithm explores rather large and unrealistic parameters for b and m. Instead, on the Saint-Gobain example, the whole shape is well fitted with very rational parameters.

3.3.3 Almost flat CDS spreads

Last, we give two examples of rather flat CDS rate curves. This kind of pattern is more uncommon and is observed in particular when a firm switches from an increasing to a decreasing curve like Saint-Gobain between 10/08/08 (Fig. 3) and 12/01/08 (Fig. 5). Flat curves are a priori not very difficult to fit since a constant intensity model can already give a first possible approximation. We show in Figure 6 the transition made by the Saint-

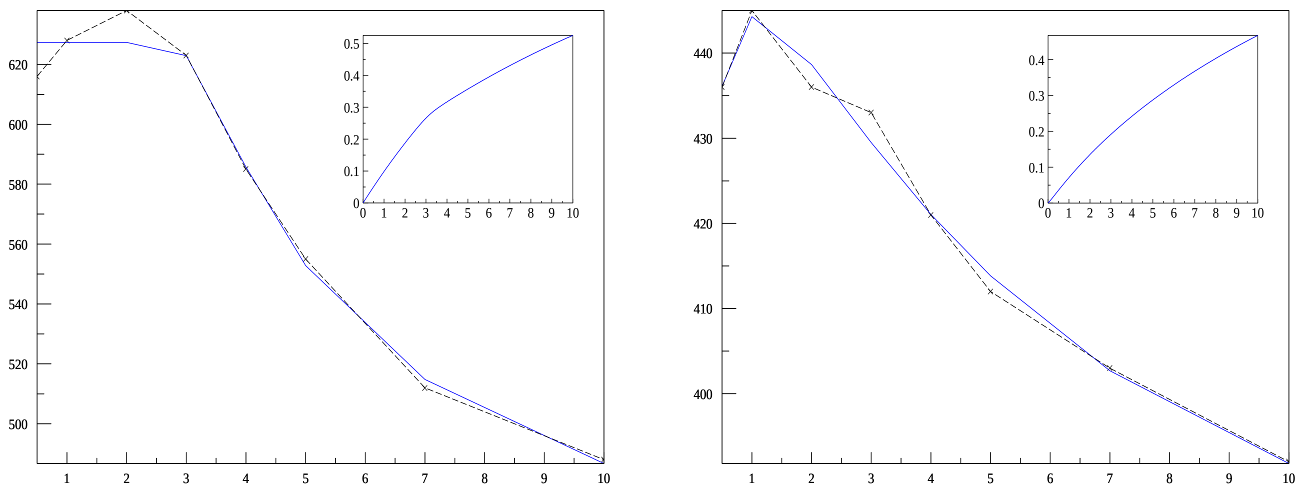

Figure 5: *Left, PSA 03/06/09:* $b = 15.55$, $m = 4.889$, $\mu_1 = 6.055 \times 10^{-2}$, $\mu_2 = 0.104$, $V_0/C = 6.32 \times 10^{-5}$, $\alpha = -3.3$. *Right, SG 12/01/08:* $b = -0.268$, $m = 0.567$, $\mu_1 = 5.46 \times 10^{-2}$, $\mu_2 = 0.154$, $V_0/C = 1.1837$, $\alpha = -0.6213$.

Gobain curve. On these flat shapes, the fitting is really good and the relative error on prices does not exceed 1%.
Let us draw a short conclusion on these calibration results. The model is able to fit a wide range of CDS data, from a very low risk level (Fig. 2) to highly stressed spreads (Fig. 4) as well as intermediate settings (Fig. 3, 5, 6) that are more frequently observed. Of course, not all the prices are perfectly matched, but the spread curves are globally well captured. Concerning the meaning of the parameters, one has to be careful since only calibrating to the CDS rates is a priori unsufficient to determine them (see Fig. 1). However, at least in the extreme settings, the values of V_0/C and α that we have obtained are as expected greater (resp. lower) than 1 and $r - \sigma^2/2$ in Fig. 2 (resp. Fig. 4), which means that the firm value gradually shifts from the μ_1 (resp. μ_2) to the μ_2 (resp. μ_1) zone.

4 Extension to multiple barriers and default intensities

In this section, we consider an extension to multiple barriers. More precisely, let us set $n \geq 2$, $C_0 = +\infty > C_1 > \cdots > C_n = 0$, $0 \leq \mu_1 < \mu_2 < \cdots < \mu_n$ and consider a model with the following default intensity:

$$\lambda_t = \sum_{i=1}^{n} \mu_i \mathbf{1}_{\{C_i\, e^{\alpha t} \leq V_t < C_{i-1}\, e^{\alpha t}\}}.$$

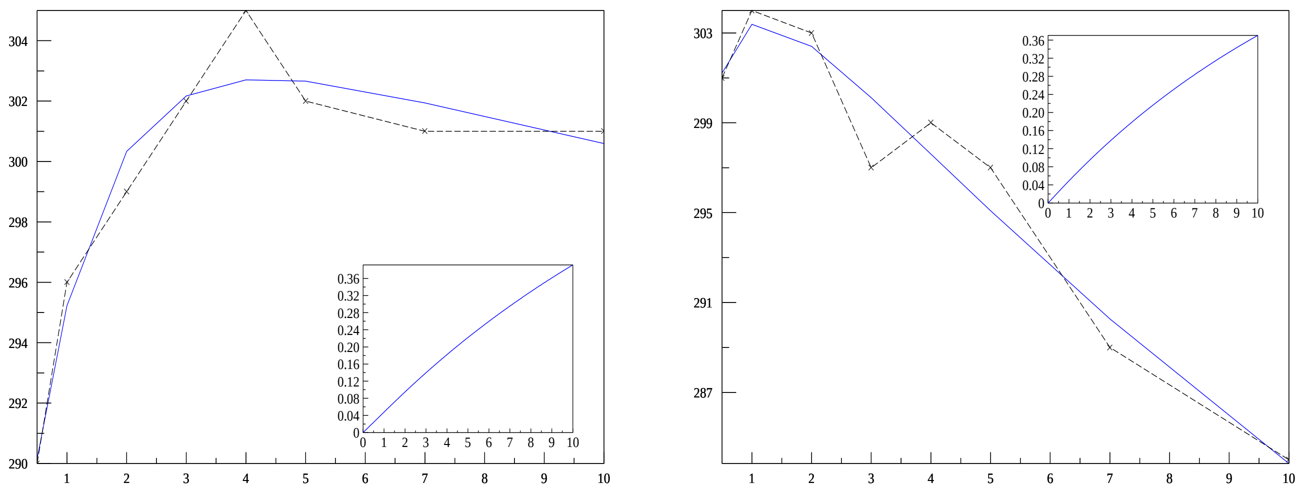

Figure 6: *Left, SG 10/21/08:* $b = -1.032$, $m = 0.493$, $\mu_1 = 4.75 \times 10^{-2}$, $\mu_2 = 9.23 \times 10^{-2}$, $V_0/C = 1.83$, $\alpha = -0.531$. *Right, SG 10/31/08:* $b = -3.42 \times 10^{-2}$, $m = 4.69 \times 10^{-2}$, $\mu_1 = 1.45 \times 10^{-2}$, $\mu_2 = 9.295 \times 10^{-2}$, $V_0/C = 1.021$, $\alpha = -0.282$.

This means that the default intensity is increased (resp. decreased) each time it crosses downward (resp. upward) a barrier. Heuristically, these constant intensities can be related to the credit grades of the firm. For a firm in difficulty, crossing downward the barriers can also represent the different credit events that precede a bankruptcy.

Theorem 4.1. *We set* $x = \frac{1}{\sigma}\log(V_0)$ *and* $x_i = \frac{1}{\sigma}\log(C_i)$ *for* $i = 0, \dots, n$. *In the above setting,* $\mathbb{P}(\tau \le t)$ *is a function of* t, x, $(x_i)_{i=1,\dots,n-1}$, m *and* $(\mu_i)_{i=1,\dots,n}$. *It is characterized by its Laplace transform that is defined for* $z \in \mathbb{C}_+$*:*

$$L_{b_i,m,\mu_i}(z) = \sum_{i=1}^{n} \mathbf{1}_{\{x_i \le x < x_{i-1}\}} \left\{ \frac{1}{z} - \frac{1}{z+\mu_i} - \beta_i^+ \, \mathrm{e}^{R_+(\mu_i)x} - \beta_i^- \, \mathrm{e}^{R_-(\mu_i)x} \right\},$$

where $R_\pm(\mu) = -m \pm \sqrt{m^2 + 2(z+\mu)}$. *The coefficients* $\beta_i = [\beta_i^- \quad \beta_i^+]'$ *are uniquely determined by the induction:*

$$\beta_i = \Pi_{i-1}\beta_1 + v_{i-1}, \quad i = 1\dots, n$$

and the conditions $\beta_1^+ = \beta_n^- = 0$. *Here,* $\Pi_0 = Id$ *and* $\Pi_i = P_i \times \cdots \times P_1$, $v_0 = 0$ *and* $v_i = A^{-1}(\mu_{i+1}, x_i)\left[\frac{1}{z+\mu_i} - \frac{1}{z+\mu_{i+1}} \quad 0\right]' + P_i v_{i-1}$ *with:*

$$\begin{aligned} P_i \;=\; & \frac{1}{[R_+(\mu_{i+1}) - R_-(\mu_{i+1})]} \times \qquad (20) \\ & \begin{bmatrix} (R_+(\mu_{i+1}) - R_-(\mu_i))\,\mathrm{e}^{x_i(R_-(\mu_i)-R_-(\mu_{i+1}))} & (R_+(\mu_{i+1}) - R_+(\mu_i))\,\mathrm{e}^{x_i(R_+(\mu_i)-R_-(\mu_{i+1}))} \\ (R_-(\mu_i) - R_-(\mu_{i+1}))\,\mathrm{e}^{x_i(R_-(\mu_i)-R_+(\mu_{i+1}))} & (R_+(\mu_i) - R_-(\mu_{i+1}))\,\mathrm{e}^{x_i(R_+(\mu_i)-R_+(\mu_{i+1}))} \end{bmatrix} \end{aligned}$$

and

$$A^{-1}(\mu_{i+1},x_i)=\frac{1}{R_+(\mu_{i+1})-R_-(\mu_{i+1})}\begin{bmatrix} R_+(\mu_{i+1})\,\mathrm{e}^{-R_-(\mu_{i+1})x_i} & -\,\mathrm{e}^{-R_-(\mu_{i+1})x_i}\\ -R_-(\mu_{i+1})\,\mathrm{e}^{-R_+(\mu_{i+1})x_i} & \mathrm{e}^{-R_+(\mu_{i+1})x_i}\end{bmatrix}.$$

Proof. We introduce for $t\geq 0$, $X^x_t=\frac{1}{\sigma}\ln(V_t\,\mathrm{e}^{-\alpha t})$, where $x=\frac{1}{\sigma}\ln(V_0)$ denotes its starting point. We have $dX^x_t=dW_t+mdt$ and therefore X^x_t is a drifted Brownian motion. The default intensity can then be written as a function of X_t:

$$\lambda_t=\sum_{i=1}^n \mu_i \mathbf{1}_{\{x_i\leq X_t<x_{i-1}\}}=:\lambda(X_t).$$

Let us introduce the survival probabilities $p(t,x)=\mathbb{P}(\tau>t)=\mathbb{E}[\exp(-\int_0^t\lambda(X^x_s)ds)]$. Now, we use a result from Kac ([12], Theorem 4.9 p.271), using in addition the Girsanov theorem when $m\neq 0$. It comes out that for $z>0$, the Laplace transform $L^c(z,x)=\int_0^\infty \mathrm{e}^{-zt}\,p(t,x)dt$ is $\mathcal{C}^1$ and piecewise $\mathcal{C}^2$ w.r.t. x, and solves:

$$\forall i,\ \frac{1}{\sigma}\ln(C_i)\leq x<\frac{1}{\sigma}\ln(C_{i-1}),\ 1-(z+\mu_i)L^c(z,x)+m\partial_x L^c(z,x)+\frac{1}{2}\partial^2_x L^c(z,x)=0. \quad (21)$$

This is an affine ODE of order 2 which admits the following solutions:

$$x_i\leq x<x_{i-1},\ L^c(z,x)=\frac{1}{z+\mu_i}+\beta_i^-\,\mathrm{e}^{R_-(\mu_i)x}+\beta_i^+\,\mathrm{e}^{R_+(\mu_i)x}.$$

Now, we write that the Laplace transform is $\mathcal{C}^1$ at x_i for $i=1,\dots,n-1$:

$$\begin{cases}\beta_i^-\,\mathrm{e}^{R_-(\mu_i)x_i}+\beta_i^+\,\mathrm{e}^{R_+(\mu_i)x_i}=\frac{1}{z+\mu_{i+1}}-\frac{1}{z+\mu_i}+\beta_{i+1}^-\,\mathrm{e}^{R_-(\mu_{i+1})x_i}+\beta_{i+1}^+\,\mathrm{e}^{R_+(\mu_{i+1})x_i}\\ \beta_i^-R_-(\mu_i)\,\mathrm{e}^{R_-(\mu_i)x_i}+\beta_i^+R_+(\mu_i)\,\mathrm{e}^{R_+(\mu_i)x_i}=\beta_{i+1}^-R_-(\mu_{i+1})\,\mathrm{e}^{R_-(\mu_{i+1})x_i}+\beta_{i+1}^+R_+(\mu_{i+1})\,\mathrm{e}^{R_+(\mu_{i+1})x_i}.\end{cases} \quad (22)$$

We rewrite this in a matrix form:

$$A(\mu_i,x_i)\begin{bmatrix}\beta_i^-\\ \beta_i^+\end{bmatrix}=\begin{bmatrix}\frac{1}{z+\mu_{i+1}}-\frac{1}{z+\mu_i}\\ 0\end{bmatrix}+A(\mu_{i+1},x_i)\begin{bmatrix}\beta_{i+1}^-\\ \beta_{i+1}^+\end{bmatrix},\ A(\mu,x)=\begin{bmatrix}\mathrm{e}^{R_-(\mu)x} & \mathrm{e}^{R_+(\mu)x}\\ R_-(\mu)\,\mathrm{e}^{R_-(\mu)x} & R_+(\mu)\,\mathrm{e}^{R_+(\mu)x}.\end{bmatrix} \quad (23)$$

We set for $i=1,\dots,n-1$, $P_i=A^{-1}(\mu_{i+1},x_i)A(\mu_i,x_i)$ that is given in explicit form in (20). We also set

$$v_0=0, v_i=A^{-1}\mu_{i+1},x_i)(\begin{bmatrix}\frac{1}{z+\mu_i}-\frac{1}{z+\mu_{i+1}} & 0\end{bmatrix}'+P_iv_{i-1} \text{ and } \Pi_0=Id, \Pi_i=P_i\dots P_1,$$

for $i=1\dots n-1$. We have that $\beta_n=\Pi_{n-1}\beta_1+v_{n-1}$. Since $L^c(z,+\infty)=1/(z+\mu_1)$ and $L^c(z,-\infty)=1/(z+\mu_n)$, we have $\beta_1^+=0$ and $\beta_n^-=0$. In particular, $(\Pi_{n-1})_{1,1}\beta_1^-+(v_{n-1})_1=0$ which uniquely determines β_1^- and gives that $(\Pi_{n-1})_{1,1}\neq 0$ since $L^c(z,x)$ is the unique solution of (21) (we can show indeed that $(\Pi_{n-1})_{1,1}>0$ because the entries of P_i are positive). Then, the coefficients β_i are also uniquely determined for $i=1,\dots,n$. In particular, when $n=2$, we find the formula stated in Theorem 1.1. Last, we point out that the formula obtained remains valid for $z\in\mathbb{C}_+$ since it is the only possible analytical extension. ☐

Remark 4.2. *The law of τ does not depend on V_0 and $C_0, \dots, C_n$ but only on $\frac{C_0}{V_0}, \dots, \frac{C_n}{V_0}$. So, we can arbitrary set $V_0 = 1$ provided that the values of the C_i are accordingly modified, which implies that $x = 0$ and $x_i = \frac{1}{\sigma}\log(C_i/V_0)$ for $i = 0, \dots, n$ in Theorem 4.1 and hence the formula for the Laplace transform of $\mathbb{P}(\tau \leq t)$ becomes*

$$L_{b_i,m,\mu_i}(z) = \sum_{i=1}^{n} \mathbf{1}_{\{x_i \leq 0 < x_{i-1}\}} \left\{ \frac{1}{z} - \frac{1}{z+\mu_i} - \beta_i^+ - \beta_i^- \right\}.$$

This remark implies that in the calibration procedure, there is no need to know σ.

5 Conclusion and further prospects

In this paper, we have proposed a very simple and natural extension of the Black-Cox model. It is an hybrid model, and contrary to hitting time models, it has a non-zero default intensity away from the threshold. Besides, the parameters have a clear heuristic meaning. The strength of this Black-Cox extension is that the cumulative distribution function of the default time remains known explicitly through its Laplace transform. This allows to instantaneously compute CDS prices and their sensitivities to the model parameters. It especially enables to get a quick way to calibrate the parameters to the CDS data. As shown in Section 3, it can correctly fit a wide range of CDS spread curves. Nonetheless, one has to be careful because even though this calibration generally leads to a correct fit of the default distribution, it may happen that the parameters themselves are not correct. Two significantly different parameter sets can give similar CDS spreads, and one has to get further information to neatly fit the parameters.

Rajouter des commentaires ici sur des recherches possibles sur d'autres calibrations: en exploitant le bilan d'une entreprise, en astreignant mu a prendre des valeurs discrètes corrsespsondant au rating...

As a continuation of this work, it would be interesting to study how this model can be used in the multi-name setting using the so-called bottom-up approach. More precisely, let us consider a basket of default times and let us assume that the underlying firm values follow a multidimensional Black-Scholes model. We have explained in this paper how it is possible to fit the CDS data of each basket component. Once we have fitted C, α, μ_1 and μ_2 for each firm, we would like to fit the whole model to multiname products such as CDO tranches. To do so, one has to calibrate the correlation matrix between the firm values and, if necessary, the dependency between the exponential variables that trigger the default times. However, the correlation matrix of the firm values is also closely related to the one of the stocks. Ideally, one would like to find a calibration procedure that is both consistent to equity and credit markets. More simply, this kind of model could make a bridge between these markets and qualitatively compare how they price the dependency between companies.

A Mathematical properties of the c.d.f. of τ

The scope of this section is to state some mathematical properties of the cumulative distribution function of τ. In particular, these properties will ensure the convergence of the two Laplace inversion algorithms considered in the paper. We will denote by $\Pi = \{(b, m, \mu_1, \mu_2),\ b, m \in \mathbb{R}, 0 \leq \mu_1 < \mu_2\}$ the set of admissible parameters.

A.1 Basic properties and regularity w.r.t parameters

First, we state a result on the monotonicity with respect to each parameter.

Proposition A.1. *For any $t \geq 0$, the function $P_{b,m,\mu_1,\mu_2}(t)$ is nondecreasing with respect to b, μ_1 and μ_2, and is nonincreasing with respect to m.*

Proof. From (3) and (26), it comes that

$$P^c_{b,m,\mu_1,\mu_2}(t) = \mathbb{E}\left[\mathrm{e}^{-\int_0^t \mu_2 \mathbf{1}_{\{W_u+mu\leq b\}}+\mu_1 \mathbf{1}_{\{W_u+mu>b\}}du}\right] = \mathrm{e}^{-\mu_1 t}\,\mathbb{E}\left[\mathrm{e}^{-\int_0^t (\mu_2-\mu-)\mathbf{1}_{\{W_u+mu\leq b\}}du}\right].$$

Using the first equality for μ_1 and μ_2, and the second one for b and m, we immediately get the result by pathwise comparison. $\square$

In the calculation of the Laplace transform in Section B, we have obtained two different formulas depending on the sign of b. However, $P_{b,m,\mu_1,\mu_2}(t)$ and its derivatives w.r.t each parameter are continuous functions of (b, m, μ_1, μ_2). This feature is important when dealing with calibration, since we will use a gradient algorithm to minimize some distance between real and theoretical prices: there is no discontinuity when crossing $b = 0$.

Proposition A.2. *The function $P_{b,m,\mu_1,\mu_2}(t)$ is continuous w.r.t. $(b, m, \mu_1, \mu_2) \in \Pi$, $t \geq 0$. It has derivative w.r.t. b, m, μ_1, μ_2, and the functions $\partial_b P_{b,m,\mu_1,\mu_2}(t)$, $\partial_m P_{b,m,\mu_1,\mu_2}(t)$, $\partial_{\mu_1} P_{b,m,\mu_1,\mu_2}(t)$, $\partial_{\mu_2} P_{b,m,\mu_1,\mu_2}(t)$ are continuous w.r.t. $(b, m, \mu_1, \mu_2) \in \Pi$, $t \geq 0$.*

Proof. From (28), we have

$$\begin{aligned} P^c_{b,m,\mu_1,\mu_2}(t) &= \mathrm{e}^{-\mu_1 t}\,\tilde{\mathbb{E}}[\mathrm{e}^{m\tilde{W}_t - m^2 t/2}\,\mathrm{e}^{-(\mu_2-\mu-)\int_0^t \mathbf{1}_{\{\tilde{W}_u\leq b\}}du}] \\ &= \mathrm{e}^{-\mu_1 t}\,\tilde{\mathbb{E}}[\mathrm{e}^{m\tilde{W}_t - m^2 t/2}\,\mathrm{e}^{-(\mu_2-\mu-)\int_{-\infty}^b \tilde{\ell}_t(x)dx}], \end{aligned}$$

where $\tilde{\ell}_t(x)$ denotes the local time associated to $(\tilde{W}_t, t \geq 0)$ and is continuous with respect to (t, x). Therefore, it is continuous w.r.t. $(b, m, \mu_1, \mu_2) \in \Pi$ and $t \geq 0$. Moreover, for each parameter, we can take the derivative in the expectation using Lebesgue's theorem and the derivative is continuous w.r.t. $(b, m, \mu_1, \mu_2) \in \Pi$ and $t \geq 0$, which yields the result. $\square$

A.2 Time regularity

To study the accuracy of the different algorithms presented in Section 3 to numerically invert the Laplace transform of τ, it is essential to know how regular the distribution function can be expected to be.

Proposition A.3. *When $b \neq 0$, the functions $P_{b,m,\mu_1,\mu_2}(t)$ and $\partial_p P_{b,m,\mu_1,\mu_2}(t)$ for $p \in \{b, m, \mu_1, \mu_2\}$ are of class $\mathcal{C}^\infty$ on $[0,\infty)$. Moreover, for any $\varepsilon > 0$, we have*

$$\forall k \in \mathbb{N}^*, P^{(k)}_{b,m,\mu_1,\mu_2}(t) \underset{t\to\infty}{=} O(\mathrm{e}^{(\varepsilon-\mu_1)t}) \text{ and } \forall k \in \mathbb{N}, \partial_p P^{(k)}_{b,m,\mu_1,\mu_2}(t) \underset{t\to\infty}{=} O(\mathrm{e}^{(\varepsilon-\mu_1)t}).$$

In particular, these functions are bounded on $\mathbb{R}_+$ when $\mu_1 > 0$.

When $b = 0$, P_{0,m,μ_1,μ_2} is of class $\mathcal{C}^1$ on $[0,\infty)$ but not $\mathcal{C}^2$ and of class $\mathcal{C}^\infty$ on $(0,\infty)$.

Remark A.4. *Since P_{b,m,μ_1,μ_2} is at least of class $\mathcal{C}^1$ on $[0,\infty)$, $\forall t < \infty \quad \mathbb{P}(\tau = t) = 0$.*

Proof of Proposition A.3. First, we consider the case $b \neq 0$.

$\boxed{b \neq 0}$: From Theorem 1.1, we know that the Laplace transform of P_{b,m,μ_1,μ_2} is given by

$$\begin{aligned} L_{b,m,\mu_1,\mu_2}(z) \quad = \quad & \mathrm{e}^{mb-|b|\sqrt{2(z+\mu_b)+m^2}}\left(\frac{1}{z+\mu_1} - \frac{1}{z+\mu_2}\right) \times \Bigg\{ -\mathbf{1}_{\{b>0\}} \\ & + \frac{-m+\sqrt{2(z+\mu_2)+m^2}}{\sqrt{2(z+\mu_1)+m^2}+\sqrt{2(z+\mu_2)+m^2}}\Bigg\} + \frac{1}{z} - \frac{1}{z+\mu_b}. \end{aligned}$$

We notice that $\frac{1}{z} - \frac{1}{z+\mu_b}$ is the Laplace transform of the cumulative density function of the exponential distribution with parameter μ_b.

For any $\varepsilon - \mu_1 > \gamma > -\mu_1$, we have

$$\begin{aligned} P_{b,m,\mu_1,\mu_2}(t) =& (1-\mathrm{e}^{-\mu_b t})\mathbf{1}_{\{t\geq 0\}} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \mathrm{e}^{(\gamma+is)t}\,\mathrm{e}^{mb-|b|\sqrt{2(\gamma+is+\mu_b)+m^2}} \\ & \left(\frac{1}{\gamma+is+\mu_1} - \frac{1}{\gamma+is+\mu_2}\right) \times \Bigg\{ -\mathbf{1}_{\{b>0\}} \\ & + \frac{-m+\sqrt{2(\gamma+is+\mu_2)+m^2}}{\sqrt{2(\gamma+is+\mu_1)+m^2}+\sqrt{2(\gamma+is+\mu_2)+m^2}}\Bigg\} ds \end{aligned}$$

The function $s \longmapsto s^k\,\mathrm{e}^{(\gamma+is)t}\,\mathrm{e}^{mb-|b|\sqrt{2(\gamma+is+\mu_b)+m^2}}\left(\frac{1}{\gamma+is+\mu_1} - \frac{1}{\gamma+is+\mu_2}\right) \times \Big\{ -\mathbf{1}_{\{b>0\}} + \frac{-m+\sqrt{2(\gamma+is+\mu_2)+m^2}}{\sqrt{2(\gamma+is+\mu_1)+m^2}+\sqrt{2(\gamma+is+\mu_2)+m^2}}\Big\}$ is integrable and continuous on $\mathbb{R}$ for all $k \in \mathbb{N}$, since $\mathcal{R}\mathrm{e}\left(\sqrt{2(\gamma+is+\mu_b)+m^2}\right) \underset{|s|\to+\infty}{\sim} \sqrt{s}$. Hence, the function P_{b,m,μ_1,μ_2} is of class $\mathcal{C}^\infty$ which implies that the random variable τ admits a density w.r.t Lebesgue's measure and moreover for every $k \in \mathbb{N}^*$ and all $t \geq 0$

$$P^{(k)}_{b,m,\mu_1,\mu_2}(t) = -\,(-\mu_b)^k\,\mathrm{e}^{-\mu_b t} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \mathrm{e}^{ist}(\gamma+is)^k\,\mathrm{e}^{\gamma t}\,\mathrm{e}^{mb-|b|\sqrt{2(\gamma+is+\mu_b)+m^2}} \left(\frac{1}{\gamma+is+\mu_1} - \frac{1}{\gamma+is+\mu_2}\right) \times \left\{ -\mathbf{1}_{\{b>0\}} + \frac{-m+\sqrt{2(\gamma+is+\mu_2)+m^2}}{\sqrt{2(\gamma+is+\mu_1)+m^2}+\sqrt{2(\gamma+is+\mu_2)+m^2}} \right\} ds$$

Using the Riemann-Lebesgue lemma ($\int_{-\infty}^{+\infty} f(s)\,\mathrm{e}^{ist}\,ds \underset{t\to+\infty}{\to} 0$ if f is integrable), we get that $P^{(k)}_{b,m,0,\mu}(t) = O(\mathrm{e}^{\gamma t})$ when $t \to \infty$. By differentiating this equation with respect to each parameter, we also get that $\partial_b P^{(k)}_{b,m,0,\mu}(t)$, $\partial_m P^{(k)}_{b,m,\mu_1,\mu_2}(t)$, $\partial_{\mu_1} P^{(k)}_{b,m,0,\mu}(t)$ and $\partial_{\mu_2} P^{(k)}_{b,m,0,\mu}(t)$ are $O(\mathrm{e}^{\gamma t})$.

$\boxed{b=0}$: Whereas when $b \neq 0$, the proof is based on the integrability of the Laplace transform given in Theorem 1.1, to treat the case $b = 0$, we use the expression of $P_{0,m,0,\mu}$ given by Equation (30). The first term in Equation (30) is obviously of class $\mathcal{C}^\infty$ on $[0,\infty)$. Using the change of variable $s = tu$, the second term of Equation (30) can be rewritten

$$\frac{\mathrm{e}^{-m^2t/2}}{\pi}\int_0^1 \frac{\mathrm{e}^{-\mu tu}}{\sqrt{u(1-u)}}du$$

and is therefore of class $\mathcal{C}^\infty$ on $[0,\infty)$. Let I_1, I_2, I_3, I_4 respectively denote the last four terms of Equation (30) that are between square brackets. Closely looking at the remaining integrals and performing the same change of variables, it clearly appears that we only have to consider two different types of integrals. For $\beta,\ \rho \in \mathbb{R}$, we introduce

$$J_1(\beta,\rho) = \int_0^1 \frac{\mathrm{e}^{\beta tu}\sqrt{t}}{\sqrt{u}}\Phi(\rho\sqrt{t}\sqrt{1-u})du \tag{24}$$

$$J_2(\beta,\rho) = \int_0^1 \frac{\mathrm{e}^{-m^2tu/2}}{\sqrt{t}\sqrt{u^3}}(1-\mathrm{e}^{-\beta tu})\Phi(\rho\sqrt{t}\sqrt{1-u})du. \tag{25}$$

An integration by parts in Equation (24) leads to

$$J_1(\beta,\rho) = \frac{\sqrt{t}}{2}\int_0^1 \frac{\mathrm{e}^{\beta tu}}{\sqrt{u}}du + \int_0^1 \left(\int_0^u \frac{\mathrm{e}^{\beta tv}}{\sqrt{v}}dv\right)\frac{\rho t\,\mathrm{e}^{-\rho^2t(1-u)/2}}{2\sqrt{2\pi}\sqrt{1-u}}du.$$

We notice that $I_1 + I_4 = \mathrm{e}^{-\mu t}(-J_1(-m^2/2, m) + J_1(-m^2/2, -m))$. Hence,

$$I_1 + I_4 = \frac{-mt\,\mathrm{e}^{-\mu t}}{\sqrt{2\pi}}\int_0^1 \left(\int_0^u \frac{\mathrm{e}^{-m^2tv/2}}{\sqrt{v}}dv\right)\frac{\mathrm{e}^{-m^2t(1-u)/2}}{\sqrt{1-u}}du.$$

This new formula makes clear that as a function of t, $I_1 + I_4$ is of class $\mathcal{C}^\infty$ on $[0,\infty)$. The term J_2 is handled by a similar integration by parts:

$$J_2(\beta,\rho) = \frac{1}{2}\int_0^1 \frac{\mathrm{e}^{-m^2tu/2}(1-\mathrm{e}^{-\beta tu})}{\sqrt{t}\sqrt{u^3}}du + \int_0^1 \left(\int_0^u \frac{\mathrm{e}^{-m^2tv/2}(1-\mathrm{e}^{-\beta tv})}{\sqrt{v^3}}dv\right)\frac{\rho}{2\sqrt{2\pi}\sqrt{1-u}}\,\mathrm{e}^{-\rho^2 t(1-u)/2}\,du$$

As a function of t, the second integral is clearly of class $\mathcal{C}^\infty$ on $[0,\infty)$ from Lebesgue's bounded convergence theorem. Noticing that $I_2 + I_3 = J_2(\mu,m) + J_2(-\mu,-m)\,\mathrm{e}^{-\mu t}$, $\int_0^1 \frac{\mathrm{e}^{-m^2tu/2}(1-\mathrm{e}^{-\beta tu})}{\sqrt{t}\sqrt{u^3}}du = \int_0^t \frac{\mathrm{e}^{-m^2s/2}(1-\mathrm{e}^{-\beta s})}{\sqrt{s^3}}ds$ and

$$\frac{d}{dt}\left(\int_0^t \frac{\mathrm{e}^{-m^2s/2}(1-\mathrm{e}^{-\mu s})}{\sqrt{s^3}}ds + \mathrm{e}^{-\mu t}\int_0^t \frac{\mathrm{e}^{-m^2s/2}(1-\mathrm{e}^{\mu s})}{\sqrt{s^3}}ds\right) = -\frac{\mu\,\mathrm{e}^{-\mu t}}{2}\int_0^t \frac{\mathrm{e}^{-m^2s/2}(1-\mathrm{e}^{\mu s})}{\sqrt{s^3}}ds$$

we get that $I_2 + I_3$ is (as a function of t) of class $\mathcal{C}^\infty$ on $(0,\infty)$ but only of class $\mathcal{C}^1$ on $[0,\infty)$ and not more. Finally, $P_{0,m,0,\mu}$ is of class $\mathcal{C}^\infty$ on $(0,\infty)$, but only of class $\mathcal{C}^1$ on the semi-closed interval $[0,\infty)$. □

B Calculation of the Laplace transform of the default time

While Theorem 1.1 can actually be deduced from Theorem 4.1, we provide in this section a standalone proof, split into three steps for the sake of clearness. The proof presented here relies on techniques developed by Chesney and al [9] and Labart and Lelong [13] for Parisian options. To get this result,

Step 1: Change of the probability measure and reduction to the case $\mu_1 = 0$

Since $V_u = V_0 \exp((r-\frac{1}{2}\sigma^2)u + \sigma W_u)$, we get $\mathbf{1}_{\{V_u \le C\,\mathrm{e}^{\alpha u}\}} = \mathbf{1}_{\{W_u + \frac{1}{\sigma}(r-\alpha-\sigma^2/2)u \le \frac{1}{\sigma}\log(C/V_0)\}}$. Let $t \ge 0$. We set for an arbitrary $T > t$,

$$b = \frac{1}{\sigma}\log(C/V_0),\ \ m = \frac{1}{\sigma}(r-\alpha-\sigma^2/2) \tag{26}$$

and we introduce a new probability $\tilde{\mathbb{P}}$ such that

$$\left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{G}_T} = \exp(-mW_T - m^2T/2).$$

Thus, $(\tilde{W}_u := W_u + mu, u \in [0,T])$ is a standard Brownian motion under $\tilde{\mathbb{P}}$, and we have

$$\tau = \inf\{t \ge 0, \int_0^t \mu_2 \mathbf{1}_{\{\tilde{W}_u \le b\}} + \mu_1 \mathbf{1}_{\{\tilde{W}_u > b\}} du \ge \xi\}. \tag{27}$$

It comes out that

$$\mathbb{P}(\tau \geq t) = \tilde{\mathbb{E}}[\exp(m\tilde{W}_t - m^2 t/2)\mathbf{1}_{\{\tau \geq t\}}] \tag{28}$$

is a function of t, b, m, μ_1 and μ_2. We introduce the following notations:

- for $t \in \mathbb{R}$, $P_{b,m,\mu_1,\mu_2}(t) = \mathbb{P}(\tau \leq t)$ (resp. $P^c_{b,m,\mu_1,\mu_2}(t) = 1 - P_{b,m,\mu_1,\mu_2}(t)$),
- for $z \in \mathbb{C}_+$, $L_{b,m,\mu_1,\mu_2}(z) = \int_0^{+\infty} \mathrm{e}^{-zt} P_{b,m,\mu_1,\mu_2}(t)dt$ (resp. $L^c_{b,m,\mu_1,\mu_2}(z) = 1/z - L_{b,m,\mu_1,\mu_2}(z)$).

Now, we make a simple remark. The default time τ defined by (3) has the same law as $\min(\xi^1/\mu_1, \inf\{t \geq 0, \int_0^t (\mu_2 - \mu_1)\mathbf{1}_{\{V_u \leq C\,\mathrm{e}^{\alpha u}\}} du \geq \xi^2\})$, where ξ^1 and ξ^2 are two independent exponential random variables with parameter 1 that are both independent of the riskless filtration $\mathcal{F}$. Therefore, we have for $t \in \mathbb{R}$ and $z \in \mathbb{C}_+$:

$$P^c_{b,m,\mu_1,\mu_2}(t) = \mathrm{e}^{-\mu_1 t} P^c_{b,m,0,\mu_2-\mu_1}(t) \quad \text{and} \quad L^c_{b,m,\mu_1,\mu_2}(z) = L^c_{b,m,0,\mu_2-\mu_1}(z+\mu_1). \tag{29}$$

Thus, it is sufficient to focus on the case $\mu_1 = 0$, and we set $\mu = \mu_2$ to continue the proof of Theorem 1.1.

Step 2: The case $b = 0$, calculation of the probability distribution of τ

For $D > 0$, we introduce

$$\tau^D = \inf\{t \geq 0, \int_0^t \mathbf{1}_{\{\tilde{W}_u \leq 0\}} du \geq D\},$$

which will be helpful for intermediate calculus. Note that the law of τ^D and τ given that $\{\xi/\mu = D\}$ are the same since ξ is independent of $(W_t, t \geq 0)$.

From Chesney and al. [9], we know the joint distribution of $(\tilde{W}_t, A^-_t := \int_0^t \mathbf{1}_{\{\tilde{W}_u \leq 0\}} du)$ for $t > 0$. For $x \in \mathbb{R}$ and $D > 0$, we have:

$$\tilde{\mathbb{P}}(\tilde{W}_t \in dx, A^-_t \leq D) = \mathbf{1}_{\{t > D\}} \Bigg(\mathbf{1}_{\{x<0\}} \left[\frac{-x}{2\pi} \int_{t-D}^t \frac{D+s-t}{\sqrt{s^3(t-s)^3}} \exp(-\frac{x^2}{2(t-s)}) ds \right]$$
$$+ \mathbf{1}_{\{x>0\}} \left[\int_0^D \frac{x \exp(-\frac{x^2}{2(t-s)})}{2\pi\sqrt{s(t-s)^3}} ds + \int_D^t \frac{xD \exp(-\frac{x^2}{2(t-s)})}{2\pi\sqrt{s^3(t-s)^3}} ds \right] \Bigg) + \mathbf{1}_{\{t \leq D\}} \frac{\exp(-\frac{x^2}{2t})}{\sqrt{2\pi t}}.$$

Observing that $\int_0^\infty x\,\mathrm{e}^{mx - \frac{x^2}{2(t-s)}} dx = (t-s)(1 + \sqrt{2\pi(t-s)} m\,\mathrm{e}^{(t-s)m^2/2}\,\Phi(m\sqrt{t-s}))$ where

Φ is the cumulative normal distribution function, we get after simplifications:

$$\begin{aligned}
\mathbb{P}(\tau^D \geq t) &= \tilde{\mathbb{E}}[\exp(m\tilde{W}_t - m^2 t/2)\mathbf{1}_{\{\tau^D \geq t\}}] = \tilde{\mathbb{E}}[\exp(m\tilde{W}_t - m^2 t/2)\mathbf{1}_{\{A_t^- \leq D\}}] \\
&= \mathbf{1}_{\{t \leq D\}} + \mathbf{1}_{\{t > D\}} \Bigg(\mathrm{e}^{-m^2 t/2} \left(\frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{D - t/2}{\sqrt{D(t-D)}} \right) \right) \\
&\quad + \frac{m}{\sqrt{2\pi}} \Bigg[\int_0^D \frac{\mathrm{e}^{-m^2 s/2}\,\Phi(m\sqrt{t-s})}{\sqrt{s}} ds + D \int_D^t \frac{\mathrm{e}^{-m^2 s/2}\,\Phi(m\sqrt{t-s})}{\sqrt{s^3}} ds \\
&\quad - \int_{t-D}^t \frac{(D+s-t)\,\mathrm{e}^{-m^2 s/2}\,\Phi(-m\sqrt{t-s})}{\sqrt{s^3}} ds \Bigg] \Bigg).
\end{aligned}$$

Now, we are able to compute the cumulative distribution function of τ: $P^c_{0,m,0,\mu}(t) = \mathbb{P}(\tau \geq t) = \int_0^{+\infty} \mathbb{P}(\tau^\delta \geq t)\mu\,\mathrm{e}^{-\mu\delta}\,d\delta = \mathrm{e}^{-\mu t} + \int_0^t \mathbb{P}(\tau^\delta \geq t)\mu\,\mathrm{e}^{-\mu\delta}\,d\delta$. An integration by parts yields that $\int_0^t \mu\,\mathrm{e}^{-\mu\delta} \arctan\left(\frac{\delta - t/2}{\sqrt{\delta(t-\delta)}}\right) d\delta = -\frac{\pi}{2}(1 + \mathrm{e}^{-\mu t}) + \int_0^t \frac{\mathrm{e}^{-\mu\delta}}{\sqrt{\delta(t-\delta)}} d\delta$. Using Fubini's theorem for the other integrals and the three following equalities $\int_s^t \mu\,\mathrm{e}^{-\mu\delta}\,d\delta = \mathrm{e}^{-\mu s} - \mathrm{e}^{-\mu t}$, $\int_0^s \mu\delta\,\mathrm{e}^{-\mu\delta}\,d\delta = \frac{1-(1+\mu s)\,\mathrm{e}^{-\mu s}}{\mu}$ and $\int_{t-s}^t (\delta - (t-s))\mu\,\mathrm{e}^{-\mu\delta}\,d\delta = \mathrm{e}^{-\mu(t-s)}\,\frac{1-(1+\mu s)\,\mathrm{e}^{-\mu s}}{\mu}$, we get:

$$\begin{aligned}
P^c_{0,m,0,\mu}(t) &= \mathrm{e}^{-\mu t}(1 - \mathrm{e}^{-m^2 t/2}) + \frac{1}{\pi} \int_0^t \frac{\mathrm{e}^{-(\mu + m^2/2)s}}{\sqrt{s}} \frac{\mathrm{e}^{-m^2(t-s)/2}}{\sqrt{t-s}} ds \qquad (30) \\
&+ \frac{m}{\sqrt{2\pi}} \Bigg[-\int_0^t \frac{\mathrm{e}^{-(\mu+m^2/2)s}}{\sqrt{s}} \Phi(m\sqrt{t-s})\,\mathrm{e}^{-\mu(t-s)}\,ds + \frac{1}{\mu} \int_0^t \frac{\mathrm{e}^{-m^2 s/2} - \mathrm{e}^{-(\mu+m^2/2)s}}{\sqrt{s^3}} \Phi(m\sqrt{t-s})\,ds \\
&- \frac{1}{\mu} \int_0^t \frac{\mathrm{e}^{-m^2 s/2} - \mathrm{e}^{-(\mu+m^2/2)s}}{\sqrt{s^3}}\,\mathrm{e}^{-\mu(t-s)}\,\Phi(-m\sqrt{t-s}) ds + \int_0^t \frac{\mathrm{e}^{-(\mu+m^2/2)s}}{\sqrt{s}} \Phi(-m\sqrt{t-s})\,\mathrm{e}^{-\mu(t-s)}\,ds \Bigg].
\end{aligned}$$

We recognize convolution products, and use the three following formulas for $z \in \mathbb{C}_+$,

$$\begin{aligned}
&\int_0^\infty t^{-1/2}\,\mathrm{e}^{-zt}\,dt = \sqrt{\frac{\pi}{z}}, \ \int_0^\infty \mathrm{e}^{-zt}\,\Phi(m\sqrt{t})dt = \frac{1}{2z} + \frac{m}{2z\sqrt{2z+m^2}}, \\
&\int_0^\infty t^{-3/2}\,\mathrm{e}^{-zt}(1 - \mathrm{e}^{-\mu t})dt = 2\sqrt{\pi}(\sqrt{z+\mu} - \sqrt{z}),
\end{aligned}$$

to get after some simplifications:

$$L^c_{0,m,0,\mu}(z) = \frac{1}{z} + \left(\frac{1}{z+\mu} - \frac{1}{z}\right) \frac{-m + \sqrt{2(z+\mu)+m^2}}{\sqrt{2z+m^2} + \sqrt{2(z+\mu)+m^2}}. \qquad (31)$$

Step 3: The cases $b < 0$ and $b > 0$

In both cases, we introduce $\tau_b = \inf\{u \geq 0, \tilde{W}_u = b\}$, the first hitting time of the barrier. We recall that $\tilde{\mathbb{E}}[\mathrm{e}^{-z\tau_b}] = \mathrm{e}^{-\sqrt{2z}|b|}$ for $z \in \mathbb{C}_+$.

$\boxed{b<0}$: Defining $\tau' = \inf\{t \geq 0, \int_0^t \mathbf{1}_{\{\tilde{W}_{\tau_b+u}-b\leq 0\}}du \geq \xi/\mu\}$, we have $\tau = \tau_b + \tau'$, and therefore $\mathbf{1}_{\{\tau\geq t\}} = \mathbf{1}_{\{\tau_b\geq t\}} + \mathbf{1}_{\{\tau_b<t\}}\mathbf{1}_{\{\tau'\geq t-\tau_b\}}$. We have:
$P^c_{b,m,0,\mu}(t) = \tilde{\mathbb{E}}[\exp(m\tilde{W}_t - m^2t/2)\mathbf{1}_{\{\tau_b\geq t\}}] + \tilde{\mathbb{E}}[\exp(m\tilde{W}_t - m^2t/2)\mathbf{1}_{\{\tau_b<t\}}\mathbf{1}_{\{\tau'\geq t-\tau_b\}}]$
$= 1 - \tilde{\mathbb{E}}[\exp(m\tilde{W}_t - m^2t/2)\mathbf{1}_{\{\tau_b<t\}}] + \tilde{\mathbb{E}}[\exp(mb - m^2\tau_b/2)\mathbf{1}_{\{\tau_b<t\}}\exp(m(\tilde{W}_{\tau_b+(t-\tau_b)} - b) - m^2(t-\tau_b)/2)\mathbf{1}_{\{\tau'\geq t-\tau_b\}}]$

Using Doob's optional sampling theorem for the first expectation and the strong Markov property in τ_b for the second one, we get that

$$P^c_{b,m,0,\mu}(t) = 1 - \tilde{\mathbb{E}}[\exp(mb - m^2\tau_b/2)\mathbf{1}_{\{\tau_b<t\}}] + \tilde{\mathbb{E}}[\exp(mb - m^2\tau_b/2)\mathbf{1}_{\{\tau_b<t\}}P^c_{0,m,0,\mu}(t-\tau_b)].$$

Then, it is easy to take the Laplace transform to get:

$$L^c_{b,m,0,\mu}(z) = \frac{1}{z} + \mathrm{e}^{mb+b\sqrt{2z+m^2}}\left(L^c_{0,m,0,\mu}(z) - \frac{1}{z}\right) \quad \text{for } b<0. \tag{32}$$

$\boxed{b>0}$: Defining $\tau'' = \inf\{t \geq 0, \int_0^t \mathbf{1}_{\{\tilde{W}_{\tau_b+u}-b\leq 0\}}du \geq \xi/\mu - \tau_b\}$, we notice that when $\tau \geq t$, we have either $\tau_b \geq t$ and then $\xi/\mu \geq t$, or $\tau_b < t$, and then $\xi/\mu \geq \tau_b$ and $\tau'' \geq t - \tau_b$:

$$\mathbf{1}_{\{\tau\geq t\}} = \mathbf{1}_{\{\tau_b\geq t\}}\mathbf{1}_{\{\xi/\mu\geq t\}} + \mathbf{1}_{\{\tau_b<t\}}\mathbf{1}_{\{\xi/\mu\geq\tau_b\}}\mathbf{1}_{\{\tau''\geq t-\tau_b\}}.$$

With the independence of ξ and $(\tilde{W}_u, u \geq 0)$, we get $P^c_{b,m,0,\mu}(t) = \mathrm{e}^{-\mu t}(1 - \tilde{\mathbb{E}}[\mathrm{e}^{mb-m^2\tau_b/2}\,\mathbf{1}_{\{\tau_b<t\}}])$ $+ \tilde{\mathbb{E}}[\mathrm{e}^{mb-m^2\tau_b/2}\,\mathbf{1}_{\{\tau_b<t\}}\mathbf{1}_{\{\xi/\mu\geq\tau_b\}}\exp(m(\tilde{W}_{\tau_b+(t-\tau_b)} - b) - m^2(t-\tau_b)/2)\mathbf{1}_{\{\tau''\geq t-\tau_b\}}]$. Now, we use the strong Markov property in τ_b and the lack of memory property of the exponential r.v.: conditioning w.r.t. $\mathcal{F}_{\tau_b}$ and $\{\xi \geq \tau_b\}$, $(\tilde{W}_{\tau_b+u} - b, u \geq 0)$ and $\xi/\mu - \tau_b$ are respectively a Brownian motion starting from 0 and an exponential r.v. with parameter μ and both are independent. It comes out that

$$\begin{aligned} P^c_{b,m,0,\mu}(t) &= \mathrm{e}^{-\mu t}(1 - \tilde{\mathbb{E}}[\mathrm{e}^{mb-m^2\tau_b/2}\,\mathbf{1}_{\{\tau_b<t\}}]) + \tilde{\mathbb{E}}[\mathrm{e}^{mb-m^2\tau_b/2}\,\mathbf{1}_{\{\tau_b<t\}}\mathbf{1}_{\{\xi/\mu\geq\tau_b\}}P^c_{0,m,0,\mu}(t-\tau_b)] \\ &= \mathrm{e}^{-\mu t}(1 - \tilde{\mathbb{E}}[\mathrm{e}^{mb-m^2\tau_b/2}\,\mathbf{1}_{\{\tau_b<t\}}]) + \tilde{\mathbb{E}}[\mathrm{e}^{mb-(\mu+m^2/2)\tau_b}\,\mathbf{1}_{\{\tau_b<t\}}P^c_{0,m,0,\mu}(t-\tau_b)]. \end{aligned}$$

Then, proceeding exactly as in the case $b<0$, we obtain

$$L^c_{b,m,0,\mu}(z) = \frac{1}{z+\mu} + \mathrm{e}^{mb-b\sqrt{2(z+\mu)+m^2}}\left(L^c_{0,m,0,\mu}(z) - \frac{1}{z+\mu}\right) \quad \text{for } b>0, \tag{33}$$

which concludes the proof of Theorem (1.1).

Remark B.1. *We also have an explicit formula for* $\mathbb{P}(\tau_b < \tau)$*, the probability that the default occurs after the firm value has reached the barrier* $C\,\mathrm{e}^{\alpha t}$*. From* (27)*, we have* $\{\tau_b < \tau\} = \{\tau_b < \xi/\mu_b\}$*, setting* μ_b *as in Theorem 1.1. Therefore, we get*

$$\begin{aligned} \mathbb{P}(\tau_b < \tau) &= \int_0^\infty \mu_b\,\mathrm{e}^{-\mu_b t}\,\mathbb{P}(\tau_b < t)dt = \int_0^\infty \mu_b\,\mathrm{e}^{-\mu_b t}\,\tilde{E}[\exp(m\tilde{W}_t - m^2t/2)\mathbf{1}_{\{\tau_b<t\}}]dt \\ &= \mathrm{e}^{mb}\,\tilde{E}[\exp((\mu_b + m^2/2)\tau_b)] = \mathrm{e}^{mb-|b|\sqrt{2\mu_b+m^2}}. \end{aligned}$$

Remark B.2. *Not surprisingly, we can also easily handle the case where the barrier follows a geometric Brownian motion, i.e.*

$$\lambda_t = \mu_2 \mathbf{1}_{\{V_t \le C\,\mathrm{e}^{(\alpha-\eta^2/2)t+\eta Z_t}\}} + \mu_1 \mathbf{1}_{\{V_t > C\,\mathrm{e}^{(\alpha-\eta^2/2)t+\eta Z_t}\}}, \text{ with } \langle W, Z\rangle_t = \rho t.$$

We exclude the trivial case $\rho = 1$ *with* $\eta = \sigma$ *and set* $\varsigma = \sqrt{\sigma^2+\eta^2-2\rho\sigma\eta} > 0$ *so that* $B_t = (\sigma W_t - \eta Z_t)/\varsigma$ *is a standard Brownian motion. Since* $\mathbf{1}_{\{V_t \le C\,\mathrm{e}^{(\alpha-\eta^2/2)t+\eta Z_t}\}} = \mathbf{1}_{\{B_t + \frac{1}{\varsigma}(r-\alpha-(\sigma^2-\eta^2)/2)t \le \frac{1}{\varsigma}\log(C/V_0)\}}$, *we can proceed like in step 1 and we get the Laplace transform of* $\mathbb{P}(\tau \le t)$ *in that case by simply taking*

$$b = \frac{1}{\varsigma}\log(C/V_0) \text{ and } m = \frac{1}{\varsigma}(r-\alpha-(\sigma^2-\eta^2)/2)$$

in formula (4)*. Said differently, considering a geometric Brownian motion barrier does not lead to a richer family of default distributions.*

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