Pricing American options with discrete dividends by binomial trees

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Abstract

We are concerned with the problem of pricing plain-vanilla options with cash dividends in a piecewise lognormal model. In the plain-vanilla case we offer a methods with provides thin upper and lower bounds of the true binomial price. I

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Introduction

The stock assets pay frequently dividends at discrete times and this produces important modifications on the numerical procedures involved in the option pricing. For plain vanilla options several close formulas and approximation techniques have been investigated in previous papers (see e.g. Haug-Haug-Lewis [5], Meyer [8], Bos-Wandemark [3], Bos-Gairat-Shepeleva [2], Beneder-Vorst [1]). These approximations, as showed in ([9]), are not very precise. Furthermore Wilmott et al. [10] presented a finite difference approach to price options in the presence of cash dividends. Vellekoop-Nieuwenhuis [9] presented a modified Cox-Ross-Rubinstein (CRR) tree [4] that overcomes the non-recombining property of the standard CRR tree when discrete dividends are considered. These algorithms are mainly based on suitable interpolation techniques at dividend dates.

We introduce different tree methods which cover European and American plain-vanilla options in the presence of discrete dividends.

In the plain-vanilla case we propose an algorithm based on the singular points approach introduced in [7]. This technique permits us to obtain a thin upper and lower bound of the true binomial price computationally efficient.

More precisely, we provide, at each time step of the tree, a continuous representation of the option price as a piecewise linear function of the stock price. This function is characterized only by a set of points, called "singular points", which can easily be computed recursively by backward induction. Although the number of singular points grows rapidly at every dividend date, their number can be drastically reduced in a straightforward way, controlling, at the same time, the error involved by the elimination procedure and providing upper and lower estimates. The control of the error allows also to get immediately the convergence of the method to the continuous value.

The paper is organized as follows: in Section 1 we present the model of the risk asset, in Section 2 we present the singular points technique. In Section 3 we introduce the singular points algorithm for pricing European and American options with cash dividends. In Section 4 we present the numerical results.

1 The model

In this paper, we consider a market model where the evolution of a risky asset, between dividend dates involving a cash dividend payment, is governed by the Black-Scholes stochastic differential equation

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t, \quad S_0 = s_0, \tag{1}$$

where $(B_t)_{0\leq t\leq T}$ is a standard Brownian motion, under the risk neutral measure Q. The nonnegative constant r is the force of interest rate and σ is the volatility of the risky asset.

We assume furthermore that the risk asset pays, at dividend dates $t_1, ..., t_{n_D}$,

$$0 = t_0 < t_1 < t_2 < ... < t_{n_D} < t_{n_D+1} = T,$$

dividend amounts

$$D_1, D_2, ..., D_{n_D},$$

respectively.

For pricing plain-vanilla and barrier options in this piecewise lognormal model with discrete dividends, we consider now a binomial approach. Let n be the number of steps of the binomial tree and $\Delta T = \frac{T}{n}$ the corresponding time-step. In order to simplify the construction of the binomial tree we assume in the sequel that $\frac{t_i - t_{i-1}}{\Delta T}$, $\quad i = 1, ..., n_D + 1$, is an integer (otherwise suitable interpolations in the time variable are required).

The standard discrete binomial process (without dividends) is given by

$$S_{(i+1)\Delta T} = S_{i\Delta T} Y_{i+1}, \quad 0 \leq i \leq n-1,$$

where the random variables $Y_1, \ldots, Y_n$ are independent and identically distributed with values in $\{d, u\}$. Let us denote by $\pi = \mathbb{P}(Y_n = u)$. The Cox-Ross-Rubinstein tree corresponds to the choice $u = \frac{1}{d} = e^{\sigma\sqrt{\Delta T}}$ and

$$\pi = \frac{e^{r\Delta T} - e^{-\sigma\sqrt{\Delta T}}}{e^{\sigma\sqrt{\Delta T}} - e^{-\sigma\sqrt{\Delta T}}}.$$

In order to take in account of the presence of dividends, at each time t_i, $i = 1, ..., n_D$, we have to subtract the corresponding dividend amount D_i at each node of the tree. Let us remark that the tree so constructed is non-recombining. In fact the presence of dividends lead to a new tree from each node on each dividend payment date.

2 The singular points approach

The pricing of an European or American options can be done by a backward dynamic programming equation using the "pure" tree algorithm (see for instance the description given in Hull [6]). However, because of the non-recombining property of the binomial tree, the straightforward implementation of the algorithm leads to an inefficient procedure. Remark that in the case $\frac{t_i - t_{i-1}}{\Delta T} = m$ for all $i = 1, ..., n_D + 1$, the computational complexity of the procedure is m^{n_D+2}.

Wilmott et al. [10] suggest to use a linear interpolation technique in order to make the tree recombining. Later Vellekoop and Nieuwenhuis [9] proved the convergence to the continuous value of a similar binomial approach both in European and American case.

Here we propose a different approach, based on the singular points technique, introduced in [7], which allows to approximate the pure binomial price with an a-priori fixed level of error. The procedure introduced in [7] can be adapted in a simple way to this context. In the sequel, for sake of completeness and in order to clarify the differences with respect to [7], we presente it in details.

According to the notations introduced in [7] we will use the next definition

Definition 1. *Let us consider a set of points: $(x_1, y_1), ..., (x_n, y_n)$, such that*

$$a = x_1 < x_2 < ... < x_n = b,$$

and the piecewise linear function $f(x)$, $x \in [a, b]$, obtained by interpolating linearly the given points. The points $(x_1, y_1), ..., (x_n, y_n)$ (which characterize completely f), will be called the <u>*singular points*</u> *of f, while $x_1, ..., x_n$ will be called the* <u>*singular values*</u> *of f.*

Let us remark that f is convex if and only the slopes

$$\alpha_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i},$$

are increasing, i.e. $\alpha_i \le \alpha_{i+1}$ for all $i = 1, ..., n - 1$.

The approach of singular point allows to construct upper and lower bounds of the option price in a simple way, as pointed out in the next remark (see also the geometrical interpretation in Figure 1 and 2).

Remark 1. *Let f be a piecewise linear and convex function defined on $[a, b]$, and let $C = \{(x_1, y_1), ..., (x_n, y_n)\}$ be the set of its singular points. Then:*
a) Removing a point (x_i, y_i), $2 \le i \le n - 1$, from the set C, the resulting piecewise linear function $\widetilde{f}$, whose set of singular points is $C \setminus \{(x_i, y_i)\}$, is again convex in $[a, b]$ and we have:

$$f(x) \le \widetilde{f}(x), \quad \forall x \in [a, b].$$

b) Let us denote by $(\overline{x}, \overline{y})$ the intersection between the straight line joining (x_{i-1}, y_{i-1}), (x_i, y_i) and the one joining (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2}), $2 \le i \le n - 2$. If we consider the new set of $n - 1$ singular points

$$\{(x_1, y_1), ..., (x_{i-1}, y_{i-1}), (\overline{x}, \overline{y}), (x_{i+2}, y_{i+2}), ..., (x_n, y_n)\},$$

the associated piecewise linear function $\widetilde{f}$ is again convex on $[a, b]$ and we have:

$$f(x) \ge \widetilde{f}(x), \quad \forall x \in [a, b].$$

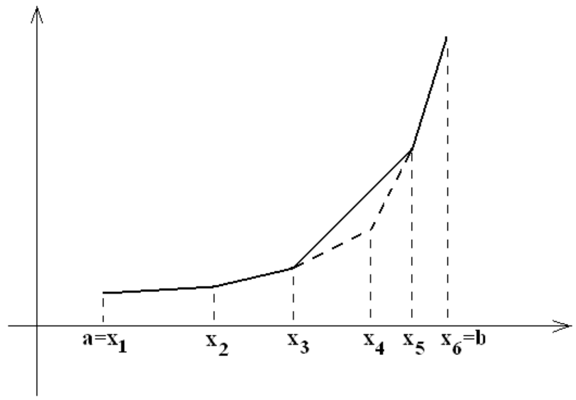

Figure 1: *Upper estimate: x_4 has been removed.*

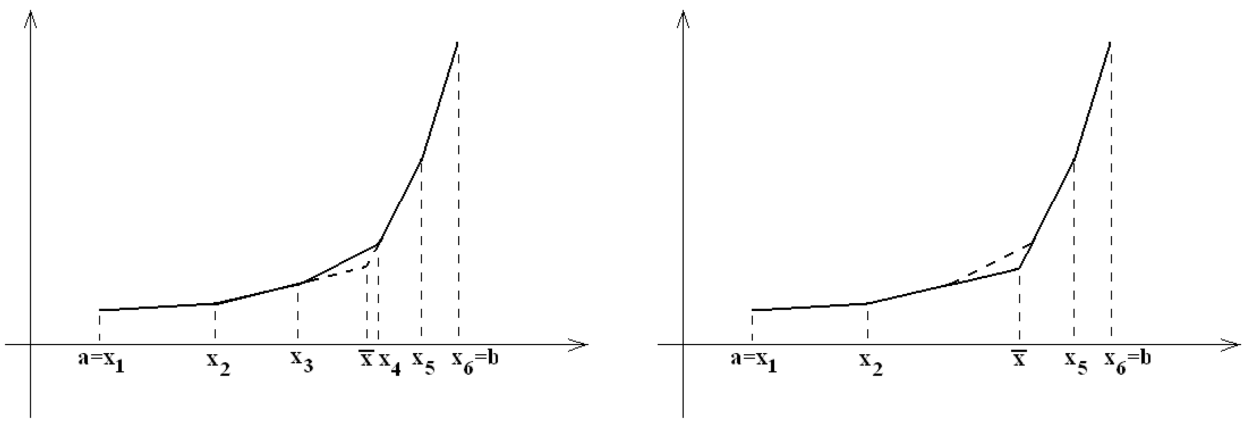

Figure 2: *Lower estimate: x_3 and x_4 have been removed, $\overline{x}$ has been inserted.*

3 Plain-vanilla options

Let us consider an European call option with discrete dividends. The singular points approach consists in a backward procedure which permits us to obtain a continuous representation of the option price at every time step as a piecewise linear continuous function of the underlying asset. Such price functions are characterized by their singular points. Hence the pricing procedure depends exclusively by the knowledge of the singular points at every time step of the tree. It is important to note that the procedure provides exactly the pure binomial value. However it permits to obtain an important improvement, in fact the binomial price can be approximated removing some singular points following the procedure described in Remark 1 and giving, in the same time, a control of the error.

We proceed now to the description of the price function $v_i(S)$ at every step of the tree. To this end we have to evaluate the minimum and the maximum of the risky asset at maturity.

Such maximum and minimum can be evaluated inductively on the tree. Denoting by S_i^{min}, S_i^{max} the minumum and the maximum value of the underlying at step i, $i = 0, ..., n$, one has

$$S_0^{min} = s_0, \qquad S_i^{min} = \begin{cases} S_{i-1}^{min} d & \text{if } i\Delta T \text{ is not a dividend date} \\ S_{i-1}^{min} d - D_j & \text{if } i\Delta T \text{ is the dividend date } t_j \end{cases}$$

$$S_0^{max} = s_0, \qquad S_i^{max} = \begin{cases} S_{i-1}^{max} u & \text{if } i\Delta T \text{ is not a dividend date} \\ S_{i-1}^{max} u - D_j & \text{if } i\Delta T \text{ is the dividend date } t_j \end{cases}.$$

At maturity, the option price, as function of the underlying asset S, is continuously defined by

$$v_n(S) = \max\{S - K, 0\}.$$

$v_n(S)$ is a piecewise linear convex function characterized by the three singular points (A_n^l, P_n^l), $l = 1, 2, 3$ given by:

$$\begin{array}{ll} A_n^1 = S_n^{min}, & P_n^1 = 0; \\ A_n^2 = K, & P_n^2 = 0; \\ A_n^3 = S_n^{max}, & P_n^3 = S_n^{max} - K. \end{array} \tag{2}$$

Clearly, if $K \notin (S_n^{min}, S_n^{max})$ the singular points reduce to two.

At step $i = n - 1$ one has

$$v_i(S) = e^{-r\Delta T}[\pi v_{i+1}(Su) + (1 - \pi) v_{i+1}(Sd)]. \tag{3}$$

We can conclude that $v_{n-1}(S)$ is piecewise linear and convex as well. The singular values of v_{n-1} are S_{n-1}^{min}, S_{n-1}^{max}, and eventually Kd, Ku if they belong to $(S_{n-1}^{min}, S_{n-1}^{max})$. In order to compute the corresponding option prices we have to apply formula (3). To this end we remark, for example, that $v_{n-1}(Ku) = e^{-r\Delta T}[\pi v_n(Ku^2) + (1-\pi) v_n(K)]$. Now $v_n(K)$ is already known, while $v_n(Ku^2)$ has to be computed by linearity. The same procedure holds true for the other singular points as well.

We then proceed iteratively in the same way for $i = n - 2, ..., 0$. More precisely we evaluate the singular values of $v_i(S)$ considering the singular values of $v_{i+1}(S)$ multiplied by the up factor u and by the down factor d. Such values become singular values of $v_i(S)$ if they belong to the domain (S_i^{min}, S_i^{max}). The evaluation of the corresponding option prices has to be done again by equation (3). As before, this formula needs the computation of $v_{i+1}(Su)$ and $v_{i+1}(Sd)$. One of them will be computed directly while the latter has to be computed by linearity.

At the dividend dates the previous procedure needs an additional treatment. Let in fact t_j a dividend date and let (A_i^1, P_i^1),...,(A_i^L, P_i^L) the singular points associated to this date and evaluated by the previous procedure. The presence of the dividend reduces the stock values of the dividend amount D_j. Going backward in time, we have to increase the singular values of this amount. Therefore the new set of singular values becomes

$$(A_i^1 + D_j, P_i^1), ..., (A_i^L + D_j, P_i^L).$$

This procedure induces a large increment of the number of singular points, in fact, due to non-recombining property, the number of singular points could double at each step following a dividend date.

Finally, at step $i = 0$ we get only one singular point: (s_0, P_0^1). P_0^1 provides the pure binomial price of the European call option with multiple discrete dividends.

In the American case the function $v_i(S)$ becomes

$$v_i(S) = max\{S - K, v_i^c(S)\}, \tag{4}$$

where

$$v_i^c(S) = e^{-r\Delta T}[\pi v_{i+1}(Su) + (1 - \pi)v_{i+1}(Sd)]$$

is the continuation value.

At the dividend date t_j, by virtue of the shifting due the dividend, $v_i^c(S)$ has to be computed by

$$v_i^c(S) = e^{-r\Delta T}[\pi v_{i+1}((S - D_j)u) + (1 - \pi)v_{i+1}((S - D_j)d)].$$

Hence at dividend dates we apply first the shift of the asset and then the early optimality. This order in the computation is due to the fact that is convenient to exercise eventually just before the dividend dates. This also implies that for the put options the order has to be inverted: after the evaluation of the continuation value we evaluate the optimality of the early exercise and then we apply the dividend shift.

At every step i $v_i(S)$ is still piecewise linear and convex, hence the procedure explained in the European case holds again. The only difference is related to the computation of the singular points. In fact at first we need to evaluate the singular points of $v_i^c(S)$. Then we have to evaluate the singular points of $v_i(S)$. By convexity, this can be done considering three possible cases:

1. $S_i^{max} - K \leq v_i^c(S_i^{max})$ then $v_i \equiv v_i^c$, so the singular points do not change;

2. $S_i^{max} - K > v_i^c(S_i^{max})$ and $S_i^{min} - K \geq v_i^c(S_i^{min})$. Then $v_i(S) \equiv S - K$, hence there are only two singular points: the extrema.

3. $S_i^{max} - K > v_i^c(S_i^{max})$ and $S_i^{min} - K < v_i^c(S_i^{min})$. Then there exists an unique value $\overline{S}$ where the continuation value coincides with the early exercise. The singular points of v_i are now: all the ones whose singular value is less than $\overline{S}$, then $(\overline{S}, \overline{S} - K)$ and $(S_i^{max}, S_i^{max} - K)$ (see Figure 3).

This argument can be applied at every step $i = n - 1, ..., 0$. This allows to compute P_0^1 which provides the pure American binomial price associated to the tree with n steps.

The technique previously presented is inefficient from a computational point of view because of the high number of singular points generated at every dividend date. However simple modifications allows to reduce drastically the number of singular points providing an upper and a lower bound of the exact binomial value.

In order to get an upper bound of the pure binomial price we just remove some singular points at every time step. Remark 1.a ensures that the value obtained in such way is an upper estimate of the pure binomial price. The criteria to remove the singular points is the same as presented in [7].

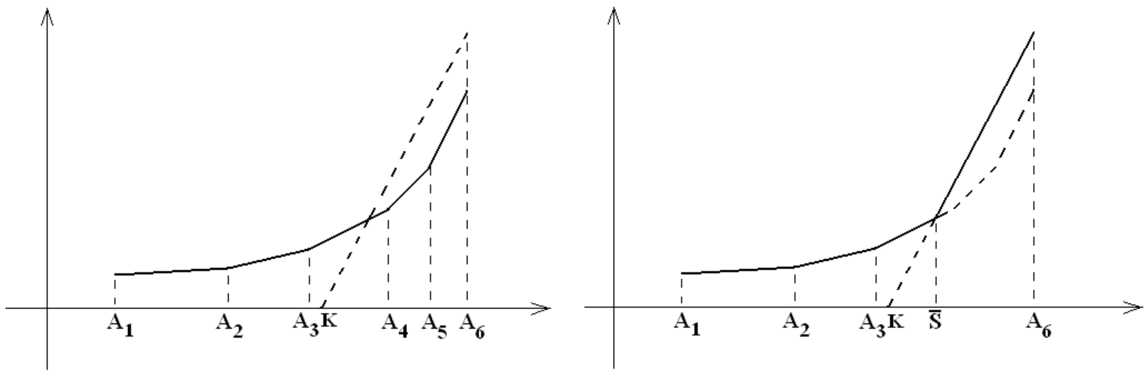

Figure 3: *The point $\overline{S}$ has been inserted, A_4 and A_5 have been removed.*

More precisely, consider the set of singular points $C = \{(A_i^1, P_i^1), ..., (A_i^L, P_i^L)\}$ at step i, and the corresponding price value function $v_i(S)$. Let $v_i'(S)$ be the price value function obtained by removing a point (A_i^l, P_i^l) from C. We have (see Figure 4)

$$|v_i(S) - v_i'(S)| \leq \epsilon_l, \qquad \forall S \in [S_i^{min}, S_i^{max}] \tag{5}$$

where

$$\epsilon_l = v_i'(A_i^l) - v_i(A_i^l). \tag{6}$$

Therefore, given a real number $h > 0$, we choose to remove the point (A_i^l, P_i^l) if $\epsilon_l < h$. Repeating this procedure sequentially at every node of the tree, avoiding the elimination of two consecutive singular points, we can conclude that the obtained upper estimate differs from the pure binomial value at most for nh.

The algorithm for the computation of the lower bound is similar and follows again by Remark 1.b (see Figure 2).

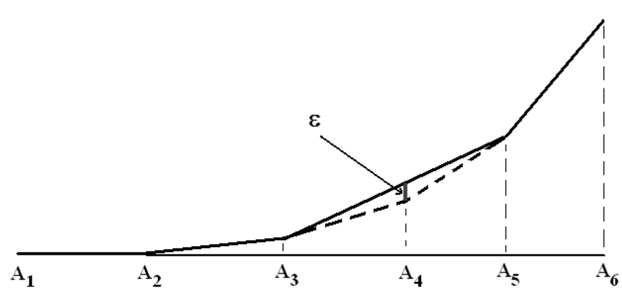

Figure 4: *If $\epsilon < h$ then A_4 will be removed.*

4 Numerical results

In this section we provide some numerical comparisons of the algorithms presented in the previous sections with the tree method given in [9] and a finite difference method implemented following [10]. For testing the efficiency we will consider a numerical experiments proposed in [9]. In that paper the authors considered a call option with time to maturity of 7 years and cash dividends equal to

$$D_1 = 6,\ D_2 = 6.5,\ D_3 = 7,\ D_4 = 7.5,\ D_5 = 8,\ D_6 = 8,\ D_7 = 8,$$

respectively at time $t_i = i - 0.5$, $i = 1, ..., 7$.

The volatility is $\sigma = 0.25$, the interest rate $r = 0.06$, the current stock price is $s_0 = 100$ and the strike price varies: $K = 70, 100, 130$.

All the computations presented in the tables have been performed in double precision on a PC with a processor Centrino at 1.6 Ghz with 512 Mb of RAM.

4.1 European and American plain-vanilla

In this case we compare the singular point algorithm described in Section 3 with the one obtained in [9] with $n = 1000$ steps (VN1000). For the singular points method we take the same number of steps and compute the upper and the lower bound provided by the method. As level of error we choose $h = \frac{T}{n\sqrt{n}}$ (SP1) and $h = 10^{-5}$ (SP2). The first choice of h produces a maximal error of $= \frac{T}{\sqrt{n}}$ which converges to 0 which implies that the estimates converge to the continuous value. The second choice is independent from the number of steps and provides an example of thin bounds of the pure binomial value (the maximal error in this case is 10^{-2}). For the comparison of the methods we take as benchmark value the one provided by Vellekoop-Nieuvenhuis [9] (VNRE) evaluated with Richardson extrapolation for $n = 64000$.

In Table 1 and in Table 2 we report respectively the price estimates for the European and American case with time of computation in parentheses (measured in seconds).

K	VN1000	$SP_1 1000$		$SP_2 1000$		VNRE
		down	up	down	up	
70	26.10	26.0770	26.0933	26.0802	26.0809	26.08
	(0.22)	(0.13)	(0.08)	(0.50)	(0.34)	
100	18.50	18.4761	18.4931	18.4795	18.4803	18.48
	(0.22)	(0.16)	(0.09)	(0.59)	(0.37)	
130	13.31	13.2771	13.2956	13.2808	13.2816	13.29
	(0.22)	(0.17)	(0.09)	(0.64)	(0.42)	

Table 1: *European Call options with discrete dividends*

K	VN1000	$SP_1$1000		$SP_2$1000		VNRE
		down	up	down	up	
70	33.40	33.4651	33.4721	33.4669	33.4673	33.47
	(0.26)	(0.08)	(0.05)	(0.31)	(0.17)	
100	20.04	20.0388	20.0535	20.0422	20.0429	20.04
	(0.26)	(0.12)	(0.06)	(0.44)	(0.28)	
130	13.76	13.7386	13.7574	13.7421	13.7429	13.75
	(0.26)	(0.16)	(0.09)	(0.57)	(0.37)	

Table 2: *American Call options with discrete dividends*

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