A dynamic Markov chain model for pricing CDO

Ahmed KEBAIER

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1 The Model

We consider a portfolio of m firms, indexed by $i \in \{1,\dots,m\}$. The evolution of the default state of the portfolio is described by a default indicator process $Y = (Y_{t,1},...,Y_{t,m})$, $t \geq 0$, defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$. We set $Y_{t,i} = 1$ if firm i has defaulted by time t and $Y_{t,i} = 0$ else, so that $Y_t \in S^Y := \{0,1\}^m$. The corresponding default times are denoted by $\tau_i := inf\{t \geq 0 : Y_{t,i} = 1\}$. We use the following notation for flipping the ith coordinate of a default state: given $y \in S^Y$ we define $y^i \in S^Y$ by

$$y^i_i := 1 - y_i \text{ and } y^i_j := y_j, j \in \{1,\dots,m\}\backslash\{i\}.$$

The default history is denoted by $(\mathcal{H}_t)$, i.e. $\mathcal{H}_t = \sigma(Y_s : s \leq t)$. An $(\mathcal{H}_t)$-adapted process $(\lambda_{t,i})$ is called the default intensity of default time τ_i (with respect to $(\mathcal{H}_t)$) if

$$Y_{t,i} - \int_0^{\tau_i\wedge t} ÿ\lambda_{s,i}ds \text{ is an } (\mathcal{H}_t)\text{-martingale.}$$

Intuitively, $ÿ\lambda_{t,i}$ gives the instantaneous chance of default of a non-defaulted firm i **given the default history up to time** t.

The default intensities $ÿ\lambda_i(t,Y_t)$ are crucial ingredients of the model. If the portfolio size N is large(such as in the pricing of typical synthetic CDO tranches) it is natural to assume that the portfolio has a homogeneous group structure. Denote the number of defaulted firms at time t by $M_t := \sum_{i=1}^N Y_{t,i}$. As discussed in [2], in a homogeneous model default intensities are necessarily of the form

$$ÿ\lambda_i(t,Y_t) = h(t,M_t) \text{ for some } h : [0,\infty)\times\{0,\dots,N\} \to \mathbb{R}_+.$$

Note that the assumption that default intensities depend on Y_t via the number of defaulted firms M_t makes sense also from an economic viewpoint, as unusually many defaults might have a negative impact on the liquidity of credit markets or on the business climate in general. In our context, we implement the following specific model

$$h(t,l) = \lambda_0 + \frac{\lambda_1}{\lambda_2}\left(\exp\left(\lambda_2\frac{(l-\mu(t))_+}{m}\wedge 0.37\right)-1\right),\ \ \lambda_0,\lambda_2 > 0,\ \lambda_1 \geq 0,$$

called *convex counterparty risk model*. The cap at 0.37 has been introduced in order to avoid an "explosion" of the intensity for high values of λ_2. we take $\mu(t) := N(1 - exp(-\textit{Index Spread}/(1-R)))$ as approximation for the expected number of defaulted firms, where R denotes the homogene recovery.

ÿNote tha λ_1 gives the slope of $h(t,l)$ at $\mu(t)$; intuitively this parameter models the strength of default interaction for "normal" realisations of M_t. The parameter $ÿ\lambda_2$ controls the degree of convexity of h and hence the tendency of the model to generate default cascades. The basic idea is simple: by increasing λ_2 we can generate occasional large clusters of defaults without affecting the left tail of the distribution of the loss process $Lt := \frac{1-R}{N}M_t$ too much; in this way we can reproduce the high spread of the CDO tranches in a way which is consistent with the observed spread of the equity tranche

2 Synthetic CDOs

Let B and A be the upper and lower attachment points of the tranche respectively. At each payment date, investors receive a coupon which is proportional to the notional of the tranche, net of the losses suffered by the credit portfolio up to that point. Under above assumptions the process $(M_t)_{t\geq 0}$ is a markov chain taking value in $\{1,\ldots,m\}$. The distribution of M_t can be determined via the following Kolmogorov forward equation

$$\frac{\partial P^M(t,s,l^1,l^2)}{\partial s} = \mathbf{1}_{\{l^2>0\}}(N-l^2+1)h(s,l^2-1)P^M(t,s,l^1,l^2-1) - (N-l^2)h(s,l^2)P^M(t,s,l^1,l^2) \quad (1)$$

with initial condition $P^M(t,t,l^1,l^2) = \mathbf{1}_{\{l^1\}}(l^2)$ for $0 \leq l^1, l^2 \leq N$. The function solving the above system and computing the quantity

$$\mathbb{E}(H_t^{A,B}) = \mathbb{E}\left[\left(\frac{1-R}{N}M_t - A\right)_+ - \left(\frac{1-R}{N}M_t - B\right)_+\right]$$

is called **double** EV_lu in the Premia code.

2.1 Premium leg and default leg

The premium leg is equal to

$$pl^{A,B} = \sum_{j=1}^{n}(T_j - T_{j-1})\mathbb{E}\left[e^{-rT_j}\left(B - A - H_{T_j}^{A,B}\right)\right],$$

where where n is the number of total payments occurring at dates $T_1,\ldots,T_n$. The default leg is equal to

$$dl^{A,B} = \sum_{j=1}^{n}\mathbb{E}\left[e^{-rT_j}\left(H_{T_j}^{A,B} - H_{T_{j-1}}^{A,B}\right)\right].$$

The function computing $pl^{A,B}$, $dl^{A,B}$ and $CDO\ Spread^{A,B} := dl/pl$ is called **static int** eber in the Premia code.

2.2 Calibration

The model is calibrated on the to 6 months of observed 5 year tranche spreads on the iTraxx Europe in the period 23.9.2005-03.03.2006 business days from November 1st to November 6th 2006. The calibrated values used in Premia are taken from Table 4 of [2]. With the values $\lambda_0 = 0.004668$, $\lambda_1 = 0.1921$ and $\lambda_2 = 20.73$, we recover the data values of table 4 except for the first tranche since, in our implemented formulas, we did not take into account the upfront payment.

References

[1] Eberlein, Ernst; Frey, Rüdiger; von Hammerstein, Ernst August. Advanced credit portfolio modeling and CDO pricing. Mathematics—key technology for the future, 253–279, Springer, Berlin, 2008.

[2] Frey, Rüdiger; Backhaus, Jochen. Pricing and hedging of portfolio credit derivatives with interacting default intensities. Int. J. Theor. Appl. Finance 11 (2008), no. 6, 611–634. 1, 2