CDOs' hedging in Markovian contagion models

1$^{\text{er}}$ mars 2012

The following method to compute the hedge of a CDO tranche is based on the article of J-P. Laurent, A. Cousin and J-D. Fermanian [1]

Premia 14

1 CDO tranche hedging

We consider a synthetic CDO of maturity T based on n companies which can make default at a random time τ_i, $1 \leq i \leq n$. We denote by $\mathcal{H}_t$ the natural filtration generated by the default times and we assume that no simultaneous default can occur.
We suppose also the existence of some $(\mathbb{P}, \mathcal{H}_t)$ intensities for the counting processes $N_i(t) = \mathbf{1}_{\{\tau_i \leq t\}}$, $i = 1, \ldots, n$ i.e. there exist some $\mathcal{H}_t$ predictable processes $(\alpha_i^{\mathbb{P}})_{(1 \leq i \leq n)}$ such that $t \mapsto N_i(t) - \int_0^t \alpha_i^{\mathbb{P}}(s) \mathrm{d}s$ are $(\mathbb{P}, \mathcal{H}_t)$-martingales.

1.1 Market assumptions

We assume that instantaneous digital default swaps are traded on the names. A such product provides a payoff of $dN_i(t) - \alpha_i(t)dt$ at $t + dt$. $dN(t)$ is the payment of the default leg while $\alpha_i(t)dt$ is the one of the payment leg.
We further assume that default-free interest rates are constant and equal to r. Hence, given some initial investment V_0 and some $\mathcal{H}_t$-predictable processes $\delta_1, \ldots, \delta_n$ associated with some self-financed trading strategy in instantaneous digital CDS, we attain at time T the payoff

$$V_0 e^{rT} + \sum_{i=1}^{n} \int_0^T \delta_i(s) e^{r(T-s)} (\mathrm{d}N_i(s) - \alpha_i(s)\mathrm{d}s) \tag{1}$$

1.2 Hedging and martingale representation theorem

>From the absence of arbitrage opportunities and under mild regularity assumptions, there exists a probability $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that the α_i are the $(\mathbb{Q}, \mathcal{H}_t)$ intensities associated with the default times. Then, considering a payoff M $\mathcal{H}_T$-measurable and $\mathbb{Q}$-integrable, the integral representation theorem

of point process martingales gives n processes $\mathcal{H}_t$-measurable $\theta_1, \dots, \theta_n$ such that :

$$M = \mathbb{E}^{\mathbb{Q}}[M] + \sum_{i=1}^{n} \int_0^T \theta_i(s)(\mathrm{d}N_i(s) - \alpha_i(s)\mathrm{d}s) \tag{2}$$

Identifying expressions (1) and (2), we obtain :

$$\delta_i(s) = \theta_i(s) e^{-r(T-s)} \quad \text{for } 0 \leq s \leq T \text{ and } \; i = 1, \dots, n$$

and an initial investment $V_0 = \mathbb{E}^{\mathbb{Q}}[Me^{-rT}]$.

1.3 The Markovian contagion model

We assume that intensities α_i depend only on the current credit status : they are deterministic functions of $N_1(t), \dots, N_n(t)$. In a homogeneous Markovian model, they take the following form : $\alpha_i^{\mathbb{Q}}(t, N_1(t), \dots, N_n(t))$. Moreover, we can specify the model by considering that intensities depend only on the number of defaults at the date t. Then, let $N(t) = \sum_{i=1}^{n} N_i(t)$, and the default intensities become $\alpha^{\mathbb{Q}}(t)(t, N(t))$. For simplicity, we will assume a constant recovery rate R. The aggregate loss at time t is given by :

$$L_t = (1 - R)\frac{N(t)}{n}$$

As a consequence of the assumption of no simultaneous defaults, L_t is the sum of the default intensities and then depends only upon the number of defaults at time t. Let $\lambda(t, N(t))$ define the risk-neutral loss intensity, we have :

$$\lambda(t, N(t)) = (n - N(t))\alpha^{\mathbb{Q}}(t)(t, N(t)) \tag{3}$$

Under those assumptions, The process $N(t)$ is Markovian (under $\mathbb{Q}$) whose generator Λ is :

$$\begin{pmatrix} -\lambda(t,0) & \lambda(t,0) & 0 & \dots & 0 \\ 0 & -\lambda(t,1) & \lambda(t,1) & 0 & \dots \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & -\lambda(t,n-1) & -\lambda(t,n-1) \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix} \tag{4}$$

1.4 Computation of the δ

We consider a European type payoff M and we denote by $V(t,.)$ its replication price at time t : this is a vector whose components are $V(t,k) = e^{-r(T-t)}\mathbb{E}[M|N(t) = k]$. By Ito's lemma we have :

$$\mathrm{d}V(t, N(t)) = \frac{\partial V(t, N(t))}{\partial t}\mathrm{d}t + \big(V(t, N(t)+1) - V(t, N(t))\big)\mathrm{d}N(t) \tag{5}$$

Since $M_t := e^{-rt}V(t, N(t))$ is a $\mathbb{Q}$-martingale, we deduce the following relation :

$$\frac{\partial V(t, N(t))}{\partial t} + \lambda(t, N(t))\big(V(t, N(t)+1) - V(t, N(t))\big) = rV(t, N(t)) \tag{6}$$

Then,

$$\begin{aligned} \mathrm{d}V(t,N(t)) &= rV(t,N(t))\mathrm{d}t + \big(V(t,N(t)+1) - V(t,N(t))\big)\big(\mathrm{d}N(t) - \lambda(t,N(t))\mathrm{d}t\big) \\ &= rV(t,N(t))\mathrm{d}t + \sum_{i=1}^{n-N(t)} \big(V(t,N(t)+1) - V(t,N(t))\big)\big(\mathrm{d}N_i(t) - \alpha^{\mathbb{Q}}(t,N(t))\mathrm{d}t\big) \end{aligned}$$

Identifying (1) and the last equality gives the hedge ratio for the company i at time t :

$$\delta_i = e^{-r(T-t)}\big(V(t,N(t)+1) - V(t,N(t))\big)(1 - N_i(t)) \tag{7}$$

We can also perfectly hedge a CDO tranche using only the index portfolio and the risk free asset and the hedge ratio is given by :

$$\delta_I = \frac{V(t,N(t)+1) - V(t,N(t))}{V_I(t,N(t)+1) - V_I(t,N(t))} \tag{8}$$

where $V_I(t,.)$ is the replication price vector of the index portfolio whose components are $V_I(t,k) = \mathbb{E}\big[1 - \frac{N(T)}{n}|N(t) = k\big]$. See [1, page 9] for more details.

2 Practical implementation

2.1 Calibration of loss intensities

Given the probabilities $p(T,k)$ of the number of default at time T (obtained from the quotes of the liquid CDO tranches, see [1, foonote 20] for more details), we can compute the loss intensities λ_k using the forward Kolmogorov equation for the Markov process $N(t)$. We assume that $p(T,k)$ can be written as $p(T,k) = \sum_{i=0}^{k} a_{k,i}e^{-\lambda_i T}$ for $k = 0,\dots,n-1$ where the $a_{k,i}$ are defined by $a_{0,0} = 1$ and $a_{k,i} = \frac{\lambda_{k-1}}{\lambda_k - \lambda_i}a_{k-1,i}$ for $i = 0,\dots,k-1$, $k = 1,\dots,n-1$ and $a_{k,k} = -\sum_{i=0}^{k-1} a_{k,i}$. Then the λ_k can be computed iteratively by solving the univariate non linear implicit equations

$$\sum_{i=0}^{k-1} a_{k-1,i}e^{-\lambda_i t}\Big(\frac{1 - e^{-(\lambda_k-\lambda_i)T}}{\lambda_k - \lambda_i}\Big) = \frac{p(T,k)}{\lambda_{k-1}},\ \ k = 1,\dots,n-1 \tag{9}$$

and using $p(0,k) = 0$ and $\lambda_0 = -\frac{\log p(T,0)}{T}$.
In Premia, the system is solved using the Newton root method available in the PNL. The default probabilities may be computed using a Gaussian Copula or calibrated from the market data.

2.2 Computation of credit δ through a recombining tree

We use a tree method to compute the price vectors $V(t,.)$ and $V_I(t,.)$ based on the approximation of the transition probabilities of the process $N(t)$, whose its generator-matrix Λ is given by (4). For an European type payoff, the price vector fulfils :

$$V(t,\cdot) = e^{-r(t'-t)} \exp^{\Lambda(t'-t)} V(t',\cdot).$$

We start by discretizing the interval $[0,T]$ using a set of node dates $t_0 = 0 < t_1 < \dots < t_N = T$, for simplicity we consider a constant time step $\Delta = t_i - t_{i-1}$.

The most simple discrete time approximation for the transition probabilities is to use the first order Taylor expansion of the exponential function : $\exp^{\Lambda(t_{i+1}-t_i)} \approx Id + \Lambda(t_i)(t_{i+1} - t_i)$. Then we obtain the following probabilities :

$$\mathbb{Q}[N(t_{i+1}) = k | N(t_i) = k] = 1 - \lambda_k \Delta$$

and

$$\mathbb{Q}[N(t_{i+1}) = k + 1 | N(t_i) = k] = \lambda_k \Delta$$

For numerical reason, we prefer use those expressions :

$$\mathbb{Q}[N(t_{i+1}) = k | N(t_i) = k] = 1 - e^{-\lambda_k \Delta} \tag{10}$$

and

$$\mathbb{Q}[N(t_{i+1}) = k + 1 | N(t_i) = k] = e^{-\lambda_k \Delta} \tag{11}$$

Computation of the CDO replication price The loss at time t is given by $L(t) = (1 - R)\frac{N(t)}{n}$. Let us consider a CDO tranche $[a, b]$, the outstanding nominal on this tranche is $O(N(t)) = b - a + (L(t) - b)^+ - (L(t) - a)^+$.
If $d(i, k)$ denotes the value at time t_i when $N(t_i) = k$ of the default payment leg of the CDO tranche, it verifies the following recurrence relation :

$$d(i, k) = e^{-r\Delta}\big((1 - e^{-\lambda_k \Delta})(d(i+1, k+1) + O(k) - O(k+1)) + e^{-\lambda_k \Delta} d(i+1, k)\big), \tag{12}$$

initialized by $d(N, k) = 0, \forall k$. Denote by $T_1, \dots, T_p$ the regular premium payment dates and assume that $\{T_1, \dots, T_p\} \subset \{t_0, \cdots, t_N\}$. $r(i, k)$ denotes the value at time t_i when $N(t_i) = k$ of the premium leg and satisfies :
if $t_{i+1} \in \{T_1, \dots, T_p\}$

$$r(i, k) = e^{-r\Delta}\big(O(k)(T_{i+1} - T_i) + (1 - e^{-\lambda_k \Delta}) r(i+1, k+1) + e^{-\lambda_k \Delta} r(i+1, k)\big) \tag{13}$$

if $t_{i+1} \notin \{T_1, \dots, T_p\}$, denotes by l the integer such that $T_l < t_{i+1} \leq T_{l+1}$

$$r(i, k) = e^{-r\Delta}\big((1 - e^{-\lambda_k \Delta})(r(i+1, k+1) + (O(k) - O(k+1))(t_{i+1} - T_l)) + e^{-\lambda_k \Delta} r(i+1, k)\big) \tag{14}$$

The spread of the CDO tranche is equal to $s = \frac{d(0,0)}{r(0,0)}$. Hence the value of the CDO tranche at time t_i when $N(t_i) = k$ is $V_{CDO}(i, k) = d(i, k) - sr(i, k)$.

Computation of the CDS index replication price Denote by $r_{IS}(i, k)$ and $d_{IS}(i, k)$ the default and premium legs of the CDS index. The default leg is the same that a [0,1] CDO tranche. $r_{IS}(i, k)$ satisfies (12) and (13) if we redefine $O(k)$ as $O(k) = 1 - \frac{k(1-R)}{n}$. The spread is $s_{IS} = \frac{d_{IS}(0,0)}{r_{IS}(0,0)}$.
The program return a $(N + 1) \times (n + 1)$ matrix where N is the size of the subdivision of $[0, T]$. The value at the intersection of the k-th row and i-th column corresponds to the hedge ratio at time t_i if k defaults occur. We have :

$$\delta(i, k) = \frac{V_{CDO}(i + 1, k + 1) - V_{CDO}(i + 1, k) + (O(k) - O(k + 1))(1 - \mathbf{1}_{t_{i+1} \notin \{T_1, \dots, T_p\}}(t_{i+1} - T_l))}{V_{IS}(i + 1, k + 1) - V_{IS}(i + 1, k) + \frac{1-R}{n} - \frac{1}{n} s_{IS} \mathbf{1}_{t_{i+1} \notin \{T_1, \dots, T_p\}}} \tag{15}$$

2.3 Parameters with Nsp

We have to fill several parameters through the Nsp interface to do the computation.

- `cdo_default_probability.dat` : this is a file containing the default probability which allows the calibration of the loss intensity.
- T : maturity of the CDO (default value $T = 5$)
- R : this is the recovery rate (default value $R = 0.4$)
- n : number of companies
- `delta` : the time step (default value $\frac{1}{365}$)

Références

[1] J.P. Laurent, A. Cousin and J-D. Fermanian. Hedging default risks of CDOs in Markovian contagion models, 2008. 1, 3