Hull and White Two-factor model

Ismail Laachir

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1 Hull and White two-factor model

Hull and White model is a short-rate model. One of its main characteristics is its ability to match the initial yield curve by using time-varying parameter. A one factor version of this model was first proposed in [1] (already implemented in Premia). In this project we consider the two-factor version proposed in [2].

Hull&White two-factor model is defined by an EDS which describes the evolution of the spot rate $r(t)$:

$$\begin{cases} dr(t) = [\theta(t) + u(t) - a\,r(t)\,]\,dt + \sigma_1\,dW_1(t) \\ \\ du(t) = -b\,u(t)\,dt + \sigma_2\,dW_2(t), \quad u(0) = 0 \end{cases}$$

The two processes W_1 and W_2 are brownian motions with instantanious correlation ρ, and θ is a deterministic function totally given by the market value of the zero coupon bonds.

Let us denote by $P_M(0,T)$ the market zero coupon bond value maturing at time T and $f_M(t) = -\frac{\partial log(P_M(0,t))}{\partial t}$ the market present instantaneous forward rate, then with an appropriate choice for the function θ (see Hull&White 1994 for details), the model exactly fits the market bonds curve and we have several analytical formulas :

Zero coupon bond at time t knowing that $r(t) = r_t$ and $u(t) = u_t$:

$$P(t,T) = A(t,T)e^{-B(t,T)\, r_t - C(t,T)\, u_t}.$$

Explicite formulations for A, B and C can be found in [2].

The price at time t for a European Call on a ZC bond :

$$C_t = \mathbb{E}_t\left[e^{-\int_t^T r(s)ds}(P(T,S)-K)_+\right] = P(t,S)\mathcal{N}(h) - KP(t,T)\mathcal{N}(h-\sigma_p).$$

Where $\mathcal{N}$ is the cumulative function of the normal law,

$$h = \frac{1}{\sigma_p} log\left(\frac{P(t,S)}{P(t,T)K}\right) + \frac{\sigma_p}{2}$$

and σ_p is given in [2].

This closed formula for european option on bond also leads to closed formula for cap and floor.

2 Trinomial Tree method

2.1 Tree for a one-factor process :

We first recall the procedure for the construction of a trinomial tree that approximates a process x of the form :

$$dx(t) = -a\, x(t)dt + \sigma\, dW(t), \quad x(0) = 0$$

Let $0 = t_0 < t_1 < .. < t_n = T$ be a time scale for our tree in $[0,T]$, $\Delta t_i = t_{i+1} - t_i$, and $x_{i,j}$ the x node value at time t_i for the j^{th} space step of the tree (starting from the down). We need then :

$$\begin{cases} \mathbb{E}\left[x(t_{i+1})|x(t_i)=x_{i,j}\right]=M_{i,j} \\ \mathbb{V}ar\left[x(t_{i+1})|x(t_i)=x_{i,j}\right]=V_{i,j}^2=V_i^2 \end{cases}$$

Knowing that x has is a gaussian process, M_{ij} and V_i can be computed :

$$\begin{cases} M_{i,j}=x_{i,j}\,e^{-a\,\Delta t_i} \\ V_i^2=\frac{\sigma^2}{2a}\left[1-e^{-a\,\Delta t_i}\right] \end{cases}$$

At time t_i, the nodes are equaly spaced, so : $x_{i,j}=j\Delta x_i$, with $\Delta x_i = V_{i-1}\sqrt{3}$.

Starting at time t_i from node $x_{i,j}$, the process can move to three node at time t_{i+1} :

$$\begin{cases} \text{Up with probability } p_u(i,j) \text{ to the node } x_{i+1,k+1} \\ \text{Middle with probability } p_m(i,j) \text{ to the node } x_{i+1,k} \\ \text{Dwon with probability } p_d(i,j) \text{ to the node } x_{i+1,k-1} \end{cases}$$

The index k is chosen so that $x_{i+1,k}$ is as close as possible to the mean $M_{i,j}$, ie :

$$k=\text{round}\left(\frac{M_{i,j}}{\Delta x_{i+1}}\right)=\text{round}\left(j\,\beta_i\right)$$

, with $\beta_i=\dfrac{\Delta x_i}{\Delta x_{i+1}}\,e^{-a\,\Delta t_i}$.

The probabilities $p_u(i,j)$, $p_m(i,j)$ adn $p_d(i,j)$ are chosen so that that conditional mean and variance of the discrete process in the tree match those of the continous process ($M_{i,j}$ and V_i). See [3] for the obtained formulas.

2.2 Tree for a two-factor process :

Now that we kown how to construct a trinomial tree for a process of the kind $dx(t)=-a\,x(t)dt+\sigma(t)\,dW(t)$, we will use this technique for the two-factor process r.

First, we consider the process x verifying the same equation as r, with $\theta=0$:

$$\begin{cases} dx(t)=\left[u(t)-a\,x(t)\,\right]dt+\sigma_1\,dW_1(t), \quad x(0)=0 \\ \\ du(t)=-b\,u(t)\,dt+\sigma_2\,dW_2(t), \quad u(0)=0 \end{cases}$$

If we suppose that $a \neq b$, then the dependance of x on u can be eliminated by defining

$$y = x + \frac{u}{b-a}$$

so that

$$\begin{cases} dy(t) = -a\,y(t)\,dt + \sigma_3\,dW_3(t), \quad y(0)=0 \\ \\ du(t) = -b\,u(t)\,dt + \sigma_2\,dW_2(t), \quad u(0)=0 \end{cases}$$

where

$$\sigma_3^2 = \sigma_3^2 + \frac{\sigma_2^2}{(b-a)^2} + \frac{2\rho\sigma_1\sigma_2}{b-a}$$

and W_3 is a brownian motion. The correlation between W_2 and W_3 is

$$\rho_{uy} = \frac{\rho\sigma_1 + \sigma_2/(b-a)}{\sigma_3}$$

The first step to construct a tree for x is to construct two trinomial trees, with the technique explained above, for the two processes u and y, then use the formula

$$x = y - \frac{u}{b-a}$$

The tree obtained for x will be a two-dimensional trinomial tree, where every node will have nine branches, result of the combination of the tree branches of u and y.

At time t_i, we define the nodes $y(i,h)$ and $u(i,l)$ so the node for x is $x(i,h,l)$. We define j the index of middle branche (in the tree of y) emanating from $y(i,h)$ and the correspanding probabilities pu, pm, pd, and define k the index of middle branche (in the tree of u) emanating from $u(i,l)$ and the correspanding probabilities qu, qm, qd. Then, starting from $x(i,h,l)$, the process move to nine branches $x(i, j+\epsilon_1, k+\epsilon_2)$, where ϵ_1 and ϵ_2 take the values 0, 1 or -1.

Finaly we have to decide the probabilities associated with every node of the nine :

In case of zero correlation between u and y (ie $\rho_{uy} = 0$), the matrix of probabilities for the nine branches of x is simply :

		u-move		
		Down	Middle	Up
	Down	$p_d q_d$	$p_d q_m$	$p_d q_u$
y-move	Middle	$p_m q_d$	$p_m q_m$	$p_m q_u$
	Up	$p_u q_d$	$p_u q_m$	$p_u q_u$

In the case of correlated processes, the elements of the matrix above are shifted in such a way that the sum of shifts in each row and column is zero (to preserve the law of u and y) and to have the good correlation ρ_{uy} between u and y. See [2] for further details.

2.3 Calibration of the tree to the market yield curve :

After the construction of the tree for the process x, the process r can be defined as $r(t) = x(t)+\alpha(t)$ where α is a deterministic function. It is calculated using the Arrow-Debreu node prices and the market price of Zero Coupon Bonds.

We denote by $Q_{i+1,j,k}$ the present value of a security that pays 1 if the node $(i+1, j, k)$ is attained and zero otherwise. These quantities are calculated recursively, knowing α_i and $Q_{i,h,l}$ for all (h, l), by :

$$Q_{i+1,j,k} = \sum_{h,l} Q_{i,h,l}\, q_i(h, l, j, k)\, \exp\{-(\alpha_i + x_{i,h,l})\,\Delta t_i\}$$

where $q_i(h, l, j, k)$ is the probability of moving from (i, h, l) to $(i+1, j, k)$. Then, α_{i+1} is calculated by solving :

$$P_M(0, t_{i+2}) = \sum_{i,j} Q_{i+1,j,k}\, \exp\{-(\alpha_{i+1} + x_{i+1,j,k})\,\Delta t_{i+1}\}$$

ie :

$$\alpha_{i+1} = \frac{1}{\Delta t_{i+1}} \ln \frac{\sum_{i,j} Q_{i+1,j,k}\, \exp\{-x_{i+1,j,k}\Delta t_{i+1}\}}{P_M(0, t_{i+2})}$$

The intial value for α and Q are : $Q_{0,0,0} = 1$ and $\alpha_0 = -\ln(P_M(0, t_1))/t_1$.

3 Pricing of a security using the tree :

Now that we have a trinomial tree of the spot rate $r_{i,j,k}$ with their transition probabilities we can compute the price $h(t, r(t), u(t))$ of any european option with payoff $H(T, r(T), u(T))$ thanks to a backward induction, starting with $h(T, r(T), u(T)) = H(T, r(T), u(T))$:

$$h_{i,j,k} = e^{-r_{i,j,k}\Delta t_i} \sum_{\epsilon_1,\epsilon_2\in\{-1,0,1\}} h_{i+1,j^*+\epsilon_2,k^*+\epsilon_2}\, q_i(j, k, j^* + \epsilon_1, k^* + \epsilon_2)$$

Where $(i+1, j^*, k^*)$ is the index of middle branche emanating from (i, j, k) and $q_i(j, k, j^* + \epsilon_1, k^* + \epsilon_2)$ is the probability of moving from (i, j, k) to $(i + 1, j^* + \epsilon_1, k^* + \epsilon_2)$.

In the case of an american payoff, we compare the result of the backward induction with the payoff $H(t_i, r_{i,j,k}, u_{i,k})$:

$$h_{i,j,k} = max\left(H(t_i, r_{i,j,k}, u_{i,k}), e^{-r_{i,j,k}\Delta t_i} \sum_{\epsilon_1,\epsilon_2\in\{-1,0,1\}} h_{i+1,j^*+\epsilon_2,k^*+\epsilon_2}\, q_i(j, k, j^* + \epsilon_1, k^* + \epsilon_2)\right)$$

References

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[2] J.Hull, A.White, *Numerical procedures for implementing term structure models II, Journal of Derivatives, Winter 1994* 1, 2, 5

[3] D. Brigo, F. Mercurio , *Interest Rate Models - Theory and Practice: With Smile, Inflation and Credit (Springer Finance)*

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References