Efficient pricing of Swing options in Lévy-driven models

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Abstract

We consider the problem of pricing Swing options with multiple exercise rights in Lévy-driven models. We propose an efficient Wiener-Hopf factorization method which solves multiple parabolic partial integro-differential equations associated to the pricing problem. We compare the proposed method with a finite difference algorithm. Both proposed deterministic methods are related to the dynamic programming principle and lead to the solution of a multiple optimal stopping problem. Numerical examples illustrate the efficiency and the precision of the proposed methods.

Keywords: Option pricing; Swing options; Finite difference methods; Wiener-Hopf factorization; American options; energy derivatives; Numerical methods for option pricing.

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1 Introduction

The motivation for the present paper comes from energy markets where financial instruments are increasingly important for the purposes of risk management. In a deregulated market, energy contracts will need to be priced according to their financial risk. Due to uncertainty of consumption and the limited fungibility of energy, new financial contracts such as Swing options have been introduced in

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the commodity market. Swing options are options with many exercise rights of American type. Their owner could exercise at many times under the condition that they respect the refracting time that separates two successive exercises. Swing options are also widely used in gas and oil markets. Thus, pricing Swing options will be more and more important.

Many numerical methods, essentially based on the solution to the dynamic programming equation, have been introduced recently in the financial literature. In the context of Swing options, two different probabilistic strategies have been developed. In the first one, Swing options are priced using an extension of the binomial tree algorithm, leading to the so-called forest tree (Lari-Lavassani *et al.* 2001, Jaillet *et al.* 2004). In the second one, Monte Carlo methods are used, in which conditional expectations are computed using either regression techniques (Barrera-Esteve *et al.* 2006) or Malliavin calculus approaches (Mnif and Zhegal 2006, Carmona and Touzi 2008). In particular, Carmona and Touzi (2008) propose a Monte Carlo approach to the problem of pricing American put options, in a finite time horizon, with multiple exercise rights in the case of geometric Brownian motion. In that paper they introduce the inductive hierarchy of Snell envelopes needed in the multiple exercise case.

The energy expenditure increases sharply with higher daily temperature variation and consequently price varies. Although these spikes of power consumption are infrequent, they have a large financial impact, so many authors propose to price Swing options in a model with jumps (see Mnif and Zhegal (2006), Wilhelm and Winter (2008)). Mnif and Zhegal (2006) extend the results of Carmona and Touzi (2008) to a market with jumps. In fact, the multiple stopping time problem for Swing options can be reduced to a cascade of single stopping time problems in a Lévy market where jumps are permitted. With regards to deterministic methods, Wilhelm and Winter (2008) develop a finite element algorithm for pricing Swing options in different models, including jump models.

In this paper, we propose two approaches to solve multiple parabolic partial integro-differential equations (PIDEs) for pricing Swing options in jump models. The first method, which is very simple and is introduced for comparison purposes, uses a finite difference scheme to solve the system of variational inequalities associated to the Swing option problem using the splitting method proposed in Barles *et al.* (1995).

The second method uses Fast Wiener-Hopf factorization (FWHF) introduced in Kudryavtsev and Levendorskiǐ (2009), where a fast and accurate numerical method for pricing barrier option for a wide class of Lévy processes was constructed. The FWHF method is based on an efficient approximation of the Wiener-Hopf factors and the Fast Fourier Transform algorithm. The advantage of the Wiener-Hopf approach over finite difference schemes in terms of accuracy and convergence property was shown in Kudryavtsev and Levendorskiǐ (2009). We will propose here a new efficient pricing algorithm for Swing options which involves dynamic programming and the solving of multiple PIDEs by the FWHF-method. We apply this algorithm for pricing Swing options where the spot electricity price is a Lévy process which allows one to take into account jump risk. Numerical results, developed as in Wilhelm and Winter (2008) in the Black-Scholes and CGMY models, show the efficiency and the accuracy of the proposed algorithms.

The rest paper is organized as follows. Section 2 is devoted to the basic facts on Lévy processes.

In Section 3 we present the multiple optimal stopping problem for Swing options. In Section 4 and Section 5 we propose, respectively, a finite difference and a Wiener-Hopf approach for pricing Swing options. The numerical results are presented in Section 6.

2 Lévy processes: basic facts

2.1 General definitions

A Lévy process is a stochastically continuous process with stationary independent increments (for general definitions, see e.g. Sato (1999)). A Lévy process may have a Gaussian component and/or pure jump component. The latter is characterized by the density of jumps, which is called the Lévy density. A Lévy process X_t can be completely specified by its characteristic exponent, ψ, definable from the equality $E[e^{i\xi X(t)}] = e^{-t\psi(\xi)}$ (we confine ourselves to the one-dimensional case).

The characteristic exponent is given by the Lévy-Khintchine formula:

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty}(1 - e^{i\xi y} + i\xi y \mathbf{1}_{|y|\leq 1})\nu(dy), \tag{2.1}$$

where $\sigma^2 \geq 0$ is the variance of the Gaussian component, and the Lévy measure $\nu(dy)$ satisfies

$$\int_{\mathbb{R}\setminus\{0\}} \min\{1, y^2\}\nu(dy) < +\infty. \tag{2.2}$$

Assume that under a risk-neutral measure chosen by the market, the price process has the dynamics $S_t = e^{X_t}$, where X_t is a certain Lévy process. Then we must have $E[e^{X_t}] < +\infty$, and, therefore, ψ must admit the analytic continuation into a strip $\Im\xi \in (-1, 0)$ and continuous continuation into the closed strip $\Im\xi \in [-1, 0]$.

The infinitesimal generator of X, denote it L, is an integro-differential operator which acts as follows:

$$Lu(x) = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}(x) + \mu\frac{\partial u}{\partial x}(x) + \int_{-\infty}^{+\infty}(u(x+y) - u(x) - y\mathbf{1}_{|y|\leq 1}\frac{\partial u}{\partial x}(x))\nu(dy). \tag{2.3}$$

The infinitesimal generator L also can be represented as a pseudo-differential operator (PDO) with the symbol $-\psi(\xi)$, i.e. $L = -\psi(D)$, where $D = -i\partial_x$. Recall that a PDO $A = a(D)$ acts as follows:

$$Au(x) = (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{ix\xi}a(\xi)\hat{u}(\xi)d\xi, \tag{2.4}$$

where $\hat{u}$ is the Fourier transform of a function u:

$$\hat{u}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi}u(x)dx.$$

Note that the inverse Fourier transform in (2.4) is defined in the classical sense only if the symbol $a(\xi)$ and function $\hat{u}(\xi)$ are sufficiently nice. In general, one defines the (inverse) Fourier transform by duality.

Further, if the riskless rate, r, is constant, and the stock does not pay dividends, then the discounted price process must be a martingale. Equivalently, the following condition (the EMM-requirement) must hold (see e.g. Boyarchenko and Levendorskiĭ (2002))

$$r + \psi(-i) = 0, \qquad (2.5)$$

which can be used to express μ via the other parameters of the Lévy process:

$$\mu = r - \frac{\sigma^2}{2} + \int_{-\infty}^{+\infty} (1 - e^y + y\mathbf{1}_{|y|\leq 1})\nu(dy). \qquad (2.6)$$

Hence, the infinitesimal generator may be rewritten as follows:

$$Lu(x) = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}(x) + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial u}{\partial x}(x) + \int_{\mathbb{R}}[u(x+y) - u(x) - (e^y - 1)\frac{\partial u}{\partial x}(x)]\nu(dy). \qquad (2.7)$$

2.2 Regular Lévy processes of exponential type

Loosely speaking, a Lévy process X is called a *Regular Lévy Process of Exponential type* (RLPE) if its Lévy density has a polynomial singularity at the origin and decays exponentially at the infinity (see Boyarchenko and Levendorskiĭ (2002))). An almost equivalent definition is: the characteristic exponent is analytic in a strip $\Im\xi \in (\lambda_-, \lambda_+)$, $\lambda_- < -1 < 0 < \lambda_+$, continuous up to the boundary of the strip, and admits the representation

$$\psi(\xi) = -i\mu\xi + \phi(\xi), \qquad (2.8)$$

where $\phi(\xi)$ stabilizes to a positively homogeneous function at the infinity:

$$\phi(\xi) \sim c_\pm|\xi|^\nu, \quad \text{as } \Re\xi \to \pm\infty, \quad \text{in the strip } \Im\xi \in (\lambda_-, \lambda_+), \qquad (2.9)$$

where $c_\pm > 0$. "Almost" means that the majority of classes of Lévy processes used in empirical studies of financial markets satisfy conditions of both definitions. These classes are: Brownian motion, Kou's model (Kou 2002), Hyperbolic processes (Eberlein and Keller 1995), Normal Inverse Gaussian processes and their generalization (Barndorff-Nielsen 1998 and Barndorff-Nielsen and Levendorskiĭ 2001), and extended Koponen's family. Koponen (1995) introduced a symmetric version; Boyarchenko and Levendorskiĭ (2000) gave a non-symmetric generalization; later a subclass of this model appeared under the name CGMY – model in Carr *et al.* (2002), and Boyarchenko and Levendorskiĭ (2002) used the name KoBoL family.

The important exception is Variance Gamma Processes (VGP; see, e.g., Madan *et al.* (1998)). VGP satisfy the conditions of the first definition but not the second one, since the characteristic exponent behaves like $const \cdot \ln|\xi|$, as $\xi \to \infty$.

Example 2.1. The characteristic exponent of a pure jump CGMY model is given by

$$\psi(\xi) = -i\mu\xi + C\Gamma(-Y)[G^Y - (G + i\xi)^Y + M^Y - (M - i\xi)^Y], \qquad (2.10)$$

where $C > 0$, $\mu \in \mathbb{R}$, $Y \in (0, 2)$, $Y \neq 1$, and $-M < -1 < 0 < G$.

Example 2.2. If Lévy measure of a jump diffusion process is given by normal distribution:

$$\nu(dx) = \frac{\lambda}{\delta\sqrt{2\pi}} \exp\left(-\frac{(x-\gamma)^2}{2\delta^2}\right) dx,$$

then we obtain Merton model. The parameter λ characterizes the intensity of jumps. The characteristic exponent of the process is of the form

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \lambda\left(1 - \exp(-\frac{\delta^2\xi^2}{2} + i\gamma\xi)\right), \tag{2.11}$$

where $\sigma, \delta, \lambda \geq 0$, $\mu, \gamma \in \mathbb{R}$.

There are two important degenerate cases:

- If the intensity of jumps $\lambda = 0$, then we obtain Black-Scholes model with $\mu = r - \frac{\sigma^2}{2}$ fixed by the EMM-requirement;

- If the intensity of jumps $\lambda > 0$ but $\delta = 0$, then we obtain a jump diffusion process with a constant jump size γ; the drift term $\mu = r - \frac{\sigma^2}{2} + \lambda(1 - e^\gamma)$ is fixed by the EMM-requirement.

2.3   The Wiener-Hopf factorization

There are several forms of the Wiener-Hopf factorization. The Wiener-Hopf factorization formula used in probability reads:

$$E[e^{i\xi X_T}] = E[e^{i\xi \bar{X}_T}]E[e^{i\xi \underline{X}_T}], \quad \forall\, \xi \in \mathbb{R}, \tag{2.12}$$

where $T \sim \text{Exp}\, q$, and $\bar{X}_t = \sup_{0 \leq s \leq t} X_s$ and $\underline{X}_t = \inf_{0 \leq s \leq t} X_s$ are the supremum and infimum processes. Introducing the notation

$$\varphi_q^+(\xi) = qE\left[\int_0^\infty e^{-qt}e^{i\xi\bar{X}_t}dt\right] = E\left[e^{i\xi\bar{X}_T}\right], \tag{2.13}$$

$$\varphi_q^-(\xi) = qE\left[\int_0^\infty e^{-qt}e^{i\xi\underline{X}_t}dt\right] = E\left[e^{i\xi\underline{X}_T}\right] \tag{2.14}$$

we can write (2.12) as

$$\frac{q}{q + \psi(\xi)} = \varphi_q^+(\xi)\varphi_q^-(\xi). \tag{2.15}$$

Equation (2.15) is a special case of the Wiener-Hopf factorization of the symbol of a PDO. In applications to Lévy processes, the symbol is $q/(q + \psi(\xi))$, and the PDO is $\mathcal{E}_q := q/(q - L) = q(q + \psi(D))^{-1}$: the normalized resolvent of the process X_t or, using the terminology of Boyarchenko and Levendorskiǐ (2005), the expected present value operator (EPV–operator) of the process X_t. The name is due to the observation that, for a stream $g(X_t)$,

$$\mathcal{E}_q g(x) = E\left[\int_0^{+\infty} qe^{-qt}g(X_t)dt \mid X_0 = x\right].$$

Introduce the following operators:

$$\mathcal{E}_q^{\pm} := \varphi_q^{\pm}(D), \tag{2.16}$$

which also admit interpretation as the EPV–operators under supremum and infimum processes. One of the basic observations in the theory of PDO is that the product of symbols corresponds to the product of operators. In our case, it follows from (2.15) that

$$\mathcal{E}_q = \mathcal{E}_q^+ \mathcal{E}_q^- = \mathcal{E}_q^- \mathcal{E}_q^+ \tag{2.17}$$

as operators in appropriate function spaces.

For a wide class of Lévy models $\mathcal{E}$ and $\mathcal{E}^{\pm}$ admit interpretation as expectation operators:

$$\mathcal{E}_q g(x) = \int_{-\infty}^{+\infty} g(x+y) P_q(y) dy, \quad \mathcal{E}_q^{\pm} g(x) = \int_{-\infty}^{+\infty} g(x+y) P_q^{\pm}(y) dy,$$

where $P_q(y)$, $P_q^{\pm}(y)$ are certain probability densities with

$$P_q^{\pm}(y) = 0, \ \forall \pm y < 0.$$

Moreover, characteristic functions of the distributions $P_q(y)$ and $P_q^{\pm}(y)$ are $q(q + \psi(\xi))^{-1}$ and $\varphi_q^{\pm}(\xi)$, respectively.

The general results in this paper are based on simple properties of the EPV operators, which are immediate from the interpretation of $\mathcal{E}^{\pm}$ as expectation operators. For details, see Boyarchenko and Levendorskiĭ (2005).

Proposition 2.1. *EPV-operators $\mathcal{E}_q^{\pm}$ enjoy the following properties*

1. *If $g(x) = 0 \ \forall \ x \geq h$, then $\forall \ x \geq h$, $(\mathcal{E}_q^+ g)(x) = 0$ and $((\mathcal{E}_q^+)^{-1} g)(x) = 0$.*

2. *If $g(x) = 0 \ \forall \ x \leq h$, then $\forall \ x \leq h$, $(\mathcal{E}_q^- g)(x) = 0$ and $((\mathcal{E}_q^-)^{-1} g)(x) = 0$.*

3. *If $g(x) \geq 0 \ \forall x$, then $(\mathcal{E}_q^+ g)(x) \geq 0$, $\forall x$. If, in addition, there exists x_0 such that $g(x) > 0$ $\forall \ x > x_0$, then $(\mathcal{E}_q^+ g)(x) > 0 \ \forall x$.*

4. *If $g(x) \geq 0 \ \forall x$, then $(\mathcal{E}_q^- g)(x) \geq 0$, $\forall x$. If, in addition, there exists x_0 such that $g(x) > 0$ $\forall \ x < x_0$, then $(\mathcal{E}_q^- g)(x) > 0 \ \forall x$.*

5. *If g is monotone, then $\mathcal{E}_q^+ g$ and $\mathcal{E}_q^- g$ are also monotone.*

6. *If g is continuous and satisfies*

$$|g(x)| \leq C(e^{\sigma_- x} + e^{\sigma_+ x}), \quad \forall \ x \in \mathbb{R}, \tag{2.18}$$

   where $\sigma_- \leq 0 \leq \sigma_+$ and C are independent of x, then $\mathcal{E}_q^+ g$ and $\mathcal{E}_q^- g$ are continuous.

3   The multiple optimal stopping problem for Swing options

We consider a price process which evolves according to the formula :

$$S_t = e^{X_t},$$

where $\{X\}_{t\geq 0}$, the driving process, is an adapted Lévy process defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions.

Let T be the option's maturity time and let $\mathcal{T}_{t,T}$ be the set of $\mathbb{F}$-stopping times with values in $[t, T]$. Consider a Swing option that gives the right to multiple exercise with $\delta > 0$ refracting period which separates two successive exercises. We consider the possibility of n put exercises. We shall denote $\mathcal{T}^n$ the collection of all vectors of stopping times $(\tau_1, \tau_2, ..., \tau_n)$, such that

- $\tau_1 \leq T$ a.s.

- $\tau_i - \tau_{i-1} \geq \delta$ on $\{\tau_{i-1} \leq T\}$ a.s., for all $i = 2, .., n$

Denote by $v^{(i)}(t, x)$ the Swing option value with the possibility of i exercises at spot level $S = e^x$ and time $t \leq T$. Following [11], the multiple exercise problem can be solved computing

$$v^{(n)}(0, x) = \sup_{(\tau_1, ..., \tau_n) \in \mathcal{T}^n} \sum_{i=1}^{n} E[e^{-r\tau_i} \phi(X_{\tau_i})] \tag{3.1}$$

where

$$\phi(x) = (K - e^x)_+$$

is the payoff function.

For solving the multiple optimal stopping problem Carmona and Touzi (2008) introduce the idea of a inductive hierarchy. In fact, they reduce the multiple stopping problem to a cascade of n optimal single stopping problems. Define the value function for $i = 1, ..., n$

$$v^{(i)}(t, x) = \sup_{\tau \in \mathcal{T}_{t,T}} E[e^{-r\tau} \phi^{(i)}(\tau, X_\tau^{t,x})] \tag{3.2}$$

where the reward function $\phi^{(i)}$ is now defined as

$$\phi^{(i)}(t, x) = \phi(x) + E[e^{-r\delta} v^{(i-1)}(t + \delta, X_{t+\delta}^{t,x})], \quad t \leq T - \delta, \tag{3.3}$$

$$\phi^{(i)}(t, x) = \phi(x), \quad t > T - \delta. \tag{3.4}$$

The problem could be solved using Monte Carlo algorithm. Let be $t_0 = 0 < t_1 < t_2 < ... < t_N = T$ a time discretization grid. The price of a Swing option can be actually computed by the backward induction procedure:

$$\begin{cases} v^{(i)}(t_N, x) = \phi(x) \\ v^{(i)}(t_{k-1}, x) = \max\left\{\phi^{(i)}(t_{k-1}, x); e^{-r(t_k - t_{k-1})} E[v^{(i)}(t_k, X_{t_k}^{t_{k-1}, x})]\right\}, k = N, ..., 1. \end{cases}$$

Carmona-Touzi (2008) and Mnif-Zeghal (2006) considered Monte Carlo Malliavin-based algorithm to compute the price, respectively, in the Black-Scholes and jump models frameworks. Barrera-Esteve *et al.* (2006) used a regression based method in order to approximate conditional expectations. In the next sections, we propose two PIDE-based approaches.

4   The finite difference scheme for pricing Swing options

We can compute the Swing option price using the formulation given in (3.2) with an analytic approach. In fact, we propose to solve the following system of variational inequalities associated to the Swing options formulation

$$
\begin{cases}
\max\left(\phi^{(i)}(t,x) - v^{(i)}(t,x), \frac{\partial v^{(i)}}{\partial t} + Lv^{(i)} - rv^{(i)}\right) = 0, \;\; (t,x) \;\; \text{in } [0,T[\times\mathbb{R}, \\
v^{(i)}(T,x) = \phi^{(i)}(T, e^x).
\end{cases}
\tag{4.1}
$$

with $i = 1, ..., n$, where the integro-differential operator L is defined in (2.7).

Now recall that for $t \leq T - \delta$

$$
\phi^{(i)}(t,x) = (K - e^x)_+ + E[e^{-r\delta} \, v^{(i-1)}(t + \delta, X_{t+\delta})].
$$

Let us define for $t \leq T - \delta$

$$
u^{(i)}(t,x) = E[e^{-r\delta} \, v^{(i)}(t + \delta, X_{t+\delta}^{t,x})].
$$

By the Feyman-Kac theorem, $u^{(i)}(t,x) = z(0,x)$, where $z(t,x)$ is the solution of the following partial integro-differential equation (PIDE)

$$
\begin{cases}
\frac{\partial z}{\partial t} + Lz - rz = 0, \;\; (t,x) \in [0,\delta[\times\mathbb{R}, \\
z(\delta, x) = v^{(i)}(t + \delta, x),
\end{cases}
\tag{4.2}
$$

which can be numerical computed using a finite difference approach. In order to price a Swing option we can therefore solve the system of variational inequalities (4.1) computing the reward payoff function $\phi^{(i)}(t,x)$ in the following way.

$$
\phi^{(i)}(t,x) = \phi(x)
$$

for $T - \delta < t \leq T$, and

$$
\phi^{(i)}(t,x) = (K - e^x)_+ + u^{(i-1)}(t,x)
$$

for $t \leq T - \delta$.

As said before, the reward payoff function can be computed numerically using a finite difference scheme. The numerical solution of each variational inequalities (4.1) requires to solve numerically each PIDE problem (4.2). In order to solve (4.1) and (4.2), we perform the following steps:

- *Localization.* It means that we choose a spatial bounded computational domain Ω_l. This implies that we have to choose some artificial boundary conditions.

- *Truncation of large jumps.* This corresponds to truncate the integration domain in the integral part.

- *Discretization.* The derivatives of the solution are replaced by usual finite differences and the integral terms are approximated using the trapezoidal rule. The problem is then solved using an implicit-explicit scheme (see Briani *et al* (2004), Cont and Voltchkova (2005) and its program implementation PREMIA). In particular, we introduce a time grid $t = s\Delta t$, $s = 0, .., M$, where $\Delta t = \frac{T}{N}$ is the time step. This produces to solve at each time step a linear system for the linear problem (4.2) and a linear complementarity problem for the non linear problem (4.1).

- *Treatment of the variational inequalities.* We solve each variational inequalities (4.1) using the splitting method of Barles *et al* (1995). The splitting methods can be viewed as an analytic version of dynamic programming. The idea contained in such scheme is to split the American problem in two steps: we construct recursively the approximate solution $v^{(i)}(s\Delta t, x)$ at each time step $s\Delta t$ starting from $v^{(i)}(N\Delta t, x) = \phi(x)$ and computing at each time step $v^{(i)}(s\Delta t, x)$ for $s = N - 1, .., 0$ as follows:

  - Compute the solution of the following linear Cauchy problem on $[s\Delta t, (s+1)\Delta t[\times\Omega_l$ using an implicit-explicit scheme:

  $$\begin{cases} \frac{\partial w^{(i)}(s\Delta t, x)}{\partial t} + Lw^{(i)}(s\Delta t, x) - rw^{(i)}(s\Delta t, x) = 0, & \text{in } [s\Delta t, (s+1)\Delta t[\times\Omega_l \\ w^{(i)}((s+1)\Delta t, x) = v^{(i)}((s+1)\Delta t, x) \end{cases}$$

  - Apply the early exercise $v^{(i)}(s\Delta t, x) = \max(w^{(i)}(s\Delta t, x), \phi^{(i)}(s\Delta t, x))$, where the reward function $\phi^{(i)}(s\Delta t, x)$ is obtained by solving the linear problem (4.2) with an implicit-explicit finite difference method.

5   Pricing Swing options by using Wiener-Hopf approach

5.1   General formulae

The method constructed in this section starts with time discretization (the method of horizontal lines). This method was introduced to finance by Carr and Faguet (1994); Carr (1998) suggested a new important probability interpretation of the method, which we call Carr's randomization. One discretizes the time $(0 =)t_0 < t_1 < \cdots < t_N(= T)$ but not the space variable. Set $v_N^i(x) = (K - e^x)_+$. For $s = N-1, N-2, \ldots, 0$, set $\Delta_s = t_{s+1} - t_s$, $q^s = r + (\Delta_s)^{-1}$, denote by $v_s^i(x)$ the Carr's randomized approximation to $v^i(t_s, x)$.

The early exercise boundary h_s^i for an interval (t_s, t_{s+1}) and $v_s^i(x)$ can be found by using backward induction. For $s = N - 1, N - 2, \ldots$, the boundary h_s^i is chosen to maximize

$$v_s^i(x) = E\left[\int_0^{\tau_s^i} e^{-q^s t} v_{s+1}^i(X_t^{0,x}) dt\right] + E\left[e^{-q^s \tau_s^i} \phi_s(X_{\tau_s^i}^{0,x})\right], \tag{5.1}$$

where τ_s^i is the hitting time of the interval of the form $(-\infty, h_s^i]$, and

$$\phi_s^{(i)}(x) = (K - e^x) + E[e^{-r\delta}v^{(i-1)}(t_s + \delta, X_{t_s+\delta}^{t_s,x})], \quad t_s \leq T - \delta,$$

and

$$\phi_s^{(i)}(x) = (K - e^x), \quad t_s > T - \delta.$$

As in Boyarchenko and Levendorskiǐ (2009), where the case of American options was considered, to derive (5.1), we replaced $\phi(x) = (K - e^x)_+$ in (3.1) with $(K - e^x)$. This is justified by a simple consideration that it is non-optimal to exercise the option when $(K - e^x) \leq 0$.

In the paper, we will use the uniform spacing, therefore, q^s and Δ_s will be independent of s and denoted q and Δt, respectively. For the case of put Swing options, v_s^i given by (5.1) is a unique solution of the boundary problem

$$
\begin{aligned}
(q - L)v_s^i(x) &= (\Delta t)^{-1}v_{s+1}^i(x), \quad x > h_s^i, & (5.2)\\
v_s^i(x) &= \phi_s^{(i)}(x), \quad\quad\quad x \leq h_s^i. & (5.3)
\end{aligned}
$$

Note that the problem (5.2)-(5.3) can be obtained by discretization of the time derivative in the generalized Black-Scholes equation (see details in Boyarchenko and Levendorskiǐ (2009) and the bibliography therein).

Let the refracting period δ be equal $k\Delta t$, where k is a certain positive integer. Then, for $i = 1, ..., n$:

$$\phi_s^{(i)}(x) = (K - e^x) + u_s^{i-1}(x), \tag{5.4}$$

where

$$
\begin{aligned}
u_s^0(x) &= 0; & (5.5)\\
u_s^i(x) &= 0, \quad t_s > T - k\Delta t; & (5.6)\\
u_s^i(x) &= E[e^{-rk\Delta t}v_{s+k}^{(i)}(X_{t_s+k}^{t_s,x})], \quad t_s \leq T - k\Delta t. & (5.7)
\end{aligned}
$$

Introduce $\tilde{v}_s^i(x) = v_s^i(x) - \phi_s^{(i)}(x)$ and substitute $v_s^i(x) = \tilde{v}_s^i(x) + \phi_s^{(i)}(x)$ into (5.2)-(5.3):

$$
\begin{aligned}
(q - L)\tilde{v}_s^i(x) &= (\Delta t)^{-1}G_s^i(x), \quad x > h_s^i, & (5.8)\\
\tilde{v}_s^i(x) &= 0, \quad x \leq h_s^i, & (5.9)
\end{aligned}
$$

where $G_s^i = \tilde{v}_{s+1}^i + \phi_{s+1}^{(i)} - \Delta t(q - L)\phi_s^{(i)}$.

Using similar arguments as Boyarchenko and Levendorskiǐ (2009), it can be shown that for $s = n-1, n-2, ..., 0$: the function G_s^i is a non-decreasing continuous function satisfying bound (2.18) with $\sigma_+ = 1$, $\sigma_- = 0$; in addition,

$$G_s^i(-\infty) < 0 < G_s^i(+\infty) = +\infty; \tag{5.10}$$

Then $G_s^i(x)$ satisfies the conditions of the Theorem 2.6 (Boyarchenko and Levendorskiǐ 2009). Due to the theorem and the Proposition 2.1, we obtain that the following statements hold.

1. the function
$$\tilde{w}_s^i := \mathcal{E}_q^+ G_s^i \tag{5.11}$$

   is continuous; it increases and satisfies (5.10);

2. the equation
$$\tilde{w}_s^i(h) = 0 \tag{5.12}$$

   has a unique solution, denote it h_s^i;

3. the hitting time of $(-\infty, h_s^i]$, $\tau(h_s^i)$, is a unique optimal stopping time;

4. (Carr's approximation to) the Swing option value with i exercise rights at the moment s is given by
$$v_s^i = (q\Delta t)^{-1}\mathcal{E}_q^- \mathbf{1}_{(h_s^i, +\infty)}\tilde{w}_s^i + \phi_s^{(i)}; \tag{5.13}$$

   equivalently,
$$\tilde{v}_s^i = (q\Delta t)^{-1}\mathcal{E}_q^- \mathbf{1}_{(h_s^i, +\infty)}\tilde{w}_s^i; \tag{5.14}$$

5. $\tilde{v}_s^i = v_s^i - \phi_s^{(i)}$ is a positive non-decreasing function that admits bound (2.18) with $\sigma_+ = 1$, $\sigma_- = 0$, and satisfies $\tilde{v}_s^i(+\infty) = +\infty$; it vanishes below h_s^i and increases on $[h_s^i, +\infty)$.

To improve the convergence we reformulate the algorithm. We take into account (5.4), then G_s^i can be rewritten as follows.

$$\begin{aligned}
G_s^i(x) &= v_{s+1}^i(x) - \Delta t(q - L)u_s^{(i-1)}(x) - \Delta t(q - L)(K - e^x) \\
&= v_{s+1}^i(x) - \tilde{u}_s^{(i-1)}(x) - (\Delta t K q - e^x),
\end{aligned} \tag{5.15}$$

where $\tilde{u}_s^i(x)$ can be approximated by the formulae

$$\begin{aligned}
\tilde{u}_s^0(x) &= 0; \tag{5.16} \\
\tilde{u}_s^i(x) &= 0, \quad t_s > T - k\Delta t; \tag{5.17} \\
\tilde{u}_s^i(x) &= E[e^{-r(k-1)\Delta t}v_{s+k}^{(i)}(X_{t_{s+k}}^{t_{s+1},x})] + o(\Delta t), \quad t_s \le T - k\Delta t. \tag{5.18}
\end{aligned}$$

Notice that we can easily compute the expectation in the RHS of (5.18) by using Fourier transform technique (see e.g. Boyarchenko and Levendorskiĭ 2002).

Now, we can rewrite (5.13) as follows.

$$v_s^i = (q\Delta t)^{-1}\mathcal{E}_q^- (\mathbf{1}_{(h_s^i, +\infty)}w_s^i + \mathbf{1}_{(-\infty; h_s^i]}w_{s,0}^i), \tag{5.19}$$

where

$$\begin{aligned}
w_s^i &= \mathcal{E}_q^+ v_{s+1}^i; \tag{5.20} \\
w_{s,0}^i &= \mathcal{E}_q^+(\tilde{u}_s^{(i-1)}(x) + \Delta t K q - e^x) = \mathcal{E}_q^+ \tilde{u}_s^{(i-1)}(x) + \Delta t K q - \varphi_q^+(-i)e^x, \tag{5.21}
\end{aligned}$$

and h_s^i is a solution to the equation

$$w_s^i = w_{s,0}^i. \tag{5.22}$$

The algorithm can be efficiently realized by using the Fast Wiener-Hopf factorization method.

5.2   The Fast Wiener-Hopf factorization method

We briefly review the framework proposed by Kudryavtsev and Levendorskiĭ (2009). The main contribution of the FWHF–method is an efficient numerical realization of EPV-operators $\mathcal{E}$, $\mathcal{E}^+$ and $\mathcal{E}^-$.

Recall that we consider the procedure for approximations of the Wiener-Hopf factors for the symbol $q/(q + \psi(\xi))$ with ψ being characteristic exponent of RLPE of order $\nu \in (0; 2]$ and exponential type $[\lambda_-; \lambda_+]$. The first ingredient is the reduction of the factorization problems to symbols of order 0, which stabilize at infinity to some constant. Introduce functions

$$\Lambda_-(\xi) \;=\; \lambda_+^{\nu_+/2}(\lambda_+ + i\xi)^{-\nu_+/2}; \tag{5.23}$$

$$\Lambda_+(\xi) \;=\; (-\lambda_-)^{\nu_-/2}(-\lambda_- - i\xi)^{-\nu_-/2}; \tag{5.24}$$

$$\Phi(\xi) \;=\; q\Big((q + \psi(\xi))\Lambda_+(\xi)\Lambda_-(\xi)\Big)^{-1}. \tag{5.25}$$

Choices of ν_+ and ν_- depend on properties of ψ, hence on order ν (see (2.8)–(2.9)) and drift μ. See details in Kudryavtsev and Levendorskiĭ (2009).

First, approximate Φ by a periodic function with a large period $2\pi/h$, which is the length of the truncated region in the frequency domain, then approximate the latter by a partial sum of the Fourier series, and, finally, use the factorization of the latter instead of the exact one.

Explicit formulae for approximations of $\varphi^\pm$ have the following form. For small positive h and large even M, set

$$b_k^h = \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} \ln \Phi(\xi) e^{-i\xi kh} d\xi, \qquad k \neq 0,$$

$$b_{h,M}^+(\xi) = \sum_{k=1}^{M/2} b_k^h (\exp(i\xi kh) - 1), \qquad b_{h,M}^-(\xi) = \sum_{k=-M/2+1}^{-1} b_k^h (\exp(i\xi kh) - 1);$$

$$\Phi^\pm(\xi) \approx \exp(b_{h,M}^\pm(\xi)), \qquad \varphi_q^\pm(\xi) = \Lambda_\pm(\xi)\Phi^\pm(\xi).$$

We can apply this realization both after the reduction to symbols of order 0 has been made, and without this reduction. In the latter case, $\Lambda_\pm = 1$, and we obtain a Poisson type approximation.

It is well-known that the limit of a sequence of the Poisson type characteristic functions is infinitely divisible characteristic function. The converse is also true. Every infinitely divisible characteristic function can be written as the limit of a sequence of finite products of Poisson type characteristic functions. Since $\psi(\xi)$ is the characteristic exponent of Lévy process, then the function $q/(q + \psi(\xi))$ is infinitely divisible characteristic function.

Approximants for EPV-operators can be efficiently computed by using the Fast Fourier Transform (FFT) for real-valued functions. Consider the algorithm of the discrete Fourier transform (DFT) defined by

$$G_l = DFT[g](l) = \sum_{k=0}^{M-1} g_k e^{2\pi ikl/M}, \quad l = 0, ..., M - 1. \tag{5.26}$$

The formula for the inverse DFT which recovers the set of g_k's exactly from G_l's is:

$$g_k = iDFT[G](k) = \frac{1}{M} \sum_{l=0}^{M-1} G_l e^{-2\pi i k l / M}, \quad k = 0, ..., M-1. \tag{5.27}$$

In our case, the data consist of a real-valued array $\{g_k\}_{k=0}^{M}$. The resulting transform satisfies $G_{M-l} = \bar{G}_l$. Since this complex-valued array has real values G_0 and $G_{M/2}$, and $M/2 - 1$ other independent complex values $G_1, ..., G_{M/2-1}$, then it has the same "degrees of freedom" as the original real data set. In this case, it is efficient to use FFT algorithm for real-valued functions (see Press *et al.* (1992) for technical details). To distinguish DFT of real functions we will use notation RDFT.

Fix the space step $h > 0$ and number of the space points $M = 2^m$. Define the partitions of the normalized log-price domain $[-\frac{Mh}{2}; \frac{Mh}{2})$ by points $x_k = -\frac{Mh}{2} + kh$, $k = 0, ..., M-1$, and the frequency domain $[-\frac{\pi}{h}; \frac{\pi}{h}]$ by points $\xi_l = \frac{2\pi l}{hM}$, $l = -M/2, ..., M/2$. Then the Fourier transform of a function g on the real line can be approximated as follows:

$$\hat{g}(\xi_l) \approx h e^{i\pi l} \overline{RDFT[g](l)}, \quad l = 0, ..., M/2.$$

Here and below, $\bar{z}$ denotes the complex conjugate of z. Using the notation $p(\xi) = q(q + \psi(\xi))^{-1}$, we can approximate $\mathcal{E}_q$:

$$(\mathcal{E}_q g)(x_k) \approx iRDFT[\bar{p} .* RDFT[g]](k), \ \ k = 0, ..., M-1. \tag{5.28}$$

Here and below, $.*$ is the element-wise multiplication of arrays that represent the functions. Further, we define

$$b_k^h \approx iRDFT[\ln \Phi](k); \ p^{\pm}(\xi_l) = \Lambda^{\pm}(\xi_l) \exp(b_{h,M}^{\pm}(\xi_l)), l = -M/2, ..., 0. \tag{5.29}$$

The action of the EPV-operator $\mathcal{E}_q^{\pm}$ is approximated as follows:

$$(\mathcal{E}_q^{\pm} g)(x_k) = iRDFT[\overline{p^{\pm}} .* RDFT[g]](k), \ \ k = 0, ..., M-1. \tag{5.30}$$

6 Numerical results

In this section we illustrate numerically the efficiency and the robustness of the proposed methods using the parameters of the numerical examples for pricing Swing options in Black-Scholes and CGMY models provided in Wilhelm and Winter (2008). We consider a put Swing option with $n = 1, 2, 3$ exercise numbers, and refracting period $\delta = 0.1$. We assume that the initial value of the stock prices is $S = 100$, the exercise price $K = 100$, the maturity $T = 1$, the force of interest rate $r = 0.05$.

We propose first to assess the numerical robustness of our algorithm in the Black-Scholes case, using the volatility $\sigma = 0.3$. In Table 1, we report the prices (with time in seconds in parenthesis) in a Black-Scholes framework using the finite difference method (FD) proposed in Section 4 and the Wiener-Hopf approach (FWFH) proposed in Section 5. Both methods use spatial discretization step $\Delta x = 0.001$ and varying number of time steps $N = 50, 100, 200$. As benchmark solutions we take the ones (B-WW) provided in Wilhelm and Winter (2008).

Table 1: Swing options prices in the Black-Scholes model

	N	n=1	n=2	n=3
FWHF	50	9.786 (0.22)	19.130 (0.7)	27.968 (1.17)
	100	9.826 (0.42)	19.190 (1.38)	28.045 (2.33)
	200	9.848 (0.83)	19.222 (2.72)	28.085 (4.59)
	B-WW	9.8700	19.2550	28.1265
FD	50	9.834 (0.12)	19.096 (5.16)	27.711 (8.10)
	100	9.864 (0.27)	19.184 (8.64)	27.925 (16.47)
	200	9.867 (0.57)	19.220 (16.68)	28.027 (33.0)

Furthermore we provide numerical results in a Lévy market model. Precisely we use the CGMY model (Carr *et al.* 2002) with $C = 1$, $G = 10$, $M = 10$, $Y = 0.5$. No comparison results are available in the paper of Wilhelm and Winter (2008), then we use as the benchmark value the FWHF method with a very fine mesh grid ($\Delta x = 0.0002$ and $N = 800$). In Table 2, we report the numerical results in the CGMY model.

Table 2: Swing options prices in the CGMY model

	N	n=1	n=2	n=3
FWHF	50	7.100 (0.22)	13.859 (0.7)	20.228 (1.17)
	100	7.131 (0.42)	13.905 (1.38)	20.287 (2.33)
	200	7.147 (0.83)	13.928 (2.72)	20.317 (4.59)
	B-FWHF	7.160	13.944	20.337
FD	50	7.173 (1.20)	13.887 (37.1)	20.102 (76.1)
	100	7.172 (2.31)	13.928 (146)	20.238 (286.5)
	200	7.171 (4.56)	13.948 (751)	20.306 (1398)

All the computations have been performed in a double precision on a Eee PC with characteristics: CPU Atom N450, 1.67 Ghz, 2Gb of RAM.

The numerical results confirm the reliability of both approaches showing the robustness of the methods. In particular, the Wiener-Hopf approach is undoubtedly very precise and efficient method for pricing Swing options in the presence of multiple jumps.

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