Connecting Discrete and Continuous Lookback or Hindsight Options under Jump-Diffusion Model

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We consider continuous lookback and hindsight options which depend on the extremal price of the underlying asset during the life of the options. Perhaps, in exponential Lévy model closed-form formulae are not, in general, available for pricing these options. Then we need to use a discrete numerical method for valuating them. In this context, we study the best way to price a continuous lookback or hindsight option using a discrete lookback or hindsight option, when the price of the underlying asset is the exponential of a finite activity Lévy process. For a general overview about this subject see [1].

1 The exponential Lévy model

The price of the underlying asset, under the risk neutral probability is modeled as followed :

$$S_t = S_0 e^{X_t}$$

where S_0 is the initial price, and X is a Lévy process with generating triplet (γ, σ, ν). It means that the carateristic function of X is given by (see [4], chapter 2)

$$\mathbb{E}e^{iuX_t} = e^{t\varphi(u)} \quad \forall u \in \mathbb{R}$$

where φ is defined by :

$$\varphi(u) = i\gamma u - \frac{\sigma^2 u^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{1}_{|x|\leq 1})\nu(dx) \tag{1.1}$$

Of course, we assume that $\left(e^{-(r-\delta)t} S_t\right)_{t\in[0,T]}$ is a martingale, where r is the continuously compounded interest rate, and δ is the continuously compounded dividend rate. The process X that we consider here is a finite activity Lévy process (i.e. $\nu(\mathbb{R}) < \infty$). Then we can write it in this form (see [2], chapter 4)

$$X_t = \gamma_0 t + \sigma B_t + \sum_{i=1}^{N_t} Y_i$$

where B is a standard Brownian motion, N is a poisson process with parameter $\lambda = \nu(\mathbb{R})$, $(Y_i)_{i\geq 1}$ are i.i.d. r.v. with law $\frac{\nu(dx)}{\nu(\mathbb{R})}$, and

$$\gamma_0 = \gamma - \int_{|x|\leq 1} x\nu(dx)$$

We define the following hypothesis :

- $(H1)$ X is a finite activity Lévy process, integrable, with $\sigma > 0$ and $\exists \alpha > 0$ such that $\mathbb{E}e^{(1+\alpha)M_T} < \infty$;

- $(H2)$ X is a finite activity Lévy process, integrable, with $\sigma > 0$;

2 The theoretical results

Let T the option maturity, $\beta_1 = 0.5826$ (see [1] for the definition of the parameter β_1) and $\Delta t = \frac{T}{n}$ the step of the fixing dates. At a given time $t \in [0, T]$, the value of a continuous lookback put option is given by (see [1] for more details)

$$V(S_+) = e^{-r(T-t)}\mathbb{E}\max\left(S_+, \sup_{t\leq u\leq T} S_u\right) - S_t e^{-\delta(T-T)}$$

where $S_+ = \sup_{0\leq u\leq t} S_u$ is the predetermined max. The call value depends similarly on the predetermined min $S_- = \inf_{0\leq u\leq t} S_u$

$$V(S_-) = S_t e^{-\delta(T-T)} - e^{-r(T-t)}\mathbb{E}\min\left(S_-, \inf_{t\leq u\leq T} S_u\right)$$

The discrete version values at the k^{th} fixing date are

$$V_n(S_+) = e^{-r\Delta(n-k)}\mathbb{E}\max\left(S_+, \max_{k\leq j\leq n} S_{j\Delta t}\right) - S_{k\Delta t}e^{-\delta(n-k)\Delta t},\ for\ the\ put$$

$$V_n(S_-) = S_{k\Delta t}e^{-\delta(n-k)\Delta t} - e^{-r\Delta(n-k)}\mathbb{E}\min\left(S_-, \min_{k\leq j\leq n} S_{j\Delta t}\right),\ for\ the\ call$$

where $S_+ = \max_{0\leq j\leq k} S_{j\Delta t}$ and $S_- = \min_{0\leq j\leq k} S_{j\Delta t}$.

Proposition 2.1. *The price of a discrete lookback option at the k^{th} fixing date and its continuous version at time $t = k\Delta t$ satisfy*

$$V_n\left(S_\pm\right) = e^{\mp\beta_1\sigma\sqrt{\frac{T}{n}}} V\left(S_\pm e^{\pm\beta_1\sigma\sqrt{\frac{T}{n}}}\right) \pm \left(e^{\mp\beta_1\sigma\sqrt{\frac{T}{n}}} - 1\right) e^{-\delta(T-t)} S_t + o\left(\frac{1}{\sqrt{n}}\right)$$
$$V\left(S_\pm\right) = e^{\pm\beta_1\sigma\sqrt{\frac{T}{n}}} V_n\left(S_\pm e^{\mp\beta_1\sigma\sqrt{\frac{T}{n}}}\right) \pm \left(e^{\pm\beta_1\sigma\sqrt{\frac{T}{n}}} - 1\right) e^{-\delta(T-t)} S_t + o\left(\frac{1}{\sqrt{n}}\right)$$

where in $\pm$ and $\mp$, the top case applies to the put and the bottom to the call. The put relations are true under $H1$, and those for the call under $H2$.

The price of a continuous hindsight call at a given time $t \in [0, T]$ with predetermined max S_+ and strike K is given by (see [1] for more details)

$$V\left(S_+, K\right) = e^{-r(T-t)} \mathbb{E}\left(\max\left(S_+, \sup_{t \le u \le T} S_u\right) - K\right)^+$$

And similarly for the put

$$V\left(S_-, K\right) = e^{-r(T-t)} \mathbb{E}\left(K - \min\left(S_-, \inf_{t \le u \le T} S_u\right)\right)^+$$

The discrete versions at the k^{th} fixing date are

$$V_n\left(S_+, K\right) = e^{-r\Delta t(n-k)} \mathbb{E}\left(\max\left(S_+, \max_{k \le j \le n} S_{j\Delta t}\right) - K\right)^+$$

and

$$V_n\left(S_-, K\right) = e^{-r\Delta t(n-k)} \mathbb{E}\left(K - \min\left(S_-, \min_{k \le j \le n} S_{j\Delta t}\right)\right)^+$$

Proposition 2.2. *The price of a discrete hindsight option at the k^{th} fixing date and its continuous version at time $t = k\Delta t$ sarisfy*

$$V_n\left(S_\pm, K\right) = e^{\mp\beta_1\sigma\sqrt{\frac{T}{n}}} V\left(S_\pm e^{\pm\beta_1\sigma\sqrt{\frac{T}{n}}}, K e^{\pm\beta_1\sigma\sqrt{\frac{T}{n}}}\right) + o\left(\frac{1}{\sqrt{n}}\right)$$
$$V\left(S_\pm, K\right) = e^{\pm\beta_1\sigma\sqrt{\frac{T}{n}}} V_n\left(S_\pm e^{\mp\beta_1\sigma\sqrt{\frac{T}{n}}}, K e^{\mp\beta_1\sigma\sqrt{\frac{T}{n}}}\right) + o\left(\frac{1}{\sqrt{n}}\right)$$

where in $\pm$ and $\mp$, the top case applies to the call and the bottom to the put. The call relations are true under $H1$, and those for the put under $H2$.

3 The variance reduction techniques

We couple two variance reduction techniques in our simulations : antithetic variates and control variates (see [3] for more details about these methods). We know that if X is a Lévy process, then (see [4], remark 45.9)

$$\sup_{0\le t\le T} X_t =^d X_T - \inf_{0\le t\le T} X_t$$
$$\inf_{0\le t\le T} X_t =^d X_T - \sup_{0\le t\le T} X_t$$

These results also hold for the discrete supremum and infimum. Then, our antithetc variates are

$$\max_{0\le k\le n} X_{\frac{kT}{n}},\ X_T - \min_{0\le k\le n} X_{\frac{kT}{n}}$$

in the case of the put lookback and the call hindsight, and

$$\min_{0\le k\le n} X_{\frac{kT}{n}},\ X_T - \max_{0\le k\le n} X_{\frac{kT}{n}}$$

in the case of the call lookback and the put hindsight, where n denote the number of discretization points. That technique permits to reduce a bit the variance. We used as control variates, the discrete option under the Black-Scholes corresponding model and the terminal price S_T. The two methods reduce the variance a lot.

References

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