A space-time finite element method for the valuation of barrier options

Jerome Busca
Université de Tours (France)

March 1, 2012

Premia 14

In terms of PDE's, the problem of evaluating a barrier option can be viewed in the following way: one has to solve the Black-Scholes equation in a (possibly time-dependent) domain. On the boundary of this domain, one sets the rebate as Dirichlet condition (on one side for single barrier options, and on both sides for double barriers).

If the barrier does not depend on time, and the spot is away from the barrier, then finite-difference methods typically perform well. However, if the spot is close to the barrier, the discontinuity due to the incompatibility between the initial condition (payoff) and the boundary condition (rebate) typically results in loss of accuracy. Moreover, if the barrier is time-dependent, an important issue is how to design the mesh, since shifts between the nodes and the barrier will produce errors. There are works in this direction (see [6] for instance), but it is clear that in this case the finite-difference approach is inappropriate. Let us emphasize that time-dependent barriers appear in pratice in several instances, e.g. when the rebate is actualized, or when it is expressed as $X(t, S) = \text{const}$ where X is a some state variable.

An alternative approach is a *space-time* finite-element method. The traditional implementations of finite-element methods in parabolic problems typically involves finite difference in time combined with finite elements in space. However, in the one-dimensional setting and for piecewise linear elements, this would not be different from finite difference. In contradistinction, the space-time finite element method involves trapezoidal elements: the nodes sit on lines $t = t_i$ and they are unequally placed in space. In this way, they can be set to match the barrier exactly (up to first order) at each time.

Moreover, the mesh can be refined near the barrier. These two features provide a simple and natural solution to solve the losses in accuracy described above. The element we choose are piecewise linear in reduced barycentric space-time coodinates, leading to a simple tridiagonal scheme that reduces to that of the finite-difference method when the mesh is straight. Is is solved using the usual LU factorization method.

Historically, this method was introduced by Bonnerot and Jamet ([1] [2]) for the Stefan problem, where its flexibility proved to be useful to follow the evolution of the free boundary.

1 The finite element method

The purpose is to discretize the Black-Scholes equation set is logarithmic variable:

$$\begin{cases} -\dfrac{\partial u}{\partial t} + (r - \dfrac{\sigma^2}{2})\dfrac{\partial u}{\partial x} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 u}{\partial x^2} - ru = 0 \\ u(0,x) = (K - e^x)_+, \end{cases} \tag{1}$$

together with the Dirichlet boundary conditions steming from the rebate on one (resp. both) side(s), and the payoff on the other side.

As is customary in finite element methods, one first needs to get a weak integral formulation of (1). First of all, one fixes $[x_{\min}, x_{\max}]$ as localization interval for the problem, and one sets, say, $\forall t \in [0,T]$, $u(t, x_{\min}) = (K - e^{x_{\min}})$, $u(t, x_{\max}) = 0$ as boundary conditions. This corresponds to an up-out put option with rebate 0.

It is known that the localized solution is close to the actual solution when $x_{\min}$ is small and $x_{\max}$ is large compared to some relevant scale of the problem.

Now, let us multiply (1) by a regular test function $\phi(t,x)$ that vanishes on $\{x = x_{\min}\}$ and $\{x = x_{\max}\}$, and integrate on the set $C_n = [t^n, t^{n+1}] \times [x_{\min}, x_{\max}]$. After some integration by part, one gets easily:

$$\begin{aligned} &-\int_{C_n} u\partial_t\phi + \frac{\sigma^2}{2}\int_{C_n} \partial_x u \partial_x \phi + \int_{x_{\min}}^{x_{\max}} u(t^{n+1},x)\phi(t^{n+1},x)dx \\ &-\int_{x_{\min}}^{x_{\max}} u(t^n,x)\phi(t^n,x)dx + \left(r - \frac{\sigma^2}{2}\right)\int_{C_n} u\partial_x\phi + r\int_{C_n} u\phi = 0. \end{aligned} \tag{2}$$

Let $k = \Delta t = t^{n+1} - t^n$ denote the time step and N be an integer. We consider a mesh made up of rows of quadrilateral elements K_i^n with vertices P_i^n, P_{i+1}^n, P_{i+1}^{n+1}, P_i^{n+1} (see figure 1). For any point $P \in K_i^n$, we use the coordinates $(\xi, \eta) \in [0,1]^2$ defined by

$$\xi = \frac{t - t^n}{k} \tag{3}$$

$$\eta = \frac{x - x_i^{n+\xi}}{x_{i+1}^{n+\xi} - x_i^{n+\xi}}, \tag{4}$$

where $x_i^{n+\xi} = (1-\xi)x_i^n + \xi x_i^{n+1}$. If $P = (t,x) \in K_i^n$ has coordinates (ξ, η), we shall write $P = (t,x) = x_{i+\eta}^{n+\xi}$ and $u_i^n = u(x_i^n)$.

To every function ψ defined in K_i^n one can associate $\hat{\psi}$ defined in $[0,1]^2$ such that $\hat{\psi}(\xi,\eta) = \psi(t,x)$. We now introduce quadrature rules for both the volume and boundary integrals appearing in (2). Let us denote by $I_{K_i^n}(\psi)$ the integral

$$\begin{aligned} I_{K_i^n}(\psi) &= \textstyle\int_{K_i^n} \psi(t,x)dt \qquad (5)\\ &= \textstyle\int_{[0,1]^2} \hat{\psi}(\xi,\eta)\hat{J}_{in}(\xi,\eta)d\xi d\eta \qquad (6)\\ &= \textstyle\frac{1}{4}\sum_{s=1}^4 \hat{\psi}\hat{J}_{in}(P_s), \qquad (7) \end{aligned}$$

where

$$\hat{J}_{in} = \frac{\partial x}{\partial \eta}\frac{\partial t}{\partial \xi} - \frac{\partial x}{\partial \xi}\frac{\partial t}{\partial \eta} \tag{8}$$

and $(P_s)_{1\le s\le 4}$ denote the vertices of the square $[0,1]^2$. Let us also denote by I_{t^n} the boundary integral

$$\begin{aligned} I_{t^n}(\psi) &= \textstyle\int_{x_i^n}^{x_{i+1}^n} \psi(t^n,x)dx \qquad (9)\\ &= (\tfrac{x_{i+1}^n - x_i^n}{2})(\psi(t^n,x_i^n) + \psi(t^n,x_{i+1}^n)). \qquad (10) \end{aligned}$$

With these notations in hand, (2) now reads

$$\begin{aligned} &\sum_{i=0}^N \Bigg(-I_{K_i^n}(u\partial_t\phi) + \frac{\sigma^2}{2} I_{K_i^n}(\partial_x u \partial_x \phi) + I_{t^{n+1}}(u(t^{n+1},x)\phi(t^{n+1},x)) \\ &-I_{t^n}(u(t^n,x)\phi(t^n,x)) + (r - \frac{\sigma^2}{2}) I_{K_i^n}(u\partial_x\phi) + rI_{K_i^n}(u\phi)\Bigg) = 0. \end{aligned} \tag{11}$$

We shall now test relation (11) with functions ϕ_{in} whose restriction to each element K_i^n belongs to the space

$$\{\phi(t,x) \mid \hat{\phi}(\xi,\eta) = \alpha_0 + \alpha_1\xi + \alpha_2\eta + \alpha_3\xi\eta\ \}, \tag{12}$$

(where α_i are arbitrary constants) and uniquely defined by the property

$$\phi_{in}(P_j^k) = \delta_{ij}\delta_{k,n+1}. \tag{13}$$

This choice leads to a fully implicit scheme.

Now, the choice

$$\phi_{in}(P_j^k) = \delta_{ij}((1-\theta)\delta_{k,n} + \theta\delta_{k,n+1}) \tag{14}$$

($\theta \in [0,1]$) leads to a so-called θ-scheme.

Note that the functions ψ_{in} assume the following form: $\hat{\psi}_{in}(\xi,\eta) = \xi(1-\eta)$ in K_i^n, $\hat{\psi}_{in}(\xi,\eta) = \xi\eta$ in K_{i-1}^n, $\hat{\psi}_{in} = 0$ elsewhere. Thus the summation in (11) is restricted to only two indices i.

As easily checked, the Jacobian term J_{in} is given by

$$\hat{J}_{in} = k(x_{i+1}^{n+\xi} - x_i^{n+\xi}) \text{ in } K_i^n \tag{15}$$

For $(t,x) \in K_i^n$ one easily computes:

$$\partial_t\hat{\psi}_{in} = \frac{1}{k}(1-\eta) + \frac{\xi}{k}\frac{x_{i+\eta}^{n+1} - x_{i+\eta}^n}{x_{i+1}^{n+\xi} - x_i^{n+\xi}} \tag{16}$$

$$\partial_x\hat{\psi}_{in} = -\frac{\xi}{x_{i+1}^{n+\xi} - x_i^{x+\xi}}. \tag{17}$$

In a similar way, one gets for $(t,x) \in K_{i-1}^n$:

$$\partial_t\hat{\psi}_{in} = \frac{\eta}{k} + \xi\left(-\frac{1}{k}\frac{x_{i-1+\eta}^{n+1} - x_{i-1+\eta}^n}{x_i^{n+\xi} - x_{i-1}^{n+\xi}}\right), \tag{18}$$

and

$$\partial_x\hat{\psi}_{in} = \xi\frac{1}{x_i^{n+\xi} - x_{i-1}^{n+\xi}} \tag{19}$$

Now one simply applies the quadrature rules (5), (9) to evaluate all terms in (11). Since this is a straightforward computation, we simply give the resulting scheme. For simplicity, we treat the totally implicit scheme, the general θ- scheme being a straightforward extension of this case.

$$\begin{gathered}
-\tfrac{1}{4}\left(\left(x_{i+1}^n - x_{i-1}^n\right)u_i^n + \left(x_{i+1}^{n+1} - x_{i+1}^n\right)u_{i+1}^{n+1} + u_i^{n+1}\left(x_{i+1}^{n+1} - x_{i-1}^{n+1}\right) - u_{i-1}^{n+1}\left(x_{i-1}^{n+1} - x_{i-1}^n\right)\right) \\
-\tfrac{\sigma^2}{4}k\left(\tfrac{u_{i+1}^{n+1} - u_i^{n+1}}{x_{i+1}^{n+1} - x_i^{n+1}} - \tfrac{u_i^{n+1} - u_{i-1}^{n+1}}{x_i^{n+1} - x_{i-1}^{n+1}}\right) + \tfrac{1}{2}\left(x_{i+1}^{n+1} - x_{i-1}^{n+1}\right)u_i^{n+1} - \tfrac{k}{4}(\left(r - \tfrac{\sigma^2}{2}\right)\left(u_{i+1}^{n+1} - u_{i-1}^{n+1}\right) \\
+\tfrac{kr}{4}u_i^{n+1}\left(x_{i+1}^{n+1} - x_{i-1}^{n+1}\right) = 0.
\end{gathered} \tag{20}$$

for all $i = 1, \cdots, N-1$, and $u_0 = (K - e^{x_{\min}})$, $u_N = 0$.

As noted above, this is a tridiagonal scheme that reduces to the finite-difference scheme when the mesh is rectangular. Hence it represents no additional computational cost in comparison to finite differences, whereas allowing much more flexibility in te choice of the mesh.

References

[1] R. Bonnerot, P. Jamet, *A second-order finite element method for the one-dimensional Stefan problem*, Int. J. Num. Meth. Eng., Vol. 8, pp. 811-820 (1974). 2

[2] R. Bonnerot, P. Jamet, *Numerical solution of the Eulerian equations of compressible flow by a finite element method which follows the free boundary and the interfaces*, J. Comp. Physics, Vol. 18, pp. 25-45 (1975). 2

[3] J. Busca, *une méthode d' éléments finis pour l'évaluation des options,* C.A.R. internal report.

[4] J. Crank, *Free and Moving Boundary Problems*, Clarendon Press, Oxford, 1982.

[5] J.N. Dewynne, S.D. Howison, I. Rupf, P. Wilmott, *Some mathematical results in the pricing of American options*, Euro. Jnl of Appl. Math. Vol. 4, pp. 381-398 (1993).

[6] F. Slougui, *Deux approches par différences finies pour résoudre les E.D.P. d'évaluation de la finance en présence de bords courbes*, projet de fin d'études, C.A.R. 1