Optimal Multiple stopping and Valuation of Swing Options in Lévy Models

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1 Introduction

Swing options are options with many exercise rights of American type. That means that the option's holder has the right of exercise or not exercise at many times under condition that he respects the refracting time that separates two successive exercises. Due to the complex patterns of consumption and the limited fungibility of energy, Swing options are usually embedded in energy contracts allowing flexibility of delivery with respect to both the timing and the amount of energy used.

Under a regulated market and since prices were set by regulator, pricing such contracts was not an issue. With the transition to a deregulated environment, energy contracts will need to be priced according to their financial risk. Consequently, pricing Swing options will be more and more important.

Many numerical recipes exist for pricing Swing options. In [9], Swing options are priced through a binomial "forest", a multiple-layer tree extension of traditional binomial tree dynamic programming approach. We also refer to [13] and [1] for other numerical pricing algorithms. However, in [5], Carmona and Touzi proposed a different method to price Swing options. They showed that pricing Swing options comes within the scope of optimal multiple stopping. They gave the solution of American put option with multiple exercise rights in the case of geometric Brownian motion. This work was followed by [4] where the problem of optimal multiple stopping was studied for general linear regular diffusions and reward function by using excessive functions.

In the energy market, the consumption is very complex because it depends on exogenous parameters like weather temperature. When we register a high temperature variation, the power consumption increases sharply and prices follow. Although these spikes of consumption are infrequent, they have a large financial impact, so pricing Swing options must take them into account. Therefore, we propose to price Swing options in a market where the price process is allowed to jump.

In this paper, we extend the results of [5] by proving the existence of a solution to the optimal multiple stopping problem in a market where jumps are permitted. To obtain the required result in our context, it was necessary to have an additional assumption on Snell envelop variation. This assumption is naturally accurate in a Lévy market. Then, we provide an explicit solution for a perpetual put Swing option, when the energy price is driven by a Lévy process with no negative jumps. The perpetual put Swing option explicit formula extends the results of [6] and [12] to the optimal multiple stopping problem.

In contrast with [5], the value function is not smooth in a Lévy market. It follows that the boundaries of each optimal exercise interval are characterized in terms of the sub-gradient value function. In the case of punctual Poisson process with no null volatility, we obtain the same results as in the continuous case of [5].

Finally, we give a numerical result for Swing options with a finite maturity. We use a Monte Carlo approximation method suggested by Lions and Regnier [11], then developed by Bouchard and Touzi [3]. Later, we apply the results of [2] to estimate the conditional expectations for the simple Lévy process using the Malliavin calculus.

This paper is organized as follows: in section 2, we formulate and solve the multiple optimal stopping problem in a Lévy model. We provide an explicit solution to the perpetual put Swing options for Lévy process with no negative jumps and we characterize the exercise boundaries regions in section **??**. Then, we study Swing options with infinite exercise rights. It is an asymptotic analysis of the optimal multiple stopping problems presented in section **??**. Finally, we provide a numerical solution for put Swing options with finite maturity using the Monte Carlo approximation method with the Malliavin calculus in section 3.

2 Problem Formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and $\mathbb{F}\{\mathcal{F}_t\}_{t \geq 0}$ a filtration which satisfies the usual conditions. Let $T \in (0, \infty)$ be the option's maturity time, $\mathcal{S}$ the set of $\mathbb{F}$-stopping times with values in $[0, T]$ and $\mathcal{S}_\sigma = \{\tau \in \mathcal{S} \; ; \; \tau \geq \sigma\}$ for every $\sigma \in \mathcal{S}$.

We shall denote by $\delta > 0$ the refracting period which seperates two succesive exercises. We also fix $l \geq 1$ the number of rights we can exercise. Now, we define by $\mathcal{S}_\sigma^{(l)}$ the set of

all vectors of stopping times :

$$\mathcal{S}_\sigma^{(l)} := \left\{ (\tau_1, ..., \tau_l) \in \mathcal{S}_\sigma^l, \ \tau_1 \in \mathcal{S}_\sigma, \ \tau_i - \tau_{i-1} \geq \delta \ on \ \{\tau_{i-1} \leq T\} \ a.s, \ \forall \ i = 2, ..., l \right\} \quad (2.1)$$

Let $X = \{X_t\}_{t \geq 0}$ be a non-negative $\mathbb{F}$-adapted process with right-continuous paths. We assume that X satisfies the integrability condition :

$$E\left[\bar{X}^p \right] < \infty \quad \text{for some} \quad p > 1, \quad \text{where} \quad \bar{X} = \sup_{t \geq 0} X_t. \quad (2.2)$$

For the valuation of Swing options, we introduce the following optimal multiple stopping problem :

$$Z_0^{(l)} := \sup_{(\tau_1, ..., \tau_l) \in \mathcal{S}_0^{(l)}} E\left[\sum_{i=1}^l X_{\tau_i} \right] \quad (2.3)$$

The optimal multiple stopping problem consist in computing the maximum expected reward $Z_0^{(l)}$ and finding the optimal exercise strategy $(\tau_1, ..., \tau_l) \in \mathcal{S}_0^{(l)}$ at which the supremum in (2.3) is attained, if such a strategy exists. Notice that Assumption (2.2) guaranties the finiteness of $Z_0^{(1)}$. As it is easily seen $Z_0^{(l)} \leq l Z_0^{(1)}$, every $Z_0^{(k)}$, $k \geq 1$, will also be finite.

To solve the optimal multiple stopping problem, we define inductively the sequence of Snell envelops :

$$Y^{(0)} = 0 \quad \text{and} \quad Y_t^{(i)} = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_t} E\left[X_\tau^{(i)} / \mathcal{F}_t \right], \quad \forall \ t \geq 0, \quad \forall \ i = 1, ..., l.$$

where the i-th exercise reward process $X^{(i)}$ is given by :

$$X_t^{(i)} = X_t + E\left[Y_{t+\delta}^{(i-1)} / \mathcal{F}_t \right] \quad \text{for } 0 \leq t \leq T - \delta \quad (2.4)$$

and, if $T < \infty$,

$$X_t^{(i)} = X_t \quad \text{for } T - \delta < t \leq T$$

Let us set:

$$\tau_1^* = \inf\{t \geq 0; Y_t^{(l)} = X_t^{(l)}\}$$

We immediately see that $\tau_1^* \leq T$ a.s. Next, for $2 \leq i \leq l$, we define

$$\tau_i^* = \inf\{t \geq \delta + \tau_{i-1}^*; Y_t^{(l-i+1)} = X_t^{(l-i+1)}\}\mathbf{1}_{\{\delta + \tau_{i-1}^* \leq T\}} + (T_+)\mathbf{1}_{\{\delta + \tau_{i-1} > T\}}.$$

Clearly, $\vec{\tau}^* = (\tau_1^*, ..., \tau_l^*) \in \mathcal{S}_0^{(l)}$.

Now, we shall prove that $Z_0^{(l)}$ can be calculated by solving inductively l single optimal stopping problems sequentially. This result is proved in [5] under the Assumption that the process X is continuous a.s and then in [4] for a process X right-continuous by using excessive functions. The existence of an optimal stopping strategy requires the left

continuity in expectation of the process X in addition to Assumption (2.2). For more details, we refer to Theorem 2.18 of [7]. This condition is not necessarily satisfied when the process i s only càdlàg as in our context (i.e under a Lévy model). In this section, we prove the result on optimal multiple stopping in a market where the price processes are càdlàg without using excessive functions.

2.1 existence of an optimal stopping strategy

In order to prove the existence of an optimal stopping strategy, we also assume that :

$$E[\Delta Y_t^{(i)}] \;\;=\;\; 0 \tag{2.5}$$

For any predictable stopping time $t \geq 0$ and for all $i = 1, ..., l$. Assumption (2.5) ensures that the iterated reward processes $X^{(i)}$ are left continuous in expectation when X is also left continuous in expectation. It will be verified for Lévy processes in the next section. In Lemma 2.1, we show that the i-th exercise reward process $X^{(i)}$ satisfies the conditions required to solve the i-th optimal stopping problem.

Lemma 2.1. *Suppose that the process X is right-continuous, left continuous in expectation and satisfies condition (2.2). Suppose further that Assumption (2.5) holds. Then, for all $i = 1, ..., l$, the process $X^{(i)}$ is right continuous, left continuous in expectation and satisfies :*

$$E\left[\left(\bar{X}^{(i)}\right)^p\right] < \infty \quad where \quad \bar{X}^{(i)} = \sup_{0 \leq t \leq T} X_t^{(i)},$$

with p given in formula (2.2)

Proof. To prove the L^p-integrability and the right continuity of the process $X^{(i)}$, we proceed exactly as in the proof of Lemma 1 in [5].

$Y^{(i)}$ is the Snell envelop of $X^{(i)}$ then, it is càdlàg and inherits the L^p-integrability of X. In fact, the martingale inequality shows that

$$E\left[\sup_{0 \leq t \leq T}(Y_t^{(i)})^p\right] \;\;\leq\;\; E\left[\sup_{0 \leq t \leq T}(\bar{X}_t^{(i)})^p\right] \leq \left(\frac{p}{p-1}\right)^p E\left[(\bar{X}^{(i)})^p\right] < \infty$$

where $\bar{X}_t^{(i)} = E[\bar{X}^{(i)}|\mathcal{F}_t]$. We deduce that $Y^{(i)}$ is uniformly integrable. Since $Y^{(i)}$ is càdlàg, then for all $(\tau_n)_{n \geq 0}$ an increasing sequence of stopping times such that $\tau_n \nearrow \tau \in \mathcal{S}$, we have

$$E\left[Y_{\tau_n}^{(i)}\right] \;\;\to\;\; E\left[Y_{\tau-}^{(i)}\right].$$

If τ is a predictable time, by Assumption (2.5), we obtain that $E[Y_{\tau-}^{(i)}] = E[Y_\tau^{(i)}]$ which provides the left continuity in expectation of $Y^{(i)}$ at τ.

Now, it remains to prove the left continuity in expectation of the iterated reward processes $X^{(i)}$. By taking the expectation in the reward process expression, at the stopping time τ_n, we get :

$$
\begin{aligned}
E\left[X^{(i)}_{\tau_n}\right] &= E\left[X_{\tau_n}\right] + E\left[E[Y^{(i-1)}_{\tau_n+\delta}/\mathcal{F}_{\tau_n}]\right] \\
&= E\left[X_{\tau_n}\right] + E\left[Y^{(i-1)}_{\tau_n+\delta}\right]
\end{aligned}
$$

we immediately see that the left continuity in expectation of $X^{(i)}$ at τ follows from the left continuity in expectation of X at τ and $Y^{(i-1)}$ at $\tau+\delta$. Since $\tau+\delta$ is a predictable time, the required result follows by using Assumption (2.5).

$\square$

We generalize Theorem 1 of [5] to càdlàg price processes. The proof of the next Theorem remains the same as in the case of continuous processes.

Theorem 2.1. *Let us assume that the process X satisfies condition* (2.2). *We also suppose that* (2.5) *holds. Then,*

$$
Z^{(l)}_0 = Y^{(l)}_0 = E\left[\sum_{i=1}^{l} X_{\tau_i^*}\right]
$$

where $(\tau_1^*, ..., \tau_l^*)$ represents the optimal exercise strategy.

Proof. In Lemma 2.1, we proved that the process $X^{(i)}$ satisfies the conditions required for the solution of the ordinary single optimal stopping problem of order i. Next, the result of the Theorem 2.1 is deduced by using the same arguments as in Theorem 1 of [5]. $\square$

2.2 The Lévy Model

In the following, we fix the payoff function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}$ by :

$$
\phi(x) = (K - x)^+
$$

for a given parameter $K > 0$. We consider a price process which evolves according to the formula :

$$
S_t = xe^{X_t}, \ x > 0,
$$

where $X = \{X\}_{t\geq 0}$, the driving process, is an adapted Lévy process defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. We shall also use the notation $S_t^{0,x}$ for S_t whenever we need to emphasize the dependence of the process S on its initial condition.

Assumption 2.1. *The probability $\mathbb{P}$ is chosen such that $\{e^{-rt}S_t\}_{t\geq 0}$ is a $\mathbb{P}$-martingale process.*

Lévy-Khinchine formula states, for a Lévy process $X = \{X\}_{t\geq 0}$:

$$E[e^{i\mu X_t}] \quad = \quad exp\{t[i\mu a - \frac{1}{2}\sigma^2\mu^2 + \int_{\mathbb{R}}(e^{i\mu y} - 1 - i\mu y\mathbf{1}_{(|y|<1)})\Pi(dy)]\}, \qquad (2.6)$$

where a and $\sigma \geq 0$ are real constants and Π is a positive measure on $\mathbb{R} \setminus \{0\}$ such that $\int(1 \wedge y^2)\Pi(dy) < +\infty$, called the Lévy measure. The triplet of parameters (a, σ, Π) completely determines the law of the process. For later use, we give the following remark :

Remark 2.1. *A Lévy process is quasi-left continuous. It means that $\Delta X_t = 0$ a.s on the set $\{t < \infty\}$, for every predictable time t. Since the exponential function is continuous, the price process is quasi-left continuous.*

Proposition 2.1. *Let X be a Lévy process characterized by the triplet (a, σ, Π) and $\phi(x) := (K-x)^+$. The reward process is then given by $Z_t := e^{-rt}\phi(S_t)$. Consequentely, Z satisfies the following conditions :*

$$Z \text{ is càdlàg}, \qquad (2.7)$$

$$Z \text{ is left continuous in expectation}, \qquad (2.8)$$

$$E\left[\bar{Z}^p\right] < \infty \text{ where } \quad \bar{Z} = \sup_{t\geq 0} Z_t \text{ and } \quad p > 1, \qquad (2.9)$$

$$E[\Delta\hat{Z}_t^{(k)}] = 0 \text{ when t is a predictable time on } \{t < +\infty\}, \qquad (2.10)$$

where $\{\hat{Z}_t^{(k)}\}_t := \{e^{-rt}v^{(k)}(S_t)\}_t$ for all $k = 1,...,l$ is the Snell envelop of the process $\{Z_t^{(k)}\}_t := \{e^{-rt}\phi^{(k)}(S_t)\}_t$.

Proof. Since $e^{-rt}\phi$ is a continuous function and X is càdlàg, it immediately follows that the process $\{Z_t\} = \{e^{-rt}\phi(S_t)\}$ is càdlàg.

The L^p-integrability of $\bar{Z}$ in condition (2.9) follows from the boundedness of ϕ.

Since X is a Lévy process, X is quasi left continuous, i.e. $\lim_{n\longrightarrow+\infty} X_{\tau_n} = X_\tau$ a.s on $\{\tau < \infty\}$, where $(\tau_n, n \in \mathbb{N})$ is an increasing sequence of stopping times with $\lim_{n\to\infty} \tau_n = \tau$ a.s. Using the boundness of Z, by the dominated convergence theorem, we deduce that $\lim_{n\to\infty} E[Z_{\tau_n}] = E[Z_\tau]$ and so, property (2.8) is proved.

It remains to prove that $E[\Delta\hat{Z}_t^{(k)}] = 0$ for all $k = 1,...,l$ and for any predictable time $t \geq 0$. Since $\hat{Z}^{(k)}$ is the Snell envelop of $Z^{(k)}$, $\hat{Z}^{(k)}$ is a super-martingale process. From the Proposition $D.2$ of [**?**], for every stopping time $\tau \geq 0$, $E[\hat{Z}_\tau^{(k)}/\mathcal{F}_{\tau-}] \leq \hat{Z}_{\tau-}^{(k)}$, which implies

$$E[\Delta\hat{Z}_\tau^{(k)}] \quad \leq \quad 0, \quad \text{for every stopping time } \tau \geq 0.$$

To obtain the reverse inequality, we set :

$$
\begin{aligned}
\hat{Z}^{(k)}(t,x) &= e^{-rt}v^{(k)}(x) \\
&= e^{-rt}\mathrm{Ess}\sup_{\tau \in \mathcal{S}_t} E\left[e^{-r(\tau-t)}\phi^{(k)}(S_\tau)/S_t = x\right] \\
&= \mathrm{Ess}\sup_{\tau \in \mathcal{S}_t} E\left[e^{-r\tau}\phi^{(k)}(S_\tau)/S_t = x\right].
\end{aligned}
$$

Now, we have to prove inductively that $\hat{Z}^{(k)}$ is Lipschitz in x. We check for $k = 1$. Let $x, y \in \mathbb{R}^+$ and $t \geq 0$. We have that :

$$
|\hat{Z}^{(1)}(t,x) - \hat{Z}^{(1)}(t,y)| = e^{-rt}|v^{(1)}(x) - v^{(1)}(y)|.
$$

>From Lemma **??**, $v^{(1)}$ is convex and non increasing on $\mathbb{R}^+$ which implies that $v^{(1)}$ is Lipschitz on any interval bounded from below. Particularly, $v^{(1)}$ is Lipschitz on $[x_1^*, +\infty)$ where x_1^* is defined in Theorem **??**. From Proposition **??**, for all $x \in [0, x_1^*]$, we have that :

$$
v^{(1)}(x) = \phi^{(1)}(x) = \phi(x).
$$

Since ϕ is Lipschitz, , convex and non increasing, we have that $v^{(1)}$ is Lipschitz on $\mathbb{R}^+$ with Lipschitz constant $|\phi'(0^+)|$.

Now, suppose that $\hat{Z}^{(k-1)}$ is Lipschitz in x and let us prove that the property is also true for $\hat{Z}^{(k)}$.

>From Lemma **??**, $\hat{Z}^{(k)}$ is a convex and non increasing function in x. It follows that it is Lipschitz on an interval bounded from below. In particular, $v^{(k)}$ is Lipschitz on $[x_k^*, +\infty)$ where x_k^* is defined in Proposition **??**. It remains to show that $\hat{Z}^{(k)}$ is also Lipschitz in $x \in [0, x_k^*]$. Recall that $v^{(k)}(x) = \phi^{(k)}(x)$ on $[0, x_k^*]$ from Proposition **??**. By using the Lipschitz property of ϕ and the induction Assumption, we obtain that : for all $x, y \in [0, x_k^*]$ and $t \geq 0$,

$$
\begin{aligned}
|\hat{Z}^{(k)}(t,x) - \hat{Z}^{(k)}(t,y)| &= e^{-rt}|v^{(k)}(x) - v^{(k)}(y)| \\
&= e^{-rt}|\phi^{(k)}(x) - \phi^{(k)}(y)| \\
&\leq e^{-rt}|\phi(x) - \phi(y)| + E|\hat{Z}^{(k-1)}(t+\delta, S_{t+\delta}^{t,x}) - \hat{Z}^{(k-1)}(t+\delta, S_{t+\delta}^{t,y})| \\
&\leq |\phi'(0^+)||x - y| + K^{(k-1)}E|S_{t+\delta}^{t,x} - S_{t+\delta}^{t,y}|
\end{aligned}
$$

where $K^{(k-1)}$ is the Lipschitz constant of $\hat{Z}^{(k-1)}$. From Assumption 2.1, we have that :

$$
\begin{aligned}
E|S_{t+\delta}^{t,x} - S_{t+\delta}^{t,y}| &= |x - y|E[S_{t+\delta}^{t,1}] \\
&= |x - y|e^{r\delta},
\end{aligned}
$$

which implies that : for all $x, y \in [0, x_k^*]$ and $t \geq 0$,

$$
\begin{aligned}
|\hat{Z}^{(k)}(t,x) - \hat{Z}^{(k)}(t,y)| &\leq |\phi'(0)||x - y| + K^{(k-1)}e^{r\delta}|x - y| \\
&= (|\phi'(0)| + K^{(k-1)}e^{r\delta})|x - y|.
\end{aligned}
$$

It follows that $\hat{Z}^{(k)}$ is Lipschitz in x with a Lipschitz constant $K^{(k)}$ equal to $|\phi'(0)| + K^{(k-1)}e^{r\delta}$:

$$|\hat{Z}^{(k)}(t,x) - \hat{Z}^{(k)}(t,y)| \leq K^{(k)}|x-y|. \tag{2.11}$$

Finally, for $t \geq 0$ a predictable time, we have that :

$$\begin{aligned} 0 \geq E[\Delta\hat{Z}_t^{(k)}] &= E[\hat{Z}^{(k)}(t,S_t) - \hat{Z}^{(k)}(t^-,S_{t-})] \\ &= E[\hat{Z}^{(k)}(t,S_t) - \hat{Z}^{(k)}(t,S_{t-})] \end{aligned}$$

by the continuity on time property of $\hat{Z}^{(k)}$. The inequality (2.11) implies that :

$$E[\hat{Z}^{(k)}(t,S_t) - \hat{Z}^{(k)}(t,S_{t-})] \geq -K^{(k)}E[\Delta S_t].$$

Relying on the quasi-left continuity of the price process S stated in Remark 2.1, we obtain :

$$\begin{aligned} 0 \geq E[\Delta\hat{Z}_t^{(k)}] &\geq -K^{(k)}E[\Delta S_t] \\ &= 0. \end{aligned}$$

which implies that $E[\Delta\hat{Z}_t^{(k)}] = 0$ when t is a predictable time on $\{t < +\infty\}$.

$\square$

3 Numerical Study for Finite Maturity Problem

The object of this section is the computation of the value function of the multiple stopping problem:

$$v^{(l)}(0,x) = \sup_{(\tau_1,\ldots,\tau_l)\in\mathcal{S}^{(l)}} \sum_{i=1}^{l} E[e^{-r\tau_i}\phi(S_{\tau_i})]$$

and the associated exercise region. We recall that $\phi(x) = (K-x)^+$ is the put payoff with strike $K > 0$ and S is the price driven by a Lévy process with no negative jumps.

We fix the option's maturity $T = 1$ and we choose the Monte Carlo numerical procedure for the implementation.

3.1 Monte Carlo Method

Let $T_n = \{t_j = j/n\}_{0 \leq j \leq n}$ be the partition of the time interval $T = [0,1]$. We denote by h the time step $\frac{T}{n}$, and by $\mathcal{S}_n$ the subset of $\mathcal{S}$ defined by

$$\mathcal{S}_n = \{\tau \in \mathcal{S} ; \tau \in T_n\}.$$

The discrete time approximation for the value function is given by :

$$v^{(k)}(t,x) = \sup_{\tau \in \mathcal{S}_n \cap \mathcal{S}_{t,T}} E[e^{-r\tau}\phi_n^{(k)}(\tau, S_\tau^{t,x})]$$

with $v^{(0)} = 0$.

The discrete time approximation for the reward function of order k for $k = 1, ..., l$ is given by :

$$\phi_n^{(k)}(t,x) = \phi(x) + e^{-r\delta}E[(v_n^{(k-1)}(t+\delta, S_\delta^{0,x})] \text{ for } t \leq T - \delta,$$

and if $T - \delta < t \leq T$,

$$\phi_n^{(k)}(t,x) = \phi(x)$$

The Snell envelop is computed by backward induction :

$$v^{(k)}(t_n, S_{t_n}) = \phi_n^{(k)}(t_n, S_{t_n})$$

and

$$v^{(k)}(t_{j-1}, S_{t_{j-1}}) = \max\left\{\phi_n^{(k)}(t_{j-1}, S_{t_{j-1}}); e^{-rh}E[v^{(k)}(t_j, S_{t_j})/\mathcal{F}_{t_{j-1}}^{(n)}]\right\},$$

where $\mathcal{F}_{t_{j-1}}^{(n)} = \sigma(S_{t_k}, k \leq j-1)$ is the discrete-time filtration. Hence :

$$E[v^{(k)}(t_j, S_{t_j})/\mathcal{F}_{t_{j-1}}^{(n)}] = E[v^{(k)}(t_j, S_{t_j})/S_{t_{j-1}}] =: \rho_n^{(k)}(t_{j-1}, S_{t_{j-1}}).$$

As in section **??**, we consider a simple Lévy process $\{X_t\}_t$ given by formula (**??**) and the price process which evolves according to the formula (**??**).

For the implementation, we simulate N independent Brownian motions and N Poisson processes as follow :

$$W_{t_{j+1}} - W_{t_j} \sim N(0,h) \quad and \quad N_{t_{j+1}} - N_{t_j} \sim P(\lambda h).$$

Then, we deduce the price paths by using the formulas (**??**) and (**??**) :

$$S_{t_j} = xe^{(r-\lambda\phi-\frac{\sigma^2}{2})t_j+\sigma W_{t_j}+\ln(1+\phi)N_{t_j}}$$

3.2 Estimation of the Conditional Expectation Using Malliavin Calculus

The implementation of the Monte Carlo numerical procedure requires the computation of many conditional expectations. Several methods can be proposed for the evaluation of these regression functions. We choose the Malliavin Calculus based Method proposed in [8] and then developed in [2]. We refer to [5] for a discussion about other methods.

We are interested in computing the conditional expectation $E[g(S_{t+h})/S_t]$ where g is a real function satisfying $E[|g(S_{t+h})|^2] < \infty$. The main idea of the Malliavin method consists in using the Malliavin integration by part formula in order to get rid of the Dirac point masses in the following expression :

$$E[g(S_{t+h})|S_t = s] = \frac{E[g(S_{t+h})\delta_s(S_t)]}{E[\delta_s(S_t)]}. \tag{3.1}$$

Notice that in our case, we have to make some relevant modifications because of the existence of a Poisson process with no negative jumps in the price's expression.

Proposition 3.1. *Let g be a real function satisfying $E[|g(S_{t+h})|^2] < \infty$ and $\{S_t\}_t$ a process driven by a simple Lévy process X :*

$$S_t = xe^{X_t}, \quad x > 0$$

where $X_t = at + \sigma W_t + \alpha N_t$ with a, α and $\sigma \geq 0$ real constants. Then,

$$E\left[g(S_{t+h})\delta_s(S_t)\right] = E\left[\mathbf{1}_{S_t \leq s} \; g(S_{t+h}) \left(-\frac{W_t}{t} + \frac{W_{t+h} - W_t}{h}\right)\right].$$

Proof.

$$E[g(S_{t+h})\delta_s(S_t)] = E[E[g(S_{t+h})\delta_s(S_t)|N_t = n, N_{t+h} = p]]$$
$$= \sum_{0 \leq n \leq p} E[g(S_{t+h})\delta_s(S_t)|N_t = n, N_{t+h} = p]P[N_t = n, N_{t+h} = p].$$

We focus on the calculation of $E[g(S_{t+h})\delta_s(S_t)|N_t = n, N_{t+h} = p]$. Let's define

$$\hat{g}_{t+h,p}(y) := g(e^{a(t+h)+\sigma y+\alpha p})$$

and

$$\hat{x}_{t,n} := \frac{1}{\sigma}(ln\frac{s}{x} - at - \alpha n)$$

we obtain :

$$E[g(S_{t+h})\delta_s(S_t)|N_t = n, N_{t+h} = p] = E[\hat{g}_{t+h,p}(W_{t+h})\delta_{\hat{x}_{t,n}}(W_t)]$$

By the Independence of Brownian motion's increments, we have :

$$E[\hat{g}_{t+h,p}(W_{t+h})\delta_{\hat{x}_{t,n}}(W_t)] = \int\int \hat{g}_{t+h,p}(w_1 + w_2)\delta_{\hat{x}_{t,n}}(w_1)\varphi(\frac{w_1}{\sqrt{t}})\varphi(\frac{w_2}{\sqrt{h}})\frac{dw_1}{\sqrt{t}}\frac{dw_2}{\sqrt{h}}$$

where φ is the density of standard one dimensional normal distribution. Recalling that $\delta_s(w_1)$ is a derivative of $-\mathbf{1}_{w_1 \leq s}$ and integrating by parts with respect to w_1 variable and then with respect to variable w_2, we get :

$$E[\hat{g}_{t+h,p}(W_{t+h})\delta_{\hat{x}_{t,n}}(W_t)] = E\left[\hat{g}_{t+h,p}(W_{t+h})\mathbf{1}_{(-\infty,\hat{x}_{t,n}]}(W_t)\left(-\frac{W_t}{t} + \frac{W_{t+h} - W_t}{h}\right)\right].$$

By denoting $A_h := -\frac{W_t}{t} + \frac{W_{t+h}-W_t}{h}$, it follows that :

$$
\begin{aligned}
E[g(S_{t+h})\delta_s(S_t)] &= \sum_{0 \le n \le p} E[\hat{g}_{t+h,p}(W_{t+h})\mathbf{1}_{(-\infty,\hat{x}_{t,n}]}(W_t)A_h]P[N_t = n, N_{t+h} = p] \\
&= \sum_{0 \le n \le p} E[g(S_{t+h})\mathbf{1}_{(-\infty,s]}(S_t)A_h]P[N_t = n, N_{t+h} = p] \\
&= E\left[g(S_{t+h})\mathbf{1}_{(-\infty,s]}(S_t)A_h\right].
\end{aligned}
$$

$\square$

Remark 3.1. *In [10], some useful formulas for computing Malliavin derivatives for simple Lévy processes are given. However, in our context, since we have to differentiate under the expectation, it was possible to condition on Poisson process and make usual computations on the Brownian motion. This provides the same result.*

3.3   Variance Reduction by Localization

By using Monte Carlo Method, we recover a convergence rate of the order $\sqrt{N}$ for the conditional expectation estimator. However, the variance of the estimator explodes as h tends to zero.

To find a remedy to this problem, we introduce localizing functions. Such functions catch the idea that the relevant information for the computation of $E[g(S_{t+h})|S_t = s]$, is located in the neighborhood of s.

Let φ be an arbitrary localizing function. By definition, φ is smooth, bounded and it satisfies $\varphi(0) = 1$. Recalling the proof of Proposition 3.1, we obtain a family of alternative representations of the conditional expectation given by (3.1) :

Proposition 3.2. *Let g be a real function satisfying $E[|g(S_{t+h})|^2] < \infty$, S a Lévy process as defined in Proposition 3.1 and φ an arbitrary localizing function. Then,*

$$
\begin{aligned}
E\left[g(S_{t+h})\delta_s(S_t)\right] &= E\left[g(S_{t+h})\delta_{\hat{x}_t}(W_t)\varphi(W_t - \hat{x}_t)\right] \\
&= E\left[\mathbf{1}_{W_t < \hat{x}_t}\, g(S_{t+h})(\varphi(W_t - \hat{x}_t)A_h - \varphi'(W_t - \hat{x}_t))\right].
\end{aligned}
$$

where $A_h = (-\frac{W_t}{t} + \frac{W_{t+h}-W_t}{h})$ and $\hat{x}_t = \frac{1}{\sigma}(ln\frac{s}{x} - at - \alpha N_t)$.

Moreover, it's possible to reduce the Monte Carlo estimator variance by a convenient choice of the localizing function.

We consider the integrated mean square error :

$$
J(\varphi) := \int_{\mathbb{R}} E\left[\mathbf{1}_{W_t < \hat{x}_t}\, g^2(S_{t+h})A_{h,\varphi}^2\right] dx. \tag{3.2}
$$

where we adopted the following notation : $A_{h,\varphi} := \varphi(W_t - \hat{x}_t)A_h - \varphi'(W_t - \hat{x}_t)$ and we are interested in minimizing J respect to the subset $\{\varphi$ smooth, bounded and $\varphi(0) = 1\}$. Following [2], we prove that the optimal localizing function is given by :

$$\varphi(x) = e^{\nu_h x} \quad where \quad \nu_h := \left(\frac{E[g^2(S_{t+h})A_h^2]}{E[g^2(S_{t+h})]}\right)^{\frac{1}{2}}.$$

Corollary 3.1. *With the same notations as in Proposition 3.2 and by choosing the optimal φ that minimizes (3.2),*

$$E\left[g(S_{t+h})\delta_s(S_t)\right] = E\left[\mathbf{1}_{W_t < \hat{x}_t} \; g(S_{t+h})e^{\nu_h(W_t-\hat{x}_t)}(A_h - \nu_h)\right].$$

where $A_h := (-\frac{W_t}{t} + \frac{W_{t+h}-W_t}{h})$ and $\nu_h := \left(\frac{E[g^2(S_{t+h})A_h^2]}{E[g^2(S_{t+h})]}\right)^{\frac{1}{2}}.$

>From (3.1), it follows that :

$$E\left[g(S_{t+h})/S_t = s\right] = \frac{E\left[\mathbf{1}_{W_t < \hat{x}_t} \; g(S_{t+h})e^{\nu_h(W_t-\hat{x}_t)}(A_h - \nu_h)\right]}{E\left[\mathbf{1}_{W_t < \hat{x}_t}e^{\nu_h(W_t-\hat{x}_t)}(A_h - \nu_h)\right]}$$

with $\hat{x}_t = \frac{1}{\sigma}(ln\frac{s}{x} - at - \alpha N_t).$

3.4 The Value Function Formula

According to the previous section, we are able to calculate the value function :

$$v^{(k)}(t_{j-1}, S_{t_{j-1}}) = \max\left\{\phi_n^{(k)}(t_{j-1}, S_{t_{j-1}}); e^{-rh}\rho_n^{(k)}(t_{j-1}, S_{t_{j-1}})\right\}.$$

Let us denote $S^{(i)}$ the i-th price simulation such that $1 \leq i \leq N$, where N is the simulation number. Then, we define the estimators of $\rho_n^{(k)}$ and $\phi_n^{(k)}$ by :

$$\tilde{\rho}_n^{(k)}(t_j, S_{t_j}) = \frac{\frac{1}{N}\sum_{i=1}^N v^{(k)}(t_{j+1}, S_{t_{j+1}}^{(i)})\mathbf{1}_{S_{t_j}^{(i)} < S_{t_j}} A_{\frac{1}{n}}^{(i)}}{\frac{1}{N}\sum_{i=1}^N \mathbf{1}_{S_{t_j}^{(i)} < S_{t_j}} A_{\frac{1}{n}}^{(i)}}$$

and

$$\tilde{\phi}_n^{(k)}(t_j, S_{t_j}) = \phi(S_{t_j}) + e^{-r\delta}\frac{\frac{1}{N}\sum_{i=1}^N v^{(k-1)}(t_{j+\delta}, S_{t_{j+\delta}}^{(i)})\mathbf{1}_{S_{t_j}^{(i)} < S_{t_j}} A_\delta^{(i)}}{\frac{1}{N}\sum_{i=1}^N \mathbf{1}_{S_{t_j}^{(i)} < S_{t_j}} A_\delta^{(i)}}\mathbf{1}_{t_j < T-\delta}.$$

where $A_h^{(i)} := e^{\nu_h(W_t^{(i)}-\hat{x}_t^{(i)})}(A_h - \nu_h)$ and $\hat{x}_t^{(i)} = \frac{1}{\sigma}(ln\frac{s}{x} - at - \alpha N_t^{(i)})$ with $W_t^{(i)}$, $N_t^{(i)}$ i-th simulations of W_t and N_t.

Notice that $\rho_n^{(k)} \leq kK$ and $\phi_n^{(k)} \leq kK$ which allows us to truncate the estimators :

$$\hat{\rho}_n^{(k)}(t_j, S_{t_j}) = \tilde{\rho}_n^{(k)}(t_j, S_{t_j})^+ \wedge kK$$
$$\hat{\phi}_n^{(k)}(t_j, S_{t_j}) = \tilde{\phi}_n^{(k)}(t_j, S_{t_j})^+ \wedge kK$$

and, hence to improve the algorithm.

3.5 Numerical Results for Swing Value Functions and Exercise Regions

The computation reported in this section uses strike price $K = 100$, initial stock price $x = 100$, maturity $T = 1$ year, refraction period $\delta = 0.1$, an interest rate $r = 0.05$, a volatility $\sigma = 0.3$, an intensity $\lambda = 1$ and jump coefficient $\phi = 0.1$. The maximal number of exercise rights is $l = 3$. We choose $n = 50$ and $N = 8000$ as in Carmona and Touzi [5]. In table 1, we present some numerical results for the put Swing price with and without jumps. If $\phi = 0$, we find similar results as in the Black and Scholes model of Carmona and Touzi [5]. When $\phi = 0.1$, we also obtain high precision for the Put Swing value.

	$\phi = 0$	$\phi = 0.1$
$v^{(1)}$ **[stand dev.]**	9.60 [.28%]	10.40 [.25%]
$v^{(2)}$ **[stand dev.]**	19.10 [.27%]	20.76 [.44%]
$v^{(3)}$ **[stand dev.]**	28.50 [.26%]	30.04 [.24%]

Table 1: Swing option values with and without jumps

In figure 1, we plot the value functions $v^{(1)}$, $v^{(2)}$ and $v^{(3)}$. We observe that these functions are convex and non-increasing. These results are consistent with our expectation. We proved them in the perpetual case.
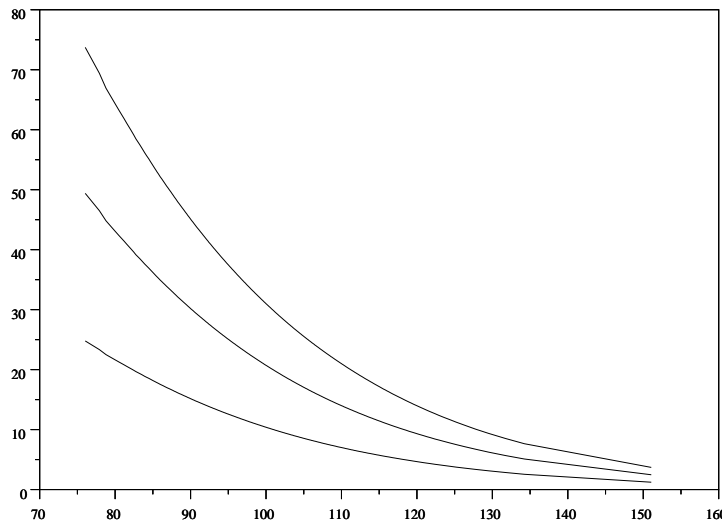

Figure 1: Plot of the value functions $v^{(k)}$, for $k = 1, 2, 3$.

As for the infinite maturity problem, figure 2 shows that Put Swing prices are higher when positive jumps occur. Even if this result is in contrast with the intuition, we explain

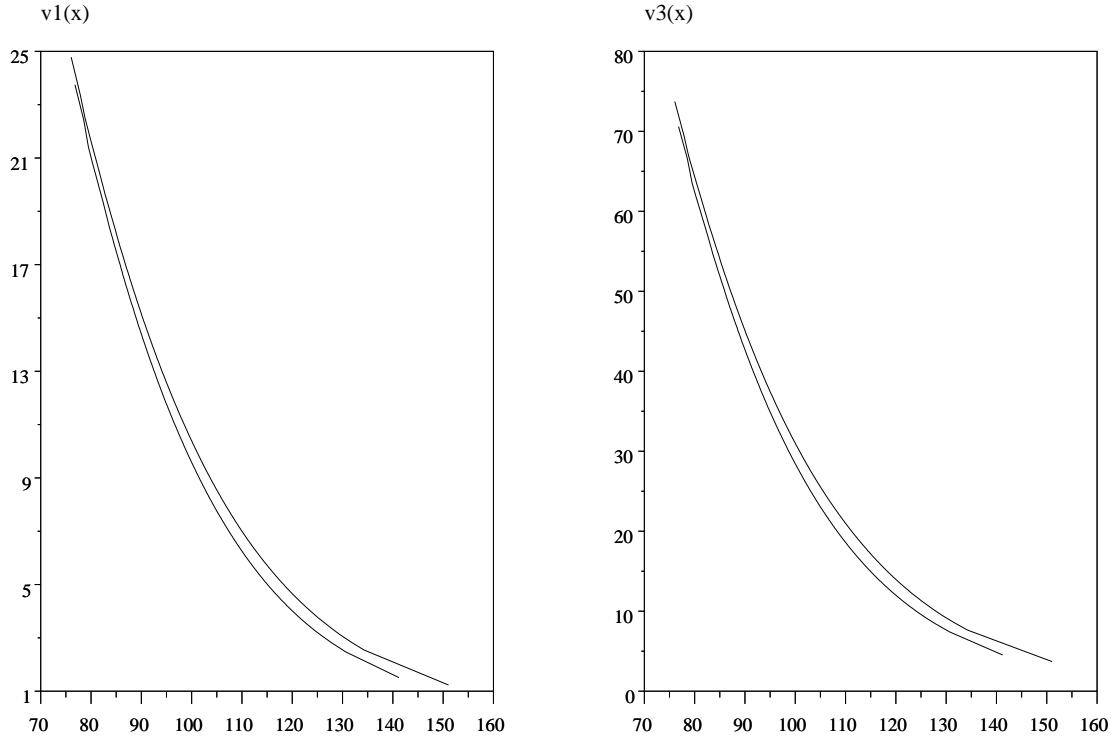that by the positive effect of positive jumps on the risk adjusted drift.

Figure 2: Comparison between Swing value functions with and without jumps

In figure 3, the exercise boundaries are time functions. They are increasing in the following sense $\hat{x}_k^*(t) \geq \hat{x}_{k-1}^*(t)$. This fact is not proved in the finite maturity case, but, it is consistent with the intuition.

4   Acknowledgments

References

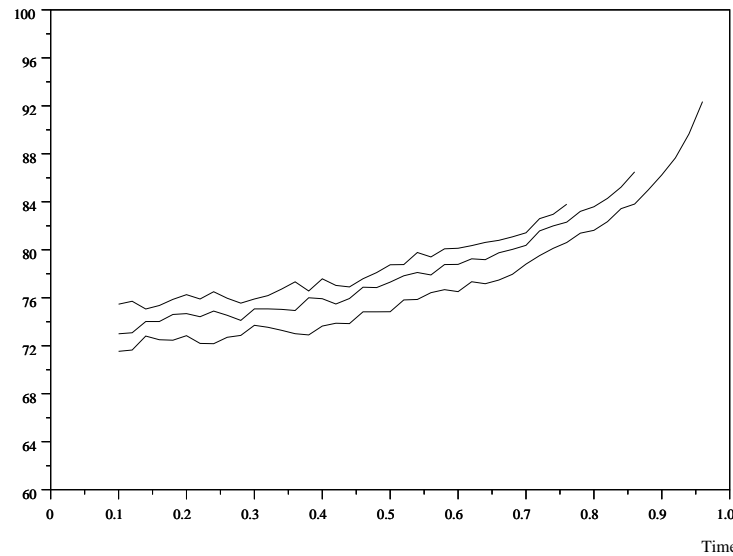[1] A. Barbieri and M.B. Garman (2002). Understanding the Valuation of Swing Contracts. *FEA Tech. Rep* 1

Figure 3: Boundaries of the exercise regions for Swing option with three exercise rights

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