mc_asiankusouka_hes

Input parameters:

- Number of iterations N
- Time Step Number M
- Generator_Type
- Increment inc

Output parameters:

- Price P
- Error Price σ_P
- Error Delta σ_D

Description:

1 Ninomiya-Victoir Scheme

See there We consider a stochastic differential equation written in the Stratonovich form

$$Y_{t,x} = x + \int_0^t V_0(Y_{s,x})ds + \sum_{i=1}^{d} \int_0^t V_i(Y_{s,x}) \circ dW_s^i \qquad Y_{0,x} = x \tag{1}$$

$$dY_{t,x} = \sum_{i=0}^{d} V_i(Y_{t,x}) \circ dW_t^i \qquad Y_{0,x} = x \tag{2}$$

where $W_t^0 = t$.

Now, given a function f with some regularity, how can one approximate efficiently $E[f(Y_{1,x})]$? It is equivalent to the following deterministic problem:

if L is the differential operator $L = V_0 + \frac{1}{2}(V_1^2 + \cdots + V_d^2)$ and u is the solution of the heat equation

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) & = -Lu(t,x) \\ u(T,x) & = f(x) \end{cases} \tag{3}$$

how does one approximate $u(1,x)$ (which is equal to $E[f(Y_{1,x})]$ by Feynman-Kac theorem).

Notation If V is a smooth vector field, i.e. an element of $C_b^\infty(R^N, R^N)$, $\exp(V)(x)$ denotes the solution at time 1 of the ordinary differential equation

$$\frac{dz_t}{dt} = V(z_t), \quad z_0 = x \tag{4}$$

for $x \in R$

Let $(\Lambda_i, Z_i)_{i \in \{1,\dots,n\}}$ be n independent random variable, where each Λ_i is a Bernoulli random variable independent of Z_i, which is a standard d-dimensional normal random variable. Define $\{\bar{X}_k^h\}_{k=0,\dots,n}$ to be a family of random variables as follows:
$\bar{X}_0^h = x$,
$\bar{X}_{k+1}^h =$

$$\begin{cases} \exp\left(\frac{V_0}{2n}\right)\exp\left(\frac{Z_k^1 V_1}{\sqrt{n}}\right)\dots\exp\left(\frac{Z_k^d V_d}{\sqrt{n}}\right)\exp\left(\frac{V_0}{2n}\right)\left(\bar{X}_k^h\right) \text{ si } \Lambda_k = +1 \\ \exp\left(\frac{V_0}{2n}\right)\exp\left(\frac{Z_k^d V_d}{\sqrt{n}}\right)\dots\exp\left(\frac{Z_k^1 V_1}{\sqrt{n}}\right)\exp\left(\frac{V_0}{2n}\right)\left(\bar{X}_k^h\right) \text{ si } \Lambda_k = -1 \end{cases} \tag{5}$$

Then, for all $\forall f \in C_b^\infty(R^N)$,

$$|Ef(X_1) - Ef(Y(1,x))| \leq \frac{C_f}{n^2} \tag{6}$$

that is, the algorithm is of order 2.

Proof

First observe that

$$Err(T,h) = Ef(X_T) - Ef(\bar{X}_T^h) = Eu(0,x) - Eu(T,\bar{X}_T^h), \quad X_0 = x \tag{7}$$

Using Taylor approximation of $Eu(0,x) - Eu(T,\bar{X}_T^h)$ we see that $Eu(0,x) - Eu(T,\bar{X}_T^h) = E\sum_{i=1}^{n}(u(ih,\bar{X}_i^h) - u((i+1)h,\bar{X}_{i+1}^h))$ where $h = \frac{1}{n}$. We add and take $u((i+1)h,\bar{X}_i^h)$ from this expression. The sum becomes
$\sum_{i=1}^{n} E\left(u(ih,\bar{X}_i^h) - u((i+1)h,\bar{X}_i^h)\right) - E\left(u((i+1)h,\bar{X}_{i+1}^h) - u((i+1)h,\bar{X}_i^h)\right) =$
$= \sum_{i=1}^{n} E(u(ih,\bar{X}_i^h) - u((i+1)h,\bar{X}_i^h))$

$$-\tfrac{1}{2}E\left[u\left((i+1)h, \exp\left(\tfrac{V_0}{2n}\right)\exp\left(\tfrac{Z_k^1V_1}{\sqrt{n}}\right)\dots\exp\left(\tfrac{Z_k^dV_d}{\sqrt{n}}\right)\exp\left(\tfrac{V_0}{2n}\right)\left(\bar{X}_i^h\right)\right)-u((i+1)h,\bar{X}_i^h)\right]$$
$$-\tfrac{1}{2}E\left[u\left((i+1)h, \exp\left(\tfrac{V_0}{2n}\right)\exp\left(\tfrac{Z_k^dV_d}{\sqrt{n}}\right)\dots\exp\left(\tfrac{Z_k^1V_1}{\sqrt{n}}\right)\exp\left(\tfrac{V_0}{2n}\right)\left(\bar{X}_i^h\right)\right)-u((i+1)h,\bar{X}_i^h)\right]$$

We consider one term of this sum. We know that

$$u(t,x) = Ef(X_T^{t,x})$$

$$u((i+1)h,x) = Ef(X_T^{(i+1)h,x})$$
$$u(ih,x) = Eu((i+1)h, X_{(i+1)h}^{ih,x})$$

By the Ito formula

$$X_{(i+1)h}^{ih,\bar{X}_i^h} = \bar{X}_i^h + \int_{ih}^{(i+1)h} b(t, X_t^{ih,\bar{X}_i^h})dt + \int_{ih}^{(i+1)h} \sigma(t, X_t^{ih,\bar{X}_i^h})dW_t$$

Then

$$\begin{aligned} u(ih,\bar{X}_i^h) = {} & Eu((i+1)h, X_{(i+1)h}^{ih,\bar{X}_i^h}) \\ = {} & E\left[u(ih,\bar{X}_i^h) + \int_{ih}^{(i+1)h} \tfrac{\partial u}{\partial x}\sigma(t, X_t^{ih,\bar{X}_i^h})dW_t + \int_{ih}^{(i+1)h} Lu(t, X_t^{ih,\bar{X}_i^h})dt\right] \end{aligned}$$

We calculer the mean of $u(ih,\bar{X}_i^h)$. The integral stochastic equals to 0. It remains to estimate

$$E\left[\int_{ih}^{(i+1)h} Lu(t, X_t^{ih,\bar{X}_i^h})dt \,\Big|\, \bar{X}_i^h\right] = \frac{1}{n}Lu(ih,\bar{X}_i^h) + \int_{ih}^{(i+1)h}\int_0^t L^2u(s, X_s^{ih,\bar{X}_i^h})ds\ dt$$

$E(u(ih,\bar{X}_i^h) - E(u((i+1)h,\bar{X}_i^h)) = \frac{Lu(ih,\bar{X}_i^h)}{n} + \frac{L^2u(ih,\bar{X}_i^h)}{2n^2} + const\ n^{-3}$ where

$$\begin{aligned} x + \tfrac{1}{n}Lf(x) + {} & \tfrac{1}{2n^2}L^2f(x) = x + \tfrac{1}{n}\left(V_0 + \tfrac{1}{2}\sum_{i=1}^d V_i^2\right)f(x) \\ & + \tfrac{1}{2n^2}\left(V_0^2 + \tfrac{1}{2}V_0\sum_{i=1}^d V_i^2 + \tfrac{1}{2}\sum_{i=1}^d V_i^2V_0 + \tfrac{1}{4}\sum_{i,j=1}^d V_i^2V_j^2\right)f(x) \end{aligned}$$

Further we apply the Taylor approximation of the ordinary differential equations

$$E\left[f\left(\exp\left(\tfrac{V_0}{2n}\right)\exp\left(\tfrac{Z_k^1V_1}{\sqrt{n}}\right)\dots\exp\left(\tfrac{Z_k^dV_d}{\sqrt{n}}\right)\exp\left(\tfrac{V_0}{2n}\right)(x)\right)\right]$$
$$-\left[x + \tfrac{1}{n}\left(V_0 + \tfrac{1}{2}\sum_{i=1}^d V_i^2\right)f(x) + \tfrac{1}{2n^2}\left(V_0^2 + \tfrac{1}{2}V_0\sum_{i=1}^d V_i^2 + \tfrac{1}{2}\sum_{i=1}^d V_i^2V_0 + \right.\right.$$

$$+\frac{1}{4}\sum_{i=1}^{d} V_i^4 + \frac{1}{2}\sum_{i<j}^{d} V_i^2 V_j^2\Bigg) f(x)\Bigg] = const\ n^{-3}$$

$$E\left[f\left(\exp\left(\frac{V_0}{2n}\right)\exp\left(\frac{Z_k^d V_d}{\sqrt{n}}\right)\ldots\exp\left(\frac{Z_k^1 V_1}{\sqrt{n}}\right)\exp\left(\frac{V_0}{2n}\right)(x)\right)\right]$$
$$-\Bigg[x + \frac{1}{n}\left(V_0 + \frac{1}{2}\sum_{i=1}^{d} V_i^2\right) f(x) + \frac{1}{2n^2}\Bigg(V_0^2 + \frac{1}{2}V_0\sum_{i=1}^{d} V_i^2 + \frac{1}{2}\sum_{i=1}^{d} V_i^2 V_0 +$$
$$+\frac{1}{4}\sum_{i=1}^{d} V_i^4 + \frac{1}{2}\sum_{i>j}^{d} V_i^2 V_j^2\Bigg) f(x)\Bigg] = const\ n^{-3}$$ Then this term is bounded by $const\ n^{-3}$, and we conclude that the Ninomiya-Victoir scheme has an order 2.

Remarks

1. Ninomiya-Victoir scheme has the same order that the Milshtein scheme. But here we haven't to calculate an integral mixed. Then this scheme is more commode in practice that Milschtein one.

2. In general, it is not always possible to obtain the closed form solution to $\exp(sV_i)$. Even in such cases, it is not difficult to implement this algorithm. All we have to do is to find an approximation of $\exp(sV_i)$. This can be achieved by Runge-Kutta method.

3. This scheme is applied for a model with Brownian motions independents.

2 Heston Model and Asian Call

The asset price Y_1 satisfies the following two factor stochastic volatility model

$$dY_1 = \mu Y_1 dt + Y_1\sqrt{Y_2}dW_t^1 \quad Y_1(0) = x_0 \tag{8}$$
$$dY_2 = \alpha(\theta - Y_2)dt + \beta\sqrt{Y_2}dW_t^2 \quad Y_2(0) = y_0 \tag{9}$$

where (W_t^1, W_t^2) is a 2-dimensional standard brownian motion with a correlation coefficient $\rho:\ dW_1\ dW_2 = \rho$
α, θ, μ are some positives constantes such that $2\alpha\theta - \beta^2 > 0$ to ensure the existence and uniqueness of a solution to stochastic differential equation. Also α is named *mean reversion*, β - *volatility of volatility*, μ- *annual interest rate*, et θ *log-run variance*.

The payoff of option is $(Y_3(T)/T - K)^+$ where

$$dY_3 = Y_1 dt, \quad Y_3(0) = 0 \tag{10}$$

The price of this option becomes $e^{-rT}(Y_3(T)/T - K)^+$.

We add two equations for reduction variance technique [1] The control variable is

$$(e^{\frac{1}{T}\int_0^T \ln(Y_4)dt} - K)^+ \tag{11}$$

where

$$\begin{aligned} dY_4 &= \mu Y_4 dt + Y_4\sqrt{e^{-\alpha t}(y_0-\theta)+\theta}dW_t^1 \quad Y_4(0)=x_0 \\ dY_5 &= \ln Y_4 dt \quad Y_5(0)=0 \end{aligned}$$

Here for Y_4 we use the same brownian motion W_t that for Y_1. In other words $Y_4 = Y_1$ where $\beta = 0$.

2.1 The mean of the control variable

$$\begin{aligned} Y_4 &= x_0 e^{\int_0^t \mu ds + \int_0^t \sqrt{e^{-\alpha s}(y_0-\theta)+\theta}dW_s - \frac{1}{2}\int_0^t (e^{-\alpha s}(y_0-\theta)+\theta)ds} \\ &= x_0 e^{\mu t - \frac{1}{2}\theta t + \frac{y_0-\theta}{2\alpha}(e^{-\alpha t}-1) + \int_0^t \sqrt{e^{-\alpha s}(y_0-\theta)+\theta}dW_s} \\ \ln Y_4 &= \ln x_0 + t(\mu - \tfrac{1}{2}\theta) + \tfrac{y_0-\theta}{2\alpha}(e^{-\alpha t}-1) + \int_0^t \sqrt{e^{-\alpha s}(y_0-\theta)+\theta}dW_s \\ \int_0^T \ln Y_4 dt &= T\ln x_0 + \tfrac{T^2}{2}(\mu - \tfrac{1}{2}\theta) - \tfrac{y_0-\theta}{2\alpha}T - \tfrac{y_0-\theta}{2\alpha^2}(e^{-\alpha T}-1) + \\ &+ \int_0^T\int_0^t \sqrt{e^{-\alpha s}(y_0-\theta)+\theta}dW_s dt \end{aligned}$$

Let $f(s) = \sqrt{e^{-\alpha s}(y_0-\theta)+\theta}$. We will calcule the integral multiple $I = \int_0^T\int_0^t f(s)dW_s dt$. We apply two times a formula of integration by parties.

$$\begin{aligned} I &= \int_0^T W_t f(t)dt - \int_0^T\int_0^t f'(s)W_s ds\ dt = \int_0^T W_t f(t)dt - \int_0^T (T-t)f'(t)W_t dt \\ &= \int_0^T W_t(f(t) - (T-t)f'(t))dt = \int_0^T (T-t)f(t)dW_t \end{aligned}$$

So I is normal random variable $\xi \sim N(0,\sigma^2)$.

[1] We would like to construct the control variable for asian call. We replace the stochastic volatility in the equation (8) by a volatility determinate, solution of the equation (9) which is $\tilde{Y}_2(t) = e^{-\alpha t}(y_0-\theta)+\theta$.

$$\begin{aligned}
\sigma^2 &= \int_0^T (T-t)^2 f^2(t)dt = \int_0^T (T-t)^2 (e^{-\alpha t}(y_0-\theta)+\theta)dt \\
&= -\theta \left.\frac{(T-t)^3}{3}\right|_0^T - \frac{(y_0-\theta)}{\alpha} e^{-\alpha t}(T-t)^2\Big|_0^T - \frac{2(y_0-\theta)}{\alpha}\int_0^T e^{-\alpha t}(T-t)dt \\
&= \frac{\theta T^3}{3} + \frac{(y_0-\theta)}{\alpha}T^2 - \frac{2(y_0-\theta)}{\alpha}\left[\frac{e^{-\alpha t}}{-\alpha}(T-t)\Big|_0^T - \int_0^T \frac{e^{-\alpha t}}{\alpha}dt\right] \\
&= \frac{\theta T^3}{3} + \frac{2(y_0-\theta)}{\alpha}\left[\frac{T^2}{2} - \frac{T}{\alpha} - \frac{1}{\alpha^2}(e^{-\alpha T}-1)\right]
\end{aligned}$$

And $Z = e^{\frac{1}{T}\int_0^T \ln X_t dt} = x_0 e^{a+\frac{1}{T}\xi}$ where $a = \frac{T}{2}(\mu - \frac{1}{2}\theta) - \frac{y_0-\theta}{2\alpha} - \frac{y_0-\theta}{2\alpha^2 T}(e^{-\alpha T}-1)$.
It remains to calcule the mean of $(Z-K)^+$.

$$\begin{aligned}
E(x_0 e^{a+\frac{1}{T}\xi} - K)^+ &= x_0 e^a E(e^{\frac{1}{T}\xi} - \frac{K}{x_0}e^{-a}) 1_{\{e^{\frac{1}{T}\xi} > \frac{K}{x_0}e^{-a}\}} \\
&= x_0 e^a \int_{T(\ln\frac{K}{x_0}-a)}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}}(e^{\frac{x}{T}} - \frac{K}{x_0}e^{-a})e^{-\frac{x^2}{2\sigma^2}}dx \\
&= x_0 e^a \int_{\frac{T}{\sigma}(\ln\frac{K}{x_0}-a)}^{+\infty} \frac{1}{\sqrt{2\pi}}(e^{\frac{\sigma y}{T}} - \frac{K}{x_0}e^{-a})e^{-\frac{y^2}{2}}dy \\
&= x_0 e^a \int_{\frac{T}{\sigma}(\ln\frac{K}{x_0}-a)}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{\sigma^2}{2T^2}} e^{-\frac{y^2 - 2\frac{\sigma}{T}y + \frac{\sigma^2}{T^2}}{2}}dy - KN(\frac{T}{\sigma}(\ln\frac{K}{x_0}-a)) \\
&= x_0 e^{a+\frac{\sigma^2}{2T^2}} N(\frac{T}{\sigma}(\ln\frac{K}{x_0}-a) - \frac{\sigma}{T}) - KN(\frac{T}{\sigma}(\ln\frac{K}{x_0}-a))
\end{aligned}$$

2.2 The functions used in Euler Scheme and in Ninomiya-Victoir Scheme.

For Euler Scheme we describe the functions $b(Y,t)$, $\sigma(Y,t)$ like

$$\begin{aligned}
Y &= (Y_1, \quad Y_2, \quad Y_3, \quad Y_4, \quad Y_5)^t \\
b(Y,t) &= (\mu Y_1, \quad \alpha(\theta - Y_2), \quad Y_1, \quad \mu Y_4, \quad \ln Y_4)^t, \\
\sigma(Y,t) &= \begin{pmatrix} Y_1\sqrt{Y_2} & 0 \\ 0 & \beta\sqrt{Y_2} \\ 0 & 0 \\ Y_4\sqrt{e^{-\alpha t}(y_0-\theta)+\theta} & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

For Ninomiya-Victoir Scheme (5) we have the functions:

$$\begin{aligned}
V_1(Y) &= (Y_1\sqrt{Y_2}, \quad 0, \quad 0, \quad Y_4\sqrt{e^{-\alpha t}(y_0-\theta)+\theta}, \quad 0)^t \\
\exp(sV_1)(Y) &= (X_1, \quad Y_2, \quad Y_3, \quad X_4, \quad Y_5)^t
\end{aligned}$$

where X_1 and X_4 arise from the equations

$$\frac{dX_1}{dt} = sX_1\sqrt{Y_2}\ ; \qquad \frac{dX_1}{X_1} = s\sqrt{Y_2}dt\ ; \qquad X_1 = X_1(0)e^{s\sqrt{Y_2}t}|_{t=1,\ X_1(0)=Y_1}$$

$$X_1 = Y_1 e^{s\sqrt{Y_2}}$$

$$\frac{dX_4}{dt} = sX_4\sqrt{e^{-\alpha t}(y_0-\theta)+\theta}\ ; \qquad \frac{dX_4}{X_4} = s\sqrt{e^{-\alpha t}(y_0-\theta)+\theta}dt$$

$$X_4 = \begin{cases} Y_4 e^{\frac{s}{\alpha}\left(2(b-a)+\sqrt{\theta}\ln\frac{(b-\sqrt{\theta})(a+\sqrt{\theta})}{(b+\sqrt{\theta})(a-\sqrt{\theta})}\right)} & \text{si } y_0 \neq \theta \\ Y_4 e^{s\sqrt{\theta}} & \text{si } y_0 = \theta \end{cases}$$

$a = \sqrt{e^{-\alpha}(y_0-\theta)+\theta}, \quad b = \sqrt{y_0}$

$$\begin{aligned} V_2(Y) &= (0, \quad \beta\sqrt{Y_2}, \quad 0, \quad 0, \quad 0)^t \\ \exp(sV_2)(Y) &= (Y_1, \quad \left(\tfrac{\beta s}{2}+\sqrt{Y_2}\right)^2, \quad Y_2, \quad Y_3, \quad Y_4, \quad Y_5)^t \end{aligned}$$

$$\begin{aligned} V_0(Y) &= (Y_1(\mu - \tfrac{1}{2}Y_2), \quad \alpha(\theta - Y_2) - \tfrac{\beta^2}{4}, \quad Y_1, \quad Y_4(\mu - \tfrac{e^{-\alpha t}(y_0-\theta)+\theta}{2}), \ln Y_4)^t \\ \exp(sV_0)(Y) &= (X_1, \quad X_2, \quad X_3, \quad X_4, \quad X_5)^t \end{aligned}$$

$$\begin{aligned} X_1 &= Y_1 \exp\left((\mu - \tfrac{J}{2})s + \tfrac{Y_2-J}{2\alpha}(e^{-\alpha s}-1)\right) \\ X_2 &= J + (Y_2 - J)e^{-\alpha s} \\ X_3 &= \begin{matrix} Y_3 + Y_1\frac{e^{As}-1}{A} + O(s^3) & \text{if } Y_2 \neq 2\mu \\ Y_3 + Y_1 s & \text{si } Y_2 = 2\mu \end{matrix} \\ X_4 &= Y_4 \exp\left(s(\mu - \tfrac{\theta}{2}) + \tfrac{s(y_0-\theta)}{2\alpha}(e^{-\alpha}-1)\right) \\ X_5 &= \begin{matrix} Y_5 & \text{if } Y_4 = 0 \\ Y_5 + s\ln X_4 - \frac{s^2}{2}(\frac{y_0-\theta}{\alpha}(1-\frac{1}{\alpha}+\frac{e^{-\alpha}}{\alpha}) - \mu + \frac{1}{2}\theta) & \text{si } Y_4 \neq 0 \end{matrix} \end{aligned}$$

$J = \theta - \frac{\beta^2}{4\alpha}, \quad A = \mu - \frac{Y_2}{2}$

We approximate

$$X_1(t) = Y_1 e^{(\mu-J/2)st + \frac{Y_2-J}{2\alpha}(e^{-s\alpha t}-1)} \approx Y_1 e^{(\mu-J/2)st + \frac{Y_2-J}{2\alpha}(-s\alpha t)} = Y_1 e^{st(\mu - Y_2/2)}$$

and then calcule X_3

References