Cap and Swaption Approximations in LIBOR Market Models with Jumps [Glasserman/Merener]

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Premia 8

1 Preliminaries and notation

The following is based on [GM]. We fix a discrete tenor structure $0 = T_0 < T_1 < \ldots < T_{M+1}$ with $T_{i+1} - T_i \equiv \delta$ and set

$$\eta(t) \ := \ \min\{i \geq 0; T_i \geq t\} \ = \ \lceil \tfrac{t}{\delta} \rceil \quad , \quad \text{hence} \quad t \in (T_{\eta(t)-1}, T_{\eta(t)}] \ , \ t > 0 \, .$$

Whenever we talk about piecewise constant functions we mean that they are constant on each interval $(T_i, T_{i+1}]$.
For $i = 0, \ldots, M$, the forward LIBOR rates for the period $[T_i, T_{i+1}]$ are denoted by $(L_t^i)_{t \in [0,T_i]}$ and

$$L_t \ = \ (L_t^{\eta(t)}, \ldots, L_t^M) \, .$$

By $P(t,T)$ we denote the price at time t of the zero-coupon bond with maturity T. Observe that we have

$$P(0, T_{M+1}) \quad = \quad \prod_{j=0}^{M} (1 \, + \, \delta \, L_0^j)^{-1} \, .$$

We will consider LIBOR market models with jumps. The source of the jumps will be compound Poisson processes; all notations concerning these compound Poisson processes as well as the used relations to Poisson measures are explained in Section 4.

2 The approximation method

Suppose that, under the spot measure, the LIBOR rates (L_t^i) are given by dynamics of the following type:

(1) $$\frac{dL_t^i}{L_{t-}^i} \quad = \quad \gamma_t^i dW_t^p \, + \, dJ_t^i \, + \, b_t^i \, dt \, ,$$

where (γ_t^i) is deterministic and (J_t^i) is a compound Poisson process

$$J_t^i \;=\; \sum_{j=1}^{N_t} H_i(X_j,\tau_j)\;.$$

We assume (X_j,τ_j) to have a deterministic Poisson measure ν^p of the type

$$\nu^p(x,t)dxdt \quad = \quad \lambda_t^p f^p(x,t)dxdt\;.$$

Furthermore, we suppose all coefficients $\gamma^i, H_i(x,\cdot), \lambda^p, f^p(x,\cdot)$ to be piecewise constant. This implies the following dynamics under the T_{i+1}-forward measure P^{i+1}:

$$\text{(2)}\qquad \tfrac{dL_t^i}{L_{t-}^i} \quad = \quad \gamma_t^i dW_t^{i+1} \;+\; dJ_t^i \;+\; a_t^i dt\;.$$

By standard no-arbitrage arguments, one can identify the drift a_t^i to be

$$\text{(3)}\qquad a_t^i \quad = \quad -\int_{\mathbb{R}_+} H_i(x,t)\,\nu^i(x,t)\,dx\,,$$

where ν^i denotes the Poisson measure of (X_j,τ_j) under P^{i+1}. By standard change-of-numeraire techniques, ν^i can be seen to be

$$\nu^i(x,t) \quad = \quad \phi_i(t,H(x,t^-),L_{t-})\,\nu^p(x,t)\;;$$

here we denote

$$\phi_i(t,H,L) \;:=\; \prod_{j=\eta(t)}^{i} \tfrac{1+\delta L^j}{1+\delta L^j(1+H_j)}\;.$$

In particular, the corresponding (non-deterministic) jump intensities λ_t^i and jump densities $f^i(x,t)$ are given by

$$\text{(4)}\qquad \lambda_t^i \quad = \quad \int_{\mathbb{R}_+} \phi_i(t,H(x,t^-),L_{t-})\,\nu^p(x,t)\,dx\;,$$

$$\text{(5)}\qquad f^i(x,t) \quad = \quad (\lambda_{t^-}^i)^{-1}\,\phi_i(t^-,H(x,t^-),L_{t-})\,\nu^p(x,t^-)\;.$$

Observe that, concerning caplet pricing, only for the caplet with maturity T_M it is reasonable to model (as we do) all the LIBOR rates $L^0,\ldots,L^M$; so we consider only this caplet. In order to compute its current price C_0^M given by

$$\text{(6)}\quad C_0^M \;=\; \delta\,P(0,T_{M+1})\,E^{M+1}((L_{T_M}^M\!-\!K)_+) \;=\; \delta\,E^p(\prod_{j=0}^{M}(1\!+\!\delta L_{T_j}^j)^{-1}(L_{T_M}^M\!-\!K)_+)\;,$$

the particular case $H_M(x,t) = x-1$ is rather pleasant since it admits an (essentially) closed formula. This can be seen from the following result which is [GM, Proposition 3.1].

Proposition 2.1. *Let the process (G_t) be given by*

$$\tfrac{dG_t}{G_{t-}} \quad = \quad \gamma_t\, dW_t \;+\; dJ_t \;+\; a_t\, dt \;,$$

where (γ_t), (a_t) are deterministic and (J_t) is a compound Poisson process

$$J_t \;=\; \sum_{j=1}^{N_t} (X_j - 1) \;.$$

We assume (X_j, τ_j) to have a deterministic Poisson measure ν of the type $\lambda_t\, f(x,t)dxdt$, where the jump densities $f(x,t)$ are supposed to be lognormal with parameters μ_t and σ_t. Suppose that the coefficients γ, a, λ, μ, σ are piecewise constant: $\gamma_t = \gamma_k$ for all $k = 0, \ldots, M-1$ and $t \in (T_k, T_{k+1}]$, etc. Then

$$E(\,(G_{T_M} - K)_+\,) \;=\; G_0\,\Pi_1 \;-\; K\,\Pi_2 \;,$$

where

$$\Pi_1 \quad = \quad \tfrac{1}{2} \;+\; \tfrac{1}{\pi} \int_0^\infty \; e^{\,B_1(u)} \sin\Big(B_2(u) \;-\; u\log(\tfrac{K}{G_0})\Big)\; \tfrac{du}{u} \;,$$

$$\Pi_2 \quad = \quad \tfrac{1}{2} \;+\; \tfrac{1}{\pi} \int_0^\infty \; e^{\,B_3(u)} \sin\Big(B_4(u) \;-\; u\log(\tfrac{K}{G_0})\Big)\; \tfrac{du}{u} \;.$$

Here we denote $m_k = e^{\mu_k + \sigma_k^2/2} - 1$, $\omega_k = \mu_k + \sigma_k^2$, $\alpha_k = a_k - \gamma_k^2/2$ and

$$B_1(u) \quad = \quad \delta \sum_{k=0}^{M-1} \lambda_k \Big(e^{\,\mu_k + \sigma_k^2(1-u^2)/2} \cos(\omega_k u) - 1 - m_k\Big) - \gamma_k^2 u^2/2 \;,$$

$$B_2(u) \quad = \quad \delta \sum_{k=0}^{M-1} \lambda_k e^{\,\mu_k + \sigma_k^2(1-u^2)/2} \sin(\omega_k u) + (\alpha_k + \gamma_k^2) u \;,$$

$$B_3(u) \quad = \quad \delta \sum_{k=0}^{M-1} \lambda_k \Big(e^{-\sigma_k^2 u^2/2} \cos(\mu_k u) - 1\Big) - \gamma_k^2 u^2/2 \;,$$

$$B_4(u) \quad = \quad \delta \sum_{k=0}^{M-1} \lambda_k e^{-\sigma_k^2 u^2/2} \sin(\mu_k u) + \alpha_k u \;.$$

Therefore, in order to approximate via Proposition 2.1 the caplet price in (6) under the dynamics (2) for L^M (which are *not* of the type considered in Proposition 2.1 !), we proceed as follows: We approximate L^M by a process $\hat{L}$ with dynamics of the type considered in Proposition 2.1, compute the $\hat{L}_{T_M}$-caplet price via Proposition 2.1 and take it as an approximation of the desired $L^M_{T_M}$-caplet price.
For this purpose, we have to approximate the non-deterministic Poisson measure $\nu^M(x,t) = \lambda_t^M f^M(x,t)$ by a deterministic (and piecewise constant) Poisson measure

$\hat{\nu}(x,t) = \hat{\lambda}_t \hat{f}(x,t)$ with lognormal jump densities $\hat{f}(x,t)$. Then we will apply Proposition 2.1 for $(\gamma_t, a_t, \nu, G_0) = (\gamma_t^M, \hat{a}_t, \hat{\nu}, L_0^M)$, where we make the following choice for the (piecewise constant) drift $\hat{a}_t$ which is obviously motivated by (3) for $i = M$:

$$\hat{a}_t \quad := \quad -\int_{\mathbb{R}_+} (x-1)\, \hat{\nu}(x,t)\, dx \ .$$

So our approximating process $\hat{L}$ will be given by the P^{M+1}-dynamics

$$\frac{d\hat{L}_t}{\hat{L}_{t-}} \quad = \quad \gamma_t^M dW_t^{M+1} \,+\, d\hat{J}_t \,+\, \hat{a}_t\, dt \quad , \quad \hat{L}_0 = L_0^M \ .$$

We have to choose jump intensities $\hat{\lambda}_t$ and lognormal jump densities $\hat{f}(x,t)$, both deterministic and piecewise constant. Firstly we take $\hat{\lambda}_t$ to be the following 'deterministic version' of λ_t^M in (4):

$$\hat{\lambda}_t \quad := \quad \int_{\mathbb{R}_+} \phi_M(t, H(x,t^-), L_0)\, \nu^p(x,t)\, dx \ .$$

Secondly, we choose the jump densities $\hat{f}(x,t)$ (more precisely, their lognormal parameters $\hat{\mu}_t$ and $\hat{\sigma}_t$) by approximately matching its first two moments with those of $f^M(x,t)$:

$$\int_{\mathbb{R}_+} H_M(x,t^-)\, f^M(x,t^-)\, dx \ \overset{!}{\approx} \ \int_{\mathbb{R}_+} (x-1)\, \hat{f}(x,t^-)\, dx \quad = \quad \hat{m}_t \ := \ e^{\hat{\mu}_t + \frac{\hat{\sigma}_t^2}{2}} - 1 \ ,$$

$$\int_{\mathbb{R}_+} H_M(x,t^-)^2\, f^M(x,t^-)\, dx \ \overset{!}{\approx} \ \int_{\mathbb{R}_+} (x-1)^2\, \hat{f}(x,t^-)\, dx \quad = \quad e^{\hat{\sigma}_t^2}(1+\hat{m}_t)^2 - 2\hat{m}_t - 1 \ .$$

Here we use the following fact:

$$\log(X) \ \sim \mathcal{N}(\mu, \sigma^2) \quad \Longrightarrow \quad E(X) \ = \ e^{\mu + \frac{\sigma^2}{2}} \quad \text{and} \quad E(X^2) \ = \ e^{2(\mu+\sigma^2)} \ .$$

Observe that, in both $\approx$-parts, the RHS is deterministic while the LHS is not. So we have to be more precise on what $\approx$ is supposed to mean:

$$\begin{aligned}
&\int_{\mathbb{R}_+} H_M(x,t^-)\, f^M(x,t^-)\, dx \\
&= \ (\lambda_{t^-}^M)^{-1} \int_{\mathbb{R}_+} H_M(x,t^-)\, \phi_M(t^-, H(x,t^-), L_{t^-})\, \nu^p(x,t^-)\, dx \qquad [\text{by } (5)] \\
&\approx \ (\hat{\lambda}_{t^-})^{-1} \int_{\mathbb{R}_+} H_M(x,t^-)\, \phi_M(t^-, H(x,t^-), L_0)\, \nu^p(x,t^-)\, dx \qquad [=: I_t] \\
&\overset{!}{=} \ \int_{\mathbb{R}_+} (x-1)\, \hat{f}(x,t^-)\, dx \quad = \quad \hat{m}_t \ .
\end{aligned}$$

For the second moment, we proceed the same way:

$$\begin{aligned}
&\int_{\mathbb{R}_+} H_M(x,t^-)^2\, f^M(x,t^-)\,dx \\
&= (\lambda^M_{t^-})^{-1} \int_{\mathbb{R}_+} H_M(x,t^-)^2\,\phi_M(t^-,H(x,t^-),L_{t^-})\,\nu^p(x,t^-)\,dx \quad [\text{by } (5)] \\
&\approx (\hat\lambda_{t^-})^{-1} \int_{\mathbb{R}_+} H_M(x,t^-)^2\,\phi_M(t^-,H(x,t^-),L_0)\,\nu^p(x,t^-)\,dx \quad [=: J_t] \\
&\overset{!}{=} \int_{\mathbb{R}_+} (x-1)^2\,\hat f(x,t^-)\,dx \;=\; e^{\hat\sigma_t^2}(1+\hat m_t)^2 - 2\hat m_t - 1\,.
\end{aligned}$$

Hence, $\hat\mu_t$ and $\hat\sigma_t$ are computed from the two identities

$$I_t \;=\; \hat m_t \quad \text{and} \quad J_t \;=\; e^{\hat\sigma_t^2}(1+\hat m_t)^2 - 2\hat m_t - 1\,.$$

This yields the following choice:

$$\hat\sigma_t^2 \;:=\; \log\left(\tfrac{J_t+1+2I_t}{(1+I_t)^2}\right) \quad \text{and} \quad \hat\mu_t \;:=\; \log(1+I_t) - \tfrac{\hat\sigma_t^2}{2}\,.$$

By changing from the spot measure not to the numeraire bond price as above but to the numeraire PVBP, one obtains by straightforward adaptions an approximation method for the prices of (European) swaptions.
The same holds if the initial dynamics are given under the forward measures (and not under the spot measure as above). Notice that, in this case, caplet prices are computed exactly via Proposition 2.1.

3 Numerical results

In all our numerical tests, the jump densities $f^p(t,\cdot)$ are supposed standard lognormal:

$$f^p(x,t) \;=\; \tfrac{1}{\sqrt{2\pi}\,x}\,\exp(-\tfrac{\log(x)^2}{2})\,.$$

According to the choice of $\hat\mu_t$ and $\hat\sigma_t$, we have

$$\hat f(x,t) \;=\; \tfrac{1}{\sqrt{2\pi}\,\hat\sigma_t x}\,\exp(-\tfrac{(\hat\mu_t-\log x)^2}{2\hat\sigma_t^2})\,.$$

3.1 Simulation

We will compute the caplet price not only by our approximation method presented in Section 2 but also by Monte Carlo simulation of the LIBOR dynamics (1) under the spot measure, i.e.

$$\frac{dL^i_t}{L^i_{t^-}} \;=\; \gamma^i_t dW^p_t + dJ^i_t + b^i_t\,dt \quad, \quad \text{where } J^i_t \;=\; \sum_{j=1}^{N_t} H_i(X_j,\tau_j)\,.$$

Analogously to the forward measure case above, one can identify by standard no-arbitrage arguments the drift b_t^i to be

$$b_t^i \;=\; \psi_i(t, L_{t-}) + a_t^i \quad , \quad \text{where } \; \psi_i(t,L) \;:=\; \delta\,\gamma_t^i \sum_{j=\eta(t)}^{i} \tfrac{\gamma_t^j L^j}{1+\delta L^j}$$

and a_t^i is as in (3). Let (ξ_l) be a sequence of independent exponential random variables and set

$$\Xi_n \;:=\; \sum_{l=1}^{n} \xi_l \quad \text{and} \quad \Lambda(t) \;:=\; \int_0^t \lambda_s^p \, ds \; .$$

We use the fact that if $(\widetilde{N}_t)$ is a standard Poisson process (with intensity 1) then we have $N_t \stackrel{\mathcal{L}}{=} \widetilde{N}_{\Lambda(t)}$.
The simulation employs a uniform time grid with step size $h = 0.01$. The following algorithm is used to pass from L_t and Ξ_{N_t} (the latter denoted by Ξ = capital ξ) to L_{t+h} and $\Xi_{N_{t+h}}$. Here we denote by $G(\,)$ and $U(\,)$ independent samples of standard Gaussian and uniform random variables.

$\Delta W = \sqrt{h}\, G(\,)$;
$\Delta J = (0, \ldots, 0)$;
while $\Xi \leq \Lambda(t+h)$:
  $X = e^{G(\,)}$;
  for $i = \eta(t+h), \ldots, M$: $\Delta J^i = \Delta J^i + H_i(X,t)$;
  $\Xi = \Xi - \log(U())$;
for $i = M, \ldots, \eta(t+h)$:
  $a = -\int H_i(x,t)\, \phi_i(t, H(x,t^-), L_t)\, \lambda_t\, f(x,t) dx$;
  $b = \psi_i(t, L_t) \, + \, a$;
  $L_t^i = L_t^i \, + \, L_t^i(\gamma_t^i \Delta W + \Delta J^i + bh)$;

The advantage of counting downwards in the second i-loop is that L_{t+h} can be stored in L_t since the functions ϕ_i and ψ_i do not use the $L^{i+1}, \ldots, L^M$.
In order to reduce the computational cost of the integral in the drift a, we will recompute it not at every time step but only at time steps with jumps and after a certain number of time steps without jumps.

3.2 Parameter values

We consider functions H_i of the following type:

$$H_i(x,t) \;=\; x^{\sigma_i(t)} \, - \, 1 \; .$$

Since we suppose all coefficients to be piecewise constant, we write similarly as in Proposition 2.1:

$$\sigma_i(t) \;=\; \sigma_{i,k}\;,\;\lambda^p_t \;=\; \lambda_k\;,\;\hat{\lambda}_t \;=\; \hat{\lambda}_k\;,\;\hat{a}_t \;=\; \hat{a}_k\;,\;\text{etc.}\quad \text{for all}\quad t\in(T_k,T_{k+1}]\;.$$

In other words, we have for all $t > 0$

$$\sigma_i(t) \;=\; \sigma_{i,\eta(t)-1}\;,\;\lambda^p_t \;=\; \lambda_{\eta(t)-1}\;,\;\hat{\lambda}_t \;=\; \hat{\lambda}_{\eta(t)-1}\;,\;\hat{a}_t \;=\; \hat{a}_{\eta(t)-1}\;,\;\text{etc.}$$

In this notation, we have the following formula for our function Λ:

$$\Lambda(t) \quad := \quad \int_0^t \lambda^p_s\,ds \quad = \quad \delta\sum_{k=0}^{\eta(t)-2}\lambda_k \;+\; (t-T_{\eta(t)-1})\lambda_{\eta(t)-1}\;.$$

As in the parameter set A in [GM, § 5.5], we consider the following values for the parameters determining the dynamics under the spot measure:

$$\delta \;=\; 0.5 \quad,\quad L^i_0 \;=\; 0.06 \quad,\quad \gamma^i_k \;=\; 0.1 \quad,\quad \lambda_k \;=\; 5{*}0.99^k \quad,\quad \sigma_{i,k} \;=\; 0.1{*}1.01^k.$$

Our results for the caplet prices are presented in the following table.

Maturity T_M	Strike K	MC (95% conf. interval)	Approximation
2	ITM 0.05	58.4357 (0.469)	58.4846
2	ATM 0.06	35.4471 (0.393)	35.523
2	OTM 0.07	20.662 (0.316)	20.7832
3	ITM 0.05	61.5082 (0.533)	61.3927
3	ATM 0.06	41.2349 (0.464)	41.1833
3	OTM 0.07	27.3235 (0.278)	27.2172
5	ITM 0.05	63.9986 (0.420)	63.7998
5	ATM 0.06	47.6215 (0.381)	47.467
5	OTM 0.07	35.4599 (0.341)	35.322

4 Some facts on compound Poisson processes and Poisson measures

Let (τ_j) be a sequence of jump times, i.e., a strictly increasing sequence of strictly positive random variables. Let (X_j) is a sequence of iiv random variables. Set

$$N_t \;:=\; \max\{\,j\,;\,\tau_j \le t\,\}\;.$$

Each process (J_t) of the type

$$J_t \;=\; \sum_{j=1}^{N_t} H(X_j, \tau_j) \quad , \text{ where } H : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R} \text{ is a bounded function}$$

is called a <u>compound Poisson process</u>. We always suppose that (X_j, τ_j) admits an intensity process. This is a (in general non-deterministic) finite Poisson measure of the type $\nu(dx,t)dt$ such that the law of X_j conditional on $\tau_j = t$ is $f(dx,t)$, where

$$f(dx,t) \;:=\; \lambda_{t^-}^{-1}\, \nu(dx, t^-) \quad \text{ and } \quad \lambda(t) \;:=\; \int_{\mathbb{R}} \nu(dx,t)\;.$$

In other words:

$$E(\, H(X_{\tau_j}, \tau_j) \,|\, \tau_j = t\,) \quad = \quad \int_{\mathbb{R}} H(x, t^-)\, f(dx, t^-)$$

for all bounded H. The λ_t are called jump intensities.

References

[GM] P. Glasserman, N. Morener; 'Cap and swaption approximations in LIBOR Market Models with jumps', to appear in Journal of Computational Finance. 1, 2, 7