ap_ju_putamer

Output parameters:

- Price
- Delta

This method, described in [1], is based on the early exercise premium formula:

$$\begin{aligned} P_A &= P_E + K(1-e^{-rT}) - S(1-e^{-\delta T}) - K\int_0^T re^{-rt}N(d_2(S,B_t,t))dt \\ &\quad + S\int_0^T \delta e^{-\delta t}N(d_1(S,B_t,t))dt \end{aligned} \tag{1}$$

where

$$\begin{aligned} d_1(x,y,t) &= \frac{log(x/y)+(r-\delta+\sigma^2/2)t}{\sigma\sqrt{t}}, \\ d_2(x,y,t) &= d_1(x,y,t)-\sigma\sqrt{t} \end{aligned}$$

P_E is the Black and Scholes (1973) price of the European put option, and B_t the exercise boundary at t.

As B_t appears only as $\log(S/B_t)$ in the definitions of d_1 and d_2, a possible approximation for the exercise boundary would be one by exponential pieces. For instance, a two-exponential pieces' approximation consists in replacing B_t by $B_{21}e^{b_{21}t}$ for $t \in [T/2;T]$, $B_{22}e^{b_{22}t}$ for $t \in [0;T/2]$.

The advantage of this method is that the integrals $\int_{t_1}^{t_2} re^{-rt}N(d_2(S,Be^{bt},t))dt$ and $\int_{t_1}^{t_2} \delta e^{-\delta t}N(d_1(S,Be^{bt},t))dt$, involved in equation (1), can be evaluated in closed form.

They become respectively $I(t_1, t_2, S, B, b, -1, r)$ and $I(t_1, t_2, S, B, b, 1, \delta)$ where I is defined by :

$$\begin{aligned} I(t_1, t_2, S, B, b, \phi, \nu) = & e^{-\nu t_1} N(z_1\sqrt{t_1} + \frac{z_2}{\sqrt{t_1}}) - e^{-\nu t_2} N(z_1\sqrt{t_2} + \frac{z_2}{\sqrt{t_2}}) \\ & + \frac{1}{2}(\frac{z_1}{z_3} + 1)e^{z_2(z_3 - z_1)}(N(z_3\sqrt{t_2} + \frac{z_2}{\sqrt{t_2}}) - N(z_3\sqrt{t_1} + \frac{z_2}{\sqrt{t_1}})) \\ & + \frac{1}{2}(\frac{z_1}{z_3} - 1)e^{-z_2(z_3 + z_1)}(N(z_3\sqrt{t_2} - \frac{z_2}{\sqrt{t_2}}) - N(z_3\sqrt{t_1} - \frac{z_2}{\sqrt{t_1}})) \end{aligned} \quad (2)$$

with

$$\begin{aligned} z_1 &= \frac{r - \delta - b + \phi\sigma^2/2}{\sigma} \\ z_2 &= \frac{\log{(S/B)}}{\sigma} \\ z_3 &= \sqrt{{z_1}^2 + 2\nu} \end{aligned}$$

By convention, when $t = 0$, $N(x\sqrt{t} + \frac{y}{\sqrt{t}}) = 0.5\ 1_{\{y=0\}} + 1_{\{y>0\}}$

/*Mathematical functions*/
/*critical price*/

It calculates the critical price with Mc Millan's method.

/*derivx*/
It computes the partial derivative of a function with respect to its first argument.

/*derivy*/
It computes the partial derivative of a function with respect to its second argument.

/*function d_1*/

/*function I*/
It is defined in the equation (3).

/*function I_S*/
It gives the partial derivative of I with respect to the spot S. We take the same convention for N as in function I.

$$\begin{aligned}
\frac{\partial I}{\partial S} &= I_S(t_1, t_2, S, B, b, \phi, \nu) \\
&= (\frac{e^{-\nu t_1}}{\sqrt{t_1}} n(z_1\sqrt{t_1} + \frac{z_2}{\sqrt{t_1}})1_{\{t_1 \neq 0\}} - \frac{e^{-\nu t_2}}{\sqrt{t_2}} n(z_1\sqrt{t_2} + \frac{z_2}{\sqrt{t_2}}))\frac{1}{\sigma S} \\
&+ \frac{1}{2\sigma S}(\frac{z_1}{z_3} + 1)(z_3 - z_1)e^{z_2(z_3 - z_1)}(N(z_3\sqrt{t_2} + \frac{z_2}{\sqrt{t_2}}) - N(z_3\sqrt{t_1} + \frac{z_2}{\sqrt{t_1}})) \\
&+ \frac{1}{2\sigma S}(\frac{z_1}{z_3} + 1)e^{z_2(z_3 - z_1)}(\frac{1}{\sqrt{t_2}} n(z_3\sqrt{t_2} + \frac{z_2}{\sqrt{t_2}}) - \frac{1}{\sqrt{t_1}} n(z_3\sqrt{t_1} + \frac{z_2}{\sqrt{t_1}})1_{\{t_1 \neq 0\}}) \\
&- \frac{1}{2\sigma S}(\frac{z_1}{z_3} - 1)(z_3 + z_1)e^{-z_2(z_3 + z_1)}(N(z_3\sqrt{t_2} - \frac{z_2}{\sqrt{t_2}}) - N(z_3\sqrt{t_1} - \frac{z_2}{\sqrt{t_1}})) \\
&- \frac{1}{2\sigma S}(\frac{z_1}{z_3} - 1)e^{-z_2(z_3 + z_1)}(\frac{1}{\sqrt{t_2}} n(z_3\sqrt{t_2} - \frac{z_2}{\sqrt{t_2}}) - \frac{1}{\sqrt{t_1}} n(z_3\sqrt{t_1} - \frac{z_2}{\sqrt{t_1}})1_{\{t_1 \neq 0\}})
\end{aligned} \tag{3}$$

/*det*/
It gives the determinant of the jacobian matrix for a couple of functions (f_1,f_2).

/*coefficients of the inverse of the jacobian matrix*/
/*coefficient 00*/
/*coefficient 01*/
/*coefficient 10*/
/*coefficient 11*/
/*inverse of the jacobian matrix*/

/*Method of Newton-Raphson*/

The algorithm of Newton-Raphson for the system $\begin{cases} f(\mathbf{x}) = 0 \\ g(\mathbf{x}) = 0 \end{cases}$, where $\mathbf{x}$ is the vector $\begin{pmatrix} x \\ y \end{pmatrix}$, is:

$$\mathbf{x_{new}} = \mathbf{x_{old}} + \delta\mathbf{x},$$

where $\delta\mathbf{x}$ is solution of $\mathbf{J}.\delta\mathbf{x} = -\mathbf{F}$.
$\mathbf{F}$ represents the vector $\begin{pmatrix} f(x_{old}) \\ g(x_{old}) \end{pmatrix}$.

$\mathbf{J}$ represents the jacobian matrix of the system.
The precision required to stop the algorithm is 10^{-7}.

The coefficients of the exponential pieces are obtained by solving the smooth fit system.

/*APPROXIMATION BY ONE EXPONENTIAL*/
$B_{11}e^{b_{11}t}$

The corresponding approximate price of the American put option is denoted by P_1.

/*APPROXIMATION BY TWO EXPONENTIAL PIECES*/
$B_{21}e^{b_{21}t}$ during $[T/2, T]$, and $B_{22}e^{b_{22}t}$ during [0,T/2]

In this case, the system given by the condition of smooth fit is:

$$\begin{aligned} K - B_{21}e^{b_{21}T/2} &= P_E(B_{21}e^{b_{21}T/2}, K, T/2) + K(1 - e^{-rT/2}) \\ &+B_{21}e^{b_{21}T/2}(1 - e^{-\delta T/2}) \\ &-KI(0, T/2, B_{21}e^{b_{21}T/2}, B_{21}e^{b_{21}T/2}, b_{21}, -1, r) \\ &+B_{21}e^{b_{21}T/2}I(0, T/2, B_{21}e^{b_{21}T/2}, B_{21}e^{b_{21}T/2}, b_{21}, 1, \delta) \end{aligned} \tag{4}$$

$$\begin{aligned} -1 &= -e^{-\delta T/2}N(-d_1(B_{21}e^{b_{21}T/2}, K, T/2)) - (1 - e^{-\delta T/2}) \\ &-KIS(0, T/2, B_{21}e^{b_{21}T/2}, B_{21}e^{b_{21}T/2}, b_{21}, -1, r) \\ &+I(0, T/2, B_{21}e^{b_{21}T/2}, B_{21}e^{b_{21}T/2}, b_{21}, 1, \delta) \\ &+B_{21}e^{b_{21}T/2}IS(0, T/2, B_{21}e^{b_{21}T/2}, B_{21}e^{b_{21}T/2}, b_{21}, 1, \delta) \end{aligned} \tag{5}$$

and the couple (B_{22},b_{22}) solution of:

$$\begin{aligned} K - B_{22} &= P_E(B_{22}, K, T) + K(1 - e^{-rT}) - B_{22}(1 - e^{-\delta T}) \\ &-KI(0, T/2, B_{22}, B_{22}, b_{22}, -1, r) \\ &+B_{22}I(0, T/2, B_{22}, B_{22}, b_{22}, 1, \delta) \\ &-KI(T/2, T, B_{22}, B_{21}, b_{21}, -1, r) \\ &+B_{22}I(T/2, T, B_{22}, B_{21}, b_{21}, 1, \delta) \end{aligned} \tag{6}$$

$$\begin{aligned} -1 &= -e^{-\delta T}N(-d1(B_{22}, K, T)) - (1 - e^{-\delta T}) \\ &-KI_S(0, T/2, B_{22}, B_{22}, b_{22}, -1, r) \\ &+I(0, T/2, B_{22}, B_{22}, b_{22}, 1, \delta) \\ &+B_{22}I_S(0, T/2, B_{22}, B_{22}, b_{22}, 1, \delta) \\ &-KI_S(T/2, T, B_{22}, B_{21}, b_{21}, -1, r) \\ &+I(T/2, T, B_{22}, B_{21}, b_{21}, 1, \delta) \\ &+B_{22}I_S(T/2, T, B_{22}, B_{21}, b_{21}, 1, \delta) \end{aligned} \tag{7}$$

Ju suggests to use the Newton-Raphson algorithm to solve these systems. To initialize this algorithm, we take the critical price, calculated by Mc Millan's method, and 0 as initial values for B_{21} and b_{21}, and the final values of B_{21} and b_{21} for the calculus of B_{22} and b_{22}.

The price P_2 of the put is given by:

$$P_2 = \begin{cases} P_E + K(1 - e^{-rT}) - S(1 - e^{-\delta T}) \\ -KI(0, T/2, S, B_{22}, b_{22}, -1, r) \\ +SI(0, T/2, S, B_{22}, b_{22}, 1, \delta) \\ -KI(T/2, T, S, B_{21}, b_{21}, -1, r) \\ +SI(T/2, T, S, B_{21}, b_{21}, 1, \delta) \ \text{ if } S > B_{22} \\ K - S \ \text{ if } S \le B_{22} \end{cases} \tag{8}$$

/*APPROXIMATION BY THREE EXPONENTIAL PIECES*/
$B_{31}e^{31*t}$ during $[2T/3; T]$, $B_{32}e^{32*t}$ during $[T/3; 2T/3]$
and $B_{33}e^{33*t}$ during $[0; T/3]$

The corresponding approximate price of the American put option is denoted by P_3.

/*PRICING*/
/*Price*/
To improve the results of the method, we make a three-point Richardson extrapolation, so the price is given by: $\mathbf{\widehat{P_A} = 4.5P_3 - 4P_2 + 0.5P_1}$
/*Delta*/
To evaluate the delta, we compute: $\frac{\widehat{P_A}(S+h) - \widehat{P_A}(S)}{h}$ with the value 10^{-7} for h.

/*PROBLEMS*/

This method does not work, when the interest rate r equals 0.

References

[1] N.JU. Pricing an american option by approximating its early exercise boundary as a multipiece exponential function. *The Review of Financial Studies*, 11, 3:627–646, 1998. 1