THE EIGENFUNCTION EXPANSION METHOD IN MULTI-FACTOR QUADRATIC TERM STRUCTURE MODELS

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We propose the eigenfunction expansion method for pricing options in quadratic terms structure models. The eigenvalues, eigenfunctions and adjoint functions are calculated using elements of the representation theory of Lie algebras not only in the self-adjoint case but in non-selfadjoint case as well; the eigenfunctions and adjoint functions are expressed in terms of the Hermite polynomials. We demonstrate that the method is efficient for pricing caps, floors and swaptions, if time to maturity is 1 year or more. We also consider subordination of the same class of models, and show that in the framework of the eigenfunction expansion approach, the subordinated models are (almost) as simple as pure Gaussian models. We study the dependence of Black implied volatilities and option prices on the type of non-Gaussian innovations.

KEY WORDS: Derivative pricing, swaptions, caps and floors, multi-factor exactly solvable models, eigenfunction expansion, continuous algebraic Riccati equations, Lyapunov equations, representation theory of Lie algebras, Hermite polynomials

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1. Introduction

1.1. In the paper, we propose an eigenfunction expansion method for pricing options in Ornstein-Uhlenbeck driven quadratic term structure models (QTSMs). Assuming that the variance-covariance matrix of the process is non-degenerate, one can make an appropriate affine transformation of the state space (see, e.g., Dai and Singleton (2000)) and reduce to the case of the simplest dynamics

$$dX_t = -\kappa X_t dt + dW_t, \tag{1.1}$$

where dW_t is the increment of the standard Wiener process. The interest rate is modelled as

$$r(X_t) = \frac{1}{2}(\Gamma X_t, X_t) + (d, X_t) + d_0, \tag{1.2}$$

where d_0, d and Γ are scalar, vector and non-negative matrix, respectively. A certain non-degeneracy condition on the pair (κ, Γ) is required. A sufficient condition is that κ is anti-stable (that is, its eigenvalues lie in the right half-plane: $\operatorname{Re}\lambda_j(\kappa) > 0$) but this condition can be relaxed.

In a special case $\Gamma = 0$ of model (1.1)–(1.2), we obtain affine term structure models (ATSMs) of class $A_0(n)$ in the classification of Dai and Singleton (2000); the case of the constant riskless rate obtains with $\Gamma = 0$ and $d = 0$. Models with non-trivial $\Gamma \geq 0$ (typically, positive-definite Γ) are referred to as quadratic term structure models. For a detailed analysis of ATSM models, see Duffie et al (2000), Dai and Singleton (2000) and Chacko and Das (2002), Duffie et al (2003), and for QTSM models, see Leippold and Wu (2002, 2003), Ahn et al. (2002, 2003), Chen and Poor (2003), Chen et al. (2003), Kim (2003) and the bibliography therein.

1.2. **General scheme of the method.** Consider a contingent claim with the maturity date T and payoff $g(X_T)$. Its price at time $t < T$ is given by

$$V(g, T; X_t, t) = E_t\left[\exp\left(-\int_t^T r(X_s)ds\right) g(X_T)\right].$$

Here E_t, the expectation operator conditioned on the information available at time t, and dynamics (1.1) are under a risk-neutral measure chosen by the market. The standard practice is to employ the Feynman-Kac theorem and reduce pricing problem to solution of the Cauchy problem for the backward parabolic equation

$$(\partial_\tau - \mathcal{L})u(x, \tau) = 0, \quad \tau > 0, \tag{1.3}$$

$$u(x, 0) = g(x), \tag{1.4}$$

where $\tau = T - t$, $\partial_\tau u = \partial u/\partial\tau$, and

$$\mathcal{L} = \frac{1}{2}(\partial_x, \partial_x) - (\kappa x, \partial_x) - r(x). \tag{1.5}$$

Here $(A, B) = \sum_{j=1}^n A_j B_j$ denotes the sum of ordered products of operators.

We solve the Cauchy problem (1.3)–(1.4) using the eigenfunction expansion method. The list of publications in mathematical finance literature, where the eigenfunction expansion technique is applied to pricing derivative securities, is fairly extensive – see Lewis (1998, 1999), Linetsky (1999, 2004a-2004e), Davidov and Linetsky (2003), Gorovoi and Linetsky (2004), Pelsser (2000), Albanese and Kuznetsov (2004) and the bibliography therein. However, in these papers, only one-factor models where studied and the well-known classical results for ODE were used. In all cases, an appropriate change of variables and unknown function reduces the stationary part of the parabolic operator to a self-adjoint operator in an appropriate Hilbert space. The same can be done here if κ in (1.1) is symmetric. In this case, an evident change of the unknown function $u(x,\tau) = \exp[(\kappa x, x)/2]v(x,\tau)$ reduces the problem (1.3)–(1.4) to the case $\kappa = 0$, which is the case of a self-adjoint infinitesimal generator. After that, an appropriate rotation in the state space leads to the case of a diagonal Γ, and separation of variables technique reduces to the one-dimensional case.

In this paper, we concentrate on the less trivial case of asymmetric κ. In Section 2, we make a conjugation with the exponential of an appropriate quadratic function, and reduce to the infinitesimal generator of the special form

$$\mathcal{L}_0 = \sum_{j=1}^{n} \mu_j \mathcal{L}_j - \sum_{j>k} b_{jk} J_{jk} - \tilde{d}_0, \tag{1.6}$$

where $J_{jk} = x_j\partial_k - x_k\partial_j$, $\mathcal{L}_j = \frac{1}{2}(\partial_{x_j}^2 - x_j^2)$, and $\tilde{d}_0$ is a constant. To construct the quadratic function, we need to solve a continuous algebraic Riccati equation (CARE) and a Lyapunov equation. Several algorithms for the solution of CARE for matrices of order greater than 1 are described in Appendix B. One algorithm reduces a CARE to calculation of eigenfunctions and eigenvectors of a $2n \times 2n$ matrix, which is a very simple computational problem for models with two or three factors, and two- or three-factor models seem to be quite sufficient for applications of QTSMs. Another algorithm uses the Newton-Kantorovich method, and, on each step of the method, a Lyapunov equation needs to be solved. The Lyapunov equation is reducible to a linear system with $2n$ unknowns (we list basic facts of the theory of Lyapunov equations in Appendix C), and hence, it is easy to solve,

In Section 3, we use the commutation relations among operators $\mathcal{L}_j$ and J_{jk}, and represent $L_2(\mathbb{R}^n)$ as the direct sum of finite-dimensional subspaces V_m which are invariant under all $\mathcal{L}_j$ and J_{jk}, hence, under $\mathcal{L}_0$. We express the eigenvalues of $\mathcal{L}_0$ and a basis in each V_m in terms of eigenvalues and eigenfunctions of the infinitesimal generator

$$\mathcal{L}_{\rm st} = \frac{1}{2}\left(\frac{d^2}{dx^2} - x^2\right) \tag{1.7}$$

of the simplest QTSM (we will call it the standard QTSM). Note that $-\mathcal{L}_{\rm st}$ is one of the most important operators in mathematical physics: the Hamiltonian of the (quantum) harmonic oscillator. The eigenvalues and eigenfunctions of the harmonic oscillator are

well-known; the former are $m+1/2$, $m=0,1,2,\dots$, and the latter are expressed in terms of the Hermite polynomials.

Thus, we represent the operator $\mathcal{L}_0$ in (1.6) as a direct sum of linear operators $\mathcal{L}^m$ acting in finite-dimensional subspaces V_m, $m=0,1,2,\dots$, where $N_m:=\dim V_m\to+\infty$ as $m\to+\infty$, and $\dim V_0=1$. Equivalently, we reduce the backward parabolic equation to a series of the Cauchy problems for systems of linear first order ODEs with constant coefficients. We construct a basis in which the matrix of $\mathcal{L}^m$ is almost diagonal (the entries vanish at distance 2 off the diagonal). We also show that the upper bound for the real parts of the eigenvalues of $\mathcal{L}^m$ is a decreasing linear function of m. These two properties make the initial problem effectively solvable by the eigenfunction expansion technique.

1.3. **Two-factor case.** In Section 4, we consider the two-factor case in more detail. We calculate explicitly the eigenvalues and eigenfunctions, which may be complex. If the canonical form of $\mathcal{L}^m$ contains Jordan blocks, we calculate adjoint functions as well. We show that the properties of the operators $\mathcal{L}^m$ are determined by two effective parameters $\alpha\ge 0$ and $\beta\ne 0$, and the following three cases are possible:

(a) $\alpha^2>\beta^2$; then all the eigenvalues of all $\mathcal{L}^m$ are real, and the eigenfunctions form a basis in $L_2(\mathbb{R}^2)$. Hence, the Cauchy problem for the backward parabolic equation reduces to a series of Cauchy problems for scalar ODEs;

(b) $\alpha^2<\beta^2$; then for $m\ge 2$ even, $\mathcal{L}^m$ has one real eigenvalue, and pair-wise distinct pairs of complex-conjugate eigenvalues. For m odd, $\mathcal{L}^m$ has only pair-wise distinct pairs of complex-conjugate eigenvalues. Using real eigenvectors, and pairs of complex-conjugate eigenvectors, we construct real solutions of the Cauchy problems;

(c) $\alpha^2=\beta^2$; then all the eigenvalues are real, and $\mathcal{L}^m$, $m\ge 1$, are similar to Jordan blocks. We calculate the eigenvalues, eigenvectors and adjoint vectors, and hence, explicitly solve the Cauchy problems for the systems of ODEs.

The constructions and proofs in Section 3 and Section 4 are variations of the well-known constructions for the quantum harmonic oscillator, and use standard tricks from the elementary representation theory of Lie algebras.

1.4. **Eigenfunction expansion method vs. Fourier transform methods.** To describe situations where the method developed in the paper can complement or successfully compete with the methods used in the literature, we recall the latter. If the payoff is the exponential of a quadratic function, then the problem (1.3)-(1.4) is reduced to a system of Riccati equations, which can be explicitly solved. In particular, the price of a zero-coupon bond and the price of a security with the complex payoff $e^{i(x,\xi)}$ can be calculated. Using the explicit formula for $V(e^{i(\cdot,\xi)},T;x,t)$ and the inverse Fourier transform, one calculates $V(g,T;x,t)$. On the theoretical level, the method is powerful, simple and straightforward, and since standard packages for the numerical calculation of the Fourier transform and its inverse (FFT and iFFT) are available, the calculation problem seems to be settled. However, in dimensions higher than 1, the resulting computational errors

or time of computation can be significant. Duffie et al. (2000) noticed in the framework of ATSMs that if the payoff $g(x)$ is of the form

$$g_{q_1,q_2}(x;y) = e^{q_1(x)}\mathbf{1}_{[0,+\infty)}(q_2(x)-y), \tag{1.8}$$

where q_j are affine polynomials and $y \in \mathbb{R}$, then, applying the Fourier transform w.r.t. y, it is possible to reduce the pricing problem to the solution of the system of Riccati equations depending on the complex parameter $\eta \in \mathbb{C}$ with the subsequent Fourier inversion. The latter can be performed using FFT in 1D. An additional advantage of this approach is that FFT calculates prices for many values of y simultaneously. Leippold and Wu (2002) and Chen et al. (2003) used the same trick in QTSM. For the case of options on a zero coupon bond, y is the log-strike, and therefore, for a given value of the stochastic factor, one can calculate option prices for many strikes simultaneously. Notice that the use of FFT implies that the integral in the pricing formula is truncated first (replaced with the integral over a finite interval), and then the efficiency of a quadrature procedure for the integral over a finite (albeit long) interval is enhanced by using FFT; the latter saves computational time but cannot improve accuracy of the truncation and a quadrature procedure. For options of long maturities, the time of calculations of the input for FFT is either very time consuming or not very accurate. The resulting errors can be significant: see the analysis in Boyarchenko and Levendorskiĭ (2006). Clearly, the eigenfunction expansion method should perform well for options of long maturities but our numerical experiments indicate that this method is fairly accurate fairly close to expiry as well.

The second advantage of the eigenfunction expansion method is that, after the eigenvalues, eigenfunctions and coefficients of the eigenfunction expansion of the payoff are calculated, one can calculate simultaneously option prices at dozens of thousand of points (x_k, τ_j) faster than FFT does for several points (x_k, τ_j).[1] For bonds and swaps, the coefficients can be calculated explicitly. For options of other types, say, caps, floors, swaptions, captions and Asian options, we calculate the coefficients numerically reducing to integrals in dimension $n-1$ or using the importance sampling technique. Notice that the Fourier transform method in dimension one can be applied to swaps, exchange options, options on bonds, caps and floors, and certain types of Asian options on yields, but not to swaptions because the latter cannot be represented as sums of contingent claims with payoffs of the form (1.8).

The natural question is: how many terms of the eigenfunction expansion are needed to achieve a reasonably small relative pricing error? Our method reduces the pricing problem to a series of systems of linear first order ODEs with constant coefficients. Therefore, it is natural to characterize the speed of convergence in terms of the size

[1]The advantage of FFT is that option prices for many strike price can be calculated simultaneously; however, as the analysis of errors of FFT in Boyarchenko and Levendorskiĭ (2006) shows, if a good accuracy of option prices is needed, then the overwhelming majority of these strike prices are not needed at all. Therefore, the advantage is not as large as it seems.

N_j of these systems, equivalently, in terms of the size of the largest system used. In the one-factor case, $N_j = 1$ for all j, in the two-factor case, $N_j = j + 1$, and $N_j = \#\{\alpha \in \mathbb{Z}^n_+ \mid \alpha_1 + \cdots + \alpha_n = j\}$ in the general case. Numerical experiments demonstrate that, if the time to maturity, T, is half a year or more, then prices of out-of-the-money call and put options, hence, caps and floors, and options with more involved payoff structure such as swaptions can be calculated with the relative error of order 0.001 if $j = 20$; at $T = 1$ (year), 10 systems produce the relative error of the same order. At $T = 10$, $j = 4$ suffices, and even the leading term of the eigenfunction expansion gives an approximation of the relative error less than 1%. (In the two-factor case, the number of terms of the asymptotic expansion is 210, 55, 10 and 1, respectively.) As far as the speed of calculations is concerned, the eigenfunction expansion method is many times faster than the Fourier transform method if it is necessary to calculate prices at many points in (x, t)-space, starting from 1 year to expiry, with relative error 0.001. Thus, for options of maturity a year or more, the eigenfunction expansion method can compete successfully with the FFT-based approach even in cases when the use of FFT in dimension one is possible.

1.5. **Non-Gaussian innovations.** The next advantage of the eigenfunction approach is that it remains computationally efficient if non-Gaussian innovations are introduced via subordination of the initial Markov pricing semigroup. This idea (subordinated processes and eigenfunction expansions), in one-factor models, was used in Albanese and Kuznetsov (2004); subordination per se was used in a number of publications, but not in the context of QTSM models. For the use of subordinated processes in financial modelling, without the eigenfunction expansion technique, see, e.g., Barndorff-Nielsen and Levendorskiĭ (2001), Carr et al. (2002), Carr and Wu (2004). One can introduce non-Gaussian innovation not by subordination but adding a jump component to the driving Gaussian process. If X_t is a Lévy-driven process of Ornstein-Uhlenbeck type, then ATSM models are (almost) as tractable as purely Gaussian ones (see Duffie et al (2000), Dai and Singleton (2000) and Chacko and Das (2002), Duffie et al. (2003)). On the contrary, exactly solvable QTSMs are known in the pure Gaussian case only; moreover, the standard simple form of the solution in Gaussian QTSMs does not admit a generalization for the case of processes with jumps as Levendorskiĭ (2005) indicates for Lévy driven QTSM, and Chen et al. (2003) prove for QTSM driven by more general Markov models. Levendorskiĭ (2005) obtains pricing formulas for QTSMs driven by Lévy processes but these formulas are approximate ones, and cannot perform well far from maturity. In the framework of the eigenfunction expansion approach, subordinated QTSMs are as tractable as Gaussian QTSMs, and at the same time, they allow one to account for jumps. We leave for the future an interesting comparison of QTSMs with jumps in Levendorskiĭ (2005) and subordinated QTSMs constructed in this paper.

Models with non-Gaussian innovations are considered in Section 5. From the analytical point of view, the only difference with the pure Gaussian case is that we replace the infinitesimal generator $\mathcal{L}$ with $\Phi(\mathcal{L})$, where Φ is the Laplace exponent of the subordinator.

Since the $\mathcal{L}$ has been (block)-diagonalized already, the solution of the Cauchy problem is straightforward. In particular, if $\mathcal{L}$ is completely diagonalizable, one needs only to replace the eigenvalues $\lambda_j(\mathcal{L})$ with $\Phi(\lambda_j(\mathcal{L}))$.

1.6. **Numerical realization.** The calculation of the coefficients of the asymptotic expansion is analyzed in Section 6. From the computational point of view, the calculation of eigenvalues and eigenvectors (and adjoint vectors) is very simple: the former are given by short explicit formulas, and the latter are of the form $P_\alpha(x)\exp(-||x||^2/2)$ (in appropriate coordinates), where P_α, $\alpha \in (\mathbb{Z}_+)^n$, are polynomials whose coefficients can be calculated recurrently starting from $P_0 = 1$. In the one-factor case, P_α are the Hermite polynomials, and in the multi-factor case, P_α are expressed in terms of the Hermite polynomials. Hence, if the payoff is the product of an exponential of a quadratic function and a polynomial, the coefficients can be calculated explicitly. In particular, we can explicitly calculate the option price. In the case of a general payoff, we need to calculate the coefficients numerically.

In Section 7, we consider numerical examples. In a two-factor QTSM, we calculate bond yields and forward rates, and then prices of bond options and swaptions. Section 8 concludes. Auxiliary results are relegated to appendices A–E. In Appendix F, we provide a detailed algorithm of our method.

2. Reduction to the infinitesimal generator of the special form

The reduction to the infinitesimal generator of the special form (1.6) is as follows. The starting observation is that the infinitesimal generator $\mathcal{L}$ of the pricing semigroup is a polynomial of degree 2 in (x, ∂_x). If $\mathcal{L}$ is self-adjoint, then one can use an appropriate symplectic transformation and diagonalize the operator. Unfortunately, this technique does not work in the non-self-adjoint case. In addition, in some cases, complete diagonalization is impossible. Hence, we need to find another route.

Step 1. The first simplification eliminates the term (d, x). This step can be interpreted as Girsanov's change of measure. We will describe this step in terms of the change of the unknown function, since the change of the unknown is a much simpler piece of mathematics, and this change will be incorporated in the change of the unknown function on the next step anyway. Let T_a be the translation operator: $T_a u(x) = u(x + a)$, and let M_b be the multiplication-by-$e^{(b,x)}$ operator. In Appendix A, we prove that given $\mathcal{L}$ of the form (1.5), there exist $a, b \in \mathbb{R}^n$ such that the transformation

$$u(x) = e^{(b,x+a)}u_1(x + a), \quad g(x) = e^{(b,x+a)}g_1(x + a) \tag{2.1}$$

kills the term (d, x). Equivalently, $\mathcal{L}_{a,b} := T_{-a}M_{-b}\mathcal{L}M_bT_a$ is of the form

$$\mathcal{L}_{a,b} = \frac{1}{2}(\partial, \partial) - (\kappa x, \partial) - \frac{1}{2}(\Gamma x, x) - \hat{d}_0, \tag{2.2}$$

where $\hat{d}_0 \in \mathbb{R}$ is given by

$$\hat{d}_0 = d_0 - (d,a) + (\Gamma a, a)/2 + (b,b)/2, \tag{2.3}$$

and the pair (a,b) is the solution of the linear system

$$\begin{cases} \kappa a + b & = 0, \\ \Gamma a - \kappa^T b - d & = 0, \end{cases} \tag{2.4}$$

whereupon

$$a = (\Gamma + \kappa^T\kappa)^{-1} d, \tag{2.5}$$

$$b = -\kappa a. \tag{2.6}$$

Since $\Gamma \geq 0$, and κ is invertible, $\Gamma + \kappa^T\kappa$ is positive-definite, and hence invertible. If κ is not stable, we need to require that $\Gamma + \kappa^T\kappa$ is positive-definite.

Step 2 is more involved. We look for symmetric matrices W and Z such that the conjugation with $e^{\Phi_0(x)}$, where $\Phi_0(x) = (Wx,x)/2$, transforms $\mathcal{L}_{a,b}$ into the following form:

$$e^{-\Phi_0(x)}\mathcal{L}_{a,b}e^{\Phi_0(x)} = \frac{1}{2}(\partial_x,\partial_x) - ((\kappa+W)x,\partial_x) - \frac{1}{2}(Zx,Zx) - \tilde{d}_0, \tag{2.7}$$

where $\tilde{d}_0$ is a constant, and $\kappa + W$ is skew-adjoint w.r.t. Z:

$$Z(\kappa+W) = -(\kappa+W)^T Z. \tag{2.8}$$

It turns out that it is simpler to find, first, the sum $Y = W + Z$ as the solution to the continuous algebraic Riccati equation (CARE)

$$Y^2 + Y\kappa + \kappa^T Y - \Gamma = 0, \tag{2.9}$$

such that $\kappa + Y$ is anti-stable, and then Z as the positive-definite solution to a Bernoulli equation

$$Z(\kappa+Y) + (\kappa+Y)^T Z - 2Z^2 = 0. \tag{2.10}$$

Equivalently, we look for a symmetric matrix Y such that the conjugation with $\exp\Phi_1(x)$, where $\Phi_1(x) = -\frac{1}{2}(Yx,x)$, eliminates the term $\frac{1}{2}(\Gamma x,x)$. We change the variables $x = x' - a$, and introduce new functions u_{11} and g_{11} so that

$$u(x,\tau) = e^{-\frac{1}{2}(Yx',x')+(b,x')}u_{11}(x',\tau), \quad g(x) = e^{-\frac{1}{2}(Yx',x')+(b,x')}g_{11}(x'), \tag{2.11}$$

and the parabolic equation for $u_{11}(x',\tau)$ has no term that is quadratic in x. To avoid the complex notation, we will denote x' by x until the end of this section.

Since $e^{-\Phi_1(x)}\partial e^{\Phi_1(x)} = \partial - Yx$, we have

$$\begin{aligned} \mathcal{L}_{a,b;-Y} : &= e^{-\Phi_1(x)}\mathcal{L}_{a,b}e^{\Phi_1(x)} \\ &= \frac{1}{2}(\partial - Yx, \partial - Yx) - (\kappa x, \partial - Yx) - \frac{1}{2}(\Gamma x,x) - \hat{d}_0 \\ &= \frac{1}{2}(\partial,\partial) - ((\kappa+Y)x,\partial) - \frac{1}{2}((\Gamma - Y\kappa - \kappa^T Y - Y^2)x,x) - \hat{d}_0 - \frac{1}{2}\mathrm{Tr}Y, \end{aligned}$$

where $\mathrm{Tr}Y$ is the trace of Y. As Y, we take the solution of (2.9), then $\mathcal{L}_{a,b;-Y}$ becomes

$$\mathcal{L}_{a,b;-Y} = \frac{1}{2}(\partial,\partial) - ((\kappa+Y)x,\partial) - \hat{d}_0 - \frac{1}{2}\mathrm{Tr}Y. \tag{2.12}$$

If $\kappa+Y$ is anti-stable, then $\mathcal{L}_{a,b;-Y}$ is the infinitesimal generator in a mean-reverting model with the constant interest rate $\hat{d}_0+\mathrm{Tr}Y/2$, variance-covariance matrix $\Sigma=I$, zero central tendency, and mean-reversion matrix $\kappa+Y$. In the one-factor case, (2.9) is easy to solve, but in the multi-factor case, (2.9) is a special case of continuous algebraic Riccati equations (CARE). In Appendix B we describe three constructions of a (unique) solution to (2.9) such that $\kappa+Y$ is anti-stable. One of these constructions uses the Newton-Kantorovich method. On each step of the method, one needs to solve the Lyapunov equation, which also appears on the next step.

Step 3. We conjugate $\mathcal{L}_{a,b;-Y}$ with $\exp\left[\frac{1}{2}(Zx,x)\right]$, where Z is symmetric. The result is

$$\mathcal{L}_{a,b;Z-Y} = \frac{1}{2}(\partial,\partial) - ((\kappa_1-Z)x,\partial) - \frac{1}{2}(\Lambda^2 x,x) - \tilde{d}_0, \tag{2.13}$$

where $\kappa_1=\kappa+Y$, $\tilde{d}_0=\hat{d}_0+\frac{1}{2}\mathrm{Tr}(Y-Z)$, and $\Lambda^2 := -Z^2+Z\kappa_1+\kappa_1^T Z$. We choose Z so that $-Z^2+Z\kappa_1+\kappa_1^T Z$ is positive-definite, and the notation Λ^2 is justified. For efficient calculation of the eigenvalues and eigenfunctions, the following property is crucial: κ_1-Z is Λ-skew-symmetric, that is,

$$\Lambda(\kappa_1-Z) = -(\kappa_1^T-Z)\Lambda. \tag{2.14}$$

Choose Z as the positive-definite solution of the Bernoulli equation (2.10); then we can set $\Lambda=Z$, and (2.14) is satisfied. If we find a positive-definite symmetric solution to the (special case of) Lyapunov equation

$$\kappa_1 X + X\kappa_1^T = 2I_n, \tag{2.15}$$

where I_n is the $n\times n$ identity matrix, then we can find $Z=X^{-1}$ as well. Since $\kappa_1=\kappa+Y$ is anti-stable, and $I_n>0$, then a unique solution of (2.15) exists, and it is positive-definite (see Appendix C). Note that (2.14) means that the matrix $B_0 := \Lambda^{1/2}(\kappa_1-Z)\Lambda^{-1/2}$ is skew-symmetric.

Step 4. We obtained the operator $\mathcal{L}_{a,b;Z-Y}$ in (2.13) after the change of variables $x'=x+a$, the data and unknown function

$$u(x,\tau) = e^{\frac{1}{2}((Z-Y)x',x')+(b,x')}v(x',\tau), \quad g(x) = e^{\frac{1}{2}((Z-Y)x',x')+(b,x')}g_2(x'). \tag{2.16}$$

To justify the application of the eigenfunction expansion technique to the Cauchy problem

$$(\partial_\tau - \mathcal{L}_{a,b;Z-Y})v(x',\tau) = 0, \quad \tau>0, \tag{2.17}$$

$$v(x',0) = g_2(x'), \tag{2.18}$$

we need to know that $Z-Y>0$, that is, $((Z-Y)x,x)\geq c||x||^2$, where $c>0$ is independent of x. Indeed, the standard pay-off functions increase at infinity not faster

than an exponential function, therefore under condition $Z-Y>0$, g_2 decreases at infinity very fast; hence, $g_2\in L_2(\mathbb{R}^n)$.

In Appendix D, we show that $Z-Y=-\frac12(H_++H_-)$, where $H_\pm$ are the so-called maximal and minimal Hermitian solutions to the continuous algebraic Riccati equation (2.9). We show that if Γ is not small w.r.t. κ or the antisymmetric part of κ is not large w.r.t. the symmetric part of κ, then for κ anti-stable, $H_++H_-<0$, and therefore, the eigenfunction expansion technique is applicable. We checked that for typical parameters values documented in empirical studies of financial markets, one of these two sufficient conditions on the parameters of the model (typically, both) holds, and hence, the method is applicable. Note also that the coefficients in the asymptotic expansion of the data can be calculated for all parameters values (see Section 6), and therefore the formal eigenfunction expansion with the truncated finite sum can be calculated in all cases, even if $Z-Y$ is not positive definite. It is feasible that the eigenfunction expansion converges (albeit not absolutely) for all pay-offs of interest and all reasonable parameters' values.

If the reader wishes to be perfectly safe, then she/he is advised to check the condition $Z-Y>0$ first: if Γ is large w.r.t. κ, and the antisymmetric part of κ is large w.r.t. the symmetric part of κ, then it may be the case that $Z-Y$ is not positive definite.

3. Reduction to systems of linear ODE

Let $\{\psi_j\}_{j=1}^n$ be an orthogonal basis of eigenvectors of Z, and let $\{\mu_j\}$ be the corresponding eigenvalues. Let y' be the coordinates of x' in the basis $\{\psi_j\}$, and C the transformation operator from the old basis to the new one: $x'=Cy'$. Then

$$\mathcal{L}_{a,b;Z-Y}=\frac12\sum_{j=1}^n\left(\partial^2_{y'_j}-\mu_j^2(y'_j)^2\right)-(C^T(\kappa_1-Z)Cy',\partial_{y'})-\tilde d_0. \tag{3.1}$$

Change the variables $x''_j=\mu_j^{1/2}y'_j$, $j=1,\dots,n$, and note that the matrix $B_0:=\Lambda^{1/2}(\kappa_1-Z)\Lambda^{-1/2}$ is skew-symmetric; hence the matrix

$$B:=C^T\Lambda^{1/2}(\kappa_1-Z)\Lambda^{-1/2}C=\operatorname{diag}(\mu_j^{1/2})\cdot C^T(\kappa_1-Z)C\cdot\operatorname{diag}(\mu_j^{-1/2})$$

is skew-symmetric as well. It follows that in the new variables,

$$\mathcal{L}_{a,b;Z-Y}=\sum_{j=1}^n\mu_j\mathcal{L}_j-\sum_{j>k}b_{jk}J_{jk}-\tilde d_0, \tag{3.2}$$

where $\mathcal{L}_j$ is the infinitesimal generator of the standard QTSM acting w.r.t. x''_j, $J_{jk}=x''_j\partial_{x''_k}-x''_k\partial_{x''_j}$, and b_{jk} are the entries of B. For $j=1,2,\dots,n$, introduce the creation and annihilation operators acting w.r.t. x''_j:

$$z_j=\frac{1}{\sqrt2}(x''_j-\partial_{x''_j}),\quad z_j^*=\frac{1}{\sqrt2}(x''_j+\partial_{x''_j}).$$

Direct calculations show that $\mathcal{L}_j = z_j z_j^* + 1/2$, $J_{jk} = z_j z_k^* - z_k z_j^*$, and the following commutation relations hold:

$$[z_j, z_k] = [z_j^*, z_k^*] = 0, \quad [z_j^*, z_k] = \delta_{jk}. \tag{3.3}$$

Here δ_{jk} is the Kronecker symbol ($\delta_{jk} = 1$ if $j = k$ and $=0$ otherwise).

Introduce functions on $\mathbb{R}$ by

$$w_n(x) = 2^{-n/2} H_n(x) e^{-x^2/2},$$

where $H_n(x) = 2xH_{n-1}(x) - H'_{n-1}(x)$ is a polynomial of degree n, and $H_0 = 1$; H_n are called the *Hermite polynomials* (see e.g. p.562 in Vilenkin (1968)). It is well-known (and can be easily verified) that $w_j(x) = z^j w_0, j = 0, 1, \ldots,$ are eigenfunctions of operator in (1.7) with corresponding eigenvalues $\nu_j = j + \frac{1}{2}$; the list $\{w_j\}_{j=0}^{+\infty}$ constitutes a basis in $L_2(\mathbb{R})$. Normalizing, one obtains an orthonormal basis $f_m = (m!\sqrt{\pi})^{-1/2} w_m, m = 0, 1, \ldots$.

For a multi-index α, define $f_\alpha := f_{\alpha_1} \otimes \cdots \otimes f_{\alpha_n}$, and introduce subspaces $V_m \subset L_2(\mathbb{R}^n)$ spanned by f_α with $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n = m$. The choice of $\{f_\alpha\}_{|\alpha|=m}$ as an orthonormal basis in V_m identifies V_m with $\mathbb{R}^{N_m}$, where $N_m = \#\{\alpha \mid |\alpha| = m\}$.

Lemma 3.1. *For $m = 0, 1, \ldots,$*

a) V_m is invariant under $\mathcal{L}_j$ and J_{jk}, hence, under $\mathcal{L}_{a,b;Z-Y}$;

b) let $\mathcal{L}^m$ be the restriction of $\mathcal{L}_{a,b;Z-Y}$ on V_m. Then the real parts of the eigenvalues of $\mathcal{L}^m$ are bounded from above by $\nu_m = -(m + n/2) \min_{1 \le j \le n} \mu_j - \tilde{d}_0$.

Proof. a) For any j, k,

$$\begin{aligned} z_j z_k^* w_\alpha &= w_{\alpha_1} \otimes \cdots \otimes z_j w_{\alpha_j} \otimes \cdots \otimes z_k^* w_{\alpha_k} \otimes \cdots \otimes w_{\alpha_n} \\ &= w_{\alpha_1} \otimes \cdots \otimes w_{\alpha_j + 1} \otimes \cdots \otimes \alpha_k w_{\alpha_k - 1} \otimes \cdots \otimes w_{\alpha_n}; \end{aligned}$$

hence, $z_j z_k^* w_\alpha \subset V_{|\alpha|}$ (if $\alpha_k = 0$, then $z_j z_k^* w_\alpha = 0 \in V_{|\alpha|}$).

b) The first sum in (3.2) is a self-adjoint operator in $L_2(\mathbb{R}^n)$, and the second sum is an antisymmetric one. Hence, their restrictions on V_m are symmetric and antisymmetric operators in $\mathbb{R}^{N_m}$; we denote them $\mathcal{L}^m_{\text{sym}}$ and $\mathcal{L}^m_{\text{as}}$, respectively. On the strength of Lemma A.1, it suffices to prove that the eigenvalues of $\mathcal{L}^m_{\text{sym}}$ are bounded from above by ν_m. But evidently, f_α, $|\alpha| = m$, are the eigenvectors of $\mathcal{L}^m_{\text{sym}}$ with eigenvalues $-\sum_{j=1}^n \mu_j(\alpha_j + 1/2) - \tilde{d}_0$, which are bounded from above by ν_m. □

The following theorem is an immediate corollary of Lemma 3.1.

Theorem 3.2. *a) $L_2(\mathbb{R}^n)$ is the direct sum of the Euclidean spaces V_m: $L_2(\mathbb{R}^n) = \oplus_{m=0}^\infty V_m$, and $\mathcal{L}_{a,b;Z-Y}$ is the direct sum of operators $\mathcal{L}^m$: $\mathcal{L}_{a,b;Z-Y} = \oplus_{m=0}^\infty \mathcal{L}^m$.*

b) Let $g_2 = \oplus_{m \ge 0} g_2^m$, where $g_2^m \in V_m \cong \mathbb{R}^{N_m}$, be the corresponding decompositions of g_2. Let $v^m(\tau) \in V_m \cong \mathbb{R}^{N_m}, \tau \ge 0$, be the solution to the Cauchy problem of the following

system of ODEs of the first order, with constant coefficients:

$$(d/d\tau - \mathcal{L}^m)v^m(\tau) = 0, \quad \tau > 0, \tag{3.4}$$

$$v^m(0) = g_2^m. \tag{3.5}$$

Then

$$v(x,\tau) = \sum_{m\geq 0} v^m(x,\tau) \tag{3.6}$$

$$= \sum_{m\geq 0} e^{\tau\mathcal{L}^m} g_2^m(x) \tag{3.7}$$

is the solution to the Cauchy problem (2.17)–(2.18)

Remark 3.1. a) In the RHS of (3.6)–(3.7), $v^m(\cdot,\tau), g_2^m \in V_m$ are regarded as functions of x, and in (3.4)–(3.5), as vectors $v^m(\tau), g_2^m \in \mathbb{R}^{N_m}$.

b) On the strength of Lemma 3.1, part b), the maximum of the real parts of the eigenvalues of $\mathcal{L}^m$ decreases as a linear function of m, therefore for large τ and m, $v^m(\tau)$ is very small. Hence, one can obtain a good approximation using a truncated finite sum in (3.6) with several terms, and each term can be calculated quite easily using standard programs for systems of linear first order ODEs or calculating the exponential of the matrix $\tau\mathcal{L}^m$.

c) $n_0 = 1$, hence the leading term v^0 is the solution to the scalar problem, and can be calculated easily; the eigenvalue with the smallest (in absolute value) real part is $\lambda_0 = -\sum_{j=1}^n \mu_j/2 - \tilde{d}_0$.

d) In the two-factor case, all $\mathcal{L}^m$ can be either diagonalized (in some cases, as operators in $\mathbb{R}^{m+1}$, in other cases as operators in $\mathbb{C}^{m+1}$), or each $\mathcal{L}^m, m \geq 1$, is similar to the $(m+1)\times(m+1)$-Jordan block. In all cases, the calculation of the exponential of the matrix-function in (3.7) is a standard exercise.

4. Two-factor models

4.1. **Reduction to an auxiliary operator and calculation of its eigenvalues, eigenvectors and adjoint vectors.** First, note that $\{w_{m-j}\otimes w_j\}_{j=0}^m$ is an orthogonal basis in $V^m \cong \mathbb{R}^{m+1}$. Next, set $\alpha = \mu_2 - \mu_1$, $\beta = 2b_{12}$, $\hat{\mathcal{L}} = \frac{1}{2}(\mu_1+\mu_2)(\mathcal{L}_1+\mathcal{L}_2)$, $J = z_1 z_2^* - z_2 z_1^*$, and then introduce

$$\tilde{\mathcal{L}} := \mathcal{L}_{a,b;Z-Y} - \hat{\mathcal{L}} + \tilde{d}_0 = \frac{1}{2}(\alpha(\mathcal{L}_2 - \mathcal{L}_1) - \beta J). \tag{4.1}$$

Finally, define the restrictions of $\hat{\mathcal{L}}$ and $\tilde{\mathcal{L}}$ on V_m by $\hat{\mathcal{L}}^m$ and $\tilde{\mathcal{L}}^m$, respectively.

Lemma 4.1. *$\tilde{\mathcal{L}}^m$ has the same eigenvectors (adjoint vectors) as $\mathcal{L}^m$, and the eigenvalues of $\mathcal{L}^m$ and $\tilde{\mathcal{L}}^m$ are related by*

$$\lambda_j(\mathcal{L}^m) = \lambda_j(\tilde{\mathcal{L}}^m) - \frac{m+1}{2}(\mu_1+\mu_2) - \tilde{d}_0. \tag{4.2}$$

Proof. Since

$$-(\mathcal{L}_1+\mathcal{L}_2)f_{m-j}\otimes f_j=(m-j+1/2+j+1/2)f_{m-j}\otimes f_j=(m+1)f_{m-j}\otimes f_j,$$

we have $\hat{\mathcal{L}}^m=-\frac{1}{2}(\mu_1+\mu_2)(m+1)I_{m+1}$, and the statement of the lemma follows. □

As it turns out, all the eigenvalues of $\tilde{\mathcal{L}}$ lie on either the real axis or imaginary one. We construct the eigenvectors from $w_0\otimes w_0\in V^0$ using auxiliary operators $Q_1^{\pm}$ and Q_2.

Set $Q_1^{\pm}=(\alpha\mp\Lambda)z_1+\beta z_2$, where $\Lambda=\sqrt{\alpha^2-\beta^2}$ is positive if $\alpha^2>\beta^2$, and has a positive imaginary part, if $\alpha^2<\beta^2$. If $(0\neq)\Lambda\in i\mathbb{R}$, then for the time being, we regard $\tilde{\mathcal{L}}$ as an operator in the complex space $L_2(\mathbb{R}^2;\mathbb{C})$, V_m as the complex space spanned by $\{f_{m-j}\otimes f_j\}_{j=0}^m$, and $\tilde{\mathcal{L}}^m$ as a linear operator in the complex space V_m.

Introduce $Q_2=\beta z_2^2+2\alpha z_1z_2+\beta z_1^2$. In Appendix A, we prove the following two simple commutation relations for operators Q_1,Q_2 and $\tilde{\mathcal{L}}$, on which the proofs of the main theorems are based:

$$\tilde{\mathcal{L}}Q_1^{\pm} = Q_1^{\pm}(\tilde{\mathcal{L}}\mp\frac{1}{2}\Lambda), \tag{4.3}$$

$$[\tilde{\mathcal{L}},Q_2] := \tilde{\mathcal{L}}Q_2-Q_2\tilde{\mathcal{L}}=0. \tag{4.4}$$

For an integer $s\geq 0$, set $Q_1^s=(Q_1^+)^s$, and $Q_1^{-s}=(Q_1^-)^s$.

Theorem 4.2. *a) $\phi_{rs}:=Q_2^rQ_1^sw_0\otimes w_0$, $s=\ldots,-1,0,1,\ldots$, $r=0,1,\ldots$, are eigenfunctions of $\tilde{\mathcal{L}}$ with corresponding eigenvalues $-s\Lambda/2$.*

b) If $\Lambda\neq 0$, then for $m=0,1,\ldots$, the list ϕ_{rs}, $r=0,1,\ldots,[m/2]$, $s=\pm(m-2r)$, is a basis in V_m.

Proof. a) It is straightforward to check that $\tilde{\mathcal{L}}$ annihilates $w_0\otimes w_0$, therefore it is an eigenvector with the eigenvalue 0. To construct the other eigenvectors, we use the commutation relations derived above. Applying (4.4) r times, we obtain $[\tilde{\mathcal{L}},Q_2^r]=0$. Therefore,

$$\tilde{\mathcal{L}}Q_2^rQ_1^sw_0\otimes w_0=Q_2^r\tilde{\mathcal{L}}Q_1^sw_0\otimes w_0.$$

Applying (4.3) s times, we find $\tilde{\mathcal{L}}Q_1^s=Q_1^s(\tilde{\mathcal{L}}-\frac{s}{2}\Lambda)$. Therefore,

$$\tilde{\mathcal{L}}Q_2^rQ_1^sw_0\otimes w_0=Q_2^r\tilde{\mathcal{L}}Q_1^sw_0\otimes w_0=Q_2^rQ_1^s(\tilde{\mathcal{L}}-\frac{s}{2}\Lambda)w_0\otimes w_0=-\frac{s}{2}\Lambda Q_2^rQ_1^sw_0\otimes w_0.$$

b) If ϕ_{rs} and $\phi_{r's'}$ are different vectors which belong to V_m, then $r\neq r'$ and $s\neq s'$. Hence, different eigenvectors from the list have different eigenvalues, which implies the linear independence of the list. Since the number of the vectors in the list is $m+1=\dim V_m$, the list constitutes a basis. □

Consider now the case when $\Lambda=0$, or, equivalently, $\beta=\pm\alpha\neq 0$. Then we will show that, in each of subspaces V_m, $\tilde{\mathcal{L}}$ is not diagonalizable but similar to the Jordan block.

Theorem 4.3. *If $\alpha = \pm\beta (\neq 0)$, then $\phi_0^m := (z_2 \pm z_1)^m w_0 \otimes w_0 \in V_m$ is the eigenvector of $\tilde{\mathcal{L}}^m$ with the eigenvalue 0, and a system of adjoint vectors for $-\tilde{\mathcal{L}}^m$ is given by*

$$\phi_j^m = \frac{1}{j!\alpha^j} z_2^j (z_2 \pm z_1)^{m-j} w_0 \otimes w_0, \tag{4.5}$$

$j = 1, 2, \dots, m$. The list $\{\phi_j^m\}_{j=0}^m$ forms a basis of V_m, and the matrix of $-\tilde{\mathcal{L}}^m$ in this basis is the $(m+1) \times (m+1)$ Jordan block.

Proof. We give the proof for the case $\beta = \alpha \neq 0$. First, notice that in this case,

$$-\tilde{\mathcal{L}} = \alpha(z_2 z_2^* - z_1 z_1^* + z_1 z_2^* - z_2 z_1^*) = \alpha(z_2 + z_1)(z_2^* - z_1^*).$$

Notice also that $[z_1 + z_2, z_2^* - z_1^*] = [z_1, z_2^*] - [z_1, z_1^*] + [z_2, z_2^*] - [z_2, z_1^*] = 0$, and therefore,

$$\begin{aligned} -\tilde{\mathcal{L}}\phi_0^m &= \alpha(z_2 + z_1)(z_2^* - z_1^*)(z_1 + z_2)^m w_0 \otimes w_0 \\ &= \alpha(z_1 + z_2)^{m+1}(z_2^* - z_1^*) w_0 \otimes w_0 \\ &= 0. \end{aligned}$$

Thus, ϕ_0^m is the eigenvector with the eigenvalue 0. To prove that (4.5) gives a system of adjoint vectors, take $k < m$, and calculate

$$\begin{aligned} -\tilde{\mathcal{L}}\phi_{k+1}^m &= \alpha(z_2 + z_1)(z_2^* - z_1^*) \frac{1}{\alpha^{k+1}(k+1)!} z_2^{k+1}(z_1 + z_2)^{m-k-1} w_0 \otimes w_0 \\ &= \frac{1}{\alpha^k (k+1)!}(z_1 + z_2)^{m-k}[z_2^{k+1}(z_2^* - z_1^*) + (k+1) z_2^n] w_0 \otimes w_0 \\ &= \frac{1}{\alpha^k k!} z_2^k (z_1 + z_2)^{m-k} w_0 \otimes w_0 \\ &= \phi_k^m. \end{aligned}$$

□

Since $\mathcal{L}^m$ and $\tilde{\mathcal{L}}^m$ differ by a multiple of the identity I_{m+1}, they have the same systems of eigenfunctions (and adjoint functions), and the corresponding eigenvalues are related by (4.2). Hence, the following statements are immediate corollaries of Theorem 4.2 and Theorem 4.3. Set $\mathcal{L} = \mathcal{L}_{a,b;Z-Y}$.

4.2. The case $\alpha^2 > \beta^2$: a basis of real eigenfunctions exists.

Theorem 4.4. *Let $\alpha^2 > \beta^2$. Then*

a) the list ϕ_{rs}, $s = 0, \pm 1, \dots$, $r = 0, 1, \dots$, of eigenfunctions of $\mathcal{L}$ constitutes a basis in $L_2(\mathbb{R}^2; \mathbb{R})$; the corresponding eigenvalues are $\lambda_{rs} = -[s\Lambda + (2r + |s| + 1)(\mu_1 + \mu_2)]/2 - \tilde{d}_0$;

b) let $g_{2,rs}$, $s = 0, \pm 1, \dots$, $r = 0, 1, \dots$, be the coefficients in the eigenfunction expansion of g_2. Then

$$v(x'', \tau) = \sum_{s=-\infty}^{+\infty} \sum_{r=0}^{\infty} e^{\lambda_{rs}\tau} g_{2,rs} \phi_{rs}(x''). \tag{4.6}$$

4.3. **The case $\alpha^2 < \beta^2$: a basis of complex eigenfunctions exists.** In this case, the statements of Theorem 4.4 can be repeated word by word, with $\mathcal{L}$ an operator in $L_2(\mathbb{R}^2;\mathbb{C})$. To enhance the speed and reduce the computational errors of a numerical realization, it may be useful (albeit by no means necessary) to recast the theorem in terms of a basis in the real space $L_2(\mathbb{R}^2;\mathbb{R})$. For $s=0$, the eigenvalues $\lambda_{r0} := -(2r+1)(\mu_1+\mu_2)/2 - \tilde{d}_0$ and eigenvectors $\phi_{r0} = Q_2^r w_0 \otimes w_0$ are real. For $s>0$, the eigenvalues $[\mp s\Lambda + (m+1)(\mu_1+\mu_2)]/2 - \tilde{d}_0$ are complex-adjoint; the corresponding eigenvectors ϕ_{rs} and $\phi_{r,-s}$ are also complex-adjoint as well. To see this, note that z_j and z_j^* act in the real space, the coefficients of Q_2 are real, and the ones of Q_1^+ and Q_1^- are complex-adjoint. Hence, $\overline{Q_2 f} = Q_2 \bar{f}$ and $\overline{Q_1^- f} = \overline{((\alpha-\Lambda)z_1 + \beta z_2)f} = ((\alpha+\Lambda)z_1 + \beta z_2)\bar{f} = Q_1^- \bar{f}$, for any function f. Since w_0 is real-valued, we conclude that $\overline{\phi_{r,s}} = Q_2^r (Q_1^-)^s w_0 \otimes w_0 = \phi_{r,-s}$. Once we have a basis of eigenvectors of the complexification of the real operator, the standard proof gives

Theorem 4.5. *Let $\alpha^2 < \beta^2$. Then*

a) vectors $\varphi_{r0} := \phi_{r0}, r = 0,1,\dots,$ and $\varphi_{rs} := \operatorname{Re}\phi_{rs}$, $\varphi_{r,-s} := \operatorname{Im}\phi_{rs}$, $r = 0,1,\dots, s = 1,2,\dots$ form a basis in $L_2(\mathbb{R}^n)$. The (infinite) matrix of $\mathcal{L}$ in this basis is block-diagonal;

b) the diagonal part consists of the diagonal elements λ_{r0} with the corresponding eigenvectors $\varphi_{r0}, r = 0,1,\dots,$ and 2×2 blocks, of the form

$$M_{rs} = \begin{bmatrix} \Lambda_{rs}^{re} & \Lambda_{rs}^{im} \\ -\Lambda_{rs}^{im} & \Lambda_{rs}^{re} \end{bmatrix}, \tag{4.7}$$

where $\Lambda_{rs}^{re} = \operatorname{Re}\lambda_{rs}, \Lambda_{rs}^{im} = \operatorname{Im}\lambda_{rs}$, and $\lambda_{rs} = -[s\Lambda + (2r+s+1)(\mu_1+\mu_2)]/2 - \tilde{d}_0$, $r = 0,1,\dots, s = 1,2,\dots$;

c) M_{rs} is the matrix of $\mathcal{L}$ in the invariant subspace spanned by

$$\varphi_{rs} := \operatorname{Re}\phi_{rs} = \frac{1}{2} Q_2^r (Q_1^s + (Q_1^-)^s) w_0 \otimes w_0$$

and

$$\varphi_{r,-s} := \operatorname{Im}\phi_{rs} = \frac{1}{2i} Q_2^r (Q_1^s - (Q_1^-)^s) w_0 \otimes w_0;$$

d) let $g_{2,rs}$ be the coefficients in the eigenfunction expansions of g_2 in the basis $\varphi_{rs}, r = 0,1,2,\dots, s = 0,\pm 1,\dots$. Then

$$\begin{aligned} v(\cdot,\tau) \;=\; & \sum_{r=0}^{\infty} e^{\lambda_{r0}\tau} g_{2,r0}\varphi_{r0}(\cdot) \\ & + \sum_{s=1}^{+\infty}\sum_{r=0}^{\infty} e^{\Lambda_{rs}^{re}\tau} \Big\{ \Big(\cos(\Lambda_{rs}^{im}\tau) g_{2,rs} + \sin(\Lambda_{rs}^{im}\tau) g_{2,r,-s}\Big)\varphi_{rs}(\cdot) \\ & \qquad + \Big(-\sin(\Lambda_{rs}^{im}\tau) g_{2,r,s} + \cos(\Lambda_{rs}^{im}\tau) g_{2,r,-s}\Big)\varphi_{r,-s}(\cdot)\Big\}. \end{aligned} \tag{4.8}$$

Remark 4.1. Notice that λ_{00} is smaller (in absolute value) than the real part of any other eigenvalue, hence the leading term of the eigenvalue expansion does not oscillate.

However, the next two terms (and infinitely many other terms) do oscillate, and, hence, a certain wiggling of the price can be observed not very far from maturity. If the oscillation is not observed in empirical data, we conclude that, when fitting the model to the data, one should impose the restriction $\alpha^2 \geq \beta^2$ on parameters of the model. On the other hand, if after the subtraction of the leading exponential term, the residue does oscillate, one must use the model under an additional restriction $\alpha^2 < \beta^2$ on the parameters of the model.

4.4. The case $\alpha^2 = \beta^2$: all $\mathcal{L}^m$ are similar to Jordan blocks.

Theorem 4.6. *Let $\alpha = \pm\beta (\neq 0)$, then*

a) vectors $\phi_j^m, m = 0, 1, 2, \dots, j = 0, 1, \dots, m$, form a basis of $L_2(\mathbb{R}^2)$.

b) the (infinite) matrix of $\mathcal{L}$ in this basis is block-diagonal, with one diagonal element $-(\mu_1+\mu_2)/2 - \tilde{d}_0$, the corresponding eigenfunction being $f_0 \otimes f_0$. The remaining diagonal pieces are $(m+1)\times(m+1)$ Jordan blocks, $m = 1, 2, \dots$. For each $m \geq 1$, the corresponding eigenfunction and adjoint functions are given by (4.5), $j = 0, 1, \dots, m$, and the eigenvalue is $\Lambda_m = -(m+1)(\mu_1+\mu_2)/2 - \tilde{d}_0$;

c) let $g_{2,j}^m$ be the coefficients in the eigenfunction expansions of g_2 in the basis $\phi_j^m, m = 0, 1, 2, \dots, j = 0, 1, \dots, m$. Then

$$v(\cdot,\tau) = e^{\Lambda_0\tau} g_0^0 w_0 \otimes w_0 + e^{\Lambda_1\tau}\left\{(g_0^1 - \tau g_1^1)\phi_0^1 + g_1^1\phi_1^1\right\} + \cdots . \tag{4.9}$$

Remark 4.2. a) To save space, we omit the following terms in the standard form of the solution of the Cauchy problem, which are especially messy when there are Jordan blocks. The reader can easily continue the expansion in (4.9).

b) The term $-e^{\Lambda_1\tau}\tau g_1^1\phi_0^1$ is hump-shaped, and if $\Lambda_1 > \Lambda_0$ is close to Λ_0, it can lead to weakly hump-shaped price curves. Weakly means that the hump can be seen after the multiplication of the price by $e^{-\Lambda_0\tau}$.

5. OU state vector with non-normal innovations

5.1. **Subordination.** Consider the subordination of the OU process in the main body of the paper. Subordination can be used to transform Markov processes into Markov processes. For details, see e.g. Chapter 6 in Sato (1999). We apply subordination to the Markov process with the infinitesimal generator (1.5). Denote this process by $X_r(t)$; its infinitesimal generator is $\mathcal{L} = L - r$.

A subordinator is a Lévy process taking values in $[0, +\infty)$, which implies that its trajectories are non-decreasing. The Laplace transform of the law of the subordinator Z can be expressed as $E[\exp(-\lambda Z_t)] = \exp(-t\Phi(\lambda))$, where $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is called the Laplace exponent of Z. The Laplace exponent is given by

$$\Phi(\lambda) = \gamma\lambda + \int_0^{+\infty} (e^{\lambda s} - 1)F(ds), \tag{5.1}$$

where $\gamma \geq 0$, and $F(dy)$ is the Lévy density of Z, which satisfies

$$\int_{-\infty}^{+\infty} \min\{1, |y|\} F(dy) < +\infty. \tag{5.2}$$

The subordinated process $Y(t) = X_r(Z(t))$ is a Markov process with the infinitesimal generator

$$\mathcal{L}_Z = \Phi(\mathcal{L}) = \gamma\mathcal{L} + \int_0^{+\infty} (e^{s\mathcal{L}} - 1) F(ds). \tag{5.3}$$

Since $\mathcal{L}_Z = \Phi(\mathcal{L})$, the backward parabolic equation for the price assumes the form

$$(\partial_\tau - \Phi(\mathcal{L}))u(x,\tau) = 0, \quad \tau > 0, \tag{5.4}$$

$$u(x,0) = g(x). \tag{5.5}$$

The function $e^{s\mathcal{L}}$ can be calculated using a basis of eigenfunctions (and adjoint functions) of $\mathcal{L}$, whereupon we find the (block-) diagonal form of $\mathcal{L}_Z = \Phi(\mathcal{L})$. After that we can apply the eigenfunction expansion method to the pricing problem (5.4)–(5.5). The first (straightforward) result is

Lemma 5.1. *If λ is a real (resp., complex) eigenvalue of $\mathcal{L}$, and f is a corresponding real (resp., complex) eigenvector, then f is an eigenvector of $\mathcal{L}_Z$ with eigenvalue $\Phi(\lambda)$.*

From the Dominant Convergence Theorem, under condition (5.2),

$$\Phi'(\lambda) = \gamma + \int_0^{+\infty} e^{s\lambda} s F(ds) \to \gamma$$

as $\operatorname{Re}\lambda \to -\infty$. It follows that if $\gamma = 0$, then the real parts of the eigenvalues $\Phi(\lambda_j)$ of $\Phi(\mathcal{L})$ decrease (to $-\infty$) slower than $\operatorname{Re}\lambda_j$, as $\operatorname{Re}\lambda_j \to -\infty$. Hence, if $\gamma = 0$, then after subordination, it may be necessary to use more terms of the asymptotic expansion than in the non-subordinated case.

Example 5.1. Let $\gamma = 0$ and

$$F(dy) = c y^{-\nu-1} e^{-py} dy, \tag{5.6}$$

where $c > 0, p > 0$, and $\nu \in (0,1)$. Then

$$\Phi(\lambda) = c\Gamma(-\nu)[(p-\lambda)^\nu - p^\nu] \sim c\Gamma(-\nu)(-\lambda)^\nu, \quad \text{as } \operatorname{Re}\lambda \to -\infty,$$

where $\Gamma(z)$ is the Gamma function. We see that for ν close to 0, many eigenvalues may be rather small, and so the corresponding terms in the eigenfunction expansion formulas cannot be neglected. In numerical experiments in Section 7, for $\nu = 0.05$, we need twice as many terms of the asymptotic expansion to ensure the same precision as in the non-subordinated case. With $\nu = 0$, (5.6) leads to the subordinator used in the variance gamma model. In this case,

$$\Phi(\lambda) = c[\ln p - \ln(p-\lambda)] \sim c\ln(-\lambda), \quad \text{as } \operatorname{Re}\lambda \to -\infty,$$

and the rate of growth (in absolute value) of the eigenvalues $\Phi(\lambda_j)$ becomes very slow.

In the limit $\nu \to 1-0$, we obtain the one-sided version of KoBoL processes of order 1: $\Phi(\lambda) = (p-\lambda)\log(p-\lambda) - p\log p$ (see equation (3.5) in Boyarchenko and Levendorskiĭ (2002)). In this case, $\Phi(\lambda) \sim (-\lambda)\log(-\lambda)$ as $\lambda \to -\infty$.

5.2. **Eigenfunction expansion method.** Now we give the reformulations of the main theorems for models in random time. The first theorem is applicable when the basis of real eigenvalues exists. In particular, this theorem is applicable to two-factor models with $\alpha^2 > \beta^2$.

Theorem 5.2. *If there exists a basis in $L_2(\mathbb{R}^n;\mathbb{R})$ of the eigenfunctions ϕ_j with the corresponding eigenvalues λ_j, then these eigenfunctions are eigenfunctions of $\mathcal{L}_Z$ with the eigenvalues $\lambda_{Z,j} = \Phi(\lambda_j(\mathcal{L}))$.*

The solution to the pricing problem is given by the same formula but with $\Phi(\lambda_j)$ instead of λ_j.

The following two theorems are analogs of Theorem 3.2, part b), and Theorem 4.5, respectively; the notation remains the same as in these two theorems.

Theorem 5.3. *Let $g_2 = \sum_{m\ge 0} g_2^m$, where $g_2^m \in V_m \cong \mathbb{R}^{N_m}$ be the decomposition of $g_2(\cdot)$ corresponding to the direct sum decomposition $L_2(\mathbb{R}^n) \cong \oplus_{m\ge 0} V_m$.*

Then

$$v(\cdot,\tau) = \sum_{m\ge 0} e^{\tau\Phi(\mathcal{L}^m)} g_2^m(\cdot) \tag{5.7}$$

is the solution to the pricing problem in random time.

Theorem 5.4. *Let $n = 2$ and $\alpha^2 < \beta^2$. Then*

a) in the basis $\varphi_{r0} := \phi_{r0}, r = 0,1,\dots,$ and $\varphi_{rs} := \operatorname{Re}\phi_{rs}$, $\varphi_{r,-s} := \operatorname{Im}\phi_{rs}$, $r = 0,1,\dots, s = 1,2,\dots$ of $L_2(\mathbb{R}^2)$, the (infinite) matrix of $\Phi(\mathcal{L})$ is block-diagonal;

b) the diagonal part consists of the diagonal elements $\Lambda_{r0} := \Phi(\lambda_{r0})$ with the corresponding eigenvectors $\varphi_{r0}, r = 0,1,\dots,$ and 2×2 diagonal blocks of the form (4.7), where $\Lambda^{re}_{rs} = \operatorname{Re}\Phi(\lambda_{rs})$, $\Lambda^{im}_{rs} = \operatorname{Im}\Phi(\lambda_{rs})$, and $\lambda_{rs} = -((2r+s+1)(\mu_1+\mu_2)/2 - \tilde{d}_0 - s\Lambda/2$, $r = 0,1,\dots, s = 1,2,\dots$;

c) matrix M_{rs} in (4.7) is the matrix of $\Phi(\mathcal{L})$ in the invariant subspace spanned by $\varphi_{rs} := \operatorname{Re}\phi_{rs}$ and $\varphi_{r,-s} := \operatorname{Im}\phi_{rs}$, respectively;

d) let $g_{2,rs}$ be the coefficients in the eigenfunction expansions of g_2 in the basis $\varphi_{rs}, r = 0,1,2,\dots, s = 0,\pm 1,\dots$. Then (4.8) holds with $\Lambda^{re}_{rs} = \operatorname{Re}\Phi(\lambda_{rs})$, $\Lambda^{im}_{rs} = \operatorname{Im}\Phi(\lambda_{rs})$.

It remains to consider two-factor models in the case $\alpha^2 = \beta^2$, when in the basis described in Theorem 4.6, $\mathcal{L}$ is block-diagonal. Then we can apply (5.7) with $\Phi(\mathcal{L}^0)$ scalar; for $m \ge 1$, $\Phi(\mathcal{L}^m)$ is the $(m+1)\times(m+1)$ Jordan block, with the eigenvalue $\Phi(\Lambda_m)$, where $\Lambda_m = -(m+1)(\mu_1+\mu_2)/2 - \tilde{d}_0$. However, calculations are easier in the basis of the eigenfunction and adjoint functions of $-\mathcal{L}^m$. Let $S_{m+1} = \sum_{j\ge 1} e_{j,j+1}$ be the matrix with the only non-zero entries (ones) above the diagonal. In the basis of the

eigenvector and adjoint vectors, $-\mathcal{L}^m = -\Lambda_m I_{m+1} + S_{m+1}$. Set $\Psi(\tau,\lambda) = \exp(\tau\Phi(\lambda))$. Since S_{m+1} is nilpotent: $S_{m+1}^{m+1} = 0$, we can use the Taylor formula and calculate

$$\begin{aligned}\exp[\tau\Phi(\mathcal{L}^m)] &= \sum_{j=0}^{m} \frac{(-1)^j}{j!}\Psi_\lambda^{(j)}(\tau,\Lambda_m)S_{m+1}^j \\ &= \exp(\tau\Phi(\Lambda_m))\sum_{j=0}^{m}\frac{(-1)^j}{j!}\Psi_\lambda^{(j)}(\tau,\Lambda_m)\exp[-\tau\Phi(\Lambda_m)]S_{m+1}^j,\end{aligned}$$

where $(-1)^j\Psi_\lambda^{(j)}(\tau,\Lambda_m)\exp[-\tau\Phi(\Lambda_m)]$ are polynomials of degree j in τ, with the coefficients expressed via derivatives of Φ evaluated at Λ_m:

$$\begin{aligned}\Psi_\lambda^{(0)}(\tau,\Lambda_m)\exp[-\tau\Phi(\Lambda_m)] &= 1,\\ \Psi_\lambda^{(1)}(\tau,\Lambda_m)\exp[-\tau\Phi(\Lambda_m)] &= -\tau\Phi'(\Lambda_m),\\ \Psi_\lambda^{(2)}(\tau,\Lambda_m)\exp[-\tau\Phi(\Lambda_m)] &= \tau^2(\Phi'(\Lambda_m))^2 + \tau\Phi''(\Lambda_m),\end{aligned}$$

etc.

6. Calculation of the expansion of a pay-off function

6.1. **General pay-off function.** All elements of all bases constructed in Section 3 and Section 4 are linear combinations of the functions

$$w_\alpha(x'') = \frac{1}{2^{|\alpha|/2}}H_\alpha(x'')e^{-||x''||^2/2}, \quad \alpha\in(\mathbb{Z}_+)^n,$$

where $H_\alpha(x'') = \prod_{j=1}^n H_j(x_j'')$, and H_j are the Hermite polynomials. The list $\{w_\alpha\}$ constitutes an orthogonal basis, and

$$||w_\alpha||^2 = \pi^{|\alpha|/2}\alpha!,$$

where $\alpha! = \prod_{j=1}^n \alpha_j!$. The coordinates of $\tilde{g}_2$ in the basis $\{w_\alpha\}$ are

$$\tilde{g}_{2;\alpha} = \frac{1}{\pi^{|\alpha|/2}\alpha!}\int_{\mathbb{R}^n}\tilde{g}_2(x'')w_\alpha(x'')dx'', \tag{6.1}$$

where $\tilde{g}_2(x'') = g_2(x')$ is the function g_2 in the coordinates x''. Therefore,

$$\tilde{g}_{2;\alpha} = \frac{1}{(2\pi)^{n/2}\alpha!}\int_{\mathbb{R}^n}H_\alpha(x'')e^{-||x''||^2/2}\tilde{g}_2(x'')dx''. \tag{6.2}$$

Recall that

$$g_2(x') = g(x'-a)\exp\left[-\frac{1}{2}((Z-Y)x',x') - (b,x')\right],$$

where a and b are given by (2.5)-(2.6), and g is the pay-off function.

Let C be the matrix of the normalized eigenvectors of Z, with eigenvalues μ_j, $j = 1,2,\dots,n$. Then $x' = C\mathrm{diag}(\mu_j^{-1/2})x'' = Z^{-1/2}Cx''$ and $x'' = C^TZ^{1/2}x'$. We make the

change of variables in (6.2). Since

$$\det\left|\frac{\partial x''}{\partial x'}\right| = \left|\det C^T Z^{1/2}\right| = \left|\det Z^{1/2}\right| = \prod_{j=1}^n \mu_j^{1/2},$$

and $||x''||^2 = (Zx', x')$, we obtain

$$\tilde{g}_{2;\alpha} = \prod_{j=1}^n \mu_j^{1/2} (\alpha!)^{-1} I_\alpha,$$

where

$$I_\alpha = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \tilde{H}_\alpha(x') e^{\frac{1}{2}(H_- x', x') - (b, x')} g(x' - a) dx', \tag{6.3}$$

$\tilde{H}_\alpha(x') = H_\alpha(x'')$, and $H_- = Y - 2Z$ is the minimal Hermitian solution to (2.9) (see Appendix D for the proof of the last equality). Since κ is anti-stable, the minimal solution is negative-definite, and therefore the integral converges for any pay-off function which increases not faster than an exponential function as $x \to \infty$. Notice that $-H_- = 2Z - Y$ is the maximal Hermitian solution to (B.1) with $A = -\kappa$, and the procedures for the construction of the maximal Hermitian solution are given in Appendix B.

We factorize $-H_- = DD^T$, change the variable $y = D^T x' + D^{-1} b$, and write I_α in the form

$$I_\alpha = \frac{\exp((H_- b, b)/2)}{\sqrt{\det(-H_-)}} \times \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-||y||^2/2} G_\alpha(y) dy, \tag{6.4}$$

where $G_\alpha(y) = \tilde{H}_\alpha(x') g(x' - a)$, and $x' = (D^T)^{-1} y - (H_-)^{-1} b$. Unless the payoff g is of a special structure which allows one to calculate the integral in (6.4) explicitly or simplify it, we use the importance sampling technique.

6.2. **Calls, puts, caps and floors.** In many cases, the payoff of a contingent claim vanishes outside a certain domain in the state space, and the domain can be rather small. Then the numerical scheme above may be non-optimal because the overwhelming majority of simulations are not needed (they contribute zero to the result). Typical examples are caps and European call options on a zero coupon bond. Pricing of puts, caps and floors can be reduced to pricing of the European call option with the payoff of the form $g(x) = (e^{q(x)} - K)_+$, where $q(x) = (ax, x)/2 + (b, x) + c$ is a quadratic polynomial, and contingent claims with payoffs of the form $e^{q(x)}$. The latter can be explicitly calculated - see e.g. Leippold and Wu (2002). Here we consider the call option. Let w_α be a normalized eigenfunction. Then we can calculate the coefficient

$$c_\alpha = \int_{\mathbb{R}^n} w_\alpha(x) (e^{q(x)} - K)_+ dx \tag{6.5}$$

using simulations of a uniform distribution, which give non-zero contributions to the integral. First, we find a matrix Σ such that $a = \Sigma\Sigma^T$, and make an affine change of

variables $x = (\Sigma^T)^{-1}x' - a^{-1}b$. Then $q(x) = (x', x')/2 - q_0$, where $q_0 = b^T a^{-1} b/2 - c$. The second change of variables $x_1' = r\cos\phi$, $x_2' = r\sin\phi$ gives

$$\begin{aligned} c_\alpha &= \frac{e^{-q_0}}{\sqrt{\det a}} \int_0^{2\pi} \int_0^{+\infty} \tilde{w}_\alpha(r,\phi) \left(e^{-r^2/2} - Ke^{q_0}\right)_+ d\left(\frac{r^2}{2}\right) d\phi \\ &= \frac{e^{-q_0}}{\sqrt{\det a}} (I_1 - Ke^{q_0} I_2), \end{aligned}$$

where $\tilde{w}(r,\phi) = w(x(x'(r,\phi)))$,

$$\begin{aligned} I_1 : &= \int_0^{2\pi} \int_0^{\sqrt{-2(\log K + q_0)}} \tilde{w}_\alpha(r,\phi) e^{-r^2/2} d\left(\frac{r^2}{2}\right) d\phi \\ &= \int_0^{2\pi} \int_0^{Ke^{q_0}} \tilde{w}_\alpha(r(\rho_1),\phi) d\rho_1 d\phi, \end{aligned}$$

where $\rho_1 = e^{-r^2/2}$, and

$$I_2 := \int_0^{2\pi} \int_0^{-\log K - q_0} \tilde{w}_\alpha(\sqrt{2\rho_2},\phi) d\rho_2 d\phi.$$

The integrals I_1 and I_2 can be calculated using simulation of the uniform distribution or other numerical integration procedures.

6.3. **Reduction to one-dimensional integrals.** Further, one can reduce calculations of integrals (6.5) over an ellipsoid in $\mathbb{R}^n$ to calculation of a number of integrals over an ellipsoid in $\mathbb{R}^{n-1}$, with new integrands being expressed in terms of the cumulative distribution of the standard normal variable. In particular, in the case $n = 2$, we obtain integrals over a segment, which can be calculated very fast and accurately. However, the number of integrals increases very fast with $|\alpha|$, and therefore, this procedure is practical for moderate $|\alpha|$ only. In the numerical examples below, we calculated c_α, $|\alpha| \le 3$, using the reduction to one-dimensional integrals, which improved the accuracy and speed. The remaining coefficients, which do not make a significant contribution to the final result, were calculated using the importance sampling technique with moderate number of simulations.

6.4. **Calculation of coefficients in closed form.** If the pay-off is of the form $g(x) = e^{q(x)}P(x)$, where q is quadratic and P is a polynomial, then the integral $\tilde{I}_\alpha$ can be calculated in closed form. Pricing of swaps can be reduced to this case. For small $|\alpha|$, one can easily write down the result by hand, but for larger $|\alpha|$, it is reasonable to write a program which will calculate the result explicitly using analytical expressions for the integrals

$$\tilde{I}(q,p,m) = \int_{-\infty}^{+\infty} x^m e^{-\frac{1}{2}q^2x^2 - px} dx. \tag{6.6}$$

If the pay-off is of the form $g_0(x) = e^{q(x)}h((l,x))$, where $l \in \mathbb{R}^n$, then making an appropriate change of variables, we can reduce the calculation of the integral in (6.3) to

calculation of the product of integrals of the form (6.6), and of an integral of the form

$$\tilde{I}(q,p,m;h)=\int_{-\infty}^{+\infty}x^m e^{-\frac{1}{2}q^2x^2-px}h(x)dx. \tag{6.7}$$

For h of a simple form such as $h(x)=(e^x-K)_+$, the integrals (6.7) can be explicitly calculated in terms of the cumulative normal distribution function and elementary functions.

6.5. **Calculation of coefficients in the asymmetric case.** The procedures described above allows us to calculate the coefficients of $\tilde{g}_2$ in the basis $\{w_\alpha\}_{|\alpha|\geq 0}$. If κ is symmetric, this basis is the basis of the eigenfunctions of $\mathcal{L}$. If κ is non-symmetric, we consider separately each invariant space V_m. In the two-factor case, the restriction $\mathcal{L}_m$ on V_m has $m+1$ linearly independent eigenvectors, which can be easily found as linear combinations of the vectors of the basis $\{w_{m-j}\otimes w_j\}_{j=0}^m$. The calculation of the coefficients of these linear combinations is a simple recurrent procedure of construction of the array of coefficients of a polynomial given the arrays of the coefficients of some other polynomials. Let T_m be the matrix of the coordinates of the eigenfunctions $\phi_{r,s}, r\geq 0, 2r+|s|=m$, in this basis, and $\tilde{g}_2^m$ the column vector of coordinates of $\tilde{g}_{2,m}$ in this basis. Finally, denote by Λ_m the diagonal form of $\mathcal{L}_m$ in the basis of the eigenfunctions $\phi_{r,s}, r\geq 0, 2r+|s|=m$, and by ϕ^m the row vector consisting of eigenfunctions. Then we calculate the m-th component of the solution using the following formula:

$$v^m(y,\tau)=\exp[-\frac{1}{2}(Yx',x')+(b,x')]\phi^m(x'')T_m e^{\Lambda_m\tau}T_m^{-1}\tilde{g}_2^m,$$

where x'' and x' are the affine functions of y specified earlier.

6.6. The final remark is: as we discovered, it is necessary to calculate all the matrices, vectors and scalars used in the explicit formulas for eigenvectors and bond prices in the expressions for the payoffs of contingent claims with very high precision. Equivalently, when solving the system of Riccati equations, it is necessary to choose very small step. We used the explicit formulas from Levin (1959) (see Boyarchenko and Levendorskiĭ (2006) for details), the built-in MATLAB procedure expm to find $A(\tau)$ and $B(\tau)$, and Simpson's rule to calculate the integral which defines the scalar $C(\tau)$. We found that step of order $2\cdot 10^{-6}$ was needed; otherwise, the relative error could increase 10-fold.

7. Numerical examples

In this section, we consider numerical examples for a two-factor model. The parameters of the process are $d_0=0.0088$,

$$\Gamma=\begin{bmatrix}0.0176 & -0.0132\\ -0.0132 & 0.1100\end{bmatrix}\quad \kappa=\begin{bmatrix}0.264 & 0\\ 0.1 & 0.66\end{bmatrix}\quad d=\begin{bmatrix}0.0066\\ -0.022\end{bmatrix}\quad x=\begin{bmatrix}0.735\\ -0.525\end{bmatrix}.$$

TABLE 1. Yields and forward rates on bonds

Maturity		Percentage Errors			
Years	N_∞	N_9	N_4	N_1	N_0
0.1	0.05183	−2.526	6.788	−535.631	144.703
0.5	0.05674	−2.536	−1.815	−68.386	37.632
1	0.06047	−2.163	−2.037	−21.967	19.063
5	0.06618	−0.231	−0.231	−0.472	1.428
10	0.06640	−0.018	−0.018	−0.024	0.176

Maturity		Percentage Errors			
Years	N_∞	N_9	N_4	N_1	N_0
0.1	0.05335	−2.658	−4.684	50.193	23.522
0.5	0.06160	−2.299	−2.984	26.122	6.994
1	0.066083	−1.366	−1.584	13.932	−0.271
5	0.067142	0.429	0.427	1.159	−1.844
10	0.06625	0.062	0.062	0.097	−0.516

Exact values and the percentage errors of approximations obtained with the EE method. Upper panel: yields; lower panel: forward rates. Parameters: $d_0 = 0.0088$, $\Gamma_{11} = 0.0176$, $\Gamma_{12} = \Gamma_{21} = -0.0132$, $\Gamma_{22} = 0.1100$, $\kappa_{11} = 0.264$, $\kappa_{21} = 0$, $\kappa_{12} = 0.1$, $\kappa_{22} = 0.66$, $d = [0.0066; -0.022]$, $x = [0.735; -0.525]$

7.1. **Yields and forward rates.** In Table 1, we show the yield, Y, and forward rate, F, for different times to maturity. N_∞ indicates that the values are calculated using the standard reduction to the Riccati equations. We also show the percentage errors of approximations F_N and Y_N. The label N_j means that the first $j+1$ invariant subspaces are used, equivalently, the first $j(j+1)/2$ terms of the eigenfunction expansion.

7.2. **Call options.** Consider the European call option with expiry date T and strike K written on a zero coupon bond maturing at time $T_1 = T + \delta$, $\delta > 0$. In Table 2, we show the option price for $\delta = 1$, several values of T: $T = 0.1, 0.5, 1, 5, 10$, and several strike prices K. We also show relative errors of different approximations. To demonstrate the accuracy of the method, we concentrate on out-of-the-money options and at-the-money options. For a chosen value of x, and the smallest strike price in our examples, the payoff equals 0.005. The other values of the strike price increase by 0.005 increments. Thus, we show one in-the-money option, one at-the-money, and 3 out-of-the-money options. We see that even for the latter options, of very small values, the relative errors of the eigenfunction expansion method are small, even very close to expiry.

In Table 3, we fix time to expiry of the option: $T = 1$, and calculate option prices on bonds of maturities $\delta = 30$ days, 90 days, half a year, one year and 2 years. The set of strike prices depends on δ so that one option is in-the-money, one at-the-money, and 3 out-of-the-money.

The time of calculation is determined, mainly, by the time needed to evaluate the integrals in formulas for the coefficients of the eigenfunction expansion. If a sufficiently large number of terms of the asymptotic expansion is chosen then the total error of the method is determined by accuracy of calculation of integrals in formulas for the coefficients c_α. To calculate the results shown in Tables 2 and 3, we calculated c_α, $|\alpha| \le 3$, using the reduction to one-dimensional integrals. Coefficients c_α with $|\alpha| \ge 4$ were calculated using the importance sampling technique; the total CPU time was 8900 sec. When we used 200k simulation for each α, the CPU time was 806 sec. to evaluate the call option price at 5 strike prices and 5 time moments; with the same number of simulations, prices at 100 strike prices and 100 time moments were calculated in 3541 sec. At time $T = 5$, the relative errors were less than 0.2%. Calculations were made in MATLAB; a PC with the characteristics Intel(R) Pentium(R) 4 CPU 2.20 GHz 256 MB was used. A detailed algorithm is given in Appendix F.

7.3. **Swaptions.** Consider a forward start payer swap settled in arrears, with notional principal 1. Let $T_0, \dots, T_{m-1}$ be reset dates, and $T_1, \dots, T_m$ settlement dates; $T_j - T_{j-1} = \delta$ is the same for all $j = 1, \dots, m$. Let ρ be a preassigned fixed rate of interest, and set $c_j = \rho\delta$ for $j = 1, \dots, m-1$, and $c_m = \tilde\delta := 1 + \rho\delta$. Then at time $t < T_0$, the price of the forward start payer swap settled in arrears is

$$FS(x,t) = B(t,T_0,x) - \sum_{j=1}^m c_j B(t,T_j,x)$$

(see, e.g., equation (16.3) in Musiela and Rutkowski (1998)). The owner of a payer swaption with strike rate ρ, maturing at time $T = T_0$, has the right to enter at time T the underlying forward payer swap settled in arrears. Therefore, the price $PS(x,t)$ of this payer swaption is the price of the European option with maturity date T and payoff function

$$g_{\rm pay}(x) = (1 - \sum_{j=1}^m c_j B(T,T_j,x))^+.$$

Similarly, the price $RS(x,t)$ of the receiver swaption is the price of the European option with maturity date T and payoff function

$$g_{\rm rec}(x) = (\sum_{j=1}^m c_j B(T,T_j,x) - 1)^+.$$

Since $FS(x,t) = PS(x,t) + RS(x,t)$, and $FS(x,t)$ can be easily calculated, we will calculate $RS(x,t)$ (it is easier to calculate the integral over a finite set).

Consider now a receiver swaption with four reset dates and $\delta := T_j - T_{j-1} = 0.25$, maturing at time $T = T_0$. In Table 4, we show the option price P for different values

TABLE 2. Call Option for $\delta := T_1 - T = 1$ and various values of T and K

$T = 0.1$			Percentage Errors				
$\log K$	Payoff*	$N_\infty^\dagger$	N_{20}	N_{10}	N_5	N_1	N_0
-0.0447	0	0.834	-3.196	74.146	215.308	150.043	204.471
-0.0499	0	1.944	-1.595	31.898	118.651	79.419	116.809
-0.0552	0	3.656	-0.630	14.857	74.680	47.034	76.357
-0.0605	0	5.984	-0.064	6.713	50.146	28.876	53.462
-0.0658	5	8.885	0.233	2.469	34.717	17.363	38.774

$T = 0.5$			Percentage Errors				
$\log K$	Payoff*	$N_\infty^\dagger$	N_{20}	N_{10}	N_5	N_1	N_0
-0.0447	0	2.638	-0.045	2.602	15.369	-4.612	11.916
-0.0499	0	4.474	-0.032	1.297	11.807	-5.461	10.268
-0.0552	0	6.698	-0.014	0.560	9.144	-6.083	8.910
-0.0605	0	9.265	0.001	0.157	7.119	-6.511	7.803
-0.0658	5	12.133	0.009	-0.056	5.542	-6.825	6.853

$T = 1$			Percentage Errors				
$\log K$	Payoff*	$N_\infty^\dagger$	N_{10}	N_5	N_3	N_1	N_0
-0.0447	0	2.914	0.262	3.138	4.446	-9.473	2.574
-0.0499	0	4.818	0.116	2.430	3.722	-8.921	2.751
-0.0552	0	7.076	0.034	1.883	3.134	-8.457	2.833
-0.0605	0	9.636	-0.009	1.461	2.657	-8.036	2.871
-0.0658	5	12.456	-0.028	1.132	2.263	-7.658	2.868

$T = 5$			Percentage Errors				
$\log K$	Payoff*	$N_\infty^\dagger$	N_5	N_3	N_2	N_1	N_0
-0.0447	0	2.324	7.017×10^{-4}	3.969×10^{-3}	0.104	-0.830	1.735
-0.0499	0	3.830	5.327×10^{-4}	3.331×10^{-3}	9.117×10^{-2}	-0.777	1.721
-0.0552	0	5.611	4.003×10^{-4}	2.795×10^{-3}	7.949×10^{-2}	-0.730	1.700
-0.0605	0	7.624	2.978×10^{-4}	2.351×10^{-3}	6.936×10^{-2}	-0.687	1.673
-0.0658	5	9.836	2.185×10^{-4}	1.984×10^{-3}	6.042×10^{-2}	-0.648	1.642

$T = 10$			Percentage Errors				
$\log K$	Payoff*	$N_\infty^\dagger$	N_5	N_3	N_2	N_1	N_0
-0.0447	0	1.686	1.757×10^{-6}	2.816×10^{-6}	9.855×10^{-4}	-0.041	0.470
-0.0499	0	2.779	1.380×10^{-6}	2.336×10^{-6}	8.591×10^{-4}	-0.038	0.460
-0.0552	0	4.069	1.074×10^{-6}	1.934×10^{-6}	7.472×10^{-4}	-0.036	0.450
-0.0605	0	5.527	8.301×10^{-7}	1.602×10^{-6}	6.503×10^{-4}	-0.034	0.439
-0.0658	5	7.128	6.361×10^{-7}	1.328×10^{-6}	5.650×10^{-4}	-0.032	0.428

$*$ In units of 10^{-3}; † In units of 10^{-3}

Exact values and the percentage errors of approximations obtained with the EE method for $T = 0.1, 0.5, 1, 5, 10$. Parameters: $d_0 = 0.0088$, $\Gamma_{11} = 0.0176$, $\Gamma_{12} = \Gamma_{21} = -0.0132$, $\Gamma_{22} = 0.1100$, $\kappa_{11} = 0.264$, $\kappa_{21} = 0$, $\kappa_{12} = 0.1$, $\kappa_{22} = 0.66$, $d = [0.0066; -0.022]$, $x = [0.735; -0.525]$

TABLE 3. Call Option for $T = 1$ and various values of $\delta := T_1 - T$ and K

$\delta = 30$ days			Percentage Errors				
$\log K$	Payoff*	$N_\infty^\dagger$	N_{10}	N_5	N_3	N_1	N_0
0.0106	0	0.0497	0.858	4.743	5.266	-13.509	-3.840
0.0057	0	0.275	0.291	2.935	3.569	-11.916	-2.662
0.0007	0	0.640	0.061	1.816	2.478	-11.029	-2.279
-0.0043	0	1.107	-0.018	1.120	1.762	-10.250	-2.005
-0.0093	5	4.452	-6.739E-03	0.106	0.548	-7.141	-0.883

$\delta = 90$ days			Percentage Errors				
$\log K$	Payoff*	$N_\infty^\dagger$	N_{10}	N_5	N_3	N_1	N_0
0.0016	0	0.073	1.016	5.647	6.224	-13.814	-3.335
-0.0034	0	0.403	0.448	3.823	4.512	-12.226	-2.150
-0.0084	0	0.936	0.217	2.695	3.411	-11.343	-1.766
-0.0135	0	3.137	-0.019	1.186	1.935	-9.844	-1.087
-0.0186	5	6.092	-0.030	0.550	1.231	-8.550	-0.593

$\delta = 0.5$			Percentage Errors				
$\log K$	Payoff*	$N_\infty^\dagger$	N_{10}	N_5	N_3	N_1	N_0
-0.0131	0	0.258	0.861	4.915	5.710	-12.525	-1.597
-0.0181	0	1.430	0.294	3.104	4.006	-10.913	-0.391
-0.0232	0	3.323	0.063	1.983	2.910	-10.016	0
-0.0284	0	5.752	-0.016	1.287	2.191	-9.229	0.280
-0.0335	5	8.590	-0.033	0.843	1.693	-8.521	0.483

$\delta = 1$			Percentage Errors				
$\log K$	Payoff*	$N_\infty^\dagger$	N_{10}	N_5	N_3	N_1	N_0
-0.0447	0	2.914	0.262	3.138	4.446	-9.473	2.574
-0.0499	0	4.818	0.116	2.430	3.722	-8.921	2.751
-0.0552	0	7.076	0.034	1.883	3.134	-8.457	2.833
-0.0605	0	9.636	-0.009	1.461	2.657	-8.036	2.871
-0.0658	5	12.456	-0.028	1.132	2.263	-7.658	2.868

$\delta = 2$			Percentage Errors				
$\log K$	Payoff*	$N_\infty^\dagger$	N_{10}	N_5	N_3	N_1	N_0
-0.1110	0	6.776	0.162	2.898	4.860	-6.657	7.591
-0.1166	0	8.957	0.088	2.434	4.307	-6.385	7.486
-0.1223	0	11.356	0.035	2.043	3.822	-6.164	7.327
-0.1279	0	13.953	0	1.706	3.383	-5.977	7.138
-0.1336	5	16.728	-0.018	1.423	3.001	-5.811	6.934

$*$ In units of 10^{-3}; † In units of 10^{-3}

Exact values and the percentage errors of approximations obtained with the EE method for $\delta := T_1 - T = 30$ days, 90 days, 0.5, 1, 2. Parameters: $d_0 = 0.0088$, $\Gamma_{11} = 0.0176$, $\Gamma_{12} = \Gamma_{21} = -0.0132$, $\Gamma_{22} = 0.1100$, $\kappa_{11} = 0.264$, $\kappa_{21} = 0$, $\kappa_{12} = 0.1$, $\kappa_{22} = 0.66$, $d = [0.0066; -0.022]$, $x = [0.735; -0.525]$, $T = 1$.

TABLE 4. Receiver Swaption: $T = 1$

			Percentage Errors			
ρ	Payoff*	$N_\infty^\dagger$	N_{20}	N_9	N_4	N_0
0.04	0	1.621	−0.006	0.463	6.830	6.083
0.05	0	5.046	0	0.097	4.239	4.786
0.06	8.070	9.794	0.001	−0.018	2.677	3.994
0.07	104.57	15.519	0	−0.045	1.688	3.396

$*$ In units of 10^{-4}; † In units of 10^{-3}

Exact values, values obtained with the EE method and the percentage errors. Parameters: $d_0 = 0.0088$, $\Gamma_{11} = 0.0176$, $\Gamma_{12} = \Gamma_{21} = -0.0132$, $\Gamma_{22} = 0.1100$, $\kappa_{11} = 0.264$, $\kappa_{21} = 0$, $\kappa_{12} = 0.1$, $\kappa_{22} = 0.66$, $d = [0.0066; -0.022]$, $x = [0.735; -0.525]$. Swap parameters: N_r=4, δ_r=0.25

of the preassigned rate of interest ρ, together with the relative errors of approximations P_N.

7.4. **Non-normal innovations.** Consider finally subordination of the same process using the subordinator with the Laplace exponent $\Phi(\lambda) = c\Gamma(-\nu)\left[(p-\lambda)^\nu - p^\nu\right]$. We fix the first two instantaneous moments m_1 and m_2 of the subordinator; then c and p are uniquely defined by the order ν. In Fig. 1, we show how the Black implied volatility curves for the call option, for $\nu = 0.05,\ 0.5,\ 0.95$, and time to expiry $T = 0.1,\ 1,\ 5,\ 10$. Subordination with $\nu = 0.05$ is close to Variance Gamma subordination, whereas the one with $\nu = 0.95$ is close to the one-sided version of KoBoL process of order $\nu = 1$: $\Phi(\lambda) = (p-\lambda)\log(p-\lambda) - p\log p$ (see equation (3.5) in Boyarchenko and Levendorskiĭ (2002)). As one would expect, the implied volatility increases as ν decreases, and the process deviates farther from the Gaussian one. We also see that the implied volatility decreases as time to maturity increases. The numerical results conform with a general conclusion about the importance of jumps for option pricing made in Johannes (2003).

In the option price space, the dependance on ν is a bit more interesting (we do not show the pictures here in order to save space). For $T = 0.1$, the price of the option is higher if ν is smaller, the opposite is true for $T = 10$. This is to be expected because as ν decreases, the relative intensity of large jumps increases. However, during a short time interval, the probability of large jumps remains too low in order to make a significant impact. For options of long maturity, the probability of a large jump during the life-time of the option becomes tangible. This causes the order of the curves to reverse.

The speed of convergence of the eigenfunction expansion decreases but even for the case $\nu = 0.05$, when the eigenvalues are increasing especially slowly, one needs only twice as many terms in order to achieve the same degree of accuracy as in the Gaussian case.

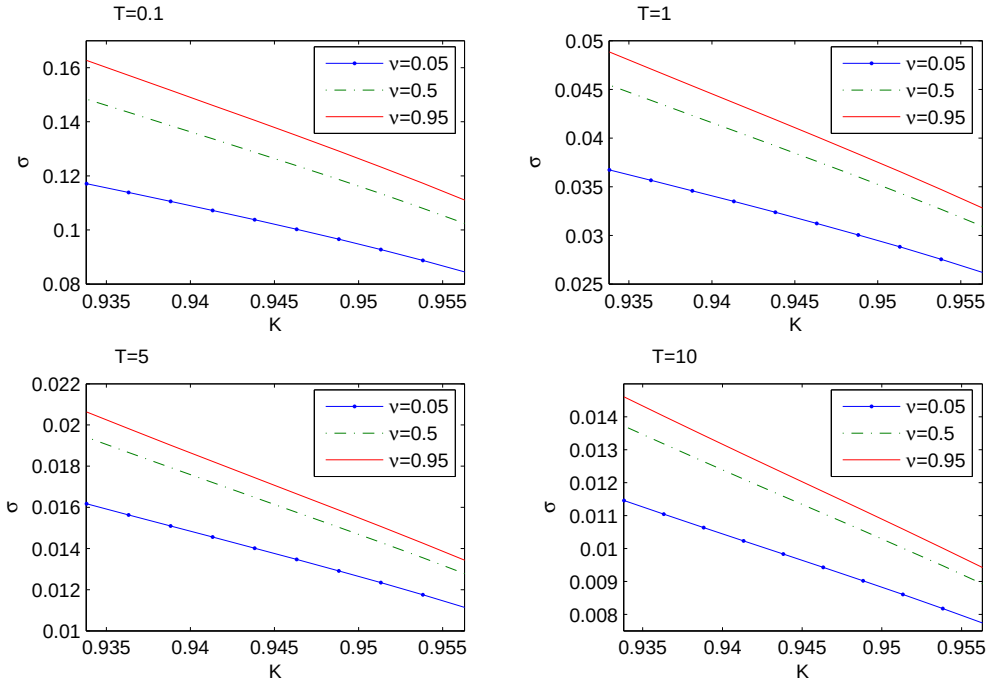

FIGURE 1. Black implied volatility of call option under subordination: dependence on the order of the subordinator, ν, and time to expiry, T. Maturity date of the bond: $T_1 = T + 1$. Parameters: $d_0 = 0.0088$, $\Gamma_{11} = 0.0176$, $\Gamma_{12} = \Gamma_{21} = -0.0132$, $\Gamma_{22} = 0.1100$, $\kappa_{11} = 0.264$, $\kappa_{21} = 0$, $\kappa_{12} = 0.1$, $\kappa_{22} = 0.66$, $d = [0.0066; -0.022]$, $x = [0.735; -0.525]$, $m_1 = 1$, $m_2 = 2$.

8. Conclusion

In the paper, we applied the eigenfunction expansion technique to models with Ornstein-Uhlenbeck state dynamics and quadratic short rate. Assuming the simplest dynamics $dX_t = -\kappa dt + dW_t$, two cases are possible: κ is symmetric, and κ is asymmetric. The symmetric case admits a straightforward reduction to the classical problem of the diagonalization of the one-dimensional harmonic oscillator, therefore we concentrate on the non-symmetric case, which leads to non-selfadjoint operators. Using the theory of continuous algebraic Riccati equations, we showed that, in this case, the pricing problem can be reduced to a sequence of systems of linear first order ODEs with constant coefficients. In the two-factor case, we showed that two cases are possible: each system can be reduced either to scalar ODEs with real or complex coefficients or to a system with a Jordan block as the matrix of coefficients. We explicitly calculated the eigenvalues and basis of eigenvectors (and adjoint vectors), which provides the reduction. Numerical examples show that the convergence of the eigenfunction expansion is good even fairly close to maturity, and excellent for options of maturity greater than 2 years. Numerical

examples are calculated for yields and forward rates, call options on a zero-coupon bond and swaptions. Notice that the prices of call options can be relatively easily calculated using the Fourier transform method because the reduction to FFT in dimension one is possible. However, this reduction is impossible in the case of swaptions, and the FFT may face certain computational difficulties even for call options. If the parameters of the computational scheme are chosen unwisely then either significant computational errors may result or CPU time may be unreasonably large. In addition, FFT exhibits a sizable systematic positive bias for out-of-the-money call and put options of short maturities (see Boyarchenko and Levendorskiĭ (2006) for details). This means that the eigenfunction expansion method is a useful addition to the existing pricing methods. We compared the performance of the eigenfunction expansion method and the improved version of the Fourier transform method constructed in Boyarchenko and Levendorskiĭ (2006), with a better speed of convergence. For options of maturity 1 year or more, the eigenfunction expansion method is sufficiently accurate and many times faster than the FFT method if one needs to calculate option prices at many points (x_j, τ_k). At time $T = 5$ or more, the relative error less than 0.2% can be achieved faster than using FFT, even if the prices at one time moment are needed.

Using subordination, we constructed exactly solvable QTSM with non-Gaussian innovations and showed that the non-Gaussian innovations matter for option pricing, and that the Black implied volatility increases as the order of the subordinator decreases. We also demonstrated that the change of option prices due to non-Gaussian innovation depends on time to maturity and the relative weight of large and small jumps. If we compare two models with the same first two instantaneous moments but different proportions of small and large jumps, then, in the model with the larger proportion of small jumps, option prices are higher close to maturity but smaller far from maturity, where the impact of very rare large jumps must be larger.

Appendix A. Proofs of technical results

Proof of (2.2)–(2.6). For a function f, let f denote the-multiplication-by-f operator. We have $M_{-b} f M_b = f$, $T_{-a} f T_a = f(\cdot - a)$, $M_{-b}\partial M_b = \partial + b$, and $T_{-a}\partial T_a = \partial$. Hence,

$$\begin{aligned} H_{a,b} &= -\frac{1}{2}(\partial + b, \partial + b) - (\theta - \kappa(x-a), \partial + b) + \frac{1}{2}(\Gamma(x-a), x-a) + (d, x-a) + d_0 \\ &= -\frac{1}{2}(\partial, \partial) + (\kappa x, \partial) + \frac{1}{2}(\Gamma x, x) + \hat{d}_0 - (b + \theta + \kappa a, \partial) + (d + \kappa^T b - \Gamma a, x), \end{aligned}$$

where $\hat{d}_0$ is given by (2.3). To kill the last two terms, we need (2.5)–(2.6).

Lemma A.1. *Let A be an operator in a real Hilbert space $\mathcal{H}$, and let*

$$(Au, u)_{\mathcal{H}} \geq c||u||^2_{\mathcal{H}}, \quad u \in \mathcal{H}. \tag{A.1}$$

Then the eigenvalues $\lambda_j(A)$ of A satisfy $\operatorname{Re}\lambda_j(A) \geq c$.

Proof. Let λ be an eigenvalue. If λ is real, the statement is evident: take the corresponding eigenfunction, and insert in (A.1). Now let $\lambda = \alpha + i\beta$ be complex. Then there exists a 2-dimensional invariant subspace V of $\mathcal{H}$ such that the eigenvalues of the restriction $A_2 = A|_V$ are $\alpha \pm i\beta$. Thus, it suffices to consider operators in $\mathbb{R}^2$. We use a basis in V which consists of eigenvectors of $A_r = \frac{1}{2}(A_2 + A_2^T)$, and calculate the eigenvalues in terms of the entries of the matrix of the operator A_2 in this basis. The diagonal part defines A_r, and the skew-diagonal part defines the skew-symmetric operator $\frac{1}{2}(A_2 - A_2^T)$. Hence, c is equal to the minimum of the diagonal elements, and the verification of the condition $\alpha \geq c$ reduces to the direct calculation of the eigenvalues. □

Proof of (4.3). We consider the sign "+". Using the commutation relations (3.3), we obtain

$$\begin{aligned}
-[2\tilde{\mathcal{L}}, \beta z_2 + (\alpha - \Lambda)z_1] &= [\alpha(-\mathcal{L}_2 + \mathcal{L}_1) + \beta J, \beta z_2 + (\alpha - \Lambda)z_1] \\
&= [\alpha(z_2 z_2^* - z_1 z_1^*) + \beta(z_1 z_2^* - z_2 z_1^*), \beta z_2 + (\alpha - \Lambda)z_1] \\
&= \alpha\beta z_2 - \alpha(\alpha - \Lambda)z_1 + \beta^2 z_1 - \beta(\alpha - \Lambda)z_2 \\
&= (\alpha\beta - \beta\alpha + \beta\Lambda)z_2 + (\beta^2 - \alpha^2 + \alpha\Lambda)z_1 \\
&= \Lambda(\beta z_2 + (\alpha - \Lambda)z_1).
\end{aligned}$$

Proof of (4.3).

$$\begin{aligned}
-2\tilde{\mathcal{L}}Q_2 &= [\alpha(z_2 z_2^* - z_1 z_1^*) + \beta(z_1 z_2^* - z_2 z_1^*)](\beta z_2^2 + 2\alpha z_1 z_2 + \beta z_1^2) \\
&= (\alpha\beta z_2^2(z_2 z_2^* - z_1 z_1^*) + 2\alpha\beta z_2^2) \\
&\quad +(\alpha\beta z_1^2(z_2 z_2^* - z_1 z_1^*) - 2\alpha\beta z_1^2) \\
&\quad +(2a^2 z_1 z_2(z_2 z_2^* - z_1 z_1^*) + 2\alpha^2 z_1 z_2 - 2\alpha^2 z_1 z_2) \\
&\quad +(\beta^2 z_2^2(z_2 z_2^* - z_1 z_1^*) + 2\beta^2 z_1 z_2) \\
&\quad +(\beta^2 z_1^2(z_2 z_2^* - z_1 z_1^*) - 2\beta^2 z_1 z_2) \\
&\quad +(2\alpha\beta z_1 z_2(z_2 z_2^* - z_1 z_1^*) + 2\alpha\beta z_1^2 - 2\alpha\beta z_2^2) \\
&= (\beta z_2^2 + 2\alpha z_1 z_2 + \beta z_1^2)[\alpha(z_2 z_2^* - z_1 z_1^*) + \beta(z_1 z_2^* - z_2 z_1^*)] \\
&= -Q_2 2\tilde{\mathcal{L}}.
\end{aligned}$$

Appendix B. Continuous algebraic Riccati equations

Equation (2.9) is a special case of continuous algebraic Riccati equations (CARE)

$$Y^2 + YA + A^T Y - \Gamma = 0, \tag{B.1}$$

with $A = \kappa$. Below, we list several definitions and facts from Ionescu et al. (1999) and Lancaster and Rodman (1995). We refer to these monographs as [IOW] and [LR], respectively. Since $\Gamma \geq 0$, a solution H_+ (resp., H_-) to (B.1) such that $A + H_+$ is anti-stable (resp., $A + H_-$ is stable) exists, and it is unique (see Theorem 9.1.1 in [LR]). These solutions are called the maximal and minimal Hermitian solutions to (B.1). The

minimal solution can be obtained as the opposite to the maximal solution to (B.1) with $-A$ instead of A. Therefore, the first two procedures of the construction of the maximal solution below, from [LR], which work for both stable and anti-stable A, can be used to construct the maximal and minimal solutions of equation (B.1). The third procedure, which we reproduce from Section 4.1 of [IOW], requires that A be anti-stable. In Section 4.3 of [IOW], the reader can find the counterpart of the third construction for the case of a stable A.

B.1. **The Newton-Kantorovich procedure.** Consider CARE (B.1) with $\Gamma \geq 0$. Let $\mathcal{R}(X) = X^2 + XA + A^T X - \Gamma$ be the Riccati function, which maps symmetric matrices to symmetric matrices. The Frechet derivative of $\mathcal{R}$ is given by

$$\mathcal{R}'_X(H) = H(A + X) + (A^T + X)H,$$

and the procedure for the solution of $\mathcal{R}(X) = 0$ is

$$X_{m+1} = X_m - (\mathcal{R}'_{X_m})^{-1}\mathcal{R}(X_m), \quad m = 0, 1, \dots. \tag{B.2}$$

The sequence converges (and fast) for any $X_0 \leq \Gamma$; with such a choice, we have $\mathcal{R}(X_m) \leq 0$, and $X_{m+1} - X_m \geq 0$ for all m. Since $\Gamma \geq 0$, we can always take $X_0 = 0$.

Equation (B.2) is equivalent to the following equation for $\Delta X_m = X_{m+1} - X_m$:

$$\Delta X_m A_m + A_m^T \Delta X_m = \mathcal{R}(X_m), \tag{B.3}$$

where $A_m = -A - X_m$ is stable. Equation (B.3) is the Lyapunov equation, which is reducible to a linear system with n^2 variables, and hence easy to solve (see Appendix C).

B.2. **Reduction to the spectral problem in dimension $2n$.** The proofs of the following facts can be found, for instance, in [LR], in a more general set-up. The argument at the end of Subsection 8.3 of [LR] shows that Y, the solution to CARE (B.1) such that $A+Y$ is anti-stable, is also the maximal hermitian solution to the same CARE, and this hermitian solution is uniquely characterized by the property that the spectrum $\sigma(A+Y)$ lies in the closed right half-plane $\operatorname{Re} z \geq 0$; since A is anti-stable, and Y is positive, the spectrum $\sigma(A+Y)$ lies in the open right half-plane. These remarks being made, we can use constructions for the hermitian matrices in Chapter 7 of [LR].

Define $2n \times 2n$ matrices

$$M = \begin{bmatrix} A & I \\ \Gamma & -A^T \end{bmatrix}, \quad \hat{H} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \quad H = \begin{bmatrix} -\Gamma & A^T \\ A & I \end{bmatrix}. \tag{B.4}$$

Direct calculations show that $\hat{H}M = -M^T\hat{H}, HM = -M^T H$, and $H = -\hat{H}M$. Since $\Gamma \geq 0$ and A is invertible, the matrix $\Gamma + A^T A$ is positive-definite, hence invertible; therefore, H and M are invertible as well. Further, $HM = -M^T H$, which means that M is H-skew-symmetric: $HMH^{-1} = -M^T$.

For any $n \times n$ matrix Y, we call the n-dimensional subspace

$$G(Y) = \mathrm{Im} \begin{bmatrix} I \\ Y \end{bmatrix} \subset \mathbb{R}^{2n}$$

the graph of Y. Also, a subspace of $\mathbb{R}^{2n}$ is called a graph subspace if it has the form $G(Y)$ for some Y. The straightforward calculation (see Proposition 7.1.1 in [LR]) shows that the graph of Y is H-invariant if and only if Y is the solution of (B.1). Notice that the same definitions apply in the complex case.

Set $S = \{\lambda \in \sigma(M) \mid \mathrm{Re}\,\lambda > 0\}$. Since $\sigma(A+Y)$ lies in the right half-plane, we have $S = \sigma(A+Y)$. Denote by $\mathcal{M}_+$ the spectral invariant subspace corresponding to the eigenvalues $\lambda \in S$. By Theorem 2.5.2 (see also Theorem 7.3.5) of [LR], there exists a unique n-dimensional M-invariant $\hat{H}$-neutral subspace $\mathcal{M}$ (that is, $(\hat{H}u, u) = 0,\ \forall\ u \in \mathcal{M}$). The subspace $\mathcal{M}$ is the graph of Y (as a subspace of $\mathbb{C}^{2n}$). Generically, $\#S = n$, and then one can construct the maximal Hermitian solution to (B.1) as follows.

1. Calculate the (complex) eigenvectors of M which correspond to the eigenvalues in the right half-plane, and write these eigenvectors as the columns of a $2n \times n$ matrix X.
2. Let X_1 be the upper $n \times n$ block, and X_2 the lower one; then X_1 is invertible, and $Y = X_2 X_1^{-1}$.

B.3. **Reduction to the Toeplitz operators.** The following procedure (equation (4.15) in [IOW]) provides an explicit formula for Y. First, introduce the operator $\mathcal{L}$ in $L_2(\mathbb{R}_+)^n$ by

$$(\mathcal{L}u)(t) = \int_0^t e^{-A(t-s)} u(s) ds,$$

and then define $\mathcal{R} := I + \mathcal{L}^*\Gamma\mathcal{L}$. Since A is anti-stable, $\mathcal{L}$ is bounded, and, since $\Gamma \geq 0$, $\mathcal{R}$ is a bounded self-adjoint operator bounded by I from below. Hence, $\mathcal{R}$ is invertible. Next, define

$$\Phi : \mathbb{R}^n \to L_2(\mathbb{R}_+)^n, \quad (\Phi\xi)(t) := e^{-At}\xi,\ t \geq 0; \tag{B.5}$$

and then $\mathcal{P}_0 := \Phi^*\Gamma\Phi$, $\mathcal{P} := \Phi^*\Gamma\mathcal{L}$. Finally, $Y := \mathcal{P}_0 - \mathcal{P}\mathcal{R}^{-1}\mathcal{P}^*$. We see that in two- and three–factor cases, it is possible to calculate Y explicitly but the calculations are rather involved.

Appendix C. Lyapunov equation

The solution to the Lyapunov equation

$$XA + A^T X = Q \tag{C.1}$$

with A stable (or anti-stable) can be found as follows (see e.g. p.86 in Kaszkurewicz (1995)). First, given a matrix $X \in \mathbb{R}^{n\times n}$, define the (column) vector $\mathrm{vec}X \in \mathbb{R}^{n^2}$ by writing the columns of X one after another. Then calculate the matrix $I_n \otimes A^T + A^T \otimes I_n \in \mathbb{R}^{n^2\times n^2}$. The matrix $I_n \otimes A^T$ is the block-diagonal matrix, with the matrix A^T for each

block on the diagonal, and the matrix $A^T \otimes I_n$ consists of blocks $a'_{jk} I_n$, where a'_{jk} are the entries of A^T. For instance, in the case $n = 2$,

$$I_2 \otimes A^T + A^T \otimes I_2 = \begin{bmatrix} 2a_{11} & a_{21} & a_{21} & 0 \\ a_{12} & a_{11}+a_{22} & 0 & a_{21} \\ a_{12} & 0 & a_{11}+a_{22} & a_{21} \\ 0 & a_{12} & a_{12} & 2a_{22} \end{bmatrix}, \tag{C.2}$$

where a_{jk} are the entries of A.

The matrix vec$X$ is found as the solution of the linear equation

$$(I_n \otimes A^T + A^T \otimes I_n)\text{vecX} = \text{vec}Q,$$

which is a simple computational task, as well as the calculation of the inverse $Y = X^{-1}$.

From Theorem 2.10.7 in Kaszkurewicz (1995), X is positive-definite, if A is anti-stable and Q is positive definite. In (2.15), $A = \kappa_1$ is anti-stable, and $Q = 2I_n$ is positive-definite, hence the solution Z is positive-definite.

Appendix D. Study of operators Y and Z

By construction, $Y = H_+$ is the maximal Hermitian solution to (2.9). The direct calculation shows that $\frac{1}{2}(H_+ - H_-) > 0$ solves the Bernoulli equation (2.10):

$$\begin{aligned} & 2\left(\frac{1}{2}(H_+ - H_-)\right)^2 - \frac{1}{2}(H_+ - H_-)(\kappa + H_+) - (\kappa^T + H_+)\frac{1}{2}(H_+ - H_-) \\ = \; & \frac{1}{2}\left[-H_+^2 - H_+\kappa - \kappa^T H_+ + H_- + H_-\kappa + \kappa^T H_-\right] \\ = \; & \frac{1}{2}(-\Gamma + \Gamma) \\ = \; & 0. \end{aligned}$$

Since the invertible solution to (2.10) is unique, we conclude that $Z = \frac{1}{2}(H_+ - H_-)$, and $Y - 2Z = H_-$ (this is the matrix which is needed to calculate the coefficients of the asymptotic expansion of g_2 in Section 6).

In order to justify the eigenfunction technique in Section 3, we need to know that $Z - Y = -\frac{1}{2}(H_+ + H_-)$ is positive definite. Consider a family of CARE with $\epsilon\Gamma$ instead of Γ, where $\epsilon > 0$. Let $H_\pm(\epsilon)$ be the corresponding maximal and minimal solutions. For small $\epsilon > 0$, $H_+(\epsilon)$ can be found as a series in ϵ, with the leading term being ϵX_0, where X_0 is the solution of the Lyapunov equation

$$\kappa X + \kappa^T X - \Gamma = 0.$$

Hence, $H_+(\epsilon)$ is small. On the other hand, for small ϵ, $H_-(\epsilon)$ is close to the negative-definite solution of the Bernoulli equation

$$X^2 + \kappa X + X\kappa^T = 0.$$

We conclude that for small $\epsilon > 0$, $H_+(\epsilon) + H_-(\epsilon)$ is negative definite. This proves that $H_+ + H_-$ is negative-definite if Γ is relatively small w.r.t. κ. We tried many pairs (κ, Γ)

with Γ not small, and $H_+ + H_-$ was negative-definite when the skew-symmetric part of κ was not large with respect to the symmetric part.

Appendix E. Solution of the system of (differential) Riccati equations

As Kim (2003), we use general results from Levin (1959). Set

$$H = \begin{bmatrix} -\kappa^T & -\Gamma \\ -I_{n\times n} & \kappa \end{bmatrix}, \tag{E.1}$$

and define $N(\tau) := \exp \tau H$. We write N in the block form

$$N(\tau) = \begin{bmatrix} N_1(\tau) & N_2(\tau) \\ N_3(\tau) & N_4(\tau) \end{bmatrix}, \tag{E.2}$$

where each block is $n \times n$ matrix. Then

$$A(\tau) = N_2(\tau)N_4^{-1}(\tau) \tag{E.3}$$

$$B(\tau) = [-\mathrm{I} \;\; A(\tau)](N(\tau) - I)H^{-1}[d \;\; 0]^T, \tag{E.4}$$

$$C(\tau) = \int_0^\tau \left[B(s)^T B(s)/2 + \mathrm{Tr}A(s)\right] ds - d_0\tau. \tag{E.5}$$

Appendix F. Algorithm

In this section, we formulate a detailed algorithm for pricing of the call option with expiry date T and strike K on the discount bond maturing at $T_1 = T + \delta$.

Step 1. Input the parameters of the process, the upper limit on the size of the blocks, N, $\delta := T_1 - T$, a grid in K, a grid in x, and a grid in T.

Step 2. Solve the system of (differential) Riccati equations on $[0, \delta]$. This block is needed to calculate the price of the bond at time 0 with maturity $\delta := T_1 - T$.

a. Define H by (E.1).

b. Select a grid for the time interval $(0, \delta)$; call the step size Δt.

c. In order to improve the accuracy, divide the first step in t in two.

d. For $s = 0, 1/2, 1$, calculate $N_j(s * \Delta t), j = 1, 2, 3, 4,$, $A_s := A(s * \Delta t)$ and $B_s := B(s * \Delta t)$ using (E.2), (E.3) and (E.4).

e. Use A_0, $A_{1/2}$, A_1, B_0, $B_{1/2}$, B_1 and Simpson's rule with step size $\Delta t/2$ to calculate $C_1 = C(\Delta t)$ given by (E.5). Set $C_0 = 0$.

f. In the cycle $s = 2 : n$, where $n = \delta/\Delta t$,

- calculate $A_2 := A(s * \Delta t)$, $B_2 := B(s * \Delta t)$ and use Simpson's rule with step size Δt to calculate $dC_s := C(s * \Delta t) - C((s - 2) * \Delta t)$;
- Set $A_0 := A_1$, $A_1 := A_2$, $B_0 := B_1$, $B_1 := B_2$, $C_0 := C_1$, $C_1 := C_1 + dC$.

g. Set $A(\delta) = A_2$, $B(\delta) = B_2$, $C(\delta) = C_1$

Step 3. Calculate a and $\hat{d}_0$ using (2.5) and (2.3).

Step 4. Calculate a unique solution Y of CARE (2.9) such that $\kappa + Y$ is anti-stable using any procedure described in Appendix B. We used Steps 1 and 2 in Appendix B.2.

Step 5. Find a unique positive-definite matrix X, which solves (2.15), using the procedure described in Appendix C, and set $Z = X^{-1}$.

Step 6. Calculate $\tilde{d}_0 = \hat{d}_0 + \frac{1}{2}\mathrm{Tr}(Y - Z)$.

Step 7. Using any standard procedure, calculate $M_1 = \mathrm{diag}(\mu_j)$, the diagonal form of Z, and C, matrix consisting of eigenvectors of Z, and set $M = M_1^{1/2}$.

Step 8. Calculate $B = M \cdot C^T(\kappa_1 - Z)C \cdot M$.

Step 9. For $m = 0, \ldots, N$, calculate the coefficients $H_m^k, k = 0, 1, \ldots, m$, of H_m, the Hermite polynomial of order m, using $H_0 = 1$ and $H_m(x) = 2xH_{m-1}(x) - H'_{m-1}(x)$.

Step 10. For $m = 0, \ldots, N$, calculate the norm c_m of w_m: $c_m = \sqrt{m!\sqrt{\pi}}$.

Step 11. (Starting with this step, we describe the algorithm for the two-factor case, real eigenvalues and no Jordan blocks). Calculate $\alpha = \mu_2 - \mu_1$, $\beta = 2b_{12}$, and check condition $\lambda^2 > \beta^2$. Set $\Lambda = \sqrt{\alpha^2 - \beta^2}$.

Step 12. For $r = 0, \ldots, \lfloor N/2 \rfloor$, use Theorem 4.2 to calculate the coefficients $\phi_{r,0}^l$ of eigenfunction expansion of $\phi_{r,0}$ in the basis $w_k \otimes w_j$:

a. Set
$$\phi_{0,0}^1 = 1,\ \phi_{1,0}^0 = \beta,\ \phi_{1,0}^1 = 2\alpha,\ \phi_{1,0}^2 = \beta.$$

b. For $2 \le r \le \lfloor N/2 \rfloor$, set
$$\phi_{r,0}^0 = \beta\phi_{r-1,0}^0,\ \phi_{r,0}^1 = 2\alpha\phi_{r-1,0}^0 + \beta\phi_{r-1,0}^1,$$
$$\phi_{r,0}^{2r-1} = 2\alpha\phi_{r-1,0}^{2r+1} + \beta\phi_{r-1,0}^{2r},\ \phi_{r,0}^{2r} = \beta\phi_{r-1,0}^{2r+1},$$
and for $2 \le l \le 2r - 2$,
$$\phi_{r,0}^l = \beta\phi_{r-1,0}^{l-2} + 2\alpha\phi_{r-1,0}^{l-1} + \beta\phi_{r-1,0}^l.$$

Step 13. For $r = 0, \ldots, \lfloor N/2 \rfloor$, calculate all eigenvalues and the coefficients $\phi_{r,s}^l$ of eigenvectors $\phi_{r,s}$, $s \neq 0$, in the basis $w_k \otimes w_j$.

a. Calculate the upper bound for $|s|$, $s^* = N - 2r$.

b. Calculate $\lambda_{r,0} = -(r + 1/2)(\mu_1 + \mu_2) - \tilde{d}_0$

c. If $s^* > 0$, then, for $0 < s \le s^*$, calculate eigenvalues $\lambda_{r,\pm s}$ and $\phi_{r,\pm s}$:

i. Set
$$\lambda_{r,\pm s} = -(r + 1/2)(\mu_{11} + \mu_{22}) - \tilde{d}_0 - s(\pm\Lambda + \mu_{11} + \mu_{22})/2.$$

ii. For $1 \le s \le s^*$ and $0 \le l \le 2r + s - 1$, set
$$\phi_{r,\pm s}^0 = (\alpha \mp \Lambda)\phi_{r,\pm(s-1)}^0,\ \phi_{r,\pm s}^{2r+s} = \beta\phi_{r,\pm(s-1)}^{2r+s},$$
$$\phi_{r,\pm s}^l = \beta\phi_{r,\pm(s-1)}^{l-1} + (\alpha \mp \Lambda)\phi_{r,\pm(s-1)}^l.$$

Step 14. Calculate vectors and scalars used on the next two steps
$$D = C^T Z^{1/2},\ A_f = Z - Y + D^T D,\ \tilde{A} = ((chol(A_f))^T)^{-1},\ \tilde{B} = A_f^{-1}\kappa a,$$
$$B^1 = \tilde{A}^T A(\delta)(\tilde{B} - a) + \tilde{A}^T B(\delta),\ A^1 = 1/2\tilde{A}^T A(\delta)\tilde{A},\ \gamma_1 = A_{22}^1 A_{11}^1 - A_{12}^1 A_{21}^1,$$
$$N_{\exp} = |\det D||\det \tilde{A}|2\pi \exp(-1/2(A_f\tilde{B}, \tilde{B}) + (\kappa a, \tilde{B})).$$

Step 15. Calculate the coefficients of eigenfunction expansion of $g_{2,\alpha}$ in the basis $w_k \otimes w_j$, for $|\alpha| \le 3$. Pick the number of steps for Simpson integration, n_z (in the call option examples in the paper, $n_z = 300$ was used), and for each K, repeat the following steps.

a. Set

$$C^1 = (A(\delta)(\tilde{B}-a), \tilde{B}-a) + (B(\delta), \tilde{B}-a) + C(\delta) - \log K,$$

$$\gamma_3 = -(B_1^1)^2/4 + C^1 A_{11}^1,\ z^{\pm} = (-\gamma_2 \pm \sqrt{\gamma_2^2 - 4\gamma_1\gamma_3})/2\gamma_1.$$

b. Set

$$f^{\pm}(z_2) = \frac{-\left((A_{12}^1 + A_{21}^1)z_2 + B_1^1\right) \pm \sqrt{\left((A_{12}^1 + A_{21}^1)z_2 + B_1^1\right)^2 - 4A_{11}^1(A_{22}^1 z_2^2 + B_2^1 z_2 + C^1)}}{2A_{11}^1}.$$

c. Set $\Delta z = (z^+ - z^-)/n_z$.

d. For $0 \le j \le 4\min(N,2)$ and $0 \le k \le 2\min(N,2)$, use Simpson's method to calculate

$$\begin{aligned}
\check{I}_{j,k} &= \int_{\mathbb{R}^2} g(\tilde{A}z + \tilde{B} - a) z_1^j z_2^k \exp(-1/2(z_1^2 + z_2^2)) dz \\
&= \int_{z^-}^{z^+} dz_2 z_2^k \exp\left(\left(-\frac{1}{2} + A_{22}^1\right) z_2^2 + B_2^1 z_2 + C^1 + \log K\right) I_j^1(z_2) \\
&- K \int_{z^-}^{z^+} dz_2 z_2^k \exp\left(-\frac{1}{2} z_2^2\right) \int_{f^-(z_2)}^{f^+(z_2)} z_1^k e^{-1/2z_1^2} dz_1,
\end{aligned}$$

where

$$I_j^1(z_2) = \int_{f^-(z_2)}^{f^+(z_2)} dz_1 z_1^j \exp\left(\left(-\frac{1}{2} + A_{11}^1\right) z_1^2 + (A_{21}^1 + A_{12}^1) z_1 z_2 + B_1^1 z_1\right).$$

e. For $0 \le j,k \le 2\min(N,2)$, calculate

$$\begin{aligned}
\tilde{I}_{j,k} &= \int_{\mathbb{R}^2} g(x'-a) \exp(-\Phi(x') - 1/2||Dx'||^2)(x_1')^j (x_2')^k dx' \\
&= \sum_{m=0}^{j} \sum_{n=0}^{k} \sum_{i=0}^{k-n} \binom{j}{m} \binom{k}{n} \binom{k-n}{i} \tilde{A}_{11}^m \tilde{A}_{21}^n \tilde{A}_{22}^i \tilde{B}_1^{j-m} \tilde{B}_2^{k-n-i} \check{I}_{m+n,i},
\end{aligned}$$

where $\Phi(x') = \frac{1}{2}((Z-Y)x', x') - (\kappa a, x')$.

f. For $0 \le j,k \le \min(N,2)$, calculate

$$\begin{aligned}
I_{j,k} &= \int_{\mathbb{R}^2} g(x'-a) \exp(-\Phi(x') - 1/2||Dx'||^2)(Dx')_1^j (Dx')_2^k |\det D| dx' \\
&= \sum_{m=0}^{j} \sum_{n=0}^{k} \binom{j}{m} \binom{k}{n} D_{11}^m D_{12}^{j-m} D_{21}^n D_{22}^{k-n} \det D \tilde{I}_{m+n,j-m+k-n}.
\end{aligned}$$

g. For $0 \le m \le \min(2,N)$ and $0 \le j \le m$, set

$$g_{2,m}^j = \sum_{n=0}^{j} \sum_{k=0}^{m-j} I_{n,k} H_j^n H_{m-j}^k / (c_j^2 c_{m-j}^2 \sqrt{2}^m).$$

Step 16. For $|\alpha| > 3$, calculate $g_{2,\alpha}$ using the importance sampling method:

a) Select the number of simulations used for calculating each integral, n_{sim}. For the call option examples in the body of the paper, $n_{sim} = 500000$ was used.

b) For $1 \le i \le n_{sim}$, simulate $z_i' \sim N(-\tilde{B}, \tilde{A})$.

c) For each K,

i. calculate $z_{i,k} = z'_{i,k} - a_k, k = 1, 2$, and $\tilde{g}_i = g(z_i)$;

ii. for $3 \le m \le N$ and $0 \le j \le m$, set

$$H^1_{i,j} = \sum_{k=0}^{j} H^k_j z^k_{i,1},\ H^2_{i,j} = \sum_{k=0}^{j} H^k_j z^k_{i,2},$$

$$g^j_{2,m} = \sum_{i=1}^{n_{sim}} \tilde{g}_i H^1_{i,m-j} H^2_{i,j} N_{\exp}/(c_j^2 c^2_{m-j} \sqrt{2}^m n_{sim}).$$

Step 17. For $0 \le m \le N$, calculate the basis of V_m, consisting of $\phi_{r,s}$, the change of variables matrix F and the corresponding eigenvalues.

a. If m is even, then for $1 \le s \le m/2$ and $0 \le j \le m$, set

$$F_{j,s} = \phi^j_{m/2-s,-2s}(((\alpha+\Lambda)^2+\beta^2))^{-s}((2\beta^2+4\alpha^2))^{(s-m/2)/2},$$

$$F_{j,s+m/2} = \phi^j_{m/2-s,2s}(((\alpha+\Lambda)^2+\beta^2))^{-s}((2\beta^2+4\alpha^2))^{(s-m/2)/2},$$

$$F_{j,m/2} = \phi_{m/2,0}((2\beta^2+4\alpha^2))^{-m/4},$$

$$LL_s = \lambda_{m/2-s,-2s},\ LL(s+m/2) = \lambda_{m/2-s,2s},\ LL(m/2) = \lambda_{m/2,0}.$$

b. If m is odd, then for $1 \le s \le (m-1)/2$ and $0 \le j \le m$, set

$$F_{j,s} = \phi^j_{m/2-s,-2s}(((\alpha+\Lambda)^2+\beta^2))^{-s}((2\beta^2+4\alpha^2))^{(s-m/2)/2},$$

$$F_{j,s+m/2} = \phi^j_{m/2-s,2s}(((\alpha+\Lambda)^2+\beta^2))^{-s}((2\beta^2+4\alpha^2))^{(s-m/2)/2},$$

$$LL_s = \lambda_{m/2-s,-2s},\ LL(s+m/2) = \lambda_{m/2-s,2s}.$$

Step 18. For x from the chosen grid, calculate $x' = x + a$, $x'' = C^T Z^{1/2} x'$.

Step 19. For $0 \le m \le N$,

a. introduce a row vector f_m with entries $f^j_m = H_{m-j}(x''_1)H_j(x''_2)/\sqrt{2}^{k+j}$, and column vector g_2 with entries $g^j_2 = g^j_{2,m}$, $0 \le j \le m$;

b. for T from the chosen grid, calculate a diagonal matrix LA with diagonal entries $LA_{j,j} = \exp(LL_j T)$, $0 \le j \le m$;

c. calculate v_m, the part of v in V_m, without the exponential factor:

$$v_m = f_m * F * LA * F^{-1} * g_2.$$

Step 20. Final step: calculate

$$\sum_{m=0}^{N} v^m = \exp[1/2(-Yx', x') - (\kappa a, x')] \sum_{m=0}^{N} v_m.$$

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