

# Pricing Discretely Monitored Asian Options under Lévy Processes

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## Abstract

We present methodologies to price discretely monitored Asian options when the underlying evolves according to a generic Lévy process. For arithmetic Asian options we solve the valuation problem by recursive integration and derive a recursive theoretical formula for the moments to check the accuracy of the results.

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## 1 Introduction

We investigate the pricing problem for Asian options monitored at discrete times. The payoff of an arithmetic (geometric) Asian option depends on the arithmetic (geometric) average value of the underlying asset price over a given time period. Asian options have been very successful in the marketplace, because they reduce the possibility of market manipulations near the expiry and they offer better hedging possibilities to firms with a stream of exposures.

Several approaches have been attempted to obtain pricing formulas for the price of Asian options, assuming a continuous-time monitoring of the underlying under the geometric Brownian motion hypothesis, see Fusai and Roncoroni (2008) for a review and numerical comparisons. Among analytical approaches, we mention the Laplace transform approach in Geman and Yor

(1993), the spectral expansion derived by Linetsky (2004), and the approximation of the average distribution by fitting integer moments in Turnbull and Wakeman (1991), Lévy (1992), Milevsky and Posner (1998) or logarithmic moments as in Fusai and Tagliani (2002). Another approach uses binomial trees, such as Gaudenzi et al. (2007). However, a large number of contracts specify discrete time monitoring, and the impact of the continuous-time assumption can be substantial for some path-dependent derivatives, see for instance the literature on lookback and barrier options, Kat (2001).

For the discrete case, Clewlow and Carverhill (1990), Andreasen (2002), Dempster et al. (1998), Zvan et al. (1999) focus their attention on the geometric Brownian motion. Benhamou (2002) enhances the algorithm of Clewlow and Carverhill (1990) based on a Fast Fourier technique and adapt it to some non-lognormal densities, like the Student  $t$ . Their approaches, although innovative, require computationally intensive numerical methods or approximations for which no clear error bound is available. Albrecher (2004) and Albrecher and Predota (2004) explore approximations based on the moments of the average, but in general it is difficult to evaluate the approximation error.

In particular, we discuss the pricing of arithmetic Asian options when the underlying asset evolves according to a generic Lévy process. We present a new numerical procedure which combines a recursive numerical quadrature with a fast Fourier transform algorithm. Our procedure is also able to provide estimates of the Greeks, such as delta and gamma.

The paper is organized as follows. In Section 2 we model the underlying process, starting from the distribution of the log-increments. In Section 3 we present the new recursive algorithm for the valuation of arithmetic Asian options. In Section 4 we discuss our numerical results with particular emphasis on the discrete monitoring feature. In Section 5 we conclude.

## 2 The process for the underlying

We are interested in pricing discrete Asian options, for which the payoff depends on the geometric or on the arithmetic average of the prices observed at equally-spaced discrete times  $t_0 \equiv 0, t_1 \equiv \Delta, \dots, t_j \equiv j\Delta, \dots$ . We denote by  $S_t$  the underlying asset price at time  $t$ . Consider the demeaned log-increments of size  $\Delta$ :

$$X_T^\Delta \equiv \ln(S_{T\Delta}) - \ln(S_{(T-1)\Delta}) - m_T^\Delta \Delta, \quad (1)$$

where  $m_T^\Delta$  is the deterministic component under the risk-neutral measure of the log-increments  $X_T^\Delta$ , whose value will be specified later on:

$$m_T^\Delta \equiv \frac{1}{\Delta} \mathbb{E} \left\{ \ln (S_{T\Delta}) - \ln (S_{(T-1)\Delta}) \right\}. \quad (2)$$

We consider the demeaned log-increments instead of the simple log-increments for future notational convenience.

As far as the underlying process is concerned, we only assume that, under the risk-neutral measure, non-overlapping log-increments be independent and that increments of equal size be identically distributed. In other words, we assume that the logarithm of the prices be a Lévy process under the risk neutral measure. Then  $m_T^\Delta \equiv m$  does not depend on either the time step  $T$  or the step size  $\Delta$  and the underlying asset price reads:

$$S_{T\Delta} = S_0 e^{mT\Delta + X_1^\Delta + \dots + X_T^\Delta}. \quad (3)$$

Lévy processes display a number of palatable features: they are the most direct generalization of the Brownian motion (BM); they are analytically tractable; Lévy processes are general enough to include a wealth of patterns and thus they account for smile and skew effects in option prices; the i.i.d. structure of the Lévy processes simplifies the estimation of the respective parameters under the real measure, see Meucci (2005). For a thorough introduction to Lévy processes with applications to finance see Geman (2002), Schoutens (2003), Cont and Tankov (2004a), Carr et al. (2003), Geman (2005).

A generic Lévy process is fully determined by the characteristic exponent of the log-increments, which is defined as the logarithm of the characteristic function:

$$\psi_\Delta(\omega) \equiv \ln \left( \mathbb{E} \left\{ e^{i\omega X_T^\Delta} \right\} \right). \quad (4)$$

In Table 1 we list a few parametric Lévy processes and their associated characteristic exponent.

**[INSERT TABLE 1 HERE]**

he Gaussian model (g) is the benchmark assumption: the ensuing process is the purely diffusive Brownian motion, which gives rise to the geometric Brownian motion (GBM) process for the price of the underlying. The model (jd) introduced by Merton (1976) and the double exponential (de) model

introduced by Kou (2002) are jump-diffusion processes that account for the presence of fat tails in the empirical distribution of the underlying<sup>1</sup>. The remaining models (nig and cgmy) are pure jump processes with finite variation that can display both finite and infinite activity<sup>2</sup>. They are subordinated Brownian motions: in other words, they can be interpreted as Brownian motions subject to a stochastic time change which is related to the level of activity in the market. In particular, stable processes (st) display the additional feature that their distribution does not depend on the monitoring interval, modulo a scale factor.

So far the drift parameter  $m$  in (3) has been left unspecified. Due to the incompleteness of the market, we have to choose a martingale measure for the risk-neutral pricing of derivatives. Except in the special case of the geometric Brownian motion, there are many equivalent measures under which the discounted price process is a martingale. Several different approaches have been suggested to select an appropriate martingale measure, but there is as yet no definitive way of pricing contingent claims in incomplete markets. A discussion on the different choices of an equivalent martingale measure with reference to Lévy Processes can be found for example in Chan (1999) and in Hubalek and Sgarra (2006). A mathematically tractable choice consists in choosing the value of  $m$  in such a way that the price  $S_t$  discounted by the money-market account  $B_t$  be a martingale, i.e.  $\mathbb{E}[S_T/B_T] = S_0/B_0, \forall T \geq 0$ , see Schoutens (2003). A simple algebraic manipulation shows that  $m$  must solve

$$m = (r - q) - \frac{\psi_{\Delta}(-i)}{\Delta}, \quad (5)$$

where  $r$  denotes the constant risk-free rate and  $q$  denotes the constant dividend payout rate. Another possible choice is represented by the Esscher

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<sup>1</sup>In these models  $\sigma^2$  represent the instantaneous variance of the diffusion part, whilst  $\lambda$  is the jump-intensity. In the jd model,  $\alpha$  and  $\delta^2$  refer respectively to the mean and the variance of the jump size. In the de model,  $p$  is the probability of a up jump, whilst  $\eta_1$  and  $\eta_2$  govern the decay of the tails of the up and down jump sizes, that are exponentially distributed.

<sup>2</sup>In particular, the nig model has stochastic time change given by an Inverse Gaussian Process  $I_t$  with parameters 1 and  $\delta\sqrt{\alpha^2 - \beta^2}$ , so that  $nig_t = \beta\delta^2 I_t + \delta W_{I_t}$ , where  $W_t$  is a Wiener process. The path behaviour of the CGMY process is determined by the  $Y$  parameter. If  $Y < 0$ , the paths have finite jumps in any finite interval; if not, the paths have infinitely many jumps in any finite time interval, i.e. the process has infinite activity. Moreover, if the  $Y$  parameter lie in the interval  $[1, 2)$ , the process is of infinite variation. See Schoutens (2003).

transform, as advocated at first in Gerber and Shiu (1994). In the following we will also specify the process for the underlying price when the underlying itself represents the numeraire, see El Karoui et al. (1995). In other words, we need to set  $m$  equal to a parameter  $\bar{m}$  such that, under the new measure, the equality  $\mathbb{E}[B_T/S_T] = B_0/S_0$  is satisfied for all  $T > 0$ . Another simple algebraic manipulation yields the appropriate value  $\bar{m} = (r - q) + \psi_\Delta(i) / \Delta$ .

### 3 Arithmetic Asian options

The payoff of an arithmetic Asian option depends on the following path-dependent random variable:

$$A_T^\Delta \equiv \frac{1}{T+1} \sum_{k=0}^T S_{\Delta k}. \quad (6)$$

Notice that we use the convention that the average starts at period  $k \equiv 0$ . In the following, we exploit a recursive formulation to price the fixed strike arithmetic Asian option. For the floating strike version see Fusai and Meucci (2007). The payoff of an arithmetic Asian call option with fixed strike  $K$  reads:

$$C_{fx}^a(K, T) \equiv \max \{A_T^\Delta - K, 0\}. \quad (7)$$

As realized in Clewlow and Carverhill (1990), the distribution of  $A_T^\Delta$  can be obtained recursively. If we define  $Z_k^\Delta \equiv m\Delta + X_k^\Delta$ , from (6) we are interested in the distribution of the following quantity

$$\sum_{k=1}^T S_{\Delta k} = e^{Z_1^\Delta} \left( 1 + e^{Z_2^\Delta} \left( \dots \left( 1 + e^{Z_T^\Delta} \right) \right) \right).$$

Starting from  $L_T^\Delta \equiv e^{Z_1^\Delta}$  and introducing recursively the quantities

$$L_k^\Delta \equiv e^{Z_k^\Delta} \left( 1 + L_{k+1}^\Delta \right), \quad k = T-1, \dots, 1, \quad (8)$$

we obtain  $A_T^\Delta \equiv S_0 \left( 1 + L_1^\Delta \right) / (T+1)$ . Therefore, the key ingredient for the computation of fixed-strike arithmetic Asian options is the density of  $L_1^\Delta$  or equivalently, the density of  $B_1^\Delta \equiv \ln(L_1^\Delta)$ . We discuss this computation in

Section 3.1. Once we obtain the density  $f_{B_1}$  we can price call options<sup>3</sup> with an additional numerical integration:

$$\mathbb{E} \{C_{fx}^a(K, T)\} = e^{-rT} \int_{\gamma}^{+\infty} \left( \frac{S_0}{T+1} (1 + e^x) - K \right) f_{B_1}(x) dx, \quad (9)$$

where  $\gamma \equiv \ln(K(T+1)/S_0 - 1)$ . Furthermore, once we have the density  $f_{B_1}$  we can compute option prices for different strikes  $K$  and spot prices  $S_0$ . From (9) we can also easily compute the Greeks. For instance, for the delta we obtain:

$$\Delta = \frac{e^{-rT} \partial \mathbb{E} \{C_{fx}^a(K, T)\}}{\partial S_0} = \frac{e^{-rT}}{T+1} \int_{\gamma}^{+\infty} (1 + e^x) f_{B_1}(x) dx. \quad (10)$$

Similarly, for the gamma we obtain:

$$\Gamma = \frac{e^{-rT} \partial^2 \mathbb{E} \{C_{fx}^a(K, T)\}}{\partial S_0^2} = e^{-rT} \left( \frac{K}{S_0} \right)^2 \frac{T+1}{K(T+1)-S_0} f_{B_1}(\gamma). \quad (11)$$

Finally, notice that the recursion (8) translates into a formula for the moments of the arithmetic average. Indeed, from the independence of  $Z_k^\Delta$  and  $L_{k+1}^\Delta$  as well as from the definition of  $Z_k^\Delta$ , we obtain:

$$\begin{aligned} \mathbb{E} \{ (L_k^\Delta)^n \} &= \mathbb{E} \{ (e^{Z_k^\Delta} (1 + L_{k+1}^\Delta))^n \} \\ &= e^{nm\Delta} \phi_{X^\Delta}(-in) \sum_{q=0}^n \binom{n}{q} \mathbb{E} \{ (L_{k+1}^\Delta)^q \}, \end{aligned} \quad (12)$$

where the recursion starts with the following initial condition:

$$\mathbb{E} \{ (L_T^\Delta)^n \} \equiv \mathbb{E} \{ e^{nZ_T^\Delta} \} = \phi_{X^\Delta}(-in). \quad (13)$$

The moments of the arithmetic average then can be computed as follows:

$$\mathbb{E} \{ (A_T^\Delta)^n \} \equiv \left( \frac{S_0}{T+1} \right)^n \sum_{j=0}^n \binom{n}{j} \mathbb{E} \{ (L_T^\Delta)^j \}. \quad (14)$$

We will use this result to verify the accuracy of our numerical method. An expression similar to (14) was obtained also in Albrecher (2004).

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<sup>3</sup>If we are interested in pricing put options, we have to integrate over the relevant domain or to exploit the put-call parity for Asian options.

### 3.1 Computation of the pricing density

Since  $Z_k^\Delta$  and  $L_{k+1}^\Delta$  are independent, the density of  $B_k \equiv \ln(L_k^\Delta) = Z_k^\Delta + \ln(1 + L_{k+1}^\Delta)$  is the convolution of the density  $f_{Z_k^\Delta}$  and that of  $\ln(1 + e^{B_{k+1}^\Delta})$ . Furthermore, since the  $Z_k^\Delta$  are i.i.d. the density  $f_{Z_k^\Delta}$  does not depend on the monitoring time index  $k$ , which we drop from the notation. With a change of variable we obtain that the density of  $f_{B_j}$  satisfies the recursion:

$$f_{B_k}(x) = \int_{-\infty}^{+\infty} f_{Z^\Delta}(x - \ln(e^y + 1)) f_{B_{k+1}}(y) dy, \quad k = T-1, \dots, 1, \quad (15)$$

where the initial condition is set as  $f_{B_T} \equiv f_{Z^\Delta}$ . For some specifications of the underlying Lévy process such as Gaussian, NIG, Double Exponential and Jump-Diffusion this density is known analytically; for other specifications, such as CGMY, it can be obtained by inverting the characteristic function of the log returns with the FFT. We remark that a recursion similar to (15) appears in Clewlow and Carverhill (1990) and then in Benhamou (2002). These authors exploit the convolution structure of the recursion to obtain the density of  $B_k^\Delta$  by applying an FFT and an inverse FFT at each monitoring date. Instead, we use the FFT once to generate the density of  $Z_k^\Delta$  given its characteristic function and then we implement a series of recursive quadratures.

We proceed by approximating the integral (15) using an  $M$ -point quadrature formula, see Press et al. (1997):

$$\begin{aligned} f_{B_k}(x) &\approx \int_l^u \varphi_{Z^\Delta}(x - \ln(e^y + 1)) f_{B_{k+1}}(y) dy \\ &\approx \sum_{j=1}^M w_j \phi(x - \ln(e^{y_j} + 1)) f_{B_{k+1}}(y_j), \end{aligned} \quad (16)$$

where  $y_j$  are the abscissas and  $w_j$  the corresponding weights in the quadrature formula. An issue in the implementation of the above procedure is the choice of the domain  $[l, u]$ . We use the results in Philips and Nelson (1995) that, for a given random variable  $X$ , yield a bound to  $\Pr(X > c)$  and to  $\Pr(X < -c)$  in terms of the integer moments of  $X$ . In our implementation we focus on the first ten integer moments of  $B_T^\Delta$  to determine  $l$  such that  $\Pr(B_T^\Delta < l) \leq 10^{-8}$  and similarly we focus on the first ten integer moments of  $L_T^\Delta \equiv \exp(B_T^\Delta)$  to determine  $u$  such that  $\Pr(L_T^\Delta > e^u) \leq 10^{-8}$  (the latter

moments are readily provided by (12)). These choices have proved sufficient to achieve accurate results. Another issue in the implementation of (16) is the choice of the quadrature rule: in our numerical implementation we adopt a Gaussian quadrature rule. To motivate this choice with respect to alternative procedures we observe that the quantity to be estimated, i.e. the option price, can be represented as a multiple integral

$$I = \int_{\ln\left(\frac{KT}{S_0}\right)}^{+\infty} dx \int_{-\infty}^{+\infty} dy_n \cdots \int_{-\infty}^{+\infty} dy_1 \zeta(x, \mathbf{y}), \quad (17)$$

where

$$\zeta(x, \mathbf{y}) \equiv \left( \frac{S_0}{T+1} (1 + e^x) - K \right) \varphi_{Z^\Delta}(x - \ln(e^{y_n} + 1)) \cdots \varphi_{Z^\Delta}(y_1). \quad (18)$$

An  $M$ -point numerical quadrature approximates (17) with the following expression:

$$\hat{I} \equiv \sum_{k_{n+1}=1}^M \sum_{k_n=1}^M \cdots \sum_{k_1=1}^M w_{k_1} \cdots w_{k_n} w_{k_{n+1}} \zeta(x_{k_{n+1}}, y_{k_1}, \dots, y_{k_n}). \quad (19)$$

The convergence rate using the trapezoid rule is  $O(J^{-2/n})$ , where  $J$  is the number of evaluations of  $\zeta(x, \mathbf{y})$ , see Haselgrove (1961). Using the Simpson rule we have an improvement to  $O(J^{-4/n})$ . A crude Monte Carlo simulation samples points uniformly and averages the function at these points, providing an approximation to (17) which is characterized by a standard error which is independent of  $n$  and of order  $O(J^{-1/2})$ : for sufficiently large values of  $n$ , this convergence rate is better than either the trapezoid or the Simpson rule. On the other hand, using an  $M$ -point Gaussian quadrature, the error is  $O(J^{-(2M+1)/2n})$ , and for  $M$  sufficiently large we obtain a faster convergence than with any of the above methods, including Monte Carlo simulations. We mention that recursive quadrature has received attention in the literature on barrier options, see Aitsahalia and Lai (1997), Sullivan (2000), Andricopoulos et al. (2003), Fusai and Recchioni (2005).

A third issue in the implementation of the recursion (16) is the computational cost. We can write that recursion in matrix form as follows:

$$\mathbf{f}_k = \mathbf{K} \mathbf{D} \mathbf{f}_{k+1}, \quad (20)$$

where  $\mathbf{f}_k$  is a vector with elements  $f_{B_k}(x_j)$ ;  $\mathbf{K}$  is an  $M \times M$  kernel whose  $(k, j)$ -th element reads  $\phi(x_k - \ln(e^{y_j} + 1))$ ; and  $\mathbf{D}$  is a diagonal matrix with



elements  $d_{jj} = w_j$ . The density at the  $n$ -th monitoring date is then given by iterating (20) starting from  $\mathbf{f}_n$ . Therefore, the solution at the  $n$ -th monitoring date for each value of  $x$  requires  $O(nM^2)$  function evaluations (matrices  $\mathbf{K}$ ), plus  $O(nM^2)$  elementary operations. Therefore the total cost is of the order of  $O(nM^2)$  elementary operations.

To summarize, the algorithm, which was implemented in C, proceeds as follows:

- Define the parametric model for log-returns as in Table 1.
- Using the FFT algorithm compute the density function of the log-returns, and assign it as initial condition  $\mathbf{f}_T$  to the recursion (20).
- Using the weights and abscissas of the Gaussian quadrature, construct the product matrix  $\mathbf{KD}$  and then iteratively compute  $\mathbf{f}_n, \mathbf{f}_{n-1}, \dots, \mathbf{f}_1$ .
- Compute the option price by numerical integration of the payoff function with the density  $\mathbf{f}_1$  as in (9).
- Check the accuracy of the results by comparing the numerical moments of the arithmetic average with the theoretical expression provided by (14).

## 4 Numerical experiments

In this section we perform numerical tests to examine the accuracy of our procedure. More precisely, we compare our recursive pricing procedure with the results of a standard Monte Carlo-based pricing with one million scenarios and using the price of the Geometric option as control variate<sup>4</sup>.

The first issue is the impact of the monitoring frequency. It is well known that for barrier options the discrepancy between option prices under continuous and discrete monitoring can be significant. Indeed, the convergence of the discrete monitored barrier option prices to the continuous case is extremely slow, of the order of  $n^{-1/2}$ , where  $n$  is the number of monitoring dates. In the sequel, we investigate if this is also the case for Asian options when the underlying follows a Lévy process.

### 4.1 Parameter settings

To perform a comparison between different Lévy models, we proceed as follows. We consider the calibration results reported in Fusai and Meucci

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<sup>4</sup>For details on the Monte Carlo simulation, the control variate and for more detailed results see Fusai and Meucci (2007).

(2007).

- The calibrated volatility parameter for the GBM is  $\hat{\sigma} = 0.17801$ .
- The calibrated parameters for the NIG process are:

$$\hat{\alpha} = 6.1882, \quad \hat{\beta} = -3.8941, \quad \hat{\delta} = 0.1622.$$

- The calibrated parameters for the CGMY process are:

$$\hat{C} = 0.0244, \quad \hat{G} = 0.0765, \quad \hat{M} = 7.5515, \quad \hat{Y} = 1.2945. \quad (21)$$

- The calibrated parameters for the Merton model are:

$$\hat{\sigma} = 0.126349, \quad \hat{\alpha} = -0.390078, \quad \hat{\lambda} = 0.174814, \quad \hat{\delta} = 0.338796.$$

- The calibrated parameters for the double-exponential model are:

$$\hat{\sigma} = 0.120381, \quad \hat{\lambda} = 0.330966, \quad \hat{p} = 0.20761, \quad \hat{\eta}_1 = 9.65997, \quad \hat{\eta}_2 = 3.13868. \quad (22)$$

Figure 1 shows the density of the log-returns for the calibrated models. The Merton jump-diffusion model, the Kou double exponential model and the CGMY densities appear very similar and remarkably different from the Gaussian case. In particular, the skewness and kurtosis parameters are respectively equal to -19.813 and 986.936 for the CGMY model, to -2.16016 and 7.46374 for the Merton model, to -2.77006 and 13.5805 for the double exponential model and to -2.13745 and 9.93736 for the NIG model.

[Insert Figure 1 here]

## 4.2 Numerical Results

We consider now arithmetic-average Asian options. In Table 2 we report the CPU time and the square root of the sum of squared errors, where the errors are the differences between the analytical moments and the numerical moments of order 0 to 5. The analytical moments are computed according to formula (14), whilst the numerical are computed in two ways: using one million Monte Carlo simulations (with and without the Geometric Average

as control variate; for these results we report the standard error) and using the density obtained by recursion (20).

The most interesting results are that, except for the NIG process, the proposed numerical integration procedure provides much more accurate estimates in lesser computational time than Monte Carlo simulation also if implemented using control variates. We stress that the Monte Carlo approach is a viable alternative only if enhanced by the control variates.

The computational cost of our technique is linear in the number of monitoring dates and quadratic in the number of quadrature points. Extremely accurate result can be obtained for the Gaussian, jump-diffusion and double exponential processes, even with a very large number of monitoring dates ( $\approx 250$ ) and a low number of nodes ( $\approx 3000$ ). Slightly less accurate results are obtained for the CGMY process, but in this case the Monte Carlo simulations do not appear to provide a reliable alternative at all. Instead, for the NIG model, our approach does not perform well, because the density peaks as we increase the monitoring frequency. In this case the Monte Carlo appears to be the only viable alternative.

[Insert Table 2 here]

In Tables 3-7 we consider prices of arithmetic Asian options. The numerical results previously presented seem to justify that model risk does not seem to be an issue (although Gaussian and NIG models produce somewhat different option prices), as important as for barrier options. Intuitively, the averaging process tapers the thickness of the tails, whilst for barrier options the model sensitivity is much higher, as different path properties are emphasized by the knock-out/in effect of the barrier, see Schoutens et al. (2004). The tapering effect for Asian options is confirmed by Figure 2, which displays the densities of the geometric and arithmetic average according to different models. As we can see, with the exception of the Gaussian and NIG cases, these densities look very similar.

Furthermore, the number of monitoring dates does not seem as crucial as for barriers. Therefore, in order to approximate the continuously monitored solution, we can use the discrete solution with a low number of monitoring dates, for which our algorithm is reliable and fast.

[Insert Table 3 here]

[Insert Table 4 here]

[Insert Table 5 here]

[Insert Table 6 here]

[Insert Table 7 here]

[Insert Figure 2 here]

Finally, in Figures 3, 4 and 5 we report the differences in prices, deltas and gammas computed according to the different models taking as benchmark the geometric Brownian motion. The differences among the Lévy models are small whilst appear remarkable if compared to the GBM case: even from a hedging perspective the effect of model risk is limited, unlike in the case of barrier options, see Schoutens et al. (2004). This is welcome news from a risk-management perspective. Clearly, it remains to be investigated the relevance of the i.i.d. assumption, that underlies all Lévy processes. This will be possibly the topic of future work, although in the non i.i.d. setting our formulae for the geometric case and the numerical approximation for the arithmetic case do not apply.

[Insert Figure 3 here]

[Insert Figure 4 here]

[Insert Figure 5 here]

## 5 Conclusions

We introduce a recursive algorithm to price arithmetic Asian options under the general assumption that the underlying evolves according to a Lévy process. Moreover, we consider discretely monitored options. However, differently from other path-dependent options like barrier and lookback, Asian option prices do not seem to be affected by the monitoring frequency. We also evaluate the impact of different model specification. As it turns out, model risk is significant in the case of the Gaussian and NIG models. It remains to be investigated the effect of stochastic volatility. Unfortunately, both the closed-form analytical formulas for the geometric Asian options and the numerical algorithm for the arithmetic Asian options rely on the i.i.d. assumption for the log-increments of the underlying: this assumption is not satisfied by non-Lévy stochastic volatility models.

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Model	$\psi_{\Delta}(\omega)$
g	$-\frac{\sigma^2}{2}\omega^2\Delta$
nig	$-\delta\Delta\left(\sqrt{\alpha^2-(\beta+i\omega)^2}-\sqrt{\alpha^2-\beta^2}\right)$
cgmy	$C\Delta\Gamma(-Y)\left((M-i\omega)^Y-M^Y+(G+i\omega)^Y-G^Y\right)$
de	$-\frac{1}{2}\sigma^2\omega^2\Delta+\lambda\Delta\left(\frac{(1-p)*\eta_2}{\eta_2+i\omega}+\frac{p*\eta_1}{\eta_1+i\omega}-1\right)$
jd	$-\frac{1}{2}\sigma^2\omega^2\Delta+\lambda\Delta\left(e^{i\omega\alpha-\frac{1}{2}\omega^2\delta^2}-1\right)$
st	$-\kappa^{\alpha} \omega ^{\alpha}\Delta\left(1-i\beta\text{sign}(\omega)\tan\left(\frac{\alpha\pi}{2}\right)\right)$

Table 1: Characteristic exponents of some parametric Lévy processes

		Monte Carlo				Number of Quadrature Points					
						7000		5000		1000	
Modello	Dates	MC	CPU	MC+CV	CPU	MSE	CPU	MSE	CPU	MSE	CPU
g	12	0.08439	63	0.03525	68	0.00000	119	0.00000	65	0.00000	6
g	50	0.97323	125	0.02433	130	0.00640	135	0.00640	52	0.00640	5
g	250	0.56911	454	0.01786	459	0.02621	250	0.02621	130	0.02537	8
nig	12	0.93767	88	0.01573	93	0.01106	67	0.01106	35	1.91168	5
nig	50	0.85467	225	0.03892	230	0.02555	97	0.06330	52	n.a.	6
nig	250	1.69247	952	0.01453	957	n.a.	252	n.a.	131	n.a.	8
cgmy	12	25.37854	105	2.84911	111	0.52170	64	0.52170	37	0.52115	8
cgmy	50	7.25580	293	0.77980	299	1.07702	95	1.07702	53	4.10542	10
cgmy	250	4.00879	997	0.43916	1005	1.18273	218	4.74037	126	n.a.	12
jd	12	0.82418	70	0.82418	75	0.03609	65	0.03609	34	0.03609	5
jd	50	1.50117	156	1.50117	161	0.04138	95	0.04138	51	0.04138	6
jd	250	1.81340	602	1.81340	607	0.04307	246	0.04307	131	6.36434	9
de	12	0.38332	73	0.38332	78	0.10211	73	0.10211	40	0.10211	6
de	50	0.90982	156	0.90982	161	0.14144	107	0.14144	57	0.14145	6
de	250	0.99756	601	0.99756	606	0.11956	261	0.11956	144	n.a.	11

Table 2: In this table we give, for different models, the 1000\*Squared Root of Sum of Squared (column ERR), the CPU time in Seconds (CPU column) for the Monte Carlo simulation (Crude and with Control Variate) and for the recursive quadrature method (with different number of grid points: from 1000 to 7000). As benchmark we have used the first five moments computed according to formula (14)

MC simulation				Numerical Quadrature		
n	K	MC+CV	se	10000	5000	1000
12	90	11.90505	0.213	11.90497	11.90498	11.90428
12	100	4.88207	0.173	4.88210	4.88212	4.88199
12	110	1.36329	0.145	1.36314	1.36314	1.36371
50	90	11.93302	0.203	11.93301	11.93299	11.93339
50	100	4.93735	0.169	4.93736	4.93738	4.93711
50	110	1.40267	0.150	1.40264	1.40262	1.40199
250	90	11.94096	0.201	11.94068	11.94069	11.94137
250	100	4.95244	0.169	4.95233	4.95239	4.94942
250	110	1.41359	0.151	1.41351	1.41350	1.41290

Table 3: Prices of arithmetic Asian options for Gaussian process. Parameters:  $S_0 = 100$ ,  $r = 0.0367$ ,  $\sigma = 0.17801$ .

MC simulation				Numerical Quadrature		
n	K	MC+CV	se	11000	5000	1000
12	90	12.62293	0.341	12.62243	12.62243	12.61315
12	100	5.06077	0.223	5.06060	5.06057	5.05723
12	110	1.01374	0.176	1.01355	1.01355	1.01379
50	90	12.66112	0.333	12.66118	12.66160	14.11925
50	100	5.10359	0.221	5.10367	5.10383	4.88735
50	110	1.03770	0.177	1.03770	1.03774	0.75191
250	90	12.67186	0.329	n.a.	n.a.	n.a.
250	100	5.11558	0.218	n.a.	n.a.	n.a.
250	110	1.04446	0.176	n.a.	n.a.	n.a.

Table 4: Prices of arithmetic Asian options for NIG process. Parameters:  $S_0 = 100$ ,  $r = 0.0367$ ,  $\alpha = 6.1882$ ,  $\beta = -3.8941$ ,  $\delta = 0.1622$ .

MC simulation				Numerical Quadrature		
n	K	MC+CV	se	10000	5000	1000
12	90	12.69114	0.360	12.70625	12.70626	12.70557
12	100	5.02787	0.211	5.03492	5.03486	5.03649
12	110	1.01895	0.156	1.02115	1.02110	1.02214
50	90	12.73548	0.371	12.73854	12.73855	12.74033
50	100	5.07403	0.223	5.07570	5.07577	5.07225
50	110	1.04594	0.165	1.04674	1.04669	1.04948
250	90	12.74864	0.393	12.74737	12.72731	n.a
250	100	5.08744	0.233	5.08694	5.07912	n.a
250	110	1.05377	0.170	1.05389	1.05251	n.a

Table 5: Prices of arithmetic Asian options for CGMY process. Parameters:  $S_0 = 100$ ,  $r = 0.0367$ ,  $C = 0.0244$ ,  $G = 0.0765$ ,  $M = 7.5515$ ,  $Y = 1.2945$ .

MC simulation				Numerical Quadrature		
n	K	MC+CV	se	10000	5000	1000
12	90	12.71298	0.379	12.71236	12.71236	12.71252
12	100	5.01729	0.243	5.01712	5.01705	5.01772
12	110	1.04141	0.183	1.04142	1.04147	1.03957
50	90	12.74420	0.371	12.74369	12.74369	12.74386
50	100	5.05849	0.243	5.05809	5.05814	5.05781
50	110	1.06886	0.184	1.06878	1.06877	1.06983
250	90	12.75141	0.358	12.75241	12.75242	12.86165
250	100	5.06910	0.232	5.06949	5.06943	5.11830
250	110	1.07632	0.177	1.07646	1.07637	1.08915

Table 6: Prices of arithmetic Asian options for Double Exponential process. Parameters:  $S_0 = 100$ ,  $r = 0.0367$ ,  $\sigma = 0.120381$ ,  $\lambda = 0.330966$ ,  $p = 0.2071$ ,  $\eta_1 = 9.65997$ ,  $\eta_2 = 3.13868$ .

MC simulation				Numerical Quadrature		
n	K	MC+CV	se	10000	5000	1000
12	90	12.71035	0.375	12.71066	12.71065	12.71038
12	100	5.01117	0.248	5.01127	5.01124	5.01008
12	110	1.05141	0.189	1.05162	1.05165	1.05140
50	90	12.74051	0.367	12.74093	12.74093	12.74112
50	100	5.05226	0.245	5.05246	5.05243	5.05337
50	110	1.07959	0.192	1.07959	1.07962	1.07966
250	90	12.74964	0.366	12.74917	12.74918	12.78492
250	100	5.06417	0.244	5.06381	5.06387	5.08129
250	110	1.08772	0.189	1.08740	1.08737	1.09043

Table 7: Prices of arithmetic Asian options for Merton Jump-Diffusion process. Parameters:  $S_0 = 100$ ,  $r = 0.0367$ ,  $\sigma = 0.126349$ ,  $\alpha = -0.390078$ ,  $\lambda = 0.174814$ ,  $\delta = 0.338796$ .

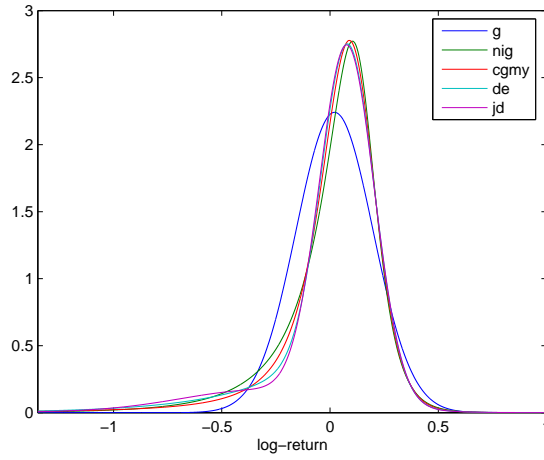


Figure 1: Densities of the log-returns at the 1 yr horizon.

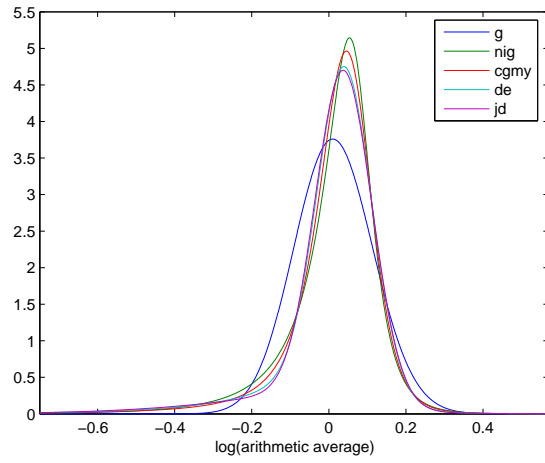


Figure 2: Density of the  $\log(\text{arithmetic mean})$  for the different Lévy models (25 monitoring dates).

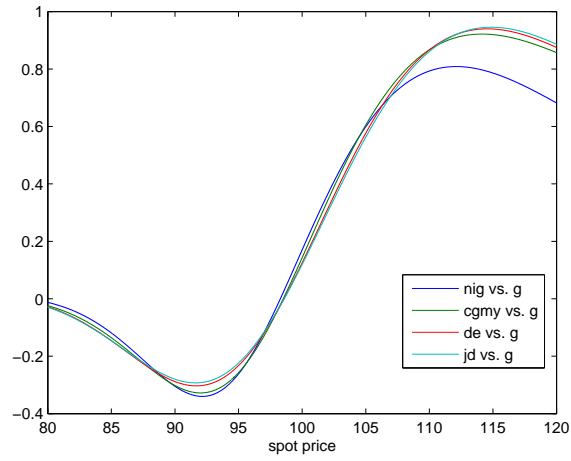


Figure 3: Price differences (Lévy model vs Gaussian) of the arithmetic Asian option under different Lévy models (25 monitoring dates).

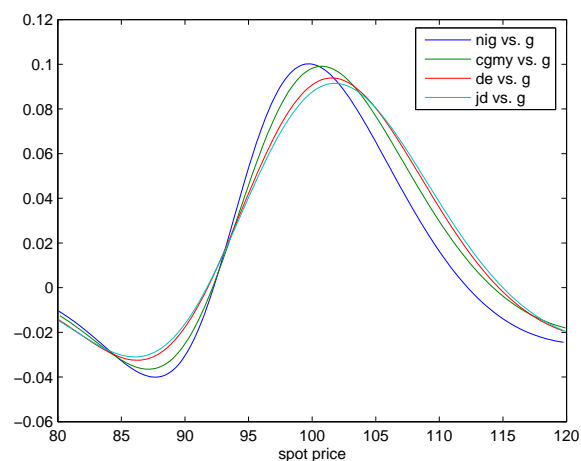


Figure 4: Delta differences (Lévy model vs Gaussian) of the arithmetic Asian option under different Lévy models (25 monitoring dates). The delta has been computed using formula (10).

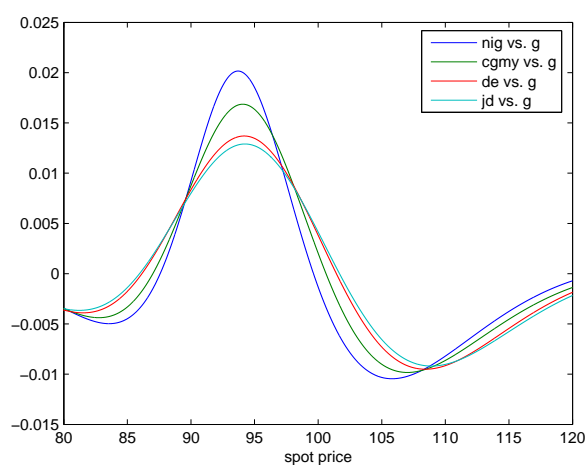


Figure 5: Gamma differences (Lévy model vs Gaussian) of the arithmetic Asian option under different Lévy models (25 monitoring dates). The gamma has been computed using formula (11).