

Premia 14

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1 Approximation Formulae of American options

In this section we present quasi-analytical solution for American options.

1.1 Barone-Adesi Whaley Approximation of the American Put

We now present MacMillan's, or Barone-Adesi and Whaley's approximation formula (see MacMillan [6], Barone-Adesi and Whaley [8]). Let $(W_t, t \geq 0)$ be a (F_t) -Brownian motion. Let r, δ and σ be three positive real numbers. Again, we suppose that the price of the risky asset obeys the Black and Scholes model under the risk neutral probability:

$$dS_t = S_t ((r - \delta)dt + \sigma dW_t).$$

Let us denote by $\phi(x) := (K - x)_+$ the payoff function of the American Put. Let

$$V_0 := \mathbb{E}(e^{-rT} \phi(S_T)),$$

be the price at time 0 of the European Put option price with maturity T and let

$$V_0^* := \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}(e^{-r\tau} \phi(S_\tau))$$

be the price at time 0 of the American Put option price with maturity T . For all smooth function f , let us recall the infinitesimal generator associated with the Black-Scholes diffusion process:

$$Af(x) := \frac{\sigma^2 x^2}{2} \frac{\partial^2 f(x)}{\partial x^2} + (r - \delta)x \frac{\partial f(x)}{\partial x}.$$

Let us recall also that $V_0 = u(0, S_0)$ where $u(t, x)$ is the classical solution to

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) + Au(t, x) - ru(t, x) = 0 & \text{in } [0, T) \times \mathbb{R}_+, \\ u(T, x) = \phi(x) \end{cases} \quad (1)$$

Suppose that there exists a nice solution $u^*(t, x)$ (which means that one can apply Itô's formula to $u^*(t, S_t)$ and that the first derivative in x of $u^*(t, x)$ is uniformly bounded in $(t, x) \in [0, T] \times \mathbb{R}_+$) to the variational inequality

$$\begin{cases} \max \left(\phi(x) - u(t, x), \frac{\partial u}{\partial t}(t, x) + Au(t, x) - ru(t, x) \right) = 0, & \text{in } [0, T) \times \mathbb{R}_+, \\ u(T, x) = \phi(x) \end{cases} \quad (2)$$

It can be shown that $V_0^* = u^*(0, S_0)$. Besides, if

$$\tau_0 := \inf\{0 \leq t \leq T, u(t, S_t) = \phi(S_t)\},$$

then $V_0^* = \mathbb{E}[e^{-r\tau_0} \phi(S_{\tau_0})]$.

The quadratic method proposed by Barone-Adesi and Whaley is based on exact solutions to approximations of the variational inequality (2).

Set $v(x) := u^*(0, x) - u(0, x)$. One approximates $v(x)$ owing to a one step time discretization (of length T) and a fully implicit method. Thus, the approximation \bar{v} of v is solution to

$$\begin{cases} -\bar{v}(x) + T(A\bar{v}(x) - r\bar{v}(x)) \leq 0 & \text{in } \mathbb{R}_+, \\ \bar{v}(x) \geq \tilde{\psi}(x) := (K - x)_+ - u(0, x) & \text{in } \mathbb{R}_+, \\ (\bar{v}(x) - \tilde{\psi}(x))(-\bar{v}(x) + T(A\bar{v}(x) - r\bar{v}(x))) = 0 & \text{in } \mathbb{R}_+. \end{cases} \quad (3)$$

There exists a continuous solution to (3) with a continuous first derivative:

$$\bar{v}(x) = \begin{cases} \lambda x^\alpha & \text{if } x \geq x^*, \\ \tilde{\psi}(x) & \text{otherwise,} \end{cases} \quad (4)$$

where λ , α and x^* , which is assumed to be lower than K , are characterized as follows:

- the constant α is such that $v(x) = x^\alpha$ is solution to

$$-v(x) + T(Av(x) - rv(x)) = 0$$

and such that $\lim_{x \rightarrow +\infty} v(x) = 0$ which implies that α must be negative.

- besides, as \bar{v} needs to be continuous with a continuous derivative, λ, α and x^* must solve the following system:

$$\begin{cases} \lambda(x^*)^\alpha = \phi(x^*) - u(0, x^*) \\ \lambda\alpha(x^*)^{(\alpha-1)} = -1 - \frac{\partial u}{\partial x}(0, x^*). \end{cases}$$

Thus, we deduce that x^* must be a solution to $f(x) = 0$ where

$$f(x) := |\alpha| \frac{K - u(0, x)}{\frac{\partial}{\partial x}u(0, x) + 1 + |\alpha|} - x.$$

There exists a unique such x^* . Indeed, from the Black and Scholes formula

$$u(0, x) = Ke^{-rT}N(-d_2) - xe^{-\delta T}N(-d_1)$$

with

$$d_1 := \frac{\log\left(\frac{x}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}, \quad N(d) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^d e^{-x^2/2} dx,$$

it is easy to check that $f(0) > 0$, $f(K) < 0$ (since $u(0, x)$ is a convex function), and that $f(x)$ is a decreasing function. Therefore, there exists a unique solution to the equation $f(x) = 0$.

To compute x^* , one can use a classical iterative method for nonlinear equations.

Similarly, the American Put value at time t is

$$u^*(t, x) = \begin{cases} u(t, x) + A_1(x/x^*)^{q_2} & \text{if } x > x^* \\ K - x & \text{otherwise} \end{cases}$$

where

$$A_1 = -\frac{x^*}{q_2}(1 - e^{-\delta(T-t)}N(-d_1(x^*)))$$

$$d_1(x) = \frac{\log\left(\frac{x}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$q_2 = \frac{-(N-1) - \sqrt{(N-1)^2 + 4M/h}}{2}$$

$$M = \frac{2r}{\sigma^2}, N = \frac{2(r-\delta)}{\sigma^2}, h = 1 - \exp^{-r(T-t)}$$

and the critical price x^* is solution of

$$K - x^* = u(t, x^*) - (1 - e^{-\delta(T-t)})N(-d_1(x^*))x^* \frac{1}{q_2}$$

Remark 1. Under the condition $\delta > 0$ which guarantees that the American Call option price is different from the European one, we proceed analogously as above to compute the American Call value at time $t, C^*(t, S_t)$. Thus,

$$C^*(t, x) = \begin{cases} C(t, x) + A_2(x/x^*)^{q_1} & \text{if } x < x^* \\ x - K & \text{otherwise} \end{cases}$$

where

$$A_2 = -\frac{x^*}{q_1}(1 - e^{-\delta(T-t)})N(-d_1(x^*))$$

$$q_1 = \frac{-(N-1) + \sqrt{(N-1)^2 + 4M/h}}{2}$$

and the critical price x^* is solution of

$$x^* - K = C(t, x^*) + (1 - e^{-\delta(T-t)})N(d_1(x^*))x^* \frac{1}{q_1}$$

1.2 MacMillan Approximation

The MacMillan's approximation formula [6] is the same of Whaley with the exponent

$$q_1 = \frac{-(N-1) + \sqrt{(N-1)^2 + 8(1+r\theta)/(\sigma^2\theta)}}{2}$$

$$q_2 = \frac{-(N-1) - \sqrt{(N-1)^2 + 8(1+r\theta)/(\sigma^2\theta)}}{2}$$

where $\theta = T - t$.

1.3 Ho-Stapleton-Subrahmanyam Approximation of the American Put

Let $u^*(t, S_t)$ the value of the American Put option on a dividend-paying stock with maturity T . Let $u(t, x)$ the value of the European Put option, $u_2(t, S_t)$ the value of a Put option which can be exercised at time $\frac{T}{2}$ and T .

The idea of this algorithm is to look at a sequences of prices of american options wick can be exercised only at discrete times taken on regular grid with n steps. Then the authors conjecture that the convergence as n goes to infinity is of the type

$$P_n = P \exp\left(-\frac{\alpha}{n}\right).$$

The Ho-Stapleton-Subrahmanyam approximation formula [9], which is a kind of Richardson extrapolation, consists in approximating the American option value by

$$u^*(t, x) = \frac{[u_2(t, x)]^2}{u(t, x)}$$

with

$$u_2(t, x) = \eta(Kw_2 - xw_1)$$

with $\eta = 1$ for a put option and $\eta = -1$ for a call option and where

$$w_1 = e^{-\delta \frac{T}{2}} N_1(-\eta d'_1) + e^{-\delta T} N_2(\eta d'_1, -\eta d''_1, -\rho)$$

$$w_2 = e^{-r \frac{T}{2}} N_1(-\eta d'_2) + e^{-r T} N_2(\eta d'_2, -\eta d''_2, -\rho)$$

with N_1 and N_2 are, respectively, the standard cumulative univariate and bivariate normal distribution, with parameters

$$d'_1 = \frac{\log\left(\frac{x}{x_1^*}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right) \frac{T}{2}}{\sigma \sqrt{\frac{T}{2}}} \quad , \quad d'_2 = d'_1 - \sigma \sqrt{\frac{T}{2}}$$

$$d''_1 = \frac{\log\left(\frac{x}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}} \quad , \quad d''_2 = d''_1 - \sigma \sqrt{T}$$

and

$$\rho = \sqrt{0.5}$$

The critical stock price, x_1^* , is obtained by solving the relationship

$$\eta(K - x_1^*) = u\left(x_1^*, \frac{T}{2}, K, r, \delta, \sigma\right)$$

with a classical iterative method for nonlinear equations.

The price of the european option $u(x_1^*, \frac{T}{2}, K, r, \delta, \sigma)$ is computed with Black-Sholes formula.

The estimated delta is obtained by differentiating to obtain

$$\frac{\partial u^*}{\partial x} = u^* \left(2 \frac{\Delta_2}{u_2} - \frac{\Delta}{u} \right)$$

where $\Delta_2 = \frac{\partial u_2}{\partial x} = -\eta w_1$ and $\Delta = \frac{\partial u}{\partial x}$ are the hedge ratios of the twice-exercisable and the European option, respectively.

1.4 Bunch-Johnson Approximation of the American Put

The Bunch-Johnson approximation [4] is a modification of of Geske-Johnson procedure [3]. Idea involved in their procedure is to construct a recursive sequence of option's prices P_n such that:

$P_1 = u(t, x)$ the value of the European Put and P_n is the value of an option which can be only exercised at time $t_n^i = \frac{iT}{n}$. The sequence $(P_n)_{n \in \mathbb{N}}$ converges to the American Put value. In case there is polynomial expansion of the price in power of $\frac{1}{n}$, it is easy (Richardson extrapolation) to get the limiting price as a linear combination of a set of P_n 's. For computing the limit, Geske-Johnson use the Richardson extrapolation as follows:

$$u^*(t, x) = P_3 + \frac{7}{2}(P_3 - P_2) - \frac{1}{2}(P_2 - P_1)$$

Bunch-Johnson prefer to use a two-points formula as follows:

$$u^*(t, x) = 2P_2 - P_1.$$

The estimated delta is obtained by differentiating to obtain

$$\frac{\partial u^*}{\partial x} = 2\Delta_2 - \Delta_1$$

where $\Delta_2 = \frac{\partial u_2}{\partial x}$ and $\Delta_1 = \frac{\partial u}{\partial x}$ are the hedge ratios of the twice-exercisable and the European option, respectively.

1.5 Bjerk Sund-Stensland Approximation

The Bjerk Sund-Stensland method is based on a exercise strategy corresponding to a flat boundary I (trigger price) [2]. The price of a call american option is

$$C = \alpha x^\beta - \alpha \phi(x, \theta, \beta, I, I) + \phi(x, \theta, 1, I, I)$$

$$-\phi(x, \theta, 1, K, I) - K\phi(x, \theta, 0, I, I) + K\phi(x, \theta, 0, K, I)$$

where

$$\alpha = (I - K)I^{-\beta}$$

$$\beta = \left(\frac{1}{2} - \frac{r - \delta}{\sigma^2}\right) + \sqrt{\left(\frac{r - \delta}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r}{\sigma^2}}$$

The function $\phi(x, t, \gamma, H, I)$ is given by

$$\phi(x, t, \gamma, I, I) = e^{\lambda x^\gamma} \left(N(d) - \left(\frac{I}{x}\right)^k N\left(d - \frac{2\log(I/x)}{\sigma\sqrt{t}}\right) \right)$$

$$\lambda = (-r + \gamma(r - \delta) + \frac{1}{2}\gamma(1 - \gamma)\sigma^2)t$$

$$d = -\frac{\log\left(\frac{x}{H}\right) + \left(r - \delta + \left(\gamma - \frac{1}{2}\sigma^2\right)\right)t}{\sigma\sqrt{t}} \quad \text{and}$$

$$k = \frac{2(r - \delta)}{\sigma^2} + (2\gamma - 1)$$

and the trigger price I is defined as

$$I = B_0 + (B_\infty - B_0)(1 - e^{h(\theta)})$$

$$h(t) = -((r - \delta)t + 2\sigma\sqrt{t}\left(\frac{B_0}{B_\infty - B_0}\right))$$

$$B_\infty = K\left(\frac{\beta}{\beta - 1}\right)$$

$$B_0 = \max\left(K, \frac{r}{\delta}K\right)$$

If $x \geq I$ it is optimal to exercise the option immediately, and the value must be equal to $x - K$.

If $b = r - \delta \geq r$ it will never be optimal to exercise the American call option before expiration, and the value can be found using the generalized Black-Scholes formula.

The value of the american put is given by the put-call transformation.

$$C(x, K, \theta, r, \delta, \sigma) = P(K, x, \theta, \delta, r, \sigma)$$

The delta is computed with a finite difference method.

1.6 Ju's piecewise exponential approximation of the exercise boundary

This method, described in [7], is based on the early exercise premium formula:

$$P_A = P_E + K(1 - e^{-rT}) - S(1 - e^{-\delta T}) - K \int_0^T r e^{-rt} N(d_2(S, B_t, t)) dt \\ + S \int_0^T \delta e^{-\delta t} N(d_1(S, B_t, t)) dt$$

where

$$d_1(x, y, t) = \frac{\log(x/y) + (r - \delta + \sigma^2/2)t}{\sigma\sqrt{t}}, \\ d_2(x, y, t) = d_1(x, y, t) - \sigma\sqrt{t},$$

P_E is the [Black and Scholes \(1973\)](#) price of the European put option, and B_t the exercise boundary at time t .

As the boundary appears only through $\log(S/B_t)$ in the definitions of d_1 and d_2 , it is reasonable to approximate the function $t \rightarrow B_t$ by exponential pieces.

The advantage of this approach is that the integrals $\int_{t_1}^{t_2} r e^{-rt} N(d_2(S, B e^{bt}, t)) dt$ and $\int_{t_1}^{t_2} \delta e^{-\delta t} N(d_1(S, B e^{bt}, t)) dt$, involved in equation (1), can be evaluated in closed form.

They become respectively $I(t_1, t_2, S, B, b, -1, r)$ and $I(t_1, t_2, S, B, b, 1, \delta)$ where I is defined by :

$$I(t_1, t_2, S, B, b, \phi, \nu) = e^{-\nu t_1} N(z_1 \sqrt{t_1} + \frac{z_2}{\sqrt{t_1}}) - e^{-\nu t_2} N(z_1 \sqrt{t_2} + \frac{z_2}{\sqrt{t_2}}) \\ + \frac{1}{2} \left(\frac{z_1}{z_3} + 1 \right) e^{z_2(z_3 - z_1)} \left(N(z_3 \sqrt{t_2} + \frac{z_2}{\sqrt{t_2}}) - N(z_3 \sqrt{t_1} + \frac{z_2}{\sqrt{t_1}}) \right) \\ + \frac{1}{2} \left(\frac{z_1}{z_3} - 1 \right) e^{-z_2(z_3 + z_1)} \left(N(z_3 \sqrt{t_2} - \frac{z_2}{\sqrt{t_2}}) - N(z_3 \sqrt{t_1} - \frac{z_2}{\sqrt{t_1}}) \right) \quad (5)$$

with

$$\begin{aligned} z_1 &= \frac{r - \delta - b + \phi\sigma^2/2}{\sigma} \\ z_2 &= \frac{\log(S/B)}{\sigma} \\ z_3 &= \sqrt{z_1^2 + 2\nu} \end{aligned}$$

By convention, when $t = 0$, $N(x\sqrt{t} + \frac{y}{\sqrt{t}}) = 0.5 \mathbf{1}_{\{y=0\}} + \mathbf{1}_{\{y>0\}}$.

Ju suggests to make the Richardson extrapolation $\hat{P}_A = \frac{9P_3}{2} - 4P_2 + \frac{P_1}{2}$ between the prices P_1 , P_2 and P_3 obtained respectively for one, two and three exponential pieces. The coefficients of the exponential functions are obtained by checking the smooth fit conditions. For instance, in the case of two pieces : $B_{21}e^{b_{21}t}$ during $[T/2, T]$ and $B_{22}e^{b_{22}t}$ during $[0, T/2]$, these conditions are :

$$\begin{aligned} K - B_{21}e^{b_{21}T/2} &= P_E(B_{21}e^{b_{21}T/2}, K, T/2) + K(1 - e^{-rT/2}) \\ &\quad + B_{21}e^{b_{21}T/2}(1 - e^{-\delta T/2}) \\ &\quad - KI(0, T/2, B_{21}e^{b_{21}T/2}, B_{21}e^{b_{21}T/2}, b_{21}, -1, r) \\ &\quad + B_{21}e^{b_{21}T/2}I(0, T/2, B_{21}e^{b_{21}T/2}, B_{21}e^{b_{21}T/2}, b_{21}, 1, \delta) \end{aligned}$$

$$\begin{aligned} -1 &= -e^{-\delta T/2}N(-d_1(B_{21}e^{b_{21}T/2}, K, T/2)) - (1 - e^{-\delta T/2}) \\ &\quad - K \frac{\partial I}{\partial S}(0, T/2, B_{21}e^{b_{21}T/2}, B_{21}e^{b_{21}T/2}, b_{21}, -1, r) \\ &\quad + I(0, T/2, B_{21}e^{b_{21}T/2}, B_{21}e^{b_{21}T/2}, b_{21}, 1, \delta) \\ &\quad + B_{21}e^{b_{21}T/2} \frac{\partial I}{\partial S}(0, T/2, B_{21}e^{b_{21}T/2}, B_{21}e^{b_{21}T/2}, b_{21}, 1, \delta) \end{aligned}$$

$$\begin{aligned} K - B_{22} &= P_E(B_{22}, K, T) + K(1 - e^{-rT}) - B_{22}(1 - e^{-\delta T}) \\ &\quad - KI(0, T/2, B_{22}, B_{22}, b_{22}, -1, r) \\ &\quad + B_{22}I(0, T/2, B_{22}, B_{22}, b_{22}, 1, \delta) \\ &\quad - KI(T/2, T, B_{22}, B_{21}, b_{21}, -1, r) \\ &\quad + B_{22}I(T/2, T, B_{22}, B_{21}, b_{21}, 1, \delta) \end{aligned}$$

$$\begin{aligned}
-1 &= -e^{-\delta T} N(-d1(B_{22}, K, T)) - (1 - e^{-\delta T}) \\
&\quad - K \frac{\partial I}{\partial S}(0, T/2, B_{22}, B_{22}, b_{22}, -1, r) \\
&\quad + I(0, T/2, B_{22}, B_{22}, b_{22}, 1, \delta) \\
&\quad + B_{22} \frac{\partial I}{\partial S}(0, T/2, B_{22}, B_{22}, b_{22}, 1, \delta) \\
&\quad - K \frac{\partial I}{\partial S}(T/2, T, B_{22}, B_{21}, b_{21}, -1, r) \\
&\quad + I(T/2, T, B_{22}, B_{21}, b_{21}, 1, \delta) \\
&\quad + B_{22} \frac{\partial I}{\partial S}(T/2, T, B_{22}, B_{21}, b_{21}, 1, \delta)
\end{aligned}$$

These systems are solved numerically thanks to the Newton-Raphson algorithm with initial values [Mc Millan](#)'s critical price and 0 for the computation of B_{21} and b_{21} , and the obtained values of B_{21} and b_{21} for the calculus of B_{22} and b_{22} .

1.7 Carr's method of randomization of the maturity

In order to approximate the price of the american put option with deterministic maturity T , Carr [1] suggests to **randomize the maturity** and consider the american put option with maturity $\tau_1 + \tau_2 + \dots + \tau_n$ where τ_1, \dots, τ_n are random variables I.I.D. according to the exponential distribution with parameter $\lambda = n/T$ and independent of the Brownian motion governing the stock dynamics $S_t^x = x \exp(\sigma W_t + (r - \delta - \frac{\sigma^2}{2})t)$. For $1 \leq k \leq n$, let $P^{(k)}$ denote the price of the option with maturity $\tau_1 + \dots + \tau_k$. The expressions for $P^{(k)}$ are obtained inductively by dynamic programming.

We first deal with $P^{(1)}$. Conditionally on $\{\tau_1 > t\}$, $\tau_1 - t$ is an exponential variable with parameter λ independent of the events occurred before time t and the price of the randomized american put option is $P^{(1)}(S_t^x)$. This option is rationally exercised if and only if $S_t^x \leq s_1$ where $s_1 = \sup\{x : P^{(1)}(x) = (K - x)_+\}$. Hence for $x \geq s_1$,

$$P^{(1)}(x) = \mathbb{E} \left(e^{-r(\tau_1 \wedge \nu_{s_1})} (K - S_{\tau_1 \wedge \nu_{s_1}}^x)^+ \right)$$

where $\nu_{s_1} = \inf\{t : S_t^x \leq s_1\}$. If $D(t, x)$ denotes the price of the Down and Out Put with barrier s_1 , rebate $K - s_1$ and maturity t , integration according to the exponential distribution of τ_1 yields

$$\forall x \geq s_1, P^{(1)}(x) = \int_0^{+\infty} \lambda e^{-\lambda t} D(t, x) dt.$$

Making use of the integration by parts formula and of the Black and Scholes P.D.E. satisfied by $D(t, x)$:

$$\begin{cases} \forall (t, x) \in (0, +\infty) \times [s_1, +\infty), D_t = \frac{\sigma^2 x^2}{2} D_{xx} + (r - \delta)x D_x - rD \\ \forall x \geq s_1, D(0, x) = (K - x)^+ \end{cases},$$

one gets the following O.D.E.:

$$\forall x \geq s_1, \frac{\sigma^2 x^2}{2} P_{xx}^{(1)}(x) + (r - \delta)x P_x^{(1)}(x) - r P^{(1)}(x) = \lambda(P^{(1)}(x) - (K - x)^+). \quad (6)$$

The boundary conditions are $\lim_{x \rightarrow +\infty} P^{(1)}(x) = 0$ and the smooth fit conditions : $\lim_{x \searrow s_1} P^{(1)}(x) = (K - s_1)$, $\lim_{x \searrow s_1} P_x^{(1)}(x) = -1$. Taking expectations in Itô's formula gives

$$\begin{aligned} \mathbb{E}(e^{-r\tau_1} P^{(1)}(S_{\tau_1}^x)) &= P^{(1)}(x) \\ &+ \mathbb{E}\left(\int_0^{\tau_1} e^{-rs} (\lambda(P^{(1)}(S_s^x) - (K - S_s^x)^+) + (\delta S_s^x - rK) 1_{\{S_s^x \leq s_1\}}) ds\right). \end{aligned}$$

$$\begin{aligned} \text{As by Fubini's theorem, } \mathbb{E}\left(\lambda \int_0^{\tau_1} e^{-rs} (P^{(1)}(S_s^x) - (K - S_s^x)^+) ds\right) \\ &= \int_0^{+\infty} \lambda e^{-\lambda t} \int_0^t \lambda e^{-rs} \mathbb{E}(P^{(1)}(S_s^x) - (K - S_s^x)^+) ds dt \\ &= \int_0^{+\infty} e^{-\lambda s} \lambda e^{-rs} \mathbb{E}(P^{(1)}(S_s^x) - (K - S_s^x)^+) ds \\ &= \mathbb{E}(e^{-r\tau_1} (P^{(1)}(S_{\tau_1}^x) - (K - S_{\tau_1}^x)^+)) \end{aligned}$$

the early exercise premium formula holds for the american put option with randomized maturity

$$P^{(1)}(x) = \mathbb{E}\left(e^{-r\tau_1} (K - S_{\tau_1}^x)^+\right) + \mathbb{E}\left(\int_0^{\tau_1} e^{-rs} (rK - \delta S_s^x) 1_{\{S_s^x \leq s_1\}} ds\right) \quad (7)$$

The price $p^{(1)}(x)$ of the european put option with random maturity τ_1 satisfies the O.D.E. :

$$\forall x > 0, \frac{\sigma^2 x^2}{2} p_{xx}^{(1)}(x) + (r - \delta)x p_x^{(1)}(x) - (r + \lambda)p^{(1)}(x) = \lambda(K - x)^+.$$

with boundary condition $\lim_{x \rightarrow +\infty} p^{(1)}(x) = 0$. Since the r.h.s. is nil for $x \geq K$, by computation of $p^{(1)}(K)$ one gets

$$\forall x \geq K, p^{(1)}(x) = \left(\frac{x}{K}\right)^{\gamma-\epsilon} p^{(1)}(K) = \left(\frac{x}{K}\right)^{\gamma-\epsilon} (qKR - \hat{q}KD),$$

where $R = \frac{\lambda}{\lambda+r}$, $D = \frac{\lambda}{\lambda+\delta}$, $\gamma = \frac{1}{2} + \frac{\delta-r}{\sigma^2}$, $\epsilon = \sqrt{\gamma^2 + 2(\lambda+r)/\sigma^2}$ and

$$p = \frac{\epsilon - \gamma}{2\epsilon}, \quad q = 1 - p, \quad \hat{p} = \frac{\epsilon - \gamma + 1}{2\epsilon}, \quad \text{and} \quad \hat{q} = 1 - \hat{p}.$$

To compute $p^{(1)}(x)$ for $x \leq K$, Carr makes use of the call-put parity $p^{(1)}(x) = c^{(1)}(x) + KR - xD$. Since the O.D.E. satisfied by the price $c^{(1)}(x)$ of the european call option with random maturity τ_1 is homogeneous for $x \leq K$, one gets $c^{(1)}(x) = (x/K)^{\gamma+\epsilon}(\hat{p}KD - pKR)$. After calculation of the present value of interests less dividends received below the critical price s_1 before τ_1

$$b^1(x) = \left(\frac{x}{s_1}\right)^{\gamma-\epsilon} \left(\frac{qKr}{\lambda+r} - \frac{\hat{q}s_1\delta}{\lambda+\delta}\right),$$

equation (7) writes

$$P^{(1)}(x) = \begin{cases} p^{(1)}(x) + b^1(x) & \text{if } s_0 = K \leq x \\ KR - xD + c^{(1)}(x) + b^1(x) & \text{if } s_1 \leq x \leq s_0 \\ K - x & \text{if } x \leq s_1. \end{cases}$$

The critical price s_1 is obtained from the value-matching condition $K - s_1 = KR - s_1D + c^{(1)}(s_1) + b^1(s_1)$. Without dividends, $D = 1$ and this equation has an explicit solution. If $\delta \neq 0$, it is solved numerically by an iterative method.

We now turn to $P^{(k)}$, $k \geq 2$. Conditionally on $t < \tau_k$, $\tau_k - t$ is an exponential variable with parameter λ ; the price of the american put option with maturity $\tau_1 + \dots + \tau_k$ is $P^{(k)}(S_t^x)$ and the exercise price is $s_k = \sup\{x : P^{(k)}(x) = (x - K)_+\}$. Hence for $x \geq s_k$, the option is equivalent to the european Down and Out option with barrier s_k , rebate $K - s_k$, payoff $P^{(k-1)}(x)$ and random maturity τ_k . Hence $P^{(k)}$ satisfies the O.D.E. analogous to (6)

$$\forall x \geq s_k, \quad \frac{\sigma^2 x^2}{2} P_{xx}^{(k)}(x) + (r - \delta)x P_x^{(k)}(x) - r P^{(k)}(x) = \lambda(P^{(k)}(x) - P^{(k-1)}(x)).$$

with boundary conditions $\lim_{x \rightarrow +\infty} P^{(k)}(x) = 0$ and the smooth fit conditions : $\lim_{x \searrow s_k} P^{(k)}(x) = (K - s_k)$, $\lim_{x \searrow s_k} P_x^{(k)}(x) = -1$. Since $\lambda = \frac{n}{T}$, this formulation shows that the randomization approach is equivalent to the approximation of the free boundary problem satisfied by the price of the american put option by a scheme implicit in the discretized time variable and with no discretization in space (semi-discretization or method of lines). By computations similar to the one made for $P^{(1)}$ one obtains $P^{(k)}$ in terms of the exercise prices s_1, \dots, s_k . The continuity conditions give an equation

that links s_1, \dots, s_{k-1} and s_k . This equation can be solved explicitly when $\delta = 0$ or numerically otherwise to get s_k .

Considering that the error is a smooth function of the time discretization parameter T/n , Carr suggests to make a three point Richardson extrapolation.

1.8 Broadie and Detemple's LBA and LUBA approximations

1.8.1 Lowerbound Approximation (LBA)

Broadie and Detemple have developped a new methode for pricing standard american options in [5]. This methode is based on the price of a european up and out call option with strike K , barrier L and rebate $(L - K)$.

This option price is given by the following formula :

$$\begin{aligned} C(x, L) = & (L - K) \left[\lambda \frac{2\phi}{\sigma^2} N(d_0) + \lambda \frac{2\phi}{\sigma^2} N(d_0 + 2f \frac{\sqrt{T}}{\sigma}) \right] \\ & + x.e^{-\delta T} [N(d_1^-(L) - \sigma\sqrt{T}) - N(d_1^-(K) - \sigma\sqrt{T})] \\ & - \lambda^{-2} \frac{x-\delta}{\sigma^2} L.e^{-\delta T} [N(d_1^+(L) - \sigma\sqrt{T}) - N(d_1^+(K) - \sigma\sqrt{T})] \\ & - K.e^{-rT} [N(d_1^-(L)) - N(d_1^-(K))] \\ & - \lambda^{1-2} \frac{x-\delta}{\sigma^2} [N(d_1^+(L)) - N(d_1^+(K))] \end{aligned}$$

Where :

$$b = \delta - r + \frac{1}{2}\sigma^2$$

$$f = \sqrt{b^2 + 2r.\sigma^2}$$

$$\phi = \frac{1}{2}(b - f)$$

$$\alpha = \frac{1}{2}(b + f)$$

$$\lambda = \frac{x}{L}$$

$$d_0 = \frac{\log(\lambda) - f(T)}{\sigma\sqrt{T}}$$

$$d_1^+(x) = \frac{\log(\lambda) - \log(L) + \log(x) + b.T}{\sigma\sqrt{T}}$$

$$d_1^-(x) = \frac{-\log(\lambda) - \log(L) + \log(x) + b.T}{\sigma\sqrt{T}}$$

Since the call up and out with rebate $(L - K)$ corresponds to exercise at the minimum of the hitting time of the boundary L and the maturity T , its price is smaller than the price of the American call option. Therefore, $C^l = \max_L C(x, L)$ provides a lower bound for the price of the American call.

So the lower bound is :

$$C^l(x) = \max_L C(x, L)$$

To obtain the approximation from this bound, Broadie and Detemple apply a coefficient λ_1 , which they have obtained after a linear regression upon 10 parameters on 2500 options.

The Lower bound approximation is :

$$C_{lba} = \lambda_1 \cdot C^l$$

1.8.2 Lower and Upperbound Approximation (LUBA)

This approximation depends on the lower bound C^l , and an upperbound C^u . It was developed by Broadie and Detemple in [5]. First, Broadie and Detemple define a function L^* as follows :

$$\forall t \in [0, T] \text{ let } L_t^* \text{ the solution of } \lim_{x \nearrow L} \frac{\partial C_t(x, L)}{\partial L} = 0$$

This function is a lower bound for the optimal exercise boundary.

From the optimal exercise boundary B , we can compute the price of the american call option using the early exercise premium formula :

$$V(x, B) = c(x) + \int_{s=0}^T [\delta \cdot x \cdot e^{-\delta \cdot s} N(d_2(x, B_s, s)) - r \cdot K \cdot e^{-r \cdot s} N(d_3(x, B_s, s))] ds$$

With :

$c(x)$ the european call option price

$$d_2(x, B_s, s) = \frac{\log(\frac{x}{B_s}) + (r - \delta + \frac{1}{2}\sigma^2)(s)}{\sigma\sqrt{s}}$$

$$d_3(x, B_s, s) = d_2(x, B_s, s) - \sigma\sqrt{s}$$

Broadie and Detemple show that since the boundary L^* is under the optimal exercise boundary B , $C^u(x) = V(x, L^*)$ is an upper bound of the price of the American call.

To obtain the approximation using both the lowerbound C^l and the upperbound C^u , Broadie and Detemple apply a coefficient λ_2 which they have obtained by making a regression upon 14 parameters on 2500 options.

The lower and upperbound approximation for the call option price is :

$$C_{luba} = \lambda_2 C^l + (1 - \lambda_2) C^u$$

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