One-factor Markov-functional interest rate models and pricing of Bermudan swaptions

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Premia 14

1 Preliminaries and notation

Most of what is presented here is taken from [HKP]. Let $P(t,T)$ denote the value at time t of a zero-coupon bond which matures and pays unity at time T. We denote by $\mathcal{F}_t$ the information available at time t from observing the values of these assets, i.e. $\mathcal{F}_t := \sigma(P(t,T); t \in \mathbb{R}_+)$. Let $(N,\mathbf{N})$ be a numeraire pair, i.e. a numeraire (N_t) and a measure $\mathbf{N}$ equivalent to the original measure such that the $\widetilde{P}(t,T) := \frac{P(t,T)}{N_t}$ are $\{\mathcal{F}_t\}$-martingales.
Given payment dates $S = (S_1, \ldots, S_M)$ and daycount fractions $\tau = (\tau_1, \ldots, \tau_M)$, we define

$$A_t^{S,\tau} \quad := \quad \sum_{j=1}^{M} \tau_j \, P(t,S_j) \qquad \text{principal value of basis point (PVBP)} \,.$$

Given, in addition, a (swap starting) date T, we define

$$R_t^{S,\tau,T} \quad := \quad \frac{P(t,T) - P(t,S_M)}{A_t^{S,\tau}} \qquad \text{swap rate} \,.$$

The corresponding (payer) swaption with maturity T and strike K is defined by the following payoff (at T) :

$$A_T^{S,\tau}\,(R_T^{S,\tau,T} - K)_+ \qquad \text{(payoff of swaption)} \,.$$

The corresponding digital (payer) swaption with maturity T and strike K is defined by the following payoff (at T) :

$$A_T^{S,\tau}\, 1_{R_T^{S,\tau,T} > K} \qquad \text{(payoff of digital swaption)} \,.$$

Note that, in the particular case $M = 1$, the quantity $R_t^{S,\tau,T}$ is nothing but the (simply compounded) forward rate as seen at time t for the period $[T,S]$.

2 The general model

For $i = 0, \ldots, m-1$, we fix payment dates $S^i = (S^i_1, \ldots, S^i_{M_i})$, daycount fractions $\tau^i = (\tau^i_1, \ldots, \tau^i_{M_i})$ and a swap starting date T_i. Now we denote

$$A^i_t \;:=\; A^{S^i,\tau^i}_t \qquad \text{and} \qquad R^i_t \;:=\; R^{S^i,\tau^i,T_i}_t \;.$$

We make the following hypotheses:

(i) (x_t) is a one-dimensional Markov process under $\mathbf{N}$ with a known law.
(ii) For all $i = 0, \ldots, m-2$, we have $R^i_{T_i} = \mathcal{R}_i(x_{T_i})$ for some strictly increasing (but apriori unknown !) function $\mathcal{R}_i$. [Here we use the fact that (x_t) is one-dimensional.]
(iii) We have $N_{T_{m-1}} = \mathcal{N}_{m-1}(x_{T_{m-1}})$ for some (known) function $\mathcal{N}_{m-1}$.
(iv) For all $i = 0, \ldots, m-2$ and $j = 1, \ldots, M_i$, we have: if $S^i_j \notin \{T_{i+1}, \ldots, T_{m-1}\}$, then $S^i_j > T_{m-1}$ and $P(T_{m-1}, S^i_j) = \mathcal{P}_{i,j}(x_{T_{m-1}})$ for some (known) function $\mathcal{P}_{i,j}$.

In order to price e.g. Bermudan swaptions with our model by using a tree for the process (x_t), it is crucial to find the functional forms $N_{T_i} = \mathcal{N}_i(x_{T_i})$ for $i = 0, \ldots, m-2$; see Section 6 for details. A first step towards these functional forms is the following lemma. We employ the usual evolution family of operators $(U_{t,s})_{t\geq s\geq 0}$ associated to the process (x_t):

$$U_{t,s}f(y) \;:=\; E^N(\, f(x_t) \,|\, x_s = y\,) \;.$$

Recall that we have the following property:

$$E^N(\, f(x_t) \,|\, \mathcal{F}_s\,) \;=\; U_{t,s}f(x_s) \;.$$

Lemma 2.1. *Let $i \in \{0, \ldots, m-2\}$. Suppose that, for all $k = i+1, \ldots, m-1$, we have $N_{T_k} = \mathcal{N}_k(x_{T_k})$ for some (known) function $\mathcal{N}_k$.*
(a) For all $j = 1, \ldots, M_i$, we have

$$\widetilde{P}(T_i, S^j_i) \;=\; \widetilde{\mathcal{P}}_{i,j}(x_{T_i})\,,\ \text{where } \widetilde{\mathcal{P}}_{i,j} \;:=\; \begin{cases} U_{T_k,T_i}\,\frac{1}{\mathcal{N}_k} & S^i_j = T_k \text{ with } k \in \{i+1, \ldots, m-1\} \\ U_{T_{m-1},T_i}\,\frac{\mathcal{P}_{i,j}}{\mathcal{N}_{m-1}} & \text{otherwise} \end{cases} \;.$$

(b) We have

$$\widetilde{A}^i_{T_i} \;=\; \widetilde{\mathcal{A}}_i(x_{T_i})\,,\ \text{where } \widetilde{\mathcal{A}}_i \;:=\; \sum_{j=1}^{M_i} \tau^i_j\, \widetilde{\mathcal{P}}_{i,j} \;.$$

Proof. (a) In the first case, the assertion follows from our hypothesis on the N_{T_k}:

$$\widetilde{P}(t, S^i_j) \;=\; E^N(\, \widetilde{P}(T_k, T_k) \,|\, \mathcal{F}_t\,) \;=\; E^N(\, \tfrac{1}{\mathcal{N}_k(x_{T_k})} \,|\, \mathcal{F}_t\,) \;=\; \Big(U_{T_k,t}\,\tfrac{1}{\mathcal{N}_k}\Big)\,(x_t)\,.$$

In the second case, the assertion is seen as follows:

$$\widetilde{P}(t, S^i_j) \;=\; E^N(\, \widetilde{P}(T_{m-1}, S^i_j) \,|\, \mathcal{F}_t\,) \;=\; E^N(\, \tfrac{\mathcal{P}_{i,j}(x_{T_{m-1}})}{\mathcal{N}_{m-1}(x_{T_{m-1}})} \,|\, \mathcal{F}_t\,) \;=\; \Big(U_{T_{m-1},t}\,\tfrac{\mathcal{P}_{i,j}}{\mathcal{N}_{m-1}}\Big)\,(x_t)\,,$$

where we used the hypotheses (iii) and (iv) in the second step.
(b) follows directly from (a) and the definition of $\widetilde{A}^i_{T_i}$:

$$\widetilde{A}^i_{T_i} \;=\; \sum_{j=1}^{M_i} \tau^i_j\, \widetilde{P}(T_i, S^i_j) \;=\; \sum_{j=1}^{M_i} \tau^i_j\, \widetilde{\mathcal{P}}_{i,j}(x_{T_i})\;. \quad \square$$

By now, we know how to compute $\widetilde{\mathcal{A}}_i$ if we have the $\mathcal{N}_{i+1}, \ldots, \mathcal{N}_{m-1}$. But how to compute $\mathcal{N}_i$ in order to pass to the next iteration step ? At first, we compute $\mathcal{R}_i$ by calibrating our model to the digital $R^i_{T_i}$-swaption. Obviously, its value at time 0 given by our model is

$$V_0^{i,N}(K) \;:=\; E^N\big(\tfrac{N_0}{N_{T_i}}\, A^i_{T_i}\, 1_{R^i_{T_i}>K}\,\big) \;=\; N_0\, E^N\big(\,\widetilde{A}^i_{T_i}\, 1_{R^i_{T_i}>K}\,\big)\;.$$

In order to represent its market value at time 0, we consider strictly decreasing functions $V_0^{i,mkt} : \mathbb{R}_+ \to \mathbb{R}_+$.

Proposition 2.2. *Let $i \in \{0, \ldots, m-2\}$. Suppose that, for all $k = i+1, \ldots, m-1$, we have $N_{T_k} = \mathcal{N}_k(x_{T_k})$ for some (known) function $\mathcal{N}_k$. Suppose furthermore that we calibrate our model to the digital $R^i_{T_i}$-swaption, i.e.*

$$V_0^{i,mkt}(K) \;=\; V_0^{i,N}(K) \quad \textit{for all strikes } K.$$

(a) We have

$$\mathcal{R}_i \;=\; \Big(V_0^{i,mkt}\Big)^{-1} \circ J_i\;, \quad \textit{where } J_i(y) \;:=\; N_0\, U_{T_i,0}\,(\widetilde{\mathcal{A}}_i\, 1_{(y,\infty)})(x_0)\;.$$

(b) We have $N_{T_i} = \mathcal{N}_i(x_{T_i})$, where the function $\mathcal{N}_i$ is given by

$$\frac{1}{\mathcal{N}_i} \quad = \quad \widetilde{\mathcal{P}}_{i,M_i} \;+\; \widetilde{\mathcal{A}}_i\, \mathcal{R}_i\;.$$

Proof. (a) is obvious in view of

$$\begin{aligned}
V_0^{i,mkt}(K) \;&=\; V_0^{i,N}(K) \;=\; N_0\, E^N\big(\,\widetilde{A}^i_{T_i}\, 1_{R^i_{T_i}>K}\,\big)\\
&=\; N_0\, E^N\big(\,\widetilde{\mathcal{A}}_i(x_{T_i})\, 1_{\mathcal{R}_i(x_{T_i})>K}\,\big) \;=\; N_0\, E^N\big(\,\widetilde{\mathcal{A}}_i(x_{T_i})\, 1_{(\mathcal{R}_i^{-1}(K),\infty)}(x_{T_i})\,\big)\\
&=\; N_0\, U_{T_i,0}\,(\widetilde{\mathcal{A}}_i\, 1_{(\mathcal{R}_i^{-1}(K),\infty)})(x_0) \;=\; J_i(\mathcal{R}_i^{-1}(K))\;,
\end{aligned}$$

where we used hypothesis (ii) in the (third and) fourth step. (b) follows directly from

$$\frac{1}{N_{T_i}} \quad = \quad \widetilde{P}(T_i, S^i_{M_i}) \;+\; \widetilde{A}^i_{T_i}\, R^i_{T_i}$$

which is just a reformulation of the definition of $R^i_{T_i}$. $\square$

Remark 2.3. Recall that if the swap rate (R^i_t) is of the type

$$dR^i_t \;=\; \widetilde{\sigma}^i\, R^i_t\, dW^{A^i}_t$$

then the value at time 0 of the digital $R^i_{T_i}$-swaption is given by Black's formula:

$$V_0^{i,A^i} \;=\; A^i_0\, E^{A^i}(1_{R^i_{T_i}>K}) \;=\; A^i_0\, \Phi\left(\frac{\log\left(\frac{R^i_0}{K}\right) - (\widetilde{\sigma}^i)^2 T_i}{\widetilde{\sigma}^i \sqrt{T_i}}\right),$$

where Φ denotes the cumulative normal distribution function. If we suppose $V_0^{i,mkt}$ to be of this type, then one easily checks that

$$\left(V_0^{i,mkt}\right)^{-1}(x) \;=\; R^i_0\, \exp\Big(-(\widetilde{\sigma}^i)^2 T_i - \widetilde{\sigma}^i \sqrt{T_i}\Phi^{-1}(\tfrac{x}{A^i_0})\Big)\ .$$

3 A LIBOR model

Here we consider the particular case of our general model where $M_i = 1$ and $S^i_1 = T_{i+1}$ for $i = 0, \ldots, m-1$ and T_m is some final payment date. In particular, hypothesis (iv) is empty. We denote

$$\widetilde{\mathcal{P}}_i \;:=\; \widetilde{\mathcal{P}}_{i,1} \qquad \text{and} \qquad \tau_i \;:=\; \tau^i_1 \;=\; \tau(T_i, T_{i+1})\ .$$

We have $A^i_t = \tau_i P(t, T_{i+1})$ and $R^i_t = R(t, T_i, T_{i+1})$, the forward rate, hence

$$\widetilde{\mathcal{P}}_i \;=\; U_{T_{i+1},T_i}\, \tfrac{1}{\mathcal{N}_{i+1}} \qquad \text{and} \qquad \widetilde{\mathcal{A}}_i \;=\; \tau_i\, \widetilde{\mathcal{P}}_i$$

in the notation of Lemma 2.1. Suppose

$$dR^{m-1}_t \;=\; \sigma^{m-1}_t\, R^{m-1}_t\, dW^N_t\,, \qquad \text{where } \sigma^{m-1}_t \;=\; \sigma e^{at} \tag{1}$$

for some $\sigma > 0$ and some mean reversion parameter a. We choose

$$N_t \;:=\; P(t, T_m) \qquad \text{and} \qquad x_t \;:=\; \int_0^t \sigma^{m-1}_s\, dW^N_s\ .$$

Then the functional form of $R^{m-1}_{T_{m-1}}$ is evident:

$$R^{m-1}_{T_{m-1}} \;=\; R^{m-1}_0 \exp\Big(-\tfrac{1}{2}\int_0^{T_{m-1}} (\sigma^{m-1}_s)^2\, ds + x_{T_{m-1}}\Big) \;=\; \mathcal{R}_{m-1}(x_{T_{m-1}})\ ,$$

where the function $\mathcal{R}_{m-1}$ is obviously given by

$$\begin{aligned}\mathcal{R}_{m-1}(x) \;&:=\; R^{m-1}_0 \exp\Big(-\tfrac{1}{2}\int_0^{T_{m-1}} (\sigma^{m-1}_s)^2\, ds + x\Big)\\ &=\; \tau^{-1}_{m-1}\left(\tfrac{P(0,T_{m-1})}{P(0,T_m)} - 1\right)\exp\Big(-\tfrac{1}{2}\Sigma^2_{T_{m-1},0} + x\Big)\ , \quad \Sigma^2_{t,s} \;:=\; \sigma^2\, \tfrac{e^{2at}-e^{2as}}{2a}\ .\end{aligned}$$

Hence, since $N_{T_{m-1}} = P(T_{m-1}, T_m) = (1 + \tau_{m-1} R^{m-1}_{T_{m-1}})^{-1}$, the functional form of $N_{T_{m-1}}$ required in hypothesis (iii) is easy to deduce: $N_{T_{m-1}} = \mathcal{N}_{m-1}(x_{T_{m-1}})$, where

$$\mathcal{N}_{m-1}(x) \;:=\; (1 + \tau_{m-1}\, \mathcal{R}_{m-1}(x))^{-1} \;=\; (1 + C_2\, e^x)^{-1} \;, \tag{2}$$

where the constant C_2 is given by

$$C_2 := \left(\tfrac{P(0,T_{m-1})}{P(0,T_m)} \;-\; 1 \right) \exp(-\tfrac{1}{2} \Sigma^2_{T_{m-1},0}) \;. \tag{3}$$

Obviously, x_t given x_s is normally distributed with mean x_s and variance $\Sigma^2_{t,s}$. In other words:

$$U_{t,s} f(y) \;=\; \tfrac{1}{\sqrt{2\pi}\Sigma_{t,s}} \int_{\mathbb{R}} f(x) \exp\Big(-\tfrac{(y-x)^2}{2\Sigma^2_{t,s}} \Big)\; dx \;.$$

For the iteration step (to deduce $\mathcal{N}_i$ from $\mathcal{N}_{i+1}$), it suffices to represent $\frac{1}{\mathcal{N}_i}$ in terms of $\widetilde{\mathcal{P}}_i$ since

$$\widetilde{\mathcal{P}}_i \;=\; U_{T_{i+1},T_i} \tfrac{1}{\mathcal{N}_{i+1}} \;. \tag{4}$$

This representation is obtained from Proposition 2.2:

$$\tfrac{1}{\mathcal{N}_i} \;=\; \widetilde{\mathcal{P}}_i \left(1 \;+\; \tau_i\, (V_0^{i,mkt})^{-1} \circ J_i \right) \;, \tag{5}$$

where the function J_i is given by

$$J_i(y) \;:=\; P(0, T_m)\, \tau_i\; U_{T_i,0}\, (\widetilde{\mathcal{P}}_i\, 1_{(y,\infty)})(0) \;. \tag{6}$$

We can summarize the algorithm for the computation of the functional forms $\mathcal{N}_{m-1}, \ldots, \mathcal{N}_0$ as follows:

1. Initialization (at time T_{m-1}): Choose $\mathcal{N}_{m-1}$ as in (2).
2. For $i = m-2, \ldots, 0$: Define $\widetilde{\mathcal{P}}_i$ as in (4) and then J as in (6). Now obtain $\mathcal{N}_i$ via (5).

Observe that the calibration instruments corresponding to the $V_0^{i,mkt}$ are the digital (T_i, T_{i+1})-caplets defined by the following payoff at T_i :

$$\tau_i\, P(T_i, T_{i+1})\, 1_{R(T_i,T_i,T_{i+1})>K} \;.$$

For $i = m-1$, it can be evaluated explicitly due to the dynamics in (1). This could be used for the choice of the parameter σ in (1).

Proposition 3.1. *The current value of the digital* (T_{m-1}, T_m)*-caplet in our LIBOR model is*

$$V_0^{m-1,N}(K) \;:=\; \tau_{m-1}\, P(0, T_m)\, \Phi\Big(\sigma_Q^{-1} [\log(\tfrac{R(0,T_{m-1},T_m)}{K}) \;-\; \tfrac{\sigma_Q^2}{2}] \Big),$$

where the parameter σ_Q is given by

$$\sigma_Q \; := \; \sigma \sqrt{\tfrac{e^{2aT_{m-1}}-1}{2a}} \; .$$

Moreover, we have for all $x \in (0, \tau_{m-1} P(0,T_m))$ that $V_0^{m-1,N}(K) = x$ if and only if

$$\sigma \; = \; \frac{\sqrt{\frac{p^2}{4} - q} - \frac{p}{2}}{\sqrt{\frac{e^{2aT_{m-1}}-1}{2a}}} \; , \qquad \text{where } p \; := \; 2\Phi^{-1}\left(\tfrac{x}{\tau_{m-1} P(0,T_m)}\right) \; , \; \text{ă}q \; := \; -2\log\left(\frac{R_0^{m-1}}{K}\right) \; .$$

The proof is straightforward and therefore omitted.

4 A (cancellable) swap model

Here we consider briefly the particular case of our general model where $M_i = m - i$ and $S_j^i = T_{i+j}$ for $i = 0, \ldots, m-1$, $j = 1, \ldots, M_i$ and T_m is some final payment date.
Since $S^i = (T_{i+1}, \ldots, T_m)$, we only have to give the functional form of $P(T_{m-1}, T_m)$ in order to check hypothesis (iv). But if we take the numeraire $N_t = P(t, T_m)$ as in the LIBOR model in Section 3, then $P(T_{m-1}, T_m) = N_{T_{m-1}} = \mathcal{N}_{m-1}(x_{T_{m-1}})$, hence hypothesis (iv) is implied by hypothesis (iii). Moreover, we have

$$A_t^i \; = \; \sum_{j=1}^{m-i} \tau_j^i \, P(t, T_{i+j}) \; . \tag{7}$$

As in the LIBOR model, we suppose

$$dR_t^{m-1} \; = \; \sigma_t^{m-1} \, R_t^{m-1} \, dW_t^N \; , \qquad \text{where } \sigma_t^{m-1} \; = \; \sigma e^{at}$$

for some $\sigma > 0$ and some mean reversion parameter a and choose as before

$$x_t \; := \; \int_0^t \sigma_s^{m-1} \, dW_s^N \; .$$

Now we can again compute the desired functional forms but, due to (7), they are more complicated than in the LIBOR model in Section 3 where we had $A_t^i = \tau_1^i P(t, T_{i+1})$.
Observe that here the natural calibration instruments are the digital (European) $(T_i, \ldots, T_{m-1})$-swaptions.

5 Numerical results: Bermudan swaption pricing in the LIBOR model

In this section, we will apply the (standard) tree method from Section 6 in order to price Bermudan swaptions in the LIBOR model of Section 3. Recall that, in this case, the calibrating instruments used in Proposition 2.2 are the digital (T_i, T_{i+1})-caplets with the following payoff at T_i :

$$\tau_i\, P(T_i, T_{i+1})\, 1_{R(T_i,T_i,T_{i+1})>K}\ .$$

Since we do not have real data for their market prices $V_0^{i,mkt}(K)$, we assume them to be given by a standard Hull-White model for the short rate (r_t):

$$dr_t \ = \ [\bar\theta_t \ - \ \bar a r_t]dt \ + \ \bar\sigma dW_t\ . \tag{8}$$

The proof of the following result on the current price of digital caplets in the Hull-White model is straight-forward and therefore omitted.

Proposition 5.1. *Consider the digital (T, S)-caplet defined by the payoff at T of*

$$\tau\, P(T, S)\, 1_{R(T,T,S)>K}\ ,$$

where τ denotes the year fraction from T to S. Its current value in the Hull-White model (8) is

$$V_0^{HW}(K) \ := \ \tau\, P(0, S)\, \Phi\Big(\, \sigma_P^{-1}[\, \log(\tfrac{R(0,T,S)\,+\,\tau^{-1}}{K\,+\,\tau^{-1}}) \ - \ \tfrac{\sigma_P^2}{2}\,]\,\Big)\,,$$

where the parameter σ_P is given by

$$\sigma_p \ := \ \bar\sigma\, \tfrac{e^{-\bar a T}-e^{-\bar a S}}{\bar a}\, \sqrt{\tfrac{e^{2\bar a T}-1}{2\bar a}}\ .$$

Moreover, we have for all $x \in (0, \tau P(0, S))$:

$$(V_0^{HW})^{-1}(x) \ = \ \tau^{-1}\, \tfrac{P(0,T)}{P(0,S)}\, \exp\Big(\, - \ \tfrac{\sigma_P^2}{2} \ - \ \sigma_P\, \Phi^{-1}(\tfrac{x}{\tau P(0,S)})\,\Big) \ - \ \tau^{-1}\ .$$

In the following, we denote

$$V_0^{i,HW}(K) \ := \ V_0^{HW}(K) \quad \text{for} \quad T = T_i\,,\ S = T_{i+1}\,,\ \tau = \tau_i\ .$$

We proceed as follows. We fix the Hull-White parameters $\bar a$ and $\bar\sigma$ and assume that the market prices $V_0^{i,mkt}(K)$ are given by the corresponding Hull-White prices:

$$V_0^{i,mkt}(K) \ = \ V_0^{i,HW}(K) \quad \text{for } i = 0, \ldots, m-2 \text{ and all } K\ .$$

Now we choose our LIBOR model parameters a and σ in (1). Then iterative calibration to the digital (T_i, T_{i+1})-caplets for $i = m-2, \dots, 0$ is used as in Proposition 2.2 [see (5) and (6)] to obtain the functional forms $\mathcal{N}_{m-2}, \dots, \mathcal{N}_0$. In other words, we suppose that

$$V_0^{i,N}(K) = V_0^{i,HW}(K) \quad \text{for } i = 0, \dots, m-2 \text{ and all } K\,.$$

Note that the iterations $i = m-2, \dots, 0$ involve (iterated) numerical integration.
Finally, will price the Bermudan (payer) swaption explained in Section 6.3: with strike K_0, with n exercise times $T_0, \dots, T_{n-1}$ and m swap payment dates $T_1, \dots, T_m$. The Bermudan swaption is priced on the one hand in our LIBOR model via a tree for the process (x_t) with N_x time steps as explained in Section 6, on the other hand in our Hull-White model via a tree for the short rate (r_t) with N_r time steps. We denote by N_{disc} the number of discretizations steps for the functional forms $\mathcal{N}_{m-2}, \dots, \mathcal{N}_0$. Our parameter values are:

$$\bar{a} = 0.1\ ,\ \bar{\sigma} = 0.01$$

$$a = \bar{a}\ ,\ \sigma = 0.09$$

$$\text{ITM: } K_0 = 0.0589092\ ,\ \text{ATM: } K_0 = 0.0687274\ ,\ \text{OTM: } K_0 = 0.0785456$$

$$n = 1, 3, 5\ ,\ m = 5\ ,\ T_i = 2 + \tfrac{i}{2}$$

Moreover, we use the standard (non-flat) PREMIA data for the intial yield curve. One obtains the following prices (given in BP); the third column of prices can be seen as Hull-White benchmarks.

n	Strike K_0	$N_x = 50\ ,\ N_{disc} = 5000$	$N_r = 150$	$N_r = 1500$
1	ITM	231.33	231.77	231.75
1	ATM	97.73	97.70	97.76
1	OTM	28.59	27.96	27.92
3	ITM	249.38	249.85	249.93
3	ATM	122.60	123.16	122.98
3	OTM	48.83	47.89	47.87
5	ITM	252.15	253.35	253.36
5	ATM	127.68	129.01	128.94
5	OTM	54.51	54.41	54.30

With only one fixed value for the LIBOR model parameters a and σ it might be hopeless to reobtain all the Hull-White prices of the rather different swaptions we consider: European ($n = 1$) and Bermudan ($n = m$) swaptions which ITM, ATM or OTM.

6 Pricing of Markov-functional Bermudan options via trees and Monte Carlo (Appendix)

Consider the Bermudan option given by the payoffs $h_0, \ldots, h_{n-1}$ at the exercise times $0 < T_0 < \ldots < T_{n-1}$. Its discounted value $\widetilde{V}_{T_0}$ at time T_0 is given by

$$\widetilde{V}_{T_0} \; = \; \sup_{\tau \in \mathcal{T}_{\{0,\ldots,n-1\}}} E(\, \widetilde{h}_\tau \,|\, \mathcal{F}_{T_0} \,) \quad \text{, where} \quad \widetilde{h}_i \; := \; \tfrac{h_i}{N_{T_i}} \;,$$

(N_t) is the numeraire and $\mathcal{T}_{\{0,\ldots,n-1\}}$ denotes the set of stopping times with values in $\{0, \ldots, n-1\}$. The discounted value $\widetilde{V}_0$ at time 0 can be computed as follows via dynamic programming:

$$\begin{aligned} \widetilde{V}_{T_{n-1}} \; &= \; \widetilde{h}_{n-1} \\ \widetilde{V}_{T_i} \; &= \; E(\, \widetilde{V}_{T_{i+1}} \,|\, \mathcal{F}_{T_i} \,) \vee \widetilde{h}_i \qquad \text{for } i = n-2, \ldots, 0 \\ \widetilde{V}_0 \; &= \; E(\, \widetilde{V}_{T_0} \,) \end{aligned}$$

Now suppose that the $\widetilde{h}_i$ have the following Markov-functional form:

$$\widetilde{h}_i \; = \; f_i(x_{T_i}) \quad \text{for} \quad i = 0, \ldots, n-1 \;. \tag{9}$$

Here (x_t) is a Markov process with values in $\mathbb{R}^D$. Then simulating (x_t) by trinomial trees or Monte Carlo yields standard methods to approximate $\widetilde{V}_0$.

6.1 Trinomial trees

Suppose ($D = 1$ and) that, for our Markov process (x_t), we are given a trinomial tree built for the time instants

$$0 = t_0 < t_1 < \ldots < t_N = T_{n-1} \;.$$

For $i = 0, \ldots, n-1$, let $t_{d(i)} = T_i$, in particular $d(n-1) = N$. Suppose that, at time t_l, the tree has S_l nodes and that, from the j-th node at time t_l, one can move to the $(k_{l,j}+1)$-th, the $k_{l,j}$-th and the $(k_{l,j}-1)$-th node at time t_{l+1}. In order to approximate the discounted present value $\widetilde{V}_0$ of the Bermudan option using our given trinomial tree, we only need (apart from the payoff functions $f_0, \ldots, f_{n-1}$) its following quantities:

- For $l = 0, \ldots, N-1$ and $j = 0, \ldots, S_l - 1$, let $p^u_{l,j}$, $p^m_{l,j}$ and $p^d_{l,j}$ be the up-, middle- and down-probability to move from the j-th node at time t_l to the $(k_{l,j}+1)$-th, the $k_{l,j}$-th and the $(k_{l,j}-1)$-th node at time t_{l+1}

- For $i = 0, \ldots, n-1$ and $j = 0, \ldots, S_{d(i)} - 1$, let $x_{d(i),j}$ be the value of x at the j-th node at time $t_{d(i)} = T_i$ (in other words, the $x_{d(i),j}$ are the values of x_{T_i} in the tree).

Then the following tree algorithm yields the approximation $\widetilde{v}^0_{0,0}$ of $\widetilde{V}_0$. The $\widetilde{v}_{l,j}$ represent the discounted value of the Bermudan option at time t_l.

1. Initialization (at time $T_{n-1} = t_{d(n-1)} = t_N$):

$$\widetilde{v}_{N,j} \ := \ f_{n-1}(x_{N,j}) \quad \text{for} \quad j = 0, \ldots, S_N - 1 \, .$$

2. For $i = n-1, \ldots, 1$:
   (a) For $l = d(i) - 1, \ldots, d(i-1)$, we set

$$\widetilde{v}_{l,j} \ := \ p^u_{l,j}\, \widetilde{v}_{l+1,k_{l,j}+1} \ + \ p^m_{l,j}\, \widetilde{v}_{l+1,k_{l,j}} \ + \ p^d_{l,j}\, \widetilde{v}_{l+1,k_{l,j}-1} \quad \text{for} \quad j = 0, \ldots, S_l - 1 \, .$$

   (b) Early exercise at $T_{i-1} = t_{d(i-1)}$:

$$\widetilde{v}_{d(i-1),j} \ := \ \widetilde{v}_{d(i-1),j} \ \vee \ f_{i-1}(x_{d(i-1),j}) \quad \text{for} \quad j = 0, \ldots, S_{d(i-1)} - 1 \, .$$

3. For $l = d(0) - 1, \ldots, 0$, we set

$$\widetilde{v}_{l,j} \ := \ p^u_{l,j}\, \widetilde{v}_{l+1,k_{l,j}+1} \ + \ p^m_{l,j}\, \widetilde{v}_{l+1,k_{l,j}} \ + \ p^d_{l,j}\, \widetilde{v}_{l+1,k_{l,j}-1} \quad \text{for} \quad j = 0, \ldots, S_l - 1 \, .$$

6.2 Monte Carlo (Longstaff-Schwartz algorithm)

Suppose that, for our Markov process (x_t), we are given M Monte Carlo samples $(x^m_{T_0}, \ldots, x^m_{T_{n-1}})$, where $m = 0, \ldots, M-1$. Suppose furthermore that, for $i = 0, \ldots, n-2$, we have suitably chosen functions $g^i_0, \ldots, g^i_{d(i)-1}$ representing a basis of a $d(i)$-dimensional subspace of $L_2(\mathbb{R}^D, \mu_i)$, where μ_i denotes the law of x_{T_i}. For $\alpha \in \mathbb{R}^{d(i)}$ and $x \in \mathbb{R}^D$, we denote $(\alpha.g^i)(x) = \sum_{j=0}^{d(i)-1} \alpha_j\, g^i_j(x)$.
Then, the following Longstaff-Schwartz algorithm approximates the current discounted value $\widetilde{V}_0$ of our Bermudan option. Here, at the i-th iteration step, $\widetilde{v}$ represents $\widetilde{V}_{T_i}$, the discounted value of the Bermudan option at T_i.

1. Initialization (at time T_{n-1}):

$$\widetilde{v}_m \ := \ f_{n-1}(x^m_{T_{n-1}}) \quad \text{for} \quad m = 0, \ldots, M-1 \, .$$

2. For $i = n-2, \ldots, 0$:
   (a) Let $\alpha \in \mathbb{R}^{d(i)}$ be the unique solution of the least square problem

$$\min_{\alpha \in \mathbb{R}^{d(i)}} \sum_{m=0}^{M-1} \Big((\alpha.g^i)(x^m_{T_i}) \ - \ \widetilde{v}_m \Big)^2 \, .$$

   (b) For $m = 0, \ldots, M-1$: if $f_i(x^m_{T_i}) > (\alpha.g^i)(x^m_{T_i})$ then $\widetilde{v}_m := f_i(x^m_{T_i})$.

3. Return the estimate $\frac{1}{M}\sum_{m=0}^{M-1} \widetilde{v}_m$ of the current discounted value $\widetilde{V}_0$.

6.2.1 Modification for large dimensions (explanatory process)

If the dimension D of our driving process (x_t) is too large $(D > 10)$, a reasonable basis g^i of functions on $\mathbb{R}^D$ would need too many functions. Hence the parameter $d(i)$ would be too large for a sufficiently fast solution of the least square problem. This difficulty arises for example in LIBOR Market models where (x_t) represents a vector of D different LIBOR rates.
In this situation, one modifies the approach from above by considering - besides the driving process (x_t) - an "explanatory process" (y_t) with values in $\mathbb{R}^d$ and $d << D$. It should be chosen such that simulating (x_t) in order to obtain our Monte Carlo samples $(x^m_{T_0}, \ldots, x^m_{T_{n-1}})$ yields also Monte Carlo samples $(y^m_{T_0}, \ldots, y^m_{T_{n-1}})$ without additional computational costs. Natural choices of (y_t) could be $y_t = W_t$ [if (x_t) is a diffusion with Brownian motion (W_t)] or $y_t = F(t, x_t)$. The latter choice is made e.g. in [PPR] where, in the LIBOR Market model situation we just mentioned, the authors consider the case $y =$swap-rate.
Suppose that, for $i = 0, \ldots, n-2$, we have suitably chosen functions $g^i_0, \ldots, g^i_{d(i)-1}$ representing a basis of a $d(i)$-dimensional subspace of $L_2(\mathbb{R}^d, \nu_i)$, where ν_i denotes the law of y_{T_i}.
Now, in the modified Longstaff-Schwartz algorithm, one only has to replace all occurences of $(\alpha.g^i)(x^m_{T_i})$ by $(\alpha.g^i)(y^m_{T_i})$.

6.3 Example: Bermudan swaptions in the Markov-functional LIBOR model

Consider an interest rate swap first resetting in T_0 and paying at $T_1, \ldots, T_m$, with fixed rate K_0 and year fractions $\tau_0, \ldots, \tau_{m-1}$. Assume that one has the right to enter the swap at the times $T_0, \ldots, T_{n-1}$, where $n \leq m$.
Then the corresponding Bermudan (payer) swaption fits in our general setting from above as the following particular case:

$$\begin{aligned} h_i &= \Big(\text{value of the interest rate swap at } T_i\Big)_+ \\ &= \Big(1 - P(T_i, T_m) \; - \; K_0 \sum_{k=i+1}^{m} \tau_{k-1} P(T_i, T_k)\Big)_+ . \end{aligned} \tag{10}$$

In the notation of our Markov-functional LIBOR model in Section 3, we can rewrite line (10) as follows:

$$\widetilde{h}_i = \Big(\tfrac{1}{N_{T_i}} - \widetilde{P}(T_i, T_m) \; - \; K_0 \sum_{k=i+1}^{m} \tau_{k-1} \widetilde{P}(T_i, T_k)\Big)_+ .$$

Since $N_t = P(t, T_m)$, we have $\widetilde{P}(T_i, T_m) = 1$. Moreover, for $k = i+1, \ldots, m-1$,

$$\widetilde{P}(T_i, T_k) \;=\; E^N(\, \widetilde{P}(T_k, T_k) \,|\, \mathcal{F}_{T_i}\,) \;=\; E^N(\, \tfrac{1}{\mathcal{N}_k(x_{T_k})} \,|\, \mathcal{F}_{T_i}\,) \;=\; (U_{T_k,T_i}\, \tfrac{1}{\mathcal{N}_k})(x_{T_i}) \,.$$

Hence, we obtain the desired Markov-functional forms in (9) as follows:

$$\widetilde{h}_i \;=\; f_i(x_{T_i}) \,,$$

where the function f_i is obviously given by

$$f_i(x) \;:=\; \Big(\tfrac{1}{\mathcal{N}_i(x)} \;-\; (1 + K\tau_{m-1}) \;-\; K_0 \sum_{k=i+1}^{m-1} \tau_{k-1}\, (U_{T_k,T_i}\, \tfrac{1}{\mathcal{N}_k})(x) \Big)_+ \,.$$

6.4 Example: (European) digital caplets in the Markov-functional LIBOR model

In order to test the calibration of our Markov-functional LIBOR model to a Hull-White model as in Section 5, one might wish to price the calibrating instruments which are the digital (T_i, T_{i+1})-caplets. This does not involve the functional forms $\mathcal{N}_0, \ldots, \mathcal{N}_{i-1}$, hence by replacing m by $m - i$ if necessary, we can assume $i = 0$. The digital (T_0, T_1)-caplet fits into our general setting from above as the following particular case: $n = 1$ (European !) and

$$h_0 \;\;=\;\; \tau_0\, P(T_0, T_1)\, 1_{R(T_0,T_0,T_1)>K} \,.$$

Since $\tau_0 R(T_0, T_0, T_1) = P(T_0, T_1)^{-1} - 1$, we can rewrite this as follows, denoting $K_1 := \tau_0 K + 1$:

$$\widetilde{h}_0 \;\;=\;\; \tau_0\, \widetilde{P}(T_0, T_1)\, 1_{P(T_0,T_1)^{-1}>K_1} \,.$$

Notice that $\widetilde{P}(T_0, T_1) = (U_{T_0,T_1}\, \frac{1}{\mathcal{N}_1})(x_{T_0}) =: \mathcal{L}(x_{T_0})$ as before and

$$P(T_0, T_1)^{-1} \;=\; P(T_0, T_m)^{-1}\widetilde{P}(T_0, T_1)^{-1} \;=\; \tfrac{1}{\mathcal{N}_0\, \mathcal{L}}(x_{T_0}) \;=:\; \mathcal{M}(x_{T_0}) \,.$$

Hence, we obtain the desired Markov-functional form in (9) as follows:

$$\widetilde{h}_0 \;=\; f_0(x_{T_0}) \,,$$

where the function f_0 is obviously given by

$$f_0(x) \;:=\; \tau_0\, \mathcal{L}(x)\, 1_{\mathcal{M}(x)>K_1} \,.$$

7 An explicit formula for $\mathcal{N}_{m-2}$ in the LIBOR model (Appendix)

The following lemma is helpful for a (more or less) explicit formula for the functional form $\mathcal{N}_{m-2}$ in the LIBOR model. It can be used to avoid the first numerical integration in the iterations. On the other hand, one needs an approximation of the cumulative normal distribution function Φ.

Lemma 7.1. *We have for all $x, y \in \mathbb{R}$:*

$$\begin{aligned} U_{t,s}(\exp 1_{(y,\infty)})(x) &= e^{\frac{1}{2}\Sigma_{t,s}^2 + x}\,\Phi(\tfrac{x-y}{\Sigma_{t,s}} + \Sigma_{t,s}) \\ U_{t,s}(1_{(y,\infty)})(x) &= \Phi(\tfrac{x-y}{\Sigma_{t,s}}) \end{aligned}$$

The proof of Lemma 7.1 is elementary and therefore omitted.

Corollary 7.2. *We have for all $x, y \in \mathbb{R}$:*

$$\begin{aligned} \widetilde{\mathcal{P}}_{m-2}(x) &= 1 + C_0\, e^x \\ J_{m-2}(y) &= P(0,T_m)\,\tau_{m-2}\Big(\,\Phi(-\tfrac{y}{\Sigma_{T_{m-2},0}})\; +\; C_1\,\Phi(-\tfrac{y}{\Sigma_{T_{m-2},0}} + \Sigma_{T_{m-2},0})\Big) \end{aligned}$$

Here we denote, using the constant C_2 from (3):

$$C_0 := C_2\,\exp(\tfrac{1}{2}\Sigma^2_{T_{m-1},T_{m-2}}) \quad \textit{and} \quad C_1 := C_0\,\exp(\tfrac{1}{2}\Sigma^2_{T_{m-2},0})\ .$$

Proof. We have $\mathcal{N}_{m-1} = (1 + C_2 \exp)^{-1}$, hence Lemma 7.1 (for $y = -\infty$) yields the first assertion:

$$\begin{aligned} \widetilde{\mathcal{P}}_{m-2}(x) &= (U_{T_{m-1},T_{m-2}}\tfrac{1}{\mathcal{N}_{m-1}})(x) = (U_{T_{m-1},T_{m-2}}(1 + C_2\exp))(x) \\ &= 1 + C_2\,e^{\frac{1}{2}\Sigma^2_{T_{m-1},T_{m-2}} + x} = 1 + C_0\,e^x\ . \end{aligned}$$

Now the second assertion can be deduced from the first and again Lemma 7.1:

$$\begin{aligned} N_0^{-1}\,J_{m-2}(y) &= U_{T_{m-2},0}(\widetilde{\mathcal{A}}_{m-2}\,1_{(y,\infty)})(x_0) = \tau_{m-2}U_{T_{m-2},0}((1 + C_0\exp)1_{(y,\infty)})(0) \\ &= \tau_{m-2}\Big(\,\Phi(-\tfrac{y}{\Sigma_{T_{m-2},0}})\; +\; C_0\,e^{\frac{1}{2}\Sigma^2_{T_{m-2},0}}\,\Phi(-\tfrac{y}{\Sigma_{T_{m-2},0}} + \Sigma_{T_{m-2},0})\Big)\ . \quad \square \end{aligned}$$

References

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