Large/Small time asymptotics for Implied volatility under the Heston model

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Premia 14

Most of what is presented here is taken from [1] and [2]

1 Small-time asymptotics for Implied Volatility

Consider the Heston stochastic volatility model for a forward price process S_t, defined by the following stochastic differential equations

$$\begin{cases} \frac{dS_t}{S_t} = \sqrt{V_t} dW_t^1, \\ dV_t = (a - bV_t)\, dt + \sigma \sqrt{V_t} dW^2, \;\; d\langle W^1, W^2 \rangle_t = \rho dt, \end{cases} \tag{1.1}$$

According to Forde and Jacquier (cf [1]), if we consider $(S =, V)$ the unique solution of (1.1) starting with (s, v), we get

$$\lim_{t \to \infty} t \log \mathbb{E}(K - S_t)_+ = -\Lambda^*(\log(\frac{K}{s})), \quad \text{pour } s > K$$

and

$$\lim_{t \to \infty} t \log \mathbb{E}(S_t - K)_+ = -\Lambda^*(\log(\frac{K}{s})), \quad \text{pour } s < K$$

where Λ^* is the Legendre transform of the function Λ defined as

$$\begin{cases} \Lambda(p) = \frac{vp}{\sigma(\sqrt{1-\rho^2}\cot(\frac{1}{2}\sigma p\sqrt{1-\rho^2})-\rho)}, & \text{pour } \ p \in]p_-, p_+[, \\ \Lambda(p) = \infty, & \text{pour } \ p \in \mathbb{R}\backslash]p_-, p_+[, \end{cases}$$

where p_- and p_+ are given by

$$p_- = \frac{\arctan\left(\frac{\sqrt{1-\rho^2}}{\rho}\right)}{\frac{1}{2}\sigma\sqrt{1-\rho^2}}\mathbb{1}_{\rho<0} - \frac{\pi}{\sigma}\mathbb{1}_{\rho=0} + \frac{-\pi + \arctan\left(\frac{\sqrt{1-\rho^2}}{\rho}\right)}{\frac{1}{2}\sigma\sqrt{1-\rho^2}}\mathbb{1}_{\rho>0}, \tag{1.2}$$

$$p_+ = \frac{\pi + \arctan\left(\frac{\sqrt{1-\rho^2}}{\rho}\right)}{\frac{1}{2}\sigma\sqrt{1-\rho^2}}\mathbb{1}_{\rho<0} + \frac{\pi}{\sigma}\mathbb{1}_{\rho=0} + \frac{\arctan\left(\frac{\sqrt{1-\rho^2}}{\rho}\right)}{\frac{1}{2}\sigma\sqrt{1-\rho^2}}\mathbb{1}_{\rho>0}, \tag{1.3}$$

The function Λ^* is given by

$$\Lambda^*(x) = xp^*(x) - \Lambda(p^*(x)), \tag{1.4}$$

where $p^*(x)$ is unique solution of

$$x = \Lambda'(p^*(x)),$$

and the function Λ' is given by

$$\Lambda'(p) = \frac{v}{\sigma(\sqrt{1-\rho^2}\cot(\frac{1}{2}\sigma p\sqrt{1-\rho^2}) - \rho)} + \frac{\sigma vp(1-\rho^2)\csc^2(\frac{1}{2}\sigma p\sqrt{1-\rho^2})}{2\sigma(\sqrt{1-\rho^2}\cot(\frac{1}{2}\sigma p\sqrt{1-\rho^2}) - \rho)^2}$$

On the other hand, if we denote by $\Sigma_t(x)$ the implied volatility defined as the unique solution of $P(t, K\ e^{-x}, v) = P_{BS}(t, K\ e^{-x}, K; \Sigma_t(x))$, où $P_{BS}(t, s, k, \Sigma) = KN(\frac{-\log(s/k)+t\Sigma/2}{\sqrt{t\Sigma}}) - sN(\frac{-\log(s/k)-t\Sigma/2}{\sqrt{t\Sigma}})$. Then we have (cf. [1], Theorem 2.4)

$$\lim_{t\to 0}\Sigma_t(x) = \frac{|x|}{\sqrt{2\Lambda^*(x)}} \tag{1.5}$$

2 Large-time asymptotics for Implied Volatility

We keep the notations of Forde-Jacquier-Mijatovic (cf. [2]). Under the assumption $b - \rho\sigma > 0$, we have for any $p \in]p_-, p_+[$,

$$V(p) = \lim_{t\to\infty} t^{-1} \log \mathbb{E}\left[\exp\left(p(X_t - x_0)\right)\right] = \frac{a}{\sigma^2}\left(b - \rho\sigma p - d(-ip)\right),$$

with

$$d(-ip) = \sqrt{(b-\rho\sigma p)^2 + \sigma^2(p-p^2)},$$

where

$$p_\pm := \left(-2b\rho + \sigma \pm \sqrt{\sigma^2 + 4b^2 - 4b\sigma\rho}\right)$$

We consider the function $p^* : \mathbb{R} \longrightarrow]p_-, p_+[$ defined as

$$p^*(x) := \frac{\sigma - 2b\rho + (a\rho + x\sigma)\left(\frac{\sigma^2+4b^2-4b\rho\sigma}{x^2\sigma^2+2xa\rho\sigma+a^2}\right)^{1/2}}{2\sigma(1-\rho^2)}, \quad \text{pour tout} \quad x \in \mathbb{R}.$$

For t large enough, we have for any $x \in \mathbb{R}$, (cf. [2])

$$\frac{1}{S_0}\mathbb{E}(S_t - S_0 e^{-x})_+ = 1 + \frac{A(0)}{\sqrt{2\pi t}} \exp\left(-(1-p^*(0))x - V^*(0)t\right)(1 + O(1/t)),$$

where V^* is the Legendre transform of V, ie $V^*(x) := \sup\left\{px - V(p), \quad p \in]p_-, p_+[\ \right\}$ and A is the function defined in a neighborhood of 0 by

$$A(x) = \frac{-1}{\sqrt{V''(p^*(x))}} \frac{U(p^*(x))}{p^*(x)(1-p^*(x))},$$

with

$$U(p) := \left(\frac{2d(-ip)}{b - \rho\sigma p + d(-ip)}\right)^{\frac{2a}{\sigma^2}} \exp\left(\frac{v}{a}V(p)\right)$$

Similarly, the implied volatility can be written as (cf. [2], Theorem 3.2)

$$\Sigma_t^2(x) = 8V^*(0) + a_1(x)/t + o(t), \tag{2.1}$$

where

$$a_1(x) = -8\log\left(-A(0)\sqrt{2V^*(0)}\right) + 4(2p^*(0)-1)x$$

In particular, for any $x \in \mathbb{R}$, we have

$$\lim_{t\to+\infty} \Sigma_t^2(x) = 8V^*(0).$$

References

[1] Forde, M. and Jaquier, A. Small-time asymptotics for implied volatility under the Heston model, with A.Jacquier, *International Journal of Theoretical and Applied Finance*, Volume 12, issue 6, pp. 861-876, (2009)

[2] Forde, F, Jacquier, A, and Mijatovic, A. Asymptotic formulae for implied volatility under the heston model. Forthcoming in Proceedings of the Royal Society A, 2009. 1, 2

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