Premia 14

1 Semi-Analytical method of Zhang

1.1 Introduction

The semi analytical method proposed by Zhang [1] consists in deriving a new analytical approximate formula for arithmetic average Asian options of European type and in computing the corresponding correction term equal to the difference between the true price and the approximation. This correction term is governed by a Partial Differential Equation with smooth coefficients and zero initial condition and is evaluated accurately by a finite differences method. We only present the case of a fixed strike option when the difference $(r-\delta)$ between the risk-free interest rate and the dividend rate is different from zero. But floating strike options are treated in a similar way. Moreover, to deal with the case $r-\delta=0$, it is enough to take the limit $r-\delta \to 0$ in the equations that we give.

1.2 Pricing and hedging approximations

The price at time $t=0$ of the arithmetic average Asian call option with maturity T fixed strike K spot S_0 is given by $C(0,S_0,0)$ where the function C satisfies the following PDE:

$$\begin{cases} \frac{\partial C}{\partial t} + S\frac{\partial C}{\partial I} + \frac{\sigma^2}{2}S^2\frac{\partial^2 C}{\partial S^2} + (r-\delta)S\frac{\partial C}{\partial S} - rC = 0 \\ \text{with terminal condition } C(S,I,T) = \max(\frac{I}{T}-K,0) \end{cases}$$

After the following change of variables

$$\begin{cases} \xi = \frac{TK-I}{S}e^{-(r-\delta)\tau} - \frac{\left(1-e^{-(r-\delta)}\right)}{r-\delta} \\ \tau = T-t \\ C(S,I,t) = e^{-\delta\tau}\frac{S}{T}f(\xi,\tau), \end{cases}$$

the function f satisfies the following PDE :

$$\begin{cases} \frac{\partial f}{\partial \tau} - \frac{\sigma^2}{2}\left(\xi + \frac{1-e^{-(r-\delta)\tau}}{r-\delta}\right)^2\frac{\partial^2 f}{\partial \xi^2} = 0 \quad \forall(\tau,\xi)\in[0,T]\times\mathbb{R} \\ \text{with the initial condition } f(\xi,0) = (-\xi)^+ \end{cases} \tag{1}$$

Zhang proposes an analytical approximation of the solution of this PDE based on the following remark

$$\frac{\partial^2 f(\xi,0)}{\partial \xi^2} = \delta_0(\xi)$$

where $\delta_0(\xi)$ is the Dirac's delta function concentrated at $\xi=0$. Therefore the diffusion effect only exists at $\xi=0$ initially, and will be significant only for small values of ξ. Dropping ξ in the diffusion coefficient in (1) we obtain the approximation $f_0(\xi,\tau)$ from the following equations :

$$\begin{cases} \frac{\partial f_0}{\partial \tau} - \frac{\sigma^2}{2}\left(\frac{1-e^{-(r-\delta)\tau}}{r-\delta}\right)^2\frac{\partial^2 f_0}{\partial \xi^2} = 0 \quad \forall\xi\in\mathbf{R} \\ f(\xi,0) = (-\xi)^+ \end{cases}$$

The following change of time variable

$$d\eta = \frac{\sigma^2}{2}\left(\frac{1-e^{-(r-\delta)\tau}}{r-\delta}\right)^2 d\tau$$

$$\eta = \frac{\sigma^2}{4(r-\delta)^3}(-3 + 2(r-\delta) + 4e^{-(r-\delta)\tau} - e^{-2(r-\delta)\tau}) \tag{2}$$

transforms the PDE into the standard heat equation

$$\begin{cases} \frac{\partial f_0}{\partial \eta} - \frac{\partial^2 f_0}{\partial \xi^2} = 0 \\ f(\xi, 0) = (-\xi)^+ \end{cases}$$

The explicit solution is

$$f_0(\eta, \xi) = -\xi N\left(-\frac{\xi}{\sqrt{2\eta}}\right) + \sqrt{\frac{\eta}{\pi}} e^{-\frac{\xi^2}{4\eta}}$$

which gives the following approximations for the price and the delta of the Asian Option :

$$C_0(S, I, t) = e^{-\delta\tau}\frac{S}{T}\left(-\xi N\left(-\frac{\xi}{\sqrt{2\eta}}\right) + \sqrt{\frac{\eta}{\pi}} e^{-\frac{\xi^2}{4\eta}}\right)$$

$$\Delta_0 = e^{-\delta\tau}\left(\frac{1 - e^{-(r-\delta)\tau}}{(r-\delta)T} N\left(-\frac{\xi}{\sqrt{2\eta}}\right) + \frac{1}{T}\sqrt{\frac{\eta}{\pi}} e^{-\frac{\xi^2}{4\eta}}\right).$$

1.3 Finite Difference Computation of the correction term

Since

$$\frac{\partial^2 f_0}{\partial \xi^2} = \frac{e^{-\frac{\xi^2}{4\eta}}}{2\sqrt{\pi\eta}}$$

the correction term $f_1(\xi, \tau) = f(\xi, \tau) - f_0(\xi, \tau)$ satisfies the following equation:

$$\begin{cases} \frac{\partial f_1}{\partial \tau} - c(\xi, \tau)\frac{\partial^2 f_1}{\partial \xi^2} = R(\xi, \tau) \\ f_1(\xi, \tau = 0) = 0 \end{cases} \tag{3}$$

where

$$c(\xi, \tau) = \frac{\sigma^2}{2}\left(\xi + \frac{1 - e^{-(r-\delta)\tau}}{r-\delta}\right)^2 \text{ and } R(\xi, \tau) = \frac{\sigma^2 \xi}{4\sqrt{\pi\eta}}\left(\xi + \frac{2}{r-\delta}\left(1 - e^{(r-\delta)\tau}\right)\right).$$

To solve this equation with a finite difference scheme, it is necessary to localize the spatial domain. The right hand hand of (3) goes to zero like $\xi^2 e^{-\xi^2/4\eta}$ as $|\xi| \to +\infty$. We choose the maximal discretization value X to satisfy $e^{-X^2/(4\eta)} = 10^{-16}$ where η is evaluated at $\tau = T$. From the definition (2) of η , for small values of $(r-\delta)$, $\eta \simeq \frac{1}{6}\sigma^2\tau^3$ which gives $X \simeq 5\sigma T^{3/2}$.

We discretize the domain $[-X, X] \times [0, T]$ into $I \times (N+1)$ points. I is the number of steps along the ξ-axis and N is the number of points along the τ-axis. The value of the unknown function $f_1(\xi, \tau)$ is written as g_i^n at the node (ξ_i, τ_n) We use a θ scheme with θ equal to 0.5. This scheme is known to be the Crank-Nicholson scheme. The partial derivatives are approximated by the formulas:

$$\frac{\partial g}{\partial \tau} = \frac{g_i^{n+1} - g_i^n}{\Delta\tau}$$

$$\frac{\partial^2 g}{\partial \xi^2} = \frac{g_{i-1}^n - 2g_i^n + g_{i+1}^n}{2(\Delta\xi)^2} + \frac{g_{i-1}^{n+1} - 2g_i^{n+1} + g_{i+1}^{n+1}}{2(\Delta\xi)^2}$$

where $\Delta\tau = T/N$ and $\Delta\xi = 2X/I$.
Denoting

$$c_i^{n+1/2} = \frac{1}{2}(c(\xi_i, \tau_{n+1}) + c(\xi_i, \tau_n))$$

$$R_i^{n+1/2} = \frac{1}{2}(R(\xi_i, \tau_{n+1}) + R(\xi_i, \tau_n)),$$

we discretize (3) in the following way

$$\frac{g_i^{n+1} - g_i^n}{\Delta\tau} - c_i^{n+1/2}\left(\frac{g_{i-1}^n - 2g_i^n + g_{i+1}^n}{2(\Delta\xi)^2} + \frac{g_{i-1}^{n+1} - 2g_i^{n+1} + g_{i+1}^{n+1}}{2(\Delta\xi)^2}\right) = R_i^{n+1/2} \tag{4}$$

We notice that this scheme evaluates the second order partial derivative w.r.t. ξ at the average of the n^{th} and the $(n+1)^{th}$ time level. The scheme is found to be consistent (thanks to the Taylor expension) with a truncation error of $O((\Delta\tau)^2, (\Delta\xi)^2)$. A rearrangement of (4) gives the algorithm:

$$\begin{aligned}
-\frac{s_i^{n+1/2} g_i^{n+1}}{2} + (1 + s_i^{n+1/2})g_i^{n+1} - \frac{s_i^{n+1/2} g_{i+1}^{n+1}}{2} \\
= \frac{s_i^{n+1/2} g_i^n}{2} + (1 - s_i^{n+1/2})g_i^n + \frac{s_i^{n+1/2} f_{i+1}^n}{2} + \Delta\tau R_i^{n+1/2}
\end{aligned}$$

with the discretization parameter $s_i^{n+1/2} = c_i^{n+1/2}\Delta\tau/(\Delta\xi)^2$.
According to the initial condition in (3), the initial vector $G^0 = (g_1^0, \dots, g_I^0) = (0, \dots, 0)$. We obtain inductively $G^{n+1} = (g_1^{n+1}, \dots, g_I^{n+1})$ from $G^n = (g_1^n, \dots, g_I^n)$ by solving by the method of Gauss the $I \times I$ tridiagonal system of equations $A^n.G^{n+1} = B^n.G^n + F^n$ where

$$\mathbf{A^n} = \begin{pmatrix}
\alpha_1^n & \beta_1^n & 0 & \dots & \dots & 0 \\
\beta_2^n & \alpha_2^n & \beta_2^n & \ddots & & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & & \ddots & \ddots & \ddots & \beta_{I-1}^n \\
0 & \dots & \dots & 0 & \beta_I^n & \alpha_I^n
\end{pmatrix}$$

$$\mathbf{B^n} = \begin{pmatrix}
\chi_1^n & \gamma_1^n & 0 & \dots & \dots & 0 \\
\gamma_2^n & \chi_2^n & \gamma_2^n & \ddots & & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & & \ddots & \ddots & \ddots & \beta_{I-1}^n \\
0 & \dots & \dots & 0 & \gamma_I^n & \chi_I^n
\end{pmatrix}$$

and

$$\begin{aligned}
\forall 1 \le i \le I, G_i^n = g_i^n, \quad F_i^n = \Delta\tau R_i^{n+1/2}, \quad \alpha_i^n = 1 + s_i^{n+1/2} \\
\beta_i^n = -0.5 s_i^{n+1/2}, \quad \chi_i^n = 1 - s_i^{n+1/2}, \quad \gamma_i^n = 0.5 s_i^{n+1/2}
\end{aligned}$$

It is possible to take advantage of the tridiagonal property of the matrices A^n and B^n to reduce the number of computations in the Gauss method.

References

[1] J.E. Zhang, A Semi-analytical Method for Pricing and Hedging Continuously-sampled Arithmetic Average Rate Options *Preprint* 2000 1