Double exponential jump model (Kou)

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The double exponential jump model, initiated by Steven KOU (see [1]), is an exponential Levy model, which is a compromise between reality and tractability. It gives an explanation of the two empirical phenomena which received much attention in financial markets : the asymmetric leptokurtic feature and the volatility smile. It permits to obtain analytical solutions to the prices of many derivatives : European call and put options; interest rate derivatives, such as swaptions, caps, floors, and bond options; as well as path-dependant options, such as perpetual American options, barrier, and lookback options .

1 The model

The behaviour of the asset price, S_t, under the risk neutral probability is modeled as followed :

$$\frac{dS_t}{S_{t^-}} = \mu dt + \sigma dW_t + d\left(\sum_{i=1}^{N_t} e^{Y_i} - 1\right) \tag{1.1}$$

Where W is a standard brownian motion, N is a poisson process with rate λ, the constants μ and $\sigma > 0$ are drift and volatility of the diffusion part and the jump sizes $\{Y_1, Y_2, \dots\}$ are i.i.d random variables with a common asymmetric double exponential distribution, of density :

$$f_Y(y) = p\eta_1 e^{-\eta_1 y} \mathbb{1}_{\{y \geq 0\}} + q\eta_2 e^{\eta_2 y} \mathbb{1}_{\{y<0\}} \tag{1.2}$$

where p,$q \geq 0$ are constants, $p + q = 1$, $\eta_1 > 1$ and $\eta_2 > 0$.
The random processes $(W_t)_{t\geq 0}$, $(N_t)_{t\geq 0}$, and random variables $\{Y_1, Y_2, \dots\}$ are independant. Furthermore we have $\mu = r - \lambda\xi$ with:

$$\xi = \frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1} - 1 \tag{1.3}$$

The condition on μ hold in order to obtain $(e^{-rt}S_t)_{t\geq 0}$ is a martingale. The caracteristic exponent G of $\log(S_t)$ (i.e. $\mathbb{E}\left[e^{\theta\log(S_t)}\right] = e^{G(\theta)t}$) is defined as :

$$G(x) = x\left(r - \frac{1}{2} - \lambda\xi\right) + \frac{1}{2}x^2\sigma^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - x} + \frac{q\eta_2}{\eta_2 + x} - 1\right)$$

The equation $G(x) = \alpha$ has exactly four roots (see [2]) : $\beta_{1,\alpha}$, $\beta_{2,\alpha}$, $-\beta_{3,\alpha}$, $-\beta_{4,\alpha}$, where

$$0 < \beta_{1,\alpha} < \beta_{2,\alpha} < \infty, \quad 0 < \beta_{3,\alpha} < \beta_{4,\alpha} < \infty. \tag{1.4}$$

2 European call and put

Let us define some special functions (see pp. 1094 and 1099 in[1]):

$$\begin{aligned}
&Hh_{-1}(x) = e^{-\frac{x^2}{2}}\\
&Hh_0(x) = \sqrt{2\pi}\Phi(-x)\\
&Hh_n(x) = \int_x^{+\infty} Hh_{n-1}(y)dy = \frac{1}{n!}\int_x^{+\infty}(t-x)^n e^{-\frac{t^2}{2}}dt \quad \forall n \geq 0\\
&I_n(c;\alpha,\beta,\gamma) = \int_c^{+\infty} e^{\alpha x}Hh_n(\beta c - \gamma)dx \quad \forall n \geq -1
\end{aligned}$$

where Φ is the standard normal cumulative distribution. Then we have :

$$nHh_n(x) = Hh_{n-2}(x) - xHh_{n-1}(x) \quad \forall n \geq 1$$

And $\forall n \geq -1$:

$$I_n(c;\alpha,\beta,\gamma) = -\frac{e^{\alpha c}}{\alpha}\sum_{i=0}^{n}\left(\frac{\beta}{\alpha}\right)^{n-i}Hh_i(\beta c - \delta) + \left(\frac{\beta}{\alpha}\right)^{n+1}\frac{\sqrt{2\pi}}{\beta}e^{\frac{\alpha\delta}{\beta}+\frac{\alpha^2}{2\beta^2}}\Phi\left(-\beta c + \delta + \frac{\alpha}{\beta}\right) \quad \beta > 0, \alpha \neq 0$$

$$I_n(c;\alpha,\beta,\gamma) = -\frac{e^{\alpha c}}{\alpha}\sum_{i=0}^{n}\left(\frac{\beta}{\alpha}\right)^{n-i}Hh_i(\beta c - \delta) - \left(\frac{\beta}{\alpha}\right)^{n+1}\frac{\sqrt{2\pi}}{\beta}e^{\frac{\alpha\delta}{\beta}+\frac{\alpha^2}{2\beta^2}}\Phi\left(\beta c - \delta - \frac{\alpha}{\beta}\right) \quad \beta < 0, \alpha < 0$$

Introduce the following notation : For any given probability P, define :

$$\psi(\mu,\sigma,\lambda,p,\eta_1,\eta_2;a,T) = \mathbb{P}\left[Z_T \geq a\right] \tag{2.5}$$

where $Z_T = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$ and Y has a double exponential distribution with density as in (1.2), and N is a poisson process with rate λ. Theorem B.1. in [1] gives us :

$$\begin{aligned}
\psi(\mu,\sigma,\lambda,p,\eta_1,\eta_2;a,T) \quad = \quad & \frac{e^{(\sigma\eta_1)^2\frac{T}{2}}}{\sigma\sqrt{2\pi T}}\sum_{n=1}^{+\infty}\pi_n\sum_{k=1}^{n}P_{n,k}\left(\sigma\sqrt{T}\eta_1\right)^k I_{k-1}\left(a - \mu T; -\eta_1, -\frac{1}{\sigma\sqrt{T}}, -\sigma\sqrt{T}\eta_1\right)\\
&+\frac{e^{(\sigma\eta_2)^2\frac{T}{2}}}{\sigma\sqrt{2\pi T}}\sum_{n=1}^{+\infty}\pi_n\sum_{k=1}^{n}Q_{n,k}\left(\sigma\sqrt{T}\eta_2\right)^k I_{k-1}\left(a - \mu T; \eta_2, \frac{1}{\sigma\sqrt{T}}, -\sigma\sqrt{T}\eta_2\right)\\
&+\pi_0 Phi\left(-\frac{a-\mu T}{\sigma\sqrt{T}}\right)
\end{aligned}$$

where

$$P_{n,i} := \sum_{j=i}^{n-1} p^j q^{n-j} \binom{n-i-1}{j-i} \left(\frac{\eta_1}{\eta_1+\eta_2}\right)^{j-i} \left(\frac{\eta_2}{\eta_1+\eta_2}\right)^{n-j}, \quad 1 \le i \le n-1$$

$$Q_{n,i} := \sum_{j=i}^{n-1} q^j p^{n-j} \binom{n-i-1}{j-i} \left(\frac{\eta_2}{\eta_1+\eta_2}\right)^{j-i} \left(\frac{\eta_1}{\eta_1+\eta_2}\right)^{n-j}, \quad 1 \le i \le n-1$$

$$P_{n,n} := p^n; \quad Q_{n,n} := q^n, \quad \pi_n = \frac{e^{-\lambda T}\lambda^n}{n!}$$

Using theorem 2, in [1], we know that the price of european call at inception and with maturity T is :

$$S_0\psi\left(r+\frac{1}{2}\sigma^2-\lambda\xi, \sigma, \widetilde{\lambda}, \widetilde{p}, \widetilde{\eta_1}, \widetilde{\eta_2}; \log\left(\frac{K}{S_0}\right), T\right) - Ke^{-rT}\psi\left(r-\frac{1}{2}\sigma^2-\lambda\xi, \sigma, \lambda, p, \eta_1, \eta_2; \log\left(\frac{K}{S_0}\right), T\right)$$

where

$$\widetilde{p} = \frac{p}{1+\xi}\frac{\eta_1}{\eta_1-1}, \quad \widetilde{\lambda} = \lambda(1+\xi), \quad \widetilde{\eta_1} = \eta_1 - 1, \quad \widetilde{\eta_2} = \eta_2 + 1$$

The put price can be obtain by using the call-put parity.

3 Finite time horizon american put option

Let $EuP(v,t)$ be the price of a european put option with initial stock price v and maturity t, $\mathbb{P}^v[S_t \le K]$ the probability that the stock price at t is below K with initial stock price v, $z = 1 - e^{-rT}$, $\beta_3 \equiv \beta_{3,r/z}$, $\beta_4 \equiv \beta_{5,r/z}$, $C_\beta = \beta_3\beta_4(1+\eta_2)$ (see (1.4)), $D_\beta = \eta_2(1+\beta_3)(1+\beta_4)$, $v_0 \equiv v_0(t) \in (0,K)$ the unique solution to the equation

$$C_\beta K - D_\beta\left(v_0 + EuP(v_0,t)\right) = \left(C_\beta - D_\beta\right) Ke^{-rT}\mathbb{P}^{v_0}\left[S_t \le K\right] \tag{3.6}$$

and

$$A = \frac{v_0^{\beta_3}}{\beta_4-\beta_3}\left\{\beta_4 K - (1+\beta_4)\left[v_0 + EuP(v_0,t)\right] + Ke^{-rT}\mathbb{P}^{v_0}\left[S_t \le K\right]\right\} > 0,$$

$$B = \frac{v_0^{\beta_4}}{\beta_3-\beta_4}\left\{\beta_4 K - (1+\beta_3)\left[v_0 + EuP(v_0,t)\right] + Ke^{-rT}\mathbb{P}^{v_0}\left[S_t \le K\right]\right\} > 0,$$

Then the price of a finite-horizon american put option with maturity t and strike K can be approximated by $\psi(S_0,t)$ which is given by (see ğ3 in [3])

$$\psi(v,t) = \begin{cases} EuP(v,t) + Av^{-\beta_3} + Bv^{-\beta_4}, & if\ v \ge v_0 \\ K - v, & if\ v \le v_0 \end{cases}$$

4 Lookback option

The price of a lookback floating strike put option is given by :

$$\begin{aligned} LP(T) &= \mathbb{E}\left[e^{-rT}\left(max\left\{M, \max_{0\leq t\leq T} S_t\right\} - S_T\right)\right] \\ &= \mathbb{E}\left[e^{-rT}\left(max\left\{M, \max_{0\leq t\leq T} S_t\right\}\right)\right] - S_0 \end{aligned}$$

where $M \geq S_0$ is a fixed constant representing the prefixed maximum at time 0. The Laplace transform of the lookback put, using notations in 1.4, is given by (see theorem 1 in [3])

$$\int_0^{+\infty} e^{-\alpha T} LP(T) dT = \frac{S_0 A_\alpha}{C_\alpha}\left(\frac{S_0}{M}\right)^{\beta_{1,\alpha+r}-1} + \frac{S_0 B_\alpha}{C_\alpha}\left(\frac{S_0}{M}\right)^{\beta_{2,\alpha+r}-1} + \frac{M}{\alpha+r} - \frac{S_0}{\alpha} \quad \forall \alpha > 0$$

where

$$\begin{aligned} A_\alpha &= \frac{(\eta_1 - \beta_{1,\alpha+r})\beta_{2,\alpha+r}}{\beta_{1,\alpha+r} - 1} \\ B_\alpha &= \frac{(\beta_{2,\alpha+r} - \eta_1)\beta_{1,\alpha+r}}{\beta_{2,\alpha+r} - 1} \\ C_\alpha &= (\alpha + r)\eta_1(\beta_{2,\alpha+r} - \beta_{1,\alpha+r}) \end{aligned}$$

The put price is obtained by using an inversion of the Laplace transform. The call option price follows just by symmetry. For the lookback fixed strike, when we have $M \geq max(S_0, K)$ for the put or $m \leq min(S_0, K)$ for the call, we get similar results to those for floatings.

5 Barrier option

Since all eight types of barrier can be solved in similar way, we focus only on the price Up and In Call option defined as followed

$$UIC = \mathbb{E}\left[e^{-rT}(S_T - K)^+ \mathbb{1}_{\{\max_{0\leq t\leq T} S_t \geq H\}}\right] \tag{5.7}$$

where $H > S_0$ is the barrier level. For any given probability P, define ;

$$\Psi(\mu, \sigma, \lambda, p, \eta_1, \eta_2; a, b, T) = \mathbb{P}\left[Z_T \geq a, \max_{0\leq t\leq T} Z_t \geq b\right] \tag{5.8}$$

where $Z_T = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$ and Y has a double exponential distribution with density as in (1.2), and N is a poisson process with rate λ. Using formula (3.1) and the result before remark 3.1 in [2], we get

$$\int_0^{+\infty} e^{-\alpha T}\mathbb{P}\left[\max_{0\leq t\leq T} Z_t \geq b\right] := \frac{1}{\alpha}\left(\frac{(\eta_1 - \beta_{1,\alpha})\beta_{2,\alpha}}{\eta_1(\beta_{2,\alpha} - \beta_{1,\alpha})} e^{-b\beta_{1,\alpha}} + \frac{(\beta_{2,\alpha} - \eta_1)\beta_{1,\alpha}}{\eta_1(\beta_{2,\alpha} - \beta_{1,\alpha})} e^{-b\beta_{2,\alpha}}\right)$$

By Inverting the Laplace transform we get $\mathbb{P}\left[\max_{0\le t\le T} Z_t \ge b\right]$, which is useful for some types of barrier options. Let us now define some functions

$$H_i(a,b,c;n) := \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{\left(\frac{1}{2}c^2-b\right)t} t^{n+\frac{i}{2}} Hh_i\left(c\sqrt{t}+\frac{a}{\sqrt{t}}\right)dt \quad i\ge -1, n\ge 0$$

$$\begin{aligned} A_\alpha &:= \mathbb{E}\left[e^{-\alpha\tau_b}\mathbb{1}_{X_{\tau_b}=b}\right] \\ &= \frac{\eta_1-\beta_{1,\alpha}}{\beta_{2,\alpha}-\beta_{1,\alpha}}e^{-b\beta_{1,\alpha}} + \frac{\beta_{2,\alpha}-\eta_1}{\beta_{2,\alpha}-\beta_{1,\alpha}}e^{-b\beta_{2,\alpha}} \end{aligned}$$

$$\begin{aligned} B_\alpha &:= \mathbb{E}\left[e^{-\alpha\tau_b}\mathbb{1}_{X_{\tau_b}>b}\right] \\ &= \frac{(\eta_1-\beta_{1,\alpha})(\beta_{2,\alpha}-\eta_1)}{\eta_1(\beta_{2,\alpha}-\beta_{1,\alpha})}\left[e^{-b\beta_{1,\alpha}}-e^{-b\beta_{2,\alpha}}\right] \end{aligned}$$

where $\tau_b = \inf\{t\ge 0;\ X_t \ge b\}$.Hh functions are defined in ğ 2, and β variables in (1.4). For $i\ge 1$, under assumption that $b>0$ and $c>-\sqrt{2b}$, we have

$$H_i(a,b,c;n) = \frac{1}{i}H_{i-2}(a,b,c;n+1) - \frac{c}{i}H_{i-1}(a,b,c;n+1) - \frac{a}{i}H_{i-1}(a,b,c;n)$$

By knowing $H_{-1}(a,b,c;n)$ and $H_0(a,b,c;n)$, this recursive formula allows us to determine all values of H_i. Lemmas A.1 and A.2 in [2] give us

$$H_{-1}(a,b,c;n) = e^{-ac-\sqrt{2a^2b}}\sqrt{\frac{1}{2b}}\left(\sqrt{\frac{a^2}{2b}}\right)^n \sum_{j=0}^{n}\frac{(-n)_j(n+1)_j}{j!\left(-2\sqrt{2a^2b}\right)^j}, \quad a\ne 0, n\ge 0$$

$$H_{-1}(a,b,c;n) = e^{-ac-\sqrt{2a^2b}}\sqrt{\frac{1}{2b}}\left(\sqrt{\frac{a^2}{2b}}\right)^n \sum_{j=0}^{-n-1}\frac{(-n)_j(n+1)_j}{j!\left(-2\sqrt{2a^2b}\right)^j}, \quad a\ne 0, n\le -1$$

$$H_{-1}(0,b,c;n) = \frac{(2n)!}{n!(4b)^n}\frac{1}{2b}, \quad n\ge 0$$

$$H_0(a,b,c;n) = \frac{c}{2(n+1)}H_{-1}(a,b,c;n+1) - \frac{a}{2(n+1)}H_{-1}(a,b,c;n), \quad b=\frac{1}{2}c^2, n\ge 0$$

And $\forall n\ge 0$ et $b\ne \frac{1}{2}c^2$

$$H_0(a,b,c;n) = \frac{n!}{\left(b-\frac{1}{2}c^2\right)^{n+1}}\sum_{i=0}^{n}\frac{\left(b-\frac{1}{2}c^2\right)^i}{i!}\left(\frac{a}{2}H_{-1}(a,b,c;i-1)-\frac{c}{2}H_{-1}(a,b,c;i)\right), \quad a>0$$

$$H_0(a,b,c;n) = \frac{n!}{\left(b-\frac{1}{2}c^2\right)^{n+1}}\left(1+\sum_{i=0}^{n}\frac{\left(b-\frac{1}{2}c^2\right)^i}{i!}\left(\frac{a}{2}H_{-1}(a,b,c;i-1)-\frac{c}{2}H_{-1}(a,b,c;i)\right)\right), \quad a<0$$

$$H_0(a,b,c;n) = \frac{n!}{\left(b-\frac{1}{2}c^2\right)^{n+1}}\left(\frac{1}{2}+\sum_{i=0}^{n}\frac{\left(b-\frac{1}{2}c^2\right)^i}{i!}\frac{c}{2}H_{-1}(a,b,c;i)\right), \quad a=0$$

where $(n)_j = n(n+1)\dots(n+j-1)$, with convention $(n)_0 = 1$.
We can now determine the exact expression of the Laplace transform of Ψ when $b > 0$ and $a \le b$(see theorem 4.1 in [2])

$$\begin{aligned}
\int_0^{+\infty} e^{-\alpha T}\mathbb{P}\left[Z_T \ge a, \max_{0\le t\le T} Z_t \ge b\right] dT &= A_\alpha \int_0^{+\infty} e^{-\alpha T}\mathbb{P}\left[Z_T \ge a-b\right] dT \\
&\quad + B_\alpha \int_0^{+\infty} e^{-\alpha T}\mathbb{P}\left[Z_T + \xi^+ \ge a-b\right] dT \\
&= (A_\alpha + B_\alpha)\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}H_0\left(-h,\gamma_\alpha,-\frac{\mu}{\sigma};n\right) \\
&\quad + e^{h\sigma\eta_1}\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{\lambda^n}{n!}\left(A_\alpha P_{n,j} + B_\alpha \overline{P}_{n,j}\right)\sum_{i=0}^{j-1}(\sigma\eta_1)^i H_i\left(h,\gamma_\alpha,c_+;n\right) \\
&\quad - e^{-h\sigma\eta_2}\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{\lambda^n}{n!}\left(A_\alpha Q_{n,j} + B_\alpha \overline{Q}_{n,j}\right)\sum_{i=0}^{j-1}(\sigma\eta_1)^i H_i\left(-h,\gamma_\alpha,c_-;n\right) \\
&\quad + e^{h\sigma\eta_1}B_\alpha\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{(\lambda)^n}{n!}(\sigma\eta_1)^i H_i\left(h,\gamma_\alpha,c_+;n\right) \\
&\quad + e^{h\sigma\eta_1}B_\alpha H_0\left(h,\gamma_\alpha,c_+;0\right)
\end{aligned}$$

where ξ^+ has an exponential law with rate η_1, matrix P and Q are as defined in ğ 2, and

$$\begin{aligned}
&\overline{P}_{n,1} := \sum_{j=i}^{n-1} Q_{n_i}\left(\frac{\eta_2}{\eta_1+\eta_2}\right)^i, \quad \overline{P}_{n,i} := P_{n,i-1}, \quad 2 \le i \le n+1 \\
&\overline{Q}_{n,i} := \sum_{j=i}^{n}\binom{n}{j} q^j p^{n-j}\binom{n-i}{j-i}\left(\frac{\eta_2}{\eta_1+\eta_2}\right)^{j-i}\left(\frac{\eta_1}{\eta_1+\eta_2}\right)^{n-j+1}, \quad 1 \le i \le n \\
&c_+ := \sigma\eta_1 + \frac{\mu}{\sigma}, \quad c_- := \sigma\eta_2 - \frac{\mu}{\sigma}, \quad \gamma_\alpha := \alpha + \lambda + \frac{\mu^2}{2\sigma^2}, \quad h := \frac{b-a}{\sigma}
\end{aligned}$$

For to get numerically $\mathbb{P}\left[Z_T \ge a, \max_{0\le t\le T} Z_t \ge b\right]$ for a given T, i find that is better to inverse the right term in the first equality above, using some propreties of the Laplace inversion. Note that $\mathbb{P}\left[Z_T \ge a-b\right]$ is given in ğ 2 and $\mathbb{P}\left[Z_T + \xi^+ \ge a-b\right]$ is given in [2] (pp. 528, formula B.5) :

$$\begin{aligned}
\mathbb{P}\left[Z_T + \xi^+ \ge a\right] &= \frac{e^{(\sigma\eta_1)^2\frac{T}{2}}}{\sigma\sqrt{2\pi T}}\sum_{n=1}^{+\infty}\pi_n\sum_{k=1}^{n+1}\overline{P}_{n,k}\left(\sigma\sqrt{T}\eta_1\right)^k I_{k-1}\left(a-\mu T; -\eta_1, -\frac{1}{\sigma\sqrt{T}}, -\sigma\sqrt{T}\eta_1\right) \\
&\quad + \frac{e^{(\sigma\eta_2)^2\frac{T}{2}}}{\sigma\sqrt{2\pi T}}\sum_{n=1}^{+\infty}\pi_n\sum_{k=1}^{n}\overline{Q}_{n,k}\left(\sigma\sqrt{T}\eta_2\right)^k I_{k-1}\left(a-\mu T; \eta_2, \frac{1}{\sigma\sqrt{T}}, -\sigma\sqrt{T}\eta_2\right) \\
&\quad + \pi_0\eta_1\frac{e^{(\sigma\eta_1)^2\frac{T}{2}}}{\sqrt{2\pi}} I_0\left(a-\mu T; -\eta_1, -\frac{1}{\sigma\sqrt{T}}, -\eta_1\sigma\sqrt{T}\right)
\end{aligned}$$

The price of the UIC option is obtained by, thanks to Kou and Wang (see theorem 2 in [3])

$$
\begin{aligned}
UIC &= S_0\Psi\left(r+\frac{1}{2}\sigma^2-\lambda\xi,\sigma,\lambda,p,\eta_1,\eta_2;\log\left(\frac{K}{S_0}\right),\log\left(\frac{H}{S_0}\right),T\right)\\
&\quad -Ke^{-rT}\Psi\left(r-\frac{1}{2}\sigma^2-\lambda\xi,\sigma,\lambda,p,\eta_1,\eta_2;\log\left(\frac{K}{S_0}\right),\log\left(\frac{H}{S_0}\right),T\right)
\end{aligned}
$$

where

$$
\widetilde{p}=\frac{p}{1+\xi}\frac{\eta_1}{\eta_1-1},\quad \widetilde{\lambda}=\lambda(1+\xi),\quad \widetilde{\eta_1}=\eta_1-1,\quad \widetilde{\eta_2}=\eta_2+1
$$

References

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