European Petroleum Survey Group

EPSG

Guidance Note Number 7

POSC literature pertaining to

Coordinate Conversions and Transformations including Formulas

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Revision history:

Version	Date	Amendments
1	December 1993	First release – POSC Epicentre
10	May 1998	Additionally issued as an EPSG guidance note.
11	November 1998	Polynomial for Spain and Tunisia Mining Grid methods added.
12	February 1999	Abridged Molodenski formulas corrected.
13	July 1999	Lambert Conic Near Conformal and American Polyconic methods added.
14	December 1999	Stereographic and Tunisia Mining Grid formulas corrected. Krovak method added.
15	June 2000	General Polynomial and Affine methods added
16	December 2000	Lambert Conformal (Belgium) remarks revised; Oblique Mercator methods consolidated and formulas added. Similarity Transformation reversibility remarks amended.
17	June 2001	Lambert Conformal, Mercator and Helmert formulas corrected.
18	August 2002	Revised to include ISO 19111 terminology. Section numbering revised. Added Preface. Lambert Conformal (West Orientated), Lambert Azimuthal Equal Area, Albers, Equidistant Cylindrical (Plate Carrée), TM zoned, Bonne, Molodenski-Badedas methods added. Errors in Transverse Mercator (South Orientated) formula corrected.
19	December 2002	Polynomial formulas amended. Formula for spherical radius in Equidistant Cylindrical projection amended. Formula for Krovak projection amended. Degree representation conversions added. Editorial amendments made to subscripts and superscripts.
20	May 2003	Font for Greek symbols in Albers section amended.
21	October 2003	Typographic errors in example for Lambert Conic (Belgium) corrected. Polar Stereographic formulae extended for secant variants. General polynomial extended to degree 13. Added Abridged Molodenski and Lambert Azimuthal Equal Area examples and Reversible polynomial formulae.
22	December 2003	Errors in FE and FN values in example for Lambert Azimuthal Equal Area corrected.
23	January 2004	Database codes for Polar Stereographic variants corrected. Degree representation conversions withdrawn.

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Preface

A **coordinate system** is set of mathematical rules for specifying how coordinates are to be assigned to points. This is unrelated to the Earth. A **coordinate reference system** (CRS) is a coordinate system related to the Earth through a **datum**. Colloquially the term coordinate system has historically been used to mean coordinate reference system.

Coordinates may be changed from one coordinate reference system to another through the application of a **coordinate operation**. Two types of coordinate operation may be distinguished:

- **coordinate conversion**, where no change of datum is involved.
- **coordinate transformation**, where the target CRS is based on a different datum to the source CRS.

A projected coordinate reference system is the result of the application of a **map projection** to a geographic coordinate reference system. A map projection is a type of coordinate conversion. It uses an identified method with specific formulas and a set of parameters specific to that coordinate conversion method.

Map projection methods are described in part 1 below. Other coordinate conversions and transformations are described in part 2.

Part 1. Map projections and their coordinate conversion formulas

1.1 Introduction

Setting aside the large number of map projection methods which may be employed for atlas maps, equally small scale illustrative exploration maps, and wall maps of the world or continental areas, EPSG provides reference parameter values for orthomorphic or conformal map projections which are used for medium or large scale topographic or exploration mapping. Here accurate positions are important and sometimes users may wish to scale accurate positions, distances or areas from the maps.

Small scale maps normally assume a spherical earth and the inaccuracies inherent in this assumption are of no consequence at the usual scale of these maps. For medium and large scale sheet maps, or maps and coordinates held digitally to a high accuracy, it is essential that due regard is paid to the actual shape of the Earth. Such coordinate reference systems are therefore invariably based on an ellipsoid and its derived map projections. The EPSG data set and this supporting conversion documentation considers only map projections for the ellipsoid.

Though not exhaustive the following list of named map projection methods are those which are most frequently encountered for medium and large scale mapping, some of them much less frequently than others since they are designed to serve only one particular country. They are grouped according to their possession of similar properties, which will be explained later. Except where indicated all are conformal.

Mercator	*Cylindrical*
with one standard parallel	
with two standard parallels	
Cassini-Soldner (N.B. not conformal)	*Transverse Cylindrical*
Transverse Mercator Group	*Transverse Cylindrical*
Transverse Mercator (including south oriented version)	
Universal Transverse Mercator	
Gauss-Kruger	
Gauss-Boaga	
Oblique Mercator Group	*Oblique Cylindrical*
Hotine Oblique Mercator	
Oblique Mercator	
Laborde Oblique Mercator	
Lambert Conical Conformal	*Conical*
with one standard parallel	
with two standard parallels	
one standard parallel with truncated formulas (N.B. not conformal)	
Stereographic	
Polar	
Oblique and equatorial	

1.2. Identification of Map Projection method

If a map or coordinate list is provided for which an EPSG listed coordinate reference system is clearly identifiable, then its name or code together with EPSG dataset version number will address the required parameters including the coordinate conversion parameter values. If the coordinate reference system is not listed it will be necessary to create a new coordinate reference system with its own coordinate conversion (parameter) values.

It may often happen that one is presented with a coordinate list or map for which the author or compiler has regrettably failed to provide any indication of parameter values or properties:- no projection name, no grid definition and no statement of ellipsoid or datum. On the map there may be no grid or graticule, or indeed neither.

In order to adequately relate the digital or displayed map data to other data it is necessary to establish the properties of the data or given map from what may be gleaned from their appearance and other information. Geographical coordinates without qualifying information do not allow identification of the coordinate reference system other than that it is a geographic one. Projected or map grid coordinates may, by virtue of the actual and relative magnitudes of the Easting and Northing and knowledge of where in the world they relate to, provide clues as to the map projection. For example eastings between say 150,000m and 850,000m, allied with 6 or 7 figure northings correlated with latitude *may* indicate a UTM grid zone.

If the map bears neither grid nor graticule it will be useless unless one can identify a number of the point features shown for which one already has coordinate data. One may then be able to superimpose and fit a rectangular grid at appropriate scale from which other coordinate data may be read. If the map carries a grid then the numerical labelling of the grid lines, the assumption that it will be conformal or orthomorphic, and prior knowledge of approximately where in the world it covers may give some indication of the type of projection, but this may not be totally definitive. If the map bears a graticule the nature of the graticule lines will give some indication of the type of projection used in its compilation. For example straight meridians and concentric parallels would suggest a conical projection or, less frequently, a polar azimuthal. If the former, and assuming that it will be orthomorphic, then it will either be with one standard parallel or two and these will have been selected in relation to the latitudinal extent of the area, very possibly those in general use for that state's mapping. If the parallels are equally spaced it will be a simple equidistant conical projection. However for large scale mapping purposes the requirement that it is conformal will dictate that the parallels will not be equally spaced and it is more than likely that it will be some form of Lambert projection with either one or two standard parallels. Unfortunately there is no easy way of detecting which, nor the values of the standard parallels. The country it comes from and its national mapping system, if known, may suggest what these are. The EPSG data set will assist but is not exhaustive.

If both meridians and parallels of latitude are straight it will be a cylindrical projection but of the normal and not the more frequent transverse or oblique variety (Figure 1 at end of section 1.3). Of the normal aspect cylindrical projections only the Mercator is conformal and it is not frequently used for topographic mapping though it is almost invariably used for the production of marine navigation charts.

If both parallels of latitude and meridians are curved the projection has numerous possibilities but a form of Transverse Mercator may well be the most likely. One may attempt to identify the projection by computing the grid values of some of the graticule intersections for several possible projections in turn, plotting these to a rounded value for the estimated scale of the map e.g. 1:50000 or 1:100000, and attempting to fit the overlaid grid plot on the graticule. Repeating this for a number of potential projections for the area may be successful in obtaining a reasonable fit. But bear in mind that paper stretch may slightly distort scale from the nominal scale of the map, and the scale factor used in the graticule to grid conversions is another variable which may take only slightly different values e.g. a

Gauss-Kruger takes a central meridian scale factor of unity while a UTM (like Gauss-Kruger, a Transverse Mercator) takes 0.9996.

Digital cartographic techniques make it relatively easy to plot grid and graticule for different projections with different parameters onto transparencies for "trial and error" overlays. The process can be time consuming so it is preferable to make maximum use of the clues which one may infer from the appearance of the map as initially presented, - its origins, its national area, and the conventional projections used for that area.

1.3. Map Projection parameters

A map projection grid is related to the geographical graticule of an ellipsoid through the definition of a coordinate conversion method and a set of parameters appropriate to that method. Differing conversion methods may require different parameters. Any one coordinate conversion method may take several different sets of associated parameter values, each set related to a particular map projection zone applying to a particular country or area of the world. Before setting out the formulas involving these parameters, which enable the coordinate conversions for the projection methods listed above, it is as well to understand the nature of these parameters.

The plane of the map and the ellipsoid surface may be assumed to have one particular point in common. This point is referred to as the **natural origin**. It is the point from which the values of both the geographical coordinates on the ellipsoid and the grid coordinates on the projection are deemed to increment or decrement for computational purposes. Alternatively it may be considered as the point which in the absence of application of false coordinates has grid coordinates of (0,0). For example, for projected coordinate reference systems using the Cassini-Soldner or Transverse Mercator methods, the natural origin is at the intersection of a chosen parallel and the chosen central meridian (see Figure 2 at end of section). The chosen parallel will frequently but not necessarily be the equator. For the stereographic projection the origin is at the centre of the projection where the plane of the map is imagined to be tangential to the ellipsoid.

Since the natural origin may be at or near the centre of the projection and under normal coordinate circumstances would thus give rise to negative coordinates over parts of the map, this origin is usually given false coordinates which are large enough to avoid this inconvenience. Hence each natural origin will normally have **False Easting, FE** and **False Northing, FN** values**.** For example, the false easting for the origins of all Universal Transverse Mercator zones is 500000m. As the UTM origin lies on the equator, areas north of the equator do not need and are not given a false northing but for mapping southern hemisphere areas the equator origin is given a false northing of 10,000,000m, thus ensuring that no point in the southern hemisphere will take a negative northing coordinate. Figure 4 illustrates the UTM arrangements.

These arrangements suggest that if there are false easting and false northing for the real or natural origin, there is also a **Grid Origin** which has coordinates (0,0). In general this point is of no consequence though its geographic position may be computed if needed. Sometimes however, rather than base the easting and northing coordinate reference system on the natural origin by giving it **FE** and **FN** values, it may be convenient to select a **False Origin** at a specific meridian/parallel intersection and attribute the false coordinates (0,0) or, more usually, $\mathbf{E_F}$ and $\mathbf{N_F}$ to this. The related easting and northing of the natural origin may then be computed if required.

The natural origin will always lie on a meridian of longitude. Longitudes are most commonly expressed relative to the **Prime Meridian** of Greenwich but some countries, particularly in former times, have preferred to relate their longitudes to a prime meridian through their national astronomic observatory, usually sited in or near their capital city, e.g. Paris for France, Bogota for Colombia. The meridian of the projection zone origin is known as the **Longitude of Origin**. For certain projection types it is often termed the **Central Meridian** or abbreviated as **CM** and provides the direction of the northing axis of the projected coordinate reference system**.**

Because of the steadily increasing distortion in the scale of the map with increasing distance from the origin, central meridian or other line on which the scale is the nominal scale of the projection, it is usual to limit the extent of a projection to within a few degrees of latitude or longitude of this point or line. Thus, for example, a UTM or other Transverse Mercator projection zone will normally extend only 2 or 3 degrees from the central meridian. For areas beyond this another **zone** of the projection, with a new origin and central meridian, needs to be used or created. The UTM system has a specified 60 numbered zones, each 6 degrees wide, covering the ellipsoid between the 84 degree North and 80 degree South latitude parallels. Other Transverse Mercator projection zones may be constructed with different central meridians, and different origins chosen to suit the countries or states for which they are used. A number of these are included in the EPSG dataset. Similarly a Lambert Conic Conformal zone distorts most rapidly in the north-south direction and may, as in Texas, be divided into latitudinal bands.

In order to further limit the scale distortion within the coverage of the zone or projection area, some projections introduce a **scale factor** at the origin (on the central meridian for Transverse Mercator projections), which has the effect of reducing the nominal scale of the map here and making it have the nominal scale some distance away. For example in the case of the UTM and some other Transverse Mercator projections a scale factor of slightly less than unity is introduced on the central meridian thus making it unity on two meridians either side of the central one, and reducing its departure from unity beyond these. The scale factor is a required parameter whether or not it is unity and is usually symbolised as $\mathbf{k_0}$**.**

Thus for projections in the Transverse Mercator group in section 1.1 above, the parameters which are required to completely and unambiguously define the projection method are:

Latitude of natural origin
Longitude of natural origin
Scale factor at natural origin
False easting
False northing

Since the UTM zones obey set rules, it is sufficient to state only the UTM zone number (or central meridian). The remaining parameters from the above list are defined by the rules.

It has been noted that the Transverse Mercator projection is employed for the topographical mapping of longitudinal bands of territories, limiting the amount of scale distortion by limiting the extent of the projection either side of the central meridian. Sometimes the shape, general trend and extent of some countries makes it preferable to apply a single zone of the same kind of projection but with its central line aligned with the trend of the territory concerned rather than with a meridian. So, instead of a meridian forming this true scale central line for one of the various forms of Transverse Mercator, or the equator forming the line for the Mercator, a line with a particular azimuth traversing the territory is chosen, and the same principles of construction are applied to derive what is now an Oblique Mercator. This projection is sometimes referred to as the Hotine Oblique Mercator after the British geodesist who set out its formulas for application to Malaysian Borneo (East Malaysia) and also West Malaysia. Laborde had previously developed the projection system for Madagascar, and Switzerland uses a similar system derived by Rosenmund.

More recently (1974) Lee has derived formulas for a minimum scale factor projection for New Zealand known as the New Zealand Map Grid. The line of minimum scale follows the general alignment of the two main islands. This resembles an Oblique Mercator projection in its effect, but is not strictly an Oblique Mercator. The additional mathematical complexity of the projection enables its derivation via an Oblique Stereographic projection, which is sometimes the way it is classified. Because of its unique formulation inclusion of the New Zealand Map Grid within international mapping software was sporadic; as a consequence New Zealand has reverted to the frequently-encountered Transverse Mercator for its most recent mapping.

The parameters required to define an Oblique Mercator projection are:

Latitude of projection centre (the origin point on the initial line)
Longitude of projection centre
Azimuth of initial line [at the projection centre]
Scale factor on initial line [at the projection centre]
Angle from Rectified to Skewed grid [at the natural origin]

and then either

False easting (easting at the projection natural origin)
False northing (northing at the projection natural origin)

or

Easting at projection centre
Northing at projection centre

The angle from rectified to skewed grid is normally applied at the natural origin of the projection, that is where the initial line of the projection intersects the aposphere. In some circumstances, for instance in the Alaskan panhandle State Plane zone, this angle is taken to be identical to the azimuth of the initial line at the projection centre. This results in grid and true north coinciding at the projection centre rather than at the natural origin as is more usual.

It is possible to define the azimuth of the initial line through the latitude and longitude of two widely spaced points along that line. This approach is not currently followed by POSC/EPSG.

For **Conical** map projections, which for the normal aspect may be considered as the projection of the ellipsoid onto an enveloping cone in contact with the ellipsoid along a parallel of latitude, the parallel of contact is known as **a standard parallel** and the scale is regarded as true along this parallel. Sometimes the cone is imagined to cut the ellipsoid with coincidence of the two surfaces along **two standard parallels**. All other parallels will be concentric with the chosen standard parallel or parallels but for the Lambert Conical Conformal will have varying separations to preserve the conformal property. All meridians will radiate with equal angular separations from the centre of the parallel circles but will be compressed from the 360 longitude degrees of the ellipsoid to a sector whose angular extent depends on the chosen standard parallel, - or both standard parallels if there are two. Of course the normal longitudinal extent of the projection will depend on the extent of the territory to be projected and will never approach 360 degrees.

As in the case of the Transverse Mercator above it is sometimes desirable to limit the maximum positive scale distortion for the one standard parallel case by distributing it more evenly over the extent of the mapped area. This may be achieved by introducing a scale factor on the standard parallel of slightly less than unity thus making it unity on two parallels either side of it. This achieves the same effect as choosing two specific standard parallels in the first place, on which the nominal scale will be preserved. The projection is then a Lambert Conical Conformal projection with two standard parallels**.** Although, strictly speaking, the scale on a standard parallel is always the nominal scale of the map and the scale factor on the one or two standard parallels should be unity, it is sometimes convenient to consider a Lambert Conical Conformal projection with one standard parallel yet which has a scale factor on the standard parallel of less than unity. This provision is allowed for by POSC/EPSG, where the single standard parallel is referred to as the **latitude of natural origin**. For an ellipsoidal projection the natural origin will fall slightly poleward of the mean of the latitudes of the two standard parallels.

A **longitude of origin** or **central meridian** will again be chosen to bisect the area of the map or, more usually, the total national map area for the country or state concerned. Where this cuts the one standard parallel will be the natural origin of the projected coordinate reference system and, as for the Transverse Mercator, it will be given a **False easting and False northing** to ensure that there are no negative coordinates within the projected area (see Figure 5). Where two standard parallels are specified a false origin may be chosen at the intersection of a specific parallel with the central meridian. This point will be given an **easting at false origin** and a **northing at false origin** to ensure that no negative coordinates will result. Figure 6 illustrates these arrangements.

It is clear that any number of Lambert projection zones may be formed according to which standard parallel or standard parallels are chosen and this is clearly exemplified by those which are used for many of the United States State Plane coordinate zones. They are normally chosen either, for one standard parallel, to approximately bisect the latitudinal extent of the country or area or, for two standard parallels, to embrace most of the latitudinal extent of the area. In the latter case the aim is to minimise the maximum scale distortion which will affect the mapped area and various formulas have been developed by different mathematicians to select the appropriate standard parallels to achieve this. Kavraisky was one mathematician who derived a recipe for choosing the standard parallels to achieve minimal scale distortion. But however the selection of the standard parallels is made the same projection formulas apply. Thus the parameters needed to specify a projection in the Lambert projection will be as follows:

For a Lambert Conical Conformal with one standard parallel (1SP),

Latitude of natural origin (the Standard Parallel)
Longitude of natural origin (the Central Meridian)
Scale factor at natural origin (on the Standard Parallel)
False easting
False northing

For a Lambert Conical Conformal with two standard parallels (2SP),

Latitude of false origin
Longitude of false origin (the Central Meridian)
Latitude of first standard parallel
Latitude of second standard parallel
Easting at false origin
Northing at false origin

where the order of the standard parallels is not material if using the formulas which follow.

The limiting case of the Lambert Conic Conformal having the apex of the cone at infinity produces a **cylindrical** projection, the Mercator. Here, for the single standard parallel case the latitude of natural origin is the equator. For the two standard parallel case the two parallels have equal latitude in the north and south hemispheres. In both one and two standard parallel cases, grid coordinates are for the natural origin at the intersection of the equator and the central meridian (see figure 1). Thus the parameters needed to specify a map projection using the Mercator map projection method will be:

For a Mercator with one standard parallel (1SP),

Latitude of natural origin (the Equator)
Longitude of natural origin (the Central Meridian)
Scale factor at natural origin (on the Equator)
False easting
False northing

For a Mercator with two standard parallels (2SP),

Latitude of first standard parallel

Longitude of natural origin (the Central Meridian)
False easting (grid coordinate at the intersection of the CM with the equator)
False northing

In the formulas that follow the absolute value of the first standard parallel must be used.

For **Azimuthal** map projections**,** which are only infrequently used for ellipsoidal topographic mapping purposes, the natural origin will be at the centre of the projection where the map plane is imagined to be tangential to the ellipsoid and which will lie at the centre of the area to be projected. The central meridian will pass through the natural origin. This point will be given a False Easting and False Northing.

The parameters needed to specify the Stereographic map projection method are:

Latitude of natural origin
Longitude of natural origin (the central meridian for the oblique case)
Scale factor at natural origin
False easting at natural origin
False northing at natural origin

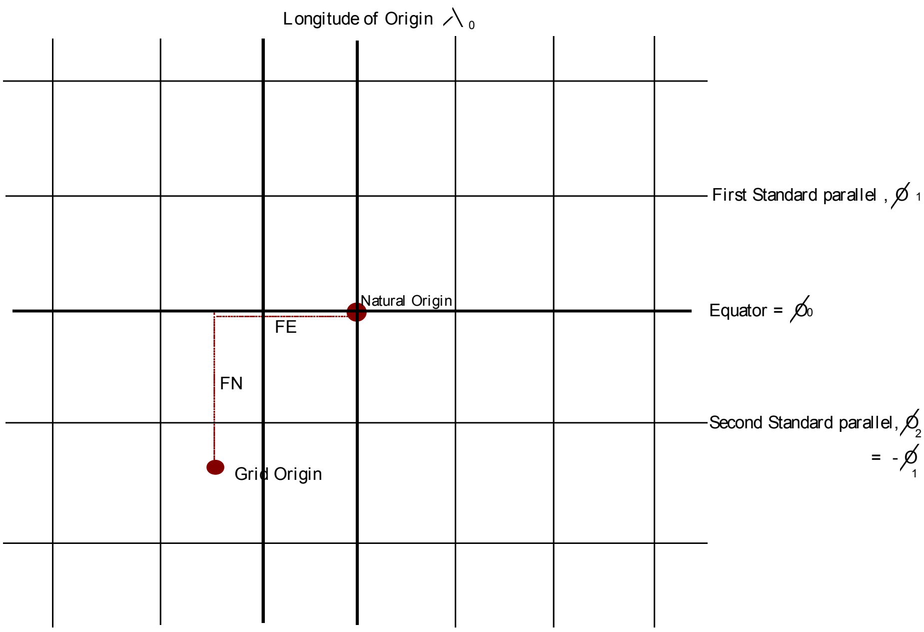

Figure 1. Key Diagram for Mercator Projection arrangements
(One and two standard parallel cases)

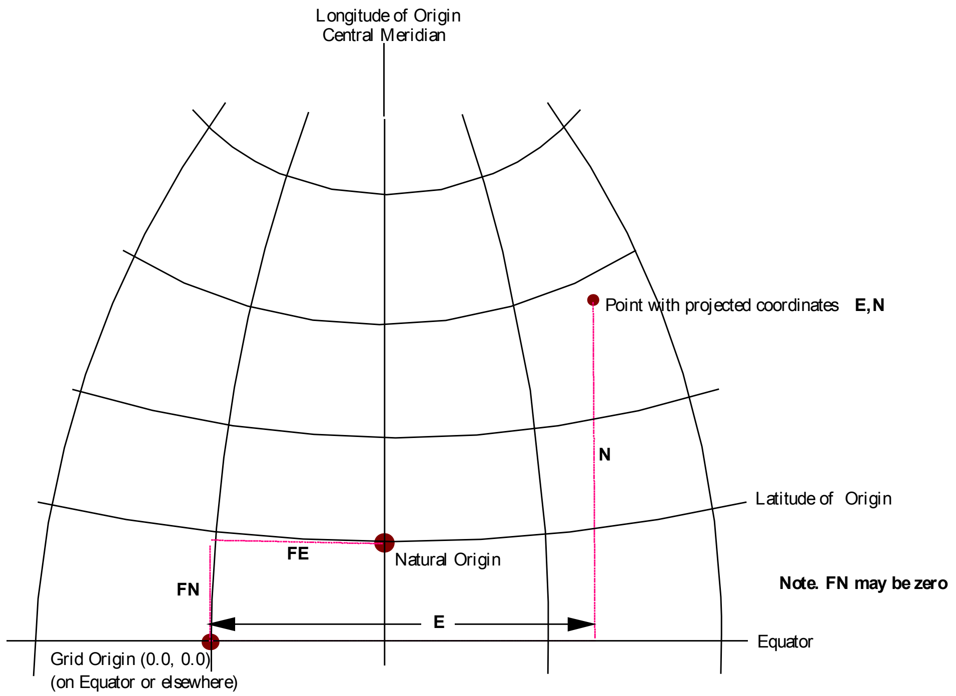

Figure 2. Key Diagram for Transverse Mercator Projection arrangements (N.Hemisphere)

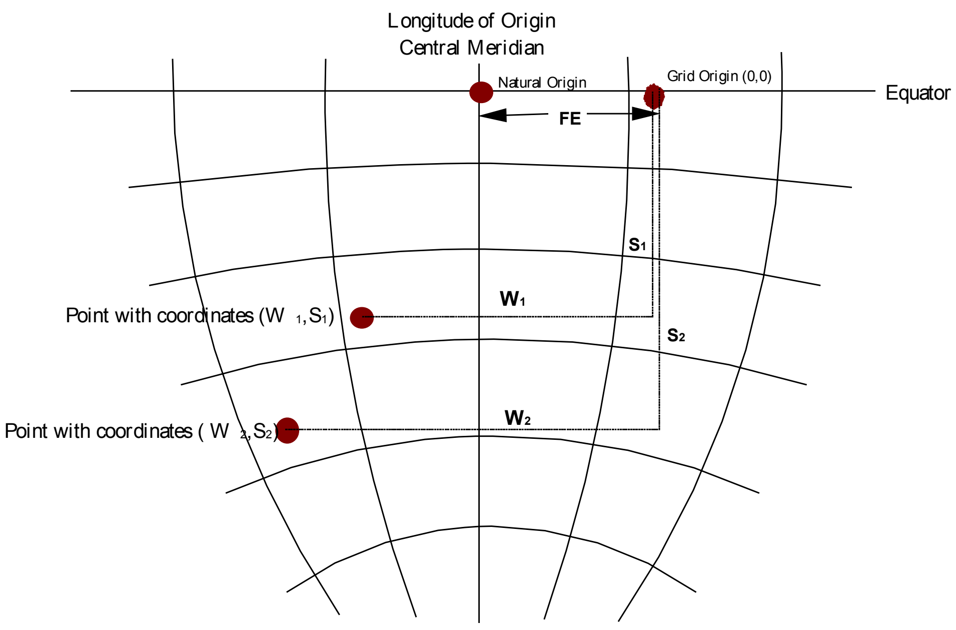

Figure 3. Key Diagram for South oriented Transverse MercatorProjection arrangements

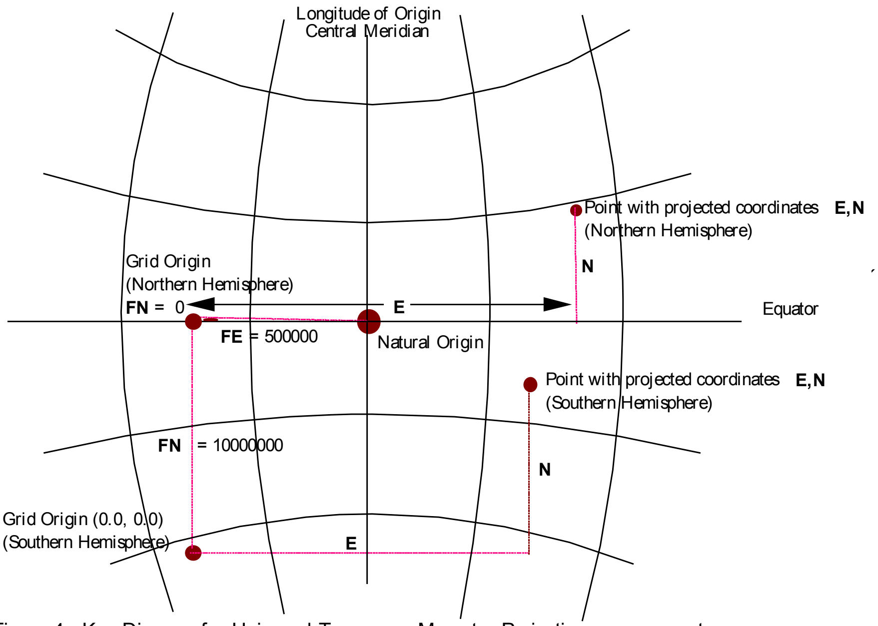

Figure 4. Key Diagram for Universal Transverse Mercator Projection arrangements
(N and S hemisphere cases)

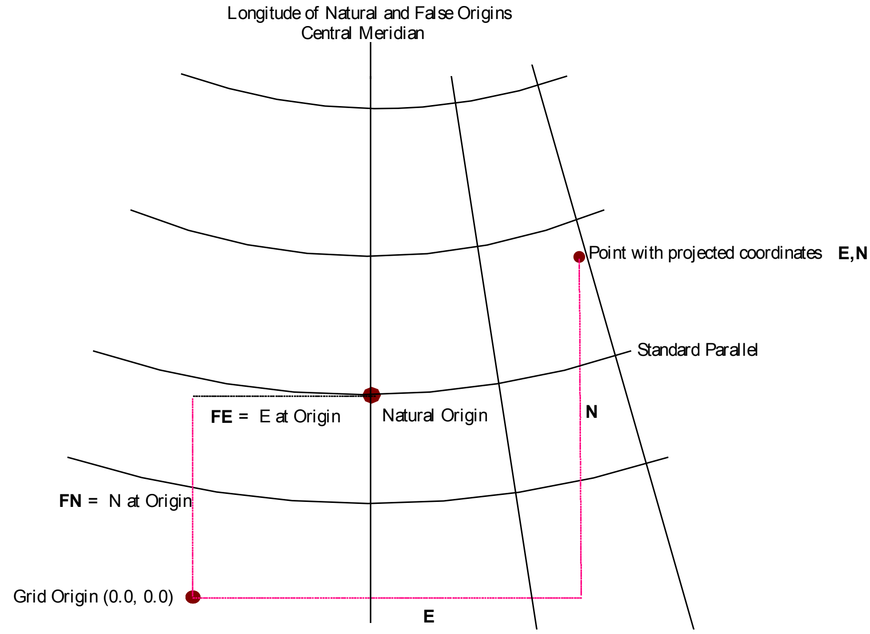

Figure 5. Key Diagram for Lambert Conical Conformal Projection with one standard parallel

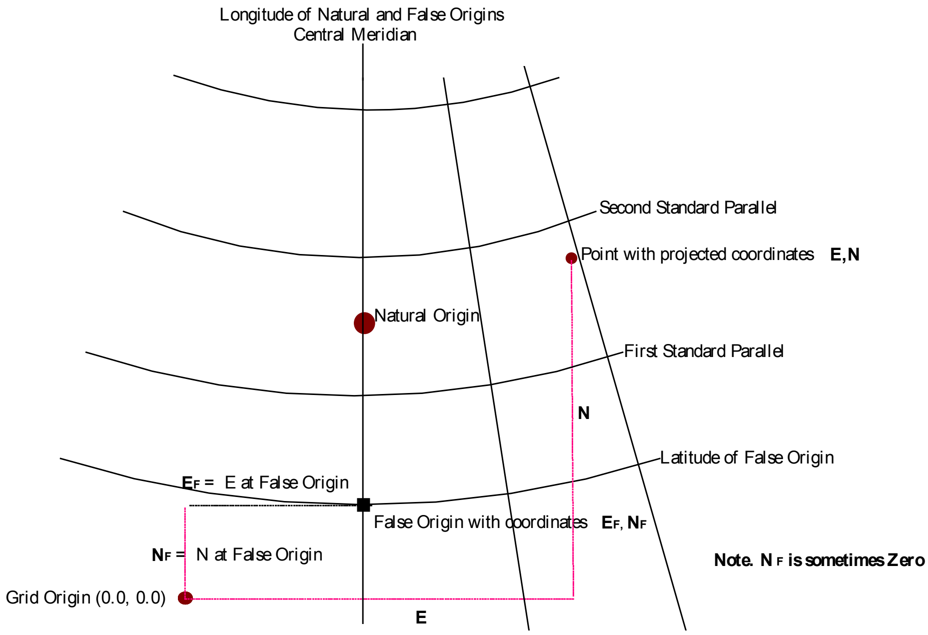

Figure 6. Key Diagram for Lambert Conical Conformal Projection with two standard parallels

TABLE 1

Summary of Coordinate Operation Parameters required for some Map Projections

	Coordinate Operation Method								
Coordinate Operstion Parameter Name	Mercator (1SP)	Mercator (2SP)	Cassini-Soldner	Transverse Mercator	Hotine Oblique Mercator	Oblique Mercator	Lambert Conical (1 SP)	Lambert Conical (2 SP)	Oblique Stereo-graphic
Latitude of false origin								1	
Longitude of false origin								2	
Latitude of 1st standard parallel		1						3	
Latitude of 2nd standard parallel								4	
Easting at false origin								5	
Northing at false origin								6	
Latitude of projection centre					1	1			
Longitude of projection centre					2	2			
Scale factor on initial line					3	3			
Azimuth of initial line					4	4			
Angle from Rectified to Skewed grid					5	5			
Easting at projection centre						6			
Northing at projection centre						7			
Latitude of natural origin	1 =equator		1	1			1		1
Longitude of natural origin	2	2	2	2			2		2
Scale factor at natural origin	3			3			3		3
False easting	4	3	3	4	6		4		4
False northing	5	4	4	5	7		5		5

1.4. Map Projection formulas

Only formulas for computation on the ellipsoid are considered. Projection formulas for the spherical earth are simpler but the spherical figure is inadequate to represent positional data with great accuracy at large map scales for the real earth. Projections of the sphere are only suitable for illustrative maps at scale of 1:1 million or less where precise positional definition is not critical.

The formulas which follow are largely adapted from "Map Projections - A Working Manual" by J.P.Snyder, published by the U.S. Geological Survey as Professional Paper No.1395[1]. As well as providing an extensive overview of most map projections in current general use, and the formulas for their construction for both the spherical and ellipsoidal earth, this excellent publication provides computational hints and details of the accuracies attainable by the formulas. It is strongly recommended that all those who have to deal with map projections for medium and large scale mapping should follow its guidance.

There are a number of different formulas available in the literature for map projections other than those quoted by Snyder. Some are closed formulas; others, for ease of calculation, may depend on series expansions and their precision will generally depend on the number of terms used for computation. Generally those formulas which follow in this chapter will provide results which are accurate to within a decimetre, which is normally adequate for exploration mapping purposes. Coordinate expression and computations for engineering operations are usually consistently performed in grid terms.

The importance of one further variable should be noted. This is the unit of linear measurement used in the definition of projected coordinate systems. For metric map projections the unit of measurement is restricted to this unit. For non-metric map projections the metric ellipsoid semi-major axis needs to be converted to the projected coordinate system linear unit before use in the formulas below. The relevant ellipsoid is obtained through the datum part of the projected coordinate reference system.

In the formulas for map projections which follow, the basic ellipsoidal parameters are represented by symbols and derived as follows:

a is the ellipsoidal semi-major axis
b is the ellipsoidal semi-minor axis
f is the flattening of the ellipsoid where $1/f = a/(a - b)$
e is the eccentricity of the ellipsoid where $e^2 = 2f - f^2$
e' is the second eccentricity where $e'^2 = e^2 /(1 - e^2)$
ρ is the radius of curvature of the meridian at latitude φ,
where $\rho = a(1 - e^2)/(1 - e^2\sin^2\varphi)^{3/2}$
ν is the radius of curvature on the prime vertical (i.e. perpendicular to the meridian) at
latitude φ, where $\nu = a /(1 - e^2\sin^2\varphi)^{1/2}$
φ is the latitude of the point to be converted, positive if north and negative if south of the equator
λ is the longitude of the point to be converted, positive if east and negative if west of the prime meridian
φ_O is the latitude of natural origin
λ_O is the longitude of natural origin (with respect to the prime meridian)
φ_F is the latitude of false origin
λ_F is the longitude of false origin (with respect to the prime meridian)
φ_C is the latitude of projection centre
λ_C is the longitude of projection centre (with respect to the prime meridian)

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[1] This was originally published with the title "Map Projections Used by the US Geological Survey". In some cases the formulas given are insufficient for global use. In these cases EPSG has modified the formulas. Note that the origin of most map projections is given false coordinates (FE and FN or E_F and N_F or E_C and N_C) to avoid negative coordinates. In the EPSG formulas these values are included where appropriate so that the projected coordinates of points result directly from the quoted formulas.

φ_1 is the latitude of first standard parallel
φ_2 is the latitude of second standard parallel
k_O is the scale factor at the natural origin
E is the Easting measured from the grid origin
N is the Northing measured from the grid origin
FE is the false easting, the Eastings value assigned to the natural origin
FN is the false northing, the Northings value assigned to the natural origin
E_F is the false easting, the Eastings value assigned to the false origin
N_F is the false northing, the Northings value assigned to the false origin
E_C is the easting at projection centre, the Eastings value assigned to the projection centre
N_C is the northing at projection centre, the Northings value assigned to the projection centre

*(Note that the origin of most map projections is given false coordinates to avoid negative coordinates. In the formulas which follow these values, (**FE** and **FN** or $\boldsymbol{E_F}$ and $\boldsymbol{N_F}$ or $\boldsymbol{E_C}$ and $\boldsymbol{N_C}$) are included where appropriate so that the projected coordinates of points result directly from the quoted formulas).*

Reversibility
Different formulas are require for forward and reverse map projection conversions: the forward formula cannot be used for the reverse conversion. However both forward and reverse formulas are explicitly given in the sections below as parts of a single conversion method. As such, map projection methods are described by EPSG as being reversible. Forward and reverse formulas for each conversion method use the projection parameters appropriate to that method with parameter values unchanged.

1.4.1. Lambert Conic Conformal

For territories with limited latitudinal extent but wide longitudinal width it may sometimes be preferred to use a single projection rather than several bands or zones of a Transverse Mercator. The Lambert Conic Conformal may often be adopted in these circumstances. But if the latitudinal extent is also large there may still be a need to use two or more zones if the scale distortion at the extremities of the one zone becomes too large to be tolerable.

Conical projections with one standard parallel are normally considered to maintain the nominal map scale along the parallel of latitude which is the line of contact between the imagined cone and the ellipsoid. For a one standard parallel Lambert the natural origin of the projected coordinate system is the intersection of the standard parallel with the longitude of origin (central meridian). See Figure 5 at end of section 1.3. To maintain the conformal property the spacing of the parallels is variable and increases with increasing distance from the standard parallel, while the meridians are all straight lines radiating from a point on the prolongation of the ellipsoid's minor axis.

Sometimes however, although a one standard parallel Lambert is normally considered to have unity scale factor on the standard parallel, a scale factor of slightly less than unity is introduced on this parallel. This is a regular feature of the mapping of some former French territories and has the effect of making the scale factor unity on two other parallels either side of the standard parallel. The projection thus, strictly speaking, becomes a Lambert Conic Conformal projection with **two** standard parallels. From the one standard parallel and its scale factor it is possible to derive the equivalent two standard parallels and then treat the projection as a two standard parallel Lambert conical conformal, but this procedure is seldom adopted. Since the two parallels obtained in this way will generally not have integer values of degrees or degrees minutes and seconds it is instead usually preferred to select two specific parallels on which the scale factor is to be unity, as for several State Plane Coordinate systems in the United States.

The choice of the two standard parallels will usually be made according to the latitudinal extent of the area which it is wished to map, the parallels usually being chosen so that they each lie a proportion inboard of the north and south margins of the mapped area. Various schemes and formulas have been developed to make this selection such that the maximum scale distortion within the mapped area is minimised, e.g. Kavraisky in 1934, but whatever two standard parallels are adopted the formulas are the same.

1.4.1.1 Lambert Conic Conformal (2SP) (EPSG coordinate operation method code 9802)

To derive the projected Easting and Northing coordinates of a point with geographical coordinates (φ,λ) the formulas for the Lambert Conic Conformal **two standard parallel case** (EPSG coordinate operation method code 9802) are:

Easting, $E = E_F + r \sin\theta$

Northing, $N = N_F + r_F - r\cos\theta$

where $m = \cos\varphi/(1 - e^2\sin^2\varphi)^{0.5}$ for m_1, φ_1, and m_2, φ_2 where φ_1 and φ_2 are the latitudes of the standard parallels

$t = \tan(\pi/4 - \varphi/2)/[(1 - e\sin\varphi)/(1 + e\sin\varphi)]^{e/2}$ for t_1, t_2, t_F and t using φ_1, φ_2, φ_F and φ respectively

$n = (\ln m_1 - \ln m_2)/(\ln t_1 - \ln t_2)$

$F = m_1/(n t_1^{\,n})$

$r = a F t^n$ for r_F and r, where r_F is the radius of the parallel of latitude of the false origin

$\theta = n(\lambda - \lambda_F)$

The reverse formulas to derive the latitude and longitude of a point from its Easting and Northing values are:

$\varphi = \pi/2 - 2\operatorname{atan}\{t'[(1 - e\sin\varphi)/(1 + e\sin\varphi)]^{e/2}\}$

$\lambda = \theta'/n + \lambda_F$

where

$r' = \pm\{(E - E_F)^2 + [r_F - (N - N_F)]^2\}^{0.5}$, taking the sign of n

$t' = (r'/(aF))^{1/n}$

$\theta' = \operatorname{atan}[(E - E_F)/(r_F - (N - N_F))]$

and n, F, and r_F are derived as for the forward calculation.

Note that the formula for φ requires iteration. First calculate t' and then a trial value for φ using $\varphi = \pi/2 - 2\operatorname{atan} t'$. Then use the full equation for φ substituting the trial value into the right hand side of the equation. Thus derive a new value for φ. Iterate the process until φ does not change significantly. The solution should quickly converge, in 3 or 4 iterations.

Example:

For Projected Coordinate Reference System: NAD27 / Texas South Central

Parameters:

Ellipsoid:	Clarke 1866	a = 6378206.400 metres	=	20925832.16 US survey feet
		1/f = 294.97870		
	then	e = 0.08227185		e^2 = 0.00676866

Latitude of false origin	φ_F	27°50'00"N	=	0.48578331 rad
Longitude of false origin	λ_F	99°00'00"W	=	-1.72787596 rad
Latitude of 1st standard parallel	φ_1	28°23'00"N	=	0.49538262 rad
Latitude of 2nd standard parallel	φ_2	30°17'00"N	=	0.52854388 rad
Easting at false origin	E_F	2000000.00	US survey feet	
Northing at false origin	N_F	0.00	US survey feet	

Forward calculation for:

Latitude	φ	=	28°30'00.00"N	= 0.49741884 rad
Longitude	λ	=	96°00'00.00"W	= -1.67551608 rad

first gives :

m_1	=	0.88046050	m_2	=	0.86428642
t	=	0.59686306	t_F	=	0.60475101
t_1	=	0.59823957	t_2	=	0.57602212
n	=	0.48991263	F	=	2.31154807
r	=	37565039.86	r_F	=	37807441.20
θ	=	0.02565177			

Then Easting E = 2963503.91 US survey feet
Northing N = 254759.80 US survey feet

Reverse calculation for same easting and northing first gives:

θ' = 0.025651765
t' = 0.59686306
r' = 37565039.86

Then Latitude φ = 28°30'00.000"N
Longitude λ = 96°00'00.000"W

1.4.1.2 Lambert Conic Conformal (1SP) (EPSG coordinate operation method code 9801)

The formulas for the two standard parallel can be used for the Lambert Conic Conformal single standard parallel case (EPSG coordinate operation method code 9801) with minor modifications. Then

$E = FE + r \sin\theta$

$N = FN + r_O - r \cos\theta$, using the natural origin rather than the false origin.

where

$n = \sin \varphi_O$

$r = aFt^n k_O$ for r_O, and r

t is found for t_O, φ_O and t, φ and m, F, and θ are found as for the two standard parallel case.

The reverse formulas for φ and λ are as for the two standard parallel case above, with n, F and r_O as before and

$\theta' = \text{atan}\{(E - FE)/[r_O - (N - FN)]\}$

$r' = \pm\{(E - FE)^2 + [r_O - (N - FN)]^2\}^{1/2}$

$t' = (r'/(ak_OF))^{1/n}$

Example:

For Projected Coordinate Reference System: JAD69 / Jamaica National Grid

Parameters:

Ellipsoid: Clarke 1866 a = 6378206.400 metres 1/f = 294.97870
then e = 0.08227185 e^2 = 0.00676866

Latitude of natural origin	φ_O	18°00'00"N	= 0.31415927 rad
Longitude of natural origin	λ_O	77°00'00"W	= -1.34390352 rad
Scale factor at natural origin	k_O	1.000000	
False easting	FE	250000.00	metres

False northing	FN	150000.00	metres

Forward calculation for:

Latitude	φ	=	17°55'55.80"N	=	0.31297535 rad
Longitude	λ	=	76°56'37.26"W	=	-1.34292061 rad

first gives

m_O	=	0.95136402	t_O	=	0.72806411
F	=	3.39591092	n	=	0.30901699
r	=	19643955.26	r_O	=	19636447.86
θ	=	0.00030374	t	=	0.728965259

Then	Easting	E	=	255966.58 metres
	Northing	N	=	142493.51 metres

Reverse calculation for the same easting and northing first gives

θ'	=	0.000303736
t'	=	0.728965259
m_O	=	0.95136402
r'	=	19643955.26

Then	Latitude	φ	=	17°55'55.80"N
	Longitude	λ	=	76°56'37.26"W

1.4.1.3 Lambert Conic Conformal (West Orientated) (EPSG coordinate operation method code 9826)

In older mapping of Denmark and Greenland the Lambert Conic Conformal is used with axes positive north and **west**. To derive the projected Westing and Northing coordinates of a point with geographical coordinates (φ, λ) the formulas are as for the standard Lambert Conic Conformal (1SP) case above (EPSG coordinate operation method code 9801) except for:

$$W = FE - r.\sin(\theta)$$

In this formula the term FE retains its definition, i.e. in the Lambert Conic Conformal (West Orientated) method it increases the Westing value at the natural origin. In this method it is effectively false westing (FW).

The reverse formulas to derive the latitude and longitude of a point from its Westing and Northing values are as for the standard Lambert Conic Conformal (1SP) case except for:

$$\theta' = \text{atan}[(FE - W)/\{r_O - (N - FN)\}]$$
$$r' = +/-[(FE - W)^2 + \{r_O - (N - FN)\}^2]^{0.5}$$

1.4.1.4 Lambert Conic Conformal (2 SP Belgium) (EPSG coordinate operation method code 9803)

In 1972, in order to retain approximately the same grid coordinates after a change of geodetic datum, a modified form of the two standard parallel case was introduced in Belgium. In 2000 this modification was replaced through use of the regular Lambert Conic Conformal (2 SP) map projection with appropriately modified parameter values.

In the 1972 modification the formulas for the regular Lambert Conic Conformal (2SP) case given above are used except for:

Easting, $E = E_F + r \sin(\theta - a)$

Northing, $N = N_F + r_F - r \cos(\theta - a)$

and for the reverse formulas

$\lambda = [(\theta' + a)/n] + \lambda_F$

where a = 29.2985 seconds.

Example:

For Projected Coordinate Reference System: Belge l972 / Belge Lambert 72

Parameters:

Ellipsoid:	International 1924	a = 6378388 metres		1/f = 297.0
	then	e = 0.08199189		e^2 = 0.006722670

Latitude of false origin	φ_F	90°00'00"N	=	1.57079633 rad
Longitude of false origin	λ_F	4°21'24.983"E	=	0.07604294 rad
Latitude of 1st standard parallel	φ_1	49°50'00"N	=	0.86975574 rad
Latitude of 2nd standard parallel	φ_2	51°10'00"N	=	0.89302680 rad
Easting at false origin	E_F	150000.01	metres	
Northing at false origin	N_F	5400088.44	metres	

Forward calculation for:

Latitude	φ	=	50°40'46.4610"N	=	0.88452540 rad
Longitude	λ	=	5°48'26.533"E	=	0.10135773 rad

first gives :

m_1	=	0.64628304	m_2	=	0.62834001
t	=	0.35913403	t_F	=	0.00
t_1	=	0.36750382	t_2	=	0.35433583
n	=	0.77164219	F	=	1.81329763
r	=	5248041.03	r_F	=	0.00
θ	=	0.01953396	a	=	0.00014204

Then	Easting	E	=	251763.20 metres
	Northing	N	=	153034.13 metres

Reverse calculation for same easting and northing first gives:

θ'	=	0.01939192
t'	=	0.35913403
r'	=	5248041.03

Then	Latitude	φ	=	50°40'46.4610"N
	Longitude	λ	=	5°48'26.533"E

1.4.2. <u>Lambert Conic Near-Conformal</u> (EPSG coordinate operation method code 9817)

The Lambert Conformal Conic with one standard parallel formulas, as published by the Army Map Service, are still in use in several countries. The AMS uses series expansion formulas for ease of computation, as was normal before the electronic computer made such approximate methods unnecessary. Where the expansion series have been carried to enough terms the results are the same as the above formulas to the centimetre level. However in some countries the expansion formulas were truncated to the third order and the map projection is not fully conformal. The full formulas are used in Libya but from 1915 for France, Morocco, Algeria, Tunisia and Syria the truncated formulas were used. In 1943 in Algeria and Tunisia, from 1948 in France, from 1953 in Morocco and from 1973 in Syria the truncated formulas were replaced with the full formulas.

To compute the Lambert Conic Near-Conformal the following formulas are used:

$E = FE + r \sin\theta$
$N = FN + M + r \sin\theta \tan(\theta / 2)$ using the natural origin rather than the false origin.

Compute constants for the ellipse:

$n = (a - b)/(a+b)$
$A' = a [1- n + 5 (n^2 - n^3) / 4 + 81 (n^4 - n^5) / 64]*\pi /180$
$B' = 3 a [n - n^2 + 7 (n^3 - n^4) / 8 + 55 n^5 / 64] / 2$
$C' = 15 a [n^2 - n^3 + 3 (n^4 - n^5) / 4] / 16$
$D' = 35 a [n^3 - n^4 + 11 n^5 / 16] / 48$
$E' = 315 a [n^4 - n^5] / 512$

Then compute the meridional arc from the equator to the parallel.
$s_O = A' \varphi_O - B' \sin 2\varphi_O + C' \sin 4\varphi_O - D' \sin 6\varphi_O + E' \sin 8\varphi_O$, where φ_O in the first term is in degrees
$s = A' \varphi - B' \sin 2\varphi + C' \sin 4\varphi - D' \sin 6\varphi + E' \sin 8\varphi$, where φ in the first term is in degrees
$m = s - s_O$
$A = 1 / (6 \rho_O \nu_O)$
$M = k_O (m + Am^3 + \mathbf{Bm^4 + Cm^5 + Dm^6})$ **This is the term that is truncated to the third order.**
M_S = M per second of arc $= M / ((\varphi - \varphi_O) * 3600)$

$\theta = (\lambda - \lambda_O) \sin \varphi_O$
$r_O = k_O \nu_O / \tan \varphi_O$
$r = r_O - M$

The reverse formulas for φ and λ from E and N with r_O and M_S as above:

$\varphi = M'/ (M_S *3600) + \varphi_O$ where φ_O and φ are in degrees
$\lambda = \lambda_O + \theta' / \sin \varphi_O$ where λ_O and λ are in radians

where
$X = E - FE$ $\quad Y = N - FN$
$\theta' = \text{atan} [X / (r_O - Y)]$
$r' = X / \sin \theta'$
$M' = r' - r_O$

Example:
For Projected Coordinate Reference System: Deir ez Zor / Levant Zone

Parameters:

Ellipsoid: Clarke 1880 (IGN) a = 6378249.2 metres 1/f = 293.46602 then $b = 6356515.000$ $n = 0.001706682563$

Latitude of natural origin	φ_O	34°39'00"N	=	0.604756586 rad
Longitude of natural origin	λ_O	37°21'00"E	=	0.651880476 rad
Scale factor at natural origin	k_O	0.99962560		
False easting	FE	300000.00	metres	
False northing	FN	300000.00	metres	

Forward calculation for:

Latitude	φ	=	37°31'17.625"N	=	0.654874806 rad
Longitude	λ	=	34°08'11.291"E	=	0.595793792 rad

first gives :

A	=	$4.1067494 * 10^{-15}$	A'	=	111131.8633
B'	=	16300.64407	C'	=	17.38751
D'	=	0.02308	E'	=	0.000033
s_O	=	3835482.233	s	=	4154101.458
m	=	318619.225			
M	=	318632.72	M_S	=	30.82262319
θ	=	-0.03188875			
r_O	=	9235264.405	r	=	8916631.685

Then Easting E = 15707.96 metres (c.f. E = 15708.00 using full formulas)
Northing N = 623165.96 metres (c.f. N = 623167.20 using full formulas)

Reverse calculation for same easting and northing first gives:

θ' = -0.03188875
r' = 8916631.685
M' = 318632.72

Then Latitude φ = 37°31'17.625"N
Longitude λ = 34°08'11.291"E

1.4.3. <u>Krovak Oblique Conformal Conic</u> (EPSG coordinate operation method code 9819)

The normal case of the Lambert Conformal conic is for the axis of the cone to be coincident with the minor axis of the ellipsoid, that is the axis of the cone is normal to the ellipsoid at a geographic pole. For the Oblique Conformal Conic the axis of the cone is normal to the ellipsoid at a defined location and its extension cuts the minor axis at a defined angle. This map projection is used in the Czech Republic and Slovakia under the name 'Krovak' projection. The map projection method is similar in principle to the Oblique Mercator (see section 1.4.5). The geographic coordinates on the ellipsoid are first reduced to conformal coordinates on the conformal (Gaussian) sphere. These spherical coordinates are then projected onto the oblique cone and converted to grid coordinates. The pseudo standard parallel is defined on the conformal sphere after its rotation, to obtain the oblique aspect of the projection. It is then the parallel on this sphere at which the map projection is true to scale; on the ellipsoid it maps as a complex curve. A scale factor may be applied to the map projection to increase the useful area of coverage.

The defining parameters for the Krovak oblique conformal conic map projection are:

φ_C = latitude of projection centre
λ_C = longitude of projection centre

α_C		= (true) azimuth of initial line passing through the projection centre
		= co-latitude of the cone axis at point of intersection with the ellipsoid
φ_1		= latitude of pseudo standard parallel
k_C		= scale factor on pseudo standard parallel
E_C		= Easting at projection centre
N_C		= Northing at projection centre

From these the following constants for the projection may be calculated :

$A = a (1 - e^2)^{0.5} / [1 - e^2 \sin^2(\varphi_C)]$

$B = \{1 + [e^2 \cos^4\varphi_C / (1 - e^2)]\}^{0.5}$

$\gamma_O = \mathrm{asin}\,[\sin(\varphi_C) / B]$

$t_O = \tan(\pi / 4 + \gamma_O / 2) \,.\, [(1 + e \sin(\varphi_C)) / (1 - e \sin(\varphi_C))]^{e.B/2} / \tan(\pi / 4 + \varphi_C / 2)^{B}$

$n = \sin(\varphi_1)$

$r_O = k_C A / \tan(\varphi_1)$

To derive the projected Easting and Northing coordinates of a point with geographical coordinates (φ,λ) the formulas for the oblique conic conformal are:

Easting: $E = E_C + r \cos\theta$
Northing: $N = N_C + r \sin\theta$

where

$U = 2\,(\mathrm{atan}\,\{ t_O \tan^B(\varphi/ 2 + \pi / 4) / [(1 + e \sin(\varphi)) / (1 - e \sin(\varphi))]^{e.B/2}\} - \pi / 4)$

$V = B\,(\lambda_C - \lambda)$

$S = \mathrm{asin}\,[\cos(\alpha_C) \sin(U) + \sin(\alpha_C) \cos(U) \cos(V)]$

$D = \mathrm{asin}\,[\cos(U) \sin(V) / \cos(S)]$

$\theta = n\,D$

$r = r_O \tan^n(\pi / 4 + \varphi_1/ 2) / \tan^n(S/2 + \pi / 4)$

Note that the terms Easting and Northing here refer to the two map grid coordinates. Their actual geographic direction depends upon the azimuth of the centre line.

The reverse formulas to derive the latitude and longitude of a point from its Easting and Northing values are:

$\varphi_j = 2\,(\mathrm{atan}\,\{ t_O^{-1/B} \tan^{1/B}(U'/2 + \pi / 4)\,[(1 + e \sin(\varphi_{j-1}))/(1 - e \sin(\varphi_{j-1}))]^{e/2}\} - \pi / 4)$

where j= 1,2 and the latitude is found by iteration.
$\lambda = \lambda_C - V' / B$
where

$r' = [(E - E_C)^2 + (N - N_C)^2]^{0.5}$

$\theta' = \mathrm{atan}\,[(E - E_C)/(N - N_C)]$

$D' = \theta' / \sin(\varphi_1)$

$S' = 2 * \{\mathrm{atan}[(r_O / r')^{1/n} \tan(\pi / 4 + \varphi_1/ 2)] - \pi / 4\}$

$U' = \mathrm{asin}\,(\cos(\alpha_C) \sin(S') - \sin(\alpha_C) \cos(S') \cos(D'))$

$V' = \mathrm{asin}\,(\cos(S') \sin(D') / \cos(U'))$

Example:

For Projected Coordinate Reference System: S-JTSK (Ferro) / Krovak

N.B. Krovak projection uses Ferro as the prime meridian. This has a longitude with reference to Greenwich of 17 degrees 40 minutes West. To apply the formulas the defining longitudes must be corrected to the Greenwich meridian.

Parameters:

Ellipsoid:	Bessel 1841	a = 6377397.155metres	1/f = 299.15281
	then	e = 0.081696831	e^2 = 0.006674372

Latitude of projection centre	φ_C	49°30'00"N	=	0.863937979 rad
Longitude of projection centre	λ_C	42°30'00" East of Ferro		
Longitude of Ferro is		17°40'00" West of Greenwich		
λ_C relative to Greenwich:		24°50'00"	=	0.433423431 rad
Azimuth of initial line	α_C	30°17'17.3031"		
Latitude of pseudo standard parallel	φ_1	78°30'00"N		
Scale factor on pseudo standard parallel	k_O	0.9999		
Easting at projection centre	E_C	0.00		metres
Northing at projection centre	N_C	0.00		metres

Projection constants:

A	=	6380703.611	B	=	1.000597498
γ_O	=	0.863239103	t_O	=	1.003419164
n	=	0.979924705	r_O	=	1298039.005

Forward calculation for:

Latitude	φ	=	50°12'32.4416"N	=	0.876312566 rad
Longitude	λ	=	16°50'59.1790"E	=	0.294083999 rad

first gives :

U = 0.875596949
V = 0.139422687
S = 1.386275049
D = 0.506554623
θ = 0.496385389
r = 1194731.014

Then "Easting" E = 1050538.643 metres
"Northing" N = 568990.997 metres

where "Easting" increases southwards and "Northing" increases westwards.

Reverse calculation for the same "Easting" and "Northing" gives

r' = 1194731.014
θ' = 0.496385389
D' = 0.506554623
S' = 1.386275049
U' = 0.875596949
V' = 0.139422687

Then by iteration

$\varphi 1$ = 0.876310601 rad
$\varphi 2$ = 0.876312560 rad
$\varphi 3$ = 0.876312566 rad

Latitude	φ	=	0.876312566 rad	=	50°12'32.4416"N
Longitude	λ	=	0.294083999 rad	=	16°50'59.1790"E

1.4.4. **Mercator** (EPSG coordinate operation method codes 9804 and 9805)

The Mercator map projection is a special limiting case of the Lambert Conic Conformal map projection with the equator as the single standard parallel. All other parallels of latitude are straight lines and the meridians are also straight lines at right angles to the equator, equally spaced. It is the basis for the transverse and oblique forms of the projection. It is little used for land mapping purposes but is in almost universal use for navigation charts. As well as being conformal, it has the particular property that straight lines drawn on it are lines of constant bearing. Thus navigators may derive their course from the angle the straight course line makes with the meridians.

In the few cases in which the Mercator projection is used for terrestrial applications or land mapping, such as in Indonesia prior to the introduction of the Universal Transverse Mercator, a scale factor may be applied to the projection. This has the same effect as choosing two standard parallels on which the true scale is maintained at equal north and south latitudes either side of the equator.

The formulas to derive projected Easting and Northing coordinates are:

For the two standard parallel case, k_O, the scale factor at the equator or natural origin, is first calculated from

$$k_O = \cos\varphi_1 / (1 - e^2 \sin^2\varphi_1)^{0.5}$$

where φ_1 is the absolute value of the first standard parallel (i.e. positive).

Then, for both one and two standard parallel cases,

$$E = FE + ak_O (\lambda - \lambda_O)$$
$$N = FN + ak_O \ln\{\tan(\pi/4 + \varphi/2)[(1 - e\sin\varphi)/(1 + e\sin\varphi)]^{(e/2)}\}$$

where symbols are as listed above and logarithms are natural.

The reverse formulas to derive latitude and longitude from E and N values are:

$$\varphi = \chi + (e^2/2 + 5e^4/24 + e^6/12 + 13e^8/360) \sin(2\chi)$$
$$+ (7e^4/48 + 29e^6/240 + 811e^8/11520) \sin(4\chi)$$
$$+ (7e^6/120 + 81e^8/1120) \sin(6\chi) + (4279e^8/161280) \sin(8\chi)$$

where $\chi = \pi/2 - 2 \operatorname{atan} t$

$t = B^{(FN-N)(a.kO)}$

B = base of the natural logarithm, 2.7182818…

and for the 2 SP case, k_O is calculated as for the forward transformation above.

$$\lambda = [(E - FE)/ak_O] + \lambda_O$$

Examples

1. Mercator (with two standard parallels) (EPSG coordinate operation method code 9805)

For Projected Coordinate Reference System: Pulkovo 1942 / Mercator Caspian Sea

Parameters:

Ellipsoid:	Krassowski 1940		a = 6378245.0 metres		1/f = 298.3
		then	e = 0.08181333		e^2 = 0.00669342

Latitude of 1st standard parallel	φ_O	42°00'00"N	=	0.73303829 rad
Longitude of natural origin	λ_O	51°00'00"E	=	0.89011792 rad
False easting	FE	300000.00	metres	
False northing	FN	300000.00	metres	

then scale factor at natural origin k_O (at latitude of natural origin at 0°N) = 0.74426089.

Forward calculation for:

Latitude	φ	=	53°00'00.00"N	=	0.9250245 rad
Longitude	λ	=	53°00'00.00"N	=	0.9250245 rad

gives	Easting	E	=	165704.29 metres
	Northing	N	=	5171848.07 metres

Reverse calculation for same easting and northing first gives:

t = 0.33639129

χ = 0.92179596

Then	Latitude	φ	=	53°00'00.000"N
	Longitude	λ	=	53°00'00.000"E

2. Mercator (1SP) (EPSG coordinate operation method code 9804)

For Projected Coordinate Reference System: Makassar / NEIEZ

Parameters:

Ellipsoid:	Bessel 1841		a = 6377397.155metres	1/f = 299.15281
		then	e = 0.081696831	

Latitude of natural origin	φ_O	0°00'00"N	=	0.0 rad
Longitude of natural origin	λ_O	110°00'00"E	=	1.91986218 rad
Scale factor at natural origin	k_O	0.997		
False easting	FE	3900000.00	metres	
False northing	FN	900000.00	metres	

Forward calculation for:

Latitude	φ	=	3°00'00.00"S	=	-0.05235988 rad
Longitude	λ	=	120°00'00.00"E	=	2.09439510 rad

gives	Easting	E	=	5009726.58 metres
	Northing	N	=	569150.82 metres

Reverse calculation for same easting and northing first gives:

t = 1.0534121

χ = -0.0520110

Then	Latitude	φ	=	3°00'00.000"S
	Longitude	λ	=	120°00'00.000"E

1.4.5. **Cassini-Soldner** (EPSG coordinate operation method code 9806)

The Cassini-Soldner projection is the ellipsoidal version of the Cassini projection for the sphere. It is not conformal but as it is relatively simple to construct it was extensively used in the last century and is still useful for mapping areas with limited longitudinal extent. It has now largely been replaced by the conformal Transverse Mercator which it resembles. Like this, it has a straight central meridian along which the scale is true, all other meridians and parallels are curved, and the scale distortion increases rapidly with increasing distance from the central meridian.

The formulas to derive projected Easting and Northing coordinates are:

$$\text{Easting, } E = FE + \nu[A - TA^3/6 - (8 - T + 8C)TA^5/120]$$

$$\text{Northing, } N = FN + M - M_O + \nu\tan\varphi[A^2/2 + (5 - T + 6C)A^4/24]$$

where $A = (\lambda - \lambda_O)\cos\varphi$
$T = \tan^2\varphi$
$C = e^2 \cos^2\varphi/(1 - e^2)$
$\nu = a /(1 - e^2\sin^2\varphi)^{0.5}$
and M, the distance along the meridian from equator to latitude φ, is given by

$$M = a[1 - e^2/4 - 3e^4/64 - 5e^6/256 - ....)\varphi - (3e^2/8 + 3e^4/32 + 45e^6/1024 + ...)\sin2\varphi + (15e^4/256 + 45e^6/1024 + ...)\sin4\varphi - (35e^6/3072 + ...)\sin6\varphi + ...]$$

with φ in radians.

M_O is the value of M calculated for the latitude of the chosen origin. This may not necessarily be chosen as the equator.

To compute latitude and longitude from Easting and Northing the reverse formulas are:

$$\varphi = \varphi_1 - (\nu_1\tan\varphi_1/\rho_1)[D^2/2 - (1 + 3T_1)D^4/24]$$

$$\lambda = \lambda_O + [D - T_1D^3/3 + (1 + 3T_1)T_1D^5/15]/\cos\varphi_1$$

where
$\nu_1 = a /(1 - e^2\sin^2\varphi_1)^{0.5}$
$\rho_1 = a(1 - e^2)/(1 - e^2\sin^2\varphi_1)^{1.5}$
φ_1 is the latitude of the point on the central meridian which has the same Northing as the point whose coordinates are sought, and is found from:

$$\varphi_1 = \mu_1 + (3e_1/2 - 27e_1^3/32 + ...)\sin2\mu_1 + (21e_1^2/16 - 55e_1^4/32 + ...)\sin4\mu_1 + (151e_1^3/96 + ...)\sin6\mu_1 + (1097e_1^4/512 - ....)\sin8\mu_1 + ...$$

where
$e_1 = [1 - (1 - e^2)^{0.5}]/[1 + (1 - e^2)^{0.5}]$
$\mu_1 = M_1/[a(1 - e^2/4 - 3e^4/64 - 5e^6/256 - ....)]$
$M_1 = M_O + (N - FN)$
$T_1 = \tan^2\varphi_1$
$D = (E - FE)/\nu_1$

Example
For Projected Coordinate Reference System: Trinidad 1903 / Trinidad Grid

Parameters:

Ellipsoid:	Clarke 1858	a = 20926348 ft	=	31706587.88 Clarke's links
		b = 20855233 ft		

then $1/f = 294.2606764$ $e^2 = 0.006785146$

Latitude of natural origin	φ_O	10°26'30"N	=	0.182241463 rad
Longitude of natural origin	λ_O	61°20'00"W	=	1.91986218 rad
False easting	FE	430000.00		Clarke's links
False northing	FN	325000.00		Clarke's links

Forward calculation for:

Latitude	φ	=	10°00'00.00"N	=	0.17453293 rad
Longitude	λ	=	62°00'00.00"W	=	-1.08210414 rad

first gives :

A	=	-0.01145876	C	=	0.00662550
T	=	0.03109120	M	=	5496860.24
ν	=	31709831.92	M_O	=	5739691.12

Then Easting E = 66644.94 Clarke's links
Northing N = 82536.22 Clarke's links

Reverse calculation for same easting and northing first gives:

e_1	=	0.00170207	D	=	-0.01145875
T_1	=	0.03109544	M_1	=	5497227.34
ν_1	=	31709832.34	μ_1	=	0.17367306
φ_1	=	0.17454458	ρ_1	=	31501122.40

Then Latitude φ = 10°00'00.000"N
Longitude λ = 62°00'00.000"W

1.4.6. Transverse Mercator

1.4.6.1 General Case (EPSG coordinate operation method code 9807)

The Transverse Mercator projection in its various forms is the most widely used projected coordinate system for world topographical and offshore mapping. All versions have the same basic characteristics and formulas. The differences which distinguish the different forms of the projection which are applied in different countries arise from variations in the choice of values for the the coordinate conversion parameters, namely the latitude of the natural origin, the longitude of the natural origin (central meridian), the scale factor at the natural origin (on the central meridian), and the values of False Easting and False Northing, which embody the units of measurement, given to the origin. Additionally there are variations in the width of the longitudinal zones for the projections used in different territories.

The following table indicates the variations in the coordinate conversion parameters which distinguish the different forms of the Transverse Mercator projection and are used in the EPSG Transverse Mercator map projection operations:

TABLE 2

Transverse Mercator

Coordinate Operation Method Name	**Areas used**	**Central meridian**	**Latitude of natural origin**	**CM Scale Factor**	**Zone width**	**False Easting**	**False Northing**

Transverse Mercator	Various, world wide	Various	Various	Various	Usually less than 6°	Various	Various
Transverse Mercator south oriented	Southern Africa	2° intervals E of 11°E	0°	1.000000	2°	0m	0m
UTM North hemisphere	World wide equator to 84 °N	6° intervals E & W of 3° E & W	Always 0°	Always 0.9996	Always 6°	500000 m	0m
UTM South hemisphere	World wide north of 80°S to equator	6° intervals E & W of 3° E & W	Always 0°	Always 0.9996	Always 6°	500000 m	10000000 m
Gauss-Kruger	Former USSR, Yugoslavia, Germany, S. America, China	Various, according to area of cover	Usually 0 °	Usually 1.000000	Usually less than 6° , often less than 4°	Various but often 500000 prefixed by zone number	Various
Gauss Boaga	Italy	Various	Various	0.9996	6°	Various	0m

The most familiar and commonly used Transverse Mercator in the oil industry is the Universal Transverse Mercator (UTM) whose natural origin lies on the equator. However, some territories use a Transverse Mercator with a natural origin at a latitude of natural origin closer to that territory.

In EPSG the coordinate conversion method is considered to be the same for all forms of the Transverse Mercator projection. The formulas to derive the projected Easting and Northing coordinates for the normal case (EPSG coordinate operation method code 9807) are in the form of a series as follows:

$$\text{Easting, } E = FE + k_O\nu[A + (1 - T + C)A^3/6 + (5 - 18T + T^2 + 72C - 58e'^2)A^5/120]$$

$$\text{Northing, } N = FN + k_O\{M - M_O + \nu\tan\varphi[A^2/2 + (5 - T + 9C + 4C^2)A^4/24 + (61 - 58T + T^2 + 600C - 330e'^2)A^6/720]\}$$

where $T = \tan^2\varphi$

$C = e^2 \cos^2\varphi/(1 - e^2)$

$A = (\lambda - \lambda_O)\cos\varphi$, with λ and λ_O in radians

$\nu = a /(1 - e^2\sin^2\varphi)^{0.5}$

$$M = a[(1 - e^2/4 - 3e^4/64 - 5e^6/256 - ....)\varphi - (3e^2/8 + 3e^4/32 + 45e^6/1024+....)\sin2\varphi + (15e^4/256 + 45e^6/1024 +.....)\sin4\varphi - (35e^6/3072 + ...)\sin6\varphi + ...]$$

with φ in radians and M_O for φ_O, the latitude of the origin, derived in the same way.

The reverse formulas to convert Easting and Northing projected coordinates to latitude and longitude are:

$$\varphi = \varphi_1 - (\nu_1\tan\varphi_1/\rho_1)[D^2/2 - (5 + 3T_1 + 10C_1 - 4C_1^2 - 9e'^2)D^4/24 + (61 + 90T_1 + 298C_1 + 45T_1^2 - 252e'^2 - 3C_1^2)D^6/720]$$

$$\lambda = \lambda_O + [D - (1 + 2T_1 + C_1)D^3/6 + (5 - 2C_1 + 28T_1 - 3C_1^2 + 8e'^2 + 24T_1^2)D^5/120] / \cos\varphi_1$$

where

$\nu_1 = a /(1 - e^2\sin^2\varphi_1)^{0.5}$

$\rho_1 = a(1 - e^2)/(1 - e^2\sin^2\varphi_1)^{1.5}$

φ_1 may be found as for the Cassini projection from:

$$\varphi_1 = \mu_1 + (3e_1/2 - 27e_1^3/32 + ...)\sin 2\mu_1 + (21e_1^2/16 - 55e_1^4/32 + ...)\sin 4\mu_1$$
$$+ (151e_1^3/96 + ...)\sin 6\mu_1 + (1097e_1^4/512 - ....)\sin 8\mu_1 + ...$$

and where

$e_1 = [1 - (1 - e^2)^{0.5}]/[1 + (1 - e^2)^{0.5}]$
$\mu_1 = M_1/[a(1 - e^2/4 - 3e^4/64 - 5e^6/256 - ....)]$
$M_1 = M_O + (N - FN)/k_0$
$T_1 = \tan^2\varphi_1$
$C_1 = e'^2\cos^2\varphi_1$
$e'^2 = e^2 /(1 - e^2)$
$D = (E - FE)/(\nu_1 k_O)$

For areas south of the equator the value of latitude φ will be negative and the formulas above, to compute the E and N, will automatically result in the correct values. Note that the false northings of the origin, if the equator, will need to be large to avoid negative northings and for the UTM projection is in fact 10,000,000m. Alternatively, as in the case of Argentina's Transverse Mercator (Gauss-Kruger) zones, the origin is at the south pole with a northings of zero. However each zone central meridian takes a false easting of 500000m prefixed by an identifying zone number. This ensures that instead of points in different zones having the same eastings, every point in the country, irrespective of its projection zone, will have a unique set of projected system coordinates. Strict application of the above formulas, with south latitudes negative, will result in the derivation of the correct Eastings and Northings.

Similarly, in applying the reverse formulas to determine a latitude south of the equator, a negative sign for φ results from a negative φ_1 which in turn results from a negative M_1.

Example
For Projected Coordinate Reference System OSGB 1936 / British National Grid

Parameters:

Ellipsoid: Airy 1830		a = 6377563.396 metres	1/f = 299.32496
	then	e^2 = 0.00667054	e'^2 = 0.00671534

Latitude of natural origin	φ_O	49°00'00"N	=	0.85521133 rad
Longitude of natural origin	λ_O	2°00'00"W	=	-0.03490659 rad
Scale factor at natural origin	k_O	0.9996012717		
False easting	FE	400000.00	metres	
False northing	FN	-100000.00	metres	

Forward calculation for:

Latitude	φ	=	50°30'00.00"N	=	0.88139127 rad
Longitude	λ	=	00°30'00.00"E	=	0.00872665 rad

first gives :

A	=	0.02775415	C	=	0.00271699
T	=	1.47160434	M	=	5596050.46
ν	=	6390266.03	M_O	=	5429228.60

Then	Easting	E	=	577274.99 metres
	Northing	N	=	69740.50 metres

Reverse calculation for same easting and northing first gives:

e_1	=	0.00167322	μ_1	=	0.87939562
M_1	=	5599036.80	ν_1	=	6390275.88

φ_1	=	0.88185987	D	=	0.02775243
ρ_1	=	6372980.21	C_1	=	0.00271391
T_1	=	1.47441726			

Then Latitude φ = 50°30'00.000"N
Longitude λ = 00°30'00.000"E

1.4.6.2 Transverse Mercator Zoned Grid System (EPSG coordinate operation method code 9824)

When the growth in distortion away from the projection origin is of concern, a projected coordinate reference system cannot be used far from its origin. A means of creating a grid system over a large area but also limiting distortion is to have several grid zones with most defining parameters being made common. Coordinates throughout the system are repeated in each zone. To make coordinates unambiguous the easting is prefixed by the relevant zone number. This procedure was adopted by German mapping in the 1930's through the Gauss-Kruger systems and later by American military mapping through the Universal Transverse Mercator (or UTM) grid system. (Note: subsequent civilian adoption of the systems usually ignores the zone prefix to easting. Where this is the case the formulas below do not apply: use the standard TM formula separately for each zone).

The parameter Longitude of natural origin (λ_O) is changed from being a defining parameter to a derived parameter, replaced by two other defining parameters, the Initial Longitude (the western limit of zone 1) (λ_I) and the Zone Width (W). Each of the remaining four Transverse Mercator defining parameters – Latitude of natural origin, Scale factor at natural origin, False easting and False northing – have the same parameter values in every zone.

The standard Transverse Mercator formulas above are modified as follows:

Zone number, Z, = $\mathrm{INT}((\lambda + \lambda_I + W) / W)$ with λ, λ_I and W in degrees.
where λ_I is the Initial Longitude of the zoned grid system
and W is the width of each zone of the zoned grid system.
If $\lambda < 0$, $\lambda = (\lambda + 360)$ degrees.

Then,
$\lambda_O = [Z * W] - [\lambda_I + (W/2)]$

For the forward calculation,
Easting, $E = Z*10^6 + FE + k_O.\nu\,[A + (1 - T + C)A^3/6 + (5 - 18T + T^2 + 72C - 58e'^2)A^5/120]$

and in the reverse calculation for longitude,
$D = (E - [FE + Z*10^6])/(\nu_1.k_O)$

1.4.6.3 Transverse Mercator (South Orientated) (EPSG coordinate operation method code 9808)
For the mapping of southern Africa a south oriented Transverse Mercator map projection method is used. Here the coordinate axes are called Westings and Southings and increment to the West and South from the origin respectively. See Figure 3 for a diagrammatic illustration. The standard Transverse Mercator above formulas need to be modified to cope with this arrangement with

$$\text{Westing, } W = FE - k_O\,\nu[A + (1 - T + C)A^3/6 + (5 - 18T + T^2 + 72C - 58e'^2)A^5/120]$$

$$\text{Southing, } S = FN - k_O\{M - M_O + \nu\tan\varphi[A^2/2 + (5 - T + 9C + 4C^2)A^4/24 + (61 - 58T + T^2 + 600C - 330e'^2)A^6/720]\}$$

In these formulas the terms FE and FN retain their definition, i.e. in the Transverse Mercator (South Orientated) method they increase the Westing and Southing value at the natural origin. In this method they are effectively false westing (FW) and false southing (FS) respectively.

For the reverse formulas, those for the standard Transverse Mercator above apply, with the exception that:

$$M_1 = M_O - (S - FN)/k_O$$

and $D = -(W - FE)/(\nu_1 k_O)$, with $\nu_1 = \nu$ for φ_1

1.4.7. <u>Oblique Mercator and Hotine Oblique Mercator</u>

(EPSG coordinate operation method codes 9815 and 9812).

It has been noted that the Transverse Mercator map projection method is employed for the topographical mapping of longitudinal bands of territories, limiting the amount of scale distortion by limiting the extent of the projection either side of the central meridian. Sometimes the shape, general trend and extent of some countries makes it preferable to apply a single zone of the same kind of projection but with its central line aligned with the trend of the territory concerned rather than with a meridian. So, instead of a meridian forming this true scale central line for one of the various forms of Transverse Mercator, or the equator forming the line for the Mercator, a line with a particular azimuth traversing the territory is chosen and the same principles of construction are applied to derive what is now an Oblique Mercator. Such a single zone projection suits areas which have a large extent in one direction but limited extent in the perpendicular direction and whose trend is oblique to the bisecting meridian - such as East and West Malaysia, Madagascar and the Alaskan panhandle. It was originally applied at the beginning of the 20th century by Rosenmund to the mapping of Switzerland, and in the 1970's adopted in Hungary. The projection's initial line may be selected as a line with a particular azimuth through a single point, normally at the centre of the mapped area, or as the geodesic line (the shortest line between two points on the ellipsoid) between two selected points.

EPSG identifies two forms of the oblique Mercator projection, differentiated only by the point at which false grid coordinates are defined. If the false grid coordinates are defined at the intersection of the initial line and the aposphere, that is at the natural origin of the coordinate system, the map projection method is known as the Hotine Oblique Mercator (EPSG coordinate operation method code 9812). If the false grid coordinates are defined at the projection centre the projection method is known as the Oblique Mercator (EPSG coordinate operation method code 9815).

Hotine projected the ellipsoid conformally onto a sphere of constant total curvature, called the 'aposphere', before projection onto the plane. This projection is sometimes referred to as the Rectified Skew Orthomorphic. Formulas, involving hyperbolic functions, were derived by Hotine. Snyder adapted these formulas to use exponential functions, thus avoiding use of Hotine's hyperbolic expressions. Alternative formulas derived by projecting the ellipsoid onto the 'conformal' sphere give identical results within the practical limits of the use of the formulas.

Snyder describes a variation of the Hotine Oblique Mercator where the initial line is defined by two points through which it passes. The latter approach is not currently followed by EPSG/POSC; it has been applied to mapping space imagery or, more frequently, for applying a geographical graticule to the imagery. However, the repeated path of the imaging satellite does not actually follow the centre lines of successive oblique cylindrical projections so a projection was derived whose centre line does follow the satellite path. This is known as the Space Oblique Mercator Projection and although it closely resembles an oblique cylindrical it is not quite conformal and has no application other than for space imagery.

The oblique Mercator co-ordinate system is defined by:

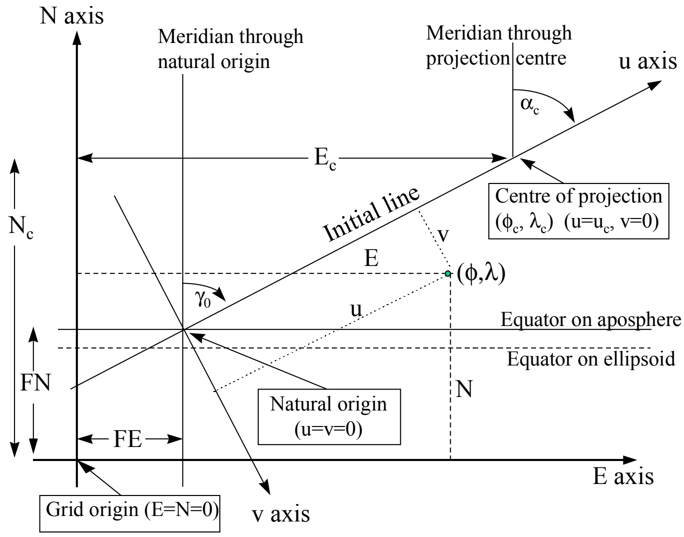

Figure 7. Key Diagram for Oblique Mercator Projection

The initial line central to the map area of given azimuth α_C passes through a defined centre of the projection (φ_C, λ_C) . The point where the projection of this line cuts the equator on the aposphere is the origin of the (u , v) co-ordinate system. The u axis is parallel to the centre line and the v axis is perpendicular to (90° clockwise from) this line.

In applying the formulas for the (Hotine) Oblique Mercator the first set of co-ordinates computed are referred to the (u, v) co-ordinate axes defined with respect to the azimuth of the centre line. These co-ordinates are then 'rectified' to the usual Easting and Northing by applying an orthogonal conversion. Hence the alternative name as the Rectified Skew Orthomorphic. In the special case of the projection covering the Alaskan panhandle the azimuth of the line at the natural origin is taken to be identical to the azimuth of the initial line at the projection centre. This results in grid and true north coinciding at the projection centre rather than at the natural origin as is more usual.

To ensure that all co-ordinates in the map area have positive grid values, false co-ordinates are applied. These may be given values (E_C , N_C) if applied at the projection centre [EPSG Oblique Mercator method] or be applied as false easting (FE) and false northing (FN) at the natural origin [EPSG Hotine Oblique Mercator method].

The formulas can be used for the following cases:

Alaska State Plane Zone 1
Hungary EOV
Madagascar Laborde Grid
East and West Malaysia Rectified Skew Orthomorphic grids
Swiss Cylindrical projection

The Swiss and Hungarian systems are a special case where the azimuth of the line through the projection centre is 90 degrees. These therefore give similar but not exactly the same results as a conventional Transverse Mercator projection.

Specific references for the formulas originally used in the individual cases of these projections are:

Switzerland: "Die Änderung des Projektionssystems der schweizerischen Landesvermessung." M. Rosenmund 1903. Also "Die projecktionen der Schweizerischen Plan und Kartenwerke." J. Bollinger 1967.
Madagascar: "La nouvelle projection du Service Geographique de Madagascar". J. Laborde 1928.
Malaysia: Series of Articles in numbers 62-66 of the Empire Survey Review of 1946 and 1947 by M. Hotine.

The defining parameters for the [Hotine] Oblique Mercator projection are:

φ_C = latitude of centre of the projection
λ_C = longitude of centre of the projection
α_C = azimuth (true) of the centre line passing through the centre of the projection
γ_C = rectified bearing of the centre line
k_C = scale factor at the centre of the projection

and either
for the Oblique Mercator:

E_C = False Easting at the centre of projection
N_C = False Northing at the centre of projection

or for the Hotine Oblique Mercator:

FE = False Easting at the natural origin
FN = False Northing at the natural origin

From these defining parameters the following constants for the map projection may be calculated :

$B = \{1 + [e^2 \cos^4\varphi_C / (1 - e^2)]\}^{0.5}$
$A = a\, B\, k_C (1 - e^2)^{0.5} / (1 - e^2 \sin^2 \varphi_C)$
$t_O = \tan(\pi / 4 - \varphi_C / 2) / [(1 - e \sin \varphi_C) / (1 + e \sin \varphi_C)]^{e/2}$
$D = B (1 - e^2)^{0.5} / [\cos \varphi_C (1 - e^2 \sin^2 \varphi_C)^{0.5}]$
To avoid problems with computation of F, if $D < 1$ make $D^2 = 1$
$F = D + (D^2 - 1)^{0.5} . \mathrm{SIGN}(\varphi_C)$
$H = F\, t_O^{B}$
$G = (F - 1/F) / 2$
$\gamma_O = \mathrm{asin}[\sin(\alpha_C) / D]$
$\lambda_O = \lambda_C - [\mathrm{asin}(G \tan\gamma_O)] / B$

Then compute the (u_C , v_C) co-ordinates for the centre point (φ_C , λ_C).
$v_C = 0$

In general
$u_C = (A / B) \mathrm{atan}[(D^2 - 1)^{0.5} / \cos(\alpha_C)]. \mathrm{SIGN}(\varphi_C)$

but for the special cases where α_c = 90 degrees (e.g. Hungary, Switzerland) then
$u_C = A (\lambda_C - \lambda_O)$

Forward case: To compute (E,N) from a given (φ,λ), for both the Hotine Oblique Mercator method and the Oblique Mercator method:

$t = \tan(\pi / 4 - \varphi / 2) / [(1 - e \sin \varphi) / (1 + e \sin \varphi)]^{e/2}$
$Q = H / t^B$
$S = (Q - 1/Q) / 2$
$T = (Q + 1/Q) / 2$

$V = \sin(B(\lambda - \lambda_O))$
$U = (-V\cos(\gamma_O) + S\sin(\gamma_O)) / T$
$v = A \ln[(1 - U) / (1 + U)] / 2B$

Then
either (a) for the Hotine Oblique Mercator (where the FE and FN values have been specified with respect to the natural origin of the (u , v) axes):
$u = A \operatorname{atan}\{(S\cos\gamma_O + V\sin\gamma_O) / \cos[B(\lambda - \lambda_O)]\} / B$

The rectified skew co-ordinates are then derived from:
$E = v\cos(\gamma_C) + u\sin(\gamma_C) + FE$
$N = u\cos(\gamma_C) - v\sin(\gamma_C) + FN$

or (b) for the Oblique Mercator (where the false easting and northing values (E_C , N_C) have been specified with respect to the centre of the projection (φ_C , λ_C) then :
$u = (A \operatorname{atan}\{(S\cos\gamma_O + V\sin\gamma_O) / \cos[B(\lambda - \lambda_O)]\} / B) - (u_C \,.\, \mathrm{SIGN}(\lambda - \lambda_C))$

The rectified skew co-ordinates are then derived from:
$E = v\cos(\gamma_C) + u\sin(\gamma_C) + E_C$
$N = u\cos(\gamma_C) - v\sin(\gamma_C) + N_C$

Reverse case: To compute (φ,λ) from a given (E,N) :

For the Hotine Oblique Mercator:
$v' = (E - FE)\cos(\gamma_C) - (N - FN)\sin(\gamma_C)$
$u' = (N - FN)\cos(\gamma_C) + (E - FE)\sin(\gamma_C)$

or for the Oblique Mercator:
$v' = (E - E_C)\cos(\gamma_C) - (N - N_C)\sin(\gamma_C)$
$u' = (N - N_C)\cos(\gamma_C) + (E - E_C)\sin(\gamma_C) + u_C$

then for both cases:
$Q' = \mathbf{e}^{-(B v' / A)}$ where $\mathbf{e}$ is the base of natural logarithms.
$S' = (Q' - 1 / Q') / 2$
$T' = (Q' + 1 / Q') / 2$
$V' = \sin(B u' / A)$
$U' = (V'\cos(\gamma_O) + S'\sin(\gamma_O)) / T'$
$t' = \{H / [(1 + U') / (1 - U')]^{0.5}\}^{1/B}$

$\chi = \pi / 2 - 2 \operatorname{atan}(t')$

$$\varphi = \chi + \sin(2\chi).(e^2 / 2 + 5e^4 / 24 + e^6 / 12 + 13e^8 / 360) + \sin(4\chi).(7e^4 / 48 + 29e^6 / 240 + 811e^8 / 11520) + \sin(6\chi).(7e^6 / 120 + 81e^8 / 1120) + \sin(8\chi).(4279e^8 / 161280)$$

$\lambda = \lambda_O - \operatorname{atan}[(S'\cos\gamma_C - V'\sin\gamma_C) / \cos(B u' / A)] / B$

Examples:
For Projected Coordinate Reference System Timbalai 1948 / R.S.O. Borneo (m) using the Oblique Mercator method. (EPSG coordinate operation method code 9815).

Parameters:

Ellipsoid:	Everest 1830 (1967 Definition)	a = 6377298.556 metres	1/f = 300.8017
	then	e = 0.081472981	e^2 = 0.006637847

Latitude of projection centre	φ_C	4°00'00"N	=	0.069813170 rad
Longitude of projection centre	λ_C	115°00'00"E	=	2.007128640 rad
Azimuth of initial line	α_C	53°18'56.9537"	=	0.930536611 rad
Angle from Rectified to Skew Grid	γ_C	53°07'48.3685"	=	0.927295218 rad
Scale factor on initial line	k_C	0.99984		
Easting at projection centre	E_C	590476.87	metres	
Northings at projection centre	N_C	442857.65	metres	

Forward calculation for:

Latitude	φ	=	5°23'14.1129"N	=	0.094025313 rad
Longitude	λ	=	115°48'19.8196"E	=	2.021187362 rad

first gives :

B	=	1.003303209	F	=	1.072121256
A	=	6376278.686	H	=	1.000002991
T_O	=	0.932946976	γ_O	=	0.927295218
D	=	1.002425787	λ_O	=	1.914373469
D^2	=	1.004857458			
u_c	=	738096.09	v_c	=	0.00
t	=	0.910700729	Q	=	1.098398182
S	=	0.093990763	T	=	1.004407419
V	=	0.106961709	U	=	0.010967247
v	=	-69702.787	u	=	163238.163

Then	Easting	E	=	679245.73 metres
	Northing	N	=	596562.78 metres

Reverse calculation for same easting and northing first gives:

v'	=	-69702.787	u'	=	901334.257
Q'	=	1.011028053			
S'	=	0.010967907	T'	=	1.000060146
V'	=	0.141349378	U'	=	0.093578324
t'	=	0.910700729	χ	=	0.093404829

Then	Latitude	φ	=	5°23'14.113"N
	Longitude	λ	=	115°48'19.820"E

If the same projection is defined using the Hotine Oblique Mercator method then:

False easting	FE	=	0.0 metres
False northing	FN	=	0.0 metres
Then	u	=	901334.257

and all other values are as for the Oblique Mercator method.

1.4.8. Stereographic

The Stereographic projection may be imagined to be a projection of the earth's surface onto a plane in contact with the earth at a single tangent point from a projection point at the opposite end of the diameter through that tangent point.

This projection is best known in its polar form and is frequently used for mapping polar areas where it complements the Universal Transverse Mercator used for lower latitudes. Its spherical form has also been widely used by the US Geological Survey for planetary mapping and the mapping at small scale of continental hydrocarbon provinces. In its transverse or oblique ellipsoidal forms it is useful for mapping limited areas centred on the point where the plane of the projection is regarded as tangential to the ellipsoid., e.g. the Netherlands. The tangent point is the origin of the projected coordinate system and the meridian through it is regarded as the central meridian. In order to reduce the scale error at the extremities of the projection area it is usual to introduce a scale factor of less than unity at the origin such that a unitary scale factor applies on a near circle centred at the origin and some distance from it.

The coordinate conversion from geographical to projected coordinates is executed via the distance and azimuth of the point from the centre point or origin. For a sphere the formulas are relatively simple. For the ellipsoid the parameters defining the conformal sphere at the tangent point as origin are first derived. The conformal latitudes and longitudes are substituted for the geodetic latitudes and longitudes of the spherical formulas for the origin and the point.

An alternative approach is given by Snyder, where, instead of defining a single conformal sphere at the origin point, the conformal latitude at each point on the ellipsoid is computed. The conformal longitude is then always equivalent to the geodetic longitude. This approach is a valid alternative to that given here, but gives slightly different results away from the origin point. The USGS formula is therefore considered by EPSG to be a different coordinate operation method to that described here.

1.4.8.1 Oblique and Equatorial Stereographic cases (EPSG coordinate operation method code 9809)

Given the geodetic origin of the projection at the tangent point (φ_O, λ_O), the parameters defining the conformal sphere are:

$$R = (\rho_O \cdot \nu_O)^{0.5}$$
$$n = \{1+[(e^2 \cos^4\varphi_O) / (1 - e^2)]\}^{0.5}$$
$$C = (n + \sin\varphi_O) (1 - \sin\chi_O) / [(n - \sin\varphi_O) (1 + \sin(\chi_O)]$$

where: $\sin \chi_O = (w_1 - 1) / (w_1 + 1)$

$w_1 = [S_1.(S_2)^e]^n$

$S_1 = (1 + \sin\varphi_O)/(1 - \sin\varphi_O)$

$S_2 = (1 - e \sin\varphi_O)/(1 + e \sin\varphi_O)$

The conformal latitude and longitude of the origin (χ_O, Λ_O) are then computed from :

$$\chi_O = \sin^{-1} [(w_2 - 1)/(w_2 + 1)]$$

where S_1 and S_2 are as above and $w_2 = c\ [S_1.(S_2)^e]^n = c.w_1$

$$\Lambda_O = \lambda_O$$

For any point with geodetic coordinates (φ,λ) the equivalent conformal latitude and longitude (χ , Λ) are then computed from

$$\Lambda = n(\lambda - \Lambda_O) + \Lambda_O$$

and

$$\chi = \sin^{-1} [(w - 1)/(w + 1)]$$

where $w = c\ [S_a.(S_a)^e]^n$

$S_a = (1 + \sin\varphi) / (1 - \sin\varphi)$

$S_b = (1 - e.\sin\varphi) / (1 + e.\sin\varphi)$

Then

$$E = FE + 2\ R\ k_O \cos\chi \sin(\Lambda - \Lambda_0) / B$$

and

$$N = FN + 2\ R\ k_O [\sin\chi \cos\chi_O - \cos\chi \sin\chi_O \cos(\Lambda - \Lambda_O)] / B$$

where $B = [1+\sin\chi \sin\chi_O + \cos\chi \cos\chi_O \cos(\Lambda - \Lambda_O)]$

The reverse formulas to compute the geodetic coordinates from the grid coordinates involves computing the conformal values, then the isometric latitude and finally the geodetic values.

The parameters of the conformal sphere and conformal latitude and longitude at the origin are computed as above. Then for any point with Stereographic grid coordinates (E,N) :

$$\chi = \chi_O + 2 \tan^{-1}\{[(N - FN) - (E - FE) \tan (j/2)] / (2\ Rk_O)\}$$

$$\Lambda = j + 2\ i + \Lambda_O$$

where $g = 2\ Rk_O \tan (\pi/4 - \chi_O/ 2)$
$h = 4\ Rk_O \tan \chi_O + g$
$i = \tan^{-1} \{(E - FE) / [h + (N - FN)]\}$
$j = \tan^{-1} \{(E - FE) / [g - (N - FN)]\} - i$

Geodetic longitude $\lambda = (\Lambda - \Lambda_O) / n + \Lambda_O$

Isometric latitude $\psi = 0.5 \ln \{(1 + \sin\chi) / [c (1 - \sin\chi)]\} / n$

First approximation $\varphi_1 = 2 \tan^{-1} \mathbf{e}^{\Psi} - \pi / 2$ where **e**=base of natural logarithms.

ψ_i = isometric latitude at φ_i

where $\psi_i= \ln\{[\tan(\varphi_i/2 +\pi/4]\ \ [(1 - e \sin\varphi_i)/(1 + e \sin\varphi_i)]^{(e/2)}\}$

Then iterate $\varphi_{i+1} = \varphi_i - (\psi_i - \psi) \cos \varphi_i (1 - e^2 \sin^2\varphi_i) / (1 - e^2)$

until the change in φ is sufficiently small.

If the projection is the equatorial case, φ_O and χ_O will be zero degrees and the formulas are simplified as a result, but the above formulas remain valid.

For the polar version, φ_O and χ_O will be 90 degrees and the formulas become indeterminate. See below for formulas for the polar case.

For stereographic projections centred on points in the southern hemisphere, the signs of E, N, λ_O and λ must be reversed to be used in the equations and φ will be negative anyway as a southerly latitude.

Example:
For Projected Coordinate Reference System: RD / Netherlands New

Parameters:

Ellipsoid: Bessel 1841 $a = 6377397.155$ metres $1/f = 299.15281$
then $e = 0.08169683$

Latitude of natural origin	φ_O	52°09'22.178"N	=	0.910296727 rad
Longitude of natural origin	λ_O	5°23'15.500"E	=	0.094032038 rad
Scale factor at natural origin	k_O	0.9999079		
False easting	FE	155000.00	metres	
False northing	FN	463000.00	metres	

Forward calculation for:

Latitude	φ	=	53°N	=	0.925024504 rad
Longitude	λ	=	6°E	=	0.104719755 rad

first gives the conformal sphere constants:

$\rho_O = 6374588.71$ $\nu_O = 6390710.613$
$R = 6382644.571$ $n = 1.000475857$ $c = 1.007576465$

where $S_1 = 8.509582274$ $S_2 = 0.878790173$ $w_1 = 8.428769183$
$\sin \chi_O = 0.787883237$

$w_2 = 8.492629457$ $\chi_O = 0.909684757$ $\Lambda_O = \lambda_O = 0.094032038$ rad

For the point (φ,λ) $\chi = 0.924394997$ $\Lambda = 0.104724841$ rad

hence $B = 1.999870665$
and $E = 196105.283$ m $N = 557057.739$ m

Reverse calculation for the same Easting and Northing (196105.28E, 557057.74N) first gives:

$g = 4379954.188$ $h = 37197327.960$ $i = 0.001102255$ $j = 0.008488122$

then $\Lambda = 0.10472467$ whence $\lambda = 0.104719584$ rad = 6°E

Also $\chi = 0.924394767$ and $\psi = 1.089495123$
Then $\varphi_1 = 0.921804948$ $\psi_1 = 1.084170164$
$\varphi_2 = 0.925031162$ $\psi_2 = 1.089506925$
$\varphi_3 = 0.925024504$ $\psi_3 = 1.089495505$
$\varphi_4 = 0.925024504$

Then Latitude φ = 53°00'00.000"N
Longitude λ = 6°00'00.000"E

1.4.8.2 Polar Stereographic

For the polar sterographic projection, three variants are recognised, differentiated by their defining parameters. In the basic variant (**variant A**) the latitude of origin is either the north or the south pole, at which is defined a scale factor at the natural origin, the meridian along which the northing axis increments and along which intersecting parallels increment *towards the north pole* (the longitude of origin), and false grid coordinates. In **variant B** instead of the scale factor at the pole being defined, the (non-polar) latitude at which the scale is unity – the standard parallel – is defined. In **variant C** the latitude of a standard parallel along which the scale is unity is defined; the intersection of this parallel with the longitude of origin is the false origin, at which grid coordinate values are defined.

Method

Parameter	Variant A (note 1)	Variant B (note 2)	Variant C (note 2)
Latitude of natural origin (φ_O)			
Latitude of standard parallel (φ_F)		x	x
Longitude of origin (λ_O)	x	x	x
Scale at natural origin (k_O)	x		
False easting (easting at natural origin = pole) (FE)	x	x	
False northing (northing at natural origin = pole) (FN)	x	x	
Easting at false origin (E_F)			x
Northing at false origin (N_F)			x

In all three variants the formulae for the **south pole case** are straightforward but some require modification for the **north pole case** to allow the longitude of origin going towards (as opposed to away from) the natural origin and for the anticlockwise increase in longitude value when viewed from the north pole (see figure 8). Several equations are common between the variants and cases.

Notes:
1. In variant A the parameter *Latitude of natural origin* is used only to identify which hemisphere case is required. The only valid entries are ±90° or equivalent in alternative angle units.
2. For variants B and C, whilst it is mathematically possible for the standard parallel to be in the opposite hemisphere to the pole at which is the projection natural origin, such an arrangement would be unsatisfactory from a cartographic perspective as the rate of change of scale would be excessive in the area of interest. EPSG therefore excludes the hemisphere of pole as a defining parameter for these variants. In the formulas that follow for these variants B and C, the hemisphere of pole is taken to be that of the hemisphere of the standard parallel.

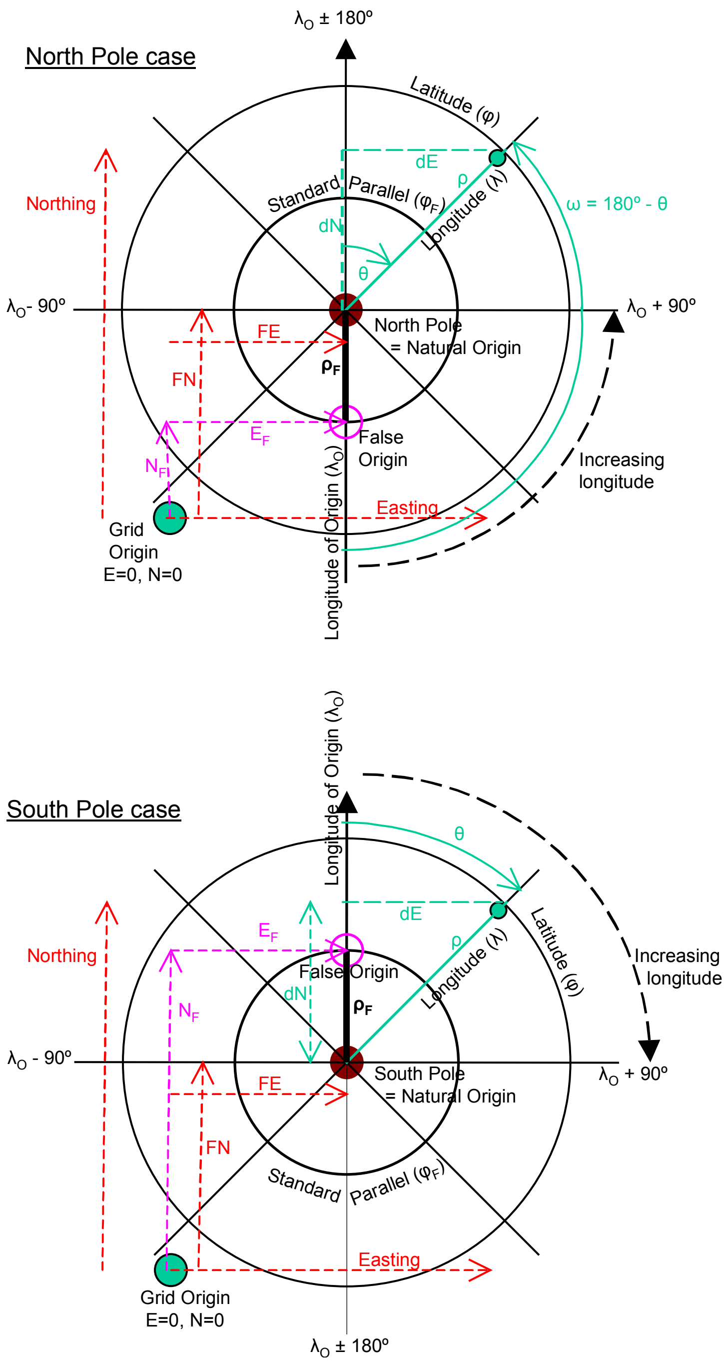

Figure 8. Key Diagram for Stereographic Projection

Polar Stereographic (Variant A) (EPSG coordinate operation method code 9810).

For the forward conversion from latitude and longitude, for the south pole case

$dE = \rho \sin(\theta)$ and $dN = \rho \cos(\theta)$

Then

$E = dE + FE = FE + \rho \sin(\lambda - \lambda_O)$

$N = dN + FN = FN + \rho \cos(\lambda - \lambda_O)$

where

$t = \tan(\pi/4 + \varphi/2) / \{[(1 + e \sin\varphi) / (1 - e \sin\varphi)]^{(e/2)}\}$

$\rho = 2\, a\, k_O\, t / \{[(1+e)^{(1+e)} (1-e)^{(1-e)}]^{0.5}\}$

For the north pole case,

$dE = \rho \sin(\theta) = \rho \sin(\omega)$.

$dN = \rho \cos(\theta) = -\rho \cos(\omega)$.

ρ and E are found as for the south pole case but

$t = \tan(\pi/4 - \varphi/2) \{[(1 + e \sin\varphi) / (1 - e \sin\varphi)]^{(e/2)}\}$

$N = FN - \rho \cos(\lambda - \lambda_O)$

For the reverse conversion from easting and northing to latitude and longitude,

$$\varphi = \chi + (e^2/2 + 5e^4/24 + e^6/12 + 13e^8/360) \sin(2\chi) + (7e^4/48 + 29e^6/240 + 811e^8/11520) \sin(4\chi) + (7e^6/120 + 81e^8/1120) \sin(6\chi) + (4279e^8/161280) \sin(8\chi)$$

where $\rho' = [(E-FE)^2 + (N - FN)^2]^{0.5}$

$t' = \rho' \{[(1+e)^{(1+e)} (1-e)^{(1-e)}]^{0.5}\} / 2 a k_O$

and for the south pole case

$\chi = 2 \operatorname{atan}(t') - \pi/2$

but for the north pole case

$\chi = \pi/2 - 2 \operatorname{atan} t'$

Then for for both north and south cases if E = FE, $\lambda = \lambda_O$

else for the south pole case

$\lambda = \lambda_O + \operatorname{atan}[(E - FE) / (N - FN)]$

and for the north pole case

$\lambda = \lambda_O + \operatorname{atan}[(E - FE) / -(N - FN)] = \lambda_O + \operatorname{atan}[(E - FE) / (FN - N)]$

Example:

For Projected Coordinate Reference System: WGS 84 / UPS North

Parameters:

Ellipsoid: WGS 84 a = 6378137.0 metres 1/f = 298.2572236

then e = 0.081819191

Latitude of natural origin	φ_O	90°00'00.000"N	=	1.570796327 rad
Longitude of origin	λ_O	0°00'00.000"E	=	0.0 rad
Scale factor at natural origin	k_O	0.994		
False easting	FE	2000000.00	metres	
False northing	FN	2000000.00	metres	

Forward calculation for:

Latitude	φ	=	73°N	=	1.274090354 rad
Longitude	λ	=	44°E	=	0.767944871 rad

t = 0.150412808

ρ = 1900814.564

whence

E = 3320416.75 m

N = 632668.43 m

Reverse calculation for the same Easting and Northing (3320416.75 E, 632668.43 N) first gives:

ρ' = 1900814.566

t' = 0.150412808

χ = 1.2722090

Then Latitude φ = 73°00'00.000"N

Longitude λ = 44°00'00.000"E

Polar Stereographic (Variant B) (EPSG coordinate operation method code 9829).

For the forward conversion from latitude and longitude:
for the south pole case

$t_F = \tan(\pi/4 + \varphi_F/2) / \{[(1 + e \sin\varphi_F) / (1 - e \sin\varphi_F)]^{(e/2)}\}$

$m_F = \cos\varphi_F / (1 - e^2 \sin^2\varphi_F)^{0.5}$

$k_O = m_F \{[(1+e)^{(1+e)} (1-e)^{(1-e)}]^{0.5}\} / (2 * t_F)$

then t, ρ, E and N are found as in the south pole case of variant A.

For the north pole case, m_F and k_O are found as for the south pole case above, but

$t_F = \tan(\pi/4 - \varphi_F/2) \{[(1 + e \sin\varphi_F) / (1 - e \sin\varphi_F)]^{(e/2)}\}$

Then t, ρ, E and N are found as in variant A.

For the reverse conversion from easting and northing to latitude and longitude, first k_O is found from m_F and t_F as in the forward conversion above, then φ and λ are found as for variant A.

Example:
For Projected Coordinate Reference System: WGS 84 / Australian Antarctic Polar Stereographic

Parameters:

Ellipsoid:	WGS 84	a = 6378137.0 metres	1/f = 298.2572236
	then	e = 0.081819191	

Latitude of standard parallel	φ_F	71°00'00.000"S	=	-1.239183769 rad
Longitude of origin	λ_O	70°00'00.000"E	=	1.221730476 rad
False easting	FE	6000000.00	metres	
False northing	FN	6000000.00	metres	

Forward calculation for:

Latitude	φ	=	75°00'00.000"S	=	-1.308996939 rad
Longitude	λ	=	120°00'00.000"E	=	2.094395102 rad

t_F = 0.168407325
m_F = 0.326546781
k_O = 0.97276901
t = 0.132508348
ρ = 1638783.238

whence

E = 7255380.79 m
N = 7053389.56 m

Reverse calculation for the same Easting and Northing (7255380.79 E, 7053389.56 N) first gives:

t_F = 0.168407325 m_F = 0.326546781 and k_O = 0.97276901

then ρ' = 1638783.236 t' = 0.132508347 χ = -1.3073146

Then Latitude φ = 75°00'00.000"S
Longitude λ = 120°00'00.000"E

Polar Stereographic (Variant C) (EPSG coordinate operation method code 9830).

For the forward conversion from latitude and longitude, for the south pole case

$E = E_F + \rho \sin(\lambda - \lambda_O)$

$N = N_F - \rho_F + \rho \cos(\lambda - \lambda_O)$

where

m_F is found as in variant B $= \cos \varphi_F \ / \ (1 - e^2 \sin^2\varphi_F)^{0.5}$

t_F is found as in variant B $= \tan(\pi/4 + \varphi_F/2) \ / \ \{[(1 + e \sin\varphi_F) / (1 - e \sin\varphi_F)]^{(e/2)}\}$

t is found as in variants A and B $= \tan(\pi/4 + \varphi/2) \ / \ \{[(1 + e \sin\varphi) / (1 - e \sin\varphi)]^{(e/2)}\}$

$\rho_F = a \, m_F$

$\rho = \rho_F \, t / t_F$

For the north pole case, m_F, ρ_F, ρ and E are found as for the south pole case but

t_F is found as in variant B $= \tan(\pi/4 - \varphi_F/2) \ \{[(1 + e \sin\varphi_F) / (1 - e \sin\varphi_F)]^{(e/2)}\}$

t is found as in variants A and B $= \tan(\pi/4 - \varphi/2) \ \{[(1 + e \sin\varphi) / (1 - e \sin\varphi)]^{(e/2)}\}$

$N = N_F + \rho_F - \rho \cos(\lambda - \lambda_O)$

For the reverse conversion from easting and northing to latitude and longitude,

$$\varphi = \chi + (e^2/2 + 5e^4/24 + e^6/12 + 13e^8/360) \sin(2\chi) + (7e^4/48 + 29e^6/240 + 811e^8/11520) \sin(4\chi) + (7e^6/120 + 81e^8/1120) \sin(6\chi) + (4279e^8/161280) \sin(8\chi)$$

(as for variants A and B)

where for the south pole case

$\rho' = [(E-E_F)^2 + (N - N_F + \rho_F)^2]^{0.5}$

$t' = \rho' \, t_F / \rho_F$

$\chi = 2 \operatorname{atan}(t') - \pi/2$

and where m_F and t_F are as for the forward conversion

For the reverse conversion north pole case, m_F, t_F and ρ_F are found as for the north pole case of the forward conversion, and

$\rho' = [(E-E_F)^2 + (N - N_F - \rho_F)^2]^{0.5}$

t' is found as for the south pole case of the reverse conversion $= \rho' \, t_F / \rho_F$

$\chi = \pi/2 - 2 \operatorname{atan} t'$

Then for for both north and south pole cases

if $E = E_F$, $\lambda = \lambda_O$

else for the south pole case

$\lambda = \lambda_O + \operatorname{atan} [(E - E_F) / (N - N_F + \rho_F)]$

and for the north pole case

$\lambda = \lambda_O + \operatorname{atan} [(E - E_F) / -(N - N_F - \rho_F)] = \lambda_O + \operatorname{atan} [(E - E_F) / (N_F + \rho_F - N)]$

Example:

For Projected Coordinate Reference System: Petrels 1972 / Terre Adelie Polar Stereographic

Parameters:

Ellipsoid:	International 1924	a = 6378388.0 metres			1/f = 297.0
	then	e = 0.081991890			
Latitude of false origin	φ_F	67°00'00.000"S	=	-1.169370599 rad	
Longitude of origin	λ_O	140°00'00.000"E	=	2.443460953 rad	
Easting at false origin	E_F	300000.00	metres		

Northing at false origin N_F 200000.00 metres

Forward calculation for:

Latitude φ = 66°36'18.820"S = -1.162480524 rad

Longitude λ = 140°04'17.040"E = 2.444707118 rad

$m_F = 0.391848769$
$\rho_F = 2499363.488$
$t_F = 0.204717630$
$t = 0.208326304$
$\rho = 2543421.183$

whence

E = 303169.52 m
N = 244055.72 m

Reverse calculation for the same Easting and Northing (303169.522 E, 244055.721 N) first gives:

$\rho' = 2543421.183$
$t' = 0.208326304$
$\chi = -1.1600190$

Then Latitude φ = 66°36'18.820"S
Longitude λ = 140°04'17.040"E

1.4.9 <u>New Zealand Map Grid</u> (EPSG coordinate operation method code 9811)

This projection system typifies the recent development in the design and formulation of map projections where, by more complex mathematics yielding formulas readily handled by modern computers, it is possible to maintain the conformal property and minimise scale distortion over the total extent of a country area regardless of shape. Thus both North and South Islands of New Zealand, previously treated not very satisfactorily in two zones of a Transverse Mercator projection, can now be projected as one zone of what resembles most closely a curved version Oblique Mercator but which, instead of being based on a minimum scale factor straight central line, has a central line which is a complex curve roughly following the trend of both North and South Islands. The projected coordinate reference system achieves this by a form of double projection where a conformal projection of the ellipsoid is first made to say an oblique Stereographic projection and then the Cauchy-Riemann equations are invoked in order to further project the rectangular coordinates on this to a modification in which lines of constant scale can be made to follow other than the normal great or small circles of Central meridians or standard parallels. The mathematical treatment of the New Zealand Map Grid is covered by a publication by New Zealand Department of Lands and Survey Technical Circular 1973/32 by I.F.Stirling.

1.4.10 <u>Tunisia Mining Grid</u> (EPSG coordinate operation method code 9816)

This grid is used as the basis for mineral leasing in Tunisia. Lease areas are approximately 2 x 2 km or 400 hectares. The corners of these blocks are defined through a six figure grid reference where the first three digits are an easting in kilometres and the last three digits are a northing. The latitudes and longitudes for block corners at 2 km intervals are tabulated in a mining decree dated 1st January 1953. From this tabulation in which geographical coordinates are given to 5 decimal places it can be seen that:

a) the minimum easting is 94 km, on which the longitude is 5.68989 grads east of Paris.
b) the maximum easting is 490 km, on which the longitude is 10.51515 grads east of Paris.
c) each 2 km grid easting interval equals 0.02437 grads.
d) the minimum northing is 40 km, on which the latitude is 33.39 grads.

e) the maximum northing is 860 km, on which the latitude is 41.6039 grads.
f) between 40 km N and 360 km N, each 2 km grid northing interval equals 0.02004 grads.
g) between 360 km N and 860 km N, each 2 km grid northing interval equals 0.02003 grads.

This grid could be considered to be two equidistant cylindrical projection zones, north and south of the 360 km northing line. However this would require the introduction of two spheres of unique dimensions. EPSG has therefore implemented the Tunisia mining grid as a coordinate conversion method in its own right. Formulas are:

Grads from Paris

φ (grads) = 36.5964 + [(N – 360) * A]
where N is in kilometres and A = 0.010015 if N > 360, else A = 0.01002.

λ_{Paris} (grads) = 7.83445 + [(E – 270) * 0.012185], where E is in kilometres.

The reverse formulas are:

E (km) = 270 + [(λ_{Paris} – 7.83445) / 0.012185] where λ_{Paris} is in grads.

N (km) = 360 + [(φ – 36.5964) / B]
where φ is in grads and B = 0.010015 if φ > 36.5964, else B = 0.01002.

Degrees from Greenwich

Modern practice in Tunisia is to quote latitude and longitude in degrees with longitudes referenced to the Greenwich meridian. The formulas required in addition to the above are:

φ_d (degrees) = (φ_g * 0.9) where φ_g is in grads.
$\lambda_{Greenwich}$ (degrees) = [(λ_{Paris} + 2.5969213) * 0.9] where λ_{Paris} is in grads.

φ_g (grads) = (φ_d / 0.9) where φ_d is in decimal degrees.
λ_{Paris} (grads) = [($\lambda_{Greenwich}$ / 0.9) – 2.5969213)] where $\lambda_{Greenwich}$ is in decimal degrees.

Example:

For grid location 302598,
Latitude φ = 36.5964 + [(598 – 360) * A]. As N > 360, A = 0.010015.
φ = 38.97997 grads = 35.08197 degrees.

Longitude λ = 7.83445 + [(E – 270) * 0.012185, where E = 302.
λ = 8.22437 grads east of Paris = 9.73916 degrees east of Greenwich.

1.4.11 **American Polyconic** (EPSG coordinate operation method code 9818)

This projection was popular before the advent of modern computers due to its ease of mechanical construction, particularly in the United States. It is neither conformal nor equal area, and is distortion-free only along the longitude of origin. A modified form of the polyconic projection was adopted in 1909 for the International Map of the World series of 1/1,000,000 scale topographic maps. A general study of the polyconic family of projections by Oscar Adams of the US Geological Survey was published in 1919 (and reprinted in 1934). The mathematical treatment of the American Polyconic is covered on pages 124-131 of USGS professional paper 1395 by J.P.Snyder.

1.4.12 **Lambert Azimuthal Equal Area** (EPSG coordinate operation method code 9820)

To derive the projected coordinates of a point, geodetic latitude (φ) is converted to authalic latitude (β). The formulae to convert geodetic latitude and longitude (φ, λ) to Easting and Northing are:

$$\text{Easting, } E = FE + \{(B \,.\, D) \,.\, [\cos \beta \,.\, \sin(\lambda - \lambda_O)]\}$$

$$\text{Northing, } N = FN + (B / D) \,.\, \{(\cos \beta_O \,.\, \sin \beta) - [\sin \beta_O \,.\, \cos \beta \,.\, \cos(\lambda - \lambda_O)]\}$$

where

$$B = R_q \,.\, (2 / \{1 + \sin \beta_O \,.\, \sin \beta + [\cos \beta_O \,.\, \cos \beta \,.\, \cos(\lambda - \lambda_O)]\})^{0.5}$$

$$D = a \,.\, [\cos \lambda_O / (1 - e^2 \sin^2 \lambda_O)^{0.5}] / (R_q \,.\, \cos \beta_O)$$

$$R_q = a \,.\, (q_P / 2)^{0.5}$$

$$\beta = \sin (q / q_P)$$

$$\beta_O = \sin (q_O / q_P)$$

$$q = (1 - e^2) \,.\, ([\sin \varphi / (1 - e^2 \sin^2 \varphi)] - \{[1/(2e)] \,.\, \ln [(1 - e \sin \varphi) / (1 + e \sin \varphi)]\})$$

$$q_O = (1 - e^2) \,.\, ([\sin \varphi_O / (1 - e^2 \sin^2 \varphi_O)] - \{[1/(2e)] \,.\, \ln [(1 - e \sin \varphi_O) / (1 + e \sin \varphi_O)]\})$$

$$q_P = (1 - e^2) \,.\, ([\sin \varphi_P / (1 - e^2 \sin^2 \varphi_P)] - \{[1/(2e)] \,.\, \ln [(1 - e \sin \varphi_P) / (1 + e \sin \varphi_P)]\})$$

The reverse formulas to derive the geodetic latitude and longitude of a point from its Easting and Northing values are:

$$\varphi = \beta' + [(e^2/3 + 31e^4/180 + 517e^6/5040) \,.\, \sin 2\beta'] + [(23e^4/360 + 251e^6/3780) \,.\, \sin 4\beta'] + [(761e^6/45360) \,.\, \sin 6\beta']$$

$$\lambda = \lambda_O + \text{atan } \{(X\text{-}FE) \,.\, \sin C / [D.\ \rho \,.\, \cos \beta_O \,.\, \cos C - D^2.\ (Y\text{-}FN) \,.\, \sin \beta_O \,.\, \sin C]\}$$

where

$$\beta' = \sin\{(\cos C \,.\, \sin \beta_O) + [(D \,.\, (Y\text{-}FN) \,.\, \sin C \,.\, \cos \beta_O) / \rho]\}$$

$$C = 2 \,.\, \sin(\rho / 2 \,.\, R_q)$$

$$\rho = \{[(X\text{-}FE)/D]^2 + [D \,.\, (Y - FN)]^2\}^{0.5}$$

and D, R_q, and β_O are as in the forward equations.

Example

For Projected Coordinate Reference System: ETRS89 / ETRS-LAEA

Parameters:

Ellipsoid:	GRS 1980		a = 6378137.0 metres		1/f = 298.2572221
		then	e = 0.081819191		

Latitude of natural origin	φ_O	53°00'00.000"N	=	0.925024504 rad
Longitude of natural origin	λ_O	9°00'00.000"E	=	0.157079633 rad
False easting	FE	4321000.00	metres	
False northing	FN	3210000.00	metres	

Forward calculation for:

Latitude	φ	=	50°00'00.000"N	=	0.872664626 rad

Longitude λ = 5°00'00.000"E = 0.087266463 rad

First gives

qP =	1.995531087	qO =	1.591111956
q =	1.525832247	Rq =	6371007.181
betaO =	0.922870909	beta =	0.870458708
D =	1.000406507	B =	6374706.698

whence

E = 4034299.86 m
N = 2884152.53 m

Reverse calculation for the same Easting and Northing (4034299.86 E, 2884152.53 N) first gives:

rho = 434042.7347
C = 0.068140987
beta' = 0.870458708

Then Latitude φ = 50°00'00.000"N
Longitude λ = 5°00'00.000"E

1.4.12.2 Lambert Azimuthal Equal Area (Spherical) (EPSG coordinate operation method code 9821)
The US National Atlas uses the spherical form of the map projection method, so exceptionally EPSG documents this. See USGS Professional Paper 1395, "Map Projections - A Working Manual" by John P. Snyder for formulas and example.

1.4.13 Albers Equal Area (EPSG coordinate operation method code 9822)

To derive the projected coordinates of a point, geodetic latitude (φ) is converted to authalic latitude (ß). The formulas to convert geodetic latitude and longitude (φ, λ) to Easting (E) and Northing (N) are:

$$\text{Easting (E)} = E_F + (\rho \cdot \sin\theta)$$
$$\text{Northing (N)} = N_F + \rho_O - (\rho \cdot \cos\theta)$$

where

$$\theta = n \cdot (\lambda - \lambda_O)$$
$$\rho = [a \cdot (C - n\,\alpha)^{0.5}] / n$$
$$\rho_O = [a \cdot (C - n\,\alpha_O)^{0.5}] / n$$

and

$$C = m_1^2 + (n \cdot \alpha_1)$$
$$n = (m_1^2 - m_2^2) / (\alpha_2 - \alpha_1)$$
$$m_1 = \cos\varphi_1 / (1 - e^2\sin^2\varphi_1)^{0.5}$$
$$m_2 = \cos\varphi_2 / (1 - e^2\sin^2\varphi_2)^{0.5}$$
$$\alpha = (1 - e^2) \cdot [(\sin\varphi / (1 - e^2\sin^2\varphi_1)] - \{[1/(2e)] \cdot \ln[(1 - e\sin\varphi_1) / (1 + e\sin\varphi_1)]\}$$
$$\alpha_O = (1 - e^2) \cdot [(\sin\varphi_O / (1 - e^2\sin^2\varphi_O)] - \{[1/(2e)] \cdot \ln[(1 - e\sin\varphi_O) / (1 + e\sin\varphi_O)]\}$$
$$\alpha_1 = (1 - e^2) \cdot [(\sin\varphi_1 / (1 - e^2\sin^2\varphi_1)] - \{[1/(2e)] \cdot \ln[(1 - e\sin\varphi_1) / (1 + e\sin\varphi_1)]\}$$
$$\alpha_2 = (1 - e^2) \cdot [(\sin\varphi_2 / (1 - e^2\sin^2\varphi_2)] - \{[1/(2e)] \cdot \ln[(1 - e\sin\varphi_2) / (1 + e\sin\varphi_2)]\}$$

The reverse formulas to derive the geodetic latitude and longitude of a point from its Easting and Northing values are:

$$\varphi = ß' + (e^2/3 + 31e^4/180 + 517e^6/5040) \cdot \sin 2ß'] + [(23e^4/360 + 251e^6/3780) \cdot \sin 4ß'] + [(761e^6/45360) \cdot \sin 6ß']$$

$\lambda = \lambda_O + (\theta / n)$

where

$\beta' = \sin(\alpha' / \{1 - [(1 - e^2) / (2 . e)] . \ln [(1 - e / (1 + e)]$

$\alpha' = [C - (\rho^2 . n^2 / a^2)] / n$

$\rho = \{(E - E_F)^2 + [\rho_O - (N - N_F)]^2 \}^{0.5}$

$\theta = \text{atan} [(E - E_F) / [\rho_O - (N - N_F)]$

and C, n and ρ_O are as in the forward equations.

Example

See USGS Professional Paper 1395, "Map Projections - A Working Manual" by John P. Snyder.

1.4.14 Equidistant Cylindrical (EPSG coordinate operation method code 9823)

This method has one of the simplest formulas available. If the latitude of natural origin (φ_O) is at the equator the method is also known as Plate Carrée. It is not used for rigorous topographic mapping because its distortion characteristics are unsuitable. Formulas are included to distinguish this map projection method from an approach sometimes mistakenly called by the same name and used for simple computer display of geographic coordinates – see Pseudo Plate Carrée below.

For the forward calculation of the Equidistant Cylindrical method:

$X = R . (\lambda - \lambda_O) . \cos(\varphi_O)$

$Y = R . \varphi$

where $R = (\rho_O^2 + \nu_O^2)^{0.5} = \{[a^2 * (1 - e^2)] / (1 - e^2 \sin^2\varphi_O)^2\}^{0.5}$
and φ_O, φ and λ are expressed in radians.

For the Equidistant Cylindrical method on a sphere (not ellipsoid), e = 0 and R = a.

For the reverse calculation:

$\varphi = Y / R$

$\lambda = \lambda_O + (X / R \cos(\varphi_O))$

where R is as for the forward method.

1.4.14.1 Pseudo Plate Carrée (EPSG coordinate operation method code 9825)

This is not a map projection in the true sense as the coordinate system units are decimal degrees and therefore of variable linear scale. It is used only for depiction of graticule (latitude/longitude) coordinates on a computer display. The origin is at latitude (φ) = 0, longitude (λ) = 0. See above for the formulas for the proper Plate Carrée map projection method.

For the forward calculation:

$X = \lambda$

$Y = \varphi$

where φ and λ are expressed in radians.

For the reverse calculation:

$\varphi = Y$
$\lambda = X$

1.4.14 **Bonne** (EPSG coordinate operation method code 9827)

The Bonne projection was frequently adopted for 18th and 19th century mapping, but being equal area rather than conformal its use for topographic mapping was replaced during the 20th century by conformal map projection methods.

The formulas to convert geodetic latitude and longitude (φ, λ) to Easting and Northing are:

$$E = (\rho \, . \, \sin T) + FE$$
$$N = (a \, . \, m_O / \sin \varphi_O - \rho \, . \, \cos T) + FN$$

where

$m = \cos\varphi / (1 - e^2\sin^2\varphi)^{0.5}$
with φ in radians and m_O for φ_O, the latitude of the origin, derived in the same way.

$$M = a[(1 - e^2/4 - 3e^4/64 - 5e^6/256 - ....)\varphi - (3e^2/8 + 3e^4/32 + 45e^6/1024 + ...)\sin 2\varphi + (15e^4/256 + 45e^6/1024 + ...)\sin 4\varphi - (35e^6/3072 + ...)\sin 6\varphi + ...]$$

with φ in radians and M_O for φ_O, the latitude of the origin, derived in the same way.

$\rho = a \, . \, m_O / \sin \varphi_O + M_O - M$
$T = a \, . \, m \, (\lambda - \lambda_O) / \rho$ with λ and λ_O in radians

For the reverse calculation:

$X = E - FE$
$Y = N - FN$
$\rho = \pm [X^2 + (a \, . \, m_O / \sin \varphi_O - Y)^2]^{0.5}$ taking the sign of φ_O
$M = a \, . \, m_O / \sin \varphi_O + M_O - \rho$
$\mu = M / [a (1 - e^2/4 - 3e^4/64 - 5e^6/256 - \ldots)]$
$e_1 = [1 - (1 - e^2)^{0.5}] / [1 + (1 - e^2)^{0.5}]$
$\varphi = \mu + (3e_1/2 - 27e_1^3/32 + ...)\sin 2\mu + (21e_1^2/16 - 55e_1^4/32 + ...)\sin 4\mu + (151e_1^3/96 + ...)\sin 6\mu + (1097e_1^4/512 - ....)\sin 8\mu + ...$
$m = \cos \varphi / (1 - e^2 \sin^2 \varphi)^{0.5}$

If φ_O is not negative

$\lambda = \lambda_O + \rho \, \{\mathrm{atan}[X / (a \, . \, m_O / \sin \varphi_O - Y)]\} / a \, . \, m$

but if φ_O is negative

$\lambda = \lambda_O + \rho \, \{\mathrm{atan}[- X / (Y - a \, . \, m_O / \sin \varphi_O)]\} / a \, . \, m$

In either case, if $\varphi = \pm 90°$, $m = 0$ and the equation for λ is indeterminate, so use $\lambda = \lambda_O$.

1.4.15.1 Bonne (South Orientated) (EPSG coordinate operation method code 9828)

In Portugal a special case of the method with coordinate system axes positive south and west has been used for older mapping. The formulas are as for the general case above except:

$$W = FE - (\rho \, . \, \sin T)$$
$$S = FN - (a \, . \, m_O / \sin \varphi_O - \rho \, . \, \cos T)$$

In these formulas the terms FE and FN retain their definition, i.e. in the Bonne (South Orientated) method they increase the Westing and Southing value at the natural origin. In this method they are effectively false westing (FW) and false southing (FS) respectively.

For the reverse formulas, those for the standard Bonne method above apply, with the exception that:

$X = FE - W$
$Y = FN - S$

Part 2: **Formulas for Coordinate Operations other than Map Projections**

2.1. **Introduction**

It has been noted earlier that it is frequently required to convert coordinates derived in one geographic coordinate reference system to values expressed in another. For example, land and marine seismic surveys are nowadays most conveniently positioned by GPS satellite in the WGS 84 geographic coordinate reference system, whereas the national geodetic reference system in use for the country concerned will probably be a much earlier coordinate reference system. It may therefore be necessary to transform the observed WGS 84 data to the national geodetic reference system in order to avoid discrepancies. This form of coordinate transformation has to be between source and target geographical coordinate reference systems with the coordinates of both expressed in geographical terms. It is not executed by a transformation between projected coordinate reference systems.

The coordinate transformations may be most readily achieved by first expressing the observed or source geographical coordinates in terms of three dimensional XYZ Cartesian values (geocentric coordinates) instead of the normal angular expressions of latitude and longitude. However in order to derive the three true cartesian coordinates of a point on the earth's surface one must recognise that as well as having a latitude and longitude the point will also have an elevation above the ellipsoid. While XYZ cartesian coordinates may be derived from a mere latitude and longitude by assuming that the point lies on the ellipsoid's surface, in fact very few earth points actually do. Therefore the height of the point must be taken into account and the height required is the height above the ellipsoid. It is this height which is delivered by GPS in relation to the WGS84 ellipsoid. Heights above other ellipsoids are not generally immediately available. Instead the height that is usually available is the gravity-related height above the national vertical datum, normally Mean Sea Level at a particular coastal point measured over a particular period. Levels will normally have been derived by conventional surveying methods and values will relate to the geoid surface. Hence in order to derive heights above the ellipsoid it is necessary to know the height of the geoid above the ellipsoid or vice versa and for most areas of the world this is not known with great precision. There exist various mathematical models of the geoid which have been derived for individual countries or parts of the world or the entire world and as satellite and terrestrial gravity data accumulates they are being steadily refined. The best available geoidal data should be used to convert gravity-related heights (surveyed other than by GPS) to ellipsoidal heights for use in a coordinate transformation.

In the event that the true ellipsoidal height of a point cannot be obtained the assumption that it is zero will often allow a coordinate transformation without introducing significant error in the horizontal coordinates.

In the early days of satellite surveying when relationships between geographic coordinate reference systems were not well defined and the data itself was not very precise, it was usual to apply merely a three parameter dX, dY, dZ shift to the XYZ coordinate set in one geographic coordinate reference system to derive those in the second geographic coordinate reference system. This assumed, generally erroneously, that the axial directions of the two ellipsoids involved were parallel. For localised work in a particular country or territory, the consequent errors introduced by this assumption were small and generally less than the observation accuracy of the data. Nevertheless, as knowledge and data has accumulated and surveying methods have become more accurate, it has become evident that a three parameter transformation is neither appropriate for world wide use, nor indeed for widespread national use if one is seeking the maximum possible accuracy from the satellite surveying and a single set of transformation parameter values.

For petroleum exploration purposes it may well be that a three parameter transformation within a particular licence area is quite adequate but it should not be assumed that the same transformation parameter values are appropriate for use with different data in an adjoining area or for another purpose.

The simplest coordinate transformation to implement involves applying shifts to the three geocentric coordinates. Molodenski developed a transformation method which applies the geocentric shifts directly to geographical coordinates. Both of these methods assume that the axes of the source and target coordinate reference systems are parallel to each other. As indicated above this assumption may not be true and consequently these transformation methods result in only moderate accuracy, especially if applied over large areas.

Improved accuracy can be obtained by applying a Helmert 7-parameter transformation to geocentric coordinates. However there are two opposing sign conventions for the three rotation parameters. In EPSG these are considered to be two different transformation methods, either a **Position Vector** transformation or a **Coordinate Frame** transformation**.** (The Position Vector transformation is elsewhere called the Bursa-Wolf transformation). It is crucial that the signs and interpretation of the transformation parameters are consistent with the convention being applied. However these methods may suffer from high correlation between the translations and rotations. An alternative approach that avoids this correlation is the **Molodenski-Badekas** transformation.

It is also possible to interpolate geographical or grid coordinate differences for points on the basis of known shifts for a number of control points in a specific defined area. One such application is the coordinate transformation introduced to enable the conversion of coordinates expressed in the North American Datum of 1927 (based on the Clarke 1966 ellipsoid) to coordinates expressed in the newer North American Datum of 1983 which takes the GRS 1980 ellipsoid. Because the North American survey control network was built by conventional terrestrial survey observations and suffers from the inevitable instrumental and adjustment shortcomings of the time, the old network, based on the non-geocentric Clarke 1866 ellipsoid and a single datum point at Meades Ranch in Kansas, is not wholly consistent when compared with data which can be more readily and accurately secured nowadays with the advantages of satellite technology, modern instrumentation, and electronic computational techniques. Hence to convert between coordinates on the old system to values in the new datum, it is not appropriate to merely apply the type of orthogonal transformation represented by the Molodenski or Helmert transformations described above. Different points in different parts of the North American continent need to undergo different positional or coordinate shifts according to their position within the continental network. This is known in EPSG as the **Bi-linear Gridded Interpolation** transformation technique. The transformation is achieved by bilinear interpolation for the new NAD 83 Latitudes and Longitudes using US Coast & Geodetic Survey NADCON control point grids. Bilinear interpolation is also used in Canada for the same purpose using the National Transformation (NT) application and grids. As with the US, longitude differences are applied to longitudes which are positive west. The NAD27 and NAD83 geographical coordinate systems documented by EPSG use the positive east longitude convention. The Canadian gridded file format has also been adopted by Australia and New Zealand. In Great Britain, bi-linear interpolation of gridded easting and northing differences is used.

Alternatively a polynomial expression with listed coefficients for both latitude and longitude may be used as the transformation method. A transformation, applicable to offshore Norway to effect the transformation between coordinates expressed in the imperfect European Datum 1950 (ED50) and the newer ED87 uses this approach. Statens Kartwerk, the Norwegian survey authority, publishes a document which lists 15 coefficients for each of separate latitude and longitude polynomial transformation formulas, involving up to fourth order expressions of latitude and longitude expressed in degrees.

If and when other countries decide to convert to using coordinate reference systems based on geocentric datums to suit satellite positional data acquisition methods, coordinate transformations similar to NADCON or the Norwegian example may well be employed to facilitate the conversion between the old terrestrial survey derived coordinates and the new geocentric datum values.

Note that it is very important to ensure that the signs of the parameter values used in the transformations are correct in respect of the transformation being executed. Preferably one should always express transformations in terms of "From".........."To"........... thus avoiding the confusion which may result from interpreting a dash as a minus sign or vice versa.

2.2 Coordinate Conversions other than Map Projections

2.2.1 Geographic/Geocentric Conversions

Latitude, **φ,** and Longitude, **λ**, in terms of a geographic coordinate reference system may be expressed in terms of a geocentric (earth centred) Cartesian coordinate system X, Y, Z with the Z axis corresponding with the Polar axis positive northwards, the X axis through the intersection of the prime meridian and equator, and the Y axis through the intersection of the equator with longitude 90°E.

Geocentric coordinate reference systems are conventionally taken to be defined with the X axis through the intersection of the Greenwich meridian and equator. This requires that the equivalent geographic coordinate reference system is based on the Greenwich meridian. In application of the formulas below, geographic coordinate systems based on a non-Greenwich prime meridian should first be transformed to their Greenwich equivalent.

If the ellipsoidal semi major axis is **a,** semi minor axis **b**, and inverse flattening **1/f**, then

$$X = (\nu + h) \cos\varphi \cos\lambda$$
$$Y = (\nu + h) \cos\varphi \sin\lambda$$
$$Z = ((1 - e^2)\,\nu + h) \sin\varphi$$

where ν is the prime vertical radius of curvature at latitude φ and is equal to
$\nu = a /(1 - e^2\sin^2\varphi)^{0.5}$,
φ and λ are respectively the latitude and longitude (related to the prime meridian) of the point,
h is height above the ellipsoid, (see note below), and
e is the eccentricity of the ellipsoid where $e^2 = (a^2 - b^2)/a^2 = 2f - f^2$

(Note that h *is the height above the ellipsoid.. This is the height value that is delivered by GPS satellite observations but is not the gravity-related height value which is normally used for national mapping and levelling operations. The gravity-related height (*H*) is usually the height above mean sea level or an alternative level reference for the country. If one starts with a gravity-related height* H*, it will be necessary to convert it to an ellipsoid height* (h) *before using the above transformation formulas.* h = H + N*, where* N *is the geoid height above the ellipsoid at the point and is sometimes negative. The geoid is a gravity surface approximating mean sea level. For the WGS 84 ellipsoid the value of* N*, representing the height of the geoid relative to the ellipsoid, can vary between values of -100m in the Sri Lanka area to +60m in the North Atlantic. Geoid heights of points above the nationally used ellipsoid may not be readily available.)*

For the reverse conversion, Cartesian coordinates in the geocentric coordinate reference system may then be converted to the geographical coordinates in terms of geographic coordinate reference system by:

$$\varphi = \text{atan}\,(Z + e^2\nu \sin\varphi) / (X^2 + Y^2)^{0.5} \text{ by iteration}$$
$$\lambda = \text{atan}\, Y/X$$
$$h = X \sec\lambda \sec\varphi - \nu$$

where λ is relative to the Greenwich prime meridian.

See section 2.3.2 below for an example.

2.3 Coordinate Transformation Formulas

2.3.1 Offsets

Several transformation methods which utilise offsets in coordinate values are recognised. These include longitude rotations, geographical coordinate offsets and vertical offsets.

Mathematically, if the origin of a one-dimensional coordinate system is shifted along the positive axis and placed at a point with ordinate A (where A>0), then the transformation formula is:

$$X_{new} = X_{old} - A$$

However it is common practice in coordinate system transformations to apply the shift as an addition, with the sign of the shift parameter value having been suitably reversed to compensate for the practice. Since 1999 this practice has been adopted by EPSG. Hence transformations allow calculation of coordinates in the target system by adding a correction parameter to the coordinate values of the point in the source system:

$$X_t = X_s + A$$

where X_s and X_t are the values of the coordinates in the source and target coordinate systems and A is the value of the transformation parameter to transform source coordinate reference system coordinate to target coordinate reference system coordinate.

2.3.1.1 Vertical Offset (EPSG coordinate operation method code 9616)

If the coordinate reference system axes are positive in opposite directions, for instance in the transformation of heights in the source vertical CRS to depths in the target vertical CRS, or in differing units, the above formula is modified to:

$$X_t = [(X_s * U_s) + (A * U_A)] * (m / U_t)$$

where
X_t = value in the target vertical coordinate reference system.
X_s = value in the source vertical coordinate reference system.
A is the value of the origin of the target system in the source system.
m is unit direction multiplier (m=1 if both systems are height or both are depth; m = –1 if one system is height and the other system is depth; the value of m is implied through the vertical coordinate reference system type attribute).
U_s U_t and U_A are unit conversion ratios to metres for the source and target systems and the offset value respectively.

2.3.2 Geocentric Translations (EPSG coordinate operation method code 9603)

If we assume that the minor axes of the ellipsoids are parallel and that the prime meridian is Greenwich, then shifts **dX, dY, dZ** in the sense from source geocentric coordinate reference system to target geocentric coordinate reference system may then be applied as

$$X_t = X_s + dX$$
$$Y_t = Y_s + dY$$
$$Z_t = Z_s + dZ$$

Example:
This example combines the geographic/geocentric conversion of section 2.2.1 above with the geocentric translation method.

Consider a North Sea point with coordinates derived by GPS satellite in the WGS84 geographical coordinate reference system, with coordinates of:

latitude φ_s = 53°48'33.82"N,

longitude λ_s = 2°07'46.38"E,
and ellipsoidal height h_s = 73.0m,

whose coordinates are required in terms of the ED50 geographical coordinate reference system which takes the International 1924 ellipsoid. The three parameter geocentric translations method's parameter values from WGS84 to ED50 for this North Sea area are given as dX = +84.87m, dY = +96.49m, dZ = +116.95m.

The WGS 84 geographical coordinates first convert to the following geocentric values using the formulas from section 2.2.1:

X_S = 3771 793.97 m
Y_S = 140 253.34 m
Z_S = 5124 304.35 m

Applying the quoted geocentric translations to these, we obtain new geocentric values now related to ED50:

X_t = 3771 793.97 + 84.87 = 3771 878.84 m
Y_t = 140 253.34 + 96.49 = 140 349.83 m
Z_t = 5124 304.35 + 116.95 = 5124 421.30 m

Using the reverse conversion given in section 2.2.1 above, these convert to ED50 values on the International 1924 ellipsoid as:
latitude φ_t = 53°48'36.565"N,
longitude λ_t = 2°07'51.477"E,
and ellipsoidal height h_t = 28.02 m,

Note that the derived height is referred to the International 1924 ellipsoidal surface and will need a further correction for the height of the geoid at this point in order to relate it to Mean Sea Level.

2.3.3 Abridged Molodenski transformation

As an alternative to the above computation of the new latitude, longitude and height above ellipsoid in discrete steps, the changes in these coordinates may be derived directly by formulas derived by Molodenski (EPSG coordinate operation method code 9604). Abridged versions of these formulas (EPSG coordinate operation method code 9605), which are quite satisfactory for three parameter transformations, are as follows:

$\varphi_t = \varphi_s + d\varphi$
$\lambda_t = \lambda_s + d\lambda$
$h_t = h_s + dh$

where

$d\varphi'' = (-dX.\sin\varphi.\cos\lambda - dY.\sin\varphi.\sin\lambda + dZ.\cos\varphi + [a.df + f.da].\sin2\varphi) / (\rho . \sin1'')$
$d\lambda'' = (-dX.\sin\lambda + dY.\cos\lambda) / (\nu.\cos\varphi . \sin 1'')$
$dh = dX.\cos\varphi.\cos\lambda + dY.\cos\varphi.\sin\lambda + dZ.\sin\varphi + (a.df + f.da).\sin^2\varphi - da$

where the dX, dY and dZ terms are as before, and ρ and ν are the meridian and prime vertical radii of curvature at the given latitude φ on the first ellipsoid (see section 1.4), da is the difference in the semi-major axes of the target and source ellipsoids [$da = a_t - a_s$] and df is the difference in the flattening of the two ellipsoids [$df = f_t - f_s = 1/(1/f_t) - 1/(1/f_s)$].

The formulas for $d\varphi$ and $d\lambda$ indicate changes in φ and λ in arc-seconds.

Example:

For the same North Sea point with coordinates derived by GPS satellite in the WGS84 geographical coordinate reference system, with coordinates of:

latitude φ_s = 53°48'33.82"N,
longitude λ_s = 2°07'46.38"E,
and ellipsoidal height h_s = 73.0m,

whose coordinates are required in terms of the ED50 geographical coordinate reference system which takes the International 1924 ellipsoid.

The three geocentric translations parameter values from WGS84 to ED50 for this North Sea area are given as dX = +84.87m, dY = +96.49m, dZ = +116.95m.
Ellipsoid Parameters are:

WGS 1984	a = 6378137.0 metres	1/f = 298.2572236
International 1924	a = 6378388.0 metres	1/f = 297.0

Then

da = 6378137 – 6378388 = –251
df = 0.003352811 - 0.003367003 = -1.41927E-05

whence

$d\varphi$ = 2.545"
$d\lambda$ = 5.097"
dh = – 44.98 m

ED50 values on the International 1924 ellipsoid are then:

latitude φ_t = 53°48'36.565"N,
longitude λ_t = 2°07'51.477"E,
and ellipsoidal height h_t = 28.02 m.

2.3.4 Helmert transformation

Transformation of coordinates from one geographic coordinate reference system into another (also colloquially known as a "datum transformation") is usually carried out as an implicit concatenation of three transformations:
[geographical to geocentric >> geocentric to geocentric >> geocentric to geographic]

The middle part of the concatenated transformation, from geocentric to geocentric, is usually described as a simplified 7-parameter Helmert transformation, expressed in matrix form with 7 parameters, in what is known as the "Bursa-Wolf" formula:

$$\begin{pmatrix} X_T \\ Y_T \\ Z_T \end{pmatrix} = M * \begin{pmatrix} 1 & -R_Z & +R_Y \\ +R_Z & 1 & -R_X \\ -R_Y & +R_X & 1 \end{pmatrix} * \begin{pmatrix} X_S \\ Y_S \\ Z_S \end{pmatrix} + \begin{pmatrix} dX \\ dY \\ dZ \end{pmatrix}$$

The parameters are commonly referred to defining the transformation "from source coordinate reference system to target coordinate reference system", whereby (X_S, Y_S, Z_S) are the coordinates of the point in the source geocentric coordinate reference system and (X_T, Y_T, Z_T) are the coordinates of the point in the target geocentric coordinate reference system. But that does not define the parameters uniquely; neither is the definition of the parameters implied in the formula, as is often believed. However, the following definition, which is consistent with the "Position Vector Transformation" convention (EPSG coordinate operation method code 9606), is common E&P survey practice, used by the International Association of Geodesy (IAG) and recommended by ISO 19111:

(dX, dY, dZ) :Translation vector, to be added to the point's position vector in the source coordinate reference system in order to transform from source system to target system; also: the coordinates of the origin of the source coordinate reference system in the target coordinate reference system.

(R_X, R_Y, R_Z) :Rotations to be applied to the point's vector. The sign convention is such that a positive rotation about an axis is defined as a clockwise rotation of the position vector when viewed from the origin of the Cartesian coordinate reference system in the positive direction of that axis; e.g. a positive rotation about the Z-axis only from source system to target system will result in a larger longitude value for the point in the target system. Although rotation angles may be quoted in any angular unit of measure, the formula as given here requires the angles to be provided in radians.

M : The scale correction to be made to the position vector in the source coordinate reference system in order to obtain the correct scale in the target coordinate reference system. $M = (1 + dS*10^{-6})$, where dS is the scale correction expressed in parts per million.

Example

This example combines the geographic/geocentric conversion of section 2.2.1 above with a position vector transformation.

Transformation from WGS 72 to WGS 84 (EPSG transformation code 1238). Transformation parameter values:

dX = 0.000 m
dY = 0.000 m
dZ = +4.5 m
R_X = 0.000 sec
R_Y = 0.000 sec
R_Z = +0.554 sec
dS = +0.219 ppm

Input point coordinate system: WGS 72 (geographic 3D)

Latitude φ_S = 55°00'00"N
Longitude λ_S = 4°00'00"E
Ellipsoidal height h_S = 0 m

Using the geographic to geocentric conversion method given in section 2.2.1, this converts to Cartesian geocentric coordinates:

X_S = 3 657 660.66 m
Y_S = 255 768.55 m
Z_S = 5 201 382.11 m

Application of the 7 parameter Position Vector Transformation (code 1238) results in:

X_T = 3 657 660.78 m
Y_T = 255 778.43 m
Z_T = 5 201 387.75 m

Using the reverse formulas for the geographic/geocentric conversion method given in section 2.2.1 this converts into:

Latitude φ_T = 55°00'00.090"N
Longitude λ_T = 4°00'00.554"E
Ellipsoidal height h_T = +3.22 m

on the WGS 84 geographic 3D coordinate reference system.

Although being common practice particularly in the European E&P industry, the Position Vector Transformation sign convention is not universally accepted. A variation on this formula is also used, particularly in the USA E&P industry. That formula is based on the same definition of translation and scale parameters, but a different definition of the rotation parameters. The associated convention is known as the "Coordinate Frame Rotation" convention (EPSG coordinate operation method code 9607). The formula is:

$$\begin{pmatrix} X_T \\ Y_T \\ Z_T \end{pmatrix} = M * \begin{pmatrix} 1 & +R_Z & -R_Y \\ -R_Z & 1 & +R_X \\ +R_Y & -R_X & 1 \end{pmatrix} * \begin{pmatrix} X_S \\ Y_S \\ Z_S \end{pmatrix} + \begin{pmatrix} dX \\ dY \\ dZ \end{pmatrix}$$

and the parameters are defined as:

(dX, dY, dZ) : Translation vector, to be added to the point's position vector in the source coordinate reference system in order to transform from source coordinate reference system to target coordinate reference system; also: the coordinates of the origin of source coordinate reference system in the target frame.

(R_X, R_Y, R_Z) : Rotations to be applied to the coordinate reference frame. The sign convention is such that a positive rotation of the frame about an axis is defined as a clockwise rotation of the coordinate reference frame when viewed from the origin of the Cartesian coordinate reference system in the positive direction of that axis, that is a positive rotation about the Z-axis only from source coordinate reference system to target coordinate reference system will result in a smaller longitude value for the point in the target coordinate reference system. Although rotation angles may be quoted in any angular unit of measure, the formula as given here requires the angles to be provided in radians.

M : The scale factor to be applied to the position vector in the source coordinate reference system in order to obtain the correct scale of the target coordinate reference system. $M = (1+dS*10^{-6})$, where dS is the scale correction expressed in parts per million.

In the absence of rotations the two formulas are identical; the difference is solely in the rotations. The name of the second method reflects this.

Note that the same rotation that is defined as positive in the first method is consequently negative in the second and vice versa. It is therefore crucial that the convention underlying the definition of the rotation parameters is clearly understood and is communicated when exchanging datum transformation parameters, so that the parameters may be associated with the correct coordinate transformation method (algorithm).

The same example as for the Position Vector Transformation can be calculated, however the following transformation parameters have to be applied to achieve the same input and output in terms of coordinate values:

Transformation parameters Coordinate Frame Rotation convention:
dX = 0.000 m
dY = 0.000 m
dZ = +4.5 m
R_X = 0.000 sec
R_Y = 0.000 sec
R_Z = – 0.554 sec
dS = +0.219 ppm

Please note that only the rotation has changed sign as compared to the Position Vector Transformation. The Position Vector convention is used by IAG and recommended by ISO 19111.

2.3.5 Molodenski-Badekas transformation

To eliminate high correlation between the translations and rotations in the derivation of parameter values for these Helmert transformation methods, instead of the rotations being derived about the geocentric coordinate reference system origin they may be derived at a location within the points used in the determination. Three additional parameters, the coordinates of the rotation point, are then required. The formula is:

$$\begin{pmatrix} X_T \\ Y_T \\ Z_T \end{pmatrix} = M * \begin{pmatrix} 1 & +R_Z & -R_Y \\ -R_Z & 1 & +R_X \\ +R_Y & -R_X & 1 \end{pmatrix} * \begin{pmatrix} X_S - X_P \\ Y_S - Y_P \\ Z_S - Z_P \end{pmatrix} + \begin{pmatrix} X_P \\ Y_P \\ Z_P \end{pmatrix} + \begin{pmatrix} dX \\ dY \\ dZ \end{pmatrix}$$

and the parameters are defined as:

(dX, dY, dZ) : Translation vector, to be added to the point's position vector in the source coordinate system in order to transform from source coordinate reference system to target coordinate reference system; also: the coordinates of the origin of source coordinate reference system in the target frame.

(R_X, R_Y, R_Z) : Rotations to be applied to the coordinate reference frame. The sign convention is such that a positive rotation of the frame about an axis is defined as a clockwise rotation of the coordinate reference frame when viewed from the origin of the Cartesian coordinate system in the positive direction of that axis, that is a positive rotation about the Z-axis only from source coordinate reference system to target coordinate reference system will result in a smaller longitude value for the point in the target coordinate reference system. Although rotation angles may be quoted in any angular unit of measure, the formula as given here requires the angles to be provided in radians.

(X_P, Y_P, Z_P) : Coordinates of the point about which the coordinate reference frame is rotated, given in the source Cartesian coordinate reference system.

M : The scale factor to be applied to the position vector in the source coordinate reference system in order to obtain the correct scale of the target coordinate reference system. $M = (1+dS*10^{-6})$, where dS is the scale correction expressed in parts per million.

Reversibility

The Molodensky-Badekas transformation in a strict mathematical sense is <u>not</u> reversible, i.e. in principle the same parameter values cannot be used to execute the reverse transformation. This is because the evaluation point coordinates are in the forward direction source coordinate reference system and the rotations have been derived about this point. They should not be applied about the point having the same coordinate values in the target coordinate reference system, as is required for the reverse transformation. However, in practical application there are exceptions when applied to the approximation of small differences between the geometry of a set of points in two different coordinate reference systems. The typical vector difference in coordinate values is in the order of $6*10^1$ to $6*10^2$ metres, whereas the evaluation point on or near the surface of the earth is $6.3*10^6$ metres from the origin of the coordinate systems at the Earth's centre. This difference of four or five orders of magnitude allows the transformation in practice to be considered reversible. Note that in the reverse transformation, only the signs of the translations and rotation parameter values are reversed; the coordinates of the evaluation point remain unchanged.

2.4 Coordinate Operation Methods that can be Conversions or Transformations

In theory, certain coordinate operation methods do not readily fit the ISO 19111 classification of being either a coordinate conversion (no change of datum involved) or a coordinate transformation. These methods change coordinates directly from one coordinate reference system to another and may be applied with or without change of datum, depending upon whether the source and target coordinate reference systems are based on the same or different datums. In practice, most usage of these methods does in fact include a change of datum. EPSG follows the general mathematical usage of these methods and classifies them as transformations.

2.4.1 Polynomial Transformations

Note: In the sections that follow, the general mathematical symbols X and Y representing the axes of a coordinate reference system must not be confused with the specific axis abbreviations or axis order in particular coordinate reference systems.

2.4.1.1 General Case

Polynomial transformations between two coordinate reference systems are typically applied in cases where one or both of the coordinate reference systems exhibits lack of homogeneity in orientation and scale. The small distortions are then approximated by polynomial functions in latitude and longitude or in easting and northing. Depending on the degree of variability in the distortions, approximation may be carried out using polynomials of degree 2, 3, or higher. In the case of transformations between two projected coordinate reference systems, the additional distortions resulting from the application of two map projections and a datum transformation can be included in a single polynomial approximation function.

Polynomial approximation functions themselves are subject to variations, as different approximation characteristics may be achieved by different polynomial functions. The simplest of all polynomials is the general polynomial function. In order to avoid problems of numerical instability this type of polynomial should be used after reducing the coordinate values in both the source and the target coordinate reference system to 'manageable' numbers, between –10 and +10 at most. This is achieved by working with offsets relative to a central evaluation point, scaled to the desired number range by applying a scaling factor to the coordinate offsets.

Hence an evaluation point is chosen in the source coordinate reference system (X_{S0}, Y_{S0}) and in the target coordinate reference system (X_{T0}, Y_{T0}). Often these two sets of coordinates do not refer to the same physical point but two points are chosen that have the same coordinate values in both the source and the target coordinate reference system. (When the two points have identical coordinates, these parameters are conveniently eliminated from the formulas, but the general case where the coordinates differ is given here).

The selection of an evaluation point in each of the two coordinate reference systems allows the point coordinates in both to be reduced as follows:

$X_S - X_{S0}$
$Y_S - Y_{S0}$

and

$X_T - X_{T0}$
$Y_T - Y_{T0}$

These coordinate differences are expressed in their own unit of measure, which may not be the same as that of the corresponding coordinate reference system.[2)]

[2)] If the source and/or the target coordinate reference system are geographical, the coordinates themselves may be expressed in sexagesimal degrees (degrees, minutes, seconds), which cannot be directly processed by a mathematical

A further reduction step is usually necessary to bring these coordinate differences into the desired numerical range by applying a scaling factor to the coordinate differences in order to reduce them to a value range that may be applied to the polynomial formulae below without introducing numerical precision errors:

$$U = m_S.(X_S - X_{S0})$$
$$V = m_S.(Y_S - Y_{S0})$$

where

X_S , Y_S are coordinates in the source coordinate reference system,
X_{S0} , Y_{S0} are coordinates of the evaluation point in the source coordinate reference system,
m_S is the scaling factor applied the coordinate differences in the source coordinate reference system.

The normalised coordinates U and V of the point whose coordinates are to be transformed are used as input to the polynomial transformation formula. In order to control the numerical range of the polynomial coefficients A_n and B_n the output coordinate differences dX and dY are multiplied by a scaling factor, m_T.

$$\begin{aligned}
m_T.dX \;=\; & A_0 + A_1.U + A_2.V + A_3.U^2 + A_4.U.V + A_5.V^2 && \text{(to degree 2)}\\
+\; & A_6.U^3 + A_7.U^2.V + A_8.U.V^2 + A_9.V^3 && \text{(degree 3 terms)}\\
+\; & A_{10}.U^4 + A_{11}.U^3.V + A_{12}.U^2.V^2 + A_{13}.U.V^3 + A_{14}.V^4 && \text{(degree 4 terms)}\\
+\; & A_{15}.U^5 + A_{16}.U^4.V + A_{17}.U^3.V^2 + A_{18}.U^2.V^3 + A_{19}.U.V^4 + A_{20}.V^5 && \text{(degree 5 terms)}\\
+\; & A_{21}.U^6 + A_{22}.U^5.V + A_{23}.U^4.V^2 + A_{24}.U^3.V^3 + A_{25}.U^2.V^4 + A_{26}.U.V^5 + && \text{(degree 6 terms)}\\
& A_{27}.V^6 &&\\
+\; & \ldots + A_{104}.V^{13} && \text{(degree 13 terms)}
\end{aligned}$$

$$\begin{aligned}
m_T.dY \;=\; & B_0 + B_1.U + B_2.V + B_3.U^2 + B_4.U.V + B_5.V^2 && \text{(to degree 2)}\\
+\; & B_6.U^3 + B_7.U^2.V + B_8.U.V^2 + B_9.V^3 && \text{(degree 3 terms)}\\
+\; & B_{10}.U^4 + B_{11}.U^3.V + B_{12}.U^2.V^2 + B_{13}.U.V^3 + B_{14}.V^4 && \text{(degree 4 terms)}\\
+\; & B_{15}.U^5 + B_{16}.U^4.V + B_{17}.U^3.V^2 + B_{18}.U^2.V^3 + B_{19}.U.V^4 + B_{20}.V^5 && \text{(degree 5 terms)}\\
+\; & B_{21}.U^6 + B_{22}.U^5.V + B_{23}.U^4.V^2 + B_{24}.U^3.V^3 + B_{25}.U^2.V^4 + B_{26}.U.V^5 + B_{27}.V^6 && \text{(degree 6 terms)}\\
+\; & \ldots + B_{104}.V^{13} && \text{(degree 13 terms)}
\end{aligned}$$

from which dX and dY are evaluated. These will be in the units of the target coordinate reference system.

In the EPSG dataset, the polynomial coefficients are given as parameters of the form Aumvn and Bumvn, where m is the power to which U is raised and n is the power to which V is raised. For example, A_{17} is represented as coordinate operation parameter Au3v2.

The relationship between the two coordinate reference systems can now be written as follows:

$$(X_T - X_{T0}) = (X_S – X_{S0}) + dX$$
$$(Y_T - Y_{T0}) = (Y_S – Y_{S0}) + dY$$

or

$$X_T = X_S – X_{S0} + X_{T0} + dX$$
$$Y_T = Y_S – Y_{S0} + Y_{T0} + dY$$

where:

X_T , Y_T are coordinates in the target coordinate reference system,
X_S , Y_S are coordinates in the source coordinate reference system,
X_{S0} , Y_{S0} are coordinates of the evaluation point in the source coordinate reference system,

formula.

X_{T0} , Y_{T0} are coordinates of the evaluation point in the target coordinate reference system,
dX, dY are derived through the scaled polynomial formulas.

Other (arguably better) approximating polynomials are described in mathematical textbooks such as *"Theory and applications of numerical analysis"*, by G.M. Phillips and P.J. Taylor (Academic Press, 1973).

Example: General polynomial of degree 6 (EPSG coordinate operation method code 9648)
For coordinate transformation TM75 to ETRS89 (1)

Ordinate 1 of evaluation point X_0 in source CRS: $X_{S0} = \varphi_{S0}$ = 53°30'00.000"N = +53.5 degrees
Ordinate 2 of evaluation point Y_0 in source CRS: $Y_{S0} = \lambda_{S0}$ = 7°42'00.000"W = – 7.7 degrees
Ordinate 1 of evaluation point X_0 in target CRS : $X_{T0} = \varphi_{T0}$ = 53°30'00.000"N = +53.5 degrees
Ordinate 2 of evaluation point Y_0 in target CRS : $Y_{S0} = \lambda_{T0}$ = 7°42'00.000"W = – 7.7 degrees
Scaling factor for source CRS coordinate differences: $m_S = 0.1$
Scaling factor for target CRS coordinate differences: $m_T = 3600$

Parameters:

$A_0 = 0.763$	$A_1 = -4.487$	….	$A_{24} = -265.898$	...	$A_{27} = 0$
$B_0 = -2.810$	$B_1 = -0.341$	….	$B_{24} = -853.950$	...	$B_{27} = 0$

Forward calculation for:

Latitude $\varphi_{TM75} = X_s$ = 55°00'00"N = + 55.000 degrees
Longitude $\lambda_{TM75} = Y_s$ = 6°30'00"W = – 6.500 degrees

$X_S - X_{S0} = \varphi_{TM75} - \varphi_{S0} = 55.0 - 53.5 = 1.5$ degrees
$Y_S - Y_{S0} = \lambda_{TM75} - \lambda_{S0} = -6.5 - (-7.7) = 1.2$ degrees

$U = m_S.(X_S - X_{S0}) = m_S.(\varphi_{TM75} - \varphi_{S0}) = 0.1*(1.5) = 0.15$
$V = m_S.(Y_S - Y_{S0}) = m_S.(\lambda_{TM75} - \lambda_{S0}) = 0.1*(1.2) = 0.12$

$dX = (A_0 + A_1.U + ... + A_{24}.U^3.V^3) / k_{TCD}$

$dX = \{0.763 + (-4.487 * 0.15) + ... + (-265.898 * 0.15^3 * 0.12^3)\}/3600$
$= 0.00003046$ degrees

$dY = (B_0 + B_1.U + ... + B_{24}.U^3.V^3) / k_{TCD}$

$dY = \{-2.81 + (-0.341 * 0.12) + ... + (-853.95 * 0.12^3 * 0.12^3)\}/3600$
$= -0.00094799$ degrees

Then Latitude $\varphi_{ETRS89} = X_T = X_S + dX = 55.0 + 0.00003046$ degrees = 55°00'10.9656"N
Longitude $\lambda_{ETRS89} = Y_T = Y_S + dY = -6.5 - 0.00094799$ degrees = 6°30'03.4128"W

2.4.1.2 Polynomial reversibility

Approximation polynomials are in a strict mathematical sense ***not reversible***, i.e. the same polynomial coefficients cannot be used to execute the reverse transformation.

In principle two options are available to execute the reverse transformation:

1. By applying a similar polynomial transformation with a different set of polynomial coefficients for the reverse polynomial transformation. This would result in a separate forward and reverse

transformation being stored in the EPSG database (or any other geodetic data repository).

2. By applying the polynomial transformation with the same coefficients but with their signs reversed and then iterate to an acceptable solution, the number of iteration steps being dependent on the desired accuracy. (Note that only the signs of the polynomial coefficients should be reversed and not the coordinates of the evaluation points or the scaling factors!) The iteration procedure is usually described by the information source of the polynomial transformation.

However, under certain conditions, described below, a satisfactory solution for the reverse transformation may be obtained using the forward coefficient values in a single step, rather than multiple step iteration. If such a solution is possible, the polynomial coordinate transformation method is classified by EPSG as a ***reversible polynomial of degree n***.

A (general) polynomial transformation is reversible only when the following conditions are met.

1. The co-ordinates of source and target evaluation point are (numerically) the same.
2. The unit of measure of the coordinate differences in source and target coordinate reference system are the same.
3. The scaling factors applied to source and target coordinate differences are the same.
4. The spatial variation of the differences between the coordinate reference systems around any given location is sufficiently small.

Clarification on conditions for polynomial reversibility:

Re 1 and 2 - In the reverse transformation the roles of the source and target coordinate reference systems are reversed. Consequently, the co-ordinates of the evaluation point in the *source* coordinate reference system become those in the *target* coordinate reference system in the reverse transformation. Usage of the same transformation parameters for the reverse transformation will therefore only be valid if the evaluation point coordinates are numerically the same in source and target coordinate reference system and in the same units. That is, $X_{S0} = X_{T0} = X_0$ and $Y_{S0} = Y_{T0} = Y_0$.

Re 3 - The same holds for the scaling factors of the source and target coordinate differences and for the units of measure of the coordinate differences. That is, $m_S = m_T = m$.

Re 4 - If conditions 1, 2 and 3 are all satisfied it then may be possible to use the forward polynomial algorithm with the forward parameters for the reverse transformation. This is the case if the spatial variations in dX and dY around any given location are sufficiently constant. The signs of the polynomial coefficients are then reversed but the scaling factor and the evaluation point coordinates retain their signs. If these spatial variations in dX and dY are too large, for the reverse transformation iteration would be necessary. It is therefore not the algorithm that determines whether a single step solution is sufficient or whether iteration is required, but the desired accuracy combined with the degree of spatial variability of dX and dY.

An example of a reversible polynomial is transformation is ED50 to ED87 (1) for the North Sea. The suitability of this transformation to be described by a reversible polynomial can easily be explained. In the first place both source and target coordinate reference systems are of type geographic 2D. The typical difference in coordinate values between ED50 and ED87 is in the order of 2 metres ($\approx 10^{-6}$ degrees) in the area of application. The polynomial functions are evaluated about central points with coordinates of 55° N, 0°E in both coordinate reference systems. The reduced coordinate differences (*in degrees*) are used as input parameters to the polynomial functions. The output coordinate differences are corrections to the input coordinate offsets of about *10^{-6} degrees*. This difference of several orders of magnitude between input and output values is the property that makes a polynomial function reversible in practice (although not in a formal mathematical sense).

The error made by the polynomial approximation formulas in calculating the reverse correction is of the same order of magnitude as the ratio of output versus input:

output error output value

$$\frac{\text{output error}}{\textit{input error}} \approx \frac{\text{output value}}{\textit{input value}} \quad (\approx 10^{-6})$$

As long as the input values (the coordinate offsets from the evaluation point) are orders of magnitude larger than the output (the corrections), and provided the coefficients are used with changed signs, the polynomial transformation may be considered to be reversible.

Hence EPSG acknowledges two classes of general polynomial functions, reversible and non-reversible, as distinguished by whether or not the coefficients may be used in both forward and reverse transformations, i.e. are reversible. EPSG does not describe the iterative solution as a separate algorithm. The iterative solution for the reverse transformation, when applicable, is deemed to be implied by the (forward) algorithm.

Example: Reversible polynomial of degree 4 (EPSG coordinate operation method code 9651)
For coordinate transformation ED50 to ED87 (1)

Ordinate 1 of evaluation point: $X_0 = \varphi_0$ = 55°00'00.000"N = +55 degrees
Ordinate 2 of evaluation point: $Y_0 = \lambda_0$ = 0°00'00.000"E = +0 degrees

Scaling factor for coordinate differences: m = 1.0

Parameters:

$A_0 = -5.56098E\text{-}06$	$A_1 = -1.55391E\text{-}06$	...	$A_{14} = -4.01383E\text{-}09$
$B_0 = +1.48944E\text{-}05$	$B_2 = +2.68191E\text{-}05$	...	$B_{14} = +7.62236E\text{-}09$

Forward calculation for:
Latitude $\varphi_{ED50} = X_s$ = 52°30'30"N = + 52.508333333 degrees
Longitude $\lambda_{ED50} = Y_s$ = 2°E = + 2.0 degrees

$U = m * (X_S - X_0) = m * (\varphi_{ED50} - \varphi_0) = 1.0 * (52.508333333 - 55.0) = -2.491666667$ degrees
$V = m * (Y_S - Y_0) = m * (\lambda_{ED50} - \lambda_0) = 1.0 * (2.0 - 0.0) = 2.0$ degrees

$dX = (A_0 + A_1.U + ... + A_{14}.V^4) / k_{CD}$
$= [-5.56098E\text{–}06 + (-1.55391E\text{-}06 * -2.491666667) + ... + (-4.01383E\text{-}09 * 2.0\verb|^|4)]/1.0$
$= -3.12958E\text{–}06$ degrees

$dY = (B_0 + B_1.U + ... + B_{14}.V^4) / k_{CD}$
$= [+1.48944E\text{–}05 + (2.68191E\text{-}05 * -2.491666667) + ... + (7.62236E\text{-}09 * 2.0\verb|^|4)]/1.0$
$= +9.80126E\text{–}06$ degrees

Then: Latitude $\varphi_{ED87} = X_T = X_S + dX$ = 52.508333333 – 3.12958E–06 degrees = 52°30'29.9887"N
Longitude $\lambda_{ED87} = Y_T = Y_S + dY$ = 2°00'00.0353"E

Reverse calculation for coordinate transformation ED50 to ED87 (1).
The transformation method for the ED50 to ED87 (1) coordinate transformation, 4th-order reversible polynomial, is reversible. The same formulas may be applied for the reverse calculation, but coefficients A_0 through A_{14} and B_0 through B_{14} are applied with reversal of their signs. Sign reversal is not applied to the coordinates of the evaluation point or scaling factor for coordinate differences. Thus:

Ordinate 1 of evaluation point: $X_0 = \varphi_0$ = 55°00'00.000"N = +55 degrees
Ordinate 2 of evaluation point: $Y_0 = \lambda_0$ = 0°00'00.000"E = +0 degrees
Scaling factor for coordinate differences: m = 1.0

$A_0 = +5.56098E\text{-}06$ $A_1 = +1.55391E\text{-}06$ ... $A_{14} = +4.01383E\text{-}09$

B_0 = –1.48944E-05 B_1 = –2.68191E-05 ... B_{14} = –7.62236E-09

Reverse calculation for:

Latitude φ_{ED87} = X_S = 52°30'29.9887"N = +52.5083301944 degrees
Longitude λ_{ED87} = Y_S = 2°00'00.0353"E = +2.0000098055 degrees

U = 1.0 * (52.5083301944 – 55.0) = – 2.4916698056 degrees
V = 1.0 * (2.0000098055 – 0.0) = 2.0000098055 degrees

$$dX = (A_0 + A_1.U + ... + A_{14}.V^4)/k$$
$$= [+5.56098E\text{-}06 + (1.55391E\text{–}06 * – 2.491666667) + ... + (4.01383E\text{-}09 * 2.0000098055\text{^}4)]/1.0$$
$$= +3.12957E\text{–}06 \text{ degrees}$$

$$dY = (B_0 + B_1.U + ... + B_{14}.V^4)/k$$
$$= [– 1.48944E\text{-}05 + (-2.68191E\text{-}05 * -2.491666667) + ... + (– 7.62236E\text{-}09 * 2.0000098055\text{^}4)]/1.0$$
$$= – 9.80124E\text{–}06 \text{ degrees}$$

Then: Latitude φ_{ED50} = $X_T = X_S + dX$ = 52.5083301944 + 3.12957E–06 degrees = 52°30'30.000"N
Longitude λ_{ED50} = $Y_T = Y_S + dY$ = = 2°00'00.000"E

2.4.1.3 Polynomial transformation with complex numbers

The relationship between two projected coordinate reference systems may be approximated more elegantly by a single polynomial regression formula written in terms of complex numbers. The advantage is that the dependence between the 'A' and 'B' coefficients (for U and V) is taken into account in the formula, resulting in fewer coefficients for the same order polynomial. A polynomial to degree 3 in complex numbers is used in Belgium. A polynomial to degree 4 in complex numbers is used in The Netherlands for transforming coordinates referenced to the Amersfoort / RD system to and from ED50 / UTM.

$$m_T.(dX + i.\ dY) = (A_1 + i.A_2).(U + i.V) + (A_3 + i.A_4).(U + i.V)^2 \quad \text{(to degree 2)}$$
$$+ (A_5 + i.A_6).(U + i.V)^3 \quad \text{(additional degree 3 terms)}$$
$$+ (A_7 + i.A_8).(U + i.V)^4 \quad \text{(additional degree 4 terms)}$$

where $U = m_S.(X_S - X_{S0})$
$V = m_S.(Y_S - Y_{S0})$

and m_S, m_T are the scaling factors for the coordinate differences in the source and target coordinate reference systems.

The polynomial to degree 4 can alternatively be expressed in matrix form as

$$\begin{pmatrix} m_T.dX \\ \\ m_T.dY \end{pmatrix} = \begin{pmatrix} +A_1 & -A_2 & +A_3 & -A_4 & +A_5 & -A_6 & +A_7 & -A_8 \\ \\ +A_2 & +A_1 & +A_4 & +A_3 & +A_6 & +A_5 & +A_8 & +A_7 \end{pmatrix} * \begin{pmatrix} U \\ V \\ U^2–V^2 \\ 2UV \\ U^3–3UV^2 \\ 3U^2V–V^3 \\ U^4–6U^2V^2 + V^4 \\ 4U^3V–4UV^3 \end{pmatrix}$$

Then as for the general polynomial case above

$X_T = X_S - X_{S0} + X_{T0} + dX$
$Y_T = Y_S - Y_{S0} + Y_{T0} + dY$

where, as above,

X_T , Y_T are coordinates in the target coordinate system,
X_S , Y_S are coordinates in the source coordinate system,
X_{S0} , Y_{S0} are coordinates of the evaluation point in the source coordinate reference system,
X_{T0} , Y_{T0} are coordinates of the evaluation point in the target coordinate reference system.

Note that the zero order coefficients of the general polynomial, A_0 and B_0, have apparently disappeared. In reality they are absorbed by the different coordinates of the source and of the target evaluation point, which in this case, are numerically very different because of the use of two different projected coordinate systems for source and target.

The transformation parameter values (the coefficients) are not reversible. For the reverse transformation a different set of parameter values are required, used within the same formulas as the forward direction.

Example: **Complex polynomial of degree 4** (EPSG coordinate operation method code 9653)
Projected coordinate transformation: Amersfoort / RD New to ED50 / UTM zone 31N (1):

EPSG coordinate transformation parameter name	**Formula symbol**	**Parameter value**	**Unit**
ordinate 1 of the evaluation point in the source CS	X_{S0}	155,000.000	metre
ordinate 2 of the evaluation point in the source CS	Y_{S0}	463,000.000	metre
ordinate 1 of the evaluation point in the target CS	X_{T0}	663,395.607	metre
ordinate 2 of the evaluation point in the target CS	Y_{T0}	5,781,194.380	metre
scaling factor for source CRS coordinate differences:	m_S	10^{-5}	
scaling factor for target CRS coordinate differences:	m_T	1.0	
A1	A_1	–51.681	coefficient
A2	A_2	+3,290.525	coefficient
A3	A_3	+20.172	coefficient
A4	A_4	+1.133	coefficient
A5	A_5	+2.075	coefficient
A6	A_6	+0.251	coefficient
A7	A_7	+0.075	coefficient
A8	A_8	-0.012	coefficient

For input point:

Easting, $X_{AMERSFOORT/RD} = X_S$ = 200,000.00 metres
Northing, $Y_{AMERSFOORT/RD} = Y_S$ = 500,000.00 metres

$U = m_S .(X_S – X_{S0}) = (200{,}000 – 155{,}000) .10^{-5} = 0.45$
$V = m_S .(Y_S – Y_{S0}) = (500{,}000 – 463{,}000) .10^{-5} = 0.37$

$dX = (–1{,}240.050) / 1.0$
$dY = (1{,}468.748) / 1.0$

Then: Easting, $\mathbf{E}_{ED50/UTM31} = X_T = X_S – X_{S0} + X_{T0} + dX$
$= 200{,}000 – 155{,}000 + 663{,}395.607 + (–1{,}240.050)$
$= 707{,}155.557$ metres

Northing, $\mathbf{N}_{ED50/UTM31N}$ $= Y_T = Y_S - Y_{S0} + Y_{T0} + dY$
$= 500,000 – 463,000 + 5,781,194.380 + 1,468.748$
$= 5,819,663.128$ metres

2.4.1.4 Polynomial transformation for Spain (EPSG coordinate operation method code 9617)
The original geographic coordinate reference system for the Spanish mainland was based on Madrid 1870 datum, Struve 1860 ellipsoid, with longitudes related to the Madrid meridian. Three second-order polynomial expressions have been empirically derived by El Servicio Geográfico del Ejército to transform geographical coordinates based on this system to equivalent values based on the European Datum of 1950 (ED50). The polynomial coefficients derived can be used to transform coordinates from the Madrid 1870 (Madrid) geographic coordinate reference system to the ED50 system. Three pairs of expressions have been derived: each pair is used to calculate the shift in latitude and longitude respectively for (i) a mean for all Spain, (ii) a better fit for the north of Spain, (iii) a better fit for the south of Spain.

The polynomial expressions are:

$d\varphi$ (arc sec) $= A_0 + (A_1*\varphi_s) + (A_2*\lambda_s) + (A_3*H_s)$
$d\lambda$ (arc sec) $= B_{00} + B_0 + (B_1*\varphi_s) + (B_2*\lambda_s) + (B_3*H_s)$

where latitude φ_s and longitude λ_s are in decimal degrees referred to the Madrid 1870 (Madrid) geographic coordinate reference system and H_s is gravity-related height in metres. B_{00} is the longitude (in seconds) of the Madrid meridian measured from the Greenwich meridian; it is the value to be applied to a longitude relative to the Madrid meridian to transform it to a longitude relative to the Greenwich meridian.

The results of these expressions are applied through the formulas:
$\varphi_{ED50} = \varphi_{M1870(M)} + d\varphi$
and $\lambda_{ED50} = \lambda_{M1870(M)} + d\lambda$.

Example:
Input point coordinate reference system: Madrid 1870 (Madrid) (geographic 2D)
Latitude φ_s = 42°38'52.77"N
= +42.647992 degrees

Longitude λ_s = 3°39'34.57"E of Madrid
= +3.659603 degrees from the Madrid meridian.

Gravity-related height H_s = 0 m

For the north zone transformation:

$A_0 = 11.328779$	$B_{00} = -13276.58$
$A_1 = -0.1674$	$B_0 = 2.5079425$
$A_2 = -0.03852$	$B_1 = 0.8352$
$A_3 = 0.0000379$	$B_2 = -0.00864$
	$B_3 = -0.0000038$

$d\varphi = +4.05$ seconds
Then latitude φ_{ED50} = 42°38'52.77"N + 4.05"
= 42°38'56.82"N

$d\lambda$ = -13270.54 seconds = -3°41'10.54"
Then longitude λ_{ED50} = 3°39'34.57"E - 3°41'10.54"
= 0°01'35.97"W of Greenwich.

2.4.2 Miscellaneous Linear Coordinate Operations

An affine 2D transformation is used for converting or transforming a coordinate reference system possibly with non-orthogonal axes and possibly different units along the two axes to an isometric coordinate reference system (i.e. a system of which the axes are orthogonal and have equal scale units). The transformation therefore involves a change of origin, differential change of axis orientation and a differential scale change. EPSG distinguishes four methods to implement this class of coordinate operation:

1) the parametric representation
2) the geometric representation, which is described in
   (a) a general case,
   (b) a simplified case in which the axes are constrained to be orthogonal, and
   (c) a further simplified case known as the Similarity Transformation.

2.4.2.1 The Affine Parametric Transformation (EPSG coordinate operation method code 9624)
Mathematical and survey literature usually provides the parametric representation of the affine transformation. The parametric algorithm is commonly used for rectification of digitised maps. It is often embedded in CAD software and Geographical Information Systems where it is frequently referred to as "rubber sheeting". Although the application of this algorithm falls outside the scope of the EPSG geodesy data set it is presented here for reasons of clarity.

The formula in matrix form is as follows:

$$\begin{pmatrix} X_T \\ Y_T \end{pmatrix} = \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} + \begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix} * \begin{pmatrix} X_S \\ Y_S \end{pmatrix}$$

or using algebraic coefficients:

$$X_T = A_0 + A_1 . X_S + A_2 . Y_S$$
$$Y_T = B_0 + B_1 . X_S + B_2 . Y_S$$

where

X_T, Y_T are the coordinates of a point P in the target coordinate reference system;
X_S, Y_S are the coordinates of P in the source coordinate reference system.

This form of describing an affine transformation is analogous to the general polynomial transformation formulas (section 3.1 above). Although it is somewhat artificial, an affine transformation could be considered to be a first order general polynomial transformation but without the reduction to source and target evaluation points.

Reversibility
The parameter values for an affine transformation cannot be used for the reverse operation. However, the reverse operation is another affine transformation using the same formulas but with different parameter values. The reverse parameter values, indicated by a prime ('), can be calculated from those of the forward operation as follows:

$D = A_1 . B_2 - A_2 . B_1$
$A_0' = (A_2 . B_0 - B_2 . A_0) / D$
$B_0' = (B_1 . A_0 - A_1 . B_0) / D$
$A_1' = +B_2 / D$

$A_2' = -A_2 / D$
$B_1' = -B_1 / D$
$B_2' = +A_1 / D$

2.4.2.2 The Affine General Geometric Transformation
(EPSG coordinate operation method code 9623)

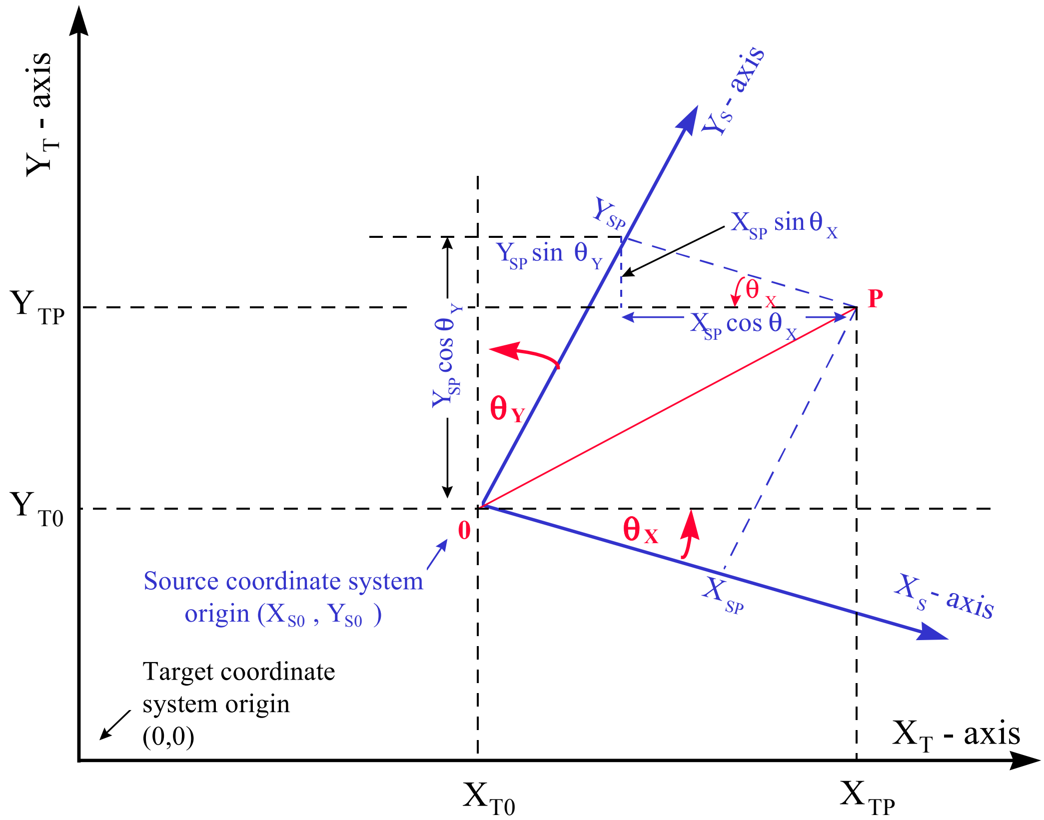

Figure 9 Geometric representation of the affine coordinate transformation
(Please note that to prevent cluttering of the figure the scale parameters of the X_s and Y_s axes have been omitted).

From the diagram above it can be seen that:

$$X_{TP} = X_{T0} + Y_{SP} . \sin \theta_Y + X_{SP} . \cos \theta_X$$
$$Y_{TP} = Y_{T0} + Y_{SP} . \cos \theta_Y - X_{SP} . \sin \theta_X$$

<u>Scale of the two coordinate reference systems</u>
The scaling of both source and target coordinate reference systems adds some complexity to this formula.

The operation will often be applied to transform an engineering coordinate reference system to a projected coordinate reference system. The engineering coordinate reference system, e.g. a seismic bin grid, may have different units of measure on its two axes. These have scale ratios of dS_X and dS_Y respective to the axes of the projected coordinate reference system.

The projected coordinate reference system is nominally defined to be in well-known units, e.g. metres. However, the distortion characteristics of the map projection only preserve true scale along certain defined lines or curves, hence the projected coordinate reference system's unit of measure is strictly speaking only valid along those lines or curves. Everywhere else its scale is distorted by the map projection. For conformal map projections the distortion at any point can be expressed by the point scale factor 'k' for that point. Please note that this point scale factor 'k' should NOT be confused with the scale factor at the natural origin of the projection, denominated by 'k_0'. For non-conformal map projections the scale distortion at a point is bearing-dependent and will not be described in this document.

It has developed as working practice to choose the origin of the source (engineering) coordinate reference system as the point in which to calculate this point scale factor 'k', although for engineering coordinate reference systems with a large coverage area a point in the middle of the coverage area may be a better choice.

Adding the scaling between each pair of axes and dropping the suffix for point P, after rearranging the terms we have the geometric representation of the general affine transformation:

$$X_T = X_{T0} + X_S . k.dS_X . \cos \theta_X + Y_S . k.dS_Y . \sin \theta_Y$$
$$Y_T = Y_{T0} - X_S . k.dS_X . \sin \theta_X + Y_S . k.dS_Y . \cos \theta_Y$$

or, in matrix form:

$$\begin{pmatrix} X_T \\ Y_T \end{pmatrix} = \begin{pmatrix} X_{T0} \\ Y_{T0} \end{pmatrix} + \begin{pmatrix} \cos \theta_X & \sin \theta_Y \\ -\sin \theta_X & \cos \theta_Y \end{pmatrix} * \begin{pmatrix} k.dS_X & 0 \\ 0 & k.dS_Y \end{pmatrix} * \begin{pmatrix} X_S \\ Y_S \end{pmatrix}$$

where:

X_{T0}, Y_{T0} = the coordinates of the origin point of the source coordinate reference system, expressed in the target coordinate reference system;
dS_X, dS_Y = the length of one unit of the source axis, expressed in units of the target axis, for the first and second source and target axis pairs respectively;
k = point scale factor of the target coordinate reference system at a chosen reference point;
θ_X, θ_Y = the angles about which the source coordinate reference system axes X_S and Y_S must be rotated to coincide with the target coordinate reference system axes X_T and Y_T respectively (counter-clockwise being positive).

Comparing the algebraic representation with the parameters of the parameteric form in section 2.3.7.1 above it can be seen that the parametric and geometric forms of the affine coordinate transformation are related as follows:

$A_0 = X_{T0}$
$A_1 = k.dS_X * \cos \theta_X$
$A_2 = k.dS_Y * \sin \theta_Y$
$B_0 = Y_{T0}$
$B_1 = -k.dS_X * \sin \theta_X$
$B_2 = k.dS_Y * \cos \theta_Y$

Reversibility
The parameters for an affine transformation cannot be used for the reverse transformation. However, the reverse coordinate operation is another affine transformation using the same formulas but with different parameters. The reverse parameter values can be calculated from the formulas provided above and applying those to the same algorithm. Alternatively the reverse operation can be described by a different formula, as shown below, using the same parameters as the forward transformation.

$$\begin{pmatrix} X_S \\ Y_S \end{pmatrix} = \frac{1}{k.D} * \begin{pmatrix} 1/dS_X & 0 \\ 0 & 1/dS_Y \end{pmatrix} * \begin{pmatrix} \cos \theta_Y & -\sin \theta_Y \\ \sin \theta_X & \cos \theta_X \end{pmatrix} * \begin{pmatrix} X_T - X_{T0} \\ Y_T - Y_{T0} \end{pmatrix}$$

where $D = \cos (\theta_X - \theta_Y)$;

or algebraically:
$X_S = [(X_T - X_{T0}) . \cos \theta_Y - (Y_T - Y_{T0}) . \sin \theta_Y] / [k.dS_X . \cos (\theta_X - \theta_Y)]$
$Y_S = [(X_T - X_{T0}) . \sin \theta_X + (Y_T - Y_{T0}) . \cos \theta_X] / [k.dS_Y . \cos (\theta_X - \theta_Y)]$

2.4.2.3 The Affine Orthogonal Geometric Transformation
(EPSG coordinate operation method code 9622)

If the source coordinate reference system happens to have orthogonal axes, that is both axes are rotated through the same angle to bring them into the direction of the orthogonal target coordinate reference system axes, i.e. $\theta_X = \theta_Y = \theta$, then the Affine General Geometric Transformation can be simplified to the Affine Orthogonal Geometric Transformation. In matrix form this is:

$$\begin{pmatrix} X_T \\ Y_T \end{pmatrix} = \begin{pmatrix} X_{T0} \\ Y_{T0} \end{pmatrix} + \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} * \begin{pmatrix} k.dS_X & 0 \\ 0 & k.dS_Y \end{pmatrix} * \begin{pmatrix} X_S \\ Y_S \end{pmatrix}$$

or algebraically:
$X_T = X_{T0} + X_S . k . dS_X . \cos \theta + Y_S . k . dS_Y . \sin \theta$
$Y_T = Y_{T0} - X_S . k . dS_X . \sin \theta + Y_S . k . dS_Y . \cos \theta$

where:

X_{T0}, Y_{T0} = the coordinates of the origin point of the source coordinate reference system, expressed in the target coordinate reference system;
dS_X , dS_Y = the length of one unit of the source axis, expressed in units of the target axis, for the X axes and the Y axes respectively;
k = point scale factor of the target coordinate reference system at a chosen reference point;
θ = the angle through which the source coordinate reference system axes must be rotated to coincide with the target coordinate reference system axes (counter-clockwise is positive). Alternatively, the bearing (clockwise positive) of the source coordinate reference system Y_S-axis measured relative to target coordinate reference system north.

For the reverse case the Affine General Geometric Transformation formulas given above can be similarly simplified by replacing θ_X and θ_Y with θ.

See below for example.

2.4.2.4 The Similarity Transformation (EPSG coordinate operation method code 9621)

If the source coordinate reference system happens to have axes of the same scale, that is both axes are scaled by the same factor to bring them into the scale of the target coordinate reference system axes (i.e. $dS_X = dS_Y = dS$) then the Affine Orthogonal Geometric Transformation can be simplified further to a Similarity Transformation.

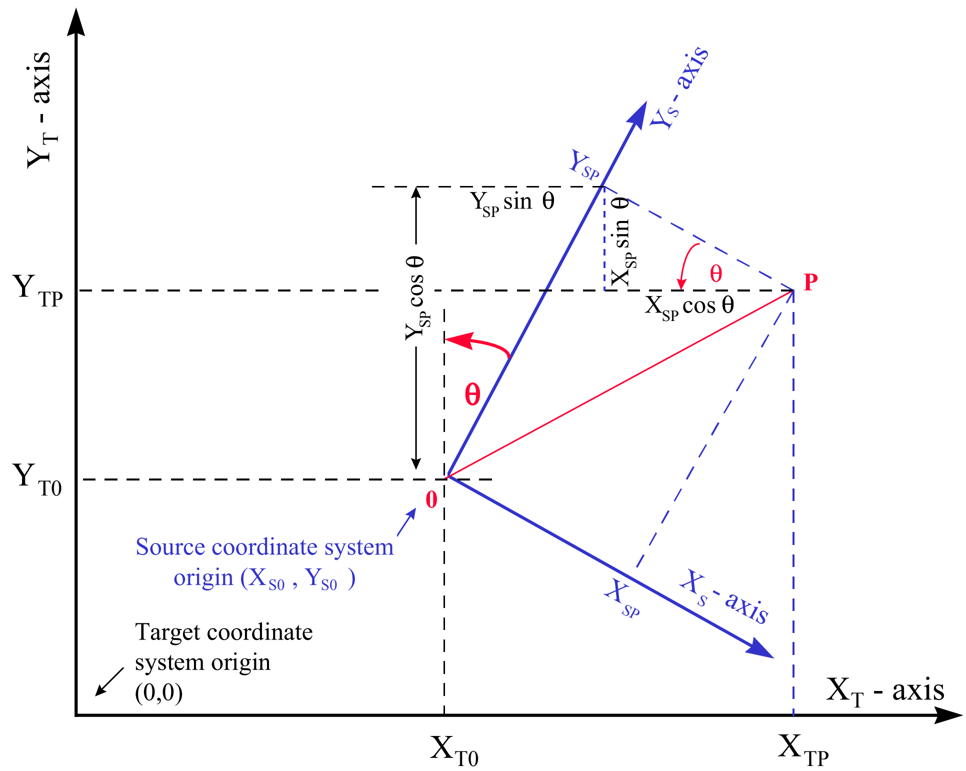

Figure 10 Similarity Transformation

From the above diagram the Similarity Transformation in algebraic form is:

$$X_{TP} = X_{T0} + Y_{SP}.\ dS\ .\ \sin\theta\ + X_{SP}.\ dS.\ \cos\theta$$
$$Y_{TP} = Y_{T0} + Y_{SP}.\ dS\ .\ \cos\theta\ - X_{SP}.\ dS.\ \sin\theta$$

Dropping the suffix for point P and rearranging the terms

$$X_T = X_{T0}\ + X_S.\ dS.\ \cos\theta\ + Y_S.\ dS\ .\ \sin\theta$$
$$Y_T = Y_{T0}\ - X_S.\ dS.\ \sin\theta\ + Y_S.\ dS\ .\ \cos\theta$$

or in matrix form:

$$\begin{pmatrix} X_T \\ Y_T \end{pmatrix} = \begin{pmatrix} X_{T0} \\ Y_{T0} \end{pmatrix} + (1 + dS) * \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} * \begin{pmatrix} X_S \\ Y_S \end{pmatrix}$$

where:

X_{T0} , Y_{T0} = the coordinates of the origin point of the source coordinate reference system expressed in the target coordinate reference system;

1+dS = the length of one unit in the source coordinate reference system expressed in units of the target coordinate reference system;

θ = the angle about which the axes of the source coordinate reference system need to be rotated to coincide with the axes of the target coordinate reference system, counter-clockwise being positive. Alternatively, the bearing of the source coordinate reference system Y_S-axis measured relative to target coordinate reference system north.

The Similarity Transformation can also be described as a special case of the Affine Parametric Transformation where coefficients $A_1 = B_2$ and $A_2 = -\ B_1$.

Reversibility

In contrast with an Affine Transformation, the Similarity Transformation parameters are reversible, but only on the condition that the scale difference between the two coordinate reference systems is small (order of several parts per million). Then dS is the deviation from unity of the ratio of the units of measure of the two coordinate reference systems. In these cases the reverse operation would require a scale correction of $1/(1+dS) \approx (1 - dS)$. This enables usage of the same scale and rotation parameters, but with reversed sign, for the reverse conversions. (The rotation angle of $+\theta$ becomes $-\theta$, which is valid for all θ). However for the reverse transformation the translation parameters, X_{T0}' and Y_{T0}', take entirely different values. Thus for all practical purposes the similarity conversion is not reversible.

When to use the Similarity Transformation

Similarity Transformations can be used when source and target coordinate reference systems
- each have orthogonal axes,
- each have the same scale along both axes,

and
- both have the same units of measure,

for example between engineering plant grids and projected coordinate reference systems.

A coordinate conversion between two systems that:
- each have orthogonal axes,

but:
- significantly different scales or units of measure,

should be defined as an Affine Orthogonal Transformation and not as a Similarity Transformation. This may be the case with seismic bin grids with square bins.

Coordinate Operations between two coordinate reference systems where in either system either the scale along axes differ or the axes are not orthogonal should be defined as an Affine General Transformation in either the parametric or geometric form.

Examples

Similarity Transformation method

Source coordinate reference system:	Astra Minas Grid (an engineering coordinate reference system)
Target coordinate reference system:	Campo Inchauspe / Argentina 2 (a projected CRS)

Note that for the Astra Minas Grid the coordinate axes are:
X (positive axis oriented ***north***)
Y (positive axis oriented ***west***)
and coordinates are quoted in that order.

whereas for Campo Inchauspe / Argentina 2 the axes are:
X (positive axis oriented ***north***)
Y (positive axis oriented ***east***)
and coordinates are quoted in that order.

Thus the Astra Minas grid X and Y axes map to the Campo Inchauspe / Argentina 2 Y and X-axes respectively. With respect to the symbols in the formulas,
X_S = Astra Minas *X*
Y_S = Astra Minas *Y*
X_T = Campo Inchauspe / Argentina 2 ***Y***
Y_T = Campo Inchauspe / Argentina 2 ***X***

Parameters of the Similarity Transformation:
X_{T0} = 2610200.48 metre

Y_{T0} = 4905282.73 metre
θ = 271°05' 30" = 271.0916667 degrees
k = 0 whence (1+k)=1.0

Forward calculation for Astra Minas point : X (north) =10000 m, Y (west) =50000 m.

X_S = Astra Minas X = 10000
Y_S = Astra Minas Y = 50000

Gauss-Kruger zone 2 Easting (Y) = $X_T = X_{T0} + X_S. dS. \cos\theta + Y_S. dS . \sin\theta$
= 2610200.48 + (50000 * 1.0 * cos(271.0916667deg))
+ (10000 * 1.0 * sin(271.0916667deg))
= 2601154.90 m.

Gauss-Kruger zone 2 Northing (X) =$Y_T = Y_{T0} - X_S. dS. \sin\theta + Y_S. dS . \cos\theta$
= 4905282.73 - (50000*1.0* sin(271.0916667deg))
+ (10000 * 1.0 * cos(271.0916667deg))
= 4955464.17 m.

<u>Affine Orthogonal Geometric Transformation method</u>
Source coordinate reference system: imaginary 3D seismic acquisition bin grid. The two axes are orthogonal, but the unit on the I-axis is 25 metres, whilst the unit on the J-axis is 12.5 metres.
The target projected coordinate reference system is WGS 84 / UTM Zone 31N and the origin of the bin grid (centre of bin 0,0) is defined at E = 456781.0, N = 5836723.0. The projected coordinate reference system point scale factor at the bin grid origin is 0.99984.
The map grid bearing of the I and J axes are 110° and 20° respectively. Thus the angle through which both the positive I and J axes need to be rotated to coincide with the positive Easting axis and Northing axis respectively is +20 degrees.

Hence:
X_{T0}, = 456 781.0 m
Y_{T0} = 5 836 723.0 m
dS_X = 25
dS_Y = 12.5
k = 0.99984
θ = +20 degrees

Forward calculation for centre of bin with coordinates: I = 300, J = 247:

X_T = Easting = $X_{T0} + X_S . k . dS_X . \cos\theta + Y_S . k . dS_Y . \sin\theta$ = 464 855.62 m.

Y_T = Northing = $Y_{T0} - X_S . k . dS_X . \sin\theta + Y_S . k . dS_Y . \cos\theta$ = 5 837 055.90 m

Reverse calculation for this point:
$X_S = [(X_T - X_{T0}) . \cos\theta_Y - (Y_T - Y_{T0}) . \sin\theta_Y] / [k . dS_X . \cos(\theta_X - \theta_Y)]$ = 230 bins

$Y_S = [(X_T - X_{T0}) . \sin\theta_X + (Y_T - Y_{T0}) . \cos\theta_X] / [k . dS_Y . \cos(\theta_X - \theta_Y)]$ = 162 bins