6.4.

1. $MSE(\hat{g})=E(\hat{g}-\theta)^2=E(\hat{g}-\tilde{g}+\tilde{g}-\theta)^2$

$$=E(\hat{g}-\tilde{g})^2+MSE(\tilde{g})+2E[(\hat{g}-\tilde{g})(\tilde{g}-\theta)]$$

注意到 $\tilde{g}=E(\hat{g}|T)$，这说明

$$E[(\hat{g}-\tilde{g})|T]=E(\hat{g}|T)-E[E(\hat{g}|T)|T]$$

$$=E(\hat{g}|T)-E(\hat{g}|T)=0$$

$$\therefore E[(\hat{g}-\tilde{g})(\tilde{g}-\theta)]=E\{E[(\hat{g}-\tilde{g})(\tilde{g}-\theta)|T]\}$$

$$=E\{(\tilde{g}-\theta)E[(\hat{g}-\tilde{g})|T]\}=0$$

$$\therefore MSE(\hat{g})=E(\hat{g}-\tilde{g})^2+MSE(\tilde{g})\geq MSE(\tilde{g})$$

4. 设 $\varphi(x_1,\dots x_n)$ 是0的任一无偏估计

$$\therefore \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\varphi\cdot(2\pi\sigma^2)^{-\frac{n}{2}}e^{-\frac{1}{2\sigma^2}\sum x_i^2+\frac{n\bar{x}}{\sigma^2}\mu-\frac{n\mu^2}{2\sigma^2}}\,dx_1\dots dx_n=0$$

两端对 μ 求导，有

$$E\left(\frac{n\bar{x}\varphi}{\sigma^2}\right)=0$$

$$\therefore E(\bar{x}\varphi)=0$$

$$Cov(\bar{x},\varphi)=E(\bar{x}\varphi)-E\bar{x}\cdot E\varphi=0$$

$\therefore \bar{x}$ 是 μ 的UMVUE

再求导，同理可得

$$E(\bar{x}^2\varphi)=0$$

进而与上同理可得

$$E(\varphi\cdot\sum_{i=1}^{n}x_i^2)=0 \quad \therefore E(S^2\varphi)=0$$

因而 $Cov(S^2,\varphi)=E(S^2\varphi)-E(S^2)E\varphi=0$

$\therefore S^2$ 是 σ^2 的UMVUE.

6. (1) $\ln L(\theta) = -n\ln g(\theta) + \left(\frac{1}{g(\theta)} - 1\right)\sum_{i=1}^{n}\ln x_i$

求导解得 $\hat{g}(\theta) = -\frac{1}{n}\sum_{i=1}^{n}\ln x_i$

(2) 令 $Y = -\ln X$

$\therefore P(Y<y) = P(-\ln x < y) = 1 - e^{-\theta y}$

$\therefore Y \sim Ga(1, \theta)$

$\therefore \hat{g}(\theta) \sim Ga(n, n\theta)$

$E\hat{g} = g(\theta) \quad Var(\hat{g}) = \frac{1}{n\theta^2}$

进而 $I(\theta) = \frac{1}{\theta^2}$

经验证

$\hat{g}(\theta) = -\frac{1}{n}\sum_{i=1}^{n}\ln x_i$ 是 $g(\theta)$ 的有效估计

10. $p(x;\lambda) = \frac{\lambda^a}{\Gamma(a)} x^{a-1} e^{-\lambda x},\ x > 0$

进而 $I(\lambda) = \frac{a}{\lambda^2}$

又 $\because E\left(\frac{\bar{x}}{a}\right) = \frac{1}{\lambda} \quad Var\left(\frac{\bar{x}}{a}\right) = \frac{1}{n a \lambda^2}$

$\therefore$ 证明了 $\frac{\bar{x}}{a}$ 是 $g(\lambda) = \frac{1}{\lambda}$ 的有效估计，即为 UMVUE

14.

(1) $p(x,\theta) = \left(\frac{1-\theta}{2}\right)^{\frac{1}{2}(x^2-x)}\left(\frac{1}{2}\right)^{1-x^2}\left(\frac{\theta}{2}\right)^{\frac{1}{2}(x^2+x)}$

解得 $\hat{\theta}_1 = \frac{1}{2} + \frac{\sum_{i=1}^{n} x_i}{2\sum_{i=1}^{n} x_i^2}$

进而有 $E\hat{\theta}_1 = \frac{1}{2} + \frac{1}{2}(\theta - \frac{1}{2})(1 - \frac{1}{2n})$

$\therefore$ 不是无偏估计

(2) $\because E(X_1) = \theta - \frac{1}{2}$

$\therefore \theta$ 的矩估计为 $\hat{\theta}_2 = \bar{X} + \frac{1}{2}$

(3) $\ln p(x;\theta) = \frac{1}{2}(x^2-x)\ln\frac{1-\theta}{2} - (1-x^2)\ln 2 + \frac{1}{2}(x^2+x)\ln\frac{\theta}{2}$

进而 $I(\theta) = \frac{1}{2\theta(1-\theta)}$

$\therefore$ 下界为 $\frac{2\theta(1-\theta)}{n}$

6.5

3.

(1) $\pi(\theta|x_1,\dots x_n)=\dfrac{h(x_1\dots x_n,\theta)}{\int h(x_1,\dots x_n,\theta)d\theta}=\dfrac{\Gamma(n+\sum_{i=1}^{n}x_i+2)}{\Gamma(n+1)\Gamma(\sum_{i=1}^{n}x_i+1)}\theta^n(1-\theta)^{\sum_{i=1}^{n}x_i}$

∴后验分布为

$Be(n+1,\sum_{i=1}^{n}x_i+1)$

(2) $\hat{\theta}_B=\dfrac{5}{5+15}=0.25$

6.

(1) $\pi(\theta|x_1\dots x_n)=\dfrac{\theta^{-2n}}{\int_{x_{(n)}}^{1}\theta^{-2n}d\theta}=\dfrac{2n-1}{\theta^{2n}(x_{(n)}^{-2n+1}-1)}$

(2) $\pi(\theta|x_1\dots x_n)=\dfrac{\theta^{-2n+2}}{\int_{x_{(n)}}^{1}\theta^{-2n+2}d\theta}=\dfrac{2n-3}{\theta^{2n-2}(x_{(n)}^{-2n+3}-1)}$

8. (1)

$h(x_1\dots x_n,\theta)=\dfrac{1}{\theta^n}\cdot\dfrac{\beta\theta_0^{\beta}}{\theta^{\beta+1}},\ \theta>\theta_0,\ x_{(n)}<\theta$

$\therefore\pi(\theta|x_1,\dots x_n)=\dfrac{\frac{1}{\theta^n}\cdot\frac{\beta\theta_0^{\beta}}{\theta^{\beta+1}}}{\int_{\max x_{(n)},\theta_0}^{+\infty}\frac{1}{\theta^n}\cdot\frac{\beta\theta_0^{\beta}}{\theta^{\beta+1}}d\theta}$

$=\dfrac{(n+\beta)[\max\{x_{(n)},\theta_0\}]^{n+\beta}}{\theta^{n+\beta+1}},\quad\theta>\max\{x_{(n)},\theta_0\}$

这是一个参数为 $n+\beta+1$ 与 $\max(x_{(n)},\theta_0)$ 的帕雷托分布

因此帕雷托分布是θ的共轭先验分布

(2) $\hat{\theta}_B=\int\theta\cdot\pi(\theta|x_1,\dots x_n)d\theta$

$=\dfrac{(n+\beta)\max\{x_{(n)},\theta_0\}}{n+\beta-1}$