习题5.4

4.

$$\sum_{i=1}^{20}\frac{(X_i-\mu)^2}{\sigma^2}\sim\chi^2(20)$$

而 $P(10\sigma^2\le\sum_{i=1}^{20}(X_i-\mu)^2\le 30\sigma^2)$

$=P(10\le\frac{\sum_{i=1}^{20}(X_i-\mu)^2}{\sigma^2}\le 30)$

$=0.8983$
(查表)

10. $\because A=\left(\frac{X_1+X_2}{X_1-X_2}\right)^2\sim F(1,1)$

$B=\frac{(X_1+X_2)^2}{(X_1-X_2)^2+(X_1+X_2)^2}=\frac{A}{1+A}$

$P(B>k)=P(A>\frac{k}{1-k})=0.05$

$\therefore k=0.9938$

11.

$c(\overline{X}-\mu_1)\sim N(0,\frac{c^2\sigma^2}{n})$

$d(\overline{Y}-\mu_2)\sim N(0,\frac{d^2\sigma^2}{m})$

$\frac{(n-1)S_x^2}{\sigma^2}\sim\chi^2(n-1)$

$\frac{(m-1)S_y^2}{\sigma^2}\sim\chi^2(m-1)$

且 $\overline{X},\overline{Y},S_x^2,S_y^2$ 相互独立

故：$c(\overline{X}-\mu_1)+d(\overline{Y}-\mu_2)\sim N(0,\frac{c^2\sigma^2}{n}+\frac{d^2\sigma^2}{m})$

$\frac{(n+m-2)S_w^2}{\sigma^2}=\frac{(n-1)S_x^2}{\sigma^2}+\frac{(m-1)S_y^2}{\sigma^2}\sim\chi^2(n+m-2)$

$\therefore t=\frac{c(\overline{X}-\mu_1)+d(\overline{Y}-\mu_2)}{S_w\sqrt{\frac{c^2}{n}+\frac{d^2}{m}}}\sim t(n+m-2)$

19.

$F(X)\sim U(0,1)$

则 $Z=-\ln Y\sim\chi^2(2)$

$U=-2\sum_{i=1}^{n}\ln F(X_i)\sim\chi^2(2n)$

5.5

9. (1) $p(x_1, \cdots x_n; \theta) = \prod_{i=1}^{n} \binom{x_i + r - 1}{r - 1} \theta^r (1-\theta)^{x_i}$

$$= \prod_{i=1}^{n} \binom{x_i + r - 1}{r - 1} \theta^{nr} (1-\theta)^{\sum_{i=1}^{n} x_i}$$

(2) $p(x_1, \dots x_n; m) = \frac{1}{m^n} I\{x_{(1)} > 1\} I\{x_{(n)} \le m\}$

(3) $p(x_1 \dots x_n, \mu, \sigma^2) = \left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi} x_i}\right) \cdot \sigma^{-n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} \ln^2 x_i + \frac{\mu}{\sigma^2}\sum_{i=1}^{n} \ln x_i - \frac{n\mu^2}{2\sigma^2}}$

(4) $p(x_1, \dots x_n; \lambda) = \prod_{i=1}^{n} [2\lambda x_i e^{-\lambda x_i^2} \cdot I\{x_i \ge 0\}] = 2^n (\prod_{i=1}^{n} x_i) \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i^2} I\{x_{(1)} \ge 0\}$

12.

$$p(x;\theta) = \begin{cases} \frac{1}{\theta} & \theta < x < 2\theta \\ 0 & \text{其他} \end{cases}$$

$\therefore p(x_1, \dots x_n; \theta) = \left(\frac{1}{\theta}\right)^n I_{\theta < x_{(1)} < x_{(n)} < 2\theta}$

令 $t_1 = x_{(1)}$, $t_2 = x_{(n)}$ 并取

$g(t;\theta) = \left(\frac{1}{\theta}\right)^n I_{\theta < t_1 < t_2 < 2\theta}$, $h(x) = 1$,

则 $T(t_1, t_2) = (x_{(1)}, x_{(n)})$ 为θ的充分统计量

15. $p(x_1, \dots x_n; \theta) = (C(\theta))^n e^{\sum_{j=1}^{n}\sum_{i=1}^{k} Q_i(\theta) T_i(x_j)} \prod_{j=1}^{n} h(x_j)$

由因子分解定理

$T(x) = \left(\sum_{j=1}^{n} T_1(x_j), \dots \sum_{j=1}^{n} T_k(x_j)\right)$ 为充分统计量

19. $\because p(x_1, \cdots x_n; \theta, \mu) = \left(\frac{1}{\theta}\right)^n e^{-\sum_{i=1}^{n} \frac{x_i - \mu}{\theta}} I_{x_{(1)} > \mu} = \left(\frac{1}{\theta}\right)^n e^{-\frac{n\bar{x} - n\mu}{\theta}} I_{x_{(1)} \ge \mu}$

令 $g(t;\theta) = \left(\frac{1}{\theta}\right)^n e^{-\frac{n\bar{x} - n\mu}{\theta}} I_{x_{(1)} > \mu}$, $h(x) = 1$

$\therefore (\bar{x}, x_{(1)})$ 是 (μ, θ)的充分统计量

6.2. 3. (1) $E(X) = \frac{N-1}{2}$

$\therefore N = 2E(X) + 1$

$\therefore \hat{N} = 2\bar{x} + 1$

(2) $E(X) = \theta^2 \sum_{k=2}^{+\infty} k(k-1)(1-\theta)^{k-2} = \frac{2}{\theta}$

$\therefore \theta = \frac{2}{E(X)}$ $\quad \hat{\theta} = \frac{2}{\bar{x}}$

4. (1) $E(X)=\int_0^{\theta}\frac{2}{\theta^2}x(\theta-x)dx=\frac{1}{3}\theta$

$\theta=3E(X)$

$\hat{\theta}=3\overline{X}$

(2) $E(X)=\int_0^1 x(\theta+1)x^{\theta}dx=\frac{\theta+1}{\theta+2}$

$\therefore\theta=\frac{1-2E(X)}{E(X)-1}$ $\therefore\hat{\theta}=\frac{1-2\overline{X}}{\overline{X}-1}$

(3) $E(X)=\int_0^1 x\sqrt{\theta}x^{\sqrt{\theta}-1}dx=\frac{\sqrt{\theta}}{\sqrt{\theta}+1}$

$\therefore\theta=\left(\frac{E(X)}{1-E(X)}\right)^2$

$\therefore\hat{\theta}=\left(\frac{\overline{X}}{1-\overline{X}}\right)^2$

(4) $E(X)=\int_{\mu}^{+\infty}x\frac{1}{\theta}e^{-\frac{x-\mu}{\theta}}=\int_0^{+\infty}t\frac{1}{\theta}e^{-\frac{t}{\theta}}dt+\int_0^{+\infty}\frac{1}{\theta}\mu e^{-\frac{t}{\theta}}dt$

$=\theta+\mu$

$E(X^2)=\int_{\mu}^{+\infty}x^2\frac{1}{\theta}e^{-\frac{x-\mu}{\theta}}dx=\int_0^{+\infty}(t+\mu)^2\frac{1}{\theta}e^{-\frac{t}{\theta}}dt$

$=2\theta^2+2\mu\theta+\mu^2$

$Var(X)=\theta^2$

$\therefore\hat{\theta}=S$

$\hat{\mu}=\overline{X}-S$

6.3 1. (1) $L(\theta)=(\sqrt{\theta})^n(x_1\ldots x_n)^{\sqrt{\theta}-1}$

$\therefore\ln L(\theta)=\frac{n}{2}\ln\theta+(\sqrt{\theta}-1)(\ln x_1+\ldots\ln x_n)$

求导解得 $\hat{\theta}=\left(\frac{1}{n}\sum_{i=1}^{n}\ln x_i\right)^{-2}$

又$\because\frac{\partial^2\ln L(\theta)}{\partial\theta^2}<0$ $\therefore\hat{\theta}$是θ的最大似然估计

(2) $L(\theta)=\theta^n C^{n\theta}(x_1\ldots x_n)^{-(\theta+1)}$

$\therefore\ln L(\theta)=n\ln\theta+n\theta\ln c-(\theta+1)(\ln x_1+\ldots\ln x_n)$

求导解得

$\hat{\theta}=\left(\frac{1}{n}\sum_{i=1}^{n}\ln x_i-\ln c\right)^{-1}$

又$\because\frac{\partial^2\ln L(\theta)}{\partial\theta^2}<0$ $\therefore\hat{\theta}$是θ的最大似然估计

3. (1) $\ln L(\theta) = -n\ln 2\theta - \frac{\sum_{i=1}^{n}|x_i|}{\theta}$

$\therefore \hat{\theta} = \frac{\sum_{i=1}^{n}|x_i|}{n}$

经验证，确为最大似然

(2) $\because$ $L(\theta)$只有两个值：0、1

当 $x_{(n)}-\frac{1}{2}<\theta<x_{(1)}+\frac{1}{2}$ 时似然函数取 1

$\therefore$ 不止一个最大似然

(3) $L(\theta) = \frac{1}{(\theta_2-\theta_1)^n} I_{\theta_1<x_{(1)}\le x_{(n)}<\theta_2}$

$\therefore$ $\theta_1 \Leftarrow x_{(1)}$

$\theta_2 \Leftarrow x_{(n)}$