1. 解: (1). $\delta J = \int_0^1 \frac{\partial (y^3 \sin x)}{\partial y} dx = 3\int_0^1 y^2(x)\sin x \cdot \delta y(x)\, dx$

(2). $\delta J = \int_0^1 \frac{\partial (y^3 x^2)}{\partial y} \cdot \delta y + \frac{\partial (y^3 x^2)}{\partial x} \delta x \, dt$

$= \int_0^1 3y^2(t) x^2(t)\, \delta y(t) + 2y^3(t) x(t)\, \delta x(t)\, dt$

2. 解: $H = L + \lambda f = \frac{1}{2}(3x^2 + u^2) + \lambda(-x+u)$

$\frac{\partial H}{\partial u} = u + \lambda = 0 \quad \therefore u = -\lambda \quad \dot{\lambda} = -\frac{\partial H}{\partial x} = -3x + \lambda \quad \dot{x} = -x + u.$

边界条件: $x(0) = x_0 \quad \lambda(1) = 0$

解方程如下: $\dot{x} = -x + u = -x - \lambda \quad \therefore \lambda = -x - \dot{x} \quad \therefore \dot{\lambda} = -\dot{x} - \ddot{x}$

$\therefore -\dot{x} - \ddot{x} = -3x - x - \dot{x}$ 即. $\ddot{x} = 4x$

$\therefore x(t) = c_1 e^{2t} + c_2 e^{-2t} \quad \therefore \dot{x}(t) = 2c_1 e^{2t} - 2c_2 e^{-2t}$

$\therefore 2c_1 e^{2t} - 2c_2 e^{-2t} = -c_1 e^{2t} - c_2 e^{-2t} - \lambda$

代入 $x(0) = x_0$. $\lambda(1) = 0$ 有:

$$\begin{cases} c_1 + c_2 = x_0 \\ 2c_1 \cdot e^2 - 2c_2 \cdot e^{-2} = -c_1 e^2 - c_2 e^{-2} \end{cases}$$

可解得: $\begin{cases} c_1 = \frac{x_0}{3e^4 + 1} \\ c_2 = \frac{3x_0}{e^{-4} + 3} \end{cases}$

$\therefore u(t) = \dot{x}(t) + x(t) = 3c_1 e^{2t} - c_2 e^{-2t} = \frac{3x_0}{3e^4+1} e^{2t} - \frac{3x_0}{e^{-4}+3} e^{-2t}$

3. 解: $H = L + \lambda f = x^2 + 4u^2 + 4\lambda u$

$\frac{\partial H}{\partial u} = 8u + 4\lambda = 0 \quad \therefore u = -\frac{\lambda}{2} \quad \dot{\lambda} = -\frac{\partial H}{\partial x} = -2x \quad \dot{x} = 4u$

边界条件: $x(0) = x_0 \quad x(T) = x_T$

解方程如下: $\dot{x} = 4u = -2\lambda \quad \therefore \dot{\lambda} = -\frac{\ddot{x}}{2} \quad \therefore -\frac{\ddot{x}}{2} = -2x$ 即 $\ddot{x} = 4x$

$\therefore x(t) = c_1 e^{2t} + c_2 e^{-2t}$

代入 $x(0) = x_0$. $x(T) = x_T$ 有:

$$\begin{cases} c_1 + c_2 = x_0 \\ c_1 e^{2T} + c_2 e^{-2T} = x_T \end{cases}$$

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可解得：
$$\begin{cases} C_1 = \dfrac{x_{0T} - x_0 e^{-2T}}{e^{2T} - e^{-2T}} \\ C_2 = \dfrac{x_0 e^{2T} - x_T}{e^{2T} - e^{-2T}} \end{cases}$$

$$\therefore u(t) = \frac{\dot{x}(t)}{4} = \frac{C_1}{2} e^{2t} - \frac{C_2}{2} e^{-2t} = \frac{e^{2t}(x_T - x_0 e^{-2T}) - e^{-2t}(x_0 e^{2T} - x_T)}{2(e^{2T} - e^{-2T})}$$

4. 解：

$$H = L + \lambda f = u^2 + \lambda u$$

$$\frac{\partial H}{\partial u} = 2u + \lambda = 0 \quad \therefore u = -\frac{\lambda}{2} \quad \dot{\lambda} = -\frac{\partial H}{\partial x} = 0 \quad \dot{x} = u$$

边界条件：$x(0) = 1$，$x(t_f) = 0$（t_f 未知），$H(t_f) = -\dfrac{\partial \phi}{\partial t_f} = -2t_f$

解方程如下：$\dot{x} = u = -\dfrac{\lambda}{2}$ $\therefore \lambda = -2\dot{x}$ $\therefore \dot{\lambda} = -2\ddot{x} = 0$ $\therefore \ddot{x} = 0$

$\therefore x(t) = C_1 t + C_2$ $\therefore x(0) = C_2 = 1$ $\therefore x(t) = C_1 t + 1$

$\therefore \dot{x}(t) = u(t) = C_1$ $\because u = -\dfrac{\lambda}{2}$ $\therefore \dot{u} = -\dfrac{\dot{\lambda}}{2} = 0$ $\therefore u(t)$ 或定.

$\therefore H = C_1^2 + C_1 \lambda$ $\therefore H(t_f) = C_1^2 + C_1 \cdot (-2u) = C_1^2 - 2C_1^2 = -C_1^2$

$\therefore -C_1^2 = -2t_f$ $\therefore C_1 = -\sqrt{2t_f}$

$\therefore x(t_f) = -\sqrt{2t_f} \cdot t_f + 1 = 0$ $\therefore t_f = 2^{-\frac{1}{3}}$，$u(t) = -2^{\frac{1}{3}}$

5. 解：

$$H = L + \lambda f = u^2 + \lambda_1 x_2 + \lambda_2 x_3 + \lambda_3 u$$

正则方程：$\dot{\lambda}_1 = -\dfrac{\partial H}{\partial x_1} = 0$ $\dot{\lambda}_2 = -\dfrac{\partial H}{\partial x_2} = -\lambda_1$ $\dot{\lambda}_3 = -\dfrac{\partial H}{\partial x_3} = -\lambda_2$　$\dot{x}_1 = x_2$，$\dot{x}_2 = x_3$，$\dot{x}_3 = u$

$|u(t)| \le 1$

边界条件：$x_1(0) = x_2(0) = x_3(0) = 0$ $\lambda_1(t_f) = 2x_1(t_f)\mu_1$ $\lambda_2(t_f) = t_f + \mu_2$ $\lambda_3(t_f) = -2x_3(t_f)\mu_2$

$$H(t_f) = -\frac{\partial \phi}{\partial t_f} - \frac{\partial g^T}{\partial t_f}\mu = -x_2(t_f) - t_f \cdot \frac{\partial x_2(t_f)}{\partial t_f} - \mu_1\left(\frac{\partial x_1^2(t_f)}{\partial t_f} - 2t_f\right) - \mu_2\left(\frac{\partial x_2(t_f)}{\partial t_f} - \frac{\partial x_3^2(t_f)}{\partial t_f}\right)$$

以上即为求解的全部必要条件

6. 解：

$$H = L + \lambda f = 2x_1^2 + \frac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2 u. \quad \frac{\partial H}{\partial u} = u + \lambda_2 = 0$$

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = -4x_1 \quad \dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -\lambda_1 \quad \dot{x}_1 = x_2 \quad \dot{x}_2 = u$$

后来发现6、7题应该用ppt中讲到的无限时间最优调节器来解，也进行了尝试，发现两种方法得出的结果一致。

边界条件：$x_1(0) = x_2(0) = 2$. $t_f = \infty$

解方程如下：$\because \dot{x}_2 = u$ $\therefore \ddot{x}_1 = \dot{x}_2 = u$ $\therefore \ddot{x}_1 + \lambda_2 = 0$ $\therefore \dddot{x}_1 + \dot{\lambda}_2 = 0$

$\therefore \dddot{x}_1 = \lambda_1$ $\therefore \ddddot{x}_1 = \dot{\lambda}_1 = -4x_1$ 即 $x_1^{(4)} = -4x_1$

$$\therefore x_1(t) = C_1 e^{-t}\cos t + C_2 e^{-t}\sin t + C_3 e^{t}\cos t + C_4 e^{t}\sin t.$$

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$\because x_1(0)=2$，$\therefore C_1+C_3=2$ 又$\because$ J最小，$\therefore \lim\limits_{t\to\infty} x(t)=0$.

$\therefore C_3=C_4=0 \quad \therefore C_1=2$

$\therefore \dot{x}_1(t)=-2e^{-t}\sin t-2e^{-t}\cos t+C_2e^{-t}\cos t-C_2e^{-t}\sin t=x_2(t)$

$\because x_2(0)=2 \quad \therefore -2+C_2=2 \quad \therefore C_2=4$

$\therefore x_2(t)=-2e^{-t}\sin t-2e^{-t}\cos t+4e^{-t}\cos t-4e^{-t}\sin t$

$=2e^{-t}\cos t-6e^{-t}\sin t$

$\therefore u(t)=\dot{x}_2(t)=-2e^{-t}\sin t-2e^{-t}\cos t-6e^{-t}\cos t+6e^{-t}\sin t$

$=4e^{-t}\sin t-8e^{-t}\cos t$

$x_1(t)=2e^{-t}\cos t+4e^{-t}\sin t$ $\therefore$ 可得最优控制律为 $u^*(t)=(2\quad 2)\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}$
$=-(2\ 2)x(t)$.

$\therefore J=\frac{1}{2}\int_0^\infty[4x_1^2(t)+u^2(t)]dt=\frac{1}{2}\int_0^\infty 16e^{-2t}\cos^2 t+64e^{-2t}\sin^2 t$
$+16e^{-2t}\sin^2 t+64e^{-2t}\cos^2 t\ dt$

$=40\int_0^\infty e^{-2t}(\sin^2 t+\cos^2 t)dt=40\int_0^\infty e^{-2t}=20$

7. 解: $H=L+\lambda f=\frac{1}{2}x^2+\frac{\rho}{2}u^2+\lambda x+\lambda u \quad \frac{\partial H}{\partial u}=\rho u+\lambda=0$.

$\dot{\lambda}=-\frac{\partial H}{\partial x}=-x-\lambda \quad \dot{x}=x+u$ 边界条件: $x(t_0)=x_0 \quad t_f=\infty$

解方程如下: $\lambda=-\rho u=-\rho(\dot{x}-x) \quad \therefore \dot{\lambda}=-\rho(\ddot{x}-\dot{x})$

$\therefore -\rho(\ddot{x}-\dot{x})=-x+\rho(\dot{x}-x)$ 即: $\ddot{x}=\frac{\rho+1}{\rho}x$

① $\rho=0$ 时. $\lambda=0$, $\dot{\lambda}=0$ $\therefore x=0\neq x_0$，舍去.

② $\rho<0$ 时. 易知J可无穷小，最优反馈无解

③ $\rho>0$ 时, $x(t)=C_1e^{kt}+C_2e^{-kt}$ 其中 $k=\sqrt{\frac{\rho+1}{\rho}}>0$.

$\because$ J最小 $\therefore \lim\limits_{t\to\infty}x(t)=0$ $\therefore C_1=0$. $u(t)=\dot{x}(t)-x(t)=-kC_2e^{-kt}-C_2e^{-kt}$

$\because x(t_0)=x_0 \therefore C_2=x_0e^{kt_0} \quad \therefore x(t)=x_0e^{k(t_0-t)}$

$u(t)=-(k+1)x_0e^{k(t_0-t)}$，其中 $k=\sqrt{\frac{\rho+1}{\rho}}$

$\therefore$ 最优控制律为:
$u^*(t)=-(1+\sqrt{\frac{\rho+1}{\rho}})x(t)$. 闭环系统响应与$\rho$关系: $x(t)=x_0e^{\sqrt{\frac{\rho+1}{\rho}}(t_0-t)}$