离散时间控制系统

为什么要研究离散时间系统

- 现代计算机控制的普遍需求
- 从新的视角理解自动控制原理

（因果系统、能控能观、最优控制等）

本章内容

— automatic control — automatic control — automatic control — automatic control — automatic control —

- 离散时间控制系统
- 采样与保持
- 离散时间系统的状态空间模型
- 离散时间系统的最优控制
- 离散时间系统 LQR 问题

离散时间控制系统

1. 离散时间过程
2. 采样控制过程

离散时间系统

动态系统的演化常常建立在离散时间之上，例如

- 疫情传播模型，时间以天为单位
- 人口模型，时间以年为单位
- 数字控制系统，时间以采样周期为单位
- 深度神经网络，“时间”以“层”为单位

举例：疫情传播模型

— automatic control — automatic control — automatic control — automatic control —

记 $\boldsymbol{I(k)}$为第$\boldsymbol{k}$天的感染总人数，每个患者每天传染$\boldsymbol{\rho}$人:

$$\boldsymbol{I(k+1)=(1+\rho)I(k) \Rightarrow I(k)=(1+\rho)^k I(0)}$$

上述模型不合理，因为总人口是有限的

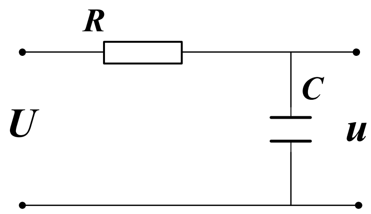
- SI（Susceptible-Infectious）模型

记总人数为N（假设不变），$\boldsymbol{i(k)}$为患者所占比例。每个感染者每天接触 $\boldsymbol{\rho}$ 个未感染者:

$$\boldsymbol{i(k+1)=i(k)+\rho i(k)[1-i(k)] \Rightarrow i(k)\rightarrow 1}$$

举例：电路模型

— automatic control — automatic control — automatic control — automatic control —



电容器上的电压是时间的连续函数：

$$u(t) = (u_0 - U)e^{-\frac{t}{RC}} + U, \quad u(0) = u_0$$

只考虑离散时刻的电压 $u(k) \leftarrow u(kT_s)$:

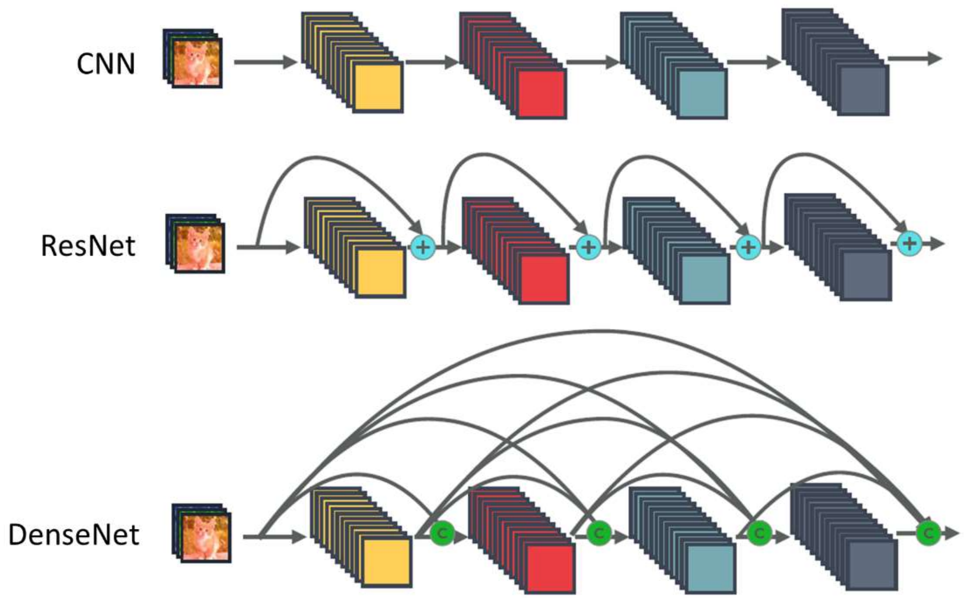
$$u(k) = (u_0 - U)e^{-\frac{kT_s}{RC}} + U \qquad (1)$$

$$u(k+1) = (u_0 - U)e^{-\frac{(k+1)T_s}{RC}} + U \qquad (2)$$

$$\xRightarrow{(2)-(1)\times e^{-\frac{T_s}{RC}}} u(k+1) = e^{-\frac{T_s}{RC}}u(k) + U\left(1 - e^{-\frac{T_s}{RC}}\right)$$

举例：深度学习中的残差神经网络

— automatic control — automatic control — automatic control — automatic control —



$$x(k+1) = x(k) + S(k)\sigma[A(k)x(k) + b(k)]$$

激活函数

采样与保持

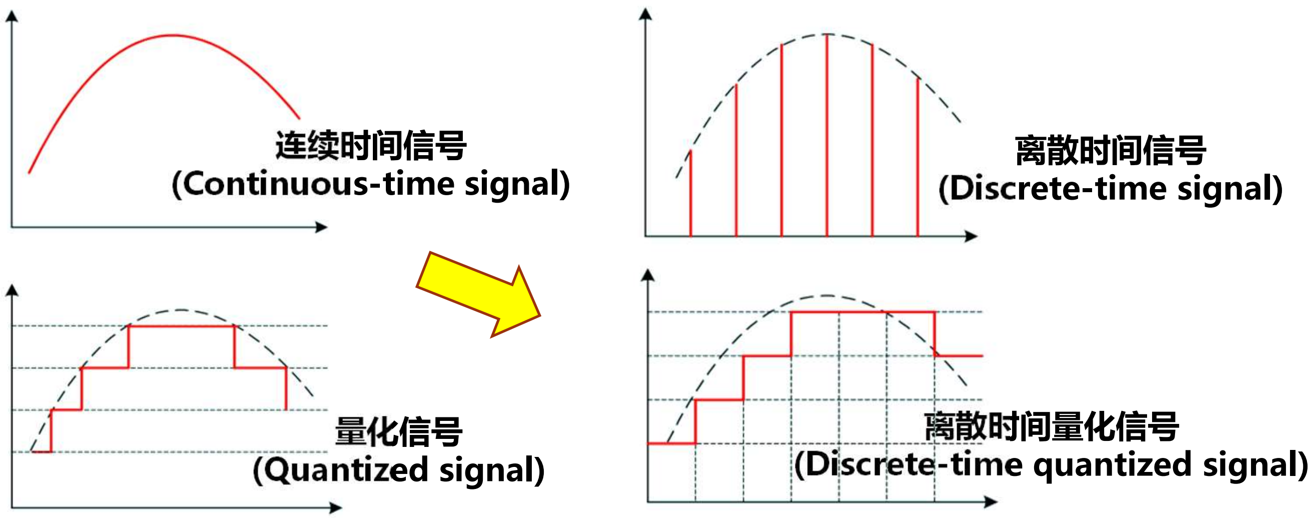
1. 采样控制系统概述
2. 采样与理想采样过程
3. 零阶保持器

参见教材 第8章， 81-91页

采样控制系统概述

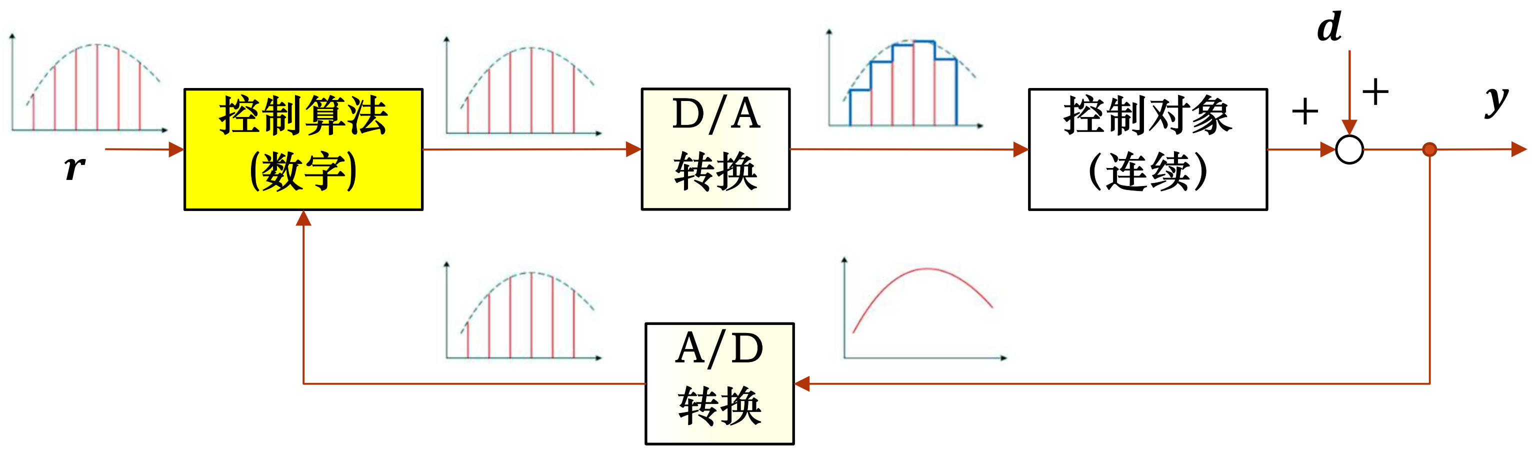
— automatic control — automatic control — automatic control — automatic control — automatic control —

数字控制器与真实物理对象之间需要信号的转换



采样（Sampling-data）控制系统

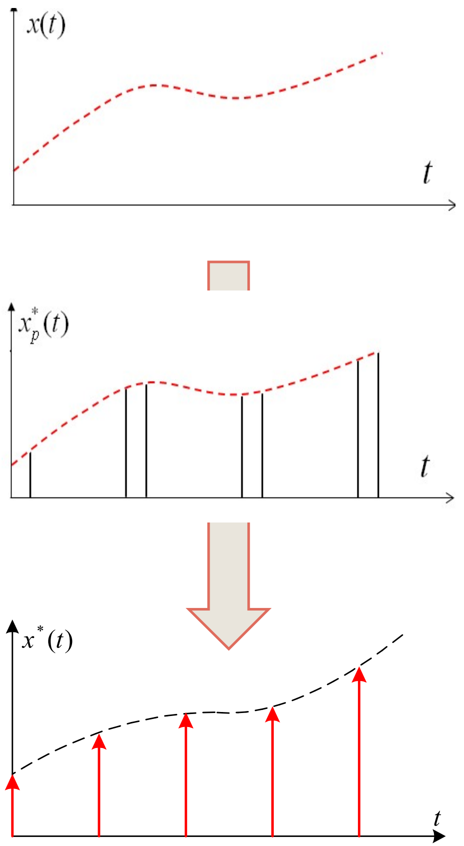
— automatic control — automatic control — automatic control — automatic control — automatic control —



- 连续时间控制对象 + 离散时间数字控制器
- A/D (采样开关)将连续时间信号抽取为离散时间信号
- D/A (采样保持)将离散时间信号恢复为连续时间信号

A/D转换：采样器（周期采样）

— automatic control — automatic control — automatic control — automatic control — automatic control —



将连续时间信号转换为以T为周期的离散时间信号：

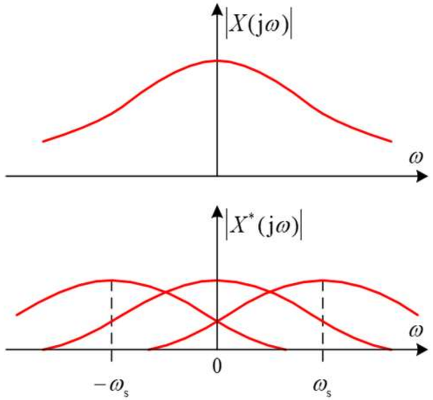
$$x^*(t) = x(t)\cdot\delta_T(t)$$

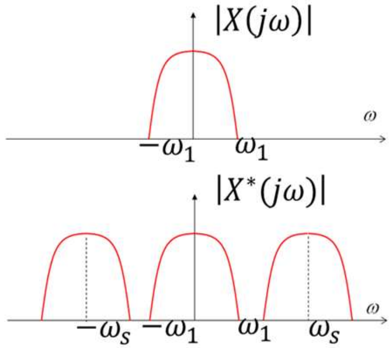
$$= \sum_{n=-\infty}^{\infty} x(t)\cdot\delta(t-nT)$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\cdot\delta(t-nT)$$

采样定理

— automatic control — automatic control — automatic control — automatic control — automatic control —





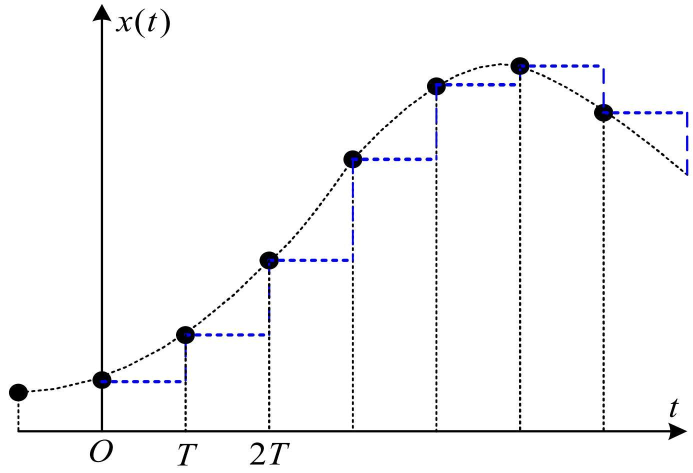
采样定理：若信号$\boldsymbol{x(t)}$带宽为$\boldsymbol{\omega_1}$，则当采样频率$\boldsymbol{\omega_s \geq 2\omega_1}$时，可根据采样后信号 $\boldsymbol{x^*(t)}$ 完全重构信号$\boldsymbol{x(t)}$。

- 实际应用时通常选择 $\boldsymbol{\omega_s \geq 10\sim20\ \omega_1}$

- 对于采样控制系统，信号最大工作频率对应于闭环系统的带宽$\boldsymbol{\omega_c}$，因此通常选择 $\boldsymbol{\omega_s \geq 10\sim20\ \omega_c}$

D/A转换：零阶保持器（ZOH）

— automatic control — automatic control — automatic control — automatic control — automatic control —



$$x(t) = x^*(nT), \qquad nT < t < (n+1)T$$

传递函数：$H_0(t) = \mathcal{L}^{-1}[h_0(t)] = \frac{1-e^{-Ts}}{s}$

离散时间系统的状态空间模型

1. 差分方程描述
2. 状态空间描述
3. 状态空间方程的解
4. 稳定性分析

参见教材 第9章， 220页；第11章，457页

线性时不变离散时间系统

— automatic control — automatic control — automatic control — automatic control — automatic control —

$\{u_1, u_2, \cdots, u_n, \cdots\}$ → 离散时间控制系统 → $\{y_1, y_2, \cdots, y_n, \cdots\}$

1） 线性叠加性【初值为零】

$$a_1\{u_1(k)\} + a_2\{u_2(k)\} \to a_1\{y_1(k)\} + a_2\{y_2(k)\}$$

2） 时不变性【与时间起点无关】

$$\{u_1, u_2, \cdots, u_n, \cdots\} \to \{y_1, y_2, \cdots, y_n, \cdots\}$$

$$\{0, 0, 0, u_1, u_2, \cdots, u_n, \cdots\} \to \{0, 0, 0, y_1, y_2, \cdots, y_n, \cdots\}$$

3） 因果性【真实物理系统】

线性时不变离散时间系统 － 差分方程

$$\boxed{\begin{aligned}&y(k+n)+a_{n-1}y(k+n-1)+\cdots+a_0y(k)\\&=b_mu(k+m)+b_{m-1}u(k+m-1)+\cdots+b_0u(k)\end{aligned}}$$

1） 线性叠加性【初值为零】

$$a_1\{u_1(k)\}+a_2\{u_2(k)\}\to a_1\{y_1(k)\}+a_2\{y_2(k)\}$$

2） 时不变性【与时间起点无关】

$$\{u_1,u_2,\cdots,u_n,\cdots\}\to\{y_1,y_2,\cdots,y_n,\cdots\}$$

$$\{0,0,0,u_1,u_2,\cdots,u_n,\cdots\}\to\{0,0,0,y_1,y_2,\cdots,y_n,\cdots\}$$

3） 因果性【真实物理系统】： $n>m$

z-传递函数模型

— automatic control — automatic control — automatic control — automatic control — automatic control —

$\boldsymbol{n}$阶线性定常离散时间系统满足差分方程：

$$y(k+n)+a_{n-1}y(k+n-1)+\cdots+a_0y(k)$$
$$=b_mu(k+m)+b_{m-1}u(k+m-1)+\cdots+b_0u(k)$$

基于z-变换 $X(z)=\sum_{k=0}^{\infty}x_kz^k$【见《信号与系统》】，

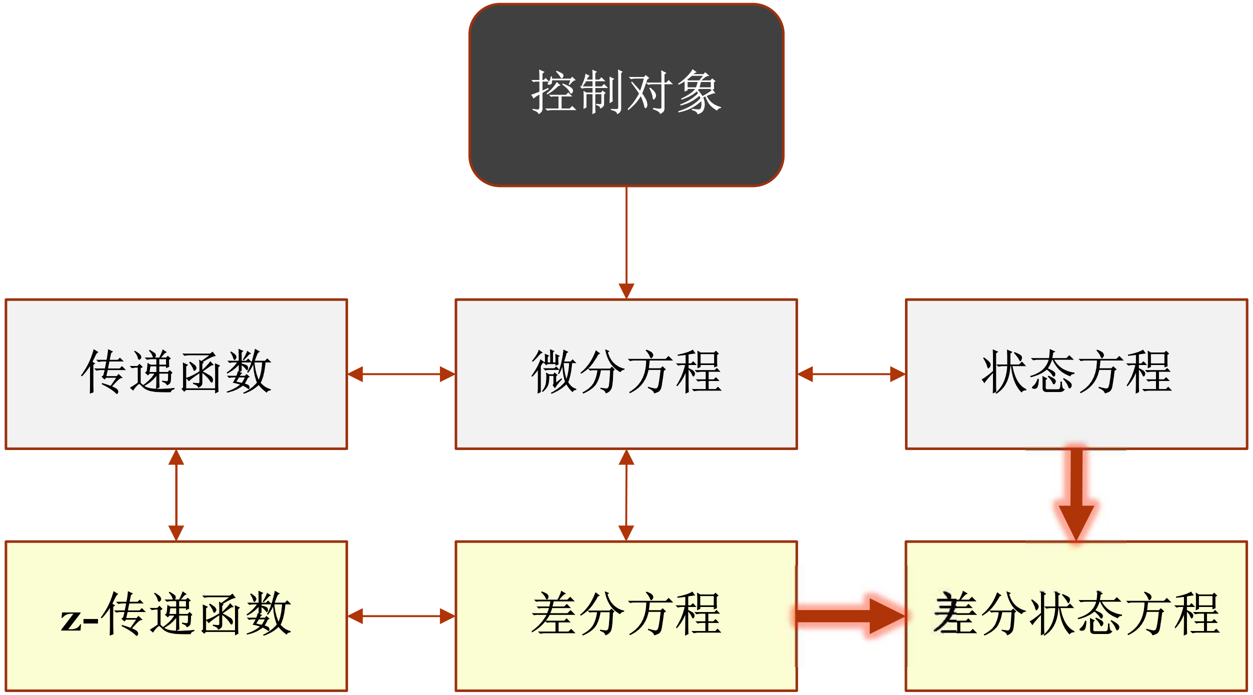
可以得到传递函数表示【初值$y(-n),\cdots,y(-1)$为零】

$$Y(z)=G(z)U(z)=\frac{z^n+a_{n-1}z^{n-1}+\cdots+a_0}{b_mz^m+\cdots+b_0}U(z)$$

- 差分方程可以通过z-变换及其反变换求解
- 连续时间传递函数与z-传递函数的转换（见教材）

离散时间控制系统的控制模型

— automatic control — automatic control — automatic control — automatic control — automatic control —



状态空间模型（从差分方程获得）

— automatic control — automatic control — automatic control — automatic control — automatic control —

以$\boldsymbol{n}$阶差分方程为例：

$$\boldsymbol{y(k)+a_{n-1}y(k-1)+\cdots+a_0y(k-n)=bu(k)}$$

令$\boldsymbol{x_1(k)=y(k-n),x_2(k)=y(k-2),\cdots,x_n(k)=y(k-1)}$，则

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_{n-1} & -a_{n-2} & -a_{n-3} & \cdots & -a_0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} u(k)$$

输出方程为 $\boldsymbol{y(k)=[1\quad 0\quad 0\quad \cdots\quad 0]x(k)}$

状态空间模型（从连续时间系统获得）

— automatic control — automatic control — automatic control — automatic control — automatic control —

考虑线性定常系统：$\dot{x}(t) = Ax(t) + Bu(t)$，其解为

$$x(t) = e^{A(t-t')}x(t') + \int_{t'}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

记采样时刻kT $(k = 0, 1, \cdots)$的状态值为$x(k)$，其中 T为采样周期。若控制输入$u(t)$在采样周期内保持不变，则

$$x(k+1) = e^{AT}x(k) + \left[\int_{kT}^{(k+1)T} e^{A[(k+1)T-\tau]}Bd\tau\right]u(k)$$

$$= e^{AT}x(k) + \left[\int_{0}^{T} e^{A\tau}Bd\tau\right]u(k)$$

$$\Rightarrow \quad x(k+1) = Gx(k) + Hu(k)$$

离散系统状态方程的解

— automatic control — automatic control — automatic control — automatic control — automatic control —

对于线性定常离散时间系统:

$$\boldsymbol{x}(k+1)=\boldsymbol{A}\boldsymbol{x}(k)+\boldsymbol{B}\boldsymbol{u}(k),\qquad \boldsymbol{x}(0)=\boldsymbol{x}_0$$

可以直接用迭代法求解:

$$\begin{aligned}\boldsymbol{x}(k)&=\boldsymbol{A}\boldsymbol{x}(k-1)+\boldsymbol{B}\boldsymbol{u}(k-1)\\&=\boldsymbol{A}[\boldsymbol{A}\boldsymbol{x}(k-2)+\boldsymbol{B}\boldsymbol{u}(k-2)]+\boldsymbol{B}\boldsymbol{u}(k-1)\\&=\boldsymbol{A}^2\boldsymbol{x}(k-2)+\boldsymbol{A}\boldsymbol{B}\boldsymbol{u}(k-2)+\boldsymbol{B}\boldsymbol{u}(k-1)\\&=\cdots\\&=\boldsymbol{A}^k\boldsymbol{x}(0)+\sum_{j=0}^{k-1}\boldsymbol{A}^{k-j-1}\boldsymbol{B}\boldsymbol{u}(j)\end{aligned}$$

离散系统状态方程的解

$$\boldsymbol{x}(k)=\boldsymbol{A}^{k}\boldsymbol{x}(0)+\sum_{j=0}^{k-1}\boldsymbol{A}^{k-j-1}\boldsymbol{B}\boldsymbol{u}(j)$$

前$\boldsymbol{k}$步的解可写作如下矩阵形式：

$$\begin{bmatrix}\boldsymbol{x}(1)\\ \boldsymbol{x}(2)\\ \vdots\\ \boldsymbol{x}(k)\end{bmatrix}=\begin{bmatrix}\boldsymbol{A}\\ \boldsymbol{A}^{2}\\ \vdots\\ \boldsymbol{A}^{k}\end{bmatrix}\boldsymbol{x}(0)+\begin{bmatrix}\boldsymbol{B} & \boldsymbol{0} & \cdots & \boldsymbol{0}\\ \boldsymbol{A}\boldsymbol{B} & \boldsymbol{B} & \cdots & \boldsymbol{0}\\ \vdots & \vdots & \ddots & \vdots\\ \boldsymbol{A}^{k-1}\boldsymbol{B} & \boldsymbol{A}^{k-2}\boldsymbol{B} & \cdots & \boldsymbol{B}\end{bmatrix}\begin{bmatrix}\boldsymbol{u}(0)\\ \boldsymbol{u}(1)\\ \vdots\\ \boldsymbol{u}(k-1)\end{bmatrix}$$

线性定常离散系统的稳定性

定理. 线性定常离散系统 $\boldsymbol{x(k+1)=Ax(k)}$ 渐近稳定

【即 $\lim\limits_{k\to\infty}\boldsymbol{x(k)=0}$】的充要条件是$\boldsymbol{A}$的特征值的模均小于1。

证明：通过变换 $\boldsymbol{T^{-1}AT=J}$ 将 $\boldsymbol{A}$ 变换为约旦标准型，其中

$$\boldsymbol{J}=\begin{bmatrix}J_1 & & \\ & J_2 & \\ & & J_3\end{bmatrix},J_i=\begin{bmatrix}\lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i\end{bmatrix},J_i^k=\begin{bmatrix}\lambda_i^k & C_k^1\lambda_i^{k-1} & \cdots & C_k^{m_i-1}\lambda_i^{k-m_i+1} \\ & \lambda_i^k & \ddots & \vdots \\ & & \ddots & C_k^1\lambda_i^k \\ & & & \lambda_i^k\end{bmatrix}$$

令$\boldsymbol{x(k)=Tz(k)}$，则 $\boldsymbol{z(k)=J^kz(0)}$，易见当且仅当$\boldsymbol{\lambda_i<1}$时，$\boldsymbol{J^k}$的所有矩阵元在 $\boldsymbol{k}\to\infty$时趋于0，即系统渐近稳定.

线性定常离散系统的稳定性

例. 考虑如下离散时间控制系统：

$$\boldsymbol{x}(\boldsymbol{k}+1)=\boldsymbol{A}\boldsymbol{x}(\boldsymbol{k}),\quad \boldsymbol{A}=\begin{bmatrix}0.5 & -0.2\\ 1.2 & 0.8\end{bmatrix}$$

试分析其稳定性。

解：解特征方程

$$\det(\lambda I-A)=\lambda^2-1.3\lambda+0.64=0$$

得特征根：

$$\lambda_{\pm}=0.65\pm 0.4664,\qquad |\lambda_{\pm}|^2=\lambda_{+}\lambda_{-}=0.64<1$$

故系统渐进稳定.

线性定常离散系统的稳定性

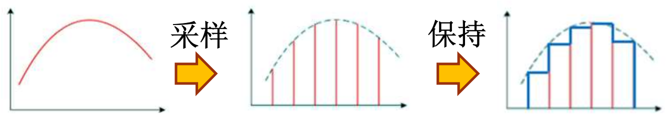
思考：

如果连续时间系统是渐进稳定的，则其离散化得到的离散时间系统模型也是渐进稳定的吗?

离散时间控制系统 回顾

— automatic control — automatic control — automatic control — automatic control —

- 离散时间系统的概念
- 采样与保持
（离散时间测控+连续时间对象）



- 离散时间系统的控制模型
  - 差分方程模型
  - 脉冲传递函数模型
  - 状态空间模型（← 连续时间模型）

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$\Rightarrow \quad x(k+1) = Gx(k) + Hu(k)$$
$$G = e^{AT}, H = \int_0^T e^{A\tau} B d\tau$$

- 离散时间系统的稳定性（极点处在单位圆内）
- 离散时间系统的能控性/能观性

离散系统的状态能控性

— automatic control — automatic control — automatic control — automatic control — automatic control —

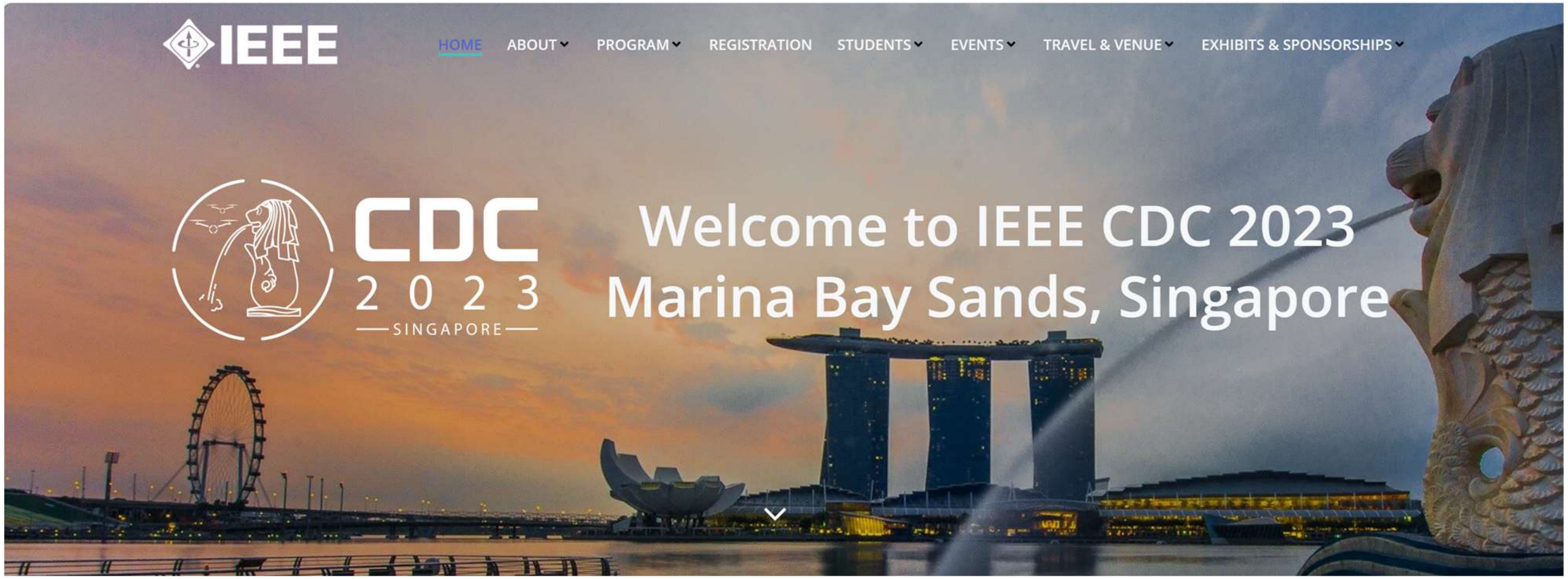
$$x(k+1) = Ax(k) + Bu(k)$$

$$\Rightarrow \quad x(n) = A^n x(0) + \sum_{j=0}^{n-1} A^{n-j-1} B u(j)$$

矩阵表示：$x(n) = A^n x(0) + \begin{bmatrix} A^{n-1}B & \cdots & AB & B \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(n-1) \end{bmatrix}$

1）能控【$\forall x(0), \exists u(\cdot) \& n, \mathrm{s.t.}\ x(n) = 0$】的条件是什么?

2）至少需要几步可以将系统从任意状态转移到原点?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GPT for Control - Panel Discussion

- A blue sky session about the impact of LLMs to control systems
- Start a discussion in the community about this emerging topic
- Explore opportunities and impact for control systems research & education
  - Can GPT / foundation models assist in control systems R&D?
  - **Can control engineers address AI safety in the LLM systems?**
  - What is the impact of GPT in control education?

Can control address LLM safety?

Taming AI Bots: Controllability of Neural States in Large Language Models

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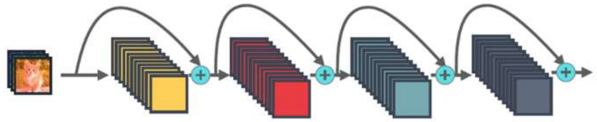
May 17, 2023

Abstract

We tackle the question of whether an agent can, by suitable choice of prompts, control an AI bot to any state. To that end, we first introduce a formal definition of "meaning" that is amenable to analysis. Then, we characterize "meaningful data" on which large language models (LLMs) are ostensibly trained, and "well-trained LLMs" through conditions that are largely met by today's LLMs. While a well-trained LLM constructs an embedding space of meanings that is Euclidean, meanings themselves do not form a vector (linear) subspace, but rather a quotient space within. We then characterize the subset of meanings that can be reached by the state of the LLMs for some input prompt, and show that a well-trained bot can reach any meaning albeit with small probability. We then introduce a stronger notion of controllability as *almost certain reachability*, and show that, when restricted to the space of meanings, an AI bot is controllable. We do so after introducing a functional characterization of attentive AI bots, and finally derive necessary and sufficient conditions for controllability. The fact that AI bots are controllable means that an adversary could steer them towards any state. However, the sampling process can be designed to counteract adverse actions and avoid reaching undesirable regions of state space before their boundary is crossed.



$$x(k+1) = x(k) + S(k)\sigma[A(k)x(k) + b(k)]$$

The fact that

AI bots are controllable

means that

an adversary could steer them

towards any state.

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Can GPT 4.0 design controllers?

Problem 2 (15 points). Consider the discrete-time system

$$x_{k+1} = Ax_k + Bu_k$$

where

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and the cost function

$$J = \sum_{k=0}^{\infty} \|x_k\|_2^2 + u_k^2$$

Note that here $Q = I$ and $R = 1$. Design an LQR feedback gain K_{lqr} for the system using the following steps.

a) (4 points) Find the 2×2 symmetric and positive definite matrix P which satisfies the *Algebraic Riccati equation* (ARE) for discrete case, which is

$$P = Q + A^T PA - A^T PB(R + B^T PB)^{-1} B^T PA$$

PROMPT

Solution 2. *a) To design an LQR (Linear Quadratic Regulator) feedback gain K_{lqr} for the given discrete-time system, we first need to find the matrix P which satisfies the Algebraic Riccati equation (ARE) for the discrete case. Given the system matrices A and B, and the cost function where $Q = I$ (the identity matrix) and $R = 1$, we can compute P from the following equation:*

$$P = Q + A^T PA - A^T PB(R + B^T PB)^{-1} B^T PA$$

In this case, the system matrices are:

$$A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad Q = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R = 1$$

Let's compute P by solving the ARE.

The matrix P, which satisfies the Algebraic Riccati equation for the given discrete-time system, is approximately:

$$P = \begin{pmatrix} 2.73205081 & -1.25944149 \times 10^{-16} \\ -1.25944149 \times 10^{-16} & 1.73205081 \end{pmatrix}$$

This matrix is symmetric and positive definite, as required. The very small off-diagonal elements (on the order of 10^{-16}) are effectively zero due to numerical precision limitations, and thus can be considered as zeros.

GPT 4.0 RESPONSE

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Impact on control education

- GPT 3.5 and GPT 4.0 were students under pseudonyms in my Linear Systems class
  - GPT 3.5 got an overall grade of 58/100 across all homeworks / exams
  - GPT 4.0 got an overall grade of 81/100 across all homeworks / exams
  - GPT 3.5 would fail our qualifying exam, GPT 4.0 is a pass
- Some remarks
  - Overall GPT did better in theoretical problems than computational ones
  - GPT does short proofs well
  - GPT struggles on longer proofs
  - GPT struggled on LTV problems
  - In computational problems, GPT performance would be boosted using a computational plugins
- Impact on control education may be more immediate

欢迎同学们尝试用**GPT**完成本次作业（但请注明并判断对错）！

Penn Engineering
UNIVERSITY of PENNSYLVANIA

离散时间系统的最优控制

最优控制问题

— automatic control — automatic control — automatic control — automatic control —

考虑如下性能指标:

$$\boldsymbol{J}=\boldsymbol{\varphi}[\boldsymbol{x}(\boldsymbol{N}),\ \boldsymbol{N}]+\sum_{k=0}^{N-1}\boldsymbol{L}[\boldsymbol{x}(\boldsymbol{k}),\ \boldsymbol{u}(\boldsymbol{k}),\ \boldsymbol{k}]$$

其中状态$\boldsymbol{x}(\boldsymbol{k})$控制$\boldsymbol{u}(\boldsymbol{k})$服从动态运动方程:

$$\boldsymbol{x}(\boldsymbol{k}+\mathbf{1})=\boldsymbol{f}[\boldsymbol{x}(\boldsymbol{k}),\ \boldsymbol{u}(\boldsymbol{k}),\ \boldsymbol{k}],\ \boldsymbol{x}(\mathbf{0})=\boldsymbol{x}_{\mathbf{0}}$$

这里，初始状态$\boldsymbol{x}(\mathbf{0})$给定，末端状态$\boldsymbol{x}(\boldsymbol{N})$自由。

问题：求解最优控制序列$\boldsymbol{u}^*(\boldsymbol{k})$，使得上述性能指标最小。

利用拉格朗日乘子求解

将状态方程看作约束，引入拉格朗日乘子$\{\boldsymbol{\lambda}(1),\cdots,\boldsymbol{\lambda}(N)\}$:

$$J\left[\underbrace{\vec{u},\vec{x},\vec{\lambda}}_{\substack{3N\text{组变量}\\ u(0),\cdots,u(N-1);\\ x(1),\cdots,x(N);\\ \lambda(1),\cdots,\lambda(N)}}\right]=\varphi[x(N),\ N]+\sum_{k=0}^{N-1}L[x(k),u(k),k]$$

$$+\sum_{k=0}^{N-1}\lambda^{\top}(k+1)\{f[x(k),\ u(k),\ k]-x(k+1)\}$$

$$=\varphi[x(N),\ N]-\sum_{k=1}^{N}\lambda^{\top}(k)x(k)+\sum_{k=0}^{N-1}H(k)$$

其中 $H(k)=L[x(k),u(k),k]+\lambda^{\top}(k+1)f[x(k),\ u(k),\ k]$.

利用拉格朗日乘子求解

— automatic control — automatic control — automatic control — automatic control —

$$J = \varphi[x(N),\ N] - \sum_{k=1}^{N} \lambda^{\top}(k)x(k) + \sum_{k=0}^{N-1} H(k)$$

$$H(k) = L[x(k), u(k), k] + \lambda^{\top}(k+1)f[x(k),\ u(k),\ k]$$

$$\left.\begin{aligned}
&\frac{\partial J}{\partial u(k)} = \frac{\partial H}{\partial u(k)} = 0 && \text{【}0 \le k \le N-1\text{】}\\
&\frac{\partial J}{\partial \lambda(k)} = \frac{\partial H(k-1)}{\partial \lambda(k)} - x(k) = 0 && \text{【}1 \le k \le N\text{】}\\
&\frac{\partial J}{\partial x(k)} = \frac{\partial H(k)}{\partial x(k)} - \lambda(k) = 0 && \text{【}1 \le k \le N-1\text{】}\\
&\frac{\partial J}{\partial x(N)} = \frac{\partial \varphi}{\partial x(N)} - \lambda(N) = 0 && \text{【}k = N]
\end{aligned}\right\} 3N\text{组方程}$$

最优控制条件

— automatic control — automatic control — automatic control — automatic control —

末时刻固定、末状态自由的离散系统最优控制必要条件如下：

$$\frac{\partial H}{\partial u(k)}=0 \quad \text{或} \quad u^*(k)=\arg\min_{u(k)\in U} H(k)$$

其中哈密顿函数：

$$H(k)=L[x(k),u(k),k]+\lambda^{\mathrm{T}}(k+1)f[x(k),u(k),k].$$

正则方程：

$$\begin{cases} x(k+1)=f[x(k),u(k),k], & x(0)=x_0 \\ \lambda(k)=\dfrac{\partial H(k)}{\partial x(k)}=\dfrac{\partial L}{\partial x(k)}+\left[\dfrac{\partial f}{\partial x(k)}\right]^{\mathrm{T}}\lambda(k+1), & \lambda(N)=\dfrac{\partial \varphi}{\partial x(N)} \end{cases}$$

示例

— automatic control — automatic control — automatic control — automatic control —

对系统 $x(k+1)=-x(k)+u(k)$, $x(0)=3$，求最优控制序列$\{u(0),u(1)\}$ 。

$$J=\frac{1}{2}x^2(2)+\frac{1}{2}\sum_{k=0}^{1}u^2(k)$$

解： $H(k)=\frac{1}{2}u^2(k)+\lambda(k+1)[u(k)-x(k)]$

- $\frac{\partial H(k)}{\partial u(k)}=u(k)+\lambda(k+1)=0\Rightarrow u(0)=-\lambda(1),u(1)=-\lambda(2)$
- $x(1)=-3+u(0),\ x(2)=3-u(0)+u(1)$
- $\lambda(k)=\frac{\partial H(k)}{\partial x(k)}=-\lambda(k+1)\Rightarrow\ \lambda(1)=-\lambda(2)$
- $\lambda(2)=\frac{\partial\varphi}{\partial x(2)}=\ x(2)$

$\Rightarrow x(2)=1,u(0)=1,u(1)=-1$

$\Rightarrow J^*=3/2$

离散时间系统的二次型最优控制

有限拍调节器问题

— automatic control — automatic control — automatic control — automatic control —

针对线性时不变受控系统：

$$\boldsymbol{x}(k+1)=\boldsymbol{A}\boldsymbol{x}(k)+\boldsymbol{B}\boldsymbol{u}(k), \quad \boldsymbol{x}(0)=\boldsymbol{x}_0$$

考虑如下最小化性能指标：

$$J=\frac{1}{2}\boldsymbol{x}^{\top}(N)\boldsymbol{F}\boldsymbol{x}(N)+\frac{1}{2}\sum_{k=0}^{N-1}[\boldsymbol{x}^{\top}(k)\boldsymbol{Q}\boldsymbol{x}(k)+\boldsymbol{u}^{\top}(k)\boldsymbol{R}\boldsymbol{u}(k)]$$

其中 $\boldsymbol{F}$ 和 $\boldsymbol{Q}$ 非负定，$\boldsymbol{R}$ 正定。

最优反馈控制

定理：对线性离散系统有限拍状态调节器问题，令$\boldsymbol{P}(\boldsymbol{k})$是Riccati矩阵递推方程

$$\boldsymbol{P}(\boldsymbol{k})=\boldsymbol{Q}+\boldsymbol{A}^{\top}[\boldsymbol{P}^{-1}(\boldsymbol{k}+1)+\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\top}]^{-1}\boldsymbol{A},\ \boldsymbol{P}(\boldsymbol{N})=\boldsymbol{F}$$

的解，其中 $\boldsymbol{k}=\boldsymbol{N}-1$， $\cdots$， 1， 0。则最优反馈控制为：

$$\boldsymbol{u}^{*}(\boldsymbol{k})=-\boldsymbol{R}^{-1}\boldsymbol{B}^{\top}(\boldsymbol{A}^{\top})^{-1}[\boldsymbol{P}(\boldsymbol{k})-\boldsymbol{Q}]\boldsymbol{x}(\boldsymbol{k})$$

或

$$\boldsymbol{u}^{*}(\boldsymbol{k})=-\boldsymbol{R}^{-1}\boldsymbol{B}^{\top}\widetilde{\boldsymbol{P}}(\boldsymbol{k})\boldsymbol{x}(\boldsymbol{k}),$$

$$\widetilde{\boldsymbol{P}}(\boldsymbol{k})=[\boldsymbol{P}^{-1}(\boldsymbol{k}+1)+\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\top}]^{-1}\boldsymbol{A},$$

且 $\boldsymbol{J}^{*}=\frac{1}{2}\boldsymbol{x}^{\top}(\boldsymbol{0})\boldsymbol{P}(\boldsymbol{0})\boldsymbol{x}(\boldsymbol{0})$.

最优反馈控制

— automatic control — automatic control — automatic control — automatic control —

证明：由最优控制条件可定义

$$\boldsymbol{H}(\boldsymbol{k}) = \frac{1}{2}[\boldsymbol{x}^\top(\boldsymbol{k})\boldsymbol{Q}\boldsymbol{x}(\boldsymbol{k}) + \boldsymbol{u}^\top(\boldsymbol{k})\boldsymbol{R}\boldsymbol{u}(\boldsymbol{k})] \\ +\boldsymbol{\lambda}^\top(\boldsymbol{k}+1)[\boldsymbol{A}\boldsymbol{x}(\boldsymbol{k}) + \boldsymbol{B}\boldsymbol{u}(\boldsymbol{k})]$$

$$0 = \frac{\partial \boldsymbol{H}(\boldsymbol{k})}{\partial \boldsymbol{u}(\boldsymbol{k})} = \boldsymbol{R}\boldsymbol{u}(\boldsymbol{k}) + \boldsymbol{B}^\top\boldsymbol{\lambda}(\boldsymbol{k}+1)$$

$$\Rightarrow \boldsymbol{u}(\boldsymbol{k}) = -\boldsymbol{R}^{-1}\boldsymbol{B}^\top\boldsymbol{\lambda}(\boldsymbol{k}+1)$$

闭环系统方程

$$\boldsymbol{x}(\boldsymbol{k}+1) = \boldsymbol{A}\boldsymbol{x}(\boldsymbol{k}) - \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top\boldsymbol{\lambda}(\boldsymbol{k}+1)$$

最优反馈控制

— automatic control — automatic control — automatic control — automatic control —

$$\boldsymbol{\lambda}(k)=\frac{\partial H(k)}{\partial \boldsymbol{x}(k)}=\boldsymbol{Q}\boldsymbol{x}(k)+\boldsymbol{A}^{\top}\boldsymbol{\lambda}(k+1)$$

令$\boldsymbol{\lambda}(k)=\boldsymbol{P}(k)\boldsymbol{x}(k)$ 并代入闭环系统方程得:

$$\boldsymbol{x}(k+1)=\boldsymbol{A}\boldsymbol{x}(k)-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\top}\boldsymbol{P}(k+1)\boldsymbol{x}(k+1)$$

$$\Rightarrow \boldsymbol{x}(k+1)=\left[\boldsymbol{I}+\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\top}\boldsymbol{P}(k+1)\right]^{-1}\boldsymbol{A}\boldsymbol{x}(k)$$

$$\begin{aligned}\boldsymbol{P}(k)\boldsymbol{x}(k)&=\boldsymbol{Q}\boldsymbol{x}(k)+\boldsymbol{A}^{\top}\boldsymbol{P}(k+1)\boldsymbol{x}(k+1)\\&=\left\{\boldsymbol{Q}+\boldsymbol{A}^{\top}\boldsymbol{P}(k+1)\left[\boldsymbol{I}+\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\top}\boldsymbol{P}(k+1)\right]^{-1}\boldsymbol{A}\right\}\boldsymbol{x}(k)\\&\Rightarrow \boldsymbol{P}(k)=\boldsymbol{Q}+\boldsymbol{A}^{\top}[\boldsymbol{P}^{-1}(k+1)+\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\top}]^{-1}\boldsymbol{A}\end{aligned}$$

最优反馈控制

— automatic control — automatic control — automatic control — automatic control —

$$u(k) = -R^{-1}B^{\top}\lambda(k+1)$$

$$\lambda(k) = Qx(k) + A^{\top}\lambda(k+1)$$

$$\begin{aligned}\Rightarrow\ \lambda(k+1) &= (A^{\top})^{-1}[\lambda(k) - Qx(k)] \\ &= (A^{\top})^{-1}[P(k) - Q]x(k)\end{aligned}$$

$$\begin{aligned}\Rightarrow u(k) &= -R^{-1}B^{\top}(A^{\top})^{-1}[P(k) - Q]x(k) \\ &= -R^{-1}(k)B^{\top}[P^{-1}(k+1) + BR^{-1}B^{\top}]^{-1}Ax(k)\end{aligned}$$

$$P(k) = Q(k) + A^{\top}[P^{-1}(k+1) + BR^{-1}B^{\top}]^{-1}A$$

示例

— automatic control — automatic control — automatic control — automatic control —

对下述系统 $\boldsymbol{x}(\boldsymbol{k}+\mathbf{1})=-\boldsymbol{x}(\boldsymbol{k})+\boldsymbol{u}(\boldsymbol{k}), \boldsymbol{x}(\mathbf{0})=\mathbf{3}$，及如下性能指标，求最优控制序列。

$$\boldsymbol{J}=\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{x}^2(2)+\frac{\mathbf{1}}{\mathbf{2}}\sum_{\boldsymbol{k}=\mathbf{0}}^{\mathbf{1}}\boldsymbol{u}^2(\boldsymbol{k})$$

解： $\boldsymbol{A}=-\mathbf{1},\quad \boldsymbol{B}=\mathbf{1},\quad \boldsymbol{F}=\mathbf{1},\quad \boldsymbol{Q}=\mathbf{0},\quad \boldsymbol{R}=\mathbf{1}$

$$\boldsymbol{P}(2)=\boldsymbol{F}=\mathbf{1},\quad \boldsymbol{P}(\boldsymbol{k})=\boldsymbol{Q}+\boldsymbol{A}^{\top}\widetilde{\boldsymbol{P}}(\boldsymbol{k})\Rightarrow\widetilde{\boldsymbol{P}}(\boldsymbol{k})=-\boldsymbol{P}(\boldsymbol{k})$$

$$\widetilde{\boldsymbol{P}}(\mathbf{1})=[\boldsymbol{P}^{-1}(2)+\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\top}]^{-1}\boldsymbol{A}=-\frac{\mathbf{1}}{\mathbf{2}},\qquad \boldsymbol{P}(\mathbf{1})=\frac{\mathbf{1}}{\mathbf{2}}$$

$$\widetilde{\boldsymbol{P}}(\mathbf{0})=[\boldsymbol{P}^{-1}(\mathbf{1})+\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\top}]^{-1}\boldsymbol{A}=-\frac{\mathbf{1}}{\mathbf{3}},\qquad \boldsymbol{P}(0)=\frac{\mathbf{1}}{\mathbf{3}}$$

示例

— automatic control — automatic control — automatic control — automatic control —

根据 $\widetilde{P}(1) = -\frac{1}{2}$，$P(1) = \frac{1}{2}$；$\widetilde{P}(0) = -\frac{1}{3}$，$P(0) = \frac{1}{3}$，$x^*(0) = 3$

$x^*(1) = -x^*(0) + u^*(0) = \left[-1 - \widetilde{P}(0)\right]x^*(1) = -\frac{2}{3}\mathbf{x}^*(0) = -2$

$x^*(2) = -x^*(1) + u^*(1) = \left[-1 - \widetilde{P}(k)\right]x^*(1) = -\frac{1}{2}x^*(1) = 1$

从而根据 $u^*(k) = -\widetilde{P}(k)x^*(k)$，得：$u^*(0) = 1$，$u^*(1) = -1$。

最优值

$$J^* = \frac{1}{2}x^*(0)P(0)x^*(0) = \frac{3}{2}$$

与 $J = \frac{1}{2}[x^*(2)]^2 + \sum_{k=0}^{1} \frac{1}{2}[u^*(k)]^2 = \frac{3}{2}$ 一致。

最优开环控制

— automatic control — automatic control — automatic control — automatic control — automatic control —

状态方程前$\boldsymbol{k}$步的解可写作如下矩阵形式：

$$\begin{bmatrix} \boldsymbol{x}(1) \\ \boldsymbol{x}(2) \\ \vdots \\ \boldsymbol{x}(\boldsymbol{k}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{A}^2 \\ \vdots \\ \boldsymbol{A}^{\boldsymbol{k}} \end{bmatrix} \boldsymbol{x}(0) + \begin{bmatrix} \boldsymbol{B} & \mathbf{0} & \cdots & \mathbf{0} \\ \boldsymbol{AB} & \boldsymbol{B} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{A}^{\boldsymbol{k}-1}\boldsymbol{B} & \boldsymbol{A}^{\boldsymbol{k}-1}\boldsymbol{B} & \cdots & \boldsymbol{B} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}(0) \\ \boldsymbol{u}(1) \\ \vdots \\ \boldsymbol{u}(\boldsymbol{k}-1) \end{bmatrix}$$

简写作 $\boldsymbol{X}(\boldsymbol{k}) = \boldsymbol{G}(\boldsymbol{k})\boldsymbol{x}_0 + \boldsymbol{H}(\boldsymbol{k})\boldsymbol{U}(\boldsymbol{k})$

指标：$\boldsymbol{J} = \frac{1}{2}\boldsymbol{x}^\top(\boldsymbol{N})\boldsymbol{F}\boldsymbol{x}(\boldsymbol{N}) + \frac{1}{2}\sum_{\boldsymbol{k}=0}^{\boldsymbol{N}-1}[\boldsymbol{x}^\top(\boldsymbol{k})\boldsymbol{Q}(\boldsymbol{k})\boldsymbol{x}(\boldsymbol{k}) + \boldsymbol{u}^\top(\boldsymbol{k})\boldsymbol{R}(\boldsymbol{k})\boldsymbol{u}(\boldsymbol{k})]$

$$= \frac{1}{2}\{\boldsymbol{X}^\top(\boldsymbol{N})\boldsymbol{Q}_{\boldsymbol{F}}(\boldsymbol{N})\boldsymbol{X}(\boldsymbol{N}) + \boldsymbol{U}^\top(\boldsymbol{N})\boldsymbol{R}(\boldsymbol{N})\boldsymbol{U}(\boldsymbol{N}) + \boldsymbol{x}_0^\top\boldsymbol{Q}\boldsymbol{x}_0\}$$

其中 $\boldsymbol{Q}_{\boldsymbol{F}}(\boldsymbol{N}) = \begin{bmatrix} \boldsymbol{Q} & & \\ & \ddots & \\ & & \boldsymbol{F} \end{bmatrix}$， $\boldsymbol{R}(\boldsymbol{N}) = \begin{bmatrix} \boldsymbol{R} & & \\ & \ddots & \\ & & \boldsymbol{R} \end{bmatrix}$.

最优开环控制的最小二乘解

— automatic control — automatic control — automatic control — automatic control — automatic control —

将 $\boldsymbol{X}(\boldsymbol{k})=\boldsymbol{G}(\boldsymbol{k})\boldsymbol{x}_0+\boldsymbol{H}(\boldsymbol{k})\boldsymbol{U}(\boldsymbol{k})$ 代入优化指标

$$\boldsymbol{J}=\frac{1}{2}\{\boldsymbol{X}^{\top}(\boldsymbol{N})\boldsymbol{Q}_{\boldsymbol{F}}(\boldsymbol{N})\boldsymbol{X}(\boldsymbol{N})+\boldsymbol{U}^{\top}(\boldsymbol{N})\boldsymbol{R}(\boldsymbol{N})\boldsymbol{U}(\boldsymbol{N})+\boldsymbol{x}_0^{\top}\boldsymbol{Q}\boldsymbol{x}_0\}$$

容易证明，使 $\boldsymbol{J}$ 最小的开环控制解为

$$\begin{aligned}&\boldsymbol{U}(\boldsymbol{N})\\&=-[\boldsymbol{R}(\boldsymbol{N})+\boldsymbol{H}^{\top}(\boldsymbol{N})\boldsymbol{Q}_{\boldsymbol{F}}(\boldsymbol{N})\boldsymbol{H}(\boldsymbol{N})]^{-1}\boldsymbol{H}^{\top}(\boldsymbol{N})\boldsymbol{Q}_{\boldsymbol{F}}(\boldsymbol{k})\boldsymbol{G}(\boldsymbol{k})\boldsymbol{x}_0\end{aligned}$$

易见最优解随状态初值变化.

示例

— automatic control — automatic control — automatic control — automatic control —

对系统 $\boldsymbol{x(k+1) = -x(k) + u(k), x(0) = 3}$，及如下性能指标，求最优控制。

$$J = \frac{1}{2}x^2(2) + \sum_{k=0}^{1} \frac{1}{2}u^2(k)$$

解：$\boldsymbol{A = -1,\quad B = 1,\quad F = 1,\quad Q = 0,\quad R = 1}$

$$X(2) = \begin{bmatrix} -3 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} U(2), \qquad X(2) = \begin{bmatrix} x(1) \\ x(2) \end{bmatrix}, U(2) = \begin{bmatrix} u(1) \\ u(2) \end{bmatrix}$$

$$J = \frac{1}{2} X^{\top}(2) \begin{bmatrix} Q & \\ & F \end{bmatrix} X(2) + \frac{1}{2} U^{\top}(2) \begin{bmatrix} R & \\ & R \end{bmatrix} U(2)$$

最优解：$U(2) = -\left(\begin{bmatrix} 1 & \\ & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^{\top} \begin{bmatrix} 0 & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \right)^{-1}$

$$\times \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^{\top} \begin{bmatrix} 0 & \\ & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

无限拍调节器问题

— automatic control — automatic control — automatic control — automatic control —

针对如下受控系统：

$$\boldsymbol{x}(k+1)=\boldsymbol{A}\boldsymbol{x}(k)+\boldsymbol{B}\boldsymbol{u}(k)$$

考虑如下最小化性能指标：

$$J=\frac{1}{2}\sum_{k=0}^{\infty}[\boldsymbol{x}^{\top}(k)\boldsymbol{Q}\boldsymbol{x}(k)+\boldsymbol{u}^{\top}(k)\boldsymbol{R}\boldsymbol{u}(k)]$$

其中$\boldsymbol{Q}$非负定，$\boldsymbol{R}$正定。

无限拍状态调节器问题

定理：若系统完全可控，则最优反馈控制如下：

$$\boldsymbol{u}^*(k) = -\boldsymbol{R}^{-1}\boldsymbol{B}^\top\widetilde{\boldsymbol{P}}\boldsymbol{x}(k)$$

其中，$\widetilde{\boldsymbol{P}}$是Riccati矩阵代数方程

$$\boldsymbol{P} = \boldsymbol{Q} + \boldsymbol{A}^\top\widetilde{\boldsymbol{P}}$$

$$\widetilde{\boldsymbol{P}} = [\boldsymbol{P}^{-1} + \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top]^{-1}\boldsymbol{A}$$

且 $J^* = \frac{1}{2}\boldsymbol{x}^\top(0)\boldsymbol{P}\boldsymbol{x}(0).$

无限拍状态调节器问题

— automatic control — automatic control — automatic control — automatic control —

证明：根据有限拍状态调节器的Riccati矩阵递推方程

$$\boldsymbol{P}(\boldsymbol{k})=\boldsymbol{Q}+\boldsymbol{A}^{\top}[\boldsymbol{P}^{-1}(\boldsymbol{k}+1)+\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\top}]^{-1}\boldsymbol{A}$$

令 $\boldsymbol{P}=\lim\limits_{\boldsymbol{k}\to\infty}\boldsymbol{P}(\boldsymbol{k})$为稳态解，可得到代数Riccati方程：

$$\boldsymbol{P}=\boldsymbol{Q}+\boldsymbol{A}^{\top}[\boldsymbol{P}^{-1}+\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\top}]^{-1}\boldsymbol{A}$$

令$\boldsymbol{P}$为代数Riccati方程的解，可得最优反馈控制：

$$\boldsymbol{u}^{*}(\boldsymbol{k})=-\boldsymbol{R}^{-1}\boldsymbol{B}^{\top}[\boldsymbol{P}^{-1}+\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\top}]^{-1}\boldsymbol{A}\boldsymbol{x}(\boldsymbol{k})$$

且 $\boldsymbol{J}^{*}=\frac{1}{2}\boldsymbol{x}^{\top}(0)\boldsymbol{P}\boldsymbol{x}(0).$

总结

— automatic control — automatic control — automatic control — automatic control —

- 离散时间系统的控制模型
- 连续时间系统模型的离散化
- 稳定性，能控性和能观性
- 离散时间系统的最优控制理论
- 离散时间系统的LQR问题

1）面向计算，概念容易理解；

2）与计算机控制紧密结合 → 模型预测控制