1. 设 $Y_i(X_i-\mu)^2$，则 $Y_i$ iid

$EY_i = E(X_i-\mu)^2 = \sigma^2$

由 khintchine：

$\frac{1}{n}\sum Y_i \xrightarrow{P} \sigma^2$

即 $\frac{1}{n}\sum_{k=1}^{n}(X_k-\mu)^2 \xrightarrow{P} \sigma^2$

2. $D(\frac{1}{n}\sum_{k=1}^{n}X_k)=\frac{1}{n^2}D(\sum_{k=1}^{n}X_k)$

$= \frac{1}{n^2}[\sum_{k=1}^{n}DX_k+\sum\sum_{i\neq j}Cov(X_i,X_j)]$

由 Cauchy：$Cov(X,Y)\le\sqrt{DXDY}$ $\quad\therefore Cov(X_i,X_j)\le C$

$D(\frac{1}{n}\sum_{k=1}^{n}X_k)=\frac{1}{n^2}[\sum_{k=1}^{n}DX_k+\sum\sum_{1<|i-j|<N}Cov(X_i,X_j)+\sum\sum_{|i-j|>N}Cov(X_i,X_j)]$

$\le \frac{1}{n^2}[NC+2N_n\cdot C+n^2\cdot\varepsilon_0\cdot C]$

$P(|\frac{1}{n}\sum_{k=1}^{n}X_k-\frac{1}{n}\sum_{k=1}^{n}EX_k|\ge\varepsilon)\le\frac{D\frac{1}{n}\sum_{k=1}^{n}X_n}{\varepsilon^2}$

$\le\frac{N(1+2n)C}{n^2\varepsilon^2}+\frac{\varepsilon_0 C}{\varepsilon^2}$

取 $\varepsilon_0=\frac{1}{n}$

故 原式 $\le\frac{N(1+2n)C}{n^2\varepsilon^2}+\frac{C}{n^2\varepsilon^2}$

$\therefore$在 $n\to+\infty$ 时

原式 $\to 0$

3. 设 $Y_k=\cos kx$

$EY_k=\int_{-\pi}^{\pi}\cos kx\cdot\frac{1}{2\pi}dx=0$

$i\neq j$：

$Cov(Y_i,Y_j)=EY_iY_j-EY_iEY_j$

$=EY_iY_j=\int_{-\pi}^{\pi}\cos ix\cos jx\,dx=0$

$DY_k=EY_k^2=\int_{-\pi}^{\pi}(\cos kx)^2dx=\int_{-\pi}^{\pi}\frac{1+\cos 2kx}{2}dx=\frac{1}{2}$

$\therefore\{Y_k\}$两两不相关且方差一致有界

由 Chebyshev 大数定律：

$\frac{S_n}{n}-E(\frac{S_n}{n})=\frac{S_n}{n}\to 0$

4. $\ln Y_n = \frac{\sum_{k=1}^{n} \ln X_k}{n}$

令 $Z_n = \ln X_n$

$P(Z_n \le z) = P(\ln X_n \le z)$

$= P(X_n \le e^z) = \int_0^{e^z} 1 dx = e^z \quad (z \le 0)$

$f_{Z_n}(z) = (e^z)' = e^z \quad (z \le 0)$

$E Z_n = \int_{-\infty}^{0} z e^z dz = -1$

$\{Z_n\}$独立同分布，由khintchine大数定律：

$$\frac{\sum_{k=1}^{n} Z_n}{n} \xrightarrow{P} -1$$

故 $\ln Y_n \xrightarrow{P} -1$ 而 $e^x$ 为连续函数

故 $e^{\ln Y_n} \xrightarrow{P} e^{-1}$

即 $Y_n \xrightarrow{P} \frac{1}{e} \quad \therefore c = \frac{1}{e}$

5.

设 $A_i = \begin{cases} 1 & 第i次试验正面朝上 \\ 0 & 第i次试验背面朝上 \end{cases}$

$EA_i = 0.5$
$DA_i = 0.25$

则 $X = \frac{\sum A_i}{n}$ 为正面朝上的频率

$EX = \frac{\sum EA_i}{n} = 0.5$

$DX = \frac{1}{4n}$

$P(0.4 < X < 0.6) = P(|X - EX| < 0.1)$

由切比雪夫不等式：

$P(|X - EX| \ge 0.1) \le \frac{DX}{0.1^2} = \frac{\frac{1}{n} DA_i}{0.1^2} = \frac{25}{n}$

$P(|X - EX| < 0.1) = 1 - P(|X - EX| \ge 0.1)$

$\ge 1 - \frac{25}{n} \ge 0.9$

故 $n \ge 250$

用正态近似估计：

$$P\left(0.4<\frac{\sum_{i=1}^{n}A_i}{n}<0.6\right)=P\left(\frac{0.4-0.5}{\frac{0.5}{\sqrt{n}}}<\frac{\frac{\sum_{i=1}^{n}A_i}{n}-0.5}{\frac{0.5}{\sqrt{n}}}<\frac{0.6-0.5}{\frac{0.5}{\sqrt{n}}}\right)$$

$$=P\left(-\frac{\sqrt{n}}{5}<Z<\frac{\sqrt{n}}{5}\right)>0.9$$

查表得：$P(Z\le 1.65)=0.9509$

$$\frac{\sqrt{n}}{5}\geq 1.65$$

$$n\geq 68$$

6. (a) 设 $A_i=\begin{cases}1 & 右移\\ -1 & 左移\end{cases}$ $A_i$独立同分布 $\begin{cases}DA_i=1\\ EA_i=0\end{cases}$

$$X_n=\sum_{i=1}^{n}A_i$$

$$DX_n=D\left(\sum_{i=1}^{n}A_i\right)=\sum_{i=1}^{n}DA_i=n$$

(b)

$$S_n^*=\frac{\sum_{i=1}^{n}A_i-n\cdot EA_i}{\sqrt{n}\sqrt{DA_i}}=\frac{\sum_{i=1}^{n}A_i}{\sqrt{n}}=\frac{X_n}{\sqrt{n}}$$

由Lindeberg – Levy 中心极限定理

$$S_n^*\xrightarrow{D}N(0,1)$$

即 $\lim\limits_{n\to\infty}F_n(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{t^2}{2}}dt$

故 $P\left(\frac{X_n}{\sqrt{n}}\le x\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{t^2}{2}}dt$

7. $X_i\overset{iid}{\sim}U(-1,1)$

$\therefore EX_i=0 \quad DX_i=\frac{1}{3}$

则 $E(\bar{X})=E\left(\frac{\sum_{i=1}^{n}X_i}{n}\right)=\frac{1}{n}\sum_{i=1}^{n}EX_i=0$

$D(\bar{X})=D\left(\frac{\sum_{i=1}^{n}X_i}{n}\right)=\frac{1}{n^2}\sum_{i=1}^{n}DX_i=\frac{1}{3n}$

8. $Cov(X_i-\bar{X},X_j-\bar{X})=Cov(X_i,X_j)-Cov(X_i,\bar{X})-Cov(\bar{X},X_j)+Cov(\bar{X},\bar{X})$

$$=0-\frac{1}{n}DX_i-\frac{1}{n}DX_j+\frac{1}{n^2}\sum_{i=1}^{n}DX_i$$

设 $\forall i$, $DX_i=\sigma^2$ 故 $Cov(X_i-\bar{X},X_j-\bar{X})=-\frac{\sigma^2}{n}$

$D(X_i-\bar{X})=D(X_j-\bar{X})=D(X_1-\bar{X})=D\left(\frac{(n-1)X_1-X_2-\cdots-X_n}{n}\right)=\frac{n-1}{n}\sigma^2$

$$r_{X_i-\bar{X},X_j-\bar{X}}=\frac{Cov(X_i-\bar{X},X_j-\bar{X})}{\sqrt{DX_{i}-\bar{X}\,DX_j-\bar{X}}}=-(n-1)^{-1}$$

9. $X_i\sim N(8,4)$ $Y_i=\frac{X_i-8}{2}\sim N(0,1)$

(1) $P(X_{(16)}>10)=1-P(X_{(16)}\le 10)=1-[P(Y_i\le 1)]^{16}=0.9370$

(2) $P(X_{(1)}>5)=[P(X_i>5)]^{16}=0.3308$

10. 令 $Z_i=X(i)$

$$\therefore\begin{cases}Z_1=y_1y_2\cdots y_n\\ Z_2=y_2\cdot y_3\cdots y_n\\ \vdots\\ Z_n=y_n\end{cases}$$

$\therefore$ Jacob 行列式为 $|J|=y_2y_3^2y_4^3\cdots y_n^{n-1}$

$$f_{Y_1\cdot Y_2\cdots Y_n}(y_1\cdots y_n)=f_{Z_1Z_2\cdots Z_n}(Z(y_1\cdots y_n),\cdots Z_n(y_1\cdots y_n)|J|)$$

$$=n!\prod_{i=1}^{n}\frac{1}{\theta}\cdot|J|=\frac{n!}{\theta^n}y_2y_3^2\cdots y_n^{n-1}$$

可分离变量 $\therefore y_1\ldots y_n$ 相互独立