35. $\eta \sim U[0,a]$
$X \sim U[\eta, a]$

$$f(x|\eta) = \frac{f(x,\eta)}{f(\eta)} = \frac{1}{a-\eta}$$

则 $E(X|\eta) = \frac{a+\eta}{2}$

$\therefore \ E(X|\eta) \sim U[\frac{a}{2}, a]$

41. $X_1 \sim E(\lambda_1)$, $X_2 \sim E(\lambda_2)$

$$f(X_1 < X_2) = \int_0^{+\infty}\int_0^{y} f_{X_1,X_2}(x,y)\,dx\,dy = 1 - \frac{\lambda_2}{\lambda_1+\lambda_2}$$

$$E(X_1 | X_1 < X_2) = \int_0^{+\infty} x f(X_1 = x | X_1 < X_2)\,dx$$

$$= \frac{1}{\lambda_1 + \lambda_2}$$

47. (1) 由课堂结论：Gamma 分布为 iid 指数分布相加结果

直接可得：$Z = X+Y$ 的概率密度函数为

$$f_Z(z) = \begin{cases} \lambda^2 z e^{-\lambda z} & , Z > 0 \\ 0 & , Z \le 0 \end{cases}$$

(2) 由对称性：

$E(X|X+Y=Z) = E(Y|X+Y=Z)$

且 $E(X+Y|X+Y=Z) = Z$

则 $E(X|X+Y=Z) = \frac{Z}{2}$

49. $\because X, Y \overset{iid}{\sim} E(1)$
$U = X+Y$, $V = \frac{X}{Y}$

$$\therefore \begin{cases} X = \frac{UV}{V+1} \\ Y = \frac{U}{V+1} \end{cases} \qquad \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{V}{V+1} & \frac{U}{(V+1)^2} \\ \frac{1}{V+1} & \frac{-U}{(V+1)^2} \end{vmatrix} = \frac{U}{(V+1)^2}$$

则 $f_{U,V}(u,v) = f_{XY}\left(\frac{uv}{v+1}, \frac{u}{v+1}\right)\frac{u}{(v+1)^2} = \frac{u}{(v+1)^2} e^{-u}$

类似地，有：$f(u) = u \cdot e^{-u}$

$f(v) = \frac{1}{(v+1)^2}$

$\therefore f(u,v) = f(u) \cdot f(v)$ $\quad \therefore$ 独立

50. $U=\min(X,Y)$ $V=\max(X,Y)$

在 $0<u\le v<1$ 时

$$f(u,v)=f(X=u,Y=v,X<Y)+f(X=v,Y=u,X\ge Y)$$

$$=\frac{12}{7}(u+v)^2$$

$$\therefore f(u,v)=\begin{cases}\frac{12}{7}(u+v)^2 & 0<u\le v<1\\ 0 & 其它\end{cases}$$

51. $X_1, X_2 \overset{iid}{\sim} U[0,1]$

$$\begin{cases}Y_1=(-2\ln X_1)^{\frac{1}{2}}\cos 2\pi X_2\\ Y_2=(-2\ln X_1)^{\frac{1}{2}}\sin 2\pi X_2\end{cases} \Rightarrow \begin{cases}X_1=e^{-\frac{Y_1^2+Y_2^2}{2}}\\ X_2=\frac{1}{2\pi}\arctan\frac{Y_2}{Y_1}\end{cases}$$

则 $$\left|\frac{\partial(X_1,X_2)}{\partial(Y_1,Y_2)}\right|=\begin{vmatrix}-Y_1e^{-\frac{Y_1^2+Y_2^2}{2}} & -Y_2e^{-\frac{Y_1^2+Y_2^2}{2}}\\ \frac{1}{2\pi}\frac{-\frac{Y_2}{Y_1^2}}{1+\left(\frac{Y_2}{Y_1}\right)^2} & \frac{1}{2\pi}\frac{\frac{1}{Y_1}}{1+\left(\frac{Y_2}{Y_1}\right)^2}\end{vmatrix}=\frac{1}{2\pi}e^{-\frac{Y_1^2+Y_2^2}{2}}$$

$$\because f(y_1,y_2)=\frac{1}{2\pi}e^{-\frac{y_1^2+y_2^2}{2}}$$

$Y_1, Y_2 \sim N(0,0,0,1,1)$

可知 Y_1, Y_2 相互独立且服从 $N(0,1)$

53. $$G_n(x)=P(Z_n\le x)=P(\max\{X_1,\dots,X_n\}\ge 1-\frac{x}{n})$$

$$=1-\prod_{i=1}^{n}P(X_i<1-\frac{x}{n})$$

$$=1-(1-\frac{x}{n})^{2n}$$

$$\lim_{n\to\infty}G_n(x)=1-e^{-2x}\ (x>0)$$

$$G(x)=\int_0^x 2e^{-2x}dx=1-e^{-2x}\ (x>0)$$

$\therefore$ 证得 $\lim\limits_{n\to\infty}G_n(x)=G(x)$

P179.

1. $X\sim U(-a,a)$

$$f(x)=\frac{1}{2a}$$

$$\varphi_X(\theta)=E(e^{i\theta x})=\int_{-a}^{a}\frac{e^{i\theta x}}{2a}dx$$

$$= \frac{1}{2ai\theta}(e^{i\theta a} - e^{i\theta a})$$

$$= \frac{e^{i\theta a} - e^{-i\theta a}}{2a\theta i} = \frac{\sin(\theta - a)}{\theta a}$$

2. $X \sim Cauchy(m, a)$

$$f(x) = \frac{a}{\pi[(x-m)^2 + a^2]} = \frac{1}{a\pi[(\frac{x-m}{a})^2 + 1]}$$

令 $y = \frac{x-m}{a}$

则 $f(y) = \frac{1}{a\pi(y^2+1)} \cdot a = \frac{1}{\pi(y^2+1)}$

则 $I(\theta) = \int_{-\infty}^{+\infty} \frac{e^{i\theta y}}{\pi(y^2+1)} dy = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\cos\theta y}{y^2+1} dy$

经求导发现：$I''(\theta) = I(\theta)$

$\lambda^2 = 1$

$\therefore \varphi(\theta) = e^{i\theta m - a|\theta|}$

6. 若随机变量Y的特征函数为$\varphi(\theta)$

且 $X \sim U(-a, a)$，则 $\varphi_X(\theta) = \frac{\sin(a\theta)}{a\theta}$

$\therefore X+Y$ 的特征函数为 $\frac{\sin(a\theta)}{a\theta}\varphi(\theta)$

$\therefore \varphi_{X+Y}(\theta) = \varphi_a(\theta)$

$\therefore \varphi_a(\theta)$为$X+Y$的特征函数

7. (a) $f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x < 0 \end{cases}$

$\therefore \varphi_X(\theta) = \int_{-\infty}^{+\infty} e^{i\theta x} f(x) dx = \frac{1}{1 - i\theta}$

(b) $\because X$与Y独立

$\therefore \varphi_{X-Y}(\theta) = \varphi_X(\theta)\varphi_{-Y}(\theta)$

$$= \frac{1}{1+\theta^2}$$

(c) $Z = X - Y$

则 $f(z) = \begin{cases} \frac{1}{2}e^{z}, & z < 0 \\ \frac{1}{2}e^{-z}, & z \geq 0 \end{cases}$

$\therefore \varphi_Z(\theta) = \int_{-\infty}^{+\infty} e^{i\theta z} f(z)\,dz$

$= \frac{1}{2}\frac{e^{(i\theta+1)z}}{i\theta+1}\Big|_{-\infty}^{0} + \frac{1}{2}\frac{e^{(i\theta-1)z}}{i\theta-1}\Big|_{0}^{+\infty}$

$= \frac{1}{2}\frac{i\theta-1-i\theta-1}{(i\theta)^2-1} = \frac{1}{\theta^2+1}$

与前相同

而 $\frac{1}{1+\theta^2} = \int_{-\infty}^{+\infty} f(x)\cos\theta x\,dx$

$= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1}{1+\theta^2}\cos\theta x\,d\theta$

$\therefore$

$f(x) = \begin{cases} e^{-x} & x>0 \\ 0 & x<0 \end{cases}$

10. 证明:

$f_{x_i}(x) = \frac{a}{\pi[(x-m)^2+a^2]}$

$\varphi_{x_i}(\theta) = e^{i\theta m - a|\theta|}$

$\varphi_X(\theta) = \prod_{i=1}^{n} \varphi_{x_i}(\frac{\theta}{n}) = \prod_{i=1}^{n} e^{i\frac{\theta}{n}\cdot m - a|\frac{\theta}{n}|}$

$= e^{i\theta m - a|\theta|} = \varphi_{x_i}(\theta)$