17. $X \sim N(\mu, \Sigma)$

$\because \Sigma$ 对称

$\therefore \exists$ 正交$Q$，对角$\Lambda$，使得 $\Sigma = Q\Lambda Q^T$

令 $A = Q^T$

$\therefore Y \sim N(0, Q^T \Sigma Q)$

而 $Q^T\Sigma Q = \Lambda$，故 $Y \sim N(0, \Lambda)$

对每个$Y_i$，$Y_i \sim N(0, \Lambda_{ii})$

18. (a) $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$

$Y = AX$，故 $Y \sim (A\mu, A\Sigma A^T)$

$\therefore \begin{pmatrix} \frac{113}{5} & \frac{112}{5} \\ \frac{112}{5} & \frac{113}{5} \end{pmatrix}$

(b) $Y \sim \left( \begin{pmatrix} 7 \\ 8 \end{pmatrix}, \begin{pmatrix} \frac{113}{5} & \frac{112}{5} \\ \frac{112}{5} & \frac{113}{5} \end{pmatrix} \right)$

而 $(Y_1, Y_2) \sim N(\mu, \mu_2, \rho, \sigma_1^2, \sigma_2^2)$

其中 $\sigma_1 = \sqrt{DY_1}$ $\sigma_2 = \sqrt{DY_2}$ $\rho = \frac{Cov(Y_1, Y_2)}{\sigma_1 \sigma_2}$

$\therefore (Y_1, Y_2) \sim N(\mu_1, \mu_2, \rho, \sigma_1^2, \sigma_2^2)$

$Y_1 + Y_2 = (1, 1)\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ $\therefore E(Y_1 + Y_2) = 15$

而 $(Y_2 | Y_1) \sim N(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(Y_1 - \mu_1), (1-\rho^2)\sigma_2^2)$

$\therefore E(Y_2 | Y_1) = 8 + \frac{112}{113}(Y_1 - 7)$

(c) $\rho = \frac{4}{5}$ $\sigma_1 = 1$ $\sigma_2 = 1$

则 $X_2 | X_1 \sim N(2 + \frac{4}{5}(x-1), \frac{9}{25})$

$E(X_2 | X_1) = \frac{4}{5}x_1 + \frac{6}{5}$

$D(X_2 | X_1) = \frac{9}{25}$

(d) $X_2 - E(X_2|X_1) = X_2 - \frac{4}{5}X_1 - \frac{6}{5}$ 为高斯分布

而 $Cov(X_2 - E(X_2|X_1), X_1) = Cov(X_2, X_1) - \frac{4}{5}Cov(X_1, X_1) = 0$

$\therefore$ 相互独立

20. (a) 必要性：

$\because$ 相同联合分布 则 $\mu_t = \mu_0$ 恒成立

而 $Cov(X_{t_i}, X_{t_j}) = Cov(X_{t_i+a}, X_{t_j+a})$

$\therefore Cov(0, X_{t-s}) = Cov(X_s, X_t)$

充分性易证

(b) $U_t = e^{-\frac{at}{2}} B e^{at}$

$E U_t = e^{\frac{at}{2}} E B e^{at}$

若 $t \geq s$：

$Cov(U_s, U_t) = Cov(e^{-\frac{as}{2}} B e^{as}, e^{-\frac{at}{2}} B e^{at})$

$= e^{-\frac{a(t-s)}{2}}$ 只与 $t-s$ 有关

$\therefore$ 由前得，为平稳的Gauss过程

21. (a)

$B_t \sim N(0,t)$

$\therefore \sum_{i=1}^{n} B_i$ 服从Gauss分布

$\therefore E(\sum_{i=1}^{n} B_i) = 0$

$D(\sum_{i=1}^{n} B_i) = \sum_{i=1}^{n} DB_i + 2\sum\sum_{0\leq i<j\leq n} Cov(B_i, B_j)$

$= \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

$\therefore N(0, \frac{n(n+1)(2n+1)}{6})$

(b) 设 $t \geq s$

$s+1 \leq t$

则 $Cov(Y_s + Y_t) = 0$

$s \leq t < s+1$

则 $Cov(Y_s, Y_t) = s-t+1$

故 $Cov(Y_s, Y_t)$ 只依赖于 $t-s$

而 $EY_t = 0$

$\therefore$ 为平稳过程

23. $B_t \sim N(0,t)$

$s < t$ 时

$\begin{pmatrix} B_s \\ B_t \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} s & s \\ s & t \end{pmatrix}\right)$

$\therefore E(B_s | B_t = x) = \frac{xs}{t}$

12. $F(x)=\int_{-\infty}^{x} f(x)\,dx = 1-e^{-(x-a)}$

$P(M_i - a \geq \varepsilon) = 1-F(a+\varepsilon) = e^{-\varepsilon}$

$P(|Y_n - a| \geq \varepsilon) = e^{-n\varepsilon}$

$\lim\limits_{n\to\infty} P(|Y_n - a| \geq \varepsilon) = \lim\limits_{n\to\infty} e^{-n\varepsilon} = 0$

$\therefore M_n \xrightarrow{P} a$

补充题

1. $\begin{pmatrix} Y \\ Z \end{pmatrix} \sim N\left(\begin{pmatrix} A\mu+B \\ C\mu+D \end{pmatrix}, \begin{pmatrix} A\Sigma A^T, & \Sigma_{YZ} \\ \Sigma_{ZY} & C\Sigma C^T \end{pmatrix}\right)$

$\Sigma_{YZ} = cov(Y, Z)$

$= \begin{pmatrix} a_1^T \Sigma C_1 & \cdots & a_1^T\Sigma C_m \\ \vdots & & \\ a_n^T \Sigma C_1 & \cdots & a_n^T \Sigma C_m \end{pmatrix} : A\Sigma C^T$

同理 $\Sigma_{ZY} = C\Sigma A^T = (A\Sigma C^T)^T$

$YZ$ 独立 $\Leftrightarrow \begin{cases} A\Sigma C^T = 0 \\ (A\Sigma C^T)^T = 0 \end{cases} \Rightarrow A\Sigma C^T = 0$

2. (1) $X = 3B_1 + 2B_2 - B_3$ 仍为 Gauss 分布

$DX = 9DB_1 + 4DB_2 + DB_3 + 12cov(B_1, B_2) - 6cov(B_1, B_3) - 4cov(B_2, B_3)$

$= 18$

$\therefore X \sim N(0, 18)$ $\varphi_X(\theta) = e^{-9\theta^2}$

(2) $cov(X, Y) = 4C+5$

$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \left[N\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} DX & cov(X,Y) \\ cov(Y,X) & DY \end{pmatrix}\right]$

$X, Y$ 独立 $\Leftrightarrow cov(X, Y) = 0$

即 $C = -\frac{5}{4}$

(3) $E(X \mid B_2 = x) = 3E(B_1 \mid B_2 = x) + 2E(B_2 \mid B_2 = x)$

$- E(B_3 \mid B_2 = x)$

则 $E(X \mid B_2 = 1) = \frac{5}{2}$

三.

由 $(X, Y) \sim N(0, 0, \rho, 1, 1)$

则 $X \sim N(0,1)$

$Y \sim N(0,1)$

则 $X+Y$, $X-Y$ 也是Gauss的

而 $Cov(X+Y, X-Y) = 0$

$\therefore X+Y$ 和 $(X-Y)^{2022}$ 独立

故 $E((X+Y)(X-Y)^{2022}) = E(X-Y)^{2022} E(X+Y) = 0$

$D(X+Y) = E(X+Y)^2 = 2+2\rho$

$D(X-Y) = 2-2\rho$

$\therefore (X+Y) \sim (0, 2+2\rho)$

$(X-Y) \sim (0, 2-2\rho)$

$\therefore P(X^2 > Y^2) = P(X+Y > 0) P(X-Y) > 0 + P(X+Y < 0) P(X-Y < 0) = \frac{1}{2}$