Technical introduction to Neural Networks

February 8, 2019

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Where σ is a non linear function, for instance a **sigmoid**.

Notation with vectors

The sum $\sum_{i=1}^{n} x_i w_i$ can also be written this way :

$$xw^T \tag{3}$$

This means a **product** of **two matrices** (a vector is also a matrix : it is just a matrix with only one line or only one column) : :

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This means a **product** of **two matrices** (a vector is also a matrix : it is just a matrix with only one line or only one column) : :

- $x = (x_1, ..., x_n)$
- $$w^T = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

Matrices

A matrix is an array used to store data. It has **lines** and **columns**

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Product of matrices

- If matrix A has p columns and matrix B has p lines, we can compute the product of the two AB of the two matrices in the following way :

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- This kind of computation is very often used
- It is way more convenient and concise to use
- We will use it when studying neural networks

Examples

$$\begin{pmatrix} 1 & 4 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = ?$$

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Examples

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

What is A^n ?

Examples

- $x = (x_1, \dots, x_n)$
- $w = (w_1, \dots, w_n)$

$$xw^T = ? \tag{8}$$

Examples

- $x = (x_1, ..., x_n)$
- $w = (w_1, ..., w_n)$

$$xw^T = \sum_{i=1}^{n} x_i w_i \tag{9}$$

Neural networks

- $\sigma(xw^T)$ allows us to compute the output of a **single neuron**
- But we will often have **several neurons** outputting a result.
- These neurons are organized in a network called neural network.

Neural networks

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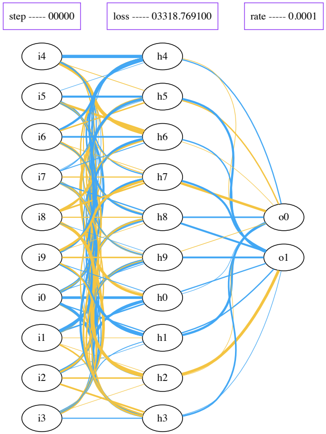
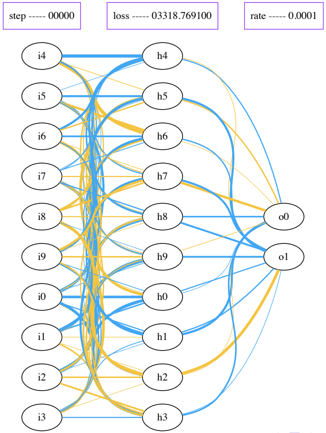


Matrices and networks

- Let $W = [W_{ij}]$ be the matrices of weights between neuron i of the left layer and neuron j of the middle layer.



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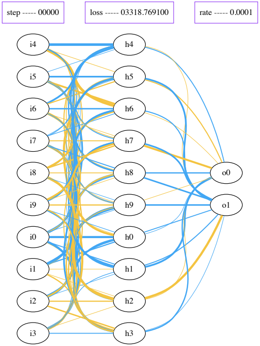
- Let $W = [W_{ij}]$ be the matrices of weights between neuron i of the left layer and neuron j of the middle layer.
- How can we write the inputs b_j of the middle layer as a function of the outpus x_k of the left layer ?



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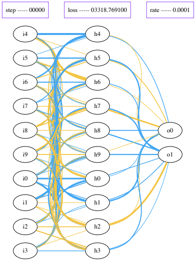
$$b_j = \sum_{k=1}^{n} x_k w_{kj} \tag{10}$$



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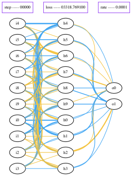
- let $w = [w_{ij}]$ be the matrices of weights between neuron i of the left layer and neuron j of the middle layer.

$$b_j = \sum_{k=1}^{n} x_k w_{kj} \tag{11}$$

- And in terms of matrices ? with :
  - $b = (b_1, ..., b_n)$
  - $x = (x_1, .., x_n)$



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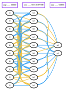
$$b = xw \tag{13}$$



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- let $w = [w_{ij}]$ be the matrices of weights between neuron i of the left layer and neuron j of the middle layer.

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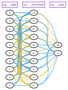
- Finally, if we want to store the outputs for several input vectors x ?



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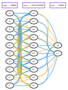
- Finally, if we want to store the outputs for several input vectors x ?
  - use matrices $x = [x_{ij}]$ and $b = [b_{ij}]$
  -
$$b = xw \tag{16}$$

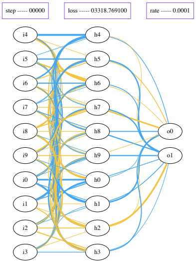


Layers

- We now know how to compute the input b of a layer as a function of the output x of the previous layer

$$b = xw \tag{17}$$

- Applying this rule **and** the non linearity σ, we can coompute the **forward propagation** of a neural network.



Layers and forward propagation

- We will use the **numpy** library to do so.
- In numpy, the **.dot** function is used to compute products of matrices
- We will use de ReLu non linearity.

Cost

- We saw how to compute the output of a neural network given an input
- But what do we want to do with the network ?

Cost

- We saw how to compute the output of a neural network given an input
- But what do we want to do with the network ?
- We want to solve a given problem, for instance a **supervised learning problem**
- This means being able to predict the output as a function of an input.

Supervised learning

- In order to learn the prediction, we **optimize** our network with training examples
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$$\sum_{\text{training samples}} \text{error}^2 \tag{18}$$

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$$SE = \sum_{\text{training samples}} (y_{\text{prediction}} - y_{\text{truth}})^2 \tag{19}$$

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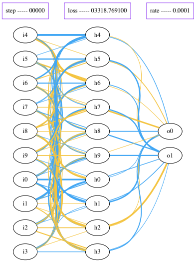
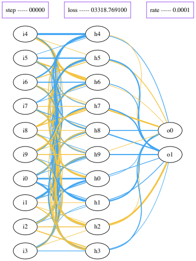
- In our examples we will use a network with three layers : input - hidden - output.



Optimization

- What are the **parameters** of the network ? ie : what we have control on and what we can change in order to minimize the error.
- The **weights matrices** w_1 and w_2.
- In our examples we will use a network with three layers : input - hidden - output.
- We wee the loss as a function $L(w1, w2)$



Gradient descent

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- If x is a local extremum, $f'(x) = 0$
- Is the reciprocal true ?

Derivation

- We can use the derivative to look for a minimum value for the function
- Example with analytic solution

gradient

- the **gradient** is similar to a derivative but in the case of a function with several inputs, such as our loss $l(w1, w2)$.
- then we store the **partial derivative** with respect to each input in a **vector** called the gradient.
- let us compute ror instance the partial derivative with respect to $w1$

gradient

- then we store the **partial derivative** with respect to each input in a **vector** called the gradient.
- let us compute ror instance the partial derivative with respect to $w1$

$$\frac{\partial L}{\partial w1} = h^T 2(y_{\text{predicted}} - y_{truth}) \tag{20}$$

(where h is the output of the relu)

Backpropagation

- By repeating the same process we can also compute the gradient with respect to $w2$.
- This is called **backpropagation**
- Knowing the gradient, we can **update the network parameters**

Application to our neural network

- We will apply this to do some supervised learning over two datasets:
  - A random dataset
  - A structured dataset

Libs

- We will need **numpy**
- **pygraphviz**
- optionally **pytorch** (less important for us today)
- It will most probably work better with python3.6

Exercise 1

- **cd neural_net**
- Use **learn_random _data** to learn to predict some output as a function of some input
- You will need to tune some hyperparameters.

Exercise 2

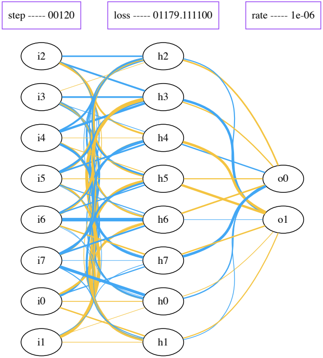
- Now make it find a local minimum that is not global

Exercise 3

- Now make it explode (diverge)

Exercise 4

- Uncomment the lines calling **show_net** so plot the evolution of the network
- You might need to use a smaller network otherwise it will be too long.



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- can we generalize what we learned ?
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- What is the test error ? (use the relevant function)

Exercise 6

- We will now use structured data
- They are artificially generated in **create_structured_data** by a function, with a noise that you can choose
- You can tune the standard deviation of the noise
- Use **create_structured_data** and **learn_structured_data** to generate a dataset and learn it in a supervised way

Exercise 7

- Make it find a local minimum

Exercise 8

- Make it explode

Exercise 9

- Uncomment the relevant stuff to visualize the evolution of the network with graphviz
- You wan choose the number of hidden layers

exercise 10

- how does the test error behave ?

exercise 10

- how does the test error behave ?
- Now it makes more sense since we worked on data with structure

With libs

- We did things manually with numpy but when the networks are large or when we need to automate the seach for good parameters, it is more convenient to work with libraries such as :
  - pytorch
  - tensorflow
  - keras

Many techniques

- There are lots of variations around neural nets
  - in the number of neurons per layer
  - type of data processed
  - relationship between weights (shared weights)
  - number of hidden layers
  - recurrent neural networks RNN
  - convolutional neural networks CNN

Stochastic gradient

- Another method to compute the gradient

Other cost functions

- Until now we used the squared error cost function
- The slowdown problem
- The Cross entropy is another possible cost function used for classification