

1 Self-paced Adaptive Bipartite Graph Learning for Consensus Clustering

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7
8
9 Consensus clustering provides an elegant framework to aggregate multiple weak clustering results to learn a consensus one that
10 is more robust and stable than a single result. However, most of the existing methods usually use all data for consensus learning,
11 whereas ignoring the side effects caused by some unreliable or difficult data. To address this issue, in this paper, we propose a
12 novel self-paced consensus clustering method with adaptive bipartite graph learning to gradually involve data from more reliable
13 to less reliable ones in consensus learning. At first, we construct an initial bipartite graph from the base results, where the nodes
14 represent the clusters and instances, and the edges indicate that an instance belongs to a cluster. Then, we adaptively learn a
15 structured bipartite graph from this initial one by self-paced learning, i.e., we automatically determine the reliability of each edge
16 with adaptive cluster similarity measuring and involve the edges in bipartite graph learning in order of their reliability. At last, we
17 obtain the final consensus result from the learned structured bipartite graph. We conduct extensive experiments on both toy and
18 benchmark data sets, and the results show the effectiveness and superiority of our method. The codes of this paper are released in
19 http://Doctor-Nobody.github.io/codes/code_SCCABG.zip.
20
21

22 CCS Concepts: • Computing methodologies → Ensemble methods.
23

24 Additional Key Words and Phrases: Consensus clustering, clustering ensemble, bipartite graph learning, self-paced learning
25

26 ACM Reference Format:

27 Peng Zhou, Xinwang Liu, Liang Du, and Xuejun Li. 2018. Self-paced Adaptive Bipartite Graph Learning for Consensus Clustering. In
28 *ACM Transactions on Knowledge Discovery from Data*, 2022. ACM, New York, NY, USA, 35 pages. <https://doi.org/10.1145/1122445.1122456>
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30

31 1 INTRODUCTION

32 Clustering is a fundamental unsupervised learning problem in machine learning and artificial intelligence, and attracts
33 increasingly more attention in recent years. However, it is well known that conventional clustering methods often
34 suffer from stability and robustness problems [49, 50]. To tackle these problems, consensus clustering is proposed.
35

36 Consensus clustering, also known as clustering ensemble, was first proposed by Strehl et al. [43], and it aims to
37 ensemble multiple weak base clustering results to obtain a robust and stable consensus clustering result. In recent years,
38 many consensus clustering methods have been proposed [29–31, 44, 52]. For example, Li et al. applied non-negative
39 matrix factorization to ensemble multiple clusterings [29]; Liu et al. applied the de-noising auto-encoder to learn the
40 consensus clustering result [31]; Tao et al. proposed a clustering ensemble method with adversarial loss [44].
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53 Although these methods achieve promising performance on clustering, they may still suffer from robustness problems.
 54 We observe that these methods always use *all* data for consensus learning. However, since the base clustering results
 55 may be imperfect, it is inappropriate to always apply all data for learning. These methods may be misled by some
 56 difficult or unreliable data in the process of the ensemble. Intuitively, in the beginning, we should not use the difficult or
 57 unreliable data for ensemble, because this early model may be too weak to handle them. Then, with consensus learning,
 58 the model becomes increasingly stronger and gradually obtains the ability to tackle those difficult and unreliable data.
 59

60 To fulfill this idea, we propose a novel Self-paced Consensus Clustering with Adaptive Bipartite Graph (SCCABG)
 61 method, which applies instances from more reliable to less reliable ones to learn the consensus result. By observing that
 62 the base clustering results can naturally be represented as a bipartite graph, where the nodes represent the instances
 63 and clusters and an edge indicates that an instance belongs to a cluster, we develop the consensus clustering method
 64 on the bipartite graph. We aim to dynamically learn a structured bipartite graph from an initial one, which contains
 65 exact c components, where c is the number of clusters. Note that, since the base results are imperfect, the edges in the
 66 initial bipartite graph are also unreliable, and thus the initial bipartite graph may not reveal such a clear clustering
 67 structure. Therefore, we plug the self-paced learning into the structured bipartite graph learning, leading to a self-paced
 68 adaptive bipartite graph learning. That is, we automatically determine the reliability of each edge, and adopt the edges
 69 in the order of their reliability to adaptively learn the structured graph. On one hand, the reliable edges are helpful to
 70 structured graph learning. On the other hand, in the process of structured graph learning, increasingly more edges
 71 become reliable. To characterize the reliability of the edges, we carefully design a regularized term with an adaptive
 72 cluster similarity measuring method. Due to the reliability characterized term, the proposed model is more robust and
 73 can even handle incomplete data. At last, we directly obtain the final consensus clustering result from the learned
 74 structured bipartite graph by finding its connective components. Therefore, the proposed method is in an end-to-end
 75 way without any uncertain postprocessing such as k-means and spectral clustering.
 76

77 Although the objective function seems complex due to the carefully designed regularized term, we develop an
 78 effective block coordinate descent algorithm to optimize it, whose convergence is theoretically guaranteed. The
 79 extensive experimental results on benchmark data sets well demonstrate the effectiveness and superiority of the
 80 proposed algorithm.

81 Notice that this work is an extension of our early work [62]. The present work adds to the conference version in
 82 some significant ways:

- 83 • Firstly, the reliability evaluation in [62] uses the fixed cluster similarity matrix of clusters, which is inappropriate
 84 for consensus learning due to the unreliability of the base clusterings. Notice that, in [62], it constructs the
 85 similarity matrix according to the base partitions. However, base partitions are often unreliable as introduced
 86 before, and thus the similarity matrix used in [62] is also unreliable and may still mislead the bipartite graph
 87 learning. To address this issue, in this paper, we design an adaptive cluster similarity measuring mechanism,
 88 i.e., the cluster similarity matrix updates with the bipartite graph learning. With the bipartite graph becoming
 89 increasingly more reliable, the similarity matrix will also become more accurate. The ablation study in our
 90 experiments demonstrates that.
- 91 • Secondly, in real applications, it often happens that some data are missing in some base results. In this paper,
 92 we also apply the proposed method to handle this incomplete consensus clustering setting.
- 93 • Thirdly, considerable new theoretical analyses and technical details are provided in this paper.

- 105 • Lastly, we extend the experiments by adding a toy example to intuitively show the effectiveness of the proposed
 106 method and adding the most recent state-of-the-art consensus clustering method for comparison on benchmark
 107 data sets. We also add some more experiments to comprehensively show the performance of our method.

109
 110 The remaining parts of this paper are organized as follows. Section 2 provides some related work. Section 3 introduces
 111 our SCCABG in detail. Section 4 shows the experimental results. Section 5 concludes this paper.
 112

113 2 RELATED WORK

115 In this section, we briefly review some related works about consensus clustering and self-paced learning. Firstly, we
 116 introduce some notations used in this paper. Boldface uppercase and lowercase letters are used to denote matrices
 117 and vectors, respectively. For a matrix \mathbf{A} , $\mathbf{A}_{\cdot i}$ and $\mathbf{A}_{i \cdot}$ are used to denote the i -th row and column vector of matrix \mathbf{A} ,
 118 respectively. A_{ij} denotes the (i, j) -th element of \mathbf{A} .
 119

121 2.1 Consensus Clustering

123 Consensus clustering, also known as clustering ensemble or clustering aggregation, aims to aggregate multiple weak
 124 base clustering results to generate a consensus and robust one. More formally, according to [43, 48, 49], given a data
 125 set $\mathcal{X} = \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ with n instances, we first generate m base partitions C^1, \dots, C^m by some standard clustering method,
 126 where C^i contains k_i clusters $\pi_1^i, \dots, \pi_{k_i}^i$, such that $\mathcal{X} = \bigcup_{j=1}^{k_i} \pi_j^i$ and $\pi_p^i \cap \pi_q^i = \emptyset$ for any $p, q \in \{1, 2, \dots, k_i\}$ and
 127 $p \neq q$. Then consensus clustering learns a consensus partition by integrating C^1, \dots, C^m .
 128

129 One related task of consensus clustering is multi-view clustering [8, 11, 21, 32, 41, 65, 67]. They integrate multiple
 130 views of data feature to generate a consensus clustering result. For example, Cai et al. integrated kmeans and multiple
 131 views fusion into a unified objective function, leading to a multi-view kmeans method [8]; Zhou et al. proposed an
 132 incremental multi-view spectral clustering to handle streaming views data [67]; Peng et al. developed a parameter-free
 133 multi-view clustering method by learning a consensus embedding of multiple views [41]; Liu et al. ensembled incomplete
 134 multi-view data to obtain a robust and consensus clustering result [32]. Note that, these multi-view methods usually
 135 take multi-view features of data as input, and they often integrate information at the data level or model level. Different
 136 from multi-view clustering, consensus clustering often ensembles at the decision level, i.e., it only takes the multiple
 137 base clustering results as input without accessing the original data features. Therefore, consensus clustering is a more
 138 challenging task. Moreover, since consensus clustering does not access the original data, it can protect the privacy of
 139 data to some extent [25].
 140

141 Since consensus clustering can often provide a more robust and stable clustering result than the single clustering
 142 methods and does not need to access original data, it attracts increasingly more attention in recent years. One of the
 143 most popular schema to ensemble the multiple base results is using the connective matrices [18, 28, 45, 47, 63, 64, 66]. For
 144 each base partition C^i , they construct an $n \times n$ connective matrix $\mathbf{H}^{(i)}$, where $H_{pq}^{(i)} = 1$ if \mathbf{x}_p and \mathbf{x}_q belong to the same
 145 cluster in C^i and $H_{pq}^{(i)} = 0$ otherwise. Then they ensemble $\mathbf{H}^{(1)}, \dots, \mathbf{H}^{(m)}$ to learn a consensus matrix \mathbf{H} and obtain the
 146 final consensus clustering result from \mathbf{H} . For example, Li et al. applied symmetric nonnegative matrix factorization to
 147 obtain the consensus results from connective matrices [28]; Zhou et al. extracted the noises on connective matrices and
 148 recover the clean ones for ensemble [66]; Tao et al. used spectral clustering to learn the consensus partition from the
 149 connective matrices [45, 47]; Zhou et al. perform self-paced learning on the connective matrices to obtain the final
 150 consensus result [63].
 151

157 Although connective matrix based methods have demonstrated promising performance in literature, they need to
 158 construct $m n \times n$ connective matrices which are inappropriate to handle large scale data sets. Therefore, many methods
 159 try to ensemble base clusterings with other data structures. For example, Strehl et al. constructed hyper-graphs from base
 160 clustering results and obtain final consensus result by partitioning the hyper-graph [43]; Zhou et al. applied an alignment
 161 method to directly combine the base clustering result matrices [69]; Liu et al. proposed a new ensemble method to
 162 handle missing values in base results, which also directly uses the result matrices [34]; Huang et al. constructed a factor
 163 graph for consensus clustering [14]; Li et al. applied multiple clustering results to measure the stability of each instance
 164 and assigned instances in the cluster according to the stability [25]; Huang et al. ensembled multiple clustering results
 165 by a fast propagation of cluster-wise similarity [17]; Bai et al. proposed a new graph based consensus clustering methods
 166 which used the k-means as the base clustering methods to handle non-linear data [4]; Abbasi et al. and Bagherinia et al.
 167 applied quality and diversity of base results to guide the consensus clustering [1, 3].
 168

169 Besides the above-mentioned works which ensemble all base clustering results, some works aim to select some
 170 informative and non-redundant base clustering results for consensus clustering. For example, Azimi et al. provided
 171 an adaptive consensus clustering selection method to select the informative base results [2]; Parvin et al. developed a
 172 weighted locally adaptive clustering method for consensus clustering selection [39, 40]; Yu et al. transferred the base
 173 results selection to feature selection and proposed a hybrid strategy to select base clusterings [55]. These methods
 174 concentrated on how to select base results. However, in this paper, we focus on how to ensemble base clustering results.
 175

176 In this paper, we learn the consensus result on a bipartite graph. Different from conventional methods, which may
 177 be misled by unreliable data, our bipartite graph learning method is in a self-paced learning framework, which could
 178 alleviate the side effects caused by unreliable data.
 179

180 2.2 Self-paced Learning

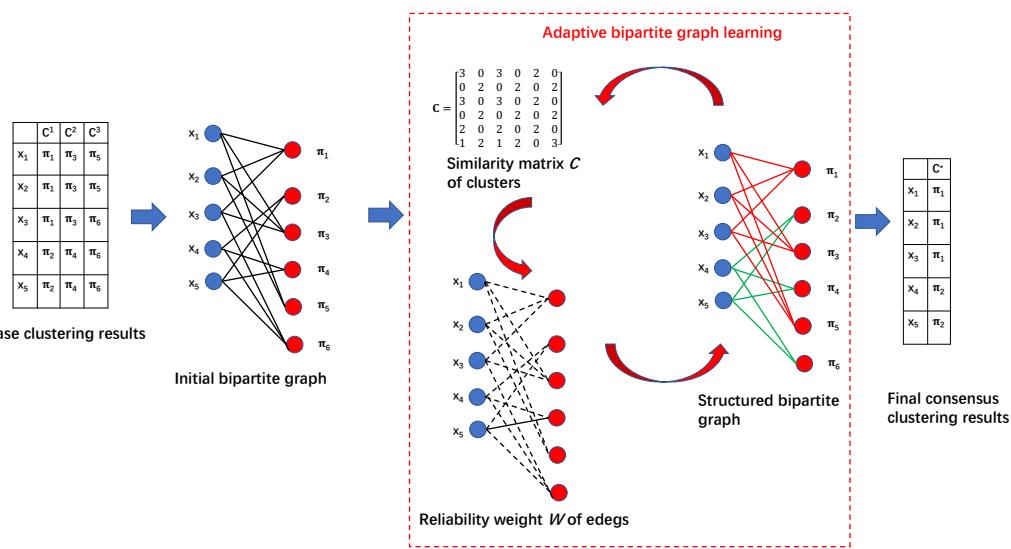
181 To mimic the learning process of humans, self-paced learning incrementally involves data in learning, where easy ones are
 182 used first and difficult ones are then involved gradually [22]. More formally, given a data set $\mathcal{X} = \{(\mathbf{x}_1, y_1), \dots, (\mathbf{x}_n, y_n)\}$,
 183 where \mathbf{x}_i is the feature vector of the i -th instance and y_i is its label, in machine learning tasks, we aim to learn a
 184 hypothesis $h(\mathbf{x}_i, \theta)$ where θ denotes the model parameters. To learn the parameters θ , we need to minimize some loss
 185 function $\mathcal{L}(h(\mathbf{x}_i, \theta), y_i)$ between the hypothesis and the ground truth. According to [56], self-paced learning introduces
 186 a weighted loss term on instances and a general regularized term on the weights as follows:
 187

$$188 \min_{\mathbf{w}, \theta} \sum_{i=1}^n w_i \mathcal{L}(h(\mathbf{x}_i, \theta), y_i) + \Omega(w_i, \lambda), \quad (1)$$

189 where w_i is the weight of the i -th instance, λ is an adaptive age parameter that grows in the process of learning, and
 190 $\Omega(w_i, \lambda)$ is the self-paced regularized term on the weights.
 191

192 In Eq. (1), when fixing θ , we denote $w_i^*(\lambda, l_i)$ as the optimum weight on \mathbf{x}_i where $l_i = \mathcal{L}(h(\mathbf{x}_i, \theta), y_i)$. Then,
 193 according to [19, 35, 58], the regularized term $\Omega(w_i, \lambda)$ should satisfy the following properties: (1) $w_i^*(\lambda, l_i)$ should
 194 decrease monotonically with l_i because instances with small loss, i.e., easy instances, should have large weights, so that
 195 they could be involved in learning early. (2) $w_i^*(\lambda, l_i)$ should increase monotonically with λ , so that with the process of
 196 learning, more and more instances will be involved in learning. Therefore, in self-paced learning, Eq. (1) is solved in an
 197 iterative way. When fixing θ , it optimizes \mathbf{w} . This process is to assign weight to each instance. When fixing \mathbf{w} , it learns
 198 the model parameter θ . This process is to train the model with easy data.
 199

209 Due to its promising performance, self-paced learning has already been adopted in many applications. For example,
 210 in [5, 6], it was used to tackle the local optimum problem in non-convex optimization; in [24, 42], it was plugged in the
 211 multi-task learning; Jiang et al. extended it into subspace learning [20]; Guo et al. applied it to deep clustering [13]; Pan
 212 et al. proposed a self-paced deep regression forests method [38]. In this paper, we will plug self-paced learning into
 213 bipartite graph learning for unsupervised consensus learning.
 214



235 Fig. 1. The framework of SCCABG. It first generates an initial bipartite graph from base results, and then adaptively learns a structured
 236 bipartite graph with self-paced learning and adaptive cluster similarity measuring. At last, it obtains the final consensus clustering
 237 result from the learned structured bipartite graph.

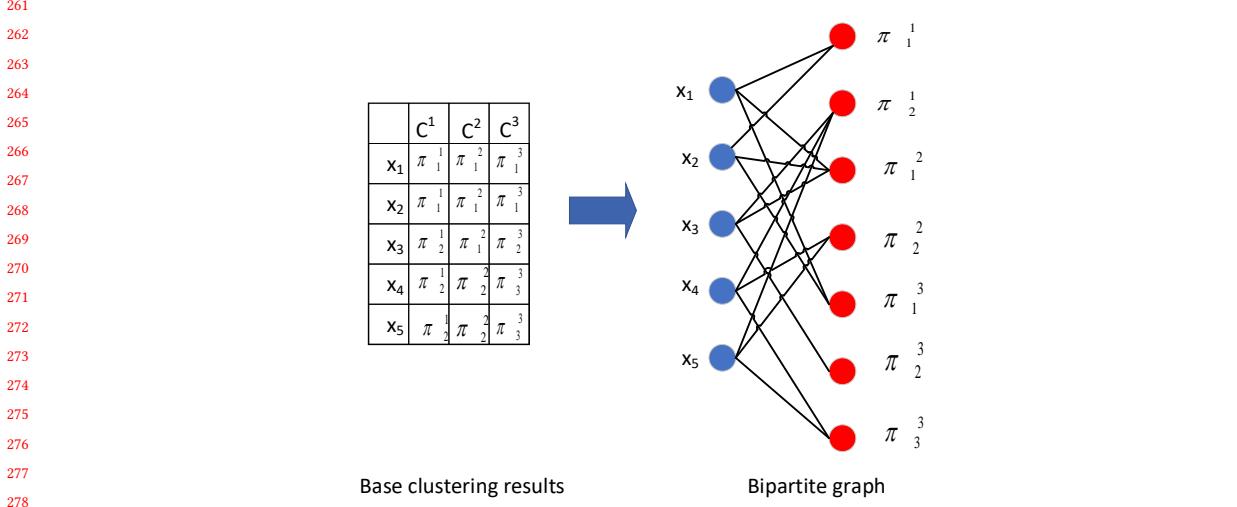
240 3 SELF-PACED CONSENSUS CLUSTERING WITH ADAPTIVE BIPARTITE GRAPH

241 In this section, we introduce the proposed SCCABG method. Figure 1 shows the framework of SCCABG. We first
 242 construct an initial bipartite graph from the given multiple base results. Then we adaptively learn a structured bipartite
 243 graph with self-paced learning and adaptive cluster similarity measuring. At last, we obtain the final clustering result
 244 directly from the structured bipartite graph. The details of each step are introduced in the following subsections.
 245

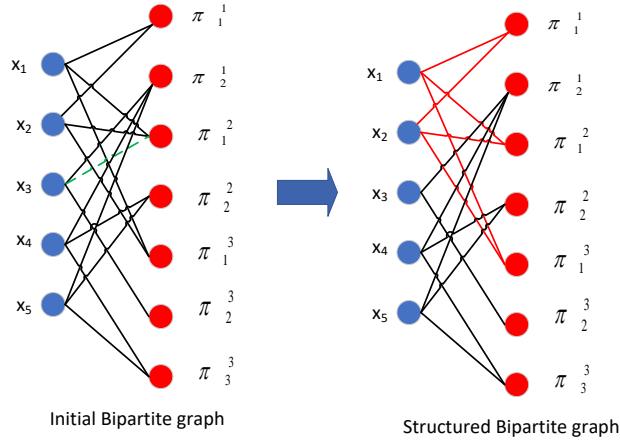
247 3.1 Initial Bipartite Graph Construction

248 Given m base clustering results $C = \{C^1, \dots, C^m\}$ of a data set with n instances $X = \{x_1, \dots, x_n\}$, where $C^i = \{\pi_1^i, \dots, \pi_{k_i}^i\}$, we first construct an initial bipartite graph $\mathcal{G} = (\mathcal{V}^1, \mathcal{V}^2, \mathcal{E})$ from it.
 249

250 In more detail, \mathcal{V}^1 contains n nodes and each node represents an instance. \mathcal{V}^2 contains $k = \sum_{i=1}^m k_i$ nodes and each
 251 node represents a cluster π_j^i ($i = 1, \dots, m$, and $j = 1, \dots, k_i$). \mathcal{E} is a set of edges which link nodes between \mathcal{V}^1 and
 252 \mathcal{V}^2 . If instance x_i belongs to the cluster π_p^q , then there is an edge between x_i and π_p^q . Figure 2 shows an illustration of
 253 constructing the bipartite graph. In this example, we have 5 instances x_1, \dots, x_5 and 3 base clusterings. For example,
 254 in the first clustering C^1 , x_1 and x_2 belong to cluster π_1^1 , and x_3, x_4 and x_5 belong to the cluster π_2^1 . The right side of
 255 Figure 2 shows the corresponding bipartite graph \mathcal{G} . \mathcal{V}^1 contains the blue nodes, \mathcal{V}^2 contains the red nodes, and \mathcal{E}
 256 denotes the set of edges. In the following, for the simplicity of notations, we use π_1, \dots, π_k to denote all base clusters.
 257



313 2 connective components (i.e., one is denoted in the red lines and the other is denoted in black lines). We find that
 314 we just need to adjust one edge in the initial bipartite graph which is denoted as the green dotted line, and we can
 315 obtain the structured one. Then clustering on \mathcal{G}' is trivial because we just need to put instances in the same connective
 316 components into the same cluster.
 317



337 Fig. 3. An illustration of constructing structured hypergraph.
 338

339 Similar to G , we can write the adjacent matrix of \mathcal{G}' as follows:
 340

$$341 \quad 342 \quad 343 \quad G' = \begin{bmatrix} 0 & S \\ S^T & 0 \end{bmatrix}, \quad (3)$$

344 where $S \in [0, 1]^{n \times k}$. To make \mathcal{G}' preserve G as well as possible, we should minimize the difference $\|S - Y\|_F^2$. Moreover,
 345 we also need to impose some constraints on S to make sure that \mathcal{G}' has c connective components.
 346

347 Given G' , we first obtain its normalized Laplacian matrix $L = I - D^{-\frac{1}{2}}G'D^{-\frac{1}{2}}$, where I is an identity matrix and
 348 D is a diagonal matrix whose diagonal element $D_{ii} = \sum_{j=1}^{k+n} G'_{ij}$. Then according to [36], we have that the number of
 349 connected components in \mathcal{G}' is equal to $n + k$ minus the rank of L , i.e., $\text{rank}(L) = n + k - c$. To this end, we obtain the
 350 following formula:
 351

$$352 \quad 353 \quad \min_S \|S - Y\|_F^2, \quad (4)$$

$$354 \quad s.t. \quad 0 \leq S_{ij} \leq 1, \quad \text{rank}(L) = n + k - c.$$

3.3 Self-paced Bipartite Graph Learning

358 Eq. (4) provides a framework to learn an adaptive structured bipartite graph. However, as introduced before, since the
 359 base clusterings may be imperfect, each edge obtained from the base clusterings may also be unreliable. To characterize
 360 the reliability of each edge, we introduce a weight matrix $W \in [0, 1]^{n \times k}$ where the larger W_{ij} is, the more reliable the
 361 corresponding edge is. With W we can integrate our consensus clustering task into a self-paced learning framework
 362 seamlessly, which involves edges gradually from more reliable ones to less reliable ones. As suggested in [19], we set
 363

365 $\Omega(w_i, \lambda)$ in Eq. (1) as $-\lambda\|\mathbf{W}\|_1$, and obtain:

$$\min_{\mathbf{S}, \mathbf{W}} \|\mathbf{W} \odot (\mathbf{S} - \mathbf{Y})\|_F^2 - \lambda\|\mathbf{W}\|_1, \quad (5)$$

$$\text{s.t. } 0 \leq S_{ij} \leq 1, \quad \text{rank}(\mathbf{L}) = n + k - c, \quad 0 \leq W_{ij} \leq 1,$$

370 where \odot is the Hadamard product, which means the element-wise production of two matrices; the second term is the
 371 self-paced regularized term, and λ is the age parameter and becomes increasingly larger in the process of optimization.
 372

373 Unfortunately, it is not enough to characterize the reliability of edges only by the first term in Eq. (5). We need to
 374 take a closer look at the \mathbf{W} . Notice that, if two clusters π_p and π_q are very similar, then for any instance \mathbf{x}_i , it is very
 375 likely that either \mathbf{x}_i belongs to both clusters or \mathbf{x}_i belongs to neither. Therefore, if $(S_{ip} - S_{iq})^2$ is large, which means
 376 \mathbf{x}_i is more likely to belong to one of the clusters, then at least one of S_{ip} and S_{iq} is unreliable, i.e., at least one of W_{ip}
 377 and W_{iq} should be small. More formally, we use the following carefully designed regularized term to characterize the
 378 reliability of edges:
 379

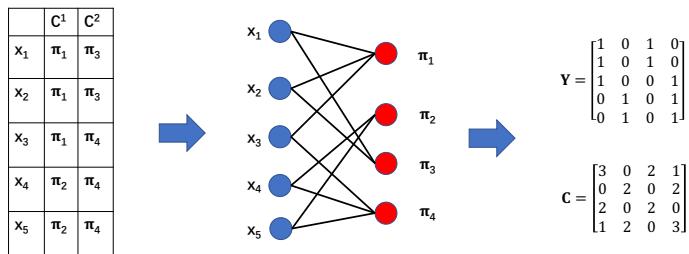
$$\min_{\mathbf{W}} \sum_{i=1}^n \sum_{p,q=1}^k C_{pq}(S_{ip} - S_{iq})^2 W_{ip} W_{iq}, \quad (6)$$

384 where C_{pq} is the (p, q) -th element in $\mathbf{C} \in \mathbb{R}^{k \times k}$, which characterizes the similarity of two clusters. In our previous
 385 work [62], we simply fix \mathbf{C} as $\mathbf{C} = \mathbf{Y}^T \mathbf{Y}$. We can find that, if C_{pq} is large (i.e., π_p and π_q are similar) and $(S_{ip} - S_{iq})^2$
 386 is large (i.e., \mathbf{x}_i only belongs to one of the clusters and does not belong to the other cluster), then the only chance to
 387 minimize Eq. (6), which is a product of C_{pq} , $(S_{ip} - S_{iq})^2$, W_{ip} and W_{iq} , is that at least one of W_{ip} and W_{iq} should be
 388 small, which means at least one of the relationship S_{ip} and S_{iq} is unreliable. Taking it into our self-paced framework
 389 (Eq. (5)), we obtain the following objective function:
 390

$$\min_{\mathbf{S}, \mathbf{W}} \|\mathbf{W} \odot (\mathbf{S} - \mathbf{Y})\|_F^2 - \lambda\|\mathbf{W}\|_1 + \gamma_1 \sum_{i=1}^n \sum_{p,q=1}^k C_{pq}(S_{ip} - S_{iq})^2 W_{ip} W_{iq} \quad (7)$$

$$\text{s.t. } 0 \leq S_{ij} \leq 1, \quad \text{rank}(\mathbf{L}) = n + k - c, \quad 0 \leq W_{ij} \leq 1,$$

395 where γ_1 is a balanced parameter. In practice, γ_1 is small to make the subproblems involving \mathbf{W} and \mathbf{S} convex.
 396



408 Fig. 4. A toy example of the reliability. There are 5 instances x_1, \dots, x_5 , wherein the first base clustering result, x_1, \dots, x_3 belong to
 409 π_1 , and x_4 and x_5 belong to π_2 . In the second base result, x_1 and x_2 belong to π_3 and the other three instances belong to π_4 . The
 410 middle side shows the initial bipartite graph, and the right side shows the corresponding \mathbf{Y} and initial similarity matrix of clusters \mathbf{C} .
 411

412 Figure 4 provides a simple toy example to show the effects of the regularized term of reliability. There are 5 instances
 413 x_1, \dots, x_5 , wherein the first base clustering result, x_1, x_2, x_3 belong to π_1 , and x_4 and x_5 belong to π_2 . In the second base
 414 result, x_1 and x_2 belong to π_3 and the other three instances belong to π_4 . Its \mathbf{Y} and initial similarity matrix of clusters \mathbf{C}
 415

are shown on the right side of Figure 4. Notice that π_1 and π_3 are similar, because the only difference between them is x_3 . π_2 and π_4 are also similar, and the only difference is also x_3 . Intuitively, since π_1 and π_3 are similar, and x_3 only belongs to π_1 but does not belong to π_3 , at least one of S_{31} and S_{33} is unreliable. Similarly, at least one of S_{32} and S_{34} is unreliable.

Considering the subproblem w.r.t. $\mathbf{W}_{3.}$, which is corresponding to x_3 , we minimize the following formula:

$$\min_{\mathbf{W}_{3.}} \|\mathbf{W}_{3.} \odot (\mathbf{S}_{3.} - \mathbf{Y}_{3.})\|_2^2 + \sum_{p,q=1}^4 C_{pq} (S_{3p} - S_{3q})^2 W_{3p} W_{3q} - \|\mathbf{W}_{3.}\|_1$$

For simplicity, we set $\gamma_1 = 1$ and $\lambda = 1$. In the first iteration, we initialize $\mathbf{S} = \mathbf{Y}$ and initialize \mathbf{C} as shown in Figure 4. Notice that $C_{12}, C_{21}, C_{23}, C_{32}, C_{34}$, and C_{43} are zeros, and $S_{31} - S_{34}$ and $S_{32} - S_{33}$ are zeros. When removing the zero terms, we have

$$\begin{aligned} \min_{\mathbf{W}_{3.}} & 4(S_{31} - S_{33})^2 W_{31} W_{33} + 4(S_{32} - S_{34})^2 W_{32} W_{34} - \|\mathbf{W}_{3.}\|_1 \\ & = 4W_{31} W_{33} + 4W_{32} W_{34} - \|\mathbf{W}_{3.}\|_1 \end{aligned}$$

Since in the first term we need to minimize $W_{31} W_{33}$, at least one of W_{31} and W_{33} should be small. Similarly, at least one of W_{32} and W_{34} should be small. In fact, when minimizing it, we obtain $W_{31} = W_{32} = W_{33} = W_{34} = 0.25$ which are all small. It is consistent with our intuition.

3.4 Adaptive Cluster Similarity Measuring

In our previous work [62], we use $\mathbf{Y}^T \mathbf{Y}$ as the cluster similarity matrix \mathbf{C} . However, as mentioned before, the base clustering results \mathbf{Y} may be unreliable, and thus $\mathbf{C} = \mathbf{Y}^T \mathbf{Y}$ is also unreliable. Since we evaluate the reliability \mathbf{W} of edges according to \mathbf{C} as shown in Eq. (6), unreliable \mathbf{C} may also mislead the evaluation of \mathbf{W} . Therefore, to better evaluate the reliability of the edges, we need a more accurate measurement of the cluster similarity.

An ideal cluster similarity measuring method is that the similarity of clusters should be adaptively adjusted during the bipartite graph learning process. On one hand, a more reliable bipartite graph can lead to a more accurate cluster similarity measuring; and on the other hand, a more accurate cluster similarity measuring can guide us to determine the reliability of edges more accurately and further be helpful to learning a better structured graph. To this end, \mathbf{C} in Eq. (7) should be learned as a variable instead of being predefined.

To adaptively learn the similarity matrix, we need some prior knowledge on \mathbf{C} . Although predefining $\mathbf{C} = \mathbf{Y}^T \mathbf{Y}$ may be inappropriate, $\mathbf{Y}^T \mathbf{Y}$ can still be regarded as a good prior on \mathbf{C} . Before graph learning, we have no extra information on \mathbf{C} except \mathbf{Y} , and thus we can initialize $\mathbf{C} = \mathbf{Y}^T \mathbf{Y}$ and hope \mathbf{C} should not be too far away from $\mathbf{Y}^T \mathbf{Y}$. Then, in the process of learning, we can adaptively fine-tune \mathbf{C} based on \mathbf{S} and \mathbf{W} . More formally, we can obtain the following objective function:

$$\begin{aligned} \min_{\mathbf{S}, \mathbf{W}, \mathbf{C}} & \|\mathbf{W} \odot (\mathbf{S} - \mathbf{Y})\|_F^2 - \lambda \|\mathbf{W}\|_1 + \gamma_1 \sum_{i=1}^n \sum_{p,q=1}^k C_{pq} (S_{ip} - S_{iq})^2 W_{ip} W_{iq} + \gamma_2 \|\mathbf{C} - \mathbf{Y}^T \mathbf{Y}\|_F^2, \\ \text{s.t. } & 0 \leq S_{ij} \leq 1, \quad \text{rank}(\mathbf{L}) = n + k - c, \quad 0 \leq W_{ij} \leq 1, \quad C_{pq} \geq 0, \quad \mathbf{C} = \mathbf{C}^T, \end{aligned} \quad (8)$$

where γ_2 is another parameter to control the prior of \mathbf{C} . Since \mathbf{C} is a similarity matrix, it should satisfy the nonnegative and symmetric constraints as denoted in Eq. (8).

469 It is worthy to take a close look at the carefully designed regularized term $\sum_{i=1}^n \sum_{p,q=1}^k C_{pq}(S_{ip} - S_{iq})^2 W_{ip} W_{iq}$ again.
 470 Although the motivation of this term is to characterize the reliability of edges, it has three functions in total:
 471

- 472 • when fixing S and C to learn W , it evaluates the reliability of edges;
- 473 • when fixing W and C to optimize S , it propagates the connection information on the bipartite graph;
- 474 • when fixing W and S to learn C , it adaptively learns the similarity of all clusters.

478 3.5 Optimization

480 Eq. (8) involves the constraint $\text{rank}(L) = n+k-c$ which is hard to optimize. We first handle this constraint. According to
 481 [37], by introducing the auxiliary orthogonal matrix $F \in \mathbb{R}^{(n+k) \times c}$ and a large enough parameter ρ , Eq. (7) is equivalent
 482 to the following formula:

$$484 \min_{S, W, C, F} \|W \odot (S - Y)\|_F^2 - \lambda \|W\|_1 + \gamma_1 \sum_{i=1}^n \sum_{p,q=1}^k C_{pq}(S_{ip} - S_{iq})^2 W_{ip} W_{iq} + \gamma_2 \|C - Y^T Y\|_F^2 + \rho \text{tr}(F^T L F) \\ 485 \quad s.t. \quad 0 \leq S_{ij} \leq 1, \quad 0 \leq W_{ij} \leq 1, \quad F^T F = I, \quad C_{pq} \geq 0, \quad C = C^T. \quad (9)$$

486 Then we optimize W , F , S , and C respectively by fixing the other variables as many other machine learning methods
 487 do [36, 37, 61].

488 3.5.1 *Optimizing W.* When optimizing W , we find that Eq. (9) can be decoupled into n independent subproblems by
 489 rows. Considering the i -th subproblem, we have

$$490 \min_{W_i} \sum_{p=1}^k W_{ip}^2 A_{ip} - \lambda \sum_{p=1}^k W_{ip} + \gamma_1 \sum_{p,q=1}^k W_{ip} B_{pq} W_{iq} \\ 491 \quad s.t. \quad 0 \leq W_{ij} \leq 1, \quad (10)$$

492 where $A_{ip} = (S_{ip} - Y_{ip})^2$ and $B_{pq} = C_{pq}(S_{ip} - S_{iq})^2$.

493 Note that, Eq. (10) is a quadratic programming problem with bounded constraint and can be solved by some standard
 494 optimization methods, such as trust-region reflective algorithm [9]. In our implementation, we use *quadprog* function
 495 provided in Matlab.

496 3.5.2 *Optimizing F.* When optimizing F , we need to solve the following subproblem:

$$497 \min_F \text{tr}(F^T L F) \\ 498 \quad s.t. \quad F^T F = I. \quad (11)$$

499 According to Ky Fan Theory [10], Eq. (11) can be solved by computing the eigen-decomposition of L . However,
 500 conducting eigen-decomposition on an $(n+k) \times (n+k)$ matrix is often in $O((n+k)^3)$ time and is very time consuming.
 501 Fortunately, since L is a Laplacian matrix of a bipartite graph, according to [36], Eq. (11) can be solved by conducting
 502 Singular Valued Decomposition (SVD) on a small rectangle matrix.

503 In more detail, define diagonal matrices $\hat{D} \in \mathbb{R}^{n \times n}$ and $\tilde{D} \in \mathbb{R}^{k \times k}$ whose diagonal elements are $\hat{D}_{ii} = \sum_{j=1}^k S_{ij}$ and
 504 $\tilde{D}_{jj} = \sum_{i=1}^n S_{ij}$ respectively, and write F as the block matrices $F = [U^T, V^T]^T$, where $U \in \mathbb{R}^{n \times c}$ and $V \in \mathbb{R}^{k \times c}$. Eq. (11)

521 can be rewritten as

$$\begin{aligned}
 & \min_{\mathbf{F}^T \mathbf{F} = \mathbf{I}} \text{tr}(\mathbf{F}^T \mathbf{L} \mathbf{F}) \\
 & \Leftrightarrow \max_{\mathbf{U}^T \mathbf{U} + \mathbf{V}^T \mathbf{V} = \mathbf{I}} \text{tr} \left(\begin{bmatrix} \mathbf{U}^T & \mathbf{V}^T \end{bmatrix} \begin{bmatrix} \hat{\mathbf{D}}^{-\frac{1}{2}} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{D}}^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{S} \\ \mathbf{S}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{D}}^{-\frac{1}{2}} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{D}}^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix} \right) \\
 & \Leftrightarrow \max_{\mathbf{U}^T \mathbf{U} + \mathbf{V}^T \mathbf{V} = \mathbf{I}} \text{tr} \left(\begin{bmatrix} \mathbf{U}^T & \mathbf{V}^T \end{bmatrix} \begin{bmatrix} \mathbf{0} & \hat{\mathbf{D}}^{-\frac{1}{2}} \mathbf{S} \tilde{\mathbf{D}}^{-\frac{1}{2}} \\ \tilde{\mathbf{D}}^{-\frac{1}{2}} \mathbf{S}^T \hat{\mathbf{D}}^{-\frac{1}{2}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix} \right) \\
 & \Leftrightarrow \max_{\mathbf{U}^T \mathbf{U} + \mathbf{V}^T \mathbf{V} = \mathbf{I}} 2 \text{tr}(\mathbf{U}^T \hat{\mathbf{D}}^{-\frac{1}{2}} \mathbf{S} \tilde{\mathbf{D}}^{-\frac{1}{2}} \mathbf{V}).
 \end{aligned} \tag{12}$$

532 Eq. (12) can be solved by the following Theorem:

533 THEOREM 1. [36] The optimal solutions to the problem (12):

$$\max_{\mathbf{U}^T \mathbf{U} + \mathbf{V}^T \mathbf{V} = \mathbf{I}} \text{tr}(\mathbf{U}^T \hat{\mathbf{D}}^{-\frac{1}{2}} \mathbf{S} \tilde{\mathbf{D}}^{-\frac{1}{2}} \mathbf{V})$$

538 are $\mathbf{U} = \frac{\sqrt{2}}{2} \mathbf{U}'$ and $\mathbf{V} = \frac{\sqrt{2}}{2} \mathbf{V}'$, where \mathbf{U}' and \mathbf{V}' are the leading c left and right singular vectors of $\hat{\mathbf{D}}^{-\frac{1}{2}} \mathbf{S} \tilde{\mathbf{D}}^{-\frac{1}{2}}$.

540 Notice that since $\hat{\mathbf{D}}^{-\frac{1}{2}} \mathbf{S} \tilde{\mathbf{D}}^{-\frac{1}{2}}$ is an $n \times k$ matrix and often has $n \gg k$, the time complexity of computing its SVD is $O(nk^2)$ which is much smaller than $O((n+k)^3)$.

543 3.5.3 Optimizing \mathbf{S} . When optimizing \mathbf{S} , notice that \mathbf{L} is relative to \mathbf{S} , and thus we should handle $\text{tr}(\mathbf{F}^T \mathbf{L} \mathbf{F})$ first. Let $\mathbf{F}' = \mathbf{D}^{-\frac{1}{2}} \mathbf{F}$, and we have

$$\begin{aligned}
 \text{tr}(\mathbf{F}^T \mathbf{L} \mathbf{F}) &= \text{tr}(\mathbf{F}^T \mathbf{D} \mathbf{F}') - \text{tr}(\mathbf{F}'^T \mathbf{G}' \mathbf{F}') \\
 &= \frac{1}{2} \left(\sum_{i=1}^c \sum_{j,l=1}^{n+k} F'_{ji}^2 G'_{jl} + \sum_{i=1}^c \sum_{j,l=1}^{n+k} F'_{li}^2 G'_{jl} - \sum_{i=1}^c \sum_{j,l=1}^{n+k} 2F'_{ji} G'_{jl} F'_{li} \right) \\
 &= \frac{1}{2} \left(\sum_{j,l=1}^{n+k} G'_{jl} \sum_{i=1}^c (F'_{ji} - F'_{li})^2 \right) \\
 &= \sum_{j=1}^n \sum_{l=n+1}^{n+k} S_{j(l-n)} \left\| \frac{\mathbf{F}_j}{\sqrt{d_j}} - \frac{\mathbf{F}_l}{\sqrt{d_l}} \right\|_2^2,
 \end{aligned} \tag{13}$$

558 where $d_j = \sum_{m=1}^k S_{jm}$ and $d_l = \sum_{m=1}^k S_{lm}$, respectively.

560 Taking Eq. (13) back into Eq. (9), we can find that Eq. (9) can also be decoupled into n subproblems by rows.
561 Considering the i -th subproblem, we have:

$$\begin{aligned}
 \min_{\mathbf{S}_{i \cdot}} \quad & \sum_{p=1}^k W_{ip}^2 (S_{ip} - Y_{ip})^2 + \gamma_1 \sum_{p,q=1}^k E_{pq} (S_{ip} - S_{iq})^2 + \rho \sum_{p=1}^k H_{ip} S_{ip} \\
 \text{s.t.} \quad & 0 \leq S_{ip} \leq 1,
 \end{aligned} \tag{14}$$

567 where $E_{pq} = C_{pq} W_{ip} W_{iq}$, and $H_{ip} = \left\| \frac{\mathbf{F}_i}{\sqrt{d_i}} - \frac{\mathbf{F}_{n+p}}{\sqrt{d_{n+p}}} \right\|_2^2$.

568 Eq. (14) is also a quadratic programming problem with bounded constraint and can be solved by the same way as
569 solving Eq. (10).

573 3.5.4 *Optimizing C.* When optimizing \mathbf{C} , we need to optimize the following problem:

$$\begin{aligned} \min_{\mathbf{C}} \quad & \gamma_1 \sum_{i=1}^n \sum_{p,q=1}^k C_{pq} (S_{ip} - S_{iq})^2 W_{ip} W_{iq} + \gamma_2 \|\mathbf{C} - \mathbf{Y}^T \mathbf{Y}\|_F^2, \\ \text{s.t.} \quad & C_{pq} \geq 0, \quad \mathbf{C} = \mathbf{C}^T. \end{aligned} \quad (15)$$

579 For simplicity, we first remove the symmetric constraint $\mathbf{C} = \mathbf{C}^T$, and then show that the learned \mathbf{C} can satisfy the
580 constraint naturally. We decouple Eq. (15) into $k \times k$ independent subproblems. Considering the (p, q) -th subproblem,
581 we obtain:

$$\begin{aligned} \min_{C_{pq}} \quad & C_{pq} G_{pq} + \tau (C_{pq} - K_{pq})^2, \\ \text{s.t.} \quad & C_{pq} \geq 0, \end{aligned} \quad (16)$$

587 where $G_{pq} = \sum_{i=1}^n (S_{ip} - S_{iq})^2 W_{ip} W_{iq}$, $\mathbf{K} = \mathbf{Y}^T \mathbf{Y}$ and $\tau = \frac{\gamma_2}{\gamma_1}$. Setting the derivative of Eq. (16) w.r.t. C_{pq} to zero, we
588 obtain:

$$C_{pq} = \frac{2\tau K_{pq} - G_{pq}}{2\tau}. \quad (17)$$

590 If $2\tau K_{pq} - G_{pq} < 0$, it is easy to verify that Eq. (16) increases monotonically in the range $[0, \infty)$, and thus the optima is
591 0. Therefore, the optima of C_{pq} is

$$C_{pq} = \max \left(\frac{2\tau K_{pq} - G_{pq}}{2\tau}, 0 \right). \quad (18)$$

593 Note that $G_{pq} = G_{qp}$ and $K_{pq} = K_{qp}$, and thus \mathbf{C} computed by Eq. (18) also satisfies that $\mathbf{C} = \mathbf{C}^T$.

600 3.6 Algorithm and Discussion

602 Algorithm 1 summarizes the whole process of SCCABG. The following Theorem provides the convergence analysis of
603 SCCABG.

605 THEOREM 2. *With bounded hyper-parameter γ_1 , Algorithm 1 always converges.*

607 PROOF. Since the self-paced parameter λ always changes, it is difficult to analyze the convergence of Algorithm 1
608 directly. To address this issue, we should focus on λ first. Notice that λ only directly influences \mathbf{W} , and thus we review
609 the solution of \mathbf{W} again. We can rewrite the subproblem w.r.t. \mathbf{W}_i . (Eq. (10)) as a more concise form:

$$\begin{aligned} \min_{\mathbf{W}_i} \quad & \mathbf{W}_i^T \mathbf{M} \mathbf{W}_i - \lambda \mathbf{1}^T \mathbf{W}_i, \\ \text{s.t.} \quad & \mathbf{0} \leq \mathbf{W}_i \leq \mathbf{1}, \end{aligned} \quad (19)$$

615 where $\mathbf{M} = \gamma_1 \mathbf{B} + \text{diag}(\mathbf{A}_{i.})$ and $\text{diag}(\mathbf{A}_{i.})$ denotes the diagonal matrix whose diagonal vector is $\mathbf{A}_{i.}$, $\mathbf{1}$ and $\mathbf{0}$ denotes
616 the vectors whose elements are all 1's and 0's, respectively. Note that for any p, q , $0 \leq S_{pq} \leq 1$, and thus $\mathbf{0} \leq \mathbf{A}_{i.} \leq \mathbf{1}$.
617 According to Eq. (18), we have $0 \leq C_{pq} \leq K_{pq}$ and $\mathbf{K} = \mathbf{Y}^T \mathbf{Y}$ which is constant, and thus all elements in \mathbf{C} have lower
618 and upper bounds. Therefore, \mathbf{B} also has lower and upper bound. With bounded γ_1 , \mathbf{M} also has a lower and upper bound.
619 Obviously, \mathbf{M} is non-negative, and thus the lower bound of the elements in \mathbf{M} is zero. Denote u as the upper bound of
620 $\text{tr}(\mathbf{M}) + \mathbf{1}^T \mathbf{M} \mathbf{1}$, i.e., $u = \sup(\text{tr}(\mathbf{M}) + \mathbf{1}^T \mathbf{M} \mathbf{1})$. We have the following lemma.

623 LEMMA 1. *When $\lambda > u$, the global optima of Eq. (19) is $\mathbf{W}_i = \mathbf{1}$.*

PROOF. We use the proof by contradiction. Here, we wish to prove that when $\lambda > u$, all elements in the global optima \mathbf{W}_i of Eq.(19) (denoted by \mathbf{w}^*) are 1's. To apply the proof by contradiction, we assume the contrary is true, i.e., there exists at least one element in \mathbf{w}^* (denoted by w_j^*), which is t where $t < 1$, and then we try to find a contradiction. To find the contradiction, we construct a new vector $\hat{\mathbf{w}}$ which is the same with \mathbf{w}^* , except that the j -th element of $\hat{\mathbf{w}}$ (denoted by \hat{w}_j) is 1 instead of t .

Now, we compute the difference between $\hat{\mathbf{w}}^T \mathbf{M} \hat{\mathbf{w}} - \lambda \mathbf{1}^T \hat{\mathbf{w}}$ and $\mathbf{w}^{*T} \mathbf{M} \mathbf{w}^* - \lambda \mathbf{1}^T \mathbf{w}^*$:

$$\begin{aligned}
& \hat{\mathbf{w}}^T \mathbf{M} \hat{\mathbf{w}} - \lambda \mathbf{1}^T \hat{\mathbf{w}} - \mathbf{w}^{*T} \mathbf{M} \mathbf{w}^* + \lambda \mathbf{1}^T \mathbf{w}^* \\
&= 2 \sum_{k \neq j} (\hat{w}_j - w_j^*) M_{kj} w_k^* + M_{jj} (\hat{w}_j^2 - (w_j^*)^2) - \lambda (\hat{w}_j - w_j^*) \\
&= 2 \sum_{k \neq j} (1 - t) M_{kj} w_k^* + M_{jj} (1 - t^2) - \lambda (1 - t) \\
&= (1 - t) (2 \sum_{k \neq j} M_{kj} w_k^* + M_{jj} (1 + t) - \lambda) \\
&< (1 - t) (2 \sum_{k \neq j} M_{kj} + 2M_{jj} - \lambda) \\
&< (1 - t) (2 \sum_{k \neq j} M_{kj} + 2M_{jj} - u) \\
&\leq 0,
\end{aligned} \tag{20}$$

where the first inequality is due to that all M_{ij} is non-negative, $t < 1$, and all $w_k^* \leq 1$. The last inequality is due to $u \geq \text{tr}(\mathbf{M}) + \mathbf{1}^T \mathbf{M} \mathbf{1}$.

Eq. (20) shows that, $\hat{\mathbf{w}}^T \mathbf{M} \hat{\mathbf{w}} - \lambda \mathbf{1}^T \hat{\mathbf{w}} < \mathbf{w}^{*T} \mathbf{M} \mathbf{w}^* - \lambda \mathbf{1}^T \mathbf{w}^*$, which means $\hat{\mathbf{w}}$ leads to a smaller objective value than the optima \mathbf{w}^* . It is a contradiction, which means the assumption (i.e., there exists at least one element in \mathbf{w}^* is not 1) is false. Therefore, all elements in \mathbf{w}^* should be 1's. This concludes the proof. \square

Now get back to the proof of Theorem 2. Note that, in Algorithm 1, we double λ in each iteration. If Algorithm 1 does not converge before $\lambda > u$, when $\lambda > u$, according to Lemma 1, all elements in \mathbf{W}_i should be 1 to obtain the minimum of Eq. (19).

Therefore, after several iterations, either Algorithm 1 converges or all elements of \mathbf{W} become 1. When all elements of \mathbf{W} reach 1, according to Algorithm 1, λ will be fixed, and thus λ and \mathbf{W} will not change. We just need to focus on \mathbf{F} , \mathbf{S} and \mathbf{C} . When updating \mathbf{F} , we find the closed-form solution of Eq. (11), which makes the objective function Eq. (9) decrease. When updating \mathbf{S} , we solve Eq. (14) by the trust region reflective method, which can also make the objective function decrease. When updating \mathbf{C} , we also obtain its closed-form solution as Eq. (18), and thus also makes the objective function decrease. Since Eq. (9) has a lower bound, Algorithm 1 always converges. \square

In fact, SCCABG often converges very fast. In our experiments, it often converges within 10 iterations.

Then, we analyze the space and time complexity of the proposed method. Since the graph we used is a bipartite graph and \mathbf{C} is a $k \times k$ matrix, the space complexity of our method is $O(nk + k^2)$.

For the time complexity, we analyze it step by step. In each iteration, when updating \mathbf{W} and \mathbf{S} , we need to solve n quadratic programming problem and each problem involves k variables, respectively. Note that, in practice, we set γ_1 a small value to make sure the quadratic programming problem is convex. Therefore, each subproblem costs $O(k^3)$ time and updating \mathbf{W} and \mathbf{S} costs $O(nk^3)$ time. Updating \mathbf{F} needs $O(nk^2)$ as introduced in the previous subsection. When

677 updating C , we need to compute G whose time complexity is $O(nk^2)$. Supposing the number of iterations is l , the whole
 678 time complexity is $O(lnk^3)$. In practice, we often have $n \gg k$, and thus the time complexity is linear with n . Notice that,
 679 when optimizing S and W , the n subproblems are independent, and thus they can be solved in parallel for a further
 680 speedup.
 681

Algorithm 1 SCCABG Algorithm

683 **Input:** m base clustering results, number of clusters c , hyper-parameters γ_1 and γ_2 .

684 **Output:** Consensus clustering results

- 685 1: Construct the initial bipartite graph from m based clustering and obtain Y , and initialize the age parameter $\lambda = 0.5$,
 - 686 $S = Y$, and $C = Y^T Y$.
 - 687 2: **while** not converge **do**
 - 688 3: Update W by solving Eq.(10).
 - 689 4: Update F by Theorem 1.
 - 690 5: Update S by solving Eq.(14).
 - 691 6: Update C by Eq.(18)
 - 692 7: Update the age parameter by $\lambda = \lambda * 2$, until all elements in W reach 1.
 - 693 8: **end while**
 - 694 9: Obtain the final bipartite graph G' from S .
 - 695 10: Obtain the final clustering result from the c connective component in G' .
-

696
 697 At last, we discuss the relationship between our self-paced consensus clustering and robust consensus clustering,
 698 which is very related to our self-paced schema. Robust consensus clustering extracts noises from original data or base
 699 results and recovers the clean results for consensus learning. For example, Tao et al. proposed spectral based robust
 700 consensus clustering methods [46, 47]; Huang et al. adopted probability trajectories to robust consensus clustering
 701 [15]; Wang et al. developed an ensemble method to handle incomplete data [52]. Although these methods often provide
 702 more robust results than conventional consensus clustering methods, they only pay attention to the outliers or noises
 703 without distinguishing between uncontaminated data. However, in our self-paced schema, the contaminated data can
 704 be viewed as the most unreliable ones. In addition, the uncontaminated data can also be handled in order of reliability.
 705 Therefore, our method provides a more sophisticated framework to handle all instances, no matter contaminated ones
 706 or uncontaminated ones. Moreover, in our framework, the reliability of each edge is changing in the process of learning.
 707 With the growth of λ , W will be increasingly large until it reaches 1, which means the edges become increasingly more
 708 reliable with learning. At last, we will obtain a reliable structured bipartite graph G' for clustering.
 709

710 **3.7 Handle Incomplete Data**

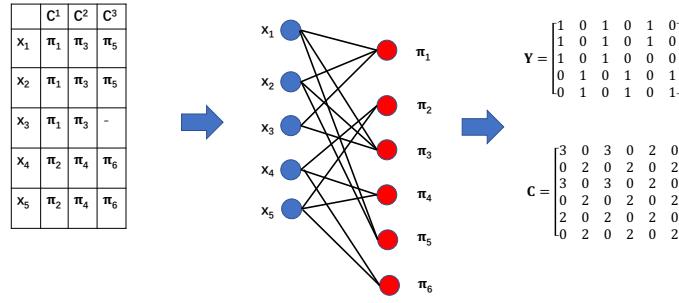
711 In many real applications, it often happens that some data are missing in some base results, especially in some federated
 712 learning scenarios. For example, in the bank system of a city, most people are the customers of only a few banks in the
 713 city. When these banks do the base clustering locally, the data of many people are missing in each base result. Then,
 714 they upload the incomplete base results to the cloud server, and learn the consensus result from the incomplete base
 715 results in the cloud server. Most existing consensus clustering methods may fail because they need complete data for
 716 consensus learning. The problem is also different from the widely studied incomplete multi-view learning [33, 53, 54, 57].
 717 Incomplete multi-view clustering needs to use the original features of data, whereas incomplete consensus clustering
 718 does not access the original features of data, which is more challenging and is quite under-explored.

719 Fortunately, since the proposed method applies self-paced learning to characterize the reliability of each edge and
 720 re-learns the bipartite graph according to its reliability, it is robust and can handle the incomplete setting easily. In more
 721

729 detail, when constructing the initial bipartite graph, we construct \mathbf{Y} directly by observed base results. For example,
 730 if \mathbf{x}_i is missing in the p -th base clustering C^p , then for any clusters in C^p (i.e., $\pi_1^p, \dots, \pi_{k_i}^p$), we do not set any edges
 731 between \mathbf{x}_i and $\pi_1^p, \dots, \pi_{k_i}^p$ initially. Then, we learn the final \mathbf{S} by optimizing Eq. (9). Due to the regularized term
 732 $\sum_{i=1}^n \sum_{p,q=1}^k C_{pq} (S_{ip} - S_{iq})^2 W_{ip} W_{iq}$, it can fill the initial bipartite graph automatically. To see this, if \mathbf{x}_i is missing in
 733 the base clustering which contains π_q , and π_q is similar to π_p (i.e., C_{pq} is large), and it is reliable that \mathbf{x}_i belongs to π_p
 734 (i.e., S_{ip} and W_{ip} are large), then by minimizing Eq. (9), S_{iq} will be large with high reliability W_{iq} , which means that in
 735 the learned bipartite graph \mathbf{S} , it is more likely that there is an edge between \mathbf{x}_i and π_q in the final graph, although \mathbf{x}_i is
 736 missing in the initial graph. Since some values are missing in the initial bipartite graph, some values in the initial \mathbf{W} are
 737 small, because they are unreliable due to the absence in the initial observation, and the initial similarity matrix \mathbf{C} is
 738 often not accurate. With the learning, more and more edges become increasingly more reliable and \mathbf{C} also becomes more
 739 accurate. Therefore, the adaptive cluster similarity measuring mechanism is also necessary to handle the incomplete
 740 data.

744

745



757 Fig. 5. A toy example of the incomplete consensus clustering. There are 5 instances $\mathbf{x}_1, \dots, \mathbf{x}_5$, wherein the first base clustering
 758 result, $\mathbf{x}_1, \dots, \mathbf{x}_3$ belong to π_1 , and \mathbf{x}_4 and \mathbf{x}_5 belong to π_2 . The second base clustering result is the same as the first one. In the third
 759 base result, \mathbf{x}_1 and \mathbf{x}_2 belong to π_5 , \mathbf{x}_4 and \mathbf{x}_5 belong to π_4 , and \mathbf{x}_3 is missing. The middle side shows the initial bipartite graph, and
 760 the right side shows the corresponding \mathbf{Y} and initial similarity matrix of clusters \mathbf{C} .

761

762

763 Figure 5 shows a simple toy example. Notice that \mathbf{x}_3 is missing in the base clustering C^3 . Therefore, in the initial
 764 bipartite graph, there are no edges between \mathbf{x}_3 and π_5 or π_6 . Despite this, the proposed one can automatically learn the
 765 relationship between \mathbf{x}_3 and π_5 or π_6 by minimizing the carefully designed term $\sum_{i=1}^n \sum_{p,q=1}^k C_{pq} (S_{ip} - S_{iq})^2 W_{ip} W_{iq}$.
 766 In the first iteration, we initialize $\mathbf{S} = \mathbf{Y}$, and then we evaluate the reliability $\mathbf{W}_{3.}$. Notice that π_5 is similar to π_1 , which
 767 leads to a large C_{15} . When minimizing $C_{15}(S_{31} - S_{35})^2 W_{31} W_{35}$ w.r.t. W_{35} , since C_{15} is large and $S_{31} = 1$ and $S_{35} = 0$, W_{35}
 768 should be small, which means $S_{35} = 0$ is unreliable and thus there may be an edge between \mathbf{x}_3 and π_5 . In fact, in the
 769 first iteration, after optimizing \mathbf{W} , we obtain $\mathbf{W}_{3.} = [1, 1, 1, 1, 0.3976, 1]$, where $W_{35} = 0.3976$ which is small.
 770

771 Then, we optimize \mathbf{S} by fixing other variables. When minimizing $C_{15}(S_{31} - S_{35})^2 W_{31} W_{35}$ w.r.t. S_{35} , since initial
 772 $S_{31} = 1$, it will pull S_{35} away from 0 to 1, i.e., it automatically fills the missing edges. In fact, after optimizing \mathbf{S} , we
 773 obtain $S_{3.} = [0.93, 0, 0.93, 0, 0.886, 0]$. Notice that although in the initial bipartite graph, S_{35} is missing and we initialize
 774 it as 0, after the first iteration, it can automatically fill it with a large value (i.e., 0.886) which means there should be
 775 an edge between \mathbf{x}_3 and π_5 . It well demonstrates that the proposed method is robust and can handle the incomplete
 776 consensus clustering setting.

777

778

779

Table 1. Description of the data sets.

	#instances	#features	#classes
ALLAML	72	7129	2
GLIOMA	50	4434	4
Tr41	878	7454	10
Tdt2	10212	36771	96
TOX	171	5748	4
K1b	2340	21839	6
Medical	706	1449	17
USPS	11000	256	10

4 EXPERIMENTS

In this section, we first use a toy example to show the effectiveness of the proposed method, and then we compare it with other state-of-the-art consensus clustering methods on several benchmark data sets.

4.1 Toy Example

Before comparing with other methods on benchmark data sets, we first provide a toy example to show the effectiveness of SCCABG intuitively.

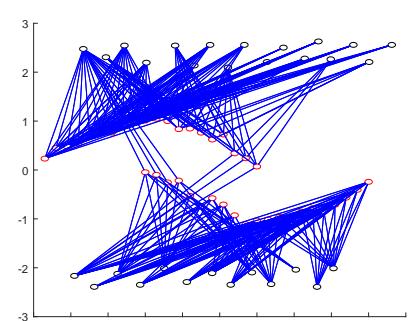
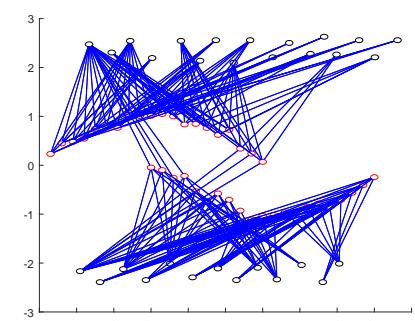
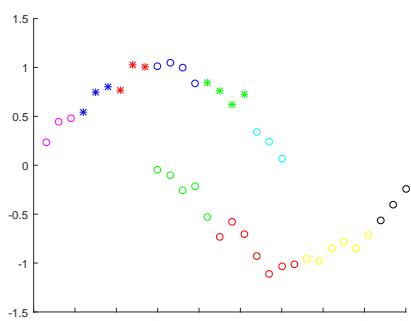
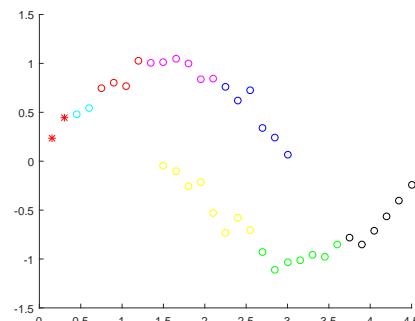
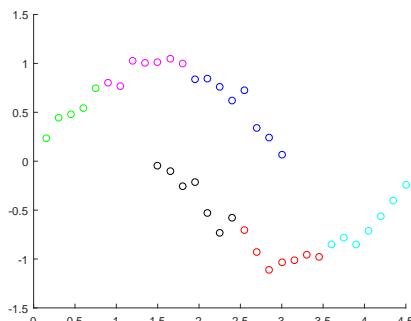
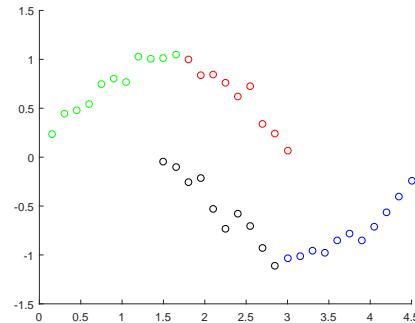
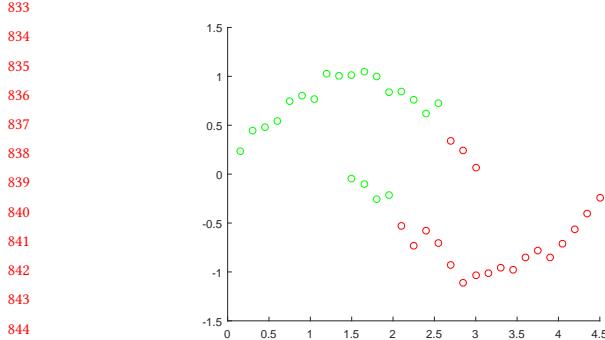
We use the two-moon data as the toy example. We run k-means 5 times with different numbers of clusters (in the range 2, 4, ⋯, 10) as the 5 base clustering results. The 5 base results are shown in Figure 6 (a)-(e). It can be seen that k-means cannot handle this non-linear manifold data well. We use the 5 base results to construct the initial bipartite graph \mathcal{G} and show it in Figure 6 (f). The red points indicate the instances and the black points indicate the clusters. We can find that, in Figure 6 (f), all instances and clusters are entangled together because of the unreliable base results. Then we run our SCCABG on \mathcal{G} to learn the structured bipartite graph \mathcal{G}' . Our SCCABG converges in 9 iterations. Figure 6 (g) and 6 (h) show the learned \mathcal{G}' in the 4-th and 9-th iteration, respectively. Note that in the 9-th iteration, our method can already discover the two-moon structure of data.

From Figure 6, we have some interesting observations. Firstly, although all base k-means methods fail on the non-linear manifold data, our consensus clustering can handle it well. Secondly, from Figure 6 (f) and (g), we find that in the first several iterations, our method prefers to propagate the connective information on the graph, so that it will add some new edges to the graph. From Figure 6 (g) and (h), it is shown that in the last several iterations, SCCABG prefers to partition the graph by removing some edges to make sure the learned graph contains just c connective components. The process is reasonable. If we partition the graph before propagating connective information, the graph will be broken into a lot of pieces and the propagation will be more difficult.

4.2 Benchmark Data Sets

We conduct experiments on the following 8 benchmark data sets:

- ALLAML [12]. It is a data set that consists of 72 bone marrow samples for leukemia diagnosis. It contains 7129 probes from 6817 human genes.
- GLIOMA [26]. It is a microarray data for glioma subtypes. It contains 50 samples with 4434 features in 4 subtypes of glioma.



17

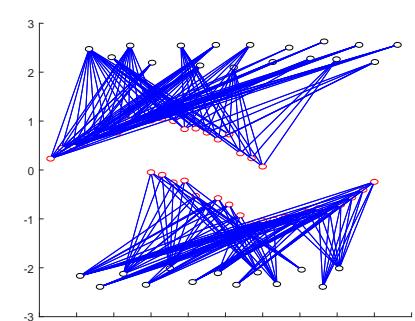


Fig. 6. Toy example results on the two-moon data. (a)-(e) show the 5 k-means base clustering results. (f) shows the initial bipartite \mathcal{G} constructed by the 5 base results. (g) shows the learned bipartite graph \mathcal{G}' in the 4-th iteration of SCCABG. (h) shows the final learned bipartite graph \mathcal{G}' of SCCABG (it is obtained in the 9-th iteration).

- Tr41 [59]. It is a text data set from Text Retrieval Conference (TREC). It contains 878 texts with 7454 features in 10 classes.
- Tdt2 [7]. It is a text data set containing 10212 documents with 36771 features in 96 categories.
- TOX [26]. It is a gene data set containing 171 instances with 5748 features in 4 classes.
- K1b [59]. It is a text data set from the WebACE project, where each document is a web page. It contains 2340 documents with 21839 features in 6 classes.
- Medical [27]. It is originally a multi-label text data set. Following [27, 66], we use the 706 instances with single label. It contains 1449 features and 17 classes.
- USPS ¹. It is an image data set containing 11000 16×16 handwritten digit images in 10 classes.

The information of these data sets is summarized in Table 1.

4.3 Experimental Setup

Following the setup in [51, 66], we also apply k-means to generate the base clustering results. In more detail, we run k-means 200 times with different random initializations to obtain 200 base results. Then we partition them into 10 subsets, with 20 in each one. Next, we run consensus clustering methods on each subset and report the average results on the 10 subsets. We compare the proposed SCCABG with the following algorithms:

- **KM**. It is the mean result of all base k-means clustering.
- **KM-best**. It is the best result among all base k-means results.
- **CSPA** [43]. Cluster-based Similarity Partitioning Algorithm (CSPA) adopts the relationship between instances to construct a measure of pairwise similarity and applies the similarity to recluster the data to obtain the final consensus clustering result.
- **HGPA** [43]. HyperGraph Partitioning Algorithm (HGPA) combines the base results to generate a hypergraph and applies a constrained minimum cut objective on the hypergraph for consensus clustering.
- **MCLA** [43]. Meta-Clustering Algorithm (MCLA) transforms the consensus clustering into a cluster correspondence problem. Then, the groups of clusters, which are called meta-clusters, are identified and combined.
- **NMFC** [28]. Nonnegative Matrix Factorization based Consensus clustering (NMFC) uses nonnegative matrix factorization to learn the consensus clustering result.
- **RCE** [66]. Robust Clustering Ensemble (RCE) explicitly extracts the noises on the connective matrices to recover the clean connective matrices, and then it ensembles the clean matrices by minimizing the KL-divergence between the base matrices and consensus matrix.
- **MEC** [45]. Multi-view Ensemble Clustering (MEC) is a robust consensus clustering method on connective matrices, which uses sparse and low-rank decomposition to integrate base clustering and extract the noises. Notice that, although it is proposed to handle multi-view clustering, since it only takes the connective matrices as inputs without access to the original data, it can be used as a baseline in our consensus clustering task.
- **LWEA** [16]. Locally Weighted Evidence Accumulation (LWEA) designs a local weighting strategy and applies a hierarchical agglomerative consensus clustering method based on such a local weighting strategy.
- **LWGP** [16]. Locally Weighted Graph Partitioning (LWGP) designs a local weighting strategy and applies a graph partition consensus clustering method on such a local weighting strategy.

¹<https://cs.nyu.edu/~roweis/data.html>

Table 2. Average ACC and standard deviation on all data sets. The bold font indicates that the difference is statistically significant (i.e., the p -value of t -test is smaller than 0.05).

Methods	ALLAML	GLIOMA	Tr41	Tdt2	Tox	K1b	Medical	USPS
KM	0.6545 ±0.0644	0.4239 ±0.0347	0.5626 ±0.0717	0.4104 ±0.0188	0.4229 ±0.0322	0.6726 ±0.0980	0.3996 ±0.0364	0.4435 ±0.0289
KM-best	0.7292 ±0.0118	0.4880 ±0.0193	0.6946 ±0.0468	0.4460 ±0.0081	0.4825 ±0.0152	0.8559 ±0.0246	0.4707 ±0.0268	0.5057 ±0.0141
CSPA [43]	0.6583 ±0.0134	0.4100 ±0.0271	0.5213 ±0.0282	0.2850 ±0.0047	0.4246 ±0.0373	0.4531 ±0.0027	0.3500 ±0.0150	0.4475 ±0138
HGPA [43]	0.5444 ±0.0403	0.4180 ±0.0394	0.4894 ±0.0549	0.2959 ±0.0041	0.3854 ±0.0286	0.5326 ±0.0469	0.2950 ±0.0283	0.1004 ±0.0000
MCLA [43]	0.6722 ±0.0149	0.4000 ±0.0133	0.5698 ±0.0392	0.4000 ±0.0088	0.4152 ±0.0242	0.7383 ±0.0913	0.4017 ±0.0197	0.4438 ±0.0239
NMFC [28]	0.6722 ±0.0149	0.4140 ±0.0212	0.6323 ±0.0370	0.3716 ±0.0169	0.4269 ±0.0226	0.5860 ±0.0348	0.3789 ±0.0183	0.4362 ±0.0320
RCE [66]	0.6708 ±0.0161	0.4260 ±0.0097	0.6391 ±0.0227	-	0.4105 ±0.0264	0.6887 ±0.0372	0.3851 ±0.0301	-
MEC [45]	0.6056 ±0.0360	0.3940 ±0.0366	0.6559 ±0.0444	-	0.4304 ±0.0310	0.8190 ±0.0901	0.3627 ±0.0167	-
LWEA [16]	0.6736 ±0.0210	0.4320 ±0.0140	0.6719 ±0.0473	0.5744 ±0.0273	0.4234 ±0.0127	0.8279 ±0.0760	0.4208 ±0.0076	0.4111 ±0.0108
LWGP [16]	0.6750 ±0.0176	0.4320 ±0.0103	0.6483 ±0.0340	0.4288 ±0.0103	0.4193 ±0.0259	0.7172 ±0.0773	0.4047 ±0.0141	0.4477 ±0.0215
RSEC [46]	0.5917 ±0.0908	0.4180 ±0.0503	0.6367 ±0.0435	0.3029 ±0.0772	0.4041 ±0.0243	0.8409 ±0.0511	0.3490 ±0.0303	0.3032 ±0.0624
DREC [60]	0.6819 ±0.0249	0.4280 ±0.0103	0.6243 ±0.0271	0.3657 ±0.0036	0.4205 ±0.0408	0.6462 ±0.0654	0.3926 ±0.0195	0.4354 ±0.0161
SPCE [63]	0.6861 ±0.0238	0.4420 ±0.0199	0.7346 ±0.0757	0.6653 ±0.0741	0.4485 ±0.0185	0.8663 ±0.0201	0.4534 ±0.0127	0.3259 ±0.0339
TRCE [64]	0.6917 ±0.0333	0.4400 ±0.0063	0.6812 ±0.0403	0.6273 ±0.0457	0.4491 ±0.0189	0.8899 ±0.0134	0.4356 ±0.0227	0.4505 ±0.0168
CESHL [68]	0.6736 ±0.0220	0.4420 ±0.0063	0.6952 ±0.0658	0.5301 ±0.0479	0.4404 ±0.0247	0.8619 ±0.0501	0.4574 ±0.0205	0.4495 ±0.0237
SCCABG	0.7139 ±0.0450	0.4620 ±0.0247	0.7244 ±0.0505	0.8350 ±0.0672	0.4427 ±0.0168	0.8881 ±0.0207	0.4720 ±0.0140	0.4620 ±0.0193

- **RSEC** [46]. Robust Spectral Ensemble Clustering (RSEC) is a spectral based robust consensus clustering method on connective matrices, which can reduce the noises on the connective matrices.
- **DREC** [60]. Dense Representation Ensemble Clustering (DREC) learns a dense representation from base results and applies it to construct a pairwise similarity matrix for the consensus clustering.
- **SPCE** [63]. Self-Paced Clustering Ensemble (SPCE) learns a consensus matrix from multiple connective matrices with self-paced multiple graph learning.
- **TRCE** [64]. Tri-level Robust Clustering Ensemble (TRCE) which learns a consensus clustering result by a multiple graph learning method. When fusing the multiple graphs, it considers three levels of robustness, i.e., the base result level, the graph level, and the data level.

Table 3. Average NMI and standard deviation on all data sets. The bold font indicates that the difference is statistically significant (i.e., the p -value of t -test is smaller than 0.05).

Methods	ALLAML	GLIOMA	Tr41	Tdt2	Tox	K1b	Medical	USPS
KM	0.0882 ±0.0490	0.1629 ±0.0391	0.5843 ±0.0512	0.6111 ±0.0072	0.1374 ±0.0397	0.5493 ±0.0608	0.4209 ±0.0286	0.4406 ±0.0166
KM-best	0.1772 ±0.0547	0.2347 ±0.0227	0.6713 ±0.0253	0.6240 ±0.0055	0.2164 ±0.0269	0.6853 ±0.0302	0.4806 ±0.0194	0.4762 ±0.0047
CSPA [43]	0.0815 ±0.0137	0.1716 ±0.0281	0.5919 ±0.0154	0.5589 ±0.0028	0.1436 ±0.0446	0.4071 ±0.0067	0.3992 ±0.0125	0.4342 ±0.0163
HGPA [43]	0.0110 ±0.0141	0.1509 ±0.0362	0.5084 ±0.0351	0.5385 ±0.0088	0.1083 ±0.0211	0.3917 ±0.0742	0.3613 ±0.0329	0.0000 ±0.0000
MCLA [43]	0.0909 ±0.0117	0.1327 ±0.0291	0.6044 ±0.0242	0.6070 ±0.0044	0.1329 ±0.0165	0.5944 ±0.0695	0.4296 ±0.0185	0.4446 ±0.0149
NMFC [28]	0.0909 ±0.0117	0.1550 ±0.0270	0.6512 ±0.0188	0.5930 ±0.0042	0.1434 ±0.0286	0.4995 ±0.0212	0.4259 ±0.0187	0.4471 ±0.0138
RCE [66]	0.0899 ±0.0125	0.1624 ±0.0163	0.6499 ±0.0183	-	0.1344 ±0.0204	0.6068 ±0.0104	0.4475 ±0.0190	-
MEC [45]	0.0485 ±0.0429	0.1312 ±0.0433	0.6758 ±0.0270	-	0.1313 ±0.0308	0.6818 ±0.0707	0.4089 ±0.0289	-
LWEA [16]	0.0935 ±0.0192	0.1686 ±0.0207	0.6666 ±0.0394	0.7183 ±0.0091	0.1236 ±0.0289	0.6948 ±0.0645	0.4185 ±0.0148	0.4211 ±0.0074
LWGP [16]	0.0932 ±0.0142	0.1682 ±0.0177	0.6535 ±0.0281	0.6266 ±0.0053	0.1333 ±0.0280	0.6115 ±0.0493	0.4266 ±0.0109	0.4452 ±0.0136
RSEC [46]	0.0495 ±0.0491	0.1544 ±0.0455	0.6449 ±0.0483	0.4670 ±0.0342	0.1184 ±0.0137	0.6615 ±0.0528	0.4036 ±0.0487	0.2774 ±0.0809
DREC [60]	0.1006 ±0.0218	0.1641 ±0.0189	0.6514 ±0.0169	0.5985 ±0.0013	0.1394 ±0.0276	0.5774 ±0.0410	0.4510 ±0.0203	0.4358 ±0.0098
SPCE [63]	0.1237 ±0.0126	0.3010 ±0.0152	0.6846 ±0.0589	0.7125 ±0.0427	0.1961 ±0.0139	0.6993 ±0.0599	0.4549 0.0130	0.3362 ±0.0494
TRCE [64]	0.1150 ±0.0278	0.2289 ±0.0193	0.6849 ±0.0333	0.7275 ±0.0236	0.1541 ±0.0298	0.7496 ±0.0218	0.4622 ±0.0266	0.4506 ±0.0103
CESHL [68]	0.0936 ±0.0197	0.1888 ±0.0118	0.6968 ±0.0471	0.6091 ±0.0585	0.1439 ±0.0285	0.7303 ±0.0557	0.3772 ±0.0119	0.4532 ±0.0176
SCCABG	0.1459 ±0.0331	0.3107 ±0.0315	0.6920 ±0.0445	0.8391 ±0.0407	0.2642 ±0.0063	0.7559 ±0.0327	0.4205 ±0.0148	0.4655 ±0.0051

- CESHL [68]. Clustering Ensemble with Structured Hypergraph Learning (CESHL) is a consensus clustering method with hypergraph learning. Different from the conventional hypergraph based method, it learns a dynamical structured hypergraph in the process of consensus learning.

For a fair comparison, we set the number of clusters to the true number of classes for all algorithms on all data sets. λ in our method is automatically adjusted as introduced in Algorithm 1. The parameter ρ is also automatically determined. In more detail, it is initialized by $\rho = 1$. Then, if the rank of L is larger than $n + k - c$, i.e., the rank constraint is not strong enough, we double it. If its rank is smaller than $n + k - c$, i.e., the rank constraint is too strong, we reduce it by half. We tune the hyper-parameter γ_1 in the range $[10^{-5}, 10^0]$, because as discussed before, γ_1 should not be too large to guarantee the convexity of the subproblems. The hyper-parameter γ_2 is tuned in $[10^{-3}, 10^3]$ by grid search. We

1041 Table 4. Average ARI and standard deviation on all data sets. The bold font indicates that the difference is statistically significant (i.e.,
 1042 the p -value of t -test is smaller than 0.05).

Methods	ALLAML	GLIOMA	Tr41	Tdt2	Tox	K1b	Medical	USPS
KM	0.0997 ±0.0103	0.0776 ±0.0060	0.4309 ±0.0256	0.2110 ±0.0025	0.1021 ±0.0381	0.4758 ±0.1210	0.2368 ±0.0310	0.2911 ±0.0047
KM-best	0.1990 ±0.0213	0.1595 ±0.0306	0.5917 ±0.0577	0.2388 ±0.0103	0.1794 ±0.0307	0.7183 ±0.0432	0.3061 ±0.0336	0.3384 ±0.0128
CSPA [43]	0.0850 ±0.0371	0.0768 ±0.0181	0.4274 ±0.0190	0.1422 ±0.0010	0.1112 ±0.0391	0.2551 ±0.0042	0.1951 ±0.0110	0.2843 ±0.0064
HGPA [43]	0.0066 ±0.0161	0.0571 ±0.0275	0.3412 ±0.0483	0.1461 ±0.0038	0.0752 ±0.0167	0.2896 ±0.0628	0.1518 ±0.0282	-0.0008 ±0.0000
MCLA [43]	0.1101 ±0.0523	0.0795 ±0.0115	0.4511 ±0.0421	0.1908 ±0.0086	0.1048 ±0.0213	0.5521 ±0.1242	0.2303 ±0.0231	0.2934 ±0.0242
NMFC [28]	0.1079 ±0.0209	0.0779 ±0.0247	0.5271 ±0.0262	0.1792 ±0.0058	0.1132 ±0.0258	0.3693 ±0.0301	0.2230 ±0.0211	0.2896 ±0.0203
RCE [66]	0.1070 ±0.0215	0.0918 ±0.0067	0.5360 ±0.0227	-	0.0992 ±0.0221	0.5138 ±0.0300	0.2344 ±0.0231	-
MEC [45]	0.0655 ±0.1012	0.0913 ±0.0219	0.5024 ±0.0760	-	0.0963 ±0.0227	0.6562 ±0.1241	0.1897 ±0.0330	-
LWEA [16]	0.1106 ±0.0302	0.0955 ±0.0112	0.5413 ±0.0528	0.4340 ±0.0020	0.1042 ±0.0181	0.6968 ±0.1142	0.2510 ±0.0060	0.3085 ±0.0147
LWGP [16]	0.1083 ±0.0268	0.0935 ±0.0067	0.5462 ±0.0338	0.2133 ±0.0073	0.1030 ±0.0252	0.5234 ±0.1120	0.2447 ±0.0101	0.3074 ±0.0155
RSEC [46]	-0.0130 ±0.0602	0.0648 ±0.0452	0.4812 ±0.0711	0.1804 ±0.0791	0.0791 ±0.0218	0.6990 ±0.0829	0.1884 ±0.0338	0.1421 ±0.0787
DREC [60]	0.1231 ±0.0361	0.0923 ±0.0073	0.5216 ±0.0293	0.1812 ±0.0038	0.1091 ±0.0312	0.4633 ±0.0661	0.2430 ±0.0190	0.3052 ±0.0167
SPCE [63]	0.1053 ±0.0623	0.1180 ±0.0160	0.6361 ±0.0931	0.4201 ±0.0791	0.1265 ±0.0251	0.7432 ±0.0601	0.2520 ±0.0088	0.0063 ±0.0200
TRCE [64]	0.1065 ±0.0570	0.0952 ±0.0150	0.5576 ±0.0698	0.2039 ±0.0113	0.0988 ±0.0230	0.6281 ±0.1785	0.2514 ±0.0182	0.2913 ±0.0186
CESHL [68]	0.0960 ±0.0503	0.0861 ±0.0346	0.4564 ±0.2368	0.2394 ±0.0895	0.0901 ±0.0276	0.4534 ±0.3724	0.2155 ±0.0767	0.2807 ±0.0498
SCCBAG	0.1917 ±0.0406	0.1280 ±0.0195	0.6150 ±0.0659	0.5281 ±0.1262	0.1161 ±0.0245	0.7914 0.0457	0.2562 ±0.0056	0.3177 ±0.0147

1078
 1079 use Accuracy (ACC), Normalized Mutual Information (NMI), Adjust Rand Index (ARI), and F1 Score (F1) to evaluate the
 1080 clustering performance. To validate the statistical significance of the results, we also do t -test on the results.
 1081

1082 The experiments are conducted using MATLAB on a PC with Windows 10, 4.2-GHz CPU, and 32-GB memory.

1084 4.4 Experimental Results

1085
 1086 Tables 2, 3, 4, and 5 show the average ACC, NMI, ARI and F1 results and standard deviation of all consensus clustering
 1087 methods, respectively. The bold font indicates that the difference is statistically significant (i.e., the p -value of t -test is
 1088 smaller than 0.05). Note that, because of their high space complexity, RCE and MEC run out of memory on the large
 1089 data set Tdt2 and USPS.
 1090

1091 From these tables, we find that:

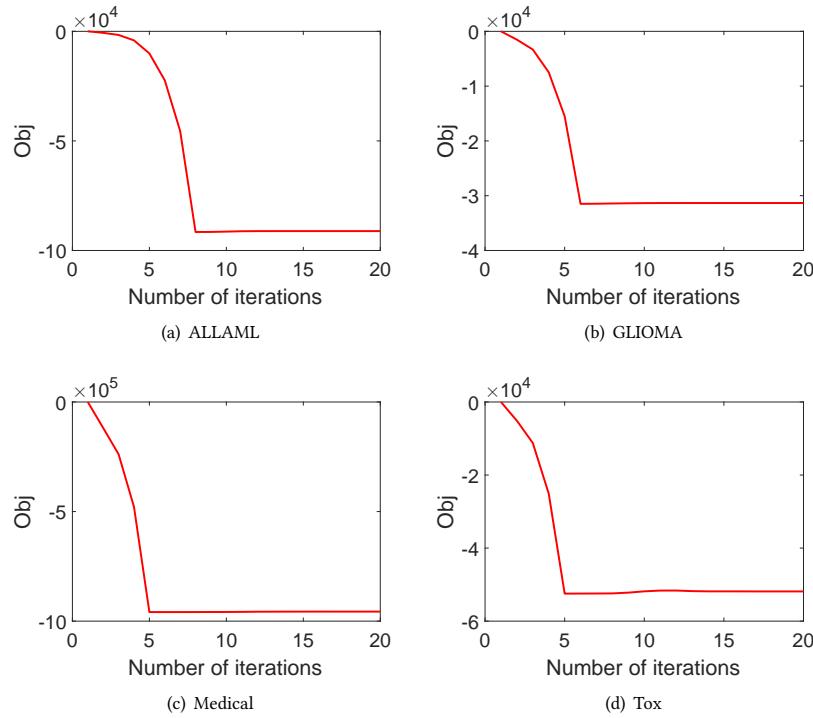
Table 5. Average F1 Score and standard deviation on all data sets. The bold font indicates that the difference is statistically significant (i.e., the p -value of t -test is smaller than 0.05).

Methods	ALLAML	GLIOMA	Tr41	Tdt2	Tox	K1b	Medical	USPS
KM	0.5779 ±0.0046	0.3812 ±0.0043	0.5217 ±0.0195	0.2361 ±0.0024	0.3516 ±0.0079	0.6373 ±0.0223	0.3298 ±0.0063	0.3657 ±0.0040
KM-best	0.6269 ±0.0117	0.4356 ±0.0254	0.6568 ±0.0488	0.2637 ±0.0105	0.4116 ±0.0296	0.8200 ±0.0300	0.3912 ±0.0261	0.4074 ±0.0114
CSPA [43]	0.5686 ±0.0175	0.3513 ±0.0162	0.4926 ±0.0246	0.1577 ±0.0010	0.3351 ±0.0280	0.4309 ±0.0062	0.2692 ±0.0089	0.3564 ±0.0058
HGPA [43]	0.5315 ±0.0076	0.3483 ±0.0119	0.4031 ±0.0371	0.1502 ±0.0025	0.2905 ±0.0228	0.4825 ±0.0321	0.2291 ±0.0152	0.1045 ±0.0000
MCLA [43]	0.5815 ±0.0247	0.3739 ±0.0145	0.5125 ±0.0426	0.1993 ±0.0114	0.3358 ±0.0203	0.6280 ±0.1019	0.3041 ±0.0136	0.3674 ±0.0205
NMFC [28]	0.5798 ±0.0104	0.3778 ±0.0157	0.5936 ±0.0445	0.1973 ±0.0612	0.3487 ±0.0174	0.5874 ±0.0678	0.2967 ±0.0199	0.3651 ±0.0178
RCE [66]	0.5794 ±0.0107	0.3836 ±0.0062	0.6060 ±0.0187	-	0.3464 ±0.0204	0.6566 ±0.0267	0.3148 ±0.0209	-
MEC [45]	0.6143 ±0.0528	0.3885 ±0.0123	0.6236 ±0.0411	-	0.3573 ±0.0206	0.7627 ±0.1009	0.2892 ±0.0284	-
LWEA [16]	0.5811 ±0.0149	0.3868 ±0.0095	0.6203 ±0.0426	0.4644 ±0.0200	0.3642 ±0.0142	0.8028 ±0.0840	0.3460 ±0.0065	0.3845 ±0.01154
LWGP [16]	0.5801 ±0.0135	0.3859 ±0.0065	0.6149 ±0.0292	0.2340 ±0.0077	0.3473 ±0.0185	0.6778 ±0.0788	0.3340 ±0.0128	0.3800 ±0.0136
RSEC [46]	0.6210 ±0.0405	0.3733 ±0.0322	0.5594 ±0.0847	0.2399 ±0.0763	0.3213 ±0.0233	0.6924 ±0.0795	0.2569 ±0.0367	0.2673 ±0.0525
DREC [60]	0.5875 ±0.0180	0.3843 ±0.0072	0.5916 ±0.0254	0.2022 ±0.0052	0.3501 ±0.0238	0.6113 ±0.0546	0.3161 ±0.0166	0.3779 ±0.0146
SPCE [63]	0.5809 ±0.0270	0.3899 ±0.0147	0.6726 ±0.0885	0.3462 ±0.0442	0.3712 ±0.0216	0.8010 ±0.0433	0.3491 ±0.0635	0.1859 ±0.0131
TRCE [64]	0.5801 ±0.0268	0.3898 ±0.0095	0.6282 ±0.0603	0.2435 ±0.0131	0.3484 ±0.0205	0.7739 ±0.0965	0.3431 ±0.0150	0.3662 ±0.0163
CESHL [68]	0.5749 ±0.0233	0.3926 ±0.0100	0.5771 ±0.1503	0.3357 ±0.0637	0.3475 ±0.0182	0.7442 ±0.1400	0.3393 ±0.0537	0.3605 ±0.0334
SCCABG	0.6547 ±0.0349	0.4372 ±0.0122	0.6850 ±0.0522	0.5831 ±0.1033	0.3967 ±0.0121	0.8737 ±0.0257	0.3665 ±0.0053	0.3911 ±0.0138

- Many consensus clustering methods, including ours, outperform the KM, which indicates the effectiveness of the consensus clustering, i.e., by integrating multiple weak base clustering results, we can learn a better consensus result.
- Compared with other consensus clustering methods, SCCABG outperforms them on most data sets, which demonstrates its superiority. Especially on the large data set Tdt2, SCCABG achieves 25.5% and 15.3% improvements compared with the second best baseline methods on ACC and NMI, respectively. Even compared with the robust methods (RCE, MEC, RSEC, and TRCE), SCCABG also performs better, because it can handle data more sophisticatedly, i.e., the self-paced framework can not only recognize noises or outliers but also handle those uncontaminated but difficult instances, as discussed in Section 3.6.

- 1145 • On most data sets, the proposed SCCABG is at least closed to KM-best. It means that SCCABG can provide
 1146 a stable good clustering result compared with base single clustering. Notice that, SCCABG only takes base
 1147 clustering results as inputs without accessing original data or labels. On some data sets, SCCABG can even
 1148 perform better than KM-best, which means SCCABG can alleviate the side effects caused by unreliable results
 1149 and apply the useful information in the unreliable ones to further improve the performance of the reliable ones.
 1150 It well demonstrates the effectiveness of our SCCABG.

1152 In Figure 7, we show the convergence curves of SCCABG on ALLAML, GLIOMA, Medical, and Tox data sets. The
 1153 results on other data sets are similar. From Figure 7, we can find that SCCABG often converges within 10 iterations,
 1154 which demonstrates the claim in Section 3.6. Notice that the curves do not look as smooth as many other machine
 1155 learning methods. It is because of the special update of λ in our algorithm. The detailed reason is as follows. At the
 1156 first several iterations, λ in the term $-\lambda\|\mathbf{W}\|_1$ is updated as $\lambda \leftarrow \lambda * 2$, and thus the objective function decreases faster
 1157 and faster with iterations. After several iterations (e.g. 5 iterations on Medical data set), all values in \mathbf{W} reach 1, and
 1158 according to our algorithm (i.e., Line 7 in Algorithm 1), λ will not change after the 5-th iteration. By the first several
 1159 iterations, the variables \mathbf{F} , \mathbf{S} , and \mathbf{C} have almost converged, and thus the objective function hardly changes in the
 1160 following iterations with a fixed λ .
 1161



1191 Fig. 7. Convergence curves on ALLAML, GLIOMA, Lung and Tox data sets.
 1192

1193 Since our method and many other methods are graph based methods, we also show the visualization of the learned
 1194 final graph matrices of our SCCABG and other graph based methods (i.e., RCE, MEC, SPCE, TRCE) in Figure 8. The
 1195

ideal graph matrix should have a clean block diagonal structure. Figure 8 (a) shows the initial graph constructed from input base clustering results directly by YY^T . The graph is not clean enough, which can be seen from the zone denoted by the red circle. Figure 8 (b)-(e) show the learned graph matrix of RCE, MEC, SPCE, and TRCE, respectively. Figure 8 (f) shows the graph constructed from the learned bipartite graph of SCCABG by SS^T . The learned graph of our method is cleaner than other methods. For example, from the zones denoted by the red circles of RCE, SPCE, and TRCE, we can see that in SCCABG, the second, third and fourth clusters can be divided more clearly. From the zone denoted by the green circle of MEC, we can find that the second cluster in MEC is fuzzier than SCCABG.

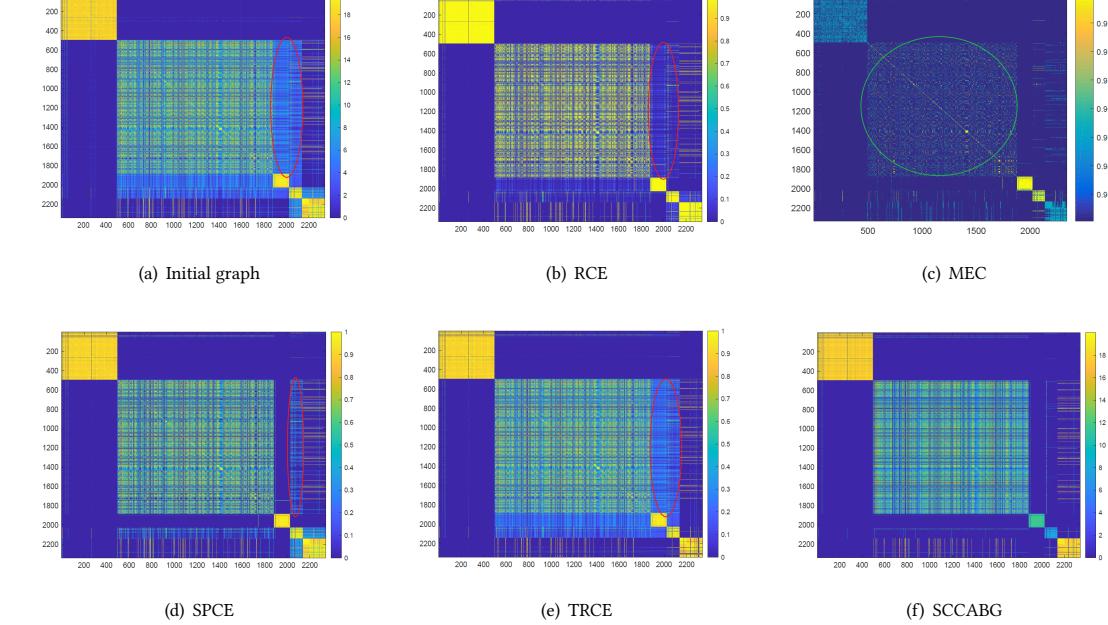


Fig. 8. The visualization of graph matrices learned by RCE, MEC, SPCE, TRCE, and SCCABG. (a) shows the initial graph matrix which is constructed by YY^T . (b)-(e) show the learned graph matrices from RCE, MEC, SPCE, and TRCE, respectively. (f) shows the graph constructed from the learned bipartite graph of SCCABG by SS^T .

Since we apply self-paced learning, we involve the weight matrix \mathbf{W} to represent the difficulty or the reliability of data. Here we show an example on USPS data set, which is a handwritten digit image data set. Notice that \mathbf{W} is an n -by- k matrix, where each element in \mathbf{W} represents the reliability of an edge in the bipartite graph. To obtain the difficulty or reliability of each data, we need to calculate the weight vector $\mathbf{w} \in [0, 1]^n$ by computing the mean of each row of \mathbf{W} . The larger w_i is, the easier or more reliable \mathbf{x}_i is and the earlier \mathbf{x}_i is involved in the consensus learning. Figure 9 shows some example images with different w_i from 1 to 0. From Figure 9, we can find that easier images have large weights, which are involved in the consensus learning earlier and harder images have small weights. For example, the last one with $w = 0$ whose true label is "6", but it is often assigned to the cluster with label "4" and thus its weight is much lower.

Fig. 9. An example of the data with its weight w on USPS data set.Table 6. Clustering results compared with degenerated versions. The bold font indicates that the difference is statistically significant (i.e., the p -value of t -test is smaller than 0.05).

Methods	Measures	ALLAML	GLIOMA	Tr41	Tdt2	Tox	K1b	Medical	USPS
SCCBG-W	ACC	0.6681 ±0.0122	0.4080 ±0.0301	0.6136 ±0.1113	0.5011 ±0.0352	0.4053 ±0.0447	0.8405 ±0.0643	0.3980 ±0.0826	0.4496 ±0.0281
	NMI	0.0894 ±0.0104	0.1567 ±0.0401	0.6039 ±0.1461	0.6433 ±0.0116	0.1239 ±0.0433	0.6888 ±0.0667	0.3220 ±0.1076	0.4509 ±0.0231
	ARI	0.1030 ±0.0174	0.0665 ±0.0504	0.4799 ±0.1493	0.2353 ±0.0342	0.0973 ±0.0306	0.6963 ±0.1030	0.2066 ±0.0673	0.3074 ±0.0161
	F1	0.5755 ±0.0084	0.3887 ±0.0074	0.5791 ±0.1049	0.2918 ±0.0278	0.3568 ±0.0239	0.8098 ±0.0766	0.3275 ±0.0469	0.3806 ±0.0141
	ACC	0.6861 ±0.0279	0.4500 ±0.0343	0.6973 ±0.0644	0.7164 ±0.0689	0.4339 ±0.0182	0.8663 ±0.0369	0.4592 ±0.0151	0.4423 ±0.0091
	NMI	0.1252 ±0.0330	0.2163 ±0.0674	0.6847 ±0.0547	0.7548 ±0.0459	0.2131 ±0.0513	0.7212 ±0.0512	0.3918 ±0.0244	0.4603 ±0.0117
	ARI	0.1284 ±0.0464	0.1064 ±0.0085	0.5622 ±0.0765	0.4931 ±0.1572	0.1030 ±0.0282	0.7617 ±0.0596	0.2484 ±0.0074	0.3081 ±0.0155
	F1	0.6441 ±0.0488	0.4143 ±0.0131	0.6410 ±0.0613	0.5460 ±0.1304	0.3837 ±0.0160	0.8550 ±0.0387	0.3609 ±0.0062	0.3813 ±0.0128
	ACC	0.7139 ±0.0450	0.4620 ±0.0247	0.7244 ±0.0505	0.8350 ±0.0672	0.4427 ±0.0168	0.8881 ±0.0207	0.4720 ±0.0140	0.4620 ±0.0193
	NMI	0.1459 ±0.0331	0.3107 ±0.0315	0.6920 ±0.0445	0.8391 ±0.0407	0.2642 ±0.0063	0.7559 ±0.0327	0.4205 ±0.0148	0.4655 ±0.0051
SCCABG	ARI	0.1917 ±0.0406	0.1280 ±0.0195	0.6150 ±0.0659	0.5281 ±0.1262	0.1161 ±0.0245	0.7914 ±0.0457	0.2562 ±0.0056	0.3177 ±0.0147
	F1	0.6547 ±0.0349	0.4372 ±0.0122	0.6850 ±0.0522	0.5831 ±0.1033	0.3967 ±0.0121	0.8737 ±0.0257	0.3665 ±0.0053	0.3911 ±0.0138

4.5 Experiments on Space and Time Consuming

In Figure 10, we show the memory used by all methods on all data sets. From Figure 10, we can find that the memory consumption of the proposed SCCABG is better than most compared methods. Notice that RCE and MEC run out-of-memory on Tdt2 and USPS data sets, and thus have no results. As introduced before, the space complexity of SCCABG is $O(nk + k^2)$, which is linear with the number of instances. However, some other ensemble methods are based on the co-association matrix, whose space complexity is $O(n^2)$. That is why some methods consume much more space than ours.

Figure 11 shows the running time of all methods on all data sets. On the small data sets (i.e., ALLAML and GLIOMA), which only contain no more than 100 instances, SCCABG is slower than other methods. Despite this, SCCABG can obtain the results within 2 seconds. Notice that the time complexity of SCCABG is $O(nk^3)$, where n is the number of instances and k is the total number of clusters. On the small data sets, n is comparable to or even smaller than k .

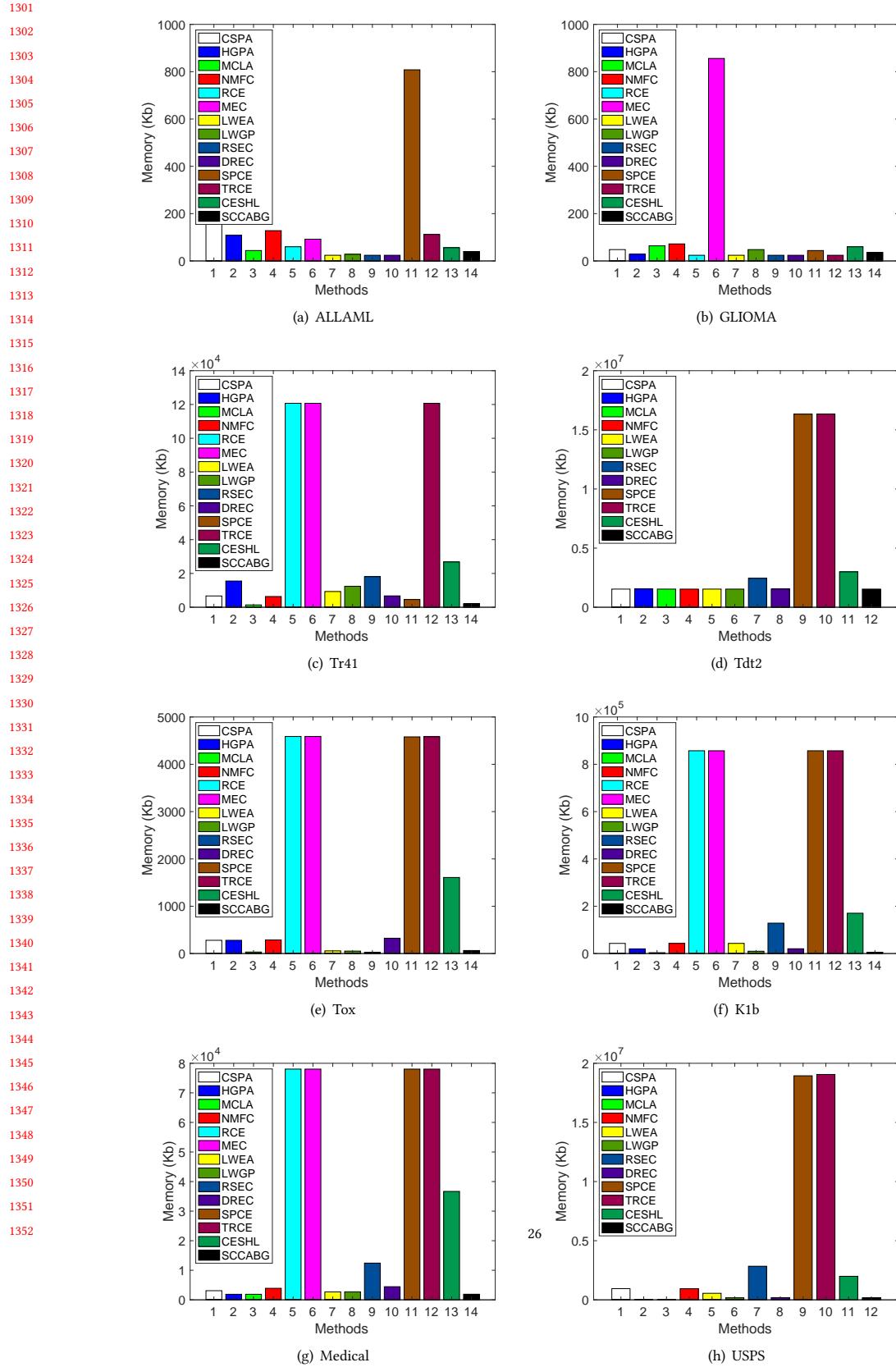


Fig. 10. Memory consumption (Kb) of all methods.

That is why SCCABG is slower than other methods on these two small data sets. However, in practice, n is often much larger than k , like other data sets we used. On these data sets, SCCABG is faster than many other ensemble methods, especially the co-association matrix based methods, whose time complexity is often square or cubic in the number of instances. For example, on the USPS data set, which contains 11000 instances, SCCABG only consumes several hundreds of seconds, whereas some compared methods cost several tens of thousands of seconds.

4.6 Ablation Study

To demonstrate the effectiveness of the self-paced learning and the adaptive cluster similarity measuring strategy, we compare our SCCABG with the following two degenerated versions:

- **SCCBG-W**, which is our method without self-paced learning. In more detail, we fix all the elements in the weight matrix \mathbf{W} as 1 and do not update them. Since we do not evaluate the reliability of edges, we also remove the self-paced regularized term and adaptive cluster similarity measuring term.
- **SCCBG** [62], which is our self-paced version with fixed cluster similarity matrix \mathbf{C} . It is also the method proposed in our previous conference version [62].

Table 6 shows the results compared with SCCBG-W and SCCBG. Compared with SCCBG-W, SCCBG often achieves better performance, which indicates the effectiveness of the self-paced learning framework. With the carefully designed self-paced regularized term, SCCBG can well evaluate the reliability of each edge (i.e., \mathbf{W}), and alleviate the side effects caused by the unreliability edges in the early model. Compared with SCCBG, SCCABG further improves the performance. It demonstrates the effectiveness of the adaptive cluster similarity measure method, i.e., adaptively updating the cluster similarity will characterize the reliability of edges better than the fixed method.

4.7 Hyper-parameter Study

In this subsection, we study the effect of the hyper-parameters γ_1 and γ_2 . As discussed before, γ_1 should be small to guarantee the convexity of the subproblems. So we tune it in $[10^{-5}, 10^0]$. We tune γ_2 in the range $[10^{-3}, 10^3]$. Figure 12 shows the ACC and NMI results on Tr41, K1b, and Medical data sets with different hyper-parameters. Results on other data sets are similar. It can be seen that, SCCABG is not sensitive with parameter $\gamma_2 \in [10^{-3}, 10^3]$ (and it often achieves the best results when $\gamma_2 \in [10^0, 10^1]$), and works well when γ_1 is in the range $[10^{-5}, 10^{-3}]$. When γ_1 grows, the performance will deteriorate, which is in line with our previous discussion. Notice that, the conference version SCCBG, which fixes $\mathbf{C} = \mathbf{Y}\mathbf{Y}^T$ and never updates \mathbf{C} , is equivalent with SCCABG with $\gamma_2 \rightarrow +\infty$. Therefore, its performance is somewhat worse than SCCABG as shown in the ablation study.

Moreover, we also show the effect of the number of base clusterings m and the number of clusters k_i in each base result. Figure 13 shows the ACC and NMI on Tr41 and K1b with $\{10, 20, \dots, 100\}$ base clusterings. Figure 14 shows the ACC and NMI on Tr41 and K1b with different numbers of clusters k_i in each base result. If the true number of classes is k , we show different k_i in the range $[k - 4, k + 4]$. For example, the real number of classes of Tr41 is 10, and thus we show k_i in $\{6, 7, 8, \dots, 14\}$; the real number of classes of K1b is 6, and we show k_i in $\{2, 3, 4, \dots, 10\}$. The results on other data sets are similar. From Figures 13 and 14, we can see that SCCABG is insensitive with m and k_i .

4.8 Experiments on Incomplete Data

As introduced in Section 3.7, the proposed method can handle incomplete data sets. To show its effectiveness, we also conduct experiments on incomplete data sets. In more detail, for each base result, we set a missing ration $r =$

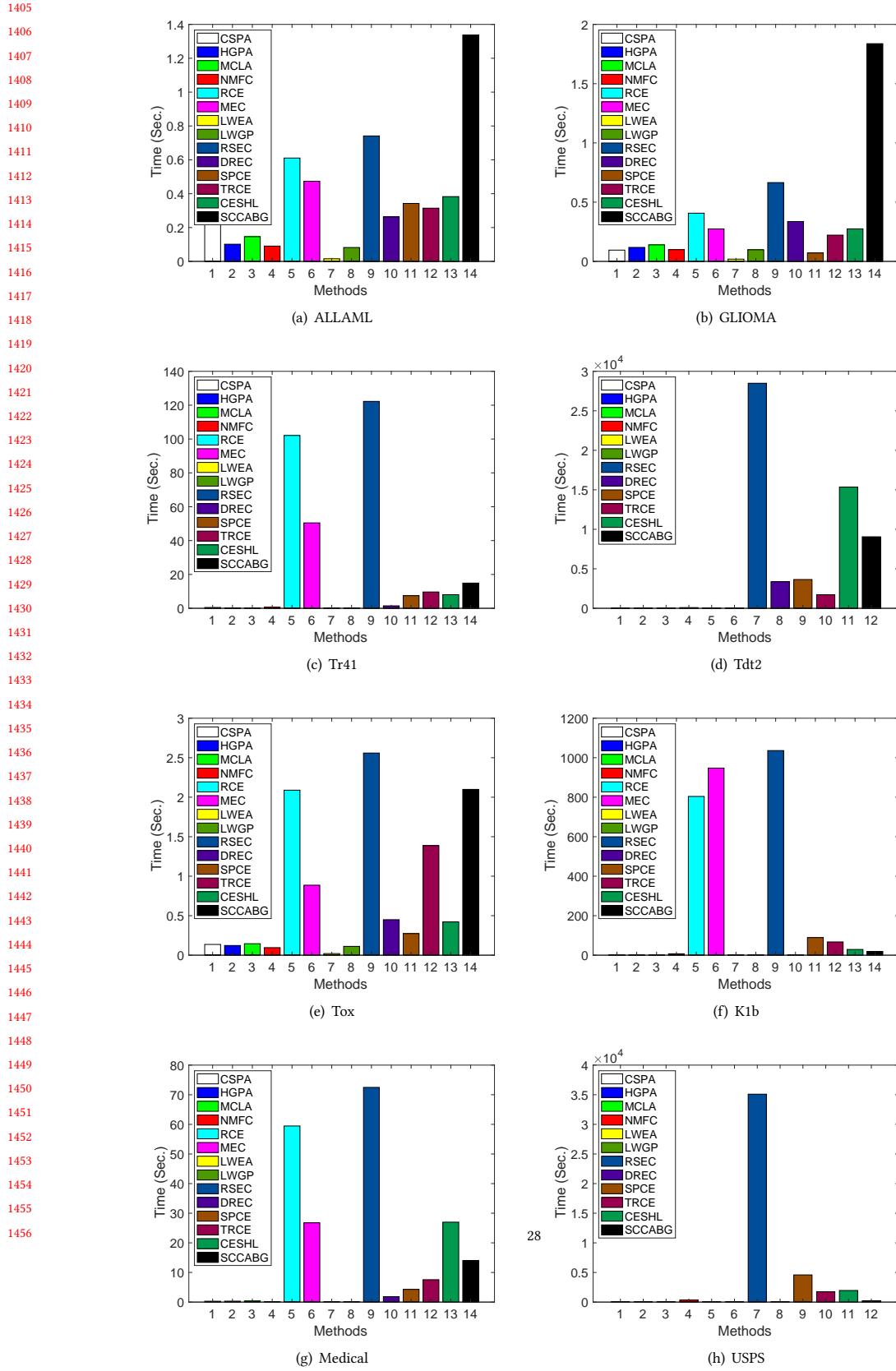
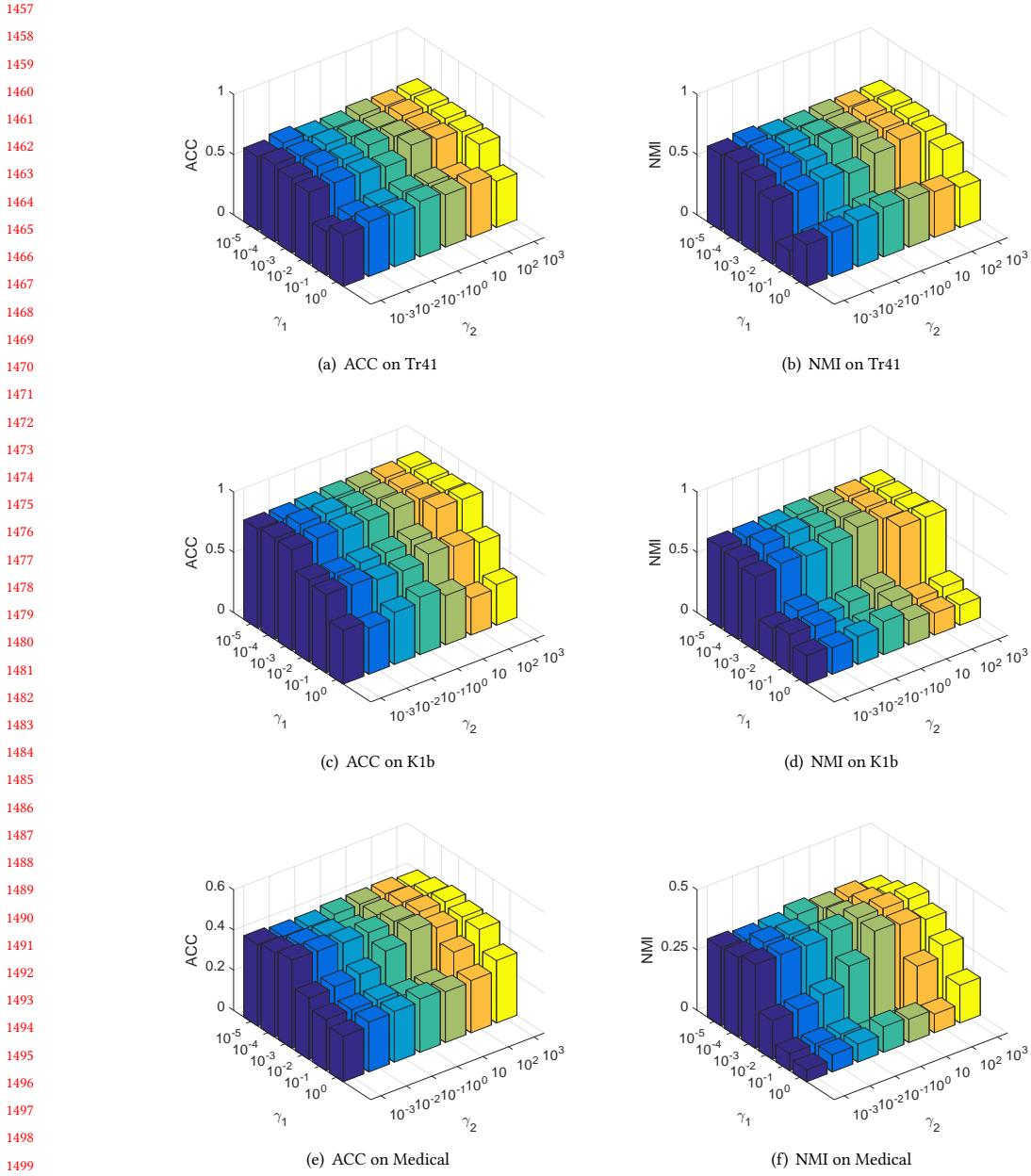
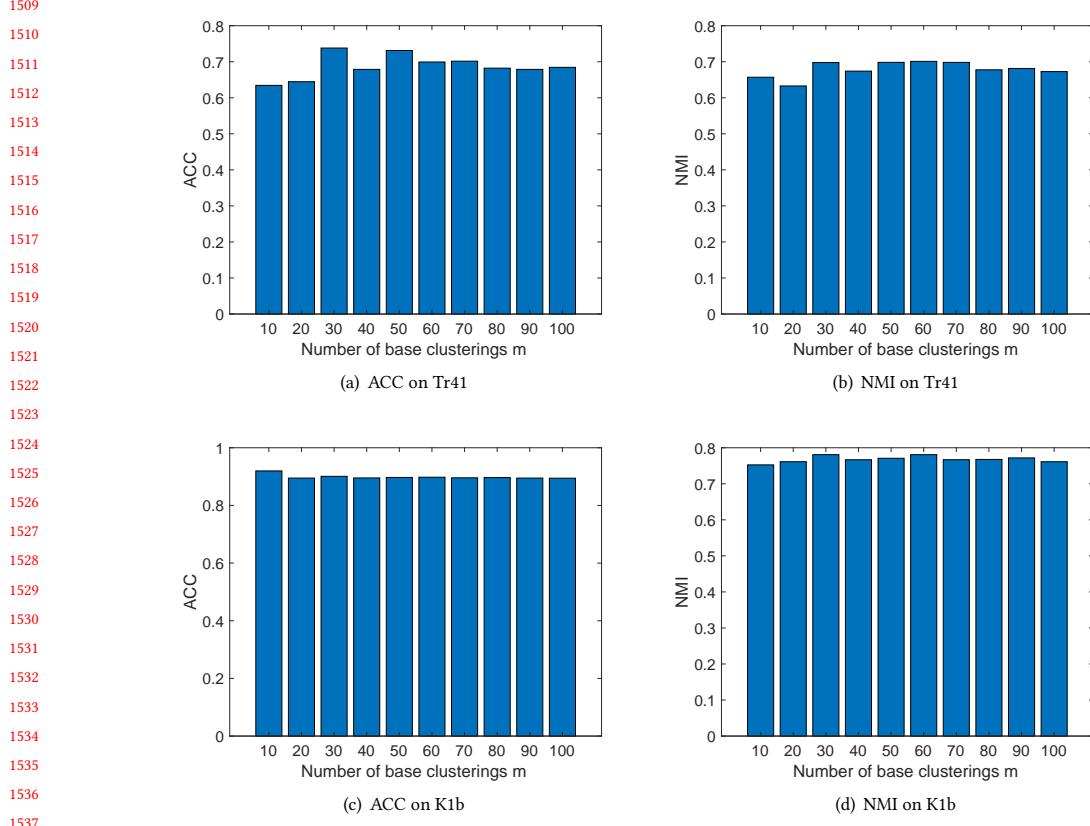


Fig. 11. Running time (Sec.) of all methods.

Fig. 12. Clustering results on different values of γ_1 and γ_2 .

$\{0, 0.1, \dots, 0.5\}$, and randomly remove instances according to r . For example, $r = 0.3$ means in each base result, there are 30% instances are missing; $r = 0$ means the base results are complete without any missing data. Notice that the compared methods cannot directly handle the missing values and need the complete data as inputs. Hence, for the

Fig. 13. Clustering results with different numbers of base clusterings m .

compared methods, we apply a random filling method to impute the missing values before doing consensus clustering. Specifically, if x_i is missing in the p -th base clustering, we randomly assign x_i to one cluster in the p -th base clustering, and then we run the compared consensus clustering methods on the filled data. Figure 15 shows the ACC results of all methods on all data sets with a missing ratio from 0 to 0.5. The results on other metrics are similar. The black solid line represents our SCCABG. From Figure 15, we find that the performance of many compared methods deteriorates rapidly with the increase of missing data. However, the performance of our method is relatively more stable on most data sets, which demonstrates that the proposed method can handle incomplete data better than the conventional consensus clustering methods.

4.9 Experiments on Large Data set

To show the effectiveness of the proposed method on the large scale data set, we also conduct experiments on MNIST data set [23]. MNIST is an image data set, which contains 70000 handwritten images in 10 categories. The size of each image is 28×28 . Following the previous experimental setup, we first also run k-means to generate 200 base results, and partition them into 10 subsets, with 20 in each one. Then, we run consensus clustering methods on each subset. Since the data set is too large for most compared methods, only HGPA, MCLA, and LWGP have a result and other methods

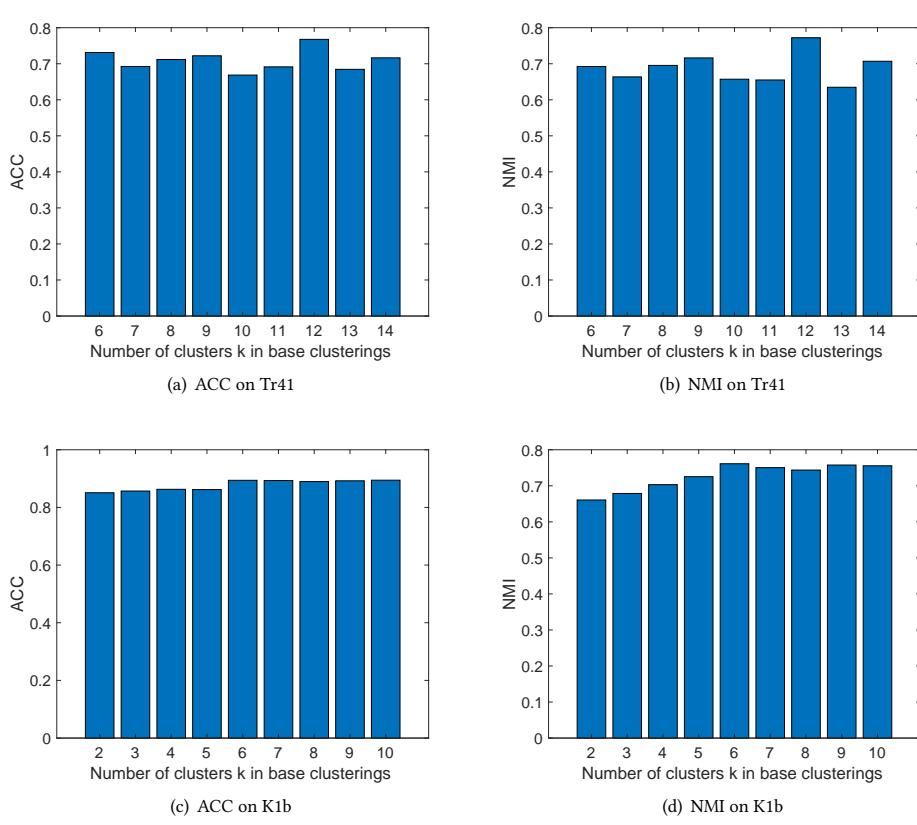


Fig. 14. Clustering results with different numbers of clusters k_i in each base clustering.

Table 7. Clustering results on MNIST. The bold font indicates that the difference is statistically significant (i.e., the p -value of t -test is smaller than 0.05).

Measures	KM	KM-best	HGPA	MCLA	LWGP	SCCABG
ACC	0.5468 ± 0.0084	0.5998 ± 0.0166	0.3042 ± 0.0369	0.5497 ± 0.0125	0.5413 ± 0.0164	0.5583 ± 0.0235
NMI	0.4933 ± 0.0041	0.5204 ± 0.0139	0.2065 ± 0.0539	0.4882 ± 0.0128	0.4899 ± 0.0030	0.4943 ± 0.0055

run out-of-memory. Table 7 shows the results. From Table 7, we find that SCCABG also outperforms HGPA, MCLA, and LWGP methods on this large data set.

5 CONCLUSION

In this paper, we proposed a novel self-paced consensus clustering method with adaptive bipartite graph learning. We constructed an initial bipartite graph with the base results, and then learned a structured bipartite graph from it adaptively. In the process of bipartite graph learning, we adopted the idea of self-paced learning, which automatically determined the reliability of each edge and involved them in bipartite graph learning in the order of reliability. At last, we directly obtained the final result by finding the connective components of the learned bipartite graph without

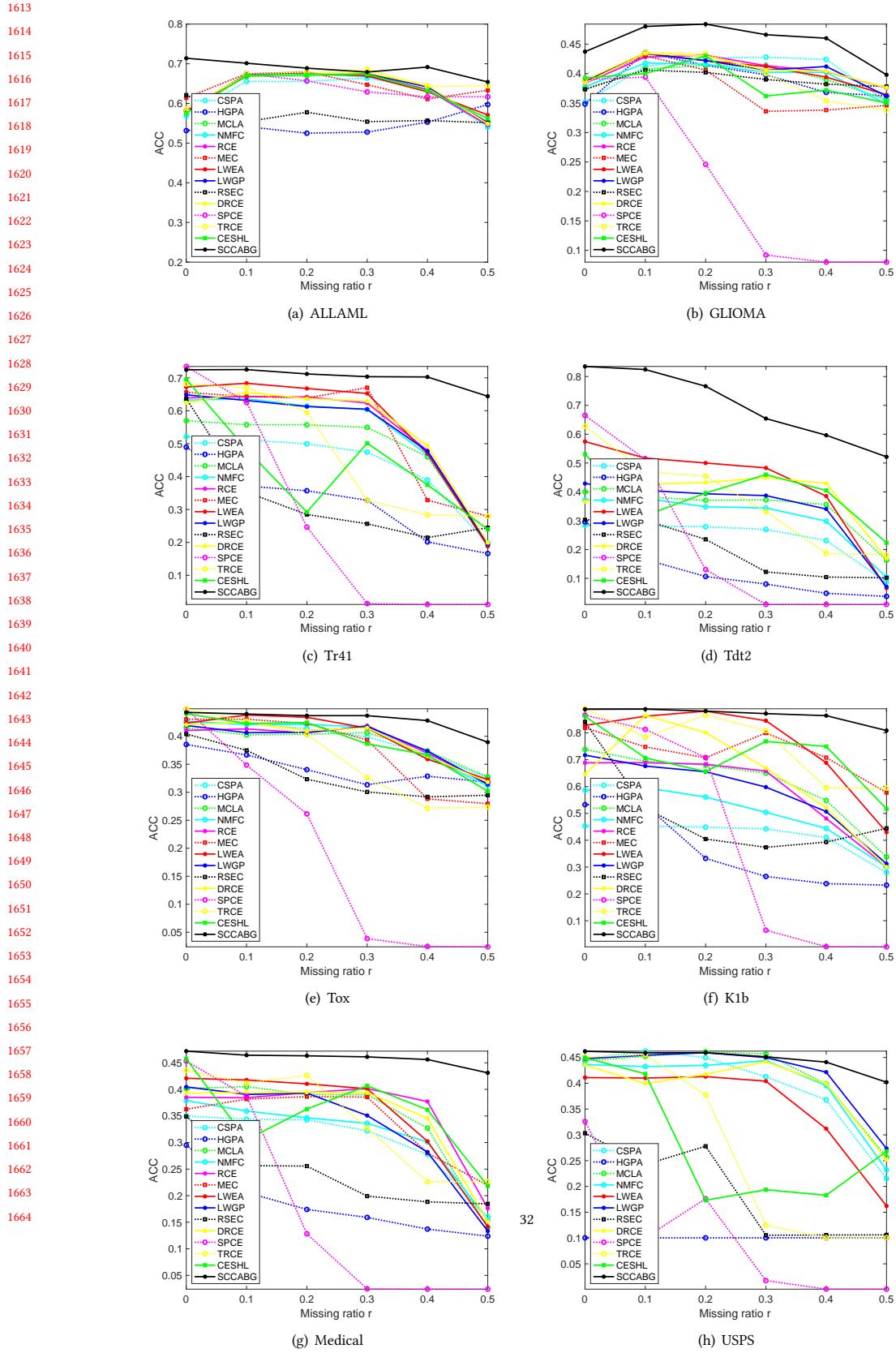


Fig. 15. ACC results on all data sets with missing ratio from 0 to 0.5.

any uncertain postprocessing. We conducted extensive experiments on toy and benchmark data sets. Compared with other state-of-the-art consensus clustering methods, our SCCABG often achieved better performance, which well demonstrated its superiority and effectiveness.

ACKNOWLEDGEMENTS

This work is supported by the National Natural Science Foundation of China grants 62176001, 61806003, 61976129, 61922088, and 61972001; and Natural Science Foundation of Anhui Province grant 1908085MF187.

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