Main Questions Part I - Power Series

Brace Goodfrey
dg681

1

a) $$= \sin\left(\frac{\pi}{6}\right) + \frac{\left(x-\frac{\pi}{6}\right)}{1!}\cos\left(\frac{\pi}{6}\right) + \frac{\left(x-\frac{\pi}{6}\right)^2}{2!}\cdot -\sin\left(\frac{\pi}{6}\right)$$

$$+ \frac{\left(x-\frac{\pi}{6}\right)^3}{3!}\cdot -\cos\left(\frac{\pi}{6}\right)$$

$$= \sin\left(\frac{\pi}{6}\right)\left[1 - \frac{\left(x-\frac{\pi}{6}\right)^2}{2}\right] + \cos\left(\frac{\pi}{6}\right)\left[\frac{x-\frac{\pi}{6}}{1} - \frac{\left(x-\frac{\pi}{6}\right)^3}{6}\right]$$

$$= \frac{1}{2}\left[1 - \frac{x^2}{2} - \frac{\pi}{6}x - \frac{\pi^2}{72}\right] +$$

$$\frac{\sqrt{3}}{2}\left[x - \frac{\pi}{6} - \frac{\left(x^3 - \frac{\pi}{2}x^2 + 3\frac{\pi^2}{72}x - \frac{\pi^3}{216}\right)}{6}\right]$$

$$= \frac{1}{2} - \frac{1}{4}x^2 - \frac{\pi}{12}x - \frac{\pi^2}{144}$$

$$+ \frac{\sqrt{3}}{2}x - \frac{\sqrt{3}\pi}{12} - \frac{\sqrt{3}}{12}x^3 + \frac{\sqrt{3}\pi}{24}x^2 - \frac{\pi^2}{288}x + \frac{\pi^3}{2592}$$

$$\rightarrow -\frac{\sqrt{3}}{12}x^3 + \left(\frac{\sqrt{3}\pi}{24} - \frac{1}{4}\right)x^2 + \left(\frac{\sqrt{3}}{2} - \frac{\pi^2}{96}\right)x + \left(\frac{1}{2} - \frac{\sqrt{3}}{12}\pi + \frac{\pi^3}{2592}\right)$$

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$\rightarrow$ b) ~~let x =~~ $31^\circ \approx \frac{\pi}{6}$ rads

$\rightarrow \sin(31^\circ) \approx \sin\left(\frac{\pi}{6}\right) \approx$ taylor series expanded around $\frac{\pi}{6}$

$\approx \underline{0.431}$

2

a) $\cos(x) \approx \cos(0) + \frac{x}{1!} \cdot -\sin(0) + \frac{x^2}{2!} \cdot -\cos(0)$

$$= \boxed{1 - \frac{x^2}{2}}$$

b) $\arcsin(x) \approx \arcsin(0) + \frac{x}{1!} \times \frac{1}{\sqrt{1-0^2}} + \frac{x^2}{2} \times 0(x\ )$

$$+ (1 - x^2)^{-\frac{3}{2}} \times \frac{x^3}{3!}$$

$$= \underline{x + \frac{x^3}{4}}$$

c) $e^x \approx \underline{1 + x}$

d) $\ln(x+1) \approx \ln(1) + \frac{x}{1!} \cdot \frac{1}{0+1}\ x - \frac{x^2}{2!}$

$$= \boxed{x - \frac{x^2}{2}}$$

3

a) $(1+x)^n$ around $x=0$

$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \dots$$

$$+ \frac{n(n-1)(n-2)\dots(n-n)}{n!}x^n + 0x^{n+1} + 0x^{n+2} \dots$$

$$\Rightarrow = \boxed{\sum_{r=0}^{n} \frac{n!}{(n-r)!\,r!}x^r}$$

b) $\underline{1 + 8x + 28x^2 + 56x^3 + 70x^4 + 56x^5 + 28x^6 + 8x^7 + x^8}$

4

a) $(1+x)^{\frac{3}{2}} = 1 + \frac{3}{2}x + \frac{\frac{3}{2}(\frac{3}{2}-1)}{2}x^2$

$$= \boxed{1 + \frac{3}{2}x + \frac{3}{8}x^2}$$

b) $= 4^{\frac{1}{2}}(1+\frac{3}{4}x)^{\frac{1}{2}} = 2(1+\frac{3}{4}x)^{\frac{1}{2}}$

$$= 2\left[1 + \frac{3}{4}\times\frac{1}{2}x + \frac{(\frac{1}{2})(\frac{1}{2}-1)}{2}(\tfrac{3}{4}x)^2\right]$$

$$= \boxed{2 + \frac{3}{4}x - \frac{9}{64}x^2}$$

c) $= 3^{-1}\left(1-\frac{1}{3}x\right)^{-1}$

$$= \frac{1}{3}\left[1+(-1)\left(-\frac{1}{3}x\right)+\frac{(-1)(-1-1)}{2}\left(-\frac{1}{3}x\right)^2\right]$$

$$= \boxed{\frac{1}{3}+\frac{1}{9}x+\frac{1}{27}x^2}$$

5

$\arctan(x)$

$$= \arctan(0) + x(\arctan^{-1})'(0) + \frac{x^2}{2!}\arctan''(0)\ldots$$

$$\cancel{= 0 + \frac{1}{1+0^2}x} +$$

$f(x) = \tan^{-1}(x) \leftarrow$ This is an ~~even~~ odd function, therefore only odd powers will be included in result

$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$

$f''(x) = -2x(1+x^2)^{-2}$

$\cancel{f'''(x) =}$

$$\rightarrow \arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5}\ldots$$

b) Error interval $= \underline{1\times10^{-11}}$

$\Rightarrow \tan^{-1} = \frac{\pi}{4}$

$4\times\tan^{-1} = \pi$

E.T $= \underline{4\times10^{-11}}$

Remainder Term $= \underline{\frac{f^{n+1}(1)}{(n+1)!}\times(x)^{n+1} \leq 4\times10^{-11}}$

$\Rightarrow f^{n+1}(1)\times x^{n+1} \leq 4\times10^{-11}(n+1)!$

→ Unsure where to go from here?

6

a) $\cancel{f(x) \approx f(x+h) + (x-(x+h))f'(x) + \frac{(x-(x+h))^2}{2!}f''(x)}$

$\cancel{= f(x+h)}$

$\cancel{f(x+h) \approx f(x) - hf'(x)}$

6

6

$\cancel{f(x) \approx f(x)+}$

$\cancel{f(x) = f(x) + f'(x) + }$

$$f(x+h) \approx f(x) + (x+h-x)f'(x) + \frac{(x+h-x)^2}{2}f''(x) \ldots$$

$$\approx \underline{f(x) + hf'(x) + h^2 f''(x) \ldots + h^n f^n(x) \ldots}$$

$$f(x_0) \approx 0$$

Let $x_1 = x_0 + h$ where $f(x_1) = 0$

$$f(x_0+h) \approx f(x_0) + hf'(x_0) + \ldots$$

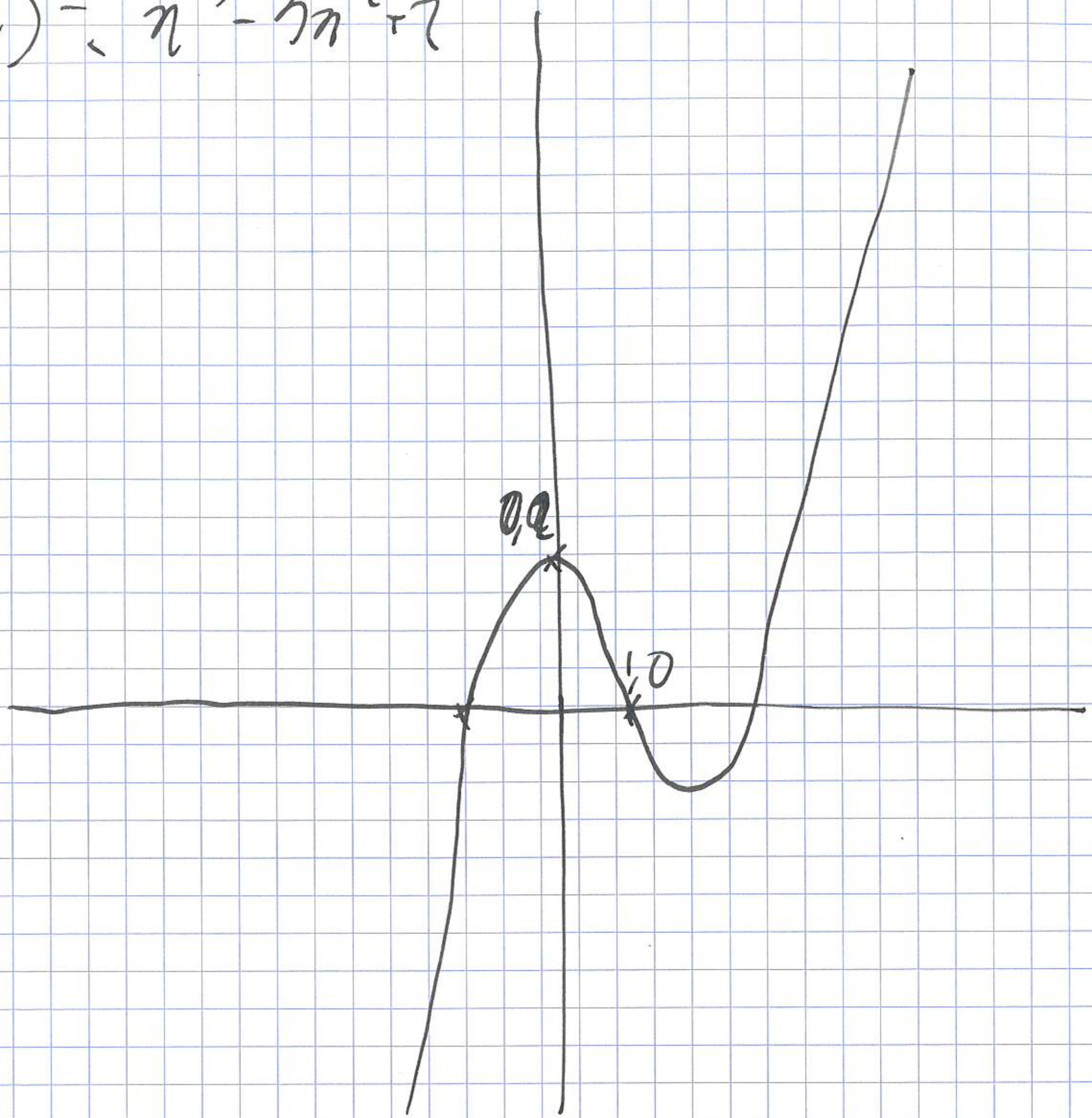
Truncate

so $f(x_0) + hf'(x_0) = 0$

$$h = \frac{-f(x_0)}{f'(x_0)}$$

so $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

b) $g(x) = x^3 - 3x^2 + 2$



c) $x_0 = 2.5$

$$x_1 = 2.5 - \frac{g(2.5)}{g'(2.5)}$$

$g(2.5) = -1.125$

~~$g'(2.5)$~~

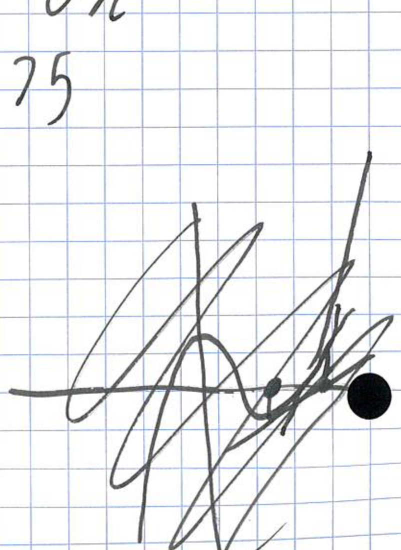
$g'(x) = 3x^2 - 6x$

$g'(2.5) = 3.75$

$$= \boxed{2.8 = x_1}$$

$$x_2 = 2.8 - \frac{g(2.8)}{g'(2.8)} \approx \boxed{\frac{383}{140}}$$

d)



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