Proving The Power Rule

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1 The Conjecture

The power rule states that for all $n \in \mathbb{Z}$, $\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$. We want to prove this from first principles.

2 The Proof

2.1 For The Naturals

Proving the power rule for $n \in \mathbb{N}, 0 \notin \mathbb{N}$ is relatively easy and just involves some simple binomial expansion.

Let $f(x) = x^n$.

We know that the derivate $f'(x)$ of $f(x)$ is defined as

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

If we plug in our $f(x) = x^n$, then we get

$$f'(x) = \lim_{h \to 0} \left(\frac{(x+h)^n - x^n}{h} \right)$$

We need to cancel the h before we let it go to 0. We can do this by expanding the binomial $(x+h)^n$ in the numerator like so:

$$(x+h)^n = x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n$$

We now have an x^n term and a $-x^n$ term in the numerator. These cancel to give us

$$\lim_{h \to 0} \left(\frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h} \right)$$

We can then factor out h from the numerator and cancel like so:

$$\lim_{h \to 0} \left(\frac{\cancel{h}\left(nx^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\right)}{\cancel{h}} \right)$$

$$= \lim_{h \to 0} \left(nx^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1} \right)$$

We can now let h go to 0 and thereby show that $f'(x) = nx^{n-1}$

$\square$

2.2 For All Integers

Proving the power rule for all $n \in \mathbb{Z}$ is a bit more complicated.

We know that the power rule would say

$$\frac{\mathrm{d}}{\mathrm{d}x}x^0 = 0x^{-1} = 0$$

We also know that x^0 is always 1, and the derivative of a constant is always 0, so the power rule holds for $n = 0$.

To prove it for negative integers, I'm going to prove that

$$\frac{\mathrm{d}}{\mathrm{d}x}x^{-n} = -nx^{-n-1}, n \in \mathbb{N}$$

because this is easier to prove and will expand the proof to all integers.

Let $f(x) = x^{-n}$.

We plug this $f(x)$ into the definition and get

$$f'(x) = \lim_{h \to 0}\left(\frac{(x+h)^{-n} - x^{-n}}{h}\right)$$

If it's possible to expand binomials with negative powers, I don't know how to do it, but I do know that $a^{-b} = \frac{1}{a^b}$, so we'll use that and focus on the numerator for now.

$$\begin{aligned}(x+h)^{-n} - x^{-n} &= \frac{1}{(x+h)^n} - \frac{1}{x^n}\\ &= \frac{x^n}{x^n(x+h)^n} - \frac{(x+h)^n}{x^n(x+h)^n}\\ &= \frac{x^n - (x+h)^n}{x^n(x+h)^n}\end{aligned}$$

We're going to expand and simplify the numerator, so for the sake of simplicity, I'm leaving the denominator unexpanded for now.

$$\begin{aligned}&\frac{x^n - (x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n)}{x^n(x+h)^n}\\ &= \frac{-nx^{n-1}h - \binom{n}{2}x^{n-2}h^2 - \cdots - nxh^{n-1} - h^n}{x^n(x+h)^n}\end{aligned}$$

Now, we're going to re-introduce h before expanding the denominator.

Dividing a fraction by h is the same as just multiplying the denominator by h.

$$\begin{aligned}&\frac{-nx^{n-1}h - \binom{n}{2}x^{n-2}h^2 - \cdots - nxh^{n-1} - h^n}{x^n(x+h)^n} \div h\\ &= \frac{-nx^{n-1}h - \binom{n}{2}x^{n-2}h^2 - \cdots - nxh^{n-1} - h^n}{hx^n(x+h)^n}\end{aligned}$$

We can factor a h out from the numerator and get

$$\frac{h(-nx^{n-1} - \binom{n}{2}x^{n-2}h - \cdots - nxh^{n-2} - h^{n-1})}{hx^n(x+h)^n}$$

We can now cancel the h and expand the denominator.

$$\frac{-nx^{n-1} - \binom{n}{2}x^{n-2}h - \cdots - nxh^{n-2} - h^{n-1}}{x^n(x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n)}$$
$$= \frac{-nx^{n-1} - \binom{n}{2}x^{n-2}h - \cdots - nxh^{n-2} - h^{n-1}}{x^{2n} + nx^{2n-1}h + \binom{n}{2}x^{2n-2}h^2 + \cdots + nx^{n+1}h^{n-1} + x^nh^n}$$

Now, we let h go to 0 to get rid of all the h terms and get left with

$$\frac{-nx^{n-1}}{x^{2n}}$$

Because $\dfrac{1}{x^{2n}} = x^{-2n}$, we can rewrite this as

$$-nx^{n-1}x^{-2n} = -nx^{n-1-2n} = -nx^{-n-1}$$

$\square$

The proof for negative integers is a bit longer and more involved. There's probably a much more elegant proof, but I'm pretty sure this one works, and I'm happy with it.

Thus, I have proved that for all $n \in \mathbb{Z}$, $\dfrac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$.