Polynomials That Sum To Pure Powers

Dyson

30th December, 2021

1 The Problem

The challenge question in Exercise 3B of the Edexcel AS and A Level Further Maths Core Pure 1 book is about polynomials that sum to pure powers of n.

Part a of this question asks for polynomials $f_2(x)$, $f_3(x)$, $f_4(x)$ such that for every $n \in \mathbb{N}$, $n > 1$,

$$\sum_{r=1}^{n} f_2(r) = n^2$$

$$\sum_{r=1}^{n} f_3(r) = n^3$$

$$\sum_{r=1}^{n} f_4(r) = n^4$$

Finding these polynomials reveals a very interesting pattern, links to the Bernoulli numbers, and has a very nice general form. And the proof of this general form simply falls out from a recurrence relation.

2 Finding Polynomials

Throughout this paper, all $n \in \mathbb{N}$, $n > 1$.

Let's first establish some basic lemmas about series.

Lemma 1 $\displaystyle\sum_{r=1}^{n}(f(r) + g(r)) = \sum_{r=1}^{n} f(r) + \sum_{r=1}^{n} g(r)$

Lemma 2 $\displaystyle\sum_{r=1}^{n} kf(r) = k\sum_{r=1}^{n} f(r)$

Lemma 3 $\displaystyle\sum_{r=1}^{n} 1 = n$

Lemma 4 $\displaystyle\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

Lemma 5 $\displaystyle\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

Lemma 6 $\displaystyle\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

2.1 Finding $f_2(r)$

We want a polynomial $f_2(r)$ such that $\sum_{r=1}^{n} f_2(r) = n^2$.

We already have, by Lemma 4,

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1) = \frac{1}{2}\left(n^2+n\right)$$

We need to get rid of the $\frac{1}{2}$, which we can do by multiplying by 2 to get

$$2\sum_{r=1}^{n} r = \sum_{r=1}^{n} 2r = n^2 + n$$

Then, we just need to get rid of the n.

We know, by Lemma 3, that $\sum_{r=1}^{n} 1 = n$, so

$$\sum_{r=1}^{n} 2r - \sum_{r=1}^{n} 1 = \sum_{r=1}^{n} (2r-1)$$
$$= n^2 + n - n = n^2$$
$$\therefore f_2(r) = 2r - 1$$

2.2 Finding $f_3(r)$

Next, we want a polynomial $f_3(r)$ such that $\sum_{r=1}^{n} f_3(r) = n^3$.

Similarly to with $f_2(r)$, by Lemma 5, we already have

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$
$$= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$
$$= \frac{1}{3}\left(n^3 + \frac{3}{2}n^2 + \frac{1}{2}n\right)$$

We can then multiply by 3 to remove the $\frac{1}{3}$ and get

$$3\sum_{r=1}^{n} r^2 = \sum_{r=1}^{n} 3r^2 = n^3 + \frac{3}{2}n^2 + \frac{1}{2}n$$

To get rid of the $\frac{1}{2}n$, we can just do

$$\sum_{r=1}^{n} 3r^2 - \sum_{r=1}^{n} \frac{1}{2} = \sum_{r=1}^{n}\left(3r^2 - \frac{1}{2}\right) = n^3 + \frac{3}{2}n^2$$

We know that $\sum_{r=1}^{n} f_2(r) = n^2$, so we simply need to subtract $\frac{3}{2}f_2(r)$ from our polynomial to get rid of the resultant $\frac{3}{2}n^2$.

$$3r^2 - \frac{1}{2} - \frac{3}{2}(2r-1) = 3r^2 - 3r + 1$$
$$\therefore f_3(r) = 3r^2 - 3r + 1$$

We can show that this is true, just to be sure of it.

$$\begin{aligned}\sum_{r=1}^{n}(3r^2-3r+1) &= 3\sum_{r=1}^{n}r^2-3\sum_{r=1}^{n}r+\sum_{r=1}^{n}1\\ &= \frac{1}{2}n(n+1)(2n+1)-\frac{3}{2}n(n+1)+n\\ &= \frac{1}{2}\left(2n^3+3n^2+n-3n^2-3n+2n\right)\\ &= \frac{1}{2}\left(2n^3\right)=n^3\end{aligned}$$

2.3 Finding $f_4(r)$

Next, we want a polynomial $f_4(r)$ such that $\sum\limits_{r=1}^{n} f_4(r)=n^4$.

We know, by Lemma 6, that

$$\begin{aligned}\sum_{r=1}^{n}r^3 &= \frac{1}{4}n^2(n+1)^2\\ &= \frac{1}{4}\left(n^4+2n^3+n^2\right)\end{aligned}$$

We can multiply by 4 to get

$$4\sum_{r=1}^{n}r^3=\sum_{r=1}^{n}4r^3=n^4+2n^3+n^2$$

We can then get n^4 on its own by subtracting $2f_3(r)$ and $f_2(r)$ from $4r^3$.

$$\begin{aligned}&4r^3-2\left(3r^2-3r+1\right)-(2r-1)\\ &=4r^3-6r^2+6r-2-2r+1\\ &=4r^3-6r^2+4r-1\\ &\therefore f_4(r)=4r^3-6r^2+4r-1\end{aligned}$$

Likewise, we can show that this is true to convince ourselves that this process works.

$$\begin{aligned}\sum_{r=1}^{n}(4r^3-6r^2+4r-1) &= 4\sum_{r=1}^{n}r^3-6\sum_{r=1}^{n}r^2+4\sum_{r=1}^{n}r-\sum_{r=1}^{n}1\\ &= n^2(n+1)^2-n(n+1)(2n+1)+2n(n+1)-n\\ &= n^4+2n^3+n^2-2n^3-3n^2-n+2n^2+2n-n\\ &= n^4+\cancel{2n^3}-\cancel{2n^3}+\cancel{3n^2}-\cancel{3n^2}+\cancel{2n}-\cancel{2n}\\ &= n^4\end{aligned}$$

3 Conjectures

We can find $f_1(r)$, where $\sum_{r=1}^{n} f_1(r) = n^1 = n$ to trivially be 1.

These are our polynomials:

$$f_1(r) = 1$$
$$f_2(r) = 2r - 1$$
$$f_3(r) = 3r^2 - 3r + 1$$
$$f_4(r) = 4r^3 - 6r^2 + 4r - 1$$

After looking at these for a while, we can notice a few things. Firstly, the constants are always ± 1, and the signs of these constants alternate with increasing degrees of polynomial. In fact, all the signs alternate.

Secondly, we can notice that the first term of $f_a(r)$ is always of the form ar^{a-1}.

However, the most interesting thing to notice with these polynomials is that the coefficients look like binomial expansions of $(r-1)^a$, albeit with the leading term removed and all the signs flipped.

We can continue this pattern to make a conjecture about $f_5(r)$. However, this notation of $f_a(r)$ is just for convenience and it doesn't make sense to directly conjecture about what $f_5(r)$ should be, so let's conjecture about its sum to n.

Conjecture 1 $\forall n \in \mathbb{N},\ n > 1,\ \sum_{r=1}^{n} (5r^4 - 10r^3 + 10r^2 - 5r + 1) = n^5$

Let's first try to prove this conjecture, and if it's true, then we can form a general conjecture for any $f_a(r)$.

3.1 Proving Conjecture 1

In order to prove Conjecture 1, we need to know the formula for $\sum_{r=1}^{n} r^4$. Wolfram Alpha says:

Lemma 7 $\displaystyle\sum_{r=1}^{n} r^4 = \frac{1}{30} n(n+1)(2n+1)\left(3n^2 + 3n - 1\right)$

This can then be expanded to give

$$\frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$
$$= \frac{1}{5}\left(n^5 + \frac{5}{2}n^4 + \frac{5}{3}n^3 - \frac{1}{6}n\right)$$

We can multiply by 5 to get

$$5\sum_{r=1}^{n} r^4 = \sum_{r=1}^{n} 5r^4 = n^5 + \frac{5}{2}n^4 + \frac{5}{3}n^3 - \frac{1}{6}n$$

Now, we just need to get rid of the other terms to get a polynomial $f_5(r)$ such that $\sum_{r=1}^{n} f_5(r) = n^5$.

$$\begin{aligned} f_5(r) &= 5r^4 - \frac{5}{2}f_4(r) - \frac{5}{3}f_3(r) + \frac{1}{6}f_1(r) \\ &= 5r^4 - \frac{5}{2}\left(4r^3 - 6r^2 + 4r - 1\right) - \frac{5}{3}\left(3r^2 - 3r + 1\right) + \frac{1}{6} \\ &= 5r^4 - 10r^3 + 10r^2 - 5r + 1 \end{aligned}$$

This shows that our previous process generates the same polynomial as the pattern would suggest. To actually prove Conjecture 1, let's test its sum and see if it gives n^5, as predicted.

$$\begin{aligned} \sum_{r=1}^{n}(5r^4 - 10r^3 + 10r^2 - 5r + 1) &= 5\sum_{r=1}^{n} r^4 - 10\sum_{r=1}^{n} r^3 + 10\sum_{r=1}^{n} r^2 - 5\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1 \\ &= \frac{1}{6}n(n+1)(2n+1)\left(3n^2+3n-1\right) - \frac{10}{4}n^2(n+1)^2 + \frac{10}{6}n(n+1)(2n+1) - \frac{5}{2}n(n+1) + n \\ &= \frac{1}{6}\left(n(n+1)(2n+1)\left(3n^2+3n-1\right) - 15n^2(n+1)^2 + 10n(n+1)(2n+1) - 15n(n+1) + 6n\right) \\ &= \frac{1}{6}\left(6n^5 + \cancel{15n^4} + 10n^3 - n - \cancel{15n^4} - 30n^3 - 15n^2 + 20n^3 + 30n^2 + 10n - 15n^2 - 15n + 6n\right) \\ &= \frac{1}{6}\left(6n^5 + \cancel{30n^3} - \cancel{30n^3} + \cancel{30n^2} - \cancel{30n^2} + \cancel{16n} - \cancel{16n}\right) \\ &= \frac{1}{6}\left(6n^5\right) = n^5 \end{aligned}$$

3.2 A general conjecture

To conjecture a general form, we have to think about how we go from $(r-1)^a$ to these polynomials.

Lets look at the example of $f_4(r)$. We want

$$4r^3 - 6r^2 + 4r - 1$$

but $(r-1)^4$ gives us

$$r^4 - 4r^3 + 6r^2 - 4r + 1$$

We have the change the signs of every term, so we get

$$-(r-1)^4 = -r^4 + 4r^3 - 6r^2 + 4r - 1$$

Then we have to remove the leading $-r^4$ and we get $-(r-1)^4 + r^4$, or more simply, $r^4 - (r-1)^4$.

This form of $r^a - (r-1)^a$ gives the results seen previously for $f_a(r)$, so we can conjecture that this pattern continues for all $a \in \mathbb{Z}^+$.[1]

But again, we don't want to directly conjecture about $f_a(r)$, so we conjecture about its sum to n.

Conjecture 2 $\forall\, a, n \in \mathbb{Z}^+,\ n > 1,\ \sum_{r=1}^{n}\left(r^a - (r-1)^a\right) = n^a$

[1] $\mathbb{Z}^+$ is simply the set of positive integers, not including 0. This is simpler than writing $a \in \mathbb{N},\ a > 0$.

4 Finding Patterns

Let's look at sums of powers of r. By Lemmas 3 to 7, we have

$$\begin{aligned}
\sum_{r=1}^{n} r^0 &= n && = n \\
\sum_{r=1}^{n} r^1 &= \frac{1}{2}n(n+1) && = \frac{1}{2}n^2 + \frac{1}{2}n \\
\sum_{r=1}^{n} r^2 &= \frac{1}{6}n(n+1)(2n+1) && = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \\
\sum_{r=1}^{n} r^3 &= \frac{1}{4}n^2(n+1)^2 && = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 \\
\sum_{r=1}^{n} r^4 &= \frac{1}{30}n(n+1)(2n+1)\left(3n^2+3n-1\right) && = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n
\end{aligned}$$

I can't see much of a pattern here, but the first term of $\sum\limits_{r=1}^{n} r^b$ always seems to be $\frac{1}{b+1}n^{b+1}$ and the second term always seems to be $\frac{1}{2}n^b$, except in the case of $b=0$.

I don't think I would able to find a formula for $\sum\limits_{r=1}^{n} r^b$ on my own, but in doing some research on this topic, I first came across Faulhaber's formula on a wiki page on brilliant.org[1]. I shall rewrite the formula here.

$$\sum_{r=1}^{n} r^b = \frac{1}{b+1}\sum_{j=0}^{b}(-1)^j\binom{b+1}{j}B_j n^{b+1-j}$$

where B_j is the jth Bernoulli number.

In doing further research, I found that this is where the Bernoulli numbers were originally found by Jacob Bernoulli in *Ars Conjectandi*, published in 1713. He was trying to find a general formula for $\sum\limits_{r=1}^{n} r^b$ and found these numbers. He could not relate them to any previously known sequence, and de Moivre named them after him. And Jacob Bernoulli mentioned Faulhaber by name in *Ars Conjectandi*, referencing his previous work in the area.[2]

People have wondered about this problem long before me, and this is where the famous Bernoulli numbers originally came from.

Bernoulli might have written $\sum\limits_{r=1}^{n} r^b$ as

$$\sum_{r=1}^{n} r^b = \sum_{r=0}^{b} \frac{B_r}{r!} b^{\underline{r-1}} n^{b-r+1}$$

where $p^{\underline{q}}$ is the falling factorial $p \times (p-1) \times (p-2) \times \cdots \times (p-q+1)$.

5 Proving Conjecture 2

I expected proving Conjecture 2 to be hard, but it's actually quite easy and doesn't require the Bernoulli numbers at all. I got these ideas from a phenomenal Mathologer video[3], in which Burkard (the presenter) talks about Bernoulli's attempts to find a general formula for $\sum\limits_{r=1}^{n} r^b$, but not before showing off a beautiful recurrence relation for finding these formulas.

I am incredibly thankful for Burkard and the people at Mathologer for producing such wonderful and engaging mathematical content. They have definitely helped to foster my love of mathematics, and I highly recommend the channel to anyone with an interest in maths. Anyway, onto the proof.

5.1 Proving Sum Formulas

We can find successive summation formulas by manipulating binomial expansions. For the sake of notation, let $\sum\limits_{r=1}^{n} r^b$ be written as S_b.

The formulas for S_0, S_1, and S_2 can be found relatively easily through many different methods, which I won't talk about here. But let's say we wanted to find a formula for S_3. How would we do this? Well, we want to relate different powers, and we can do that nicely with some binomial expansion.

To find S_3, we look at $(r-1)^4$. This expands to give $r^4 - 4r^3 + 6r^2 - 4r + 1$. We can manipulate this and get

$$\begin{aligned}(r-1)^4 &= r^4 - 4r^3 + 6r^2 - 4r + 1 \\ \implies 4r^3 - 6r^2 + 4r - 1 &= r^4 - (r-1)^4\end{aligned}$$

We can now substitute different values for r and get a list of equations.

$$\begin{gathered}4 \cdot 1^3 - 6 \cdot 1^2 + 4 \cdot 1^1 - 1 \cdot 1^0 = 1^4 - \cancel{(1-1)^4} \\ 4 \cdot 2^3 - 6 \cdot 2^2 + 4 \cdot 2^1 - 1 \cdot 2^0 = 2^4 - (2-1)^4 \\ 4 \cdot 3^3 - 6 \cdot 3^2 + 4 \cdot 3^1 - 1 \cdot 3^0 = 3^4 - (3-1)^4 \\ \vdots \\ 4 \cdot n^3 - 6 \cdot n^2 + 4 \cdot n^1 - 1 \cdot n^0 = n^4 - (n-1)^4\end{gathered}$$

We can now sum these equations. On the RHS, the terms in brackets are cancelled out by the first term on the RHS in the line above. $-(2-1)^4$ cancels with 1^4, $-(3-1)^4$ cancels with 2^4, etc. and we end up with just n^4.

On the LHS, we simply get $4S_3 - 6S_2 + 4S_1 - 1S_0$. This means that

$$4S_3 - 6S_2 + 4S_1 - 1S_0 = n^4$$

Since we know S_0, S_1, and S_2, we can simply solve for S_3 and get

$$\begin{aligned}S_3 &= \frac{n^4 + 6S_2 - 4S_1 + 1S_0}{4} \\ &= \frac{n^4 + n(n+1)(2n+1) - 2n(n+1) + n}{4} \\ &= \frac{n(n^3 + (n+1)(2n+1) - 2(n+1) + 1)}{4} \\ &= \frac{n(n^3 + 2n^2 + 3n + 1 - 2n - 2 + 1)}{4} \\ &= \frac{n(n^3 + 2n^2 + n)}{4} \\ &= \frac{n^2(n+1)^2}{4}\end{aligned}$$

This is the formula given by Lemma 6, so we know we've done this correctly.

It can be seen that this process continues and can be used to generate $S_b \ \forall \ b \in \mathbb{Z}^+$, but I will prove it rigorously.

We can expand a general $(r-1)^{b+1}$ like so,

$$(r-1)^{b+1} = r^{b+1} - (b+1)r^b + \binom{b+1}{2}r^{b-1} - \cdots \pm \binom{b+1}{b-1}r^2 \mp (b+1)r \pm 1$$

$$\implies (b+1)r^b - \binom{b+1}{2}r^{b-1} + \cdots \mp \binom{b+1}{b-1}r^2 \pm (b+1)r \mp 1 = r^{b+1} - (r-1)^{b+1}$$

We can then sum up many substitutions up to n.

$$(b+1)1^b - \binom{b+1}{2}1^{b-1} + \cdots \mp \binom{b+1}{b-1}1^2 \pm (b+1)1 \mp 1 = 1^{b+1} - \cancel{(1-1)^{b+1}}$$

$$(b+1)2^b - \binom{b+1}{2}2^{b-1} + \cdots \mp \binom{b+1}{b-1}2^2 \pm (b+1)2 \mp 1 = 2^{b+1} - (2-1)^{b+1}$$

$$(b+1)3^b - \binom{b+1}{2}3^{b-1} + \cdots \mp \binom{b+1}{b-1}3^2 \pm (b+1)3 \mp 1 = 3^{b+1} - (3-1)^{b+1}$$

$$\vdots$$

$$(b+1)n^b - \binom{b+1}{2}n^{b-1} + \cdots \mp \binom{b+1}{b-1}n^2 \pm (b+1)n \mp 1 = n^{b+1} - (n-1)^{b+1}$$

Again, all but one term on the RHS cancel out and we just get n^{b+1}.

On the LHS, we get

$$(b+1)S_b - \binom{b+1}{2}S_{b-1} + \cdots \mp \binom{b+1}{b-1}S_2 \pm (b+1)S_1 \mp S_0$$

Then, knowing all S_c for $c < b$, we can solve for S_b.

$$(b+1)S_b - \binom{b+1}{2}S_{b-1} + \cdots \mp \binom{b+1}{b-1}S_2 \pm (b+1)S_1 \mp S_0 = n^{b+1}$$

$$\implies S_b = \frac{n^{b+1} + \binom{b+1}{2}S_{b-1} - \cdots \pm \binom{b+1}{b-1}S_2 \mp (b+1)S_1 \pm S_0}{b+1}$$

Now that we have a way of generating every S_b, we can use that to prove Conjecture 2. It's actually remarkably simple.

5.2 Proving The Conjecture

Let's reuse our example and backtrack a little to when we had $4S_3 - 6S_2 + 4S_1 - 1S_0 = n^4$. This equation is all we need. We don't even need to know any sum formulas. We can just expand the S_b terms to their equivalent sums and use Lemmas 1 and 2 to manipulate this.

$$4S_3 - 6S_2 + 4S_1 - 1S_0 = n^4$$

$$\implies 4\sum_{r=1}^{n} r^3 - 6\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r^1 - 1\sum_{r=1}^{n} r^0 = n^4$$

$$\implies \sum_{r=1}^{n} 4r^3 + \sum_{r=1}^{n} -6r^2 + \sum_{r=1}^{n} 4r + \sum_{r=1}^{n} -1 = n^4$$

$$\implies \sum_{r=1}^{n} (4r^3 - 6r^2 + 4r - 1) = n^4$$

This is exactly $f_4(r)$. This process can be used to find any S_b, and thus we can do this with any $f_a(r)$.

In our process of generating S_b, $a = b + 1$. We can use this to rewrite the equation in terms of a.

$$aS_{a-1} - \binom{a}{2}S_{a-2} + \cdots \mp \binom{a}{a-2}S_2 \pm aS_1 \mp S_0 = n^a$$

$$\implies a\sum_{r=1}^{n} r^{a-1} - \binom{a}{2}\sum_{r=1}^{n} r^{a-2} + \cdots \mp \binom{a}{a-2}\sum_{r=1}^{n} r^2 \pm a\sum_{r=1}^{n} r \mp \sum_{r=1}^{n} 1 = n^a$$

$$\implies \sum_{r=1}^{n} ar^{a-1} + \sum_{r=1}^{n} -\binom{a}{2}r^{a-2} + \cdots + \sum_{r=1}^{n} \mp\binom{a}{a-2}r^2 + \sum_{r=1}^{n} \pm ar + \sum_{r=1}^{n} \mp 1 = n^a$$

$$\implies \sum_{r=1}^{n}\left(ar^{a-1} - \binom{a}{2}r^{a-2} + \cdots \mp \binom{a}{a-2}r^2 \pm ar \mp 1\right) = n^a$$

We want to show that this expression in the sum on the LHS is $r^a - (r-1)^a$, so we can just expand this binomial and show that it's equal to the expression in the sum.

$$r^a - (r-1)^a = \cancel{r^a} - \cancel{r^a} + ar^{a-1} - \binom{a}{2}r^{a-2} + \cdots \mp \binom{a}{a-2}r^2 \pm ar \mp 1$$

$$\therefore \sum_{r=1}^{n}\left(ar^{a-1} - \binom{a}{2}r^{a-2} + \cdots \mp \binom{a}{a-2}r^2 \pm ar \mp 1\right) = \sum_{r=1}^{n}\left(r^a - (r-1)^a\right)$$

$$\therefore \sum_{r=1}^{n}\left(r^a - (r-1)^a\right) = n^a$$

Thus, we have proven Conjecture 2.

□

6 Conclusion

This challenge question is the first question that's inspired me to do proper research on a topic that I didn't know much about at the time, and then write up a full paper on it. Originally, this paper just answered the original question and posited Conjecture 2, and I was going to leave it at that because I assumed I didn't have the tools to prove it at the time. But then when I actually looked into it, the proof basically fell out of a recurrence relationship between sums. All I had to do was make that proof rigorous.

I assume there's a name for this polynomial $f_a(r)$ in the context of these sums, where $\sum_{r=1}^{n} f_a(r) = n^a$, but I don't know what it's called. So for now, I'm going to claim this little corner of mathematics for myself. In the context of these sums, I will call it the *Dyson Sum Function*.

References

[1] *Sum of n, n², or n³*. Brilliant. URL: `https://brilliant.org/wiki/sum-of-n-n2-or-n3/`.

[2] *Bernoulli number*. URL: `https://en.wikipedia.org/wiki/Bernoulli_number`.

[3] Mathologer. *Power sum MASTER CLASS: How to sum quadrillions of powers ... by hand! (Euler-Maclaurin formula)*. Oct. 26, 2019. URL: `https://www.youtube.com/watch?v=fw1kRz83Fj0`.