Differential Equations

Contents

First Order Homogeneous

Separation of variables

The simplest technique for solving first order homogeneous differential equations is to separate the variables. This involves treating the dx and dy as their own variables, rearranging everything to get all y terms on one side and all x terms on the other, and then adding integral signs. For example,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x} \implies \frac{1}{y}\,\mathrm{d}y = -\frac{1}{x}\,\mathrm{d}x \implies \int \frac{1}{y}\,\mathrm{d}y = \int -\frac{1}{x}\,\mathrm{d}x$$

$$\implies \ln|y| = -\ln|x| + c \implies \ln|y| + \ln|x| = c \implies \ln|xy| = c$$

$$\implies xy = \pm A \implies y = \pm\frac{A}{x}$$

This technique is easy, but very very rarely possible.

Reverse product rule

Sometimes the LHS (the y and y' terms in this case) can be seen as the result of the product rule applied to some function of x and y. For example,

$$x^3y' + 3x^2y = \sin x \implies \frac{\mathrm{d}}{\mathrm{d}x}(x^3y) = \sin x \implies x^3y = \int \sin x\,\mathrm{d}x$$

$$\implies x^3y = -\cos x + c \implies y = \frac{-\cos x + c}{x^3}$$

Again, this technique is easy, but very rarely possible.

Integrating factor

The reverse product rule is not always applicable, but you can always multiply a first order linear homogeneous differential equation by some function of x, called an **integrating factor (IF)**, to make the reverse product rule applicable.

This **IF** could be anything, and you may be able to find a simple one, but there is a general formula:

Remember

$$y' + P(x)y = Q(x) \implies \textbf{IF} = e^{\int P(x)\,\mathrm{d}x}$$

For example,

$$y' + \frac{3y}{x} = \frac{\sin x}{x^3} \implies \textbf{IF} = x^3 \text{ (by inspection) or } \textbf{IF} = e^{\int \frac{3}{x}\,\mathrm{d}x} = e^{3\ln|x|} = x^3$$

$$x^3y' + 3x^2y = \sin x \implies y = \frac{-\cos x + c}{x^3} \text{ as shown above}$$