Maths Crib Sheets

Contents

Binomial Expansion

Definitions

The factorial $n! \equiv n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$.

The falling factorial $n^{\underline{k}} \equiv n \times (n-1) \times (n-2) \times \cdots \times (n-(k-2)) \times (n-(k-1))$. It has k terms.

$$\boxed{0! = n^{\underline{0}} = 1}$$

The choose function ${}^nC_r \equiv \boxed{\binom{n}{r} \equiv \frac{n!}{r!(n-r)!}}$

Expansions

For a natural number n, the expansion of $(a+b)^n$ is

$$a^n + na^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots \binom{n}{r}a^{n-r}b^r + \cdots b^n$$

In general, $\boxed{(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r}b^r \ \ (n \in \mathbb{N})}$

That's true if n is a natural number, but there is a version that works for all real numbers. For an expression $(a+bx)^n$, it should first be normalised to $a^n(1+\frac{b}{a}x)^n$. Let $y = \frac{b}{a}x$. Then the expansion of $(1+y)^n$ is given by

$$1 + ny + \frac{n(n-1)}{2!}y^2 + \frac{n(n-1)(n-2)}{3!}y^3 + \cdots + \frac{n(n-1)(n-2)\cdots(n-(r-1))}{r!}y^r + \cdots$$

In general, $\boxed{(a+bx)^n = a^n \sum_{r=0}^{\infty} \frac{n^{\underline{r}}}{r!} \left(\frac{b}{a}x\right)^r \ \ (n \in \mathbb{R})}$

Calculus

Elementary Derivatives

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

f	C	x^n	$\sin x$	$\cos x$	a^x	$\ln x$
f'	0	nx^{n-1}	$\cos x$	$-\sin x$	$a^x \ln a$	$\frac{1}{x}$

Composition Laws

Let f and g be differentiable functions over x.

The $'$ mark denotes the derivative with respect to x, so $f' = \dfrac{df}{dx}$ and $g' = \dfrac{dg}{dx}$.

The $\circ$ symbol denotes function composition, so $(f \circ g)(x) = f(g(x))$.

$$(f \pm g)' = f' \pm g'$$

$$(fg)' = fg' + f'g$$

$$(f \circ g)' = (f' \circ g)g'$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

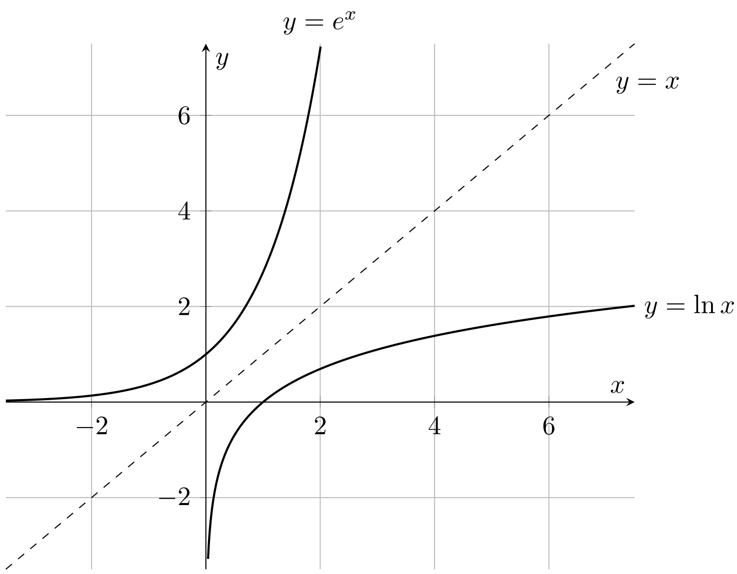
Integral Tricks

For integrals of the form on the left, consider the function on the right.

$\int kf'f^n \ dx$	f^{n+1}
$\int k\frac{f'}{f} \ dx$	$\ln \lvert f \rvert$

Exponentials and Logarithms

Exponentials



$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}e^{kx} = ke^{kx}$$

$$\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$$

$\ln x$ is the inverse of e^x, meaning its graph is reflected in the line $y = x$.

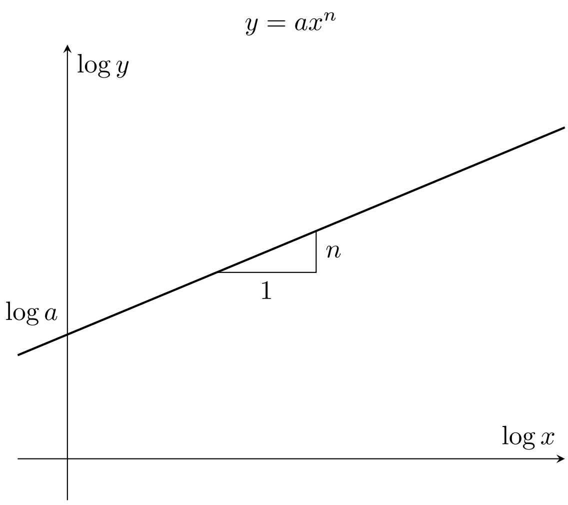
Log Laws

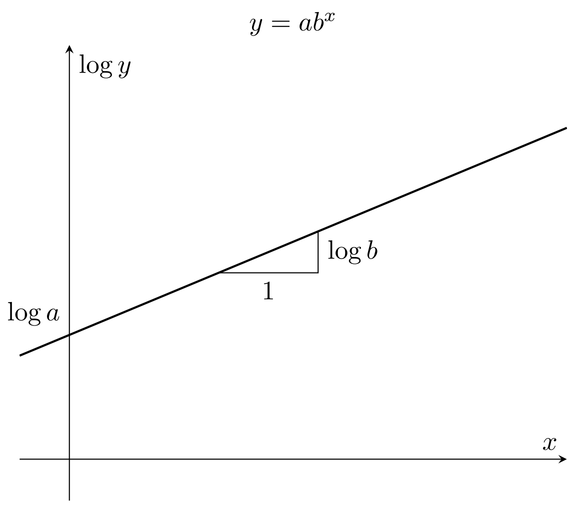
$$\log_a b \equiv \frac{\ln b}{\ln a}$$

$$\log xy \equiv \log x + \log y \qquad \log \frac{x}{y} \equiv \log x - \log y \qquad \log x^y \equiv y \log x$$

$$\log_a a \equiv 1 \qquad \log 1 \equiv 0 \qquad \log \frac{1}{x} \equiv -\log x$$

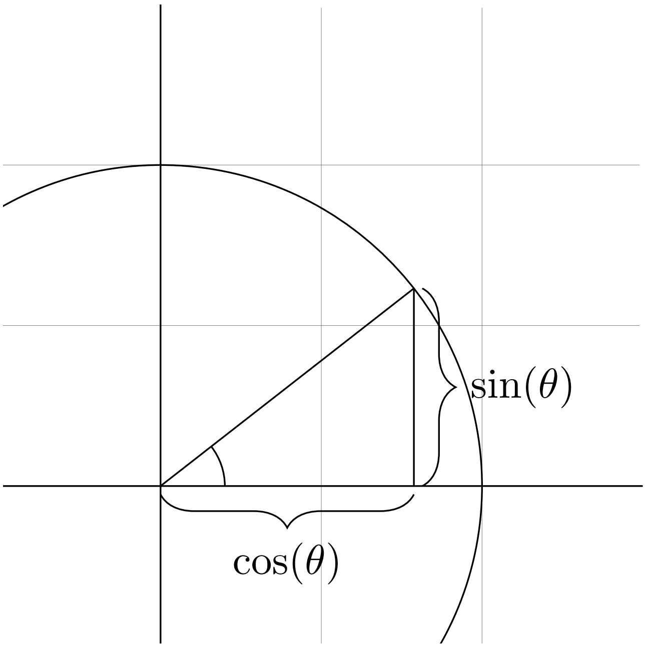
Log Plots





Trigonometry

Definitions



$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

$$\sec\theta \equiv \frac{1}{\cos\theta}$$

$$\operatorname{cosec}\theta \equiv \frac{1}{\sin\theta}$$

$$\cot\theta \equiv \frac{1}{\tan\theta} \equiv \frac{\cos\theta}{\sin\theta}$$

Identities

$$\sin^2\theta + \cos^2\theta \equiv 1$$

$$1 + \tan^2\theta \equiv \sec^2\theta$$

$$1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$$

$$\sin 2\theta \equiv 2\sin\theta\cos\theta$$

$$\sin(\alpha \pm \beta) \equiv \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\tan 2\theta \equiv \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\cos(\alpha \pm \beta) \equiv \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) \equiv \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

$$\begin{aligned}\cos 2\theta &\equiv \cos^2\theta - \sin^2\theta \\ &\equiv 2\cos^2\theta - 1 \\ &\equiv 1 - 2\sin^2\theta\end{aligned}$$

Calculus

$$\frac{d}{dx}\sin x = \cos x$$

$$\int \sin x \ dx = -\cos x + C$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\int \cos x \ dx = \sin x + C$$

$$\frac{d}{dx}\tan x = \sec^2 x$$

$$\int \tan x \ dx = \ln|\sec x| + C$$

$$\frac{d}{dx}\sec x = \sec x \tan x$$

$$\int \sec x \ dx = \ln|\tan x + \sec x| + C$$

$$\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x$$

$$\int \operatorname{cosec} x \ dx = -\ln|\cot x + \operatorname{cosec} x| + C$$

$$\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$$

$$\int \cot x \ dx = \ln|\sin x| + C$$

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\int \arcsin x \ dx = x\arcsin x + \sqrt{1-x^2} + C$$

$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$$

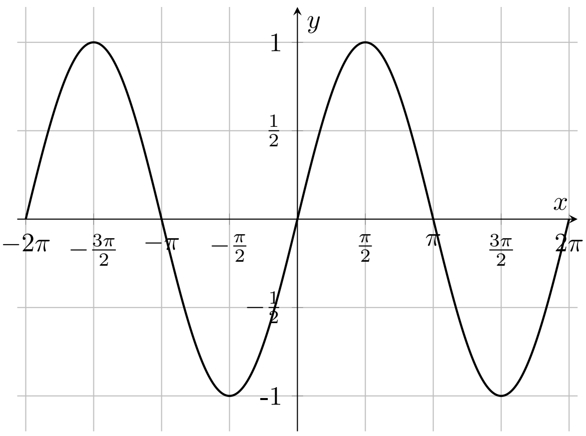
$$\int \arccos x \ dx = x\arccos x - \sqrt{1-x^2} + C$$

$$\frac{d}{dx}\arctan x = \frac{1}{x^2+1}$$

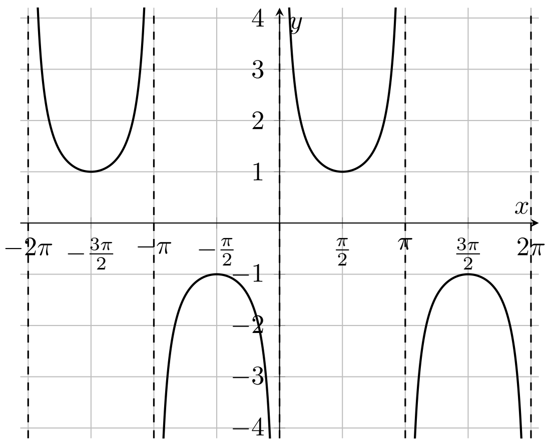
$$\int \arctan x \ dx = x\arctan x - \frac{\ln(x^2+1)}{2} + C$$

Graphs

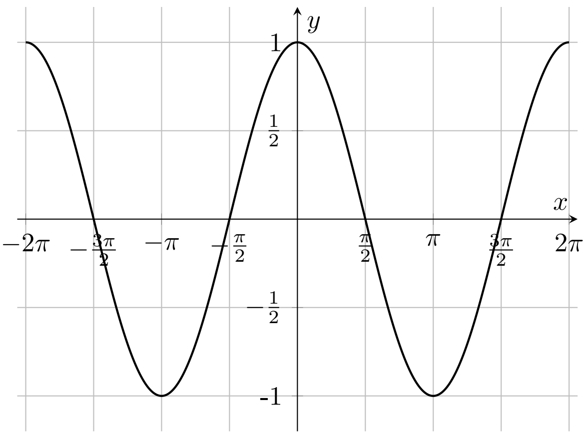
$y = \sin(x)$



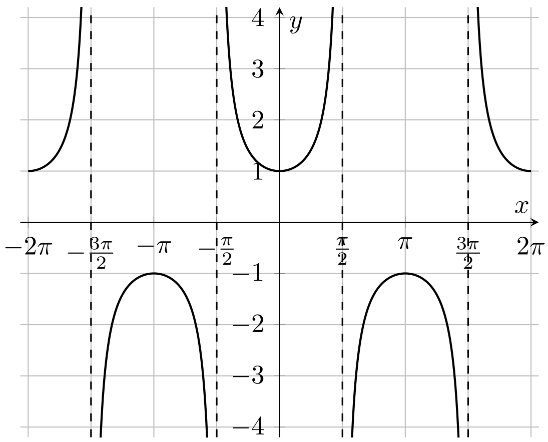
$y = \mathrm{cosec}(x)$



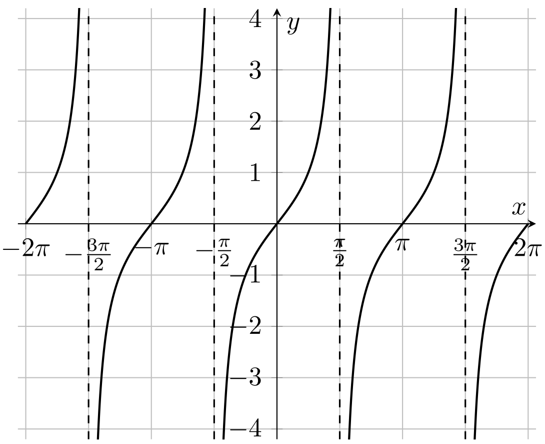
$y = \cos(x)$



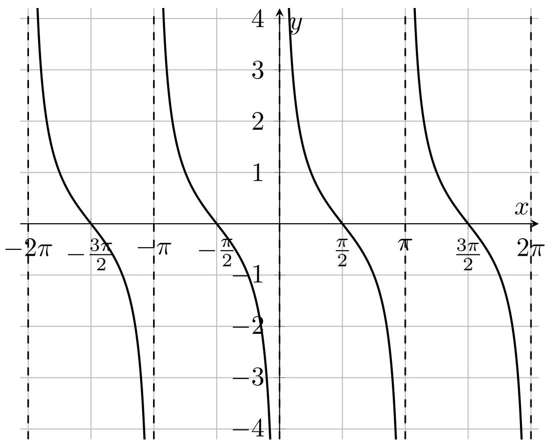
$y = \sec(x)$



$y = \tan(x)$



$y = \cot(x)$



Probability Distributions

Binomial

The binomial distribution is used to model a situation with a fixed number of independent trials each with a constant probability of success.

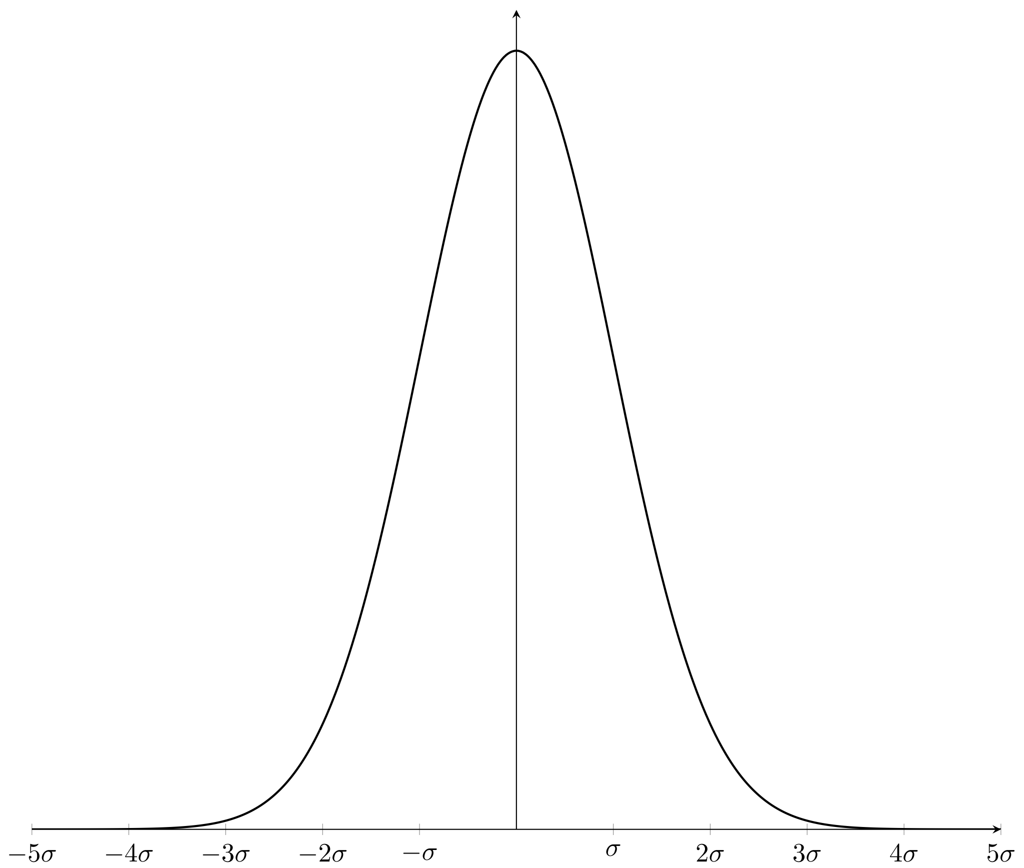
You can model X as a binomial distribution if:

- There a fixed number of trials, n
- Each trial must succeed or fail
- There is a fixed probability of success, p
- Each trial is independent

$$\text{If } X \sim B(n, p), \text{ then } \boxed{P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}} \; (0 \le x \le n)$$

Normal

The normal distribution $X \sim N(\mu, \sigma^2)$ is symmetrical, meaning the mean and median are equal.



Probability Distributions

Poisson

The Poisson distribution is used to model a situation where an event occurs at a fixed rate.

You can model X as a Poisson distribution if:

- The events must occur independently
- They must occur singly in space or time
- The events must occur at a constant average rate

$$\text{If } X \sim Po(\lambda), \text{ then } \boxed{P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}} \; (x \geq 0)$$

Geometric

The Geometric distribution is used to model a situation where you try an event several times until a success occurs, and you want to know how many tries it will take.

You can model X as a Geometric distribution if:

- Each attempt is independent
- Each attempt has the same probability

$$\text{If } X \sim Geo(p), \text{ then } \boxed{P(X = x) = p(1-p)^{x-1}} \; (x > 0)$$

- $\boxed{P(X \leq x) = 1 - (1-p)^x}$
- $P(X > x) = (1-p)^x$
- $\boxed{P(X \geq x) = (1-p)^{x-1}}$
- $P(X < x) = 1 - (1-p)^{x-1}$

Problem Solving Matters

General Tips

- Be lazy; only do necessary work
- Write in sentences to explain (especially in proofs)
- Avoid long and/or complicated calculations
- Draw diagrams and make them big
- In diagrams, label things and add lines
- Look for similar shapes (often triangles)

Tips For Sketching Graphs

- Look for symmetries
- Think about periodicity
- Look for turning points (0 derivative)
- Look for asymptotes
- Try values of x like 0, 1, -1, etc.
- If there's a trig function involved, try multiples of π
- See what happens when x tends to 0 or $\pm\infty$

Things To Remember

- $\log_a b \times \log_b a = 1$
- $\log_{a^c} b^c = \log_a b$
- When graphing $y^2 = f(x)$, draw the positive branch of $y = \sqrt{f(x)}$ *and* reflect it in the x axis
- $\log x$ is negative when $0 < x < 1$