Quarter Circles in a Semi Circle

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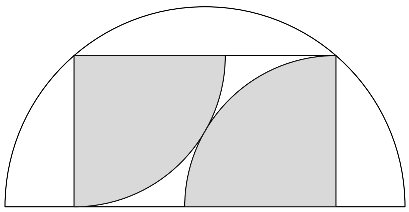
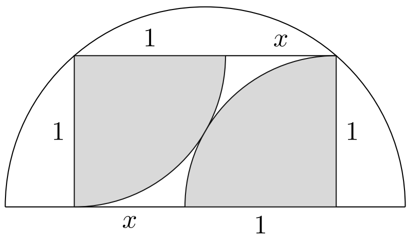
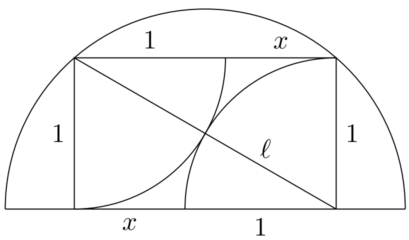
This question wants the fraction of the area of two quarter circles divided by the area of the semi circle that bounds them.



We can begin by defining the radius of the quarter circles to be 1. We can also call the remaining length along the bottom of the rectangle x.



Now, to find x, we can define the diagonal length ℓ in two ways. I'll remove the shading to make things easier to see.



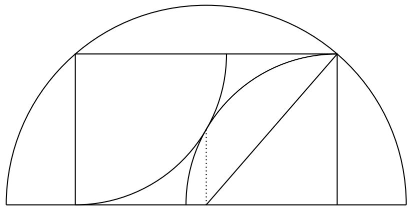
Because the two quarter circles are centred at opposite corners of the rectangle, the diagonal line ℓ must pass through their kissing point. This means that ℓ is the sum of the quarter circles' radii, so $\ell = 2$.

Alternatively, we can use Pythagoras to get $\ell = \sqrt{1^2 + (x+1)^2} = \sqrt{x^2 + 2x + 2}$. Therefore, we can say that $2 = \sqrt{x^2 + 2x + 2}$. We can then solve for x.

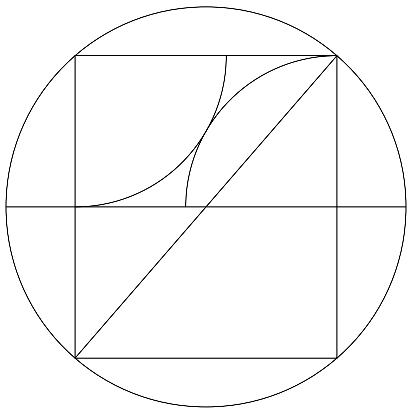
$$4 = x^2 + 2x + 2$$
$$x^2 + 2x - 2 = 0$$
$$x = -1 \pm \sqrt{3}$$

x is a distance, so it must be positive, which means that $x = -1 + \sqrt{3}$.

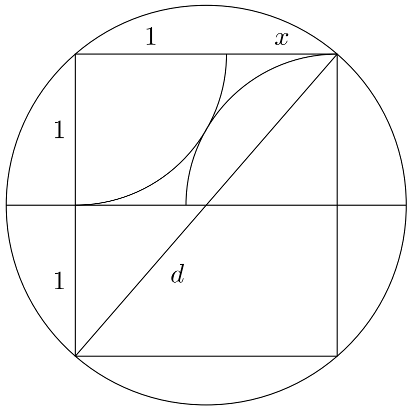
Before computing any areas to get our answer, we need to be able to find the area of the semi circle. To do this, we need the radius. To find the radius, we need the midpoint, which is the projection of the kissing point onto the horizontal line.



It's quite hard to find the midpoint. However, it's quite easy to simply reflect the circle and bounding box in the horizontal axis and continue this line to be a diameter.



We can then add our lengths back on to the diagram and use Pythagoras to find this diameter.



We can now see that the diameter d is simply $\sqrt{2^2 + (x+1)^2}$. Since $x = -1 + \sqrt{3}$, we know that $x + 1 = \sqrt{3}$, so $d = \sqrt{4+3} = \sqrt{7}$. That means that the radius of the big circle is $\frac{\sqrt{7}}{2}$.

Now we can compute the fraction of area that we want. The area of the two quarter circles is $2 \times \frac{\pi}{4} \times 1^2 = \frac{\pi}{2}$. The area of the original semi circle is $\frac{\pi}{2} \times \left(\frac{\sqrt{7}}{2}\right)^2 = \frac{7\pi}{8}$.

We divide these areas and we get $\dfrac{\pi}{2} \div \dfrac{7\pi}{8} = \dfrac{4}{7} \approx 57.1\%$, which is our final answer.