Differential Equations

Contents

First Order Homogeneous

Separation of variables

The simplest technique for solving first order homogeneous differential equations is to separate the variables. This involves treating the dx and dy as their own variables, rearranging everything to get all y terms on one side and all x terms on the other, and then adding integral signs. For example,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x} \implies \frac{1}{y}\,\mathrm{d}y = -\frac{1}{x}\,\mathrm{d}x \implies \int \frac{1}{y}\,\mathrm{d}y = \int -\frac{1}{x}\,\mathrm{d}x$$

$$\implies \ln|y| = -\ln|x| + c \implies \ln|y| + \ln|x| = c \implies \ln|xy| = c$$

$$\implies xy = \pm A \implies y = \pm\frac{A}{x}$$

This technique is easy, but very very rarely possible.

Reverse product rule

Sometimes the LHS (the y and y' terms in this case) can be seen as the result of the product rule applied to some function of x and y. For example,

$$x^3y' + 3x^2y = \sin x \implies \frac{\mathrm{d}}{\mathrm{d}x}(x^3y) = \sin x \implies x^3y = \int \sin x\,\mathrm{d}x$$

$$\implies x^3y = -\cos x + c \implies y = \frac{-\cos x + c}{x^3}$$

Again, this technique is easy, but very rarely possible.

Integrating factor

The reverse product rule is not always applicable, but you can always multiply a first order linear homogeneous differential equation by some function of x, called an **integrating factor (IF)**, to make the reverse product rule applicable.

This **IF** could be anything, and you may be able to find a simple one, but there is a general formula:

Remember

$$y' + P(x)y = Q(x) \implies \textbf{IF} = e^{\int P(x)\,\mathrm{d}x}$$

For example,

$$y' + \frac{3y}{x} = \frac{\sin x}{x^3} \implies \textbf{IF} = x^3 \text{ (by inspection) or } \textbf{IF} = e^{\int \frac{3}{x}\,\mathrm{d}x} = e^{3\ln|x|} = x^3$$

$$x^3y' + 3x^2y = \sin x \implies y = \frac{-\cos x + c}{x^3} \text{ as shown above}$$

Second Order Homogeneous

To solve a general second order homogeneous DE of the form $ay'' + by' + cy = 0$, you can use the **auxiliary equation** $am^2 + bm + c = 0$. Find the solutions to the auxiliary equation and they will tell you the general solutions to the differential equation.

Remember

$$\begin{aligned}
\text{2 real roots } \alpha \text{ and } \beta &\implies y = Ae^{\alpha x} + Be^{\beta x} \\
\text{1 real repeated root } \alpha &\implies y = (A + Bx)e^{\alpha x} \\
\text{2 complex roots } p \pm qi &\implies y = e^{px}(A\cos qx + B\sin qx)
\end{aligned}$$

Second Order Non-homogeneous

To solve a DE of the form $ay'' + by' + cy = f(x)$, first solve the corresponding homogeneous equation $ay'' + by' + c = 0$. The general solution to this equation is know as the **complementary function (CF)**.

You then need to find the **particular integral (PI)**, which is a function that satisfies the original DE. The form of the PI depends on the form of $f(x)$. The PI will typically be of the same form as $f(x)$ but with different coefficients. However, if $f(x)$ is a single trig function like $\sin kx$, then the PI should be of the form $\lambda \cos kx + \mu \sin kx$.

If the **PI** is contained as a term in the **CF**, then you should multiply the **PI** by x.

To find the coefficients of the PI, differentiate it twice and sub the derivatives into the original DE and equate coefficients to solve simultaneously.

Remember

The general solution is $y = \mathbf{CF} + \mathbf{PI}$