Series Polynomials

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The challenge question in Exercise 3B of the Edexcel AS and A Level Further Maths Core book is about series sums of polynomials.

Part a of this question asks for polynomials $f_2(x)$, $f_3(x)$, $f_4(x)$ such that for every $n \in \mathbb{N}$, $n > 1$,

$$\sum_{r=1}^{n} f_2(r) = n^2$$
$$\sum_{r=1}^{n} f_3(r) = n^3$$
$$\sum_{r=1}^{n} f_4(r) = n^4$$

Finding these polynomials reveals a very interesting pattern.

1 Finding Polynomials

Throughout this paper, all $n \in \mathbb{N}$, $n > 1$.

Let's first establish some basic lemmas about series.

Lemma 1 $\sum_{r=1}^{n}(f(r) + g(r)) = \sum_{r=1}^{n} f(r) + \sum_{r=1}^{n} g(r)$

Lemma 2 $\sum_{r=1}^{n} kf(r) = k \sum_{r=1}^{n} f(r)$

Lemma 3 $\sum_{r=1}^{n} 1 = n$

Lemma 4 $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

Lemma 5 $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

Lemma 6 $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

1.1 Finding $f_2(r)$

We want a polynomial $f_2(r)$ such that $\sum_{r=1}^{n} f_2(r) = n^2$.

We already have, by Lemma 4,

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1) = \frac{1}{2}\left(n^2+n\right)$$

We need to get rid of the $\frac{1}{2}$, which we can do by multiplying by 2 to get

$$2\sum_{r=1}^{n} r = \sum_{r=1}^{n} 2r = n^2+n$$

Then, we just need to get rid of the n.

We know, by Lemma 3, that $\sum_{r=1}^{n} 1 = n$, so

$$\sum_{r=1}^{n} 2r - \sum_{r=1}^{n} 1 = \sum_{r=1}^{n}(2r-1) = n^2$$
$$\therefore f_2(r) = 2r-1$$

1.2 Finding $f_3(r)$

Next, we want a polynomial $f_3(r)$ such that $\sum_{r=1}^{n} f_3(r) = n^3$.

Similarly to with $f_2(r)$, by Lemma 5, we already have

$$\begin{aligned}\sum_{r=1}^{n} r^2 &= \frac{1}{6}n(n+1)(2n+1)\\ &= \frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n\\ &= \frac{1}{3}\left(n^3+\frac{3}{2}n^2+\frac{1}{2}n\right)\end{aligned}$$

We can then multiply by 3 to remove the $\frac{1}{3}$ and get

$$3\sum_{r=1}^{n} r^2 = \sum_{r=1}^{n} 3r^2 = n^3+\frac{3}{2}n^2+\frac{1}{2}n$$

To get rid of the $\frac{1}{2}n$, we can just do

$$\sum_{r=1}^{n} 3r^2 - \sum_{r=1}^{n}\frac{1}{2} = \sum_{r=1}^{n}\left(3r^2-\frac{1}{2}\right) = n^3+\frac{3}{2}n^2$$

We know that $\sum_{r=1}^{n} f_2(r) = n^2$, so we simply need to subtract $\frac{3}{2}f_2(r)$ from our polynomial to get rid of the resultant $\frac{3}{2}n^2$.

$$3r^2-\frac{1}{2}-\frac{3}{2}(2r-1) = 3r^2-3r+1$$
$$\therefore f_3(r) = 3r^2-3r+1$$

1.3 Finding $f_4(r)$

Next, we want a polynomial $f_4(r)$ such that $\sum\limits_{r=1}^{n} f_4(r) = n^4$.

We know, by Lemma 6, that

$$\begin{aligned}\sum_{r=1}^{n} r^3 &= \frac{1}{4}n^2(n+1)^2 \\ &= \frac{1}{4}\left(n^4 + 2n^3 + n^2\right)\end{aligned}$$

We can multiply by 4 to get

$$4\sum_{r=1}^{n} r^3 = \sum_{r=1}^{n} 4r^3 = n^4 + 2n^3 + n^2$$

We can then get n^4 on its own by subtracting $2f_3(r)$ and $f_2(r)$ from $4r^3$.

$$\begin{gathered}4r^3 - 2\left(3r^2 - 3r + 1\right) - (2r - 1) \\ = 4r^3 - 6r^2 + 6r - 2 - 2r + 1 \\ = 4r^3 - 6r^2 + 4r - 1 \\ \therefore f_4(r) = 4r^3 - 6r^2 + 4r - 1\end{gathered}$$

2 Finding Patterns

We can find $f_1(r)$, where $\sum\limits_{r=1}^{n} f_1(r) = n^1 = n$ to trivially be 1.

These are our polynomials:

$$\begin{aligned}&f_1(r) = 1 \\ &f_2(r) = 2r - 1 \\ &f_3(r) = 3r^2 - 3r + 1 \\ &f_4(r) = 4r^3 - 6r^2 + 4r - 1\end{aligned}$$

After looking at these for a while, we can notice a few things. Firstly, the constants are always ± 1, and the signs of these constants alternate with increasing degrees of polynomial. In fact, all the signs alternate.

Secondly, we can notice that the first term of $f_a(r)$ is always of the form ar^{a-1}.

However, the most interesting thing to notice with these polynomials is that the coefficients look like binomial expansions of $(r-1)^a$, albeit with the leading term removed and the signs flipped.

We can continue this pattern to make a conjecture about $f_5(r)$.

Conjecture 1 $f_5(r) = 5r^4 - 10r^3 + 10r^2 - 5r + 1$

To find a general form, we have to think about how we go from $(r-1)^a$ to these polynomials.

Lets look at the example of $f_4(r)$.

$$(r-1)^4 = r^4 - 4r^3 + 6r^2 - 4r + 1$$

We have the change the signs of all of these, so we get

$$-(r-1)^4 = -r^4 + 4r^3 - 6r^2 + 4r - 1$$

Then we have to remove the leading $-r^4$ and we get $-(r-1)^4 + r^4$, or more simply, $r^4 - (r-1)^4$.

This form of $r^a - (r-1)^a$ gives the results seen previously for $f_a(r)$, so we can conjecture that this pattern continues for all $a \in \mathbb{Z}^+$.

$\mathbb{Z}^+$ is simply the set of positive integers, not including 0. This is simpler than writing $a \in \mathbb{N}, a > 0$.

Rather than conjecturing about $f_a(r)$, we want to conjecture about its sum to n.

Conjecture 2 $\forall a, n \in \mathbb{Z}^+, n > 1, \displaystyle\sum_{r=1}^{n} (r^a - (r-1)^a) = n^a$

2.1 Proving Conjecture 1

In order to prove Conjecture 1, we need to know the formula for $\sum_{r=1}^{n} r^4$. Wolfram Alpha says:

Lemma 7 $\sum_{r=1}^{n} r^4 = \frac{1}{30} n(n+1)(2n+1)\left(3n^2 + 3n - 1\right)$

This can then be expanded to give

$$\frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$
$$= \frac{1}{5}\left(n^5 + \frac{5}{2}n^4 + \frac{5}{3}n^3 - \frac{1}{6}n\right)$$

We can multiply by 5 to get

$$5\sum_{r=1}^{n} r^4 = \sum_{r=1}^{n} 5r^4 = n^5 + \frac{5}{2}n^4 + \frac{5}{3}n^3 - \frac{1}{6}n$$

Now, we just need to get rid of the other terms to get a polynomial $f_5(r)$ such that $\sum_{r=1}^{n} f_5(r) = n^5$.

$$f_5(r) = 5r^4 - \frac{5}{2}f_4(r) - \frac{5}{3}f_3(r) + \frac{1}{6}f_1(r)$$
$$= 5r^4 - \frac{5}{2}\left(4r^3 - 6r^2 + 4r - 1\right) - \frac{5}{3}\left(3r^2 - 3r + 1\right) + \frac{1}{6}$$
$$= 5r^4 - 10r^3 + 10r^2 - 5r + 1$$

This proves Conjecture 1.

However, to prove Conjecture 2, I think I need to find more patterns.

3 Finding Patterns (again)

Let's look at sums of powers of r. By Lemmas 3 to 7, we have

$$\begin{aligned}
\sum_{r=1}^{n} r^0 &= n && = n \\
\sum_{r=1}^{n} r^1 &= \frac{1}{2}n(n+1) && = \frac{1}{2}n^2 + \frac{1}{2}n \\
\sum_{r=1}^{n} r^2 &= \frac{1}{6}n(n+1)(2n+1) && = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \\
\sum_{r=1}^{n} r^3 &= \frac{1}{4}n^2(n+1)^2 && = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 \\
\sum_{r=1}^{n} r^4 &= \frac{1}{30}n(n+1)(2n+1)\left(3n^2+3n-1\right) && = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n
\end{aligned}$$

I can't see much of a pattern here, but the first term of $\sum\limits_{r=1}^{n} r^b$ always seems to be $\frac{1}{b+1}n^{b+1}$ and the second term always seems to be $\frac{1}{2}n^b$, except in the case of $b = 0$.

I don't think I would able to find a formula for $\sum\limits_{r=1}^{n} r^b$ on my own, but in doing some research on this topic, I found this wiki page on brilliant.org about sums which mentions Faulhaber's formula. I shall rewrite it here.

$$\sum_{r=1}^{n} r^b = \frac{1}{b+1}\sum_{j=0}^{b}(-1)^j \binom{b+1}{j} B_j n^{b+1-j}$$

where B_j is the jth Bernoulli number.

In doing further research, I found that this is were the Bernoulli numbers where found by Jakob Bernoulli in *Ars Conjectandi*, published in 1713. He was trying to find a general formula for $\sum\limits_{r=1}^{n} r^b$ and found these numbers. He could not relate them to anything, and de Moivre named them after him.[1]

Not only that, but Jakob Bernoulli mentioned Faulhaber by name in *Ars Conjectandi*. People have wondered about this problem long before me, and this is where the famous Bernoulli numbers originally came from.

Bernoulli would have written $\sum\limits_{r=1}^{n} r^b$ as

$$\sum_{r=1}^{n} r^b = \sum_{r=0}^{b} \frac{B_r}{r!} b^{\underline{r-1}} n^{b-r+1}$$

where $p^{\underline{q}}$ is the falling factorial $p \times (p-1) \times (p-2) \times \cdots \times (p-q+1)$.

4 Proving Conjecture 2

Proving Conjecture 2 will undoubtedly be hard, and will definitely require knowledge of the Bernoulli numbers. Unfortunately, there seems to be no easy formula for B_k. The Bernoulli numbers are usually either defined with a recurrence relation, or with a generating function, or in terms of things like the Riemann Zeta function. So, nothing friendly. Unfortunately, it seems like proving this conjecture is a bit out of my reach currently. But I posited it, and I want to prove it, even if it takes ages.

References

[1] *Bernoulli number*. URL: https://en.wikipedia.org/wiki/Bernoulli_number.