A Circle and a Hexagon

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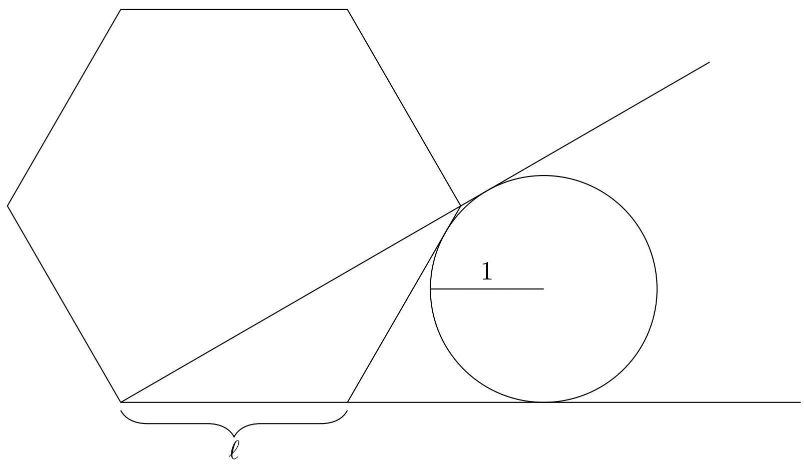
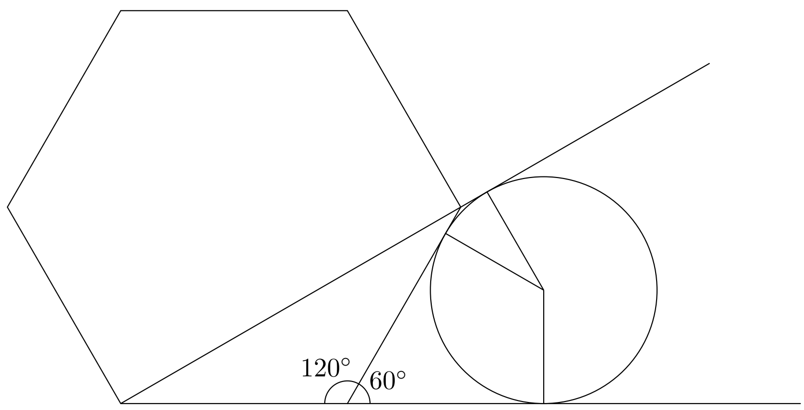
5th November, 2021

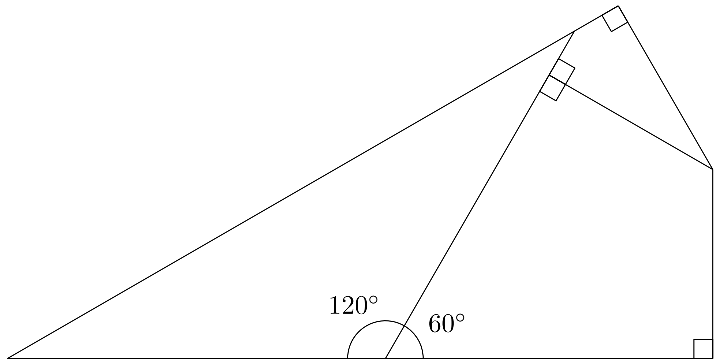
This question places a unit circle tangent to a hexagon and a line, and asks for the side length ℓ of the hexagon.



We can begin by drawing some of the radii of this circle. We can observe that the interior angle of the hexagon is $120°$ and thus find the other angle on the straight line to be $60°$.



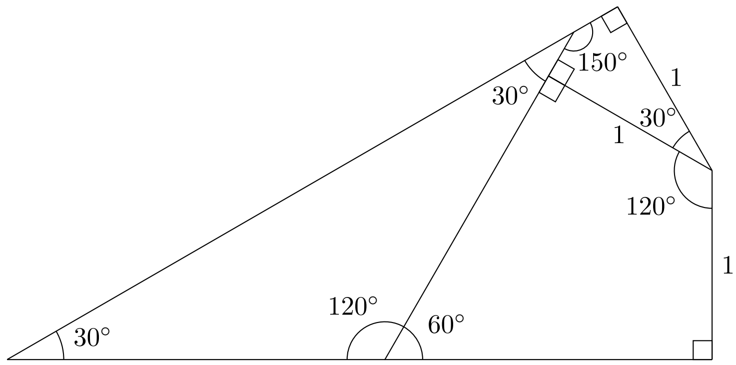
We can get rid of the hexagon and the circle, because we now only care about the kites and triangles.



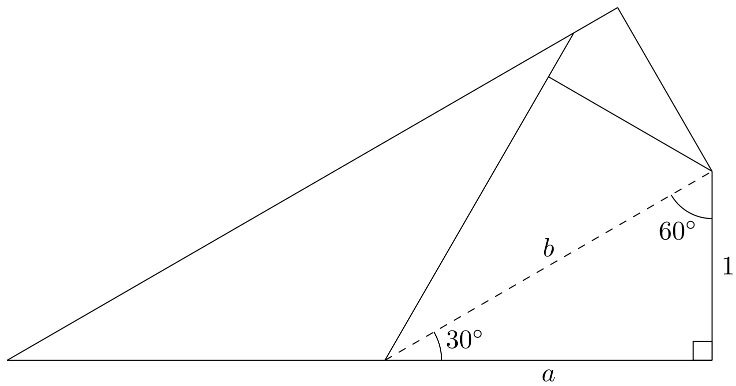
We know that the two line segments either side of the $120°$ angle are ℓ, so this triangle is isosceles. That means the other two angles are $30°$.

We can also find the angle opposite $60°$ to be $120°$ because we know all the other angles in the kite.

At the top, we know the angle in the isosceles triangle is $30°$, so the other angle on this straight line must be $150°$. We can also then find its opposite angle to be $30°$ by angles in a kite again.



We can then split the larger kite into two identical triangles and examine one.



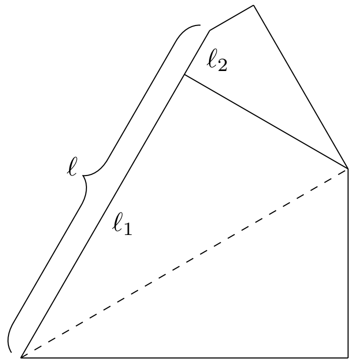
We can use the sine rule to find a and b like so.

$$\frac{1}{\sin 30°} = \frac{a}{\sin 60°} = \frac{b}{\sin 90°}$$

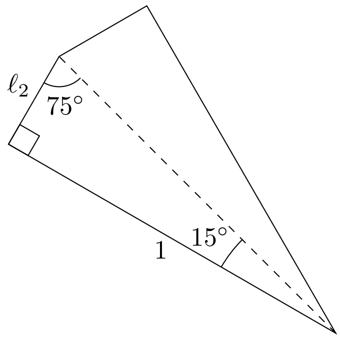
$$\implies a = \frac{\sin 60°}{\sin 30°} = \sqrt{3}$$

$$\implies b = \frac{\sin 90°}{\sin 30°} = 2$$

Now, we can examine our diagram once more and see that we have the length ℓ split between two line segments. We can call these two smaller lengths ℓ_1 and ℓ_2.



Since the two triangles of the larger kite are identical, we know that $\ell_1 = \sqrt{3}$. We can find ℓ_2 by the same process with the other kite.



Again, using the sine rule, we find that

$$\frac{1}{\sin 75°} = \frac{\ell_2}{\sin 15°}$$
$$\implies \ell_2 = \frac{\sin 15°}{\sin 75°} = 2 - \sqrt{3}$$

Therefore,

$$\ell = \ell_1 + \ell_2 = \sqrt{3} + 2 - \sqrt{3} = 2$$

□