

# Complex Numbers

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# Basics

The imaginary number  $i$  is defined to satisfy

$$i^2 \equiv -1$$

.

A complex number  $a + bi$  is the sum of a real number and an imaginary number (which is a real multiple of  $i$ ).

Complex numbers are added element-by-element, and multiplied by expanding brackets à la foil.

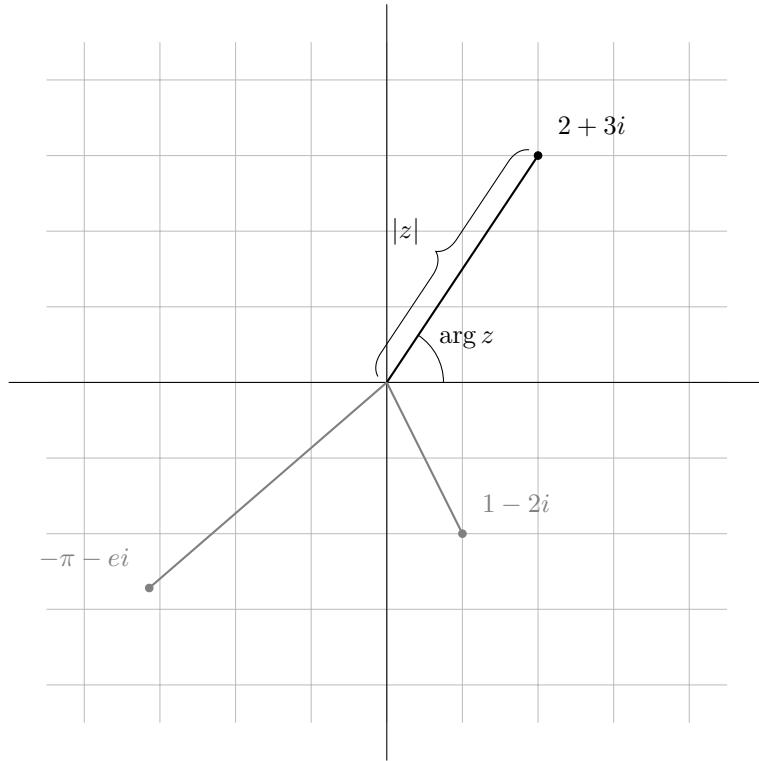
The *complex conjugate* for a complex number  $z = a + bi$  is  $z^* = a - bi$ .

For a polynomial  $f(x)$  with real coefficients, complex roots must occur in conjugate pairs.

# Argand Diagrams

## The Diagram

An Argand diagram is a way of representing complex numbers on a 2D plane. The horizontal axis is the real numbers, and the vertical axis is the imaginary numbers.



The modulus  $|z|$  of a complex number  $z$  is the distance from the point to the origin. The argument  $\arg z$  is the angle between the vector line and the positive real axis.

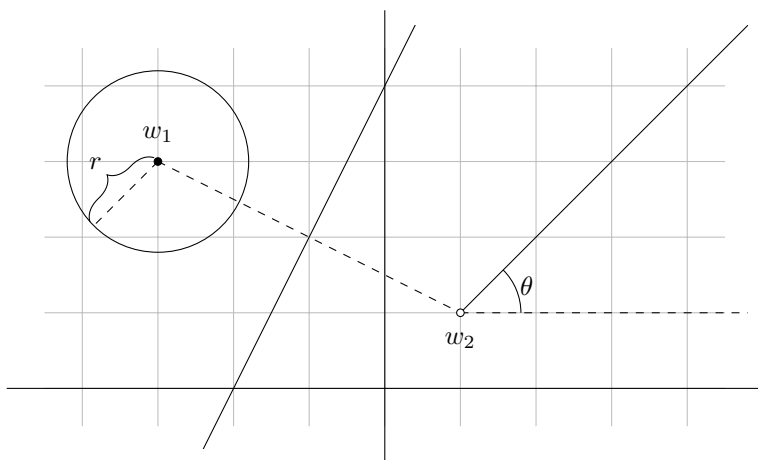
For a modulus  $|z| = r$  and an argument  $\arg z = \theta$ , the modulus-argument form of  $z$  is  $r(\cos \theta + i \sin \theta)$ .

Multiplication is unaffected by the modulus, so  $|zw| = |z||w|$  and  $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$ .

Multiplication is additive over  $\arg$ , so  $\arg(zw) = \arg z + \arg w$  and  $\arg\left(\frac{z}{w}\right) = \arg z - \arg w$ .

$|z - w|$  is the distance between  $z$  and  $w$ .

## Loci



For a complex number  $w_1 = x + yi$ , the locus of points  $|z - w_1| = r \Leftrightarrow |z - (x + yi)| = r$  is a circle of radius  $r$  around the point  $w_1$ .

$|z - w_1| = |z - w_2|$  is the perpendicular bisector of the line joining  $w_1$  and  $w_2$ .

$\arg(z - w_2) = \theta$  is a half-line from, but not including, the point  $w_2$  making an angle  $\theta$  from the real axis.

# Sums of Series

Questions in this topic give you a geometric series with complex terms. You have to use the geometric series formulae and then simplify the result.

Here's an example:

$$S = e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \dots + e^{8i\theta} \quad (\theta \neq 2n\pi)$$

We want to condense the series into a single expression, so we start by applying the geometric series formula.

$$S_n = \frac{a(r^n - 1)}{r - 1} \implies S = \frac{e^{i\theta}((e^{i\theta})^8 - 1)}{e^{i\theta} - 1} = \frac{e^{i\theta}(e^{8i\theta} - 1)}{e^{i\theta} - 1}$$

We're going to use Technique 1 to simplify this, so we multiply the top and bottom by  $e^{-\frac{i\theta}{2}}$ .

$$\frac{e^{\frac{i\theta}{2}}(e^{8i\theta} - 1)}{e^{\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}}} = \frac{e^{\frac{i\theta}{2}}(e^{8i\theta} - 1)}{2i \sin \frac{\theta}{2}}$$

Now we can factorise the numerator using Technique 3 and simplify it.

$$\frac{e^{\frac{i\theta}{2}} e^{4i\theta} (e^{4i\theta} - e^{-4i\theta})}{2i \sin \frac{\theta}{2}} = \frac{e^{\frac{9i\theta}{2}} (2i \sin 4\theta)}{2i \sin \frac{\theta}{2}} = \frac{e^{\frac{9i\theta}{2}} \sin 4\theta}{\sin \frac{\theta}{2}}$$

Now you could split the numerator into real and imaginary parts if the questions asked for it.

## Techniques

The aim of the process is normally to get something of the form  $z^n \pm z^{-n}$ .

When  $z = e^{i\theta}$ , we know that  $z^n = e^{ni\theta}$  and

$$z^n + z^{-n} = 2 \cos n\theta$$

$$z^n - z^{-n} = 2i \sin n\theta$$

### Technique 1: Denominators

$$\frac{\dots}{e^{ni\theta} \pm 1} = \frac{\dots}{e^{ni\theta} \pm 1} \times \frac{e^{-\frac{ni\theta}{2}}}{e^{-\frac{ni\theta}{2}}} = \frac{e^{-\frac{ni\theta}{2}}(\dots)}{e^{\frac{ni\theta}{2}} \pm e^{-\frac{ni\theta}{2}}}$$

When the denominator is of the form  $e^{ni\theta} \pm 1$ , we can multiply the top and bottom of the fraction by the negative half power of  $e$ . This gives us  $z^n \pm z^{-n}$  in the denominator, which we can get rid of with ease.

### Technique 2: Denominator General Case

$$\frac{\dots}{e^{ni\theta} + k} = \frac{\dots}{e^{ni\theta} + k} \times \frac{e^{-ni\theta} + k}{e^{-ni\theta} + k} = \frac{(e^{-ni\theta} + k)(\dots)}{(e^{ni\theta} + k)(e^{-ni\theta} + k)}$$

Always factor the denominator to be of the form  $e^{ni\theta} + k$ . If  $k = \pm 1$ , then use Technique 1. Otherwise, multiply the top and bottom by the bottom, but with the sign of the power flipped. The sign between terms stays the same, but the power of  $e$  gets negated (and not halved).

The denominator will then expand to create something that can be simplified with little difficulty.

### **Technique 3: Numerators**

$$\boxed{\frac{e^{ni\theta} \pm 1}{\dots} = \frac{e^{\frac{ni\theta}{2}} \left( e^{\frac{ni\theta}{2}} \pm e^{\frac{-ni\theta}{2}} \right)}{\dots}}$$

Once the denominator is in a manageable form, we can then apply this technique to the numerator.

When you've got something of the form  $e^{ni\theta} \pm 1$ , you can factor out  $e$  to half of the power to get  $e^{\frac{ni\theta}{2}} \left( e^{\frac{ni\theta}{2}} \pm e^{\frac{-ni\theta}{2}} \right)$ . The inside of the bracket is now of the form  $z^n \pm z^{-n}$ , so we can simplify it.