Complex Numbers

Contents

Basics

The imaginary number i is defined to satisfy

$$\boxed{i^2 \equiv -1}$$

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A complex number $a + bi$ is the sum of a real number and an imaginary number (which is a real multiple of i).

Complex numbers are added element-by-element, and multiplied by expanding brackets à la foil.

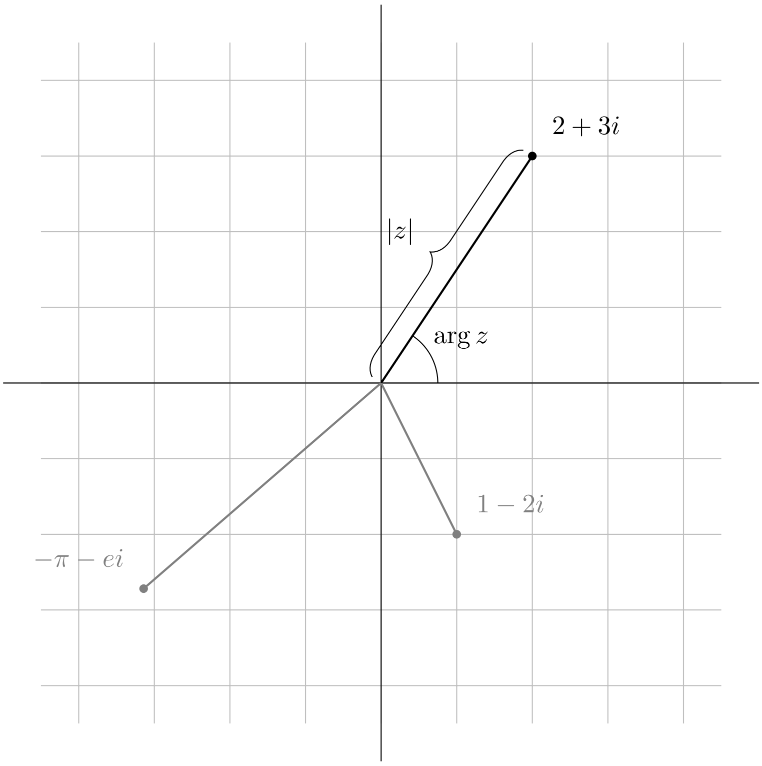
The *complex conjugate* for a complex number $z = a + bi$ is $z^* = a - bi$.

For a polynomial $f(x)$ with real coefficients, complex roots must occur in conjugate pairs.

Argand Diagrams

The Diagram

An Argand diagram is a way of representing complex numbers on a 2D plane. The horizontal axis is the real numbers, and the vertical axis is the imaginary numbers.



The modulus $|z|$ of a complex number z is the distance from the point to the origin. The argument $\arg z$ is the angle between the vector line and the positive real axis.

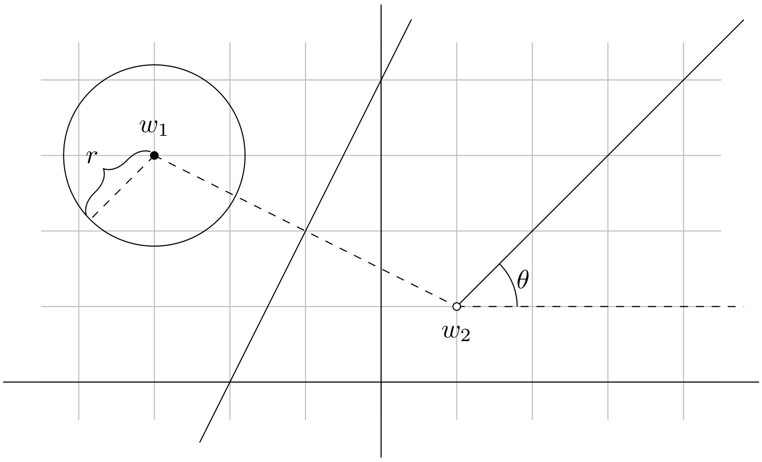
For a modulus $|z| = r$ and an argument $\arg z = \theta$, the modulus-argument form of z is $\boxed{r(\cos\theta + i\sin\theta)}$.

Multiplication is unaffected by the modulus, so $|zw| = |z||w|$ and $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$.

Multiplication is additive over arg, so $\arg(zw) = \arg z + \arg w$ and $\arg\left(\dfrac{z}{w}\right) = \arg z - \arg w$.

$|z-w|$ is the distance between z and w.

Loci



For a complex number $w_1 = x + yi$, the locus of points $|z - w_1| = r \Leftrightarrow |z - (x+yi)| = r$ is a circle of radius r around the point w_1.

$|z - w_1| = |z - w_2|$ is the perpendicular bisector of the line joining w_1 and w_2.

$\arg(z - w_2) = \theta$ is a half-line from, but not including, the point w_2 making an angle θ from the real axis.

De Moivre's Theorem

De Moivre's theorem can be used to convert between sines and cosines with multiples of θ, and power series of sines and cosines. The theorem itself is:

$$\boxed{\big(r(\cos\theta + i\sin\theta)\big)^n = r^n(\cos n\theta + i\sin n\theta)}$$

A special case, where $r = 1$,

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$$

If we let $z = e^{i\theta} = \cos\theta + i\sin\theta$, then we can derive the identities

$$\boxed{z^n + z^{-n} = 2\cos n\theta} \qquad \boxed{z^n - z^{-n} = 2i\sin n\theta}$$

Multiple to Power Series

When we want to show something like $\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$, then we know that $(\cos\theta + i\sin\theta)^6 = \cos 6\theta + i\sin 6\theta$, so we can expand the bracket on the left and then take the real part to get $\cos 6\theta$.

For this example, that goes as follows:

$$\begin{aligned}
(\cos\theta + i\sin\theta)^6 &= \cos^6\theta + 6i\cos^5\theta\sin\theta - 15\cos^4\theta\sin^2\theta - 20i\cos^3\theta\sin^3\theta \\
&\quad +15\cos^2\theta\sin\theta + 6i\cos\theta\sin^5\theta - \sin^6\theta \\
\implies \cos 6\theta &= \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta \\
&= \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3 \\
&= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1
\end{aligned}$$

Of course, you could this with other powers, or you could take the imaginary part to get sin.

Power to Multiple Series

When we want to show something like $\cos^5\theta = \frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta$, then we know that $(z + z^{-1})^5 = (2\cos\theta)^5 = 32\cos^5\theta$, so we can expand the bracket and collect pairs, then divide everything by 32.

For this example, that goes as follows:

$$\begin{aligned}
(z + z^{-1})^5 &= z^5 + 5z^3 + 10z + 10z^{-1} + 5z^{-3} + z^{-5} \\
&= (z^5 + z^{-5}) + 5(z^3 + z^{-3}) + 10(z + z^{-1}) \\
\implies 32\cos^5\theta &= 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta \\
\implies \cos^5\theta &= \frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta
\end{aligned}$$

Of course, you can do this with other multiples and powers, or use $(z - z^{-1})^n$ for sin.

This form of de Moivre's theorem is especially useful for computing integrals which involve higher powers of trig functions.

Sums of Series

Questions in this topic give you a geometric series with complex terms. You have to use the geometric series formulae and then simplify the result.

Here's an example:

$$S = e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \cdots + e^{8i\theta} \ (\theta \neq 2n\pi)$$

We want to condense the series into a single expression, so we start by applying the geometric series formula.

$$S_n = \frac{a(r^n - 1)}{r - 1} \implies S = \frac{e^{i\theta}\left((e^{i\theta})^8 - 1\right)}{e^{i\theta} - 1} = \frac{e^{i\theta}\left(e^{8i\theta} - 1\right)}{e^{i\theta} - 1}$$

We're going to use Technique 1 to simplify this, so we multiply the top and bottom by $e^{-\frac{i\theta}{2}}$.

$$\frac{e^{\frac{i\theta}{2}}\left(e^{8i\theta} - 1\right)}{e^{\frac{i\theta}{2}} - e^{\frac{i\theta}{2}}} = \frac{e^{\frac{i\theta}{2}}\left(e^{8i\theta} - 1\right)}{2i\sin\frac{\theta}{2}}$$

Now we can factorise the numerator using Technique 3 and simplify it.

$$\frac{e^{\frac{i\theta}{2}}e^{4i\theta}\left(e^{4i\theta} - e^{-4i\theta}\right)}{2i\sin\frac{\theta}{2}} = \frac{e^{\frac{9i\theta}{2}}(2i\sin 4\theta)}{2i\sin\frac{\theta}{2}} = \frac{e^{\frac{9i\theta}{2}}\sin 4\theta}{\sin\frac{\theta}{2}}$$

Now you could split the numerator into real and imaginary parts if the questions asked for it.

Techniques

The aim of the process is normally to get something of the form $z^n \pm z^{-n}$.

Technique 1: Denominators

$$\boxed{\frac{\cdots}{e^{ni\theta} \pm 1} = \frac{\cdots}{e^{ni\theta} \pm 1} \times \frac{e^{\frac{-ni\theta}{2}}}{e^{\frac{-ni\theta}{2}}} = \frac{e^{\frac{-ni\theta}{2}}(\cdots)}{e^{\frac{ni\theta}{2}} \pm e^{\frac{-ni\theta}{2}}}}$$

When the denominator is of the form $e^{ni\theta} \pm 1$, we can multiply the top and bottom of the fraction by the negative half power of e. This gives us $z^n \pm z^{-n}$ in the denominator, which we can get rid of with ease.

Technique 2: Denominator General Case

$$\boxed{\frac{\cdots}{e^{ni\theta} + k} = \frac{\cdots}{e^{ni\theta} + k} \times \frac{e^{-ni\theta} + k}{e^{-ni\theta} + k} = \frac{(e^{-ni\theta} + k)(\cdots)}{(e^{ni\theta} + k)(e^{-ni\theta} + k)}}$$

Always factor the denominator to be of the form $e^{ni\theta} + k$. If $k = \pm 1$, then use Technique 1. Otherwise, multiply the top and bottom by the bottom, but with the sign of the power flipped. The sign between terms stays the same, but the power of e gets negated (and not halved).

The denominator will then expand to create something that can be simplified with little difficulty.

Technique 3: Numerators

$$\boxed{\frac{e^{ni\theta} \pm 1}{\dots} = \frac{e^{\frac{ni\theta}{2}}\left(e^{\frac{ni\theta}{2}} \pm e^{\frac{-ni\theta}{2}}\right)}{\dots}}$$

Once the denominator is in a manageable form, we can then apply this technique to the numerator.

When you've got something of the form $e^{ni\theta} \pm 1$, you can factor out e to half of the power to get $e^{\frac{ni\theta}{2}}\left(e^{\frac{ni\theta}{2}} \pm e^{\frac{-ni\theta}{2}}\right)$. The inside of the bracket is now of the form $z^n \pm z^{-n}$, so we can simplify it.