

Graphs

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Basics

Definitions

Graph	A collection of nodes and edges
Node (or vertex)	A point on a graph
Edge (or arc)	A line segment connecting two nodes
Weighted graph (or network)	A graph with weights associated with its edges
Directed graph (or digraph)	A graph where the edges only go one way
Subgraph	A subset of the nodes and vertices from the original graph
Degree (or valency, or order)	The number of edges connected to a given vertex
Walk	A route through a graph, connecting vertices along edges
Path	A walk in which no <i>vertex</i> is visited more than once
Trail	A walk in which no <i>edge</i> is visited more than once
Eulerian circuit	A trail which starts where it ends and traverses every edge
Cycle	A walk which ends where it started, and no other vertex is visited more than once
Hamiltonian cycle	A cycle that includes every vertex
Connected (vertices)	There is a path between the vertices
Connected (graph)	All the vertices in the graph are connected
Loop	An edge that starts and finishes at the same vertex
Simple graph	A graph where there are no loops and there is at most one edge connecting any pair of vertices
Tree	A connected graph with no cycles
Spanning tree (of graph \mathbf{G})	A subgraph of \mathbf{G} which contains all vertices and is a tree
Complete graph	A graph in which every vertex is directly connected by a single edge to all other vertices
Isomorphic graphs	Graphs which represent the same information but are drawn differently

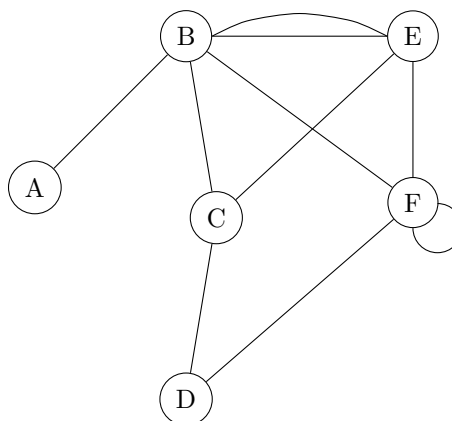
Matrices

A graph can be represented by an adjacency matrix, and a weighted graph can be represented by a distance matrix. The edge or distance *from* A *to* B is the number on row A and column B - read from the left.

Adjacency matrices

Each entry in an adjacency matrix describes the number of edges joining the corresponding nodes.

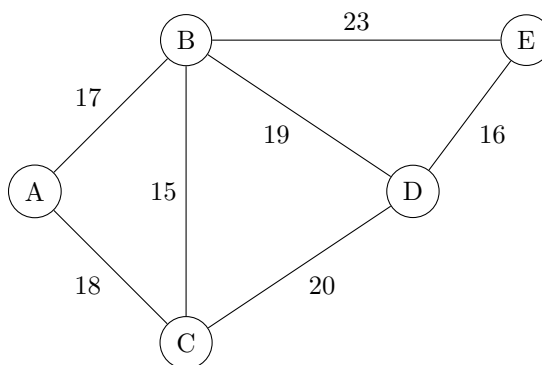
	A	B	C	D	E	F
A	0	1	0	0	0	0
B	1	0	1	0	2	1
C	0	1	0	1	1	0
D	0	0	1	0	0	1
E	0	2	1	0	0	1
F	0	1	0	1	1	2



Distance matrices

Each entry in a distance matrix describes the weight of the edge joining the corresponding nodes, if any.

	A	B	C	D	E
A	—	17	18	—	—
B	17	—	15	19	23
C	18	15	—	20	—
D	—	19	20	—	16
E	—	23	—	16	—



Minimum Spanning Trees

A minimum spanning tree (MST) is a subgraph with contains all nodes but no cycles, and has the minimum total length.

Kruskal's algorithm

An algorithm to find an MST is Kruskal's algorithm. It goes as follows:

1. Sort all the edges into order of ascending weight
2. Select the edge of least weight to start the tree
3. Consider the next shortest edge. If it would form a cycle, then reject it, else add it to the tree
4. Repeat step 3 until all nodes have been added

Prim's algorithm

Another algorithm to find an MST is Prim's algorithm. It goes as follows:

1. Choose an arbitrary vertex to start the tree
2. Select the edge of least weight which joins a vertex in the tree to a vertex not yet in the tree
3. Repeat step 2 until all nodes have been added

Prim's algorithm on a distance matrix

To apply Prim's algorithm to a distance matrix, use the following algorithm:

1. Choose an arbitrary vertex to start the tree
2. Delete the **row** of the matrix for the chosen vertex
3. Number the **column** of the matrix for the chosen vertex
4. Put a ring around the smallest undeleted entry in the numbered columns
5. The ringed entry becomes the next edge to join the tree
6. Repeat steps 2 to 5 until all rows have been deleted

Shortest Path

Dijkstra's algorithm

Dijkstra's algorithm can be used to find the shortest path between two chosen vertices in a network.

Dijkstra uses boxes like this in place of nodes to make the working clear:

Vertex name	Order of labelling	Final label
Working values		

To find the shortest path from S to T , use the following algorithm:

1. Label the start vertex, S with the final label 0
2. The vertex which just received its final label is X . For every unlabelled vertex Y which is connected to X :
 - (a) The new working value of Y is the final label of X plus the distance between X and Y
 - (b) But only replace Y 's working value if the new one is smaller
3. Look at all vertices without final labels. Choose the one with the smallest working value. This becomes the final label for this vertex
4. Repeat steps 2 and 3 until the destination vertex T receives a final label
5. To find the shortest path, trace back from T to S . Include an edge only when its weight is the difference of the final labels of the vertices on each end of it

Floyd's algorithm

Floyd's algorithm can be used to find the shortest path between every pair of vertices in a network. It produces two tables - the first shows the minimum distance between any two vertices, and the second can be used to find the route.

1. Complete an initial **distance table** for the network. If there is no direct route from one vertex to another, label the distance with ∞
2. Complete an initial **route table** by making every entry in a column the same as the label at the top of the column
3. For the first iteration, copy the values in the first row and column in the tables into a new pair of tables. Shade these values
4. Consider each unshaded position in turn:
 - (a) Compare the value in this position in the previous table to the sum of the corresponding shaded values. Choose the smaller of the two options
 - (b) If you chose the sum, then copy the letter of this vertex into the corresponding position in the route table
 - (c) Repeat until all unshaded cells have been filled in
5. For the second iteration, copy the second row and column from the previous tables into a new pair of tables and shade them. Repeat step 4
6. Continue iterating like this until you've done every row and column