Differential Equations

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First Order Homogeneous

Separation of variables

The simplest technique for solving first order homogeneous differential equations is to separate the variables. This involves treating the $\mathrm{d}x$ and $\mathrm{d}y$ as their own variables, rearranging everything to get all y terms on one side and all x terms on the other, and then adding integral signs. For example,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x} \implies \frac{1}{y}\,\mathrm{d}y = -\frac{1}{x}\,\mathrm{d}x \implies \int \frac{1}{y}\,\mathrm{d}y = \int -\frac{1}{x}\,\mathrm{d}x$$

$$\implies \ln|y| = -\ln|x| + c \implies \ln|y| + \ln|x| = c \implies \ln|xy| = c$$

$$\implies xy = \pm A \implies y = \pm\frac{A}{x}$$

This technique is easy, but very very rarely possible.

Reverse product rule

Sometimes the LHS (the y and y' terms in this case) can be seen as the result of the product rule applied to some function of x and y. For example,

$$x^3 y' + 3x^2 y = \sin x \implies \frac{\mathrm{d}}{\mathrm{d}x}(x^3 y) = \sin x \implies x^3 y = \int \sin x\,\mathrm{d}x$$

$$\implies x^3 y = -\cos x + c \implies y = \frac{-\cos x + c}{x^3}$$

Again, this technique is easy, but very rarely possible.

Integrating factor

The reverse product rule is not always applicable, but you can always multiply a first order linear homogeneous differential equation by some function of x, called an **integrating factor (IF)**, to make the reverse product rule applicable.

This **IF** could be anything, and you may be able to find a simple one, but there is a general formula:

$$\boxed{\;\text{Remember}\;\\ y' + P(x)y = Q(x) \implies \textbf{IF} = e^{\int P(x)\,\mathrm{d}x}\;}$$

For example,

$$y' + \frac{3y}{x} = \frac{\sin x}{x^3} \implies \textbf{IF} = x^3 \text{ (by inspection) or } \textbf{IF} = e^{\int \frac{3}{x}\,\mathrm{d}x} = e^{3\ln|x|} = x^3$$

$$x^3 y' + 3x^2 y = \sin x \implies y = \frac{-\cos x + c}{x^3} \text{ as shown above}$$

Second Order Homogeneous

To solve a general second order homogeneous DE of the form $ay'' + by' + cy = 0$, you can use the **auxiliary equation** $am^2 + bm + c = 0$. Find the solutions to the auxiliary equation and they will tell you the general solutions to the differential equation.

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Remember

2 real roots α and $\beta \implies y = Ae^{\alpha x} + Be^{\beta x}$

1 real repeated root $\alpha \implies y = (A + Bx)e^{\alpha x}$

2 complex roots $p \pm qi \implies y = e^{px}(A \cos qx + B \sin qx)$

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Second Order Non-homogeneous

To solve a DE of the form $ay'' + by' + cy = f(x)$, first solve the corresponding homogeneous equation $ay'' + by' + c = 0$. The general solution to this equation is know as the **complementary function (CF)**.

You then need to find the **particular integral (PI)**, which is a function that satisfies the original DE. The form of the PI depends on the form of $f(x)$. The PI will typically be of the same form as $f(x)$ but with different coefficients. However, if $f(x)$ is a single trig function like $\sin kx$, then the PI should be of the form $\lambda \cos kx + \mu \sin kx$.

If the **PI** is contained as a term in the **CF**, then you should multiply the **PI** by x.

To find the coefficients of the PI, differentiate it twice and sub the derivatives into the original DE and equate coefficients to solve simultaneously.

The general solution is

Remember

$$y = \mathbf{CF} + \mathbf{PI}$$

Harmonic Motion

For basically all differential equations describing physical motion, we use x for displacement, $\dot{x}$ for velocity, and $\ddot{x}$ for acceleration. All are functions of time t.

Simple

Simple harmonic motion means a particles oscillates around a fixed point O, always accelerating towards it. Taking O as the origin of our reference frame, we get the fact that the displacement of any particle moving in simple harmonic motion can be described with

> **Remember**
>
> $$\ddot{x} = -\omega^2 x$$

If you only need $\dot{x}$, then you can use the facts that $\dot{x} = v$ and $\ddot{x} = \dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{\mathrm{d}v}{\mathrm{d}x} \times \dfrac{\mathrm{d}x}{\mathrm{d}t} = v\dfrac{\mathrm{d}v}{\mathrm{d}x}$.

> **Remember**
>
> $$\ddot{x} = v\frac{\mathrm{d}v}{\mathrm{d}x}$$

This equation will allow you to separate the variables and find v quite easily.

If you need to find $\ddot{x}$, however, you will need to use the techniques for second order homogeneous ODEs.

The ODE $\ddot{x} = -\omega^2 x$ will have the general solution $x = A\sin\omega t + B\cos\omega t$. This proof is left as an exercise to the reader. Once you've found a particular solution, you can simplify it to $R\sin(\omega t + \alpha)$ using trig rules. Then we get:

> **Remember**
>
> $$\text{Amplitude} = R = \sqrt{A^2 + B^2}$$
> $$\text{Period} = \frac{2\pi}{\omega}$$