# Differential Equations

## Contents

# First Order Homogeneous

## *Separation of variables*

The simplest technique for solving first order homogeneous differential equations is to separate the variables. This involves treating the $\mathrm{d}x$ and $\mathrm{d}y$ as their own variables, rearranging everything to get all $y$ terms on one side and all $x$ terms on the other, and then adding integral signs. For example,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x} \implies \frac{1}{y}\,\mathrm{d}y = -\frac{1}{x}\,\mathrm{d}x \implies \int \frac{1}{y}\,\mathrm{d}y = \int -\frac{1}{x}\,\mathrm{d}x$$

$$\implies \ln|y| = -\ln|x| + c \implies \ln|y| + \ln|x| = c \implies \ln|xy| = c$$

$$\implies xy = \pm A \implies y = \pm\frac{A}{x}$$

This technique is easy, but very very rarely possible.

## *Reverse product rule*

Sometimes the LHS (the $y$ and $y'$ terms in this case) can be seen as the result of the product rule applied to some function of $x$ and $y$. For example,

$$x^3y' + 3x^2y = \sin x \implies \frac{\mathrm{d}}{\mathrm{d}x}(x^3y) = \sin x \implies x^3y = \int \sin x\,\mathrm{d}x$$

$$\implies x^3y = -\cos x + c \implies y = \frac{-\cos x + c}{x^3}$$

Again, this technique is easy, but very rarely possible.

## *Integrating factor*

The reverse product rule is not always applicable, but you can always multiply a first order linear homogeneous differential equation by some function of $x$, called an **integrating factor (IF)**, to make the reverse product rule applicable.

This **IF** could be anything, and you may be able to find a simple one, but there is a general formula:

Remember

$$y' + P(x)y = Q(x) \implies \textbf{IF} = e^{\int P(x)\,\mathrm{d}x}$$

For example,

$$y' + \frac{3y}{x} = \frac{\sin x}{x^3} \implies \textbf{IF} = x^3 \text{ (by inspection) or } \textbf{IF} = e^{\int \frac{3}{x}\,\mathrm{d}x} = e^{3\ln|x|} = x^3$$

$$x^3y' + 3x^2y = \sin x \implies y = \frac{-\cos x + c}{x^3} \text{ as shown above}$$

# Second Order Homogeneous

To solve a general second order homogeneous DE of the form $ay'' + by' + cy = 0$, you can use the **auxiliary equation** $am^2 + bm + c = 0$. Find the solutions to the auxiliary equation and they will tell you the general solutions to the differential equation.

Remember

$$\begin{aligned}
\text{2 real roots } \alpha \text{ and } \beta &\implies y = Ae^{\alpha x} + Be^{\beta x} \\
\text{1 real repeated root } \alpha &\implies y = (A + Bx)e^{\alpha x} \\
\text{2 complex roots } p \pm qi &\implies y = e^{px}(A\cos qx + B\sin qx)
\end{aligned}$$

# Second Order Non-homogeneous

To solve a DE of the form $ay'' + by' + cy = f(x)$, first solve the corresponding homogeneous equation $ay'' + by' + c = 0$. The general solution to this equation is know as the **complementary function (CF)**.

You then need to find the **particular integral (PI)**, which is a function that satisfies the original DE. The form of the PI depends on the form of $f(x)$. The PI will typically be of the same form as $f(x)$ but with different coefficients. However, if $f(x)$ is a single trig function like $\sin kx$, then the PI should be of the form $\lambda \cos kx + \mu \sin kx$.

If the **PI** is contained as a term in the **CF**, then you should multiply the **PI** by $x$.

To find the coefficients of the PI, differentiate it twice and sub the derivatives into the original DE and equate coefficients to solve simultaneously.

Remember

The general solution is $y = \mathbf{CF} + \mathbf{PI}$

# Harmonic Motion

For basically all differential equations describing physical motion, we use $x$ for displacement, $\dot{x}$ for velocity, and $\ddot{x}$ for acceleration. All are functions of time $t$.

## *Simple*

Simple harmonic motion means a particles oscillates around a fixed point $O$, always accelerating towards it. Taking $O$ as the origin of our reference frame, we get the fact that the displacement of any particle moving in simple harmonic motion can be described with

Remember

$$\ddot{x} = -\omega^2 x$$

Remember

$$\ddot{x} = v\frac{\mathrm{d}v}{\mathrm{d}x}$$

If you only need $\dot{x}$, then you can use the facts that $\dot{x} = v$ and $\ddot{x} = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}x}$. This equation will allow you to separate the variables and find $v$ quite easily.

If you need to find $\ddot{x}$, however, you will need to use the techniques for second order homogeneous ODEs.

The ODE $\ddot{x} = -\omega^2 x$ will have the general solution $x = A\sin\omega t + B\cos\omega t$. This proof is left as an exercise to the reader. Once you've found a particular solution, you can simplify it to $R\sin(\omega t + \alpha)$ using trig rules. Then we get:

Remember

$$\text{Amplitude} = R = \sqrt{A^2 + B^2}$$

$$\text{Period} = \frac{2\pi}{\omega}$$

## *Damped*

Damped harmonic motion is when a force acts to slow the particle down. It can be modelled with the equation

Remember

$$\ddot{x} + k\dot{x} + \omega^2 x = 0$$

When $k^2 > 4\omega^2$, there are two real roots to the auxiliary equation and there is **heavy damping**.

When $k^2 = 4\omega^2$, there are two repeated roots to the auxiliary equation and there is **critical damping**.

When $k^2 < 4\omega^2$, there are two complex roots to the auxiliary equation and there is **light damping**.

For light damping, the particle oscillates but gradually slows down. The period of oscillations can be calculated. For critical or heavy damping, the particle slows down and never oscillates.

## *Forced*

Forced harmonic motion is when a force acts to force the particle to oscillate at a frequency other than a natural one. It can be modelled with the equation

Remember

$$\ddot{x} + k\dot{x} + \omega^2 x = f(t)$$

# Coupled Simultaneous ODEs

In the real world, there are often multiple systems that depend on each other. These are best modelled by **coupled simultaneous differential equations**. For example, we can model the populations of grizzly bears $x$ and salmon $y$ with the equations:

$$\dot{x} = ax + by + f(t) \tag{1}$$

$$\dot{y} = cx + dy + g(t) \tag{2}$$

You can solve coupled first-order linear simultaneous ODEs by eliminating one of the dependent variables to form a second order ODE.

When trying to find a second order ODE for $x$, solve (1) for $y$ and differentiate it.

$$\begin{aligned}
\dot{x} &= ax + by + f(t) \\
y &= \frac{1}{b}(\dot{x} - ax - f(t)) \qquad (3) \\
\dot{y} &= \frac{1}{b}(\ddot{x} - a\dot{x} - \dot{f}(t))
\end{aligned}$$

Now replace the LHS with (2).

$$cx + dy + g(t) = \frac{1}{b}(\ddot{x} - a\dot{x} - \dot{f}(t))$$

And sub in (3) in place of $y$ to get a second order ODE containing just $x$ and $t$.

$$\begin{aligned}
cx + \frac{d}{b}\big(\dot{x} - ax - f(t)\big) + g(t) &= \frac{1}{b}(\ddot{x} - a\dot{x} - \dot{f}(t)) \\
\therefore\ \frac{1}{b}\ddot{x} - \frac{a+d}{b}\dot{x} + \left(\frac{ad}{b} - c\right)x &= \frac{1}{b}\dot{f}(t) - \frac{d}{b}f(t) + g(t)
\end{aligned}$$

Don't try to memorise this result; just learn the process.

Once you've solved this to find $x$, don't repeat the whole process for $y$. Just use the $x$ you found, differentiate it to get $\dot{x}$ and plug those into (3).