Differential Equations

Contents

First Order Homogeneous

Separation of variables

The simplest technique for solving first order homogeneous differential equations is to separate the variables. This involves treating the $\mathrm{d}x$ and $\mathrm{d}y$ as their own variables, rearranging everything to get all y terms on one side and all x terms on the other, and then adding integral signs. For example,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x} \implies \frac{1}{y}\,\mathrm{d}y = -\frac{1}{x}\,\mathrm{d}x \implies \int \frac{1}{y}\,\mathrm{d}y = \int -\frac{1}{x}\,\mathrm{d}x$$
$$\implies \ln|y| = -\ln|x| + c \implies \ln|y| + \ln|x| = c \implies \ln|xy| = c$$
$$\implies xy = \pm A \implies y = \pm\frac{A}{x}$$

This technique is easy, but very very rarely possible.

Reverse product rule

Sometimes the LHS (the y and y' terms in this case) can be seen as the result of the product rule applied to some function of x and y. For example,

$$x^3y' + 3x^2y = \sin x \implies \frac{\mathrm{d}}{\mathrm{d}x}(x^3y) = \sin x \implies x^3y = \int \sin x\,\mathrm{d}x$$
$$\implies x^3y = -\cos x + c \implies y = \frac{-\cos x + c}{x^3}$$

Again, this technique is easy, but very rarely possible.

Integrating factor

The reverse product rule is not always applicable, but you can always multiply a first order linear homogeneous differential equation by some function of x, called an **integrating factor (IF)**, to make the reverse product rule applicable.

This **IF** could be anything, and you may be able to find a simple one, but there is a general formula:

Remember

$$y' + P(x)y = Q(x) \implies \textbf{IF} = e^{\int P(x)\,\mathrm{d}x}$$

For example,

$$y' + \frac{3y}{x} = \frac{\sin x}{x^3} \implies \textbf{IF} = x^3 \text{ (by inspection) or } \textbf{IF} = e^{\int \frac{3}{x}\,\mathrm{d}x} = e^{3\ln|x|} = x^3$$
$$x^3y' + 3x^2y = \sin x \implies y = \frac{-\cos x + c}{x^3} \text{ as shown above}$$

Second Order Homogeneous

To solve a general second order homogeneous DE of the form $ay'' + by' + cy = 0$, you can use the **auxiliary equation** $am^2 + bm + c = 0$. Find the solutions to the auxiliary equation and they will tell you the general solutions to the differential equation.

Remember

$$\begin{aligned} \text{2 real roots } \alpha \text{ and } \beta &\implies y = Ae^{\alpha x} + Be^{\beta x} \\ \text{1 real repeated root } \alpha &\implies y = (A + Bx)e^{\alpha x} \\ \text{2 complex roots } p \pm qi &\implies y = e^{px}(A\cos qx + B\sin qx) \end{aligned}$$

Second Order Non-homogeneous

To solve a DE of the form $ay'' + by' + cy = f(x)$, first solve the corresponding homogeneous equation $ay'' + by' + c = 0$. The general solution to this equation is know as the **complementary function (CF)**.

You then need to find the **particular integral (PI)**, which is a function that satisfies the original DE. The form of the PI depends on the form of $f(x)$. The PI will typically be of the same form as $f(x)$ but with different coefficients. However, if $f(x)$ is a single trig function like $\sin kx$, then the PI should be of the form $\lambda \cos kx + \mu \sin kx$.

If the **PI** is contained as a term in the **CF**, then you should multiply the **PI** by x.

To find the coefficients of the PI, differentiate it twice and sub the derivatives into the original DE and equate coefficients to solve simultaneously.

Remember

The general solution is $y = \mathbf{CF} + \mathbf{PI}$

Harmonic Motion

For basically all differential equations describing physical motion, we use x for displacement, $\dot{x}$ for velocity, and $\ddot{x}$ for acceleration. All are functions of time t.

Simple

Simple harmonic motion means a particles oscillates around a fixed point O, always accelerating towards it. Taking O as the origin of our reference frame, we get the fact that the displacement of any particle moving in simple harmonic motion can be described with

Remember

$$\ddot{x} = -\omega^2 x$$

If you only need $\dot{x}$, then you can use the facts that $\dot{x} = v$ and $\ddot{x} = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}x}$. This equation will allow you to separate the variables and find v quite easily.

Remember

$$\ddot{x} = v\frac{\mathrm{d}v}{\mathrm{d}x}$$

If you need to find $\ddot{x}$, however, you will need to use the techniques for second order homogeneous ODEs.

The ODE $\ddot{x} = -\omega^2 x$ will have the general solution $x = A\sin\omega t + B\cos\omega t$. This proof is left as an exercise to the reader. Once you've found a particular solution, you can simplify it to $R\sin(\omega t + \alpha)$ using trig rules. Then we get:

Remember

$$\text{Amplitude} = R = \sqrt{A^2 + B^2}$$
$$\text{Period} = \frac{2\pi}{\omega}$$

Damped

Damped harmonic motion is when a force acts to slow the particle down. It can be modelled with the equation

Remember

$$\ddot{x} + k\dot{x} + \omega^2 x = 0$$

When $k^2 > 4\omega^2$, there are two real roots to the auxiliary equation and there is **heavy damping**.

When $k^2 = 4\omega^2$, there are two repeated roots to the auxiliary equation and there is **critical damping**.

When $k^2 < 4\omega^2$, there are two complex roots to the auxiliary equation and there is **light damping**.

For light damping, the particle oscillates but gradually slows down. The period of oscillations can be calculated. For critical or heavy damping, the particle slows down and never oscillates.

Forced

Forced harmonic motion is when a force acts to force the particle to oscillate at a frequency other than a natural one. It can be modelled with the equation

Remember

$$\ddot{x} + k\dot{x} + \omega^2 x = f(t)$$