Core Pure

Contents

Matrices

Multiplication

The size of a matrix is described as $h \times w$, so $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ is 2×3 but $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ is 3×2. To multiply a $h_1 \times w_1$ matrix by a $h_2 \times w_2$ matrix, we require that $w_1 = h_2$; the product matrix will then be of size $h_1 \times w_2$.

Addition of matrices is done element-wise, and requires both matrices to be of the same size.

Multiplication is more complicated. Use the rows of the matrix on the left and the columns of the matrix on the right, take their dot product, and put the answer in the corresponding slot of the answer matrix.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} \boxed{ae+bg} & \\ & \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & \boxed{af+bh} \\ & \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ \boxed{ce+dg} & \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & \boxed{cf+dh} \end{pmatrix}$$

Determinants

The determinant of a 2×2 matrix can be found by taking the product of the leading diagonal and subtracting the product of the other diagonal. To remember which way round it is, I remember that the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has determinant 1.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \boxed{ad} - \boxed{bc}$$

The determinant of a 3×3 matrix is a little more complicated, but still quite simple. Take the top row, and multiply each element by the determinant of its minor matrix, flipping the signs for each term.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \mathrm{a} \begin{vmatrix} e & f \\ h & i \end{vmatrix} - \mathrm{b} \begin{vmatrix} d & f \\ g & i \end{vmatrix} + \mathrm{c} \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \mathrm{a} \begin{vmatrix} e & f \\ h & i \end{vmatrix} - \mathrm{b} \begin{vmatrix} d & f \\ g & i \end{vmatrix} + \mathrm{c} \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \mathrm{a} \begin{vmatrix} e & f \\ h & i \end{vmatrix} - \mathrm{b} \begin{vmatrix} d & f \\ g & i \end{vmatrix} + \mathrm{c} \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Standard results

Reflection in $y = x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
Reflection in $y = -x$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
2D rotation	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
3D rotation around x-axis	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$
3D rotation around y-axis	$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$
3D rotation around z-axis	$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Inverses

The inverse of a 2×2 matrix is quite easy to find. Remember that $\mathbf{I}^{-1} = \mathbf{I}$, not $-\mathbf{I}$, so flip the leading diagonal and negate the opposite diagonal.

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

The inverse of a 3×3 matrix is significantly more complicated. For a 3×3 matrix $\mathbf{A}$, $\boxed{\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \mathbf{C}^{\mathrm{T}}}$, where $\mathbf{C}$ is the matrix of cofactors, formed by the matrix of minors $\mathbf{M}$.

Here's an example:

Step 1

The matrix $\mathbf{A} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 4 & 1 \\ 2 & -1 & 0 \end{pmatrix}$. Begin by finding its determinant, as explained previously.

$\det \mathbf{A} = 1 \begin{vmatrix} 4 & 1 \\ -1 & 0 \end{vmatrix} - 3 \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} + 1 \begin{vmatrix} 0 & 4 \\ 2 & -1 \end{vmatrix} = 1 + 6 - 8 = -1.$

Step 2

Keep the determinant handy; we'll need it later. Now we want to find the matrix of minors, $\mathbf{M}$. For each element in the original matrix $\mathbf{A}$, its equivalent element in the matrix $\mathbf{M}$ is found by crossing out the row and column containing that element in $\mathbf{A}$ and taking the determinant of the remaining elements. Like what we did to find the determinant, but without the sign flipping and multiplying by the coefficients. Therefore,

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 4 & 1 \\ -1 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 4 \\ 2 & -1 \end{vmatrix} \\[2ex] \begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} \\[2ex] \begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 1 & -2 & -8 \\ 1 & -2 & -7 \\ -1 & 1 & 4 \end{pmatrix}$$

Step 3

Now, we find the matrix of cofactors, $\mathbf{C}$, by flipping the signs of certain elements in the matrix of minors, $\mathbf{M}$.

Use the pattern $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$ to find that $\mathbf{C} = \begin{pmatrix} 1 & 2 & -8 \\ -1 & -2 & 7 \\ -1 & -1 & 4 \end{pmatrix}$.

Step 4

Take the transpose of $\mathbf{C}$ to get $\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 1 & -1 & -1 \\ 2 & -2 & 1 \\ -8 & 7 & 4 \end{pmatrix}$.

Step 5

Use the formula to find $\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -1 & 1 & 1 \\ -2 & 2 & 1 \\ 8 & -7 & -4 \end{pmatrix}$.