Series Polynomials

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The challenge question in Exercise 3B is about series sums of polynomials.

Part a of this question asks for polynomials $f_2(x)$, $f_3(x)$, $f_4(x)$ such that for every $n \in \mathbb{N}$,

$$\sum_{r=1}^{n} f_2(r) = n^2$$
$$\sum_{r=1}^{n} f_3(r) = n^3$$
$$\sum_{r=1}^{n} f_4(r) = n^4$$

Finding these polynomials reveals a very interesting pattern.

1 Finding Polynomials

Throughout this paper, all $n \in \mathbb{N}$.

Let's first establish some basic lemmas about series.

Lemma 1 $\sum_{r=1}^{n} (f(r) + g(r)) = \sum_{r=1}^{n} f(r) + \sum_{r=1}^{n} g(r)$

Lemma 2 $\sum_{r=1}^{n} kf(r) = k \sum_{r=1}^{n} f(r)$

Lemma 3 $\sum_{r=1}^{n} 1 = n$

Lemma 4 $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

Lemma 5 $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

Lemma 6 $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

1.1 Finding $f_2(r)$

We want a polynomial $f_2(r)$ such that $\sum_{r=1}^{n} f_2(r) = n^2$.

We already have, by Lemma 4,

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1) = \frac{1}{2}\left(n^2 + n\right)$$

We need to get rid of the $\frac{1}{2}$, which we can do by multiplying by 2 to get

$$2\sum_{r=1}^{n} r = \sum_{r=1}^{n} 2r = n^2 + n$$

Then, we just need to get rid of the n.

We know that $\sum\limits_{r=1}^{n} 1 = n$, so

$$\sum_{r=1}^{n} 2r - \sum_{r=1}^{n} 1 = \sum_{r=1}^{n}(2r-1) = n^2$$
$$\therefore f_2(r) = 2r - 1$$

1.2 Finding $f_3(r)$

Next, we want a polynomial $f_3(r)$ such that $\sum\limits_{r=1}^{n} f_3(r) = n^3$.

Similarly to with $f_2(r)$, by Lemma 5, we already have

$$\begin{aligned}\sum_{r=1}^{n} r^2 &= \frac{1}{6}n(n+1)(2n+1) \\ &= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \\ &= \frac{1}{3}\left(n^3 + \frac{3}{2}n^2 + \frac{1}{2}n\right)\end{aligned}$$

We can then multiply by 3 to remove the $\frac{1}{3}$ and get

$$3\sum_{r=1}^{n} r^2 = \sum_{r=1}^{n} 3r^2 = n^3 + \frac{3}{2}n^2 + \frac{1}{2}n$$

To get rid of the $\frac{1}{2}n$, we can just do

$$\sum_{r=1}^{n} 3r^2 - \sum_{r=1}^{n} \frac{1}{2} = \sum_{r=1}^{n}\left(3r^2 - \frac{1}{2}\right) = n^3 + \frac{3}{2}n^2$$

We know that $\sum\limits_{r=1}^{n} f_2(r) = n^2$, so we simply need to subtract $\frac{3}{2}f_2(r)$ from our polynomial to get rid of the resultant $\frac{3}{2}n^2$.

$$3r^2 - \frac{1}{2} - \frac{3}{2}(2r-1) = 3r^2 - 3r + 1$$
$$\therefore f_3(r) = 3r^2 - 3r + 1$$

1.3 Finding $f_4(r)$

Next, we want a polynomial $f_4(r)$ such that $\sum_{r=1}^{n} f_4(r) = n^4$.

We know, by Lemma 6, that

$$\begin{aligned}\sum_{r=1}^{n} r^3 &= \frac{1}{4}n^2(n+1)^2 \\ &= \frac{1}{4}\left(n^4 + 2n^3 + n^2\right)\end{aligned}$$

We can multiply by 4 to get

$$4\sum_{r=1}^{n} r^3 = \sum_{r=1}^{n} 4r^3 = n^4 + 2n^3 + n^2$$

We can then get n^4 on its own by subtracting $2f_3(r)$ and $f_2(r)$ from $4r^3$.

$$\begin{gathered}4r^3 - 2\left(3r^2 - 3r + 1\right) - (2r - 1) \\ = 4r^3 - 6r^2 + 6r - 2 - 2r + 1 \\ = 4r^3 - 6r^2 + 4r - 1 \\ \therefore f_4(r) = 4r^3 - 6r^2 + 4r - 1\end{gathered}$$

2 Finding Patterns

We can find $f_1(r)$, where $\sum_{r=1}^{n} f_1(r) = n^1 = n$ to trivially be 1.

These are our polynomials:

$f_1(r) = 1$

$f_2(r) = 2r - 1$

$f_3(r) = 3r^2 - 3r + 1$

$f_4(r) = 4r^3 - 6r^2 + 4r - 1$

After looking at these for a while, we can notice a few things. Firstly, the constants are always ± 1, and the signs of these constants alternate with increasing degrees of polynomial. In fact, all the signs alternate.

Secondly, we can notice that the first term of $f_m(r)$ is always of the form mr^{m-1}.

However, the most interesting thing to notice with these polynomials is that the coefficients look like binomial expansions, albeit with the leading term removed.

Conjecture 1 $f_5(r) = 5r^4 - 10r^3 + 10r^2 - 5r + 1$

And, more generally,

Conjecture 2 $f_m(r) = r^m - (r-1)^m$

This form expands to give us the full binomial expansion with alternating signs, but we subtract this from r^m to flip all the signs and remove the r^m term.

Conjecture 2 states:

$$\forall m, n \in \mathbb{N}, \sum_{r=1}^{n} \left(r^m - (r-1)^m\right) = n^m$$

In order to prove Conjecture 1, we need to know the formula for $\sum_{r=1}^{n} r^4$. Wolfram Alpha says:

Lemma 7 $\sum_{r=1}^{n} r^4 = \frac{1}{30}n(n+1)(2n+1)\left(3n^2+3n-1\right)$

This can also be expanded to give

$$\begin{aligned} &\frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n \\ &= \frac{1}{5}\left(n^5 + \frac{5}{2}n^4 + \frac{5}{3}n^3 - \frac{1}{6}n\right) \end{aligned}$$

We can multiply by 5 to get

$$5\sum_{r=1}^{n} r^4 = \sum_{r=1}^{n} 5r^4 = n^5 + \frac{5}{2}n^4 + \frac{5}{3}n^3 - \frac{1}{6}n$$

Now, we just need to get rid of the other terms to get a polynomial $f_5(r)$ such that $\sum_{r=1}^{n} f_5(r) = n^5$.

$$\begin{aligned} f_5(r) &= 5r^4 - \frac{5}{2}f_4(r) - \frac{5}{3}f_3(r) + \frac{1}{6}f_1(r) \\ &= 5r^4 - \frac{5}{2}\left(4r^3 - 6r^2 + 4r - 1\right) - \frac{5}{3}\left(3r^2 - 3r + 1\right) + \frac{1}{6} \\ &= 5r^4 - 10r^3 + 10r^2 - 5r + 1 \end{aligned}$$

This proves Conjecture 1.

However, to prove Conjecture 2, I think I need to find more patterns.

3 Finding Patterns (again)

Let's look at sums of powers of r. By Lemmas 3 to 7, we have

$$\begin{aligned} \sum_{r=1}^{n} r^0 &= n &&= n \\ \sum_{r=1}^{n} r^1 &= \frac{1}{2}n(n+1) &&= \frac{1}{2}n^2 + \frac{1}{2}n \\ \sum_{r=1}^{n} r^2 &= \frac{1}{6}n(n+1)(2n+1) &&= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \\ \sum_{r=1}^{n} r^3 &= \frac{1}{4}n^2(n+1)^2 &&= \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 \\ \sum_{r=1}^{n} r^4 &= \frac{1}{30}n(n+1)(2n+1)\left(3n^2+3n-1\right) &&= \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n \end{aligned}$$

I can't see much of a pattern here, but the first term of $\sum_{r=1}^{n} r^m$ always seems to be $\frac{1}{m+1}n^{m+1}$ and the second term always seems to be $\frac{1}{2}n^m$, except in the case of $m = 0$.

Finding a general formula for $\sum_{r=1}^{n} r^m$ seems quite hard. In fact, Wolfram Alpha evaluates it as $H_n^{(-m)}$, where $H_n^{(k)}$ is the *generalised harmonic number*. These numbers are related to the harmonic series, defined as $\sum_{n=1}^{\infty} \frac{1}{n}$.

$H_n^{(k)}$ can be defined as

$$H_n^{(k)} = \sum_{r=1}^{n} r^{-k}$$

This means that

$$H_n^{(-k)} = \sum_{r=1}^{n} r^{k}$$

This is just rephrasing the same thing. This is useless. Thanks Wolfram Alpha.

Asking SymPy to evaluate `Sum(r**m - (r - 1)**m, (r, 1, n))` gives $-0^m + n^m$, so there's clearly an algorithm to do this, but I don't know how to prove this result. I also don't know why it returns -0^m as part of the answer.

I believe that proving Conjecture 2 is possible, but I don't think I currently have the tools to do so.

4 Part *b*

This question does have a part b, which I should probably address quickly. It says:

Hence, show that for any linear, quadratic, or cubic polynomial $h(x)$, there exists a polynomial $g(x)$ such that $\sum_{r=1}^{n} g(r) = n(h(n))$.

If we let

$$h(n) = an^3 + bn^2 + cn + d$$

then

$$n(h(n)) = an^4 + bn^3 + cn^2 + dn$$

Then, $g(r)$ is simply

$$af_4(r) + bf_3(r) + cf_2(r) + df_1(r)$$

This idea extends to linear and quadratic polynomials $h(x)$ and if we assume Conjecture 2 to be true, meaning we can find any $f_m(r)$, then this idea extends to any degree polynomial.