MA144 Methods of Mathematical Modelling 2, Assignment 1

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Question 1

Consider the polar curve with equation $r = f(\theta)$, where $\theta \in [a, b]$ and f is some function of θ.

Q1 (a)

Using the arc-length formula for parametric curves (in the lecture notes), show that the arc length of the polar curve is given by the integral

$$\int_a^b \sqrt{\left(\frac{\mathrm{d}f}{\mathrm{d}\theta}\right)^2 + f^2} \, \mathrm{d}\theta.$$

The polar curve with equation $r = f(\theta)$ can be parametrised as $\underline{r}(\theta) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$. Then

$$\frac{\mathrm{d}\underline{r}}{\mathrm{d}\theta} = \left(\frac{\mathrm{d}f}{\mathrm{d}\theta}(\theta)\cos\theta - f(\theta)\sin\theta, \frac{\mathrm{d}f}{\mathrm{d}\theta}(\theta)\sin\theta + f(\theta)\cos\theta\right)$$

Therefore

$$\begin{aligned}
\|\underline{r}'(\theta)\| &= \sqrt{(f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2} \\
&= \sqrt{f'(\theta)^2\cos^2\theta - 2f(\theta)f'(\theta)\cos\theta\sin\theta + f(\theta)^2\sin^2\theta} \\
&\quad \overline{+f'(\theta)^2\sin^2 + 2f(\theta)f'(\theta)\cos\theta\sin\theta + f(\theta)^2\cos^2\theta} \\
&= \sqrt{f'(\theta)^2(\cos^2\theta + \sin^2\theta) + f(\theta)^2(\cos^2\theta + \sin^2\theta)} \\
&= \sqrt{f'(\theta)^2 + f(\theta)^2} \\
&= \sqrt{\left(\frac{\mathrm{d}f}{\mathrm{d}\theta}\right)^2 + f^2}
\end{aligned}$$

Therefore the arc length of the curve is

$$s = \int_a^b \sqrt{\left(\frac{\mathrm{d}f}{\mathrm{d}\theta}\right)^2 + f^2}\ \mathrm{d}\theta$$

as required.

Q1 (b)

Sketch the closed curve with polar equation $r = 1 + \cos\theta$. Find its arc length.

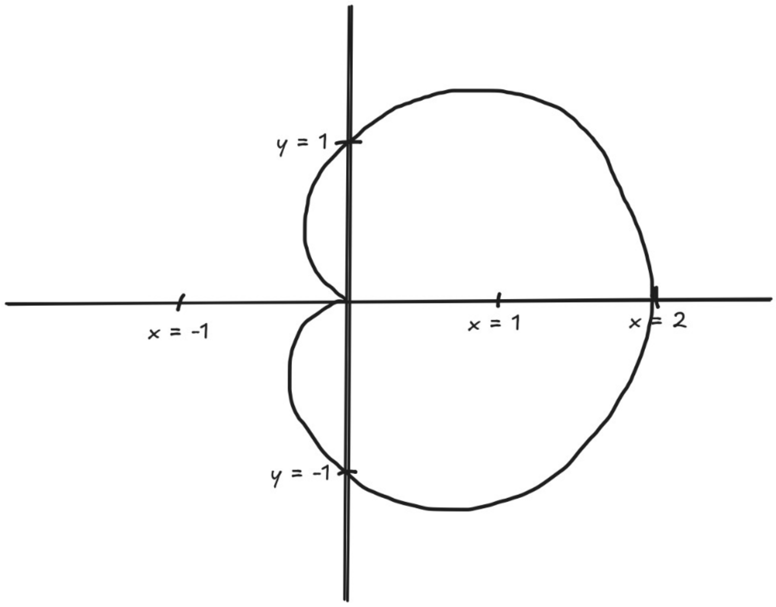

Figure 1: A sketch of $r = 1 + \cos\theta$

$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\sin\theta$ so the arc length is

$$\begin{aligned} s &= \int_0^{2\pi} \sqrt{\sin^2\theta + (1+\cos\theta)^2}\ \mathrm{d}\theta \\ &= \int_0^{2\pi} \sqrt{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}\ \mathrm{d}\theta \\ &= \int_0^{2\pi} \sqrt{2 + 2\cos\theta}\ \mathrm{d}\theta \end{aligned}$$

I don't know how to do this integral but apparently it's 4π.

Q1 (c)

Consider the curve with polar equation $r = 1 + \cos k\theta$, where $\theta \in \mathbb{R}$. Give a value of the constant k such that the curve is not simple. Justify your answer.

The curve given by $r = 1 + \cos k\theta$ is non-simple (self-intersecting) whenever k is not an integer.

Question 2

Consider the curve $\mathcal{C}$ with equation

$$4y^2 - 9x^2 = 1, \quad y > 0$$

Q2 (a)

Parametrise the curve in terms of hyperbolic functions of t. Is this a regular parametrisation of the curve?

Since $\cosh^2 t - \sinh^2 t = 1$, we can let $y = \frac{1}{2}\cosh t$ and $x = \frac{1}{3}\sinh t$. Then we'll have the equation of $\mathcal{C}$. Therefore we can parametrise $\mathcal{C}$ as $\underline{r}(t) = \left(\frac{1}{3}\sinh t, \frac{1}{2}\cosh t\right)$.

Since $\frac{1}{2}\cosh t > 0$ for all $t \in (-\infty, \infty)$, the requirement of $y > 0$ is satisfied by $t \in (-\infty, \infty)$, so that's the range of this parametrisation.

Q2 (b)

Sketch the curve described by your parametrisation (include an arrow to indicate its orientation). Give the equations of any asymptotes.

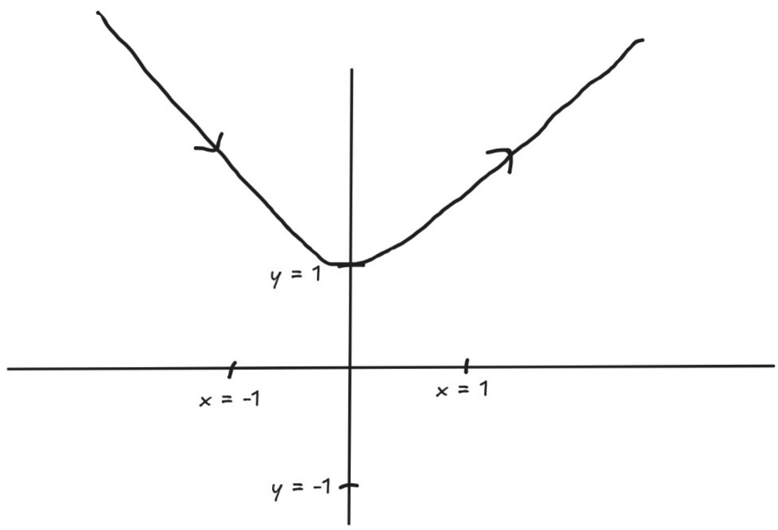

Figure 2: A sketch of $\underline{r}(t) = \left(\frac{1}{3}\sinh t, \frac{1}{2}\cosh t\right)$

There is a turning point at $\left(0, \frac{1}{2}\right)$ and asymptotes are $y = \frac{3}{2}x$ and $y = -\frac{3}{2}x$.

Q2 (c)

Write down a parametrisation of the curve using parameter u such that $0 < u < 1$.

To reparametrise with u such that the bounds on the parametrisation are $u \in (0,1)$, we want to adjust t in the old parametrisation. We want $u = \frac{1}{2}$ when $t = 0$, $u \to 1$ as $t \to \infty$, and $u \to 0$ as $t \to \infty$. We want an increasing, sigmoid-shaped curve with horizontal asymptotes at $y = 0$ and $y = 1$. tanh almost fits this shape, but needs minor adjustments.

Let $u = \dfrac{1 + \tanh t}{2}$. Then $u = \frac{1}{2}$ when $t = 0$, $u \to 1$ as $t \to \infty$, and $u \to 0$ as $t \to \infty$, as required. We rearrange to get $t = \operatorname{artanh}(2u - 1)$ and plug this into the old parametrisation.

Therefore $\mathcal{C}$ can also be parametrised as

$$\left(\frac{1}{3}\sinh\left(\operatorname{artanh}(2u-1)\right), \frac{1}{2}\cosh\left(\operatorname{artanh}(2u-1)\right)\right) \quad u \in (0,1)$$

We can also remove the hyperbolic functions and write it as

$$\left(\frac{2u-1}{6\sqrt{u-u^2}}, \frac{1}{4\sqrt{u-u^2}}\right) \quad u \in (0,1)$$

Question 3

A circle radius $r < 1$ is rolling on the inside of a circle radius 1, centred at the origin O. The centre C of the smaller circle is initially at $(1-r, 0)$. Let θ be the angle subtended by the line OC measured with respect to the positive x axis. The point P, initially at $(1, 0)$, traces out a curve as the smaller circle rolls inside the unit circle.

The curve traced out by P has the parametric equations:

$$x(\theta) = (1-r)\cos\theta + r\cos\left(\frac{1-r}{r}\theta\right),$$
$$y(\theta) = (1-r)\sin\theta - r\cos\left(\frac{1-r}{r}\theta\right).$$

Q3 (a)

Use Python to plot the curves corresponding to $r = \frac{1}{2}, \frac{1}{4}, \frac{2}{3}$, and $\frac{1}{\sqrt{2}}$, where $\theta \in [0, 10\pi]$. Plot them in 4 separate figures.

Q3 (b)

Make one conjecture on the appearance of the curve for an arbitrary $r \in \mathbb{R}$.

I conjecture that for any $r \in \mathbb{R}$, the curve will be closed if and only if $r \in \mathbb{Q}$.

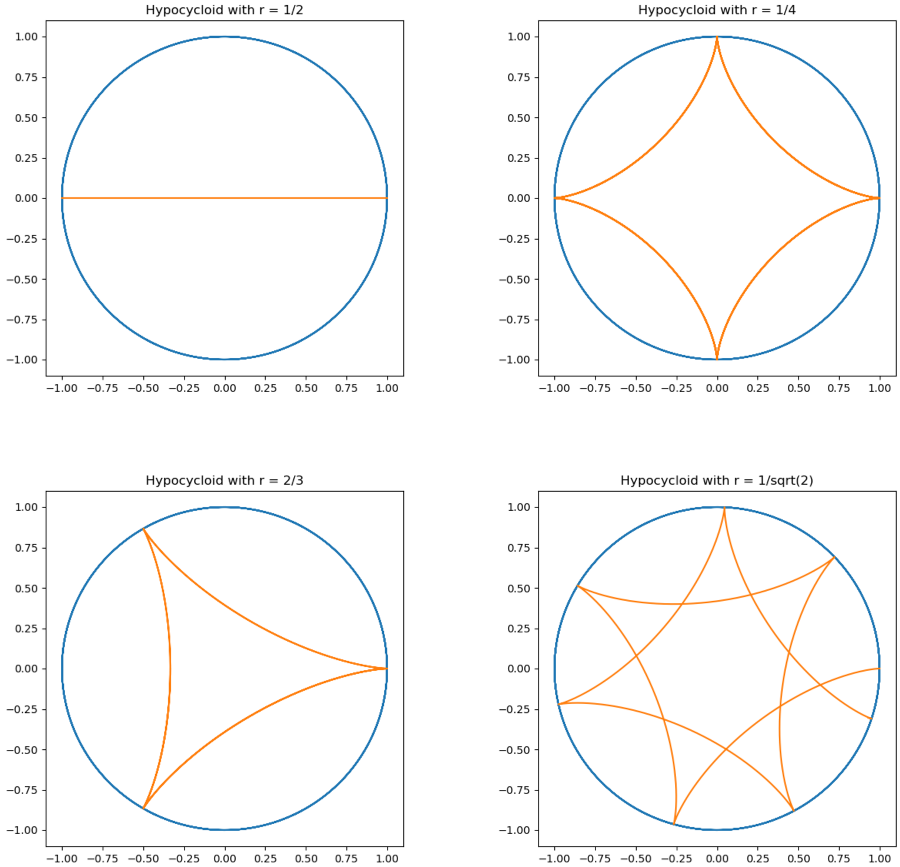

Figure 3: Plots of hypocycloids for various values of r

```python
#!/usr/bin/env python3

from pathlib import Path

import matplotlib.pyplot as plt
import numpy as np


def generate_plot(r: float, title_text: str, file_number: int) -> None:
    """Generate the plot for given radius."""
    theta = np.linspace(0, 10 * np.pi, 100_000)
    x = (1 - r) * np.cos(theta) + r * np.cos((1 - r) / r * theta)
    y = (1 - r) * np.sin(theta) - r * np.sin((1 - r) / r * theta)

    plt.figure(figsize=(6, 6))
    plt.plot(np.cos(theta), np.sin(theta))
    plt.plot(x, y)
    plt.title(f"Hypocycloid with r = {title_text}")
    plt.xlim(-1.1, 1.1)
    plt.ylim(-1.1, 1.1)

    plt.savefig(Path(__file__).parent.parent / "imgs" / f"Q3a-{file_number}.png")
    plt.clf()


def main() -> None:
    """Generate the plots for the necessary radii."""
    for r, title_text, file_number in [
        (1 / 2, "1/2", 1),
        (1 / 4, "1/4", 2),
        (2 / 3, "2/3", 3),
        (1 / np.sqrt(2), "1/sqrt(2)", 4),
    ]:
        generate_plot(r, title_text, file_number)


if __name__ == "__main__":
    main()
```

Figure 4: The code used to generate the plots in Figure 3. The code can also be found on GitHub