MA144 Methods of Mathematical Modelling 2, Assignment 4

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Question 1

In Assignment 1, you plotted the curve C parametrised by

$$x(\theta) = (1-r)\cos\theta + r\cos\left(\frac{1-r}{r}\theta\right)$$
$$y(\theta) = (1-r)\sin\theta - r\sin\left(\frac{1-r}{r}\theta\right)$$

Let $r = \dfrac{1}{k}$ and $k \in \mathbb{N}$. Use Green's Theorem to find the area enclosed by the curve.

Also let D be the area enclosed by C. We want the area of D, which is $\iint_D \mathrm{d}x\,\mathrm{d}y$. Green's Theorem states that

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y = \oint_C (P\,\mathrm{d}x + Q\,\mathrm{d}y)$$

So if we just choose P and Q such that $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1$, then we can apply Green's Theorem to simplify the double integral to a single integral. We will choose $Q(x,y) = x$ and $P(x,y) = 0$.

Therefore the area is

$$
\begin{aligned}
\oint_C x \,\mathrm{d}y &= \oint_C x(\theta)\frac{\mathrm{d}y}{\mathrm{d}\theta}\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \left(\frac{k-1}{k}\cos\theta + \frac{1}{k}\cos((k-1)\theta)\right) \\
&\quad \times \left(\frac{k-1}{k}\cos\theta - \frac{k-1}{k}\cos((k-1)\theta)\right)\mathrm{d}\theta \\
&= \int_0^{2\pi} \left(\frac{(k-1)^2}{k^2}\cos^2\theta - \frac{(k-1)^2}{k^2}\cos\theta\cos((k-1)\theta)\right. \\
&\quad \left. + \frac{k-1}{k^2}\cos\theta\cos((k-1)\theta) - \frac{k-1}{k^2}\cos^2((k-1)\theta)\right)\mathrm{d}\theta \\
&= \frac{k-1}{k^2}\int_0^{2\pi} \Big((k-1)\cos^2\theta + (2-k)\cos\theta\cos((k-1)\theta) \\
&\quad - \cos^2((k-1)\theta)\Big)\,\mathrm{d}\theta \\
&= \frac{k-1}{2k^2}\int_0^{2\pi} \Big((k-1)(\cos 2\theta + 1) + 2(2-k)\cos\theta\cos((k-1)\theta) \\
&\quad - \cos(2(k-1)\theta + 1)\Big)\,\mathrm{d}\theta \\
&= \frac{k-1}{2k^2}\left[\frac{k-1}{2}\sin 2\theta - \frac{1}{2k-2}\sin((2k-2)\theta) + (k-2)\theta\right]_0^{2\pi} \\
&\quad + \frac{k-1}{2k^2}\int_0^{2\pi} 2(2-k)\cos\theta\cos((k-1)\theta)\,\mathrm{d}\theta \\
&= \frac{k-1}{2k^2}(k-2)2\pi \\
&\quad + \frac{k-1}{2k^2}\int_0^{2\pi} \left(\cos\left((k-1)\theta + \theta\right) + \cos\left((k-1)\theta - \theta\right)\right)\mathrm{d}\theta \\
&= \frac{\pi(k-1)(k-2)}{k^2} + \frac{k-1}{2k^2}\int_0^{2\pi} \left(\cos k\theta + \cos\left((k-2)\theta\right)\right)\mathrm{d}\theta \\
&= \frac{\pi(k-1)(k-2)}{k^2} + \frac{k-1}{2k^2}\left[\frac{1}{k}\sin k\theta + \frac{1}{k-2}\sin((k-2)\theta)\right]_0^{2\pi} \\
&= \frac{\pi(k-1)(k-2)}{k^2}
\end{aligned}
$$

See also this Desmos graph that I made while doing this question.

Question 2

Consider the vector field $\underline{F}(x, y, z) = \begin{pmatrix} y^2 \\ x \\ z \end{pmatrix}$ and the surface S parametrised by

$$\underline{r}(u, v) = \begin{pmatrix} u \cos v \\ u \sin v \\ 3 - u \sin v \end{pmatrix}$$

where $u \in [0, 1]$ and $v \in [0, 2\pi]$.

Q2 (a)

Recall the formula from the lecture notes:

$$\underline{\hat{n}}\, \mathrm{d}S = \pm(\underline{r}_u \times \underline{r}_v)\, \mathrm{d}u\, \mathrm{d}v. \qquad (*)$$

Use this to evaluate

$$\iint_S \nabla \times \underline{F} \cdot \underline{\hat{n}}\, \mathrm{d}S,$$

where $\underline{\hat{n}}$ is the upward-pointing unit normal. Explain briefly how you chose the sign in equation $(*)$.

It is fairly simple to notice by inspection that $\underline{r}(u, v)$ parametrises the section of the plane $z = 3 - y$ where $x^2 + y^2 \leq 1$. Therefore the outward-pointing unit normal is in this case upward-pointing, so it will be the branch with positive z component.

First, we need the curl of $\underline{F}$:

$$\nabla \times \underline{F} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} y^2 \\ x \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -0 \\ 1 - 2y \end{pmatrix}$$

To find the normal vector, we also need the partial derivatives of $\underline{r}$:

$$\underline{r}_u(u, v) = \begin{pmatrix} \cos v \\ \sin v \\ -\sin v \end{pmatrix} \qquad \underline{r}_v(u, v) = \begin{pmatrix} -u \sin v \\ u \cos v \\ u \cos v \end{pmatrix}$$

Then we get

$$\begin{aligned}
\underline{\hat{n}}\,\mathrm{d}S &= \pm(\underline{r}_u \times \underline{r}_v)\,\mathrm{d}u\,\mathrm{d}v \\
&= \pm \begin{pmatrix} 2u \sin v \cos v \\ -(u\cos^2 v - u \sin^2 v) \\ u\cos^2 v + u\sin^2 v \end{pmatrix} \mathrm{d}u\,\mathrm{d}v \\
&= \pm \begin{pmatrix} 2u \sin v \cos v \\ -u\cos 2v \\ u \end{pmatrix} \mathrm{d}u\,\mathrm{d}v
\end{aligned}$$

We want the z component to be positive as discussed earlier, so we choose the positive branch.

Therefore

$$\begin{aligned}
\iint_S \nabla \times \underline{F} \cdot \underline{\hat{n}}\,\mathrm{d}S &= \iint_S \begin{pmatrix} 0 \\ 0 \\ 1 - 2u\sin v \end{pmatrix} \cdot \begin{pmatrix} 2u \sin v \cos v \\ -u\cos 2v \\ u \end{pmatrix} \mathrm{d}u\,\mathrm{d}v \\
&= \iint_S (u - 2u^2 \sin v)\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_0^{2\pi} \int_0^1 \left(u - 2u^2 \sin v\right)\, \mathrm{d}u\ \mathrm{d}v \\
&= \int_0^{2\pi} \left[\frac{1}{2}u^2 - \frac{2}{3}u^3 \sin v\right]_{u=0}^1 \mathrm{d}v \\
&= \int_0^{2\pi} \left(\frac{1}{2} - \frac{2}{3}\sin v\right)\, \mathrm{d}v \\
&= \left[\frac{1}{2}v + \frac{2}{3}\cos v\right]_0^{2\pi} \\
&= \pi + \frac{2}{3} - \frac{2}{3} \\
&= \pi
\end{aligned}$$

Q2 (b)

Verify that Stokes' Theorem holds in this situation.

Stokes' Theorem states the integral we found in part **(a)** is equal to $\oint_C \underline{F} \cdot \mathrm{d}\underline{r}$ where C is the boundary curve of S.

The geometry of S is very simple, so it's clear that C is just the level set where

$u = 1$. Therefore we can parametrise C as

$$\underline{r}(t) = \begin{pmatrix} \cos t \\ \sin t \\ 3 - \sin t \end{pmatrix}$$

Then we get

$$\frac{\mathrm{d}\underline{r}}{\mathrm{d}t} = \begin{pmatrix} -\sin t \\ \cos t \\ \cos t \end{pmatrix}$$

And so,

$$\begin{aligned}
\oint_C \underline{F} \cdot \mathrm{d}\underline{r} &= \oint_C \underline{F} \cdot \frac{\mathrm{d}\underline{r}}{\mathrm{d}t}\,\mathrm{d}t \\
&= \int_0^{2\pi} \begin{pmatrix} y^2 \\ x \\ z \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ \cos t \end{pmatrix} \mathrm{d}t \\
&= \int_0^{2\pi} \begin{pmatrix} \sin^2 t \\ \cos t \\ 3 - \sin t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ \cos t \end{pmatrix} \mathrm{d}t \\
&= \int_0^{2\pi} (-\sin^3 t + \cos^2 t + 3\cos t - \sin t \cos t)\,\mathrm{d}t \\
&= \int_0^{2\pi} \left(-\sin t(1 - \cos^2 t) + \frac{1}{2}(\cos 2t + 1) + 3\cos t - \sin t \cos t\right) \mathrm{d}t \\
&= \int_0^{2\pi} \left(-\sin t + \sin t \cos^2 t + \frac{1}{2}\cos 2t + \frac{1}{2} + 3\cos t - \sin t \cos t\right) \mathrm{d}t \\
&= \left[\cos t - \frac{1}{3}\cos^3 t + \frac{1}{4}\sin 2t + \frac{t}{2} + 3\sin t - \frac{1}{2}\sin^2 t\right]_0^{2\pi} \\
&= 1 - \frac{1}{3} + \pi - \left(1 - \frac{1}{3}\right) \\
&= \pi
\end{aligned}$$

This matches the integral from part **(a)**, so Stokes' Theorem holds in this situation.

Question 3

A *torus* has the following parametrisation:

$$\underline{r}(u, v) = \begin{pmatrix} (2 + \cos u)\cos v \\ (2 + \cos u)\sin v \\ \sin u \end{pmatrix} \qquad u, v \in [0, 2\pi]$$

Q3 (a)

Plot the torus using Python. Make sure the aspect ratio of your plot is roughly equal in all x, y, z directions so the surface looks like a doughnut and doesn't appear stretched. Label the axes.

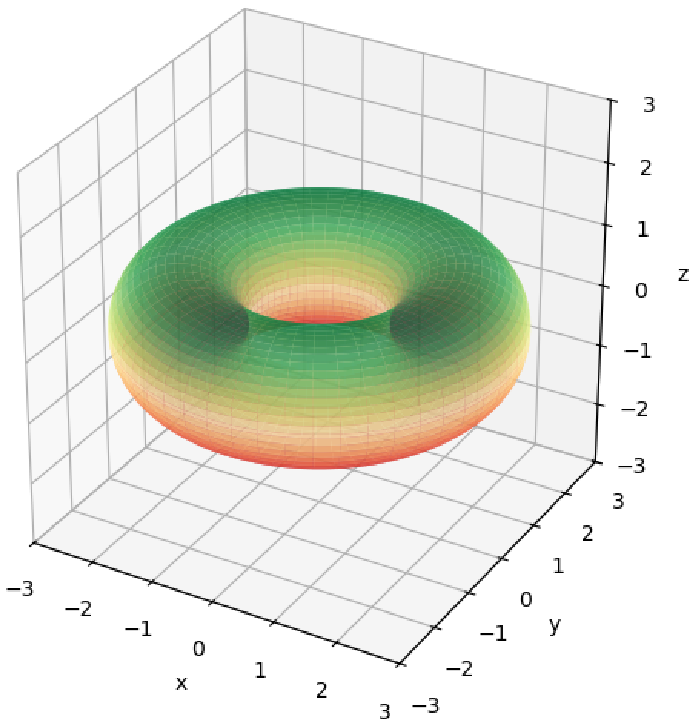

Figure 1: A torus, plotted with matplotlib in Python

```python
#!/usr/bin/env python3

from pathlib import Path

import matplotlib.pyplot as plt
import numpy as np


def main() -> None:
    """Plot the torus."""
    N = 100
    u = np.linspace(0, 2 * np.pi, N)
    v = np.linspace(0, 2 * np.pi, N)

    U, V = np.meshgrid(u, v)

    fig = plt.figure(figsize=(6, 6))
    ax = fig.add_subplot(projection="3d")
    ax.set_box_aspect((1, 1, 1))
    ax.set_aspect("equal")

    ax.set_xlabel("x")
    ax.set_ylabel("y")
    ax.set_zlabel("z")

    ax.set_xlim(-3, 3)
    ax.set_ylim(-3, 3)
    ax.set_zlim(-3, 3)

    ax.plot_surface(
        (2 + np.cos(U)) * np.cos(V),
        (2 + np.cos(U)) * np.sin(V),
        np.sin(U),
        alpha=0.7,
        cmap="RdYlGn",
    )

    plt.savefig(Path(__file__).parent.parent / "imgs" / "Q3a-torus.png")
    plt.clf()


if __name__ == "__main__":
    main()
```

Figure 2: The code used to generate the plot in Figure 1. The code can also be found on GitHub

Q3 (b)

Using the Divergence Theorem with $\underline{F}(x, y, z) = (x, 0, 0)$, calculate the volume of the torus.

The divergence of $\underline{F}$ is $\nabla \cdot \underline{F} = 1$. By the Divergence Theorem,

$$\iiint_V \nabla \cdot \underline{F} \, \mathrm{d}V = \iint_S \underline{F} \cdot \underline{\hat{n}} \, \mathrm{d}S$$

Therefore the volume of the torus is $\displaystyle\iint_S \underline{F} \cdot \underline{\hat{n}} \, \mathrm{d}S$.

To find the normal vector, we want the partial derivatives of $\underline{r}$:

$$\underline{r}_u = \begin{pmatrix} -\sin u \cos v \\ -\sin u \sin v \\ \cos u \end{pmatrix} \qquad \underline{r}_v = \begin{pmatrix} -(2 + \cos u) \sin v \\ (2 + \cos u) \cos v \\ 0 \end{pmatrix}$$

Then we cross them and get

$$\begin{aligned} \underline{r}_u \times \underline{r}_v &= \begin{pmatrix} -(2 + \cos u) \cos v \cos u \\ -(2 + \cos u) \sin v \cos u \\ -(2 + \cos u) \cos^2 v \sin u - (2 + \cos u) \sin^2 v \sin u \end{pmatrix} \\ &= \begin{pmatrix} -(2 + \cos u) \cos v \cos u \\ -(2 + \cos u) \sin v \cos u \\ -(2 + \cos u) \sin u \end{pmatrix} \end{aligned}$$

Just by thinking about the geometry of the parametrisation, we can see that as u increases, a point moves around the small circle anti-clockwise (in the poloidal direction) and as v increases, a point moves around the big circle anti-clockwise (in the toroidal direction).

Therefore, by the right hand rule, $\underline{r}_u \times \underline{r}_v$ will point inwards, so we want the negative version.

Therefore

$$\begin{aligned}
\iiint_V \mathrm{d}V &= \iiint_V \nabla \cdot \underline{F} \, \mathrm{d}V \\
&= \iint_S \underline{F} \cdot \underline{\hat{n}} \, \mathrm{d}S \\
&= \iint_S \underline{F} \cdot \frac{-(\underline{r}_u \times \underline{r}_v)}{\|\underline{r}_u \times \underline{r}_v\|} \|\underline{r}_u \times \underline{r}_v\| \, \mathrm{d}u \, \mathrm{d}v \\
&= \iint_S \underline{F} \cdot (-(\underline{r}_u \times \underline{r}_v)) \, \mathrm{d}u \, \mathrm{d}v \\
&= \iint_S \begin{pmatrix} (2+\cos u)\cos v \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} (2+\cos u)\cos v \cos u \\ (2+\cos u)\sin v \cos u \\ (2+\cos u)\sin u \end{pmatrix} \mathrm{d}u \, \mathrm{d}v \\
&= \iint_S (2+\cos u)^2 \cos^2 v \cos u \, \mathrm{d}u \, \mathrm{d}v \\
&= \int_0^{2\pi} \int_0^{2\pi} \cos^2 v (4\cos u + 4\cos^2 u + \cos^3 u) \ \mathrm{d}u \ \mathrm{d}v \\
&= \int_0^{2\pi} \cos^2 v \ \mathrm{d}v \int_0^{2\pi} (4\cos u + 4\cos^2 u + \cos^3 u) \ \mathrm{d}u \\
&= \frac{1}{2} \int_0^{2\pi} (\cos 2v + 1) \ \mathrm{d}v \int_0^{2\pi} (4\cos u + 2\cos 2u + 2 + \cos u (1 - \sin^2 u)) \ \mathrm{d}u \\
&= \frac{1}{2} \left[\frac{1}{2} \sin 2v + v \right]_0^{2\pi} \int_0^{2\pi} (2 + 5\cos u + 2\cos 2u - \cos u \sin^2 u) \ \mathrm{d}u \\
&= \frac{1}{2} (2\pi) \left[2u + 5\sin u + \sin 2u + \frac{1}{3} \sin^3 u \right]_0^{2\pi} \\
&= \pi (4\pi) \\
&= 4\pi^2
\end{aligned}$$

The torus in the question appears to have major radius $R = 2$ and minor radius $r = 1$, so my volume calculation matches the expected volume of $2\pi^2 R r^2 = 4\pi^2$ in this case.