MA146 Methods of Mathematical Modelling 1, Assignment 1

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Question 1

> For the following models, classify the system (discrete or continuous, deterministic or stochastic) and the questions (forward, inverse, or control problem)

Q1 (a)

> Suppose that the amount $u(t)$ of a drug in the blood stream satisfies a differential equation of the form
>
> $$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = s(t) - ru(t).$$
>
> Here, $s(t) \geq 0$ is the supply and $r > 0$ is a factor related to the degradation of the drug.
>
> From some blood samples taken at different times, therapists measure the drug amount and then want to determine r.

This is a continuous, deterministic system and an inverse problem.

Q1 (b)

> An internet server receives requests at random times and stores them in a queue. They are processed at a rate that depends on the number of requests in the queue (by allocating more computing nodes when the number is high).
>
> The managers want to allocate computing power such that all requests are processed within a certain time of arrival with a probability above 95%.

This is a discrete, stochastic system and a control problem.

Q1 (c)

> We consider how a piece of information spreads in a social network. By i_k we denoted the number of participants that are reached at a time $t_k = k\Delta t$ where $\Delta t > 0$ is a given time interval. We believe that
>
> $$i_k = i_{k-1} + ri_{k-1}(b - i_{k-1})$$
>
> Here, $r \in \mathbb{N}$ is a growth factor and b represents the maximal number of participants, both of which are known.
>
> We'd like to know how many participants are in the know after one hour, half a day, and two weeks.

This is a discrete, stochastic system and a forward problem.

Question 2

In this question we stuy linear second order equations with constant coefficients of the form

$$a\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) + b\frac{\mathrm{d}}{\mathrm{d}x}(t) + cx(t) = 0, \quad t \in \mathbb{R} \tag{1}$$

where $a, b, c \in \mathbb{R}$ are given numbers and $a \neq 0$. One way to find solutions of (1) is by substituting $x(t) = \mathrm{e}^{\lambda t}$ for some $\lambda \in \mathbb{C}$ into the equation. This yields $a\lambda^2 \mathrm{e}^{\lambda t} + b\lambda \mathrm{e}^{\lambda t} + c\mathrm{e}^{\lambda t} = 0$.

After dividing by $\mathrm{e}^{\lambda t} \neq 0$ we obtain the *auxiliary equation*

$$a\lambda^2 + b\lambda + c = 0.$$

Its solutions are given by

$$\lambda_{1,2} = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}.$$

If $b^2 - 4ac > 0$ then the solutions $\lambda_1 \neq \lambda_2$ are real. Two solutions to (1) then are given by $x_1(t) = \mathrm{e}^{\lambda_1 t}$ and $x_2(t) = \mathrm{e}^{\lambda_2 t}$.

The *complementary function* (or *general solution*)

$$x(t) = l_1 x_1(t) + l_2 x_2(t) = l_1 \mathrm{e}^{\lambda_1 t} + l_2 \mathrm{e}^{\lambda_2 r}, \quad l_1, l_2 \in \mathbb{R}$$

describes all solutions to (1).

If $b^2 - 4ac < 0$ then we write $\lambda_1 = -\frac{b}{2a} + \mathbf{i}\omega \in \mathbb{C}$ with $\omega = \frac{1}{2a}\sqrt{4ac - b^2} \in \mathbb{R}$ and note that $\lambda_2 = -\frac{b}{2a} - \mathbf{i}\omega = \overline{\lambda_1}$. Two real-values solutions of (1) then are $x_1(t) = \mathrm{e}^{-\frac{b}{2a}t}\cos(\omega t)$ and $x_2(t) = \mathrm{e}^{-\frac{b}{2a}t}\sin(\omega t)$, and the *complementary function* is given by

$$x(t) = l_1 x_1(t) + l_2 x_2(t) = l_1 \mathrm{e}^{-\frac{b}{2a}t}\cos(\omega t) + l_2 \mathrm{e}^{-\frac{b}{2a}t}\sin(\omega t), \quad l_1, l_2 \in \mathbb{R}$$

Q2 (a)

In the case $b^2 - 4ac < 0$ with complex solutions of the auxiliary equation, show that $x_2(t)$ is indeed a solution of (1).

We have $x_2(t) = e^{-\frac{b}{2a}t}\sin(\omega t)$ and derivatives

$$x_2'(t) = e^{-\frac{b}{2a}t}\omega\cos(\omega t) - \frac{b}{2a}e^{-\frac{b}{2a}t}\sin(\omega t)$$
$$= e^{-\frac{b}{2a}t}\left(\omega\cos(\omega t) - \frac{b}{2a}\sin(\omega t)\right)$$

$$x_2''(t) = e^{-\frac{b}{2a}t}\left(-\omega^2 - \frac{b\omega}{2a}\cos(\omega t) - \frac{b}{2a}\left(\omega\cos(\omega t) - \frac{b}{2a}\sin(\omega t)\right)\right)$$
$$= e^{-\frac{b}{2a}t}\left(\left(\frac{b^2}{4a^2} - \omega^2\right)\sin(\omega t) - \frac{b\omega}{a}\cos(\omega t)\right)$$

We plug these into (1) and get

$$e^{-\frac{b}{2a}t}\left(a\left(\left(\frac{b^2}{4a^2} - \omega^2\right)\sin(\omega t) - \frac{b\omega}{a}\cos(\omega t)\right)\right.$$
$$\left. + b\left(\omega\cos(\omega t) - \frac{b}{2a}\sin(\omega t)\right) + c\sin(\omega t)\right)$$
$$= e^{-\frac{b}{2a}t}\left(\frac{b^2}{4a} - a\omega^2 - \frac{b^2}{2a} + c\right)\sin(\omega t)$$
$$= e^{-\frac{b}{2a}t}\left(-\frac{b^2}{4a} - a\omega^2 + c\right)\sin(\omega t)$$

Recall that $\omega = \frac{1}{2a}\sqrt{4ac - b^2}$ so $\omega^2 = \frac{1}{4a^2}(4ac - b^2)$. Therefore (1) becomes

$$e^{-\frac{b}{2a}t}\left(-\frac{b^2}{4a} - \frac{4ac}{4a} + \frac{b^2}{4a} + c\right)\sin(\omega t)$$
$$= e^{-\frac{b}{2a}t} \times 0 \times \sin(\omega t)$$
$$= 0$$

as expected.

Q2 (b)

Find the complementary function of (1) with $a = 2$, $b = -4$, and $c = -6$.

$$b^2 - 4ac = 16 + 8 \times 6$$
$$= 64$$
$$\lambda_{1,2} = \frac{4 \pm \sqrt{64}}{4}$$
$$= 1 \pm 2$$
$$x(t) = l_1 e^{3t} + l_2 e^{-t}$$

Q2 (c)

Find the complementary function of (1) with $a = -2$, $b = 3$, and $c = -\frac{5}{4}$.

$$b^2 - 4ac = 9 - 5 \times 2$$
$$= -1$$
$$\lambda_{1,2} = \frac{3}{4} \pm \mathbf{i}\frac{1}{-4}\sqrt{1}$$
$$= \frac{3}{4} \mp \mathbf{i}\frac{2}{4}$$
$$x(t) = e^{\frac{3}{4}t}\left(l_1 \cos\frac{-t}{4} + l_2 \sin\frac{-t}{4}\right)$$

Q2 (d)

Find the complementary function solution of (1) with $a = -2$, $b = 0$, and $c = -18$ and then find numbers l_1, l_2 such that the solution also satisfies

$$x(2\pi) = 15, \quad \frac{d}{dt}x(2\pi) = 12.$$

$$b^2 - 4ac = 0 - 8 \times 18$$
$$= -144$$
$$\lambda_{1,2} = \frac{0 \pm 12\mathbf{i}}{-4}$$
$$= \pm 3\mathbf{i}$$
$$x(t) = e^{0t}\left(l_1 \sin 3t + l_2 \cos 3t\right)$$

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$$x(t) = l_1 \sin 3t + l_2 \cos 3t$$
$$x'(t) = 3l_1 \cos 3t - 3l_2 \sin 3t$$
$$x(2\pi) = l_1 \sin 6\pi + l_2 \cos 6\pi$$
$$= 0 + l_2$$
$$l_2 = 15$$
$$x'(2\pi) = 3l_1 \cos 6\pi - 3l_2 \sin 6\pi$$
$$= 3l_1 + 0$$
$$3l_1 = 12$$
$$l_1 = 4$$
$$\therefore \ x(t) = 4 \sin 3t + 15 \cos 3t$$

Q2 (e)

Download the Jupyter notebook `MA146_Assignment1.ipynb`. The problem from part **(d)** is symbolically solved and its solution is displayed.

Run the notebook with the parameters $a = 3$, $b = 12$, $c = 12$ and the initial conditions $x(0) = -4$, $\frac{\mathrm{d}}{\mathrm{d}x}(0) = 2$.

Submit the code where the initial condition is implemented and the output that displays the solution.

Furthermore, retrieve the solution $x(t)$ from the notebook and show with 'pen and paper' that it indeed satisfies the differential equation (1).

The solution given by the notebook is
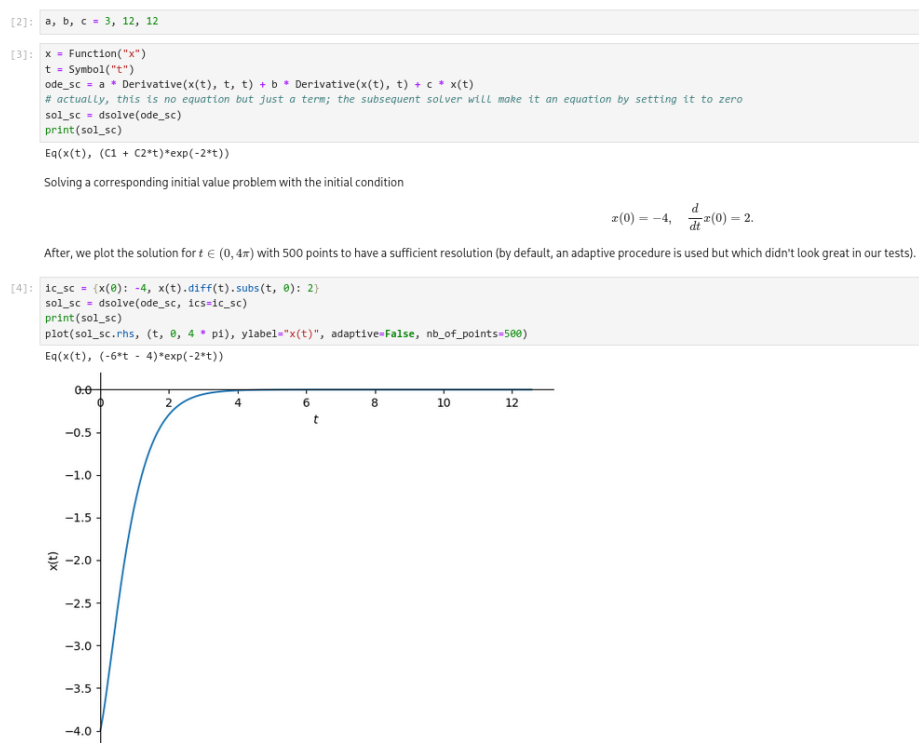
$$x(t) = (-6t - 4)\mathrm{e}^{-2t} = -(6t + 4)\mathrm{e}^{-2t}$$

We differentiate twice and get

$$x'(t) = 2(6t + 4)\mathrm{e}^{-2t} - 6\mathrm{e}^{-2t}$$
$$= (12t + 2)\mathrm{e}^{-2t}$$
$$x''(t) = -2(12t + 2)\mathrm{e}^{-2t} + 12\mathrm{e}^{-2t}$$
$$= (-24t + 8)\mathrm{e}^{-2t}$$

Plugging this into (1) gives

$$3(-24t + 8)\mathrm{e}^{-2t} + 12(12t + 2)\mathrm{e}^{-2t} - 12(6t + 4)\mathrm{e}^{-2t}$$
$$= \mathrm{e}^{-2t}\left(-72t + 24 + 144t + 24 - 72t - 48\right)$$
$$= \mathrm{e}^{-2t} \times 0$$
$$= 0$$

as expected.

```
[2]: a, b, c = 3, 12, 12
```

```
[3]: x = Function("x")
     t = Symbol("t")
     ode_sc = a * Derivative(x(t), t, t) + b * Derivative(x(t), t) + c * x(t)
     # actually, this is no equation but just a term; the subsequent solver will make it an equation by setting it to zero
     sol_sc = dsolve(ode_sc)
     print(sol_sc)
```

Eq(x(t), (C1 + C2*t)*exp(-2*t))

Solving a corresponding initial value problem with the initial condition

$$x(0) = -4, \quad \frac{d}{dt}x(0) = 2.$$

After, we plot the solution for $t \in (0, 4\pi)$ with 500 points to have a sufficient resolution (by default, an adaptive procedure is used but which didn't look great in our tests).

```
[4]: ic_sc = {x(0): -4, x(t).diff(t).subs(t, 0): 2}
     sol_sc = dsolve(ode_sc, ics=ic_sc)
     print(sol_sc)
     plot(sol_sc.rhs, (t, 0, 4 * pi), ylabel="x(t)", adaptive=False, nb_of_points=500)
```

Eq(x(t), (-6*t - 4)*exp(-2*t))

Figure 1: The code for **Q2 (e)**

Q2 (f)

Consider now the parameters $a = 1.0$, $b = 0.2$, and $c = 8.0$. Use the notebook to solve the differential equation three times with initial data $x(0) \in \{1.5, 2.0, 2.7\}$ and always $\frac{\mathrm{d}}{\mathrm{d}t}x(0) = 0$.

Generate a parametric plot displaying all solutions for $t \in (0, 4\pi)$ in the same graph.

```
[4]:  def get_ics(x0):
          return {x(0): x0, x(t).diff(t).subs(t, 0): 0}

      initial_x0_list = [1.5, 2, 2.7]

      sols_sc = [dsolve(ode_sc, ics=get_ics(x0)) for x0 in initial_x0_list]
      print(sols_sc)
      # plot(sol_sc.rhs, (t, 0, 4 * pi), ylabel="x(t)", adaptive=False, nb_of_points=500)
```

```
[Eq(x(t), (-0.053066185325792*sin(2.82665880502051*t) + 1.5*cos(2.82665880502051*t))*exp(0.
35864255*sin(2.82665880502051*t) + 2.7*cos(2.82665880502051*t))*exp(0.1*t))]
```

Retrieve the derivative of the solution and plot it (in orange) together with the solution (in blue as before).

Another popular way to illustrate the solution is via a parametric plot in the phase space. We will look at it

```
[5]:  plot_parametric(*[(sol.rhs, sol.rhs.diff(t)) for sol in sols_sc], (t, 0, 4 * pi))
```
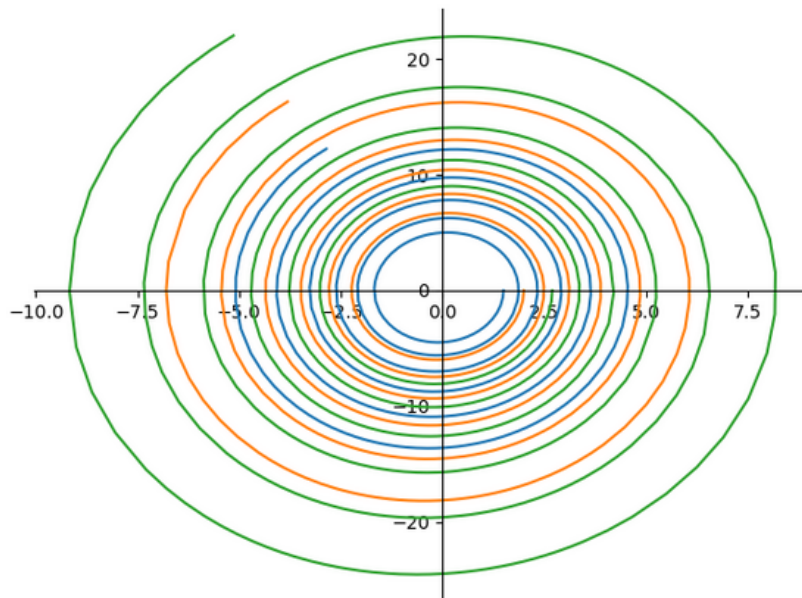

Figure 2: The code for **Q2 (f)**

Question 3

Let us work out the height $h(t)$ as a function of time t during a bungee jump.

Assuming that the jumper just drops the initial velocity is zero, $\dot{h}(0) = 0$, and to have a height reference we also set the initial height to zero, $h(0) = 0$. By Newton's law

$$m\ddot{h}(t) = f(t), \quad t > 0$$

where $m > 0$ is the mass of the jumper (assumed a point mass, we don't account for their shape) and f is the force acting on them. We neglect air resistance and for now assume that the cliff or bridge is high enough so that the jumper could enjoy it a second time (so $h(t)$ is unlimited from below).

Denote the length of the rope with $\ell > 0$. Whilst the jumper is in free fall, gravitation is the only force acting on them, hence

$$f(t) = -mg, \quad \text{when } h(t) \geq -\ell$$

with the gravitation constant g. Once the rope gets extended, i.e. when $h(t) \leq -\ell$, it exerts a fall-damping tension force that we assume to be proportional to the extension.

Denoting the proportionality factor by $k > 0$ the force then is

$$f(t) = k(-\ell - h(t)) - mg \quad \text{when } h(t) \leq -\ell.$$

Q3 (a)

Find $h(t)$ as a function of m, g, ℓ, k during the first free fall stage and the subsequent re-bounce stage.

Jumpers usually bounce back and get into a free fall stage a few times, but which we ignore here for simplicity.

Hint: for the second stage you could try with a solution of the form as in the previous question plus a constant.

We will start by considering free fall. We know $m\ddot{h}(t) = f(t) = -mg$, so $\ddot{h}(t) = -g$. Integrating twice, we get that

$$h(t) = -gt^2 + C_1 t + C_2$$

and the intitial conditions of $h(0) = \dot{h}(0) = 0$ tell us that $C_1 = C_2 = 0$, so

$$h(t) = -gt^2 \qquad \text{(in free fall)}$$

Now we will consider the re-bounce. We derive the ODE like so:

$$m\ddot{h}(t) = f(t) = k\left(-\ell - h(t)\right) - mg$$

$$\ddot{h}(t) = \frac{k}{m}\left(-\ell - h(t)\right) - g$$

$$\ddot{h}(t) = -\frac{k\ell}{m} - h(t)\frac{k}{m} - g$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}h(t) + \frac{k}{m}h(t) = -\frac{k\ell}{m} - g \qquad\qquad (*)$$

The auxiliary equation is $\lambda^2 + \dfrac{k}{m} = 0$ which has solutions $\lambda_{1,2} = \pm\sqrt{-\dfrac{k}{m}}$. And $k, m > 0$ so $\lambda_{1,2} = \pm\mathbf{i}\sqrt{\dfrac{k}{m}}$.

Therefore the complementary function is

$$h(t) = C_1 \mathrm{e}^{\mathbf{i}t\sqrt{\frac{k}{m}}} + C_2 \mathrm{e}^{-\mathbf{i}t\sqrt{\frac{k}{m}}}$$

$$= C_1\left(\cos\sqrt{\frac{k}{m}}t + \mathbf{i}\sin\sqrt{\frac{k}{m}}t\right) + C_2\left(\cos\sqrt{\frac{k}{m}}t - \mathbf{i}\sin\sqrt{\frac{k}{m}}t\right)$$

$$= (C_1 + C_2)\cos\sqrt{\frac{k}{m}}t + (C_1 - C_2)\mathbf{i}\sin\sqrt{\frac{k}{m}}t$$

$$= A\cos\sqrt{\frac{k}{m}}t + B\sin\sqrt{\frac{k}{m}}t$$

The particular solution wants $h(t)$ to be some constant to reflect the constant $-\dfrac{k\ell}{m} - g$ on the right hand side of (*). Let $h(t)$ be some constant C. Then $\ddot{h}(t) = 0$. We plug that into (*) and get

$$0 + \frac{k}{m}C = -\frac{k\ell}{m} - g$$

so then $C = -\ell - \dfrac{mg}{k}$.

We can then add the particular solution to the complementary function and get

$$h(t) = A\cos\sqrt{\frac{k}{m}}t + B\sin\sqrt{\frac{k}{m}}t - \ell - \frac{mg}{k}$$

Then we can use the initial conditions of $h(0) = \dot{h}(0) = 0$ to get the final solution.

$$h(0) = A\cos 0 + B\sin 0 - \ell - \frac{mg}{k}$$

$$0 = A - \ell - \frac{mg}{k}$$

$$A = \ell + \frac{mg}{k}$$

$$\dot{h}(t) = -A\sqrt{\frac{k}{m}}\sin\sqrt{\frac{k}{m}}t + B\sqrt{\frac{k}{m}}\cos\sqrt{\frac{k}{m}}t$$

$$\dot{h}(0) = -A\sqrt{\frac{k}{m}}\sin 0 + B\sqrt{\frac{k}{m}}\cos 0$$

$$0 = \sqrt{\frac{k}{m}}B$$

$$B = 0$$

$$\therefore\ h(t) = \left(\ell + \frac{mg}{k}\right)\cos\sqrt{\frac{k}{m}}t - \ell - \frac{mg}{k}$$

$$= \left(\ell + \frac{mg}{k}\right)\left(\cos\sqrt{\frac{k}{m}}t - 1\right)$$

Q3 (b)

This part is not for credit.

Suppose that the cliff's/bridge's height is limited to $h_m < 0$. How long may the rope at most be to avoid hitting the ground?

In the re-bounce stage, $h(t)$ is basically just a scaled cosine function. That means the minimum of $h(t)$ in this stage is

$$\left(\ell + \frac{mg}{k}\right)(-1 - 1) = -2\left(\ell + \frac{mg}{k}\right)$$

If we want the jumper to just touch the floor in the worst case, then

$$h_m \leq -2\left(\ell + \frac{mg}{k}\right)$$

We can then simply rearrange to find ℓ.

$$-\frac{h_m}{2} \geq \ell + \frac{mg}{k}$$

$$\therefore\ \ell \leq -\left(\frac{h_m}{2} + \frac{mg}{k}\right)$$