MA2K4 Numerical Methods and Computing, Assignment 1

Dyson Dyson

Question 1

For a vector $x \in \mathbb{R}^m$ and a matrix $A \in \mathbb{R}^{n\times n}$, and any norm $\|\cdot\|$ on $\mathbb{R}^n$, show that

$$\|A\|_{\text{op}} = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|}$$

is a norm on $\mathbb{R}^{n\times n}$.

We need to show three things:

1) $\|A\|_{\text{op}} \geq 0$ for all $A \in \mathbb{R}^{n\times n}$, and $\|A\|_{\text{op}} = 0$ if and only if $A = 0$;
2) $\|\lambda A\|_{\text{op}} = |\lambda| \, \|A\|_{\text{op}}$ for all $A \in \mathbb{R}^{n\times n}$ and $\lambda \in \mathbb{R}$;
3) $\|A + B\|_{\text{op}} \leq \|A\|_{\text{op}} + \|B\|_{\text{op}}$ for all $A, B \in \mathbb{R}^{n\times n}$.

Since $\|\cdot\|$ on $\mathbb{R}^m$ has the first property and we take the supremum over $x \neq 0$, we see that $\|\cdot\|_{\text{op}} \geq 0$ has the first property.

As before, $\|\cdot\|$ on $\mathbb{R}^m$ has the second property, so

$$\begin{aligned}
\|\lambda A\|_{\text{op}} &= \sup_{x\neq 0} \frac{\|\lambda Ax\|}{\|x\|} \\
&= \sup_{x\neq 0} \frac{|\lambda| \, \|Ax\|}{\|x\|} \\
&= \sup_{x\neq 0} |\lambda| \frac{\|Ax\|}{\|x\|} \\
&= |\lambda| \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|} \\
&= |\lambda| \|A\|_{\text{op}}
\end{aligned}$$

as required.

Since $\|\cdot\|$ on $\mathbb{R}^m$ has the triangle inequality, and since $\|\cdot\|_{\mathrm{op}}$ has property 2,

$$\begin{aligned}\|(A+B)x\| &= \|Ax + Bx\| \\ &\leq \|Ax\| + \|Bx\| \\ &\leq \big(\|A\|_{\mathrm{op}} + \|B\|_{\mathrm{op}}\big)\|x\|,\end{aligned}$$

and therefore

$$\begin{aligned}\|A+B\|_{\mathrm{op}} &= \sup_{x\neq 0} \frac{\|(A+B)x\|}{\|x\|} \\ &\leq \|A\|_{\mathrm{op}} + \|B\|_{\mathrm{op}}.\end{aligned}$$

Question 2

Compute the condition number $K(d)$ in the $|\cdot|$-norm on $\mathbb{R}$ for the following problems:

(a) $x - a^d = 0$ with $a > 0$;

(b) the area of a triangle with all three sides having lengths d;

where d is the datum, and x is the 'unknown'. Comment on values of d for which the problem is ill-posed. Compute the condition number via the derivative of the resolvent.

Q2 (a)

Since a is a fixed constant, the problem is always well-posed.

We want $x = a^d$ and so $x = G(d) = a^d$. Then $G'(d) = a^d \log a$. Then we have

$$\begin{aligned} K(d) &\approx |G'(d)| \frac{|d|}{|G(d)|} \\ &= |a^d \log a| \frac{|d|}{|a^d|} \\ &= |d \log a|. \end{aligned}$$

Q2 (b)

The problem is well-posed when $d > 0$.

Any triangle with all three sides of equal length must be equilateral, so its area is $x = G(d) = \dfrac{d^2\sqrt{3}}{4}$. Then $G'(d) = \dfrac{d\sqrt{3}}{2}$. Then we have

$$\begin{aligned} K(d) &\approx |G'(d)| \frac{|d|}{|G(d)|} \\ &= \left|\frac{d\sqrt{3}}{2}\right| \frac{|d|}{\left|\frac{d^2\sqrt{3}}{4}\right|} \\ &= \frac{|d|\sqrt{3}}{2} \frac{4|d|}{d^2\sqrt{3}} \\ &= \frac{4d^2\sqrt{3}}{2d^2\sqrt{3}} \\ &= 2. \end{aligned}$$

Question 3

For the system of equations

$$\begin{aligned} x + dy &= 0 \\ dx + dy &= 1, \end{aligned} \tag{1}$$

consider the following questions:

(a) Considering the vector (x, y) as the unknown and d as given datum, for what values of d is the problem well-posed?

(b) Now, consider only x the unknown. For what values of d does a unique solution exist for this problem?

(c) Again, considering x the unknown, what is the condition number $K(d)$ in the $|\cdot|$-norm on $\mathbb{R}$?

(d) Still considering x the unknown, for what values of d is the problem well-conditioned? In this case, we demand the condition number to be smaller than 10.

Compute the condition number via the derivative of the resolvent.

Q3 (a)

The problem is well-posed when the matrix $\begin{pmatrix} 1 & d \\ d & d \end{pmatrix}$ has non-zero determinant, so whenever $d - d^2 \neq 0$. We can factorise this into $d(1-d) \neq 0$, so we only need to require that $d \neq 0$ and $d \neq 1$. Any other value of d gives a well-posed problem.

Q3 (b)

If x is the only unknown then we need $x = -dy$ to satisfy the first equation. Then we need $-d^2y + dy = 1$ for the second equation, so we get

$$\begin{aligned} yd^2 - yd + 1 = 0 \\ d &= \frac{y \pm \sqrt{y^2 - 4y}}{2y} \\ &= \frac{y \pm \sqrt{y(y-4)}}{2y} \end{aligned}$$

as the only values of d that admit a unique solution.

Q3 (c)

There are two resolvent maps, which must agree. In the first case, $x = G(d) = -dy$. Then $G'(d) = -y$ so the condition number is

$$\begin{aligned} K(d) &\approx |G'(d)| \frac{|d|}{|G(d)|} \\ &= |-y| \frac{|d|}{|-dy|} \\ &= |y| \frac{|d|}{|d|\,|y|} \\ &= 1 \end{aligned}$$

In the second case, $x = G(d) = \frac{1}{d} - y$. Then $G'(d) = -\frac{1}{d^2}$ so the condition number is

$$\begin{aligned} K(d) &\approx |G'(d)| \frac{|d|}{|G(d)|} \\ &= \left| -\frac{1}{d^2} \right| \frac{|d|}{\left| \frac{1}{d} - y \right|} \\ &= \frac{1}{|d|^2} \frac{|d|}{\left| \frac{1}{d} - y \right|} \\ &= \frac{1}{|d| \left| \frac{1}{d} - y \right|} \\ &= \frac{1}{|1 - dy|} \end{aligned}$$

Of course these condition number must agree, so we require that $|1 - dy| = 1$, so $dy = 0$ or $dy = 2$.

Q3 (d)

The condition number is 1 and 1 is always less than 10, so the problem is always well-conditioned.

Question 4

Consider the problem of finding the solution to a quadratic equation

$$x^2 + 2px - q = 0,$$

which has the solutions

$$x_\pm = -p \pm \sqrt{p^2 + q}.$$

Consider each solution as a separate problem on $\mathbb{R}$. Study the conditioning of these two problems:

(a) Consider first p as the datum to compute $K(p)$ in the $|\cdot|$-norm on $\mathbb{R}$. Simplify as much as possible to show that it does not depend on whether one considers x_+ or x_-.

(b) Now, consider instead q as the datum to compute $K(q)$ in the $|\cdot|$-norm on $\mathbb{R}$.

Compute the condition number via the derivative of the resolvent.

Q4 (a)

We shall first consider x_+. Then

$$G_+(p) = -p + \sqrt{p^2 + q}$$

and

$$G'_+(p) = -1 + \frac{p}{\sqrt{p^2 + q}}.$$

Therefore

$$
\begin{aligned}
K_+(p) &\approx \left|-1 + \frac{p}{\sqrt{p^2+q}}\right| \frac{|p|}{\left|-p+\sqrt{p^2+q}\right|} \\
&= \frac{\left|-1 + \frac{p}{\sqrt{p^2+q}}\right| |p|}{\left|-p+\sqrt{p^2+q}\right|} \\
&= \frac{\left|-p + \frac{p^2}{\sqrt{p^2+q}}\right|}{\left|-p+\sqrt{p^2+q}\right|} \\
&= \left|\frac{-p + \frac{p^2}{\sqrt{p^2+q}}}{-p+\sqrt{p^2+q}}\right| \\
&= \left|\frac{p}{\sqrt{p^2+q}}\right|
\end{aligned}
$$

And in the case of x_-, we get

$$G_-(p) = -p - \sqrt{p^2+q}$$

and

$$G'_-(p) = -1 - \frac{p}{\sqrt{p^2+q}}.$$

Therefore

$$
\begin{aligned}
K_-(p) &\approx \left| -1 - \frac{p}{\sqrt{p^2+q}} \right| \frac{|p|}{\left| -p - \sqrt{p^2+q} \right|} \\
&= \frac{\left| -1 - \frac{p}{\sqrt{p^2+q}} \right| |p|}{\left| -p - \sqrt{p^2+q} \right|} \\
&= \frac{\left| -p - \frac{p^2}{\sqrt{p^2+q}} \right|}{\left| -p - \sqrt{p^2+q} \right|} \\
&= \frac{\left| p + \frac{p^2}{\sqrt{p^2+q}} \right|}{\left| p + \sqrt{p^2+q} \right|} \\
&= \left| \frac{p + \frac{p^2}{\sqrt{p^2+q}}}{p + \sqrt{p^2+q}} \right| \\
&= \left| \frac{p}{\sqrt{p^2+q}} \right|
\end{aligned}
$$

And therefore $K_+(p) = K_-(p)$.

Q4 (b)

Again, we shall first consider x_+. Then

$$G_+(q) = -p + \sqrt{p^2+q}$$

and

$$G'_+(q) = \frac{1}{2\sqrt{p^2+q}}.$$

Therefore

$$
\begin{aligned}
K_+(q) &\approx \left| \frac{1}{2\sqrt{p^2+q}} \right| \frac{|q|}{\left| -p + \sqrt{p^2+q} \right|} \\
&= \frac{|q|}{\left| 2\sqrt{p^2+q}\left(-p + \sqrt{p^2+q} \right) \right|} \\
&= \frac{|q|}{2\left| -p\sqrt{p^2+q} + p^2 + q \right|}
\end{aligned}
$$

And in the case of x_-, we get

$$G_-(q) = -p - \sqrt{p^2+q}$$

and

$$G'_-(q) = -\frac{1}{2\sqrt{p^2+q}}.$$

Therefore

$$\begin{aligned}
K_-(q) &\approx \left|-\frac{1}{2\sqrt{p^2+q}}\right| \frac{|q|}{\left|-p-\sqrt{p^2+q}\right|} \\
&= \left|\frac{1}{2\sqrt{p^2+q}}\right| \frac{|q|}{\left|p+\sqrt{p^2+q}\right|} \\
&= \frac{|q|}{\left|2\sqrt{p^2+q}\left(p+\sqrt{p^2+q}\right)\right|} \\
&= \frac{|q|}{2\left|p\sqrt{p^2+q}+p^2+q\right|}
\end{aligned}$$

Question 5

Write down the linear interpolant $p_1(x)$ for the function $f(x) = x^3 - x$, using the nodes $x_0 = 0$ and $x_1 = a$.

Denote by

$$e(x) = f(x) - p_1(x)$$

the error that we produced in the linear interpolation. Show that this error fulfils

$$e(x) = \frac{1}{2} f''(\xi) x(x - a)$$

for the unique $\xi = \frac{1}{3}(x + a)$.

$$\begin{aligned}
p_1(x) &= f(x_0) + \left(\frac{f(x_1) - f(x_0)}{x_1 - x_0} \right) (x - x_0) \\
&= f(0) + \left(\frac{f(a) - f(0)}{a - 0} \right) (x - 0) \\
&= \left(\frac{f(a)}{a} \right) x \\
&= \frac{x(a^3 - a)}{a} \\
&= x(a^2 - 1)
\end{aligned}$$

We will need the second derivative.

$$\begin{aligned}
f'(x) &= 3x^2 - 1 \\
f''(x) &= 6x
\end{aligned}$$

Then we get

$$\begin{aligned}
e(x) &= f(x) - p_1(x) \\
&= x^3 - x - x(a^2 - 1) \\
&= x^3 - x - xa^2 + x \\
&= x^3 - xa^2 \\
&= x(x^2 - a^2) \\
&= x(x + a)(x - a) \\
&= \frac{1}{2} f''(\xi) x(x - a) \\
&= 3\xi x(x - a),
\end{aligned}$$

which implies that $\xi = \frac{1}{3}(x + a)$ as required.

Question 6

Repeat this calculation for the function $f(x) = (x-a)^4$ for the same nodes, and show that in this case there are two possible values for ξ. Give their values.

$$\begin{aligned}
p_1(x) &= f(x_0) + \left(\frac{f(x_1) - f(x_0)}{x_1 - x_0}\right)(x - x_0) \\
&= f(0) + \left(\frac{f(a) - f(0)}{a - 0}\right)(x - 0) \\
&= f(0) + \left(\frac{-f(0)}{a}\right)x \\
&= (-a)^4 + \frac{-x(-a)^4}{a} \\
&= a^4 - xa^3
\end{aligned}$$

The second derivative is

$$\begin{aligned}
f'(x) &= 4(x-a)^3 \\
f''(x) &= 12(x-a)^2
\end{aligned}$$

And so the error is

$$\begin{aligned}
e(x) &= f(x) - p_1(x) \\
&= (x-a)^4 - a^4 + xa^3 \\
&= x^4 - 4x^3a + 6x^2a^2 - 4xa^3 + a^4 - a^4 + xa^3 \\
&= x^4 - 4x^3a + 6x^2a^2 - 3xa^3 \\
&= x(x^3 - 4x^2a + 6xa^2 - 3a^3) \\
&= \frac{1}{2}f''(\xi)x(x-a) \\
&= \frac{1}{2}\cdot 12(\xi - a)^2 x(x-a) \\
&= 6(\xi - a)^2 x(x-a) \\
&= 6(\xi^2 - 2\xi a + a^2)x(x-a)
\end{aligned}$$