MA146 Assignment 1

Q1

(a) Continuous, deterministic system and an inverse problem.

(b) Discrete, stochastic system and a control problem.

(c) Discrete, stochastic system and a forward problem.

Q2

(a) $b^2 - 4ac < 0 \quad x_2(t) = e^{-\frac{b}{2a}t} \sin(\omega t)$

$$x_2'(t) = e^{-\frac{b}{2a}t}\omega\cos(\omega t) - \frac{b}{2a}e^{-\frac{b}{2a}t}\sin(\omega t)$$

$$= e^{-\frac{b}{2a}t}\left(\omega\cos(\omega t) - \frac{b}{2a}\sin(\omega t)\right)$$

$$x_2''(t) = e^{-\frac{b}{2a}t}\Big(-\omega^2\sin(\omega t) - \frac{b\omega}{2a}\cos\omega t - \frac{b}{2a}\left(\omega\cos(\omega t) - \frac{b}{2a}\sin(\omega t)\right)\Big)$$

$$= e^{-\frac{b}{2a}t}\left(\left(\frac{b^2}{4a^2} - \omega^2\right)\sin\omega t - \frac{b}{a}\omega\cos\omega t\right)$$

We plug these ~~into~~ ~~the~~ ODE and get

$$e^{-\frac{b}{2a}t}\left(a\left(\frac{b^2}{4a^2} - \omega^2\right)\sin\omega t - b\omega\cos\omega t + b\omega\cos\omega t - \frac{b^2}{2a}\sin\omega t + c\sin\omega t\right)$$

$$= e^{-\frac{b}{2a}t}\cancel{\frac{b^2-2b^2}{4a}}\left(\frac{b^2}{4a} - \frac{b^2}{2a} - a\omega^2 + c\right)\sin\omega t$$

Q2 (a) cont.

$$= e^{-\frac{b}{2a}t}\left(\frac{-b^2}{4a} - a\omega^2 + c\right)\sin\omega t$$

Recall ~~that~~ $\omega = \frac{1}{2a}\sqrt{4ac - b^2}$ $\therefore \omega^2 = \frac{1}{4a^2}(4ac - b^2)$

$$\therefore ODE = e^{-\frac{b}{2a}t}\left(\frac{-b^2}{4a} - \frac{4ac}{4a} + \frac{b^2}{4a} + c\right)\sin\omega t$$

$$= e^{-\frac{b}{2a}t} \times 0 \times \sin\omega t = 0 \quad \text{as expected.}$$

(b) $a=2, b=-4, c=-6$

$b^2 - 4ac = 16 + 8\times 6 = 64$ $\quad \lambda_{1,2} = \frac{4 \pm \sqrt{64}}{4} = 1 \pm 2$

$\therefore x(t) = C_1 e^{3t} + C_2 e^{-t}$

(c) $a = -2, b = 3, c = \frac{-5}{4}$

$b^2 - 4ac = 9 - 5\times 2 = -1$ $\quad \lambda_{1,2} = \frac{3}{4} \pm i\frac{1}{-4}\sqrt{1}$

$\omega = \frac{-1}{4} \therefore \lambda_{1,2} = \frac{3}{4} \pm i\frac{-1}{4}$

$\therefore x(t) = e^{\frac{3}{4}t}\left(C_1\cos\frac{-t}{4} + C_2\sin\frac{-t}{4}\right) = e^{\frac{3}{4}t}\left(C_1\cos\frac{-t}{4} + C_2\sin\frac{-t}{4}\right)$

(d) $b^2 - 4ac = -8\times 18 = -144$ $\quad \lambda_{1,2} = \frac{0 \pm 12i}{-4} = \pm 3i$

$a = -2, b = 0, c = -18$

~~$x(t) = C_1 e^{3t} + C_2 e^{-3t}$~~ $\therefore x(t) = e^{0t}(C_1\sin 3t + C_2\cos 3t)$

Q2 (d) cont.

$x(t) = c_1 \sin 3t + c_2 \cos 3t$

$x'(t) = 3c_1 \cos 3t - 3c_2 \sin 3t$

$x'(2\pi) = 3c_1 \cos 6\pi - 3c_2 \sin 6\pi = 3c_1 - 0 = 12 \therefore c_1 = 4$

$x(2\pi) = c_1 \sin 6\pi + c_2 \cos 6\pi = 0 + c_2 = 15 \therefore c_2 = 15$

$\therefore x(t) = 4 \sin 3t + 15 \cos 3t$

(e) $a = 3, b = 12, c = 12$ Jupyter notebook says

$x(t) = (-6t - 4)e^{-2t}$
$= -(6t + 4)e^{-2t}$

$\therefore x'(t) = 2(6t + 4)e^{-2t} - 6e^{-2t}$
$= (12t + 2)e^{-2t}$

$x''(t) = -2(12t + 2)e^{-2t} + 12e^{-2t} = (-24t + 8)e^{-2t}$

$\cancel{\therefore ODE = -12(6t+4)e^{-2t} + 12(12t+2)e^{-2t}}$

$ODE = 3(-24t + 8)e^{-2t} + 12(12t + 2)e^{-2t} - 12(6t + 4)e^{-2t}$
$= e^{-2t}(-72t + 24 + 144t + 24 - 72t - 48)$
$= e^{-2t} \times 0 = 0$ as expected

[2]:
```python
a, b, c = 3, 12, 12
```

[3]:
```python
x = Function("x")
t = Symbol("t")
ode_sc = a * Derivative(x(t), t, t) + b * Derivative(x(t), t) + c * x(t)
# actually, this is no equation but just a term; the subsequent solver will make it an equation by setting it to zero
sol_sc = dsolve(ode_sc)
print(sol_sc)
```

```
Eq(x(t), (C1 + C2*t)*exp(-2*t))
```

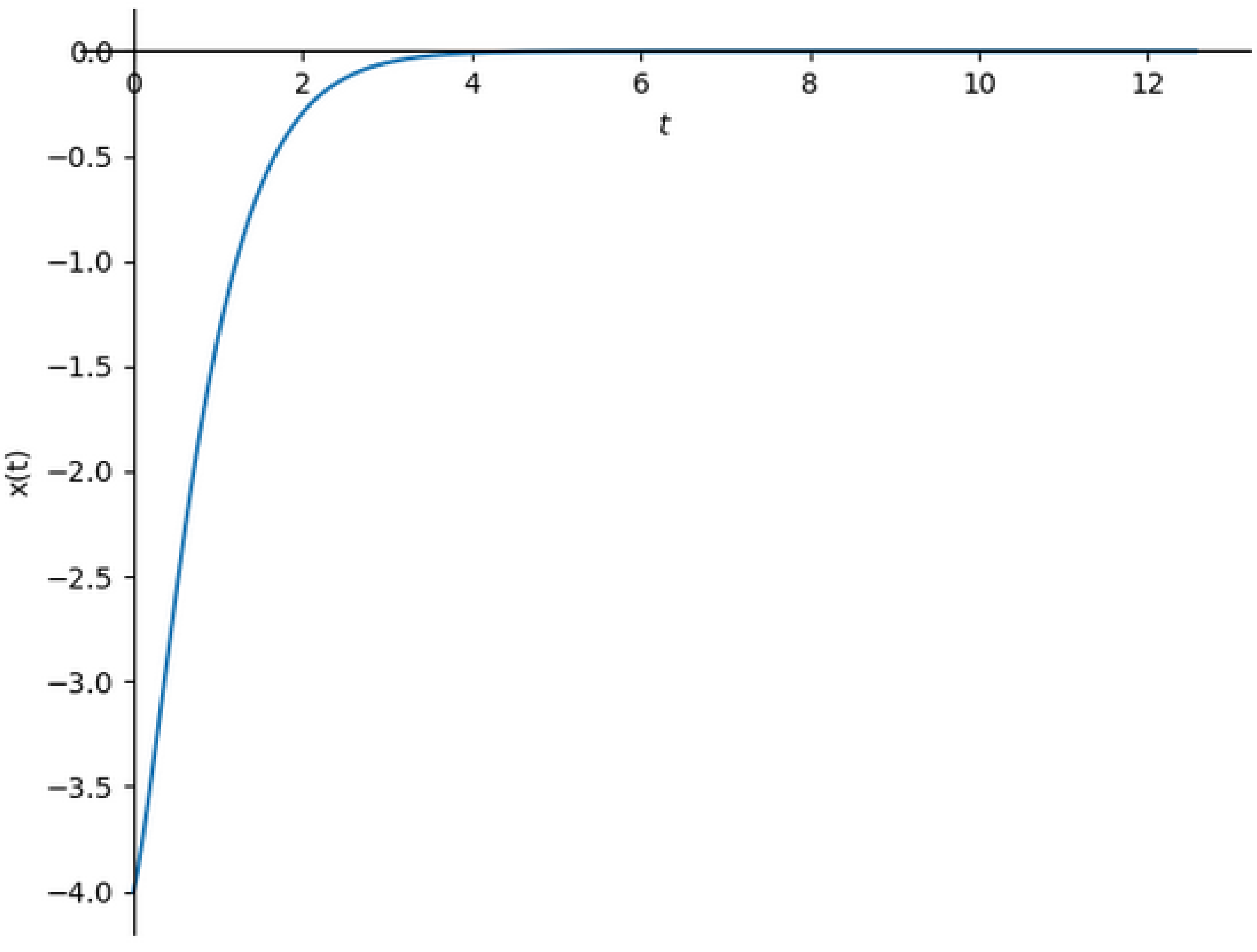
Solving a corresponding initial value problem with the initial condition

$$x(0) = -4, \quad \frac{d}{dt}x(0) = 2.$$

After, we plot the solution for $t \in (0, 4\pi)$ with 500 points to have a sufficient resolution (by default, an adaptive procedure is used but which didn't look great in our tests).

[4]:
```python
ic_sc = {x(0): -4, x(t).diff(t).subs(t, 0): 2}
sol_sc = dsolve(ode_sc, ics=ic_sc)
print(sol_sc)
plot(sol_sc.rhs, (t, 0, 4 * pi), ylabel="x(t)", adaptive=False, nb_of_points=500)
```

```
Eq(x(t), (-6*t - 4)*exp(-2*t))
```



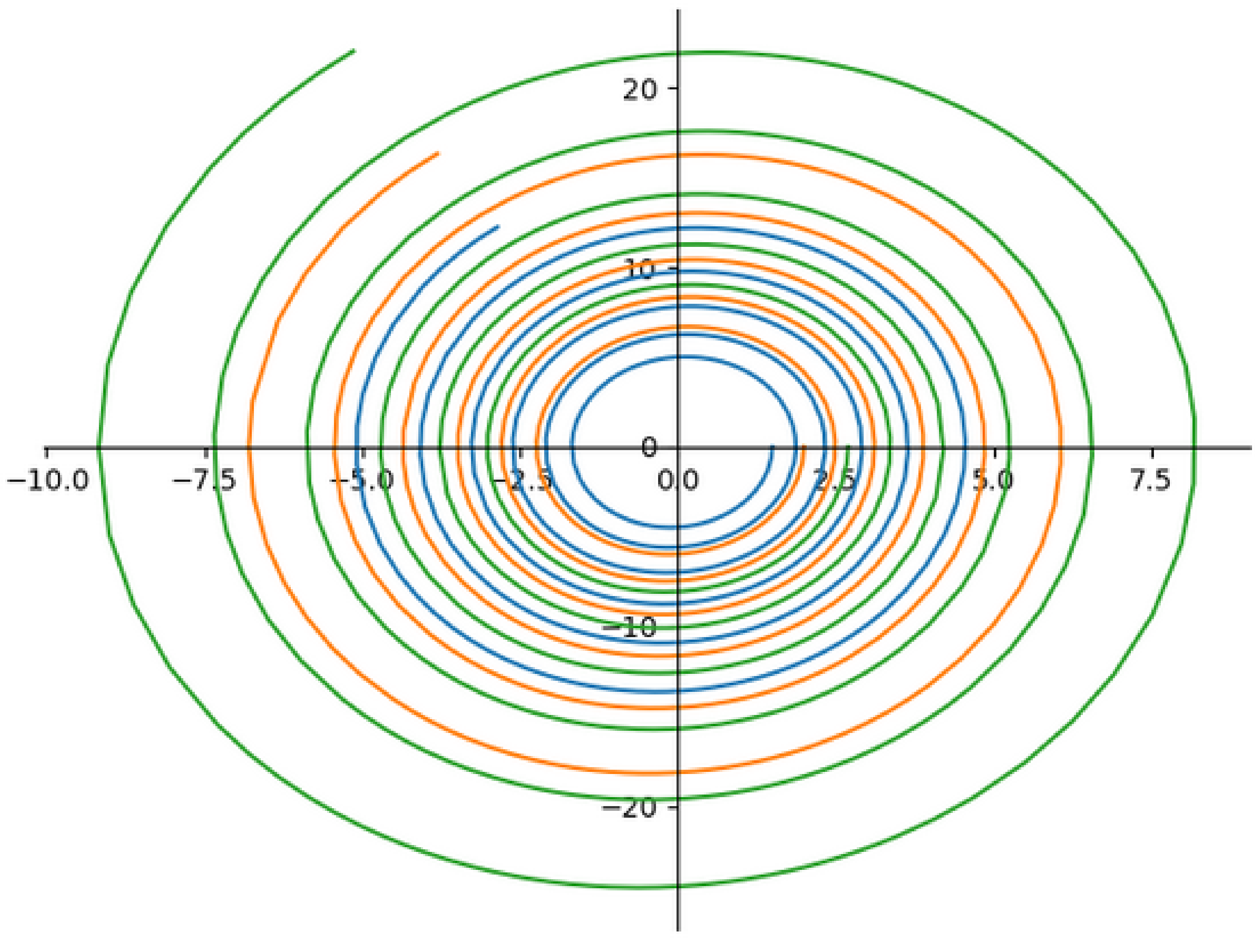
[4]:
```python
def get_ics(x0):
    return {x(0): x0, x(t).diff(t).subs(t, 0): 0}

initial_x0_list = [1.5, 2, 2.7]

sols_sc = [dsolve(ode_sc, ics=get_ics(x0)) for x0 in initial_x0_list]
print(sols_sc)
# plot(sol_sc.rhs, (t, 0, 4 * pi), ylabel="x(t)", adaptive=False, nb_of_points=500)
```

```
[Eq(x(t), (-0.053066185325792*sin(2.82665880502051*t) + 1.5*cos(2.82665880502051*t))*exp(0.
35864255*sin(2.82665880502051*t) + 2.7*cos(2.82665880502051*t))*exp(0.1*t))]
```

Retrieve the derivative of the solution and plot it (in orange) together with the solution (in blue as before).

Another popular way to illustrate the solution is via a parametric plot in the phase space. We will look at it

[5]:
```python
plot_parametric(*[(sol.rhs, sol.rhs.diff(t)) for sol in sols_sc], (t, 0, 4 * pi))
```



~~Q2 (f)~~

Q3 (a) During free fall: $\boxed{h(0)=0}$ $\boxed{\dot{h}(0)=0}$

$$m\ddot{h}(t) = f(t) = -mg$$

$$\Rightarrow \ddot{h}(t) = -g$$

$\therefore$ In free fall, $\boxed{h(t) = -gt^2}$ when $h(t) \geq -l$.

We don't have to worry about terms in t or constants because of the initial conditions being 0. They are both satisfied by $h(t) = -gt^2$.

During re-bounce: $(h(t) \leq -l)$

$$m\ddot{h}(t) = f(t) = k(-l - h(t)) - mg$$

$$\Rightarrow \ddot{h}(t) = \frac{k}{m}(-l - h(t)) - g$$

$$\Rightarrow \frac{d^2}{dt^2}h(t) + \frac{k}{m}h(t) = -\frac{kl}{m} - g$$

~~$\therefore$ $\lambda^2 + \frac{k}{m} = 0$ is the auxiliary equation, so $\lambda = \pm\sqrt{\frac{-k}{m}}$~~

~~$k, m > 0$, so $\lambda = \pm i\sqrt{\frac{k}{m}}$~~

~~$\therefore$ $\lambda^2 + \frac{k}{m} = -\frac{kl}{m} - g$ is the auxiliary equation~~

I was right the first time.

$\lambda^2 + \frac{k}{m} = 0$ is the auxiliary equation for the complementary function.

Q3 (a) $\therefore \lambda = \pm\sqrt{\frac{-k}{m}}$ $k, m > 0$, so $\lambda = \pm i\sqrt{\frac{k}{m}}$

Therefore the complementary function is

$$h(t) = C_1 e^{i\sqrt{k/m}\,t} + C_2 e^{-i\sqrt{k/m}\,t}$$

$$= C_1\left(\cos\sqrt{\tfrac{k}{m}}t + i\sin\sqrt{\tfrac{k}{m}}t\right) + C_2\left(\cos\sqrt{\tfrac{k}{m}}t - i\sin\sqrt{\tfrac{k}{m}}t\right)$$

$$= (C_1 + C_2)\cos\sqrt{\tfrac{k}{m}}t + (C_1 - C_2)i\sin\sqrt{\tfrac{k}{m}}t$$

I'm not sure why, but we always drop the i here to get

$$h(t) = A\cos\sqrt{\tfrac{k}{m}}t + B\sin\sqrt{\tfrac{k}{m}}t$$

The particular solution wants $h(t)$ to be some constant, to reflect the constant on the RHS of ODE, $\frac{-kl}{m} - g$.

Let $h(t) = C$. Then $h''(t) = 0$.

$$\therefore 0 + \frac{k}{m}C = -\frac{k}{m}l - g \Rightarrow C = -l - \frac{mg}{k}$$

$$\therefore h(t) = A\cos\sqrt{\tfrac{k}{m}}t + B\sin\sqrt{\tfrac{k}{m}}t - l - \frac{mg}{k}$$

$$= A\cos\left(t\sqrt{\tfrac{k}{m}}\right) + B\sin\left(t\sqrt{\tfrac{k}{m}}\right) - l - \frac{mg}{k}$$

$$h(0) = A + 0 - l - \frac{mg}{k} = 0 \Rightarrow A = l + \frac{mg}{k}$$

$$h'(t) = -A\sqrt{\tfrac{k}{m}}\sin\left(t\sqrt{\tfrac{k}{m}}\right) + B\sqrt{\tfrac{k}{m}}\cos\left(t\sqrt{\tfrac{k}{m}}\right)$$

$$h'(0) = 0 + B\sqrt{\tfrac{k}{m}} = 0 \Rightarrow B = 0 \text{ since } k, m > 0$$

$$\therefore \boxed{h(t) = \left(l + \frac{mg}{k}\right)\cos\left(t\sqrt{\tfrac{k}{m}}\right) - l - \frac{mg}{k}}$$

Q3 (b) $h(t)$ in the re-bounce stage is basically just a cosine function, stretched and ~~scale~~ scaled. That means the minimum of $h(t)$ is $-l-\frac{mg}{k}-\left(l+\frac{mg}{k}\right)$

$$=-2\left(l+\frac{mg}{k}\right)$$

If we want the jumper to just touch the floor in the worst case, ~~then~~

$$h_m \leq -2\left(l+\frac{mg}{k}\right)$$

~~that~~ We can then simply rearrange to find l.

$$-\frac{h_m}{2} \geq l+\frac{mg}{k} \quad \therefore \cancel{\boxed{l \geq -\left(\frac{h_m}{2}+\frac{mg}{k}\right)}}$$

$$\therefore \boxed{l \leq -\left(\frac{h_m}{2}+\frac{mg}{k}\right)}$$