MA270 Analysis 3, Assignment 4

Dyson Dyson

Question 1

> *Definition*: A function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to admit a global minimum at (x_0, y_0) if $f(x_0, y_0) \leq f(x_1, y_1)$ for every $(x_1, y_1) \in \mathbb{R}^2$. It is said to admit a unique global minimum if there exists $(x_0, y_0) \in \mathbb{R}^2$ such that f admits a global minimum at (x_0, y_0) and for every $(x_1, y_1) \in \mathbb{R}^2$, $f(x_0, y_0) = f(x_1, y_1)$ implies $x_0 = x_1$ and $y_0 = y_1$.
>
> Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x^2 + y^2 - 2x - 4y$. Show that this function admits a unique global minimum on $\mathbb{R}^2$ and calculate the minimum of f.

The shape of f is a positive paraboloid so we expect a unique global minimum, which will be the only stationary point.

$$\begin{aligned}\nabla f(x, y) &= \begin{pmatrix} \partial_x f \\ \partial_y f \end{pmatrix} \\ &= \begin{pmatrix} 2x - 2 \\ 2y - 4 \end{pmatrix}\end{aligned}$$

Clearly this equals $\underline{0}$ only at $(1, 2)$, so this is our minimum point. The minimum value is

$$\begin{aligned}f(1, 2) &= 1^2 + 2^2 - 2 - 4(2) \\ &= 1 + 4 - 2 - 8 \\ &= -5.\end{aligned}$$

Question 2

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(0,0) = 0$ and $f(x,y) = \dfrac{\sin(xy)}{|x|+|y|}$ if $(x,y) \neq (0,0)$.

(a) Show that f is continuous on $\mathbb{R}^2$.

Hint: To show continuity at zero, we can show $|f(x,y)| \leq \min\{|x|,|y|\}$ for every $(x,y) \in \mathbb{R}^2$ and to show this we can use the inequality $|\sin(x)| \leq |x|$ valid for every $x \in \mathbb{R}$.

(b) Show that for any $x \geq 0$, $\partial_2 f(x,0)$ exists.

(c) Show that f is not continuously differentiable at $(0,0)$.

Q2 (a)

I don't know how to do this, sorry.

Q2 (b)

$$
\begin{aligned}
\partial_2 f &= \partial_y \left(\frac{\sin(xy)}{|x|+|y|} \right) \\
&= \partial_y \left(\sin(xy)(|x|+|y|)^{-1} \right) \\
&= \sin(xy) \left(\partial_y (|x|+|y|)^{-1} \right) + (|x|+|y|)^{-1} \left(\partial_y \sin(xy) \right) \\
&= \sin(xy) \left(-\operatorname{sgn}(y)(|x|+|y|)^{-2} \right) + \frac{x\cos(xy)}{|x|+|y|} \\
&= \frac{-\sin(xy)\operatorname{sgn}(y)}{(|x|+|y|)^2} + \frac{x\cos(xy)}{|x|+|y|} \\
\partial_2 f(x,0) &= -0 + \frac{x\cos(0)}{|x|+0} \\
&= \frac{x}{|x|} \\
&= \operatorname{sgn}(x)
\end{aligned}
$$

So $\partial_2 f(x,0) = 1$ for all $x > 0$.

f is constant 0 along the y-axis, so $\partial_2 f(0,0) = 0$. Therefore $\partial_2 f(x,0)$ exists for any $x \geq 0$.

Q2 (c)

By part **(b)**, $\partial_2 f(x,0) = -1$ for all $x < 0$. Therefore

$$\partial_2 f(x,0) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases},$$

so $\partial_2 f$ is not continuous along the x-axis and therefore cannot be continuously Fréchet differentiable at $(0,0)$.

Question 3

Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a fixed linear map, and define

$$f : \mathbb{R}^n \to \mathbb{R}, \qquad f(x) = \|Ax\|^2.$$

(a) For any $h \in \mathbb{R}^n$, compute $Df(x)(h)$ in terms of A, h, and x.

(b) Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a C^1 curve. Use the chain rule to compute $\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))$ in terms of $\gamma(t)$, $\gamma'(t)$, A and A^T, where A^T is by definition the unique linear map $\mathbb{R}^m \to \mathbb{R}^n$ satisfying $A^T x \cdot y = x \cdot Ay$ for every $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$.

(c) Show that the critical points of f (points where $\nabla f(x) = 0$) are precisely those x with $Ax = 0$.

Q3 (a)

We can use the relation between the Fréchet derivative and the directional derivative, and the definition of the directional derivative:

$$Df(x)(h) = \partial_h f(x) = \lim_{t \to 0} \frac{f(x + th) - f(x)}{t}.$$

Then we just compute

$$\begin{aligned}
Df(x)(h) &= \lim_{t \to 0} \frac{f(x + th) - f(x)}{t} \\
&= \lim_{t \to 0} \frac{\|A(x + th)\|^2 - \|Ax\|^2}{t} \\
&= \lim_{t \to 0} \frac{\|Ax + tAh\|^2 - \|Ax\|^2}{t} \\
&= \lim_{t \to 0} \frac{\|Ax\|^2 + 2|t|\|Ax\|\|Ah\| + |t|^2\|Ah\|^2 - \|Ax\|^2}{t} \\
&= \lim_{t \to 0} \left(2\,\|Ax\|^2\|Ah\|^2 + |t|\,\|Ah\|^2\right) \\
&= 2\,\|Ax\|^2\|Ah\|^2 \\
&= 2f(x)f(h)
\end{aligned}$$

Q3 (b)

By Example 6.24 in the notes,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)) &= \nabla f(\gamma(t)) \cdot \gamma'(t) \\
&= A(\gamma(t)) \cdot \gamma'(t)
\end{aligned}$$

Q3 (c)

The points where $\nabla f(x) = 0$ are the points where $f(x) = 0$, and so $Ax = 0$.

Question 4

Definition: A function $f : \mathbb{R}^n \to \mathbb{R}^k$ is said to be L-Lipschitz for a constant $L \geq 0$ if for every $x, y \in \mathbb{R}^n$, $\|f(x) - f(y)\| \leq L\|x - y\|$.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function (i.e. continuously differentiable). Suppose that there exists a constant $L \geq 0$ such that for every $x \in \mathbb{R}^n$, we have $\|\nabla f(x)\| \leq L$.

(a) Let $x, y \in \mathbb{R}^n$. Show that

$$\int_0^1 \nabla f(x + t(y - x)) \cdot (y - x)\ \mathrm{d}t = f(y) - f(x).$$

(b) Deduce that f is L-Lipschitz.

Q4 (a)

$$\begin{aligned}
\int_0^1 \nabla f(x + t(y - x)) \cdot (y - x)\ \mathrm{d}t &= \int_0^1 \sum_{i=1}^n (y_i - x_i)\, \partial_i f(x_i + t(y_i - x_i))\ \mathrm{d}t \\
&= \sum_{i=1}^n (y_i - x_i) \int_0^1 \partial_i f(x_i + t(y_i - x_i))\ \mathrm{d}t
\end{aligned}$$

I'm lost, sorry.