MA141 Analysis 1, Assignment 1

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Question 1

> Use induction to prove Bernoulli's inequality: if $x > -1$ then for every $n \in N$, $(1+x)^n \geq 1 + nx$. (Where do you use the fact that $x > -1$?)

Bernoulli's inequality states: if $x > -1$ then for every $n \in \mathbb{N}$, $(1+x)^n \geq 1+nx$. We will prove this by induction.

The base case is $n = 0$. We get $(1+x)^0 = 1$ and $1 + 0 \times x = 1$. Clearly $1 \geq 1$, so the base case holds.

Now assume that we know the inequality holds for some $n = k$, so $(1+x)^k \geq 1 + kx$. Then

$$
\begin{aligned}
(1+x)^{k+1} &= \underbrace{(1+x)^k}_{\geq 1+kx}\underbrace{(1+x)}_{>0} \\
&\geq (1+kx)(1+x) && \text{since } 1+x > 0 \\
&= 1 + kx + x + kx^2 \\
&= 1 + (k+1)x + kx^2 \\
&\geq 1 + (k+1)x && \text{since } kx^2 \geq 0
\end{aligned}
$$

Question 5

> Identify the greatest lower bound and least upper bound for each of the following sets, and prove that they are indeed the GLB and LUB; say whether these bounds are elements of the set.

For each part, I shall use S to refer to the set in question.

Q5 (a) $\{x : 0 \leq x \leq 1\}$

The greatest lower bound is 0, which is in the set. Suppose we have some other lower bound $\ell > 0$. We know that $0 \in S$ and $0 < \ell$, so ℓ cannot be a lower bound.

Likewise, the least upper bound is 1, which is in the set. Suppose we have some other upper bound $\ell < 1$. We know that $1 \in S$ and $\ell < 1$, so ℓ cannot be an upper bound.

Q5 (b) $\{x : 0 < x < 1\}$

The greatest lower bound is 0, which is not in the set. Suppose we have some other lower bound $\ell > 0$. We know from the Archimedean property of real numbers that for any real number $\varepsilon > 0$, we can find a natural number n such that $0 < \frac{1}{n} < \varepsilon$. Thus, we can find an n such that $0 < \frac{1}{n} < \ell$, so $\frac{1}{n}$ is less than ℓ but also in S. That means that ℓ cannot be a lower bound.

The least upper bound is 1, which is not in the set. Suppose we have some other upper bound $\ell < 1$ and a real number $\varepsilon > 0$. If ε is sufficiently small, then $\ell + \varepsilon < 1$, so $\ell + \varepsilon \in S$. That means that ℓ cannot be an upper bound, since $\ell + \varepsilon$ is an upper bound $> \ell$. Therefore there cannot exist an upper bound $\ell < 1$, so 1 is the least upper bound.

Q5 (c) $\left\{1 + \dfrac{1}{n} : n \in \mathbb{N}\right\}$

We can enumerate this set as something like

$$\left\{1 + 1, 1 + \frac{1}{2}, 1 + \frac{1}{3}, 1 + \frac{1}{4}, 1 + \frac{1}{5}, \dots\right\}$$

The greatest lower bound is 1, which is not in the set. 1 is a lower bound since $\frac{1}{n} > 0 \; \forall \, n \in \mathbb{N}$, so $1 + \frac{1}{n} > 1 \; \forall \, n \in \mathbb{N}$. Suppose we have some other lower bound $\ell > 1$. By the Archimedean property of real numbers, we can find a natural number k such that $\frac{1}{k} < \ell - 1$. Therefore $1 + \frac{1}{k} < \ell$, and since k is a natural number, we know that $1 + \frac{1}{k} \in S$. Therefore $1 + \frac{1}{k}$ is a lower bound which is smaller than ℓ, so ℓ cannot be a lower bound.

The least upper bound is 2, which is in the set. The first element of the set is 2, and every other element is $1 + \frac{1}{n}$, where $n > 1$. We can show that this is always less than 2 when $n > 1$.

$$\begin{aligned} & n > 1 \\ \implies & 1 > \frac{1}{n} \\ \implies & 2 > 1 + \frac{1}{n} \end{aligned}$$

Therefore 2 is an upper bound and since it's in the set, it is also the least upper bound.

Q5 (d) $\left\{2 - \dfrac{1}{n} : n \in \mathbb{N}\right\}$

We can enumerate this set as something like

$$\left\{2 - 1, 2 - \frac{1}{2}, 2 - \frac{1}{3}, 2 - \frac{1}{4}, \ldots\right\}$$

The greatest lower bound is 1, which is in the set. 1 is a lower bound since $\frac{1}{n} \leq 1 \ \forall \ n \in \mathbb{N}$, so $2 - \frac{1}{n} \geq 1$. Suppose we have some lower bound $\ell > 1$. ℓ cannot be a lower bound since $1 \in S$ and $1 < \ell$. Therefore we cannot have a lower bound > 1, so 1 is the greatest lower bound.

The least upper bound is 2, which is not in the set. 2 is an upper bound since $\frac{1}{n} > 0 \ \forall \ n \in \mathbb{N}$, so $2 - \frac{1}{n} < 2$. Suppose we have some other upper bound $\ell < 2$. The Archimedean property of real numbers tells us that we can find a natural number k such that $0 < \frac{1}{k} < 2 - \ell$. Therefore $\ell + \frac{1}{k} < 2$, so $\ell + \frac{1}{k}$ is an upper bound of S which is $> \ell$, so ℓ cannot be an upper bound. Therefore 2 is the least upper bound.

Q5 (e) $\left\{1 + \dfrac{(-1)^n}{n} : n \in \mathbb{N}\right\}$

We can enumerate this set as something like

$$\left\{1 - \frac{1}{1}, 1 + \frac{1}{2}, 1 - \frac{1}{3}, 1 + \frac{1}{4}, 1 - \frac{1}{5}, 1 + \frac{1}{6}, \ldots\right\}$$

The greatest lower bound is 0, which is in the set. 0 is a lower bound since $0 < \frac{1}{n} \leq 1 \ \forall \ n \in \mathbb{N}$, so $0 \leq 1 \pm \frac{1}{n} \leq 2$. Therefore every element of S is ≥ 0. We cannot have a lower bound $\ell > 0$, since $0 \in S$. So any $\ell > 0$ cannot be a lower bound, so 0 must be the greatest lower bound.

The least upper bound is $\frac{3}{2}$, which is in the set. We can discount all the odd n values from the set, since they result in $1 - \frac{1}{n}$, which will always be < 1. Therefore to find the upper bound, we only have to focus on the even values of n, which result is $1 + \frac{1}{n}$. These values are $1 + \frac{1}{2}, 1 + \frac{1}{4}, 1 + \frac{1}{6}, \ldots$ and it should be clear to see that the largest of these is $1 + \frac{1}{2} = \frac{3}{2}$. Therefore $\frac{3}{2}$ is an upper bound. We cannot have another upper bound $\ell < \frac{3}{2}$ because $\frac{3}{2} \in S$, so ℓ could never be an upper bound. Therefore $\frac{3}{2}$ is the least upper bound.

Q5 (f) $\{q < 0 : q^2 < 4, q \in \mathbb{Q}\}$

We can rewrite this set as something like $\{q \in \mathbb{Q} : -2 < q < 0\}$.

The greatest lower bound is -2, which is not in the set. This is a lower bound because the condition $q^2 < 4$ is equivalent to $-2 < q < 2$, so we need $-2 < q \ \forall \ q \in S$. Suppose we have some lower bound $\ell > -2$. The Archimedean property of real numbers tells us that we can find a natural number n such that $\frac{1}{n} < \ell + 2$, therefore $-2 < \ell - \frac{1}{n}$. Since ℓ and $\frac{1}{n}$ and both rational, their difference is rational. Therefore $\ell - \frac{1}{n} \in S$, but $\ell - \frac{1}{n} < \ell$, so ℓ cannot be a lower bound. Therefore -2 is the greatest lower bound.

The least upper bound is 0, which is not in the set. This is an upper bound because the definition of S directly tells us that $q < 0 \ \forall \ q \in S$. Suppose we have an upper bound $-2 < \ell < 0$. The Archimedean property of real numbers tells us that we can find a natural number n such that $0 < \frac{1}{n} < -\ell$. Therefore $\ell < -\frac{1}{n} < 0$. Since $\ell > -2$, $-\frac{1}{n} > -2$, so $-\frac{1}{n}$ is in S, but it's bigger than ℓ. Therefore ℓ cannot be an upper bound, so 0 is the least upper bound.

Question 7

The integer part (or 'floor') of a rational number x, written $\lfloor x \rfloor$, is defined as

$$\lfloor x \rfloor = \text{largest integer } n \in \mathbb{Z} \text{ such that } n \leq x;$$

Use the Least Upper Bound Property to show that this quantity exists and satisfies

$$x - 1 < \lfloor x \rfloor \leq x. \tag{1}$$

Hint: Consider the set $S = \{m \in \mathbb{Z} \colon m \leq x\}$. Show that it has a least upper bound r, and use Lemma 1.6 with $t = r-1$ to find $n \in S$ that satisfies the requirements for $\lfloor x \rfloor$ in (1).

We will consider the set $S = \{m \in \mathbb{Z} \colon m \leq x\}$. This is clearly bounded above by x, so by the *Least Upper Bound Axiom*, we know that S has a least upper bound $r = \sup S$. In the case of $x \in \mathbb{Z}$, we can see that $r = x$.

Then Lemma 1.6 tells us that $r = \sup S$ if and only if r is an upper bound for S and for every $t < r$, there exists $s \in S$ such that $s > t$.

We already know that $r = \sup S$ by the *Least Upper Bound Axiom*. That means we also know that for every $t < r, \exists s \in S$ such that $s > t$. For the sake of satisfying equation (1), we will choose $t = r - 1$. Therefore we know that there exists some element $n \in S$ such that $r - 1 < n$.

Since $n \in S$, we also know that $n \leq x$. Therefore $r - 1 < n \leq x$. Call this element $n = \lfloor x \rfloor$ and we can conclude that $r - 1 < \lfloor x \rfloor \leq x$.

This isn't quite equation (1), but I'm not sure how to finish off the argument. It's intuitive to me that $r = \lfloor x \rfloor = \sup S$ and $x - 1 < \lfloor x \rfloor$, but I don't know how to formalise those ideas into a proper argument.

We could set $t = x - 1$ and then prove that $x - 1 < r$, but that doesn't really help me prove anything, and the problem sheet suggests $t = r - 1$ anyway, so I'm not sure how to finish this argument.

Question 9

> By following the argument of Proposition 1.7 we can show that for any $q \in \mathbb{N}$ and $y > 0$ there exists $x \in \mathbb{R}$ such that $x^q = y$. Take $y > 1$ and consider the set
> $$S = \{x \in \mathbb{R} \colon x \geq 0 \text{ with } x^q < y\}$$

Q9 (i)

> Show that S is non-empty and bounded above. It follows from the Least Upper Bound Axiom that S has a supremum: set $r = \sup S$, and note (why?) that $r \geq 1$.

S is non-empty, since $0 \in S$, and it is bounded above since y is finite, so there will eventually be some x such that $x^q > y$. That x will be greater than the upper bound of S, so S must be bounded above.

Thus, from the *Least Upper Bound Axiom*, we know that S has a supremum, $r = \sup S$. 1 will always be $\in S$, since $1^q < y$ for any q when $y > 1$. Thus, $r \geq 1$.

Q9 (ii)

> Use the binomial expansion to show that if $x \geq 1$ and $0 < \varepsilon < 1$ then $(x+\varepsilon)^q \leq x^q(1 + 2^q\varepsilon)$.
>
> Hint: $(1+1)^q = 2^q$. If you cannot do this part of the question, you can still use the result to try part **(iii)**.

The binomial expansion of $(x+\varepsilon)^q$ gives

$$\begin{aligned}(x+\varepsilon)^q &= \sum_{k=0}^{q} \binom{q}{k} x^{q-k}\varepsilon^k \\ &= x^q + qx^{q-1}\varepsilon + \binom{q}{2} x^{q-2}\varepsilon^2 + \cdots + qx\varepsilon^{q-1} + \varepsilon^q\end{aligned}$$

We can then factor this to get

$$\begin{aligned}(x+\varepsilon)^q &= x^q \sum_{k=0}^{q} \binom{q}{k} \frac{1}{x^k} \varepsilon^k \\ &= x^q \sum_{k=0}^{q} \binom{q}{k} \left(\frac{\varepsilon}{x}\right)^k \\ &= x^q \left(1 + \frac{\varepsilon}{x}\right)^q\end{aligned}$$

Somehow we conclude that $(x+\varepsilon)^q \leq x^q\left(1 + 2^q\varepsilon\right)$.

Q9 (iii)

> Suppose that $r^q < y$. Use part **(ii)** to show that $(r+\varepsilon)^q < y$ for some sufficiently small $\varepsilon > 0$, and hence deduce a contradiction with the fact that r is an upper bound for S.

Suppose that $r^q < y$ and let $0 < \varepsilon < 1$.

By part *(ii)*, we know $(r+\varepsilon)^q \leq r^q(1 + 2^q\varepsilon)$. Thus if $r^q < y$, then $(r+\varepsilon)^q \leq r^q(1+2^q\varepsilon) < y(1+2^q\varepsilon)$.

For sufficiently small ε, $(1+2^q\varepsilon) \approx 1$, but I don't know how to make the jump and show that $(r+\varepsilon)^q < y$.

Since $(r+\varepsilon)^q < y$, $r+\varepsilon \in S$. But $\varepsilon > 0$, so $r + \varepsilon > r$. Thus, we have found an element of S which is greater than r. That's a contradiction, since $r = \sup S$. Therefore, we know that $r^q < y$ must be false.

Q9 (iv)

> Suppose that $r^q > y$. Use Bernoulli's Inequality from **Q1** to show that $(r-\varepsilon)^q > y$ for some sufficiently small $\varepsilon > 0$, and hence deduce a contradiction with the fact that r is the least upper bound for S.

Suppose that $r^q > y$ and let $0 < \varepsilon < 1$.

Bernoulli's Inequality tells us that if $x > -1$, then $\forall\, n \in \mathbb{N}$, $(1+x)^n \geq 1 + nx$. Since $0 < \varepsilon < 1$, we know that $-\varepsilon > -1$ and $q \in \mathbb{N}$. Therefore Bernoulli's Inequality tells us that $(1-\varepsilon)^q \geq 1 - q\varepsilon$.

I have no idea how to complete this argument, but I know it ends by showing that $(r-\varepsilon)^q > y$.

Since $\varepsilon > 0$, $r - \varepsilon < r$ but $r - \varepsilon \notin S$. Additionally, $r \notin S$ since $r^q > y$, but every $s \in S$ requires $s^q < y$. Thus, we have found a better upper bound for S. $r - \varepsilon$ is an upper bound for S which is smaller that r, so r cannot be the supremum of S. This is a contradiction, therefore $r^q > y$ must be false.

Thus, since $r^q \not< y$ and $r^q \not> y$, we must conclude that $r^q = y$. □