MA243 Geometry, Assignment 1

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Question 1

Write down all the symmetries of the rectangle X in $\mathbb{R}^2$ with vertices at

$$(1,0),(1,4),(3,0),(3,4)$$

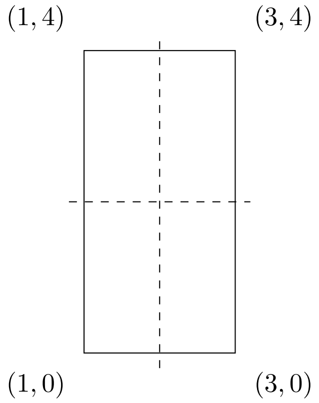
That is,

$$X = \{(x,y) \in \mathbb{R}^2 : 1 \leq x \leq 3, 0 \leq y \leq 4\}$$

Write all your symmetries as affine transformations of the form

$$T(\mathbf{v}) = A\mathbf{v} + \mathbf{b}$$

where A is a 2×2 matrix and $\mathbf{b}$ is a vector in $\mathbb{R}^2$.



The rectangle X has 4 symmetries, the identity, a rotation by π followed by a translation, a reflection in $y = 2$, and a reflection in $x = 2$. We shall write these as affine transformations with matrices.

The identity is of course

$$T(\mathbf{v}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{v} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The rotation and translation is

$$T(\mathbf{v}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{v} + \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

The reflection in $y = 2$ is

$$T(\mathbf{v}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{v} + \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

And the reflection in $x = 2$ is

$$T(\mathbf{v}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{v} + \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

Question 2

Suppose that a function $f\colon X \times X \to \mathbb{R}$ is non-degenerate, symmetric, and satisfies the triangle inequality. Then

(a) Prove that the image of f is in $[0, \infty)$ (hence f is a metric).

(b) When is f injective? Explain.

(c) Suppose $X = \mathbb{R}$. Does f have to be surjective onto $[0, \infty)$? Either prove this or give a counterexample.

Q2 (a)

Since the codomain of f is $\mathbb{R}$, we know the image of f will be a subset of $\mathbb{R}$, so we only need to prove $f \geq 0$.

Suppose there exist $a, b, c \in X$ with $a \neq b \neq c$ and $f(a, b) < 0$. Since f satisfies the triangle inequality, we know three things:

$$\begin{aligned} f(a,b) &\leq f(a,c) + f(b,c) \\ f(b,c) &\leq f(a,b) + f(a,c) \\ f(a,c) &\leq f(a,b) + f(b,c) \end{aligned}$$

We can apply the fact that $f(a, b) < 0$ to the last two lines and observe

$$\begin{aligned} f(b,c) &< f(a,c) \\ f(a,c) &< f(b,c) \end{aligned}$$

This, however, is a contradiction. Therefore our initial assumption must be false, so in fact, there does not exist $a, b \in X$ with $f(a, b) < 0$. Therefore $f \geq 0$.

Q2 (b)

f can only be injective if X is finite, or a 1-dimensional Euclidean space, but it is not guaranteed to be injective. If X were any higher dimension, then rotations would prevent injectivity.

Q2 (c)

A counterexample is the function $f_b : X \to \mathbb{R}$ defined by

$$f_b(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Question 3

Let $\delta : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ be a function defined by

$$\delta\big((x_1, y_1), (x_2, y_2)\big) = \sqrt{(x_1 - x_2)^2 + 2(y_1 - y_2)^2}$$

Is $(\mathbb{R}^2, \delta)$ a metric space? Prove it is, or explain why it isn't.

For $(\mathbb{R}^2, \delta)$ to be a metric space, δ must be a metric. This means is must be non-degenerate, symmetric, and satisfy the triangle inequality.

The only way to make $\delta = 0$ is to make everything under the square root equal to 0, which means making $x_1 - x_2 = 0$ and $y_1 - y_2 = 0$. That means $x_1 = x_2$ and $y_1 = y_2$, so $\delta(\underline{u}, \underline{v}) = 0$ only when $\underline{u} = \underline{v}$ and therefore δ is non-degenerate.

Since the terms $x_1 - x_2$ and $y_1 - y_2$ are both squared, their order is unimportant, so δ is symmetric.

For the triangle inequality, let $\underline{x}, \underline{y}, \underline{z} \in \mathbb{R}^n$. We can use translation invariance to move $\underline{z}$ to the origin, then let $\underline{x}' = \underline{x} - \underline{z}$, $\underline{y}' = \underline{y} - \underline{z}$, $\underline{z}' = \underline{z} - \underline{z} = \underline{0}$.

Now we want to show that $\delta(\underline{x}, \underline{y}) \leq \delta(\underline{x}, \underline{z}) + \delta(\underline{y}, \underline{z})$. By translation invariance again, this is equivalent to $\delta(\underline{x}', \underline{y}') \leq \delta(\underline{x}', \underline{0}) + \delta(\underline{y}', \underline{0})$, which is equivalent to $\|\underline{x}' - \underline{y}'\| \leq \|\underline{x}'\| + \|\underline{y}'\|$. Since both sides are non-negative, it suffices to square both sides and show the resulting equation holds.

$$\begin{aligned}
\big(\|\underline{x}'\| + \|\underline{y}'\|\big)^2 &= \|\underline{x}'\|^2 + 2\|\underline{x}'\|\,\|\underline{y}'\| + \|\underline{x}'\|^2 \\
&\geq \|\underline{x}'\|^2 + 2|\underline{x}' \cdot \underline{y}'| + \|\underline{x}'\|^2 \\
&\geq \|\underline{x}'\|^2 - 2(\underline{x}' \cdot \underline{y}') + \|\underline{x}'\|^2 \\
&= \underline{x}' \cdot \underline{x}' - 2(\underline{x}' \cdot \underline{y}') + \underline{y}' \cdot \underline{y}' \\
&= (\underline{x}' - \underline{y}') \cdot (\underline{x}' - \underline{y}') \\
&= \|\underline{x}' - \underline{y}'\|^2
\end{aligned}$$

Since δ is non-degenerate, symmetric, and preserves the triangle inequality, it is a metric and therefore $(\mathbb{R}^2, \delta)$ is a metric space.

$\square$

Question 4

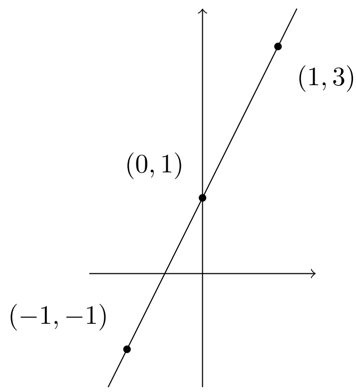
Let L be the line in $\mathbb{R}^2$ given by

$$L = \left\{(x, y) \in \mathbb{R}^2 \colon y - 2x = 1\right\}$$

Define a metric d on L given by the restriction of the Euclidean metric on $\mathbb{R}^2$ to L. That is, for points (x_1, y_2), (x_2, y_2), we set

$$d\big((x_1, y_1), (x_2, y_2)\big) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Find an isometry $f \colon L \to \mathbb{R}$, where $\mathbb{R}$ has the usual Euclidean metric, $d_1(x, y) = |x - y|$.



Consider two points $(x_1, 2x_1 + 1)$ and $(x_2, 2x_2 + 1)$ on L. The distance between them is

$$\begin{aligned} d\big((x_1, 2x_1 + 1), (x_2, 2x_2 + 1)\big) &= \sqrt{(x_1 - x_2)^2 + (2x_1 + 1 - 2x_2 - 1)^2} \\ &= \sqrt{(x_1 - x_2)^2 + (2(x_1 - x_2))^2} \\ &= \sqrt{(x_1 - x_2)^2 + 4(x_1 - x_2)^2} \\ &= \sqrt{5(x_1 - x_2)^2} \\ &= \sqrt{5}\,|x_1 - x_2| \end{aligned}$$

This is the normal Euclidean metric scaled by $\sqrt{5}$, so we just project L down onto the x-axis and apply this scaling factor. So let $f(x, y) = x\sqrt{5}$.