MA150 Algebra 2, Assignment 2

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Question 5

Let $P = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $Q = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}$ and let L be the line between P and Q we can parametrise L as $\overrightarrow{P} + \lambda \overrightarrow{PQ}$. $\overrightarrow{PQ} = \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix}$ so L can be parametrised as

$$L = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} : \lambda \in \mathbb{R} \right\}$$

To describe L in terms of two implicit equations, we need two planes. We will start with the plane Π_1 through P, Q, and the origin. The vectors $\overrightarrow{P}$ and $\overrightarrow{Q}$ will both be in Π_1 so a normal vector is

$$\underline{n} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ -11 \\ 5 \end{pmatrix}$$

Therefore any point on Π_1 will satisfy

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 7 \\ -11 \\ 5 \end{pmatrix} = 0$$

So we get the equation $7x - 11y + 5z = 0$.

Note that the point $R = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ does not satisfy this equation and thus is not on Π_1. So we can define Π_2 as the plane containing P, Q, and R.

Two vectors in Π_2 are $\overrightarrow{RP} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$ and $\overrightarrow{RQ} = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix}$. Then we can find a normal vector

$$\underline{n} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ -9 \\ 6 \end{pmatrix}$$

R is a point on Π_2 and $\overrightarrow{R} \cdot \underline{n} = 7$. Therefore any point on Π_2 will satisfy

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 7 \\ -9 \\ 6 \end{pmatrix} = 7$$

So we get the equation $7x - 9y + 6z = 7$.

Therefore the line L can be described by the pair of equations

$$\begin{aligned} 7x - 11y + 5z &= 0 \\ 7x - 9y - 6z &= 7 \end{aligned}$$

We can check and indeed, P and Q both satisfy both of these equations.

Question 6

$$A = \begin{pmatrix} 1 & -3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ -2 & 6 & 1 & 0 \end{pmatrix}$$

$$A_{13}(2) \implies \begin{pmatrix} 1 & -3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$A_{23}(-1) \implies \begin{pmatrix} 1 & -3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Question 7

Q7 (a)

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$. Only square matrices can be invertible, so A is not invertible.

Q7 (b)

$$
\begin{aligned}
(A \mid I) &= \left(\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 6 & 8 & 0 & 1 \end{array}\right) \\
M_1\left(\frac{1}{2}\right) &\implies \left(\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 6 & 8 & 0 & 1 \end{array}\right) \\
A_{12}\left(-6\right) &\implies \left(\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & -4 & -3 & 1 \end{array}\right) \\
M_2\left(-\frac{1}{4}\right) &\implies \left(\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{4} & -\frac{1}{4} \end{array}\right) \\
A_{21}\left(-2\right) &\implies \left(\begin{array}{cc|cc} 1 & 0 & -1 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} & -\frac{1}{4} \end{array}\right)
\end{aligned}
$$

Therefore $A^{-1} = A_{21}(-2)\, M_2\left(-\frac{1}{4}\right)\, A_{12}(-6)\, M_1\left(\frac{1}{2}\right) = \begin{pmatrix} -1 & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} \end{pmatrix}$.

Q7 (c)

$$
\begin{aligned}
(A \mid I) &= \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 2 & -3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \\
A_{12}(-2) &\implies \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \\
A_{21}(2) &\implies \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)
\end{aligned}
$$

Therefore $A^{-1} = A_{21}(2)\, A_{12}(-2) = \begin{pmatrix} -3 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Question 8

Q8 (a)

The system produces this augmented matrix:

$$\left(\begin{array}{cccccc|c} 0 & 6 & 2 & 0 & -1 & 0 & 1 \\ 0 & 0 & 4 & 0 & 1 & 0 & -1 \\ -2 & 1 & 0 & 4 & 1 & -1 & 0 \\ -3 & 0 & 1 & 6 & 2 & 0 & 4 \end{array}\right)$$

Q8 (b)

The row reduction process for this system goes as follows:

$$\begin{aligned}
&\left(\begin{array}{cccccc|c} 0 & 6 & 2 & 0 & -1 & 0 & 1 \\ 0 & 0 & 4 & 0 & 1 & 0 & -1 \\ -2 & 1 & 0 & 4 & 1 & -1 & 0 \\ -3 & 0 & 1 & 6 & 2 & 0 & 4 \end{array}\right) \\
S_{13} \implies &\left(\begin{array}{cccccc|c} -2 & 1 & 0 & 4 & 1 & -1 & 0 \\ 0 & 0 & 4 & 0 & 1 & 0 & -1 \\ 0 & 6 & 2 & 0 & -1 & 0 & 1 \\ -3 & 0 & 1 & 6 & 2 & 0 & 4 \end{array}\right) \\
M_1\left(-\frac{1}{2}\right) \implies &\left(\begin{array}{cccccc|c} 1 & -\frac{1}{2} & 0 & -2 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 4 & 0 & 1 & 0 & -1 \\ 0 & 6 & 2 & 0 & -1 & 0 & 1 \\ -3 & 0 & 1 & 6 & 2 & 0 & 4 \end{array}\right) \\
A_{14}(3) \implies &\left(\begin{array}{cccccc|c} 1 & -\frac{1}{2} & 0 & -2 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 4 & 0 & 1 & 0 & -1 \\ 0 & 6 & 2 & 0 & -1 & 0 & 1 \\ 0 & -\frac{3}{2} & 1 & 0 & \frac{1}{2} & \frac{3}{2} & 4 \end{array}\right)
\end{aligned}$$

$$S_{24} \implies \left(\begin{array}{cccccc|c} 1 & -\frac{1}{2} & 0 & -2 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{3}{2} & 1 & 0 & \frac{1}{2} & \frac{3}{2} & 4 \\ 0 & 6 & 2 & 0 & -1 & 0 & 1 \\ 0 & 0 & 4 & 0 & 1 & 0 & -1 \end{array}\right)$$

$$M_2\left(-\frac{2}{3}\right) \implies \left(\begin{array}{cccccc|c} 1 & -\frac{1}{2} & 0 & -2 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} & 0 & -\frac{1}{3} & -1 & -\frac{8}{3} \\ 0 & 6 & 2 & 0 & -1 & 0 & 1 \\ 0 & 0 & 4 & 0 & 1 & 0 & -1 \end{array}\right)$$

$$A_{21}\left(\frac{1}{2}\right) \implies \left(\begin{array}{cccccc|c} 1 & 0 & -\frac{1}{3} & -2 & -\frac{2}{3} & 0 & -\frac{4}{3} \\ 0 & 1 & -\frac{2}{3} & 0 & -\frac{1}{3} & -1 & -\frac{8}{3} \\ 0 & 6 & 2 & 0 & -1 & 0 & 1 \\ 0 & 0 & 4 & 0 & 1 & 0 & -1 \end{array}\right)$$

$$A_{23}(-6) \implies \left(\begin{array}{cccccc|c} 1 & 0 & -\frac{1}{3} & -2 & -\frac{2}{3} & 0 & -\frac{4}{3} \\ 0 & 1 & -\frac{2}{3} & 0 & -\frac{1}{3} & -1 & -\frac{8}{3} \\ 0 & 0 & 6 & 0 & 1 & 6 & 17 \\ 0 & 0 & 4 & 0 & 1 & 0 & -1 \end{array}\right)$$

$$M_3\left(\frac{1}{6}\right) \implies \left(\begin{array}{cccccc|c} 1 & 0 & -\frac{1}{3} & -2 & -\frac{2}{3} & 0 & -\frac{4}{3} \\ 0 & 1 & -\frac{2}{3} & 0 & -\frac{1}{3} & -1 & -\frac{8}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{6} & 1 & \frac{17}{6} \\ 0 & 0 & 4 & 0 & 1 & 0 & -1 \end{array}\right)$$

$$A_{34}(-4) \implies \left(\begin{array}{cccccc|c} 1 & 0 & -\frac{1}{3} & -2 & -\frac{2}{3} & 0 & -\frac{4}{3} \\ 0 & 1 & -\frac{2}{3} & 0 & -\frac{1}{3} & -1 & -\frac{8}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{6} & 1 & \frac{17}{6} \\ 0 & 0 & 0 & 0 & \frac{1}{3} & -4 & -\frac{37}{3} \end{array}\right)$$

$$A_{31}\left(\frac{1}{3}\right) \implies \left(\begin{array}{cccccc|c} 1 & 0 & 0 & -2 & -\frac{11}{18} & \frac{1}{3} & -\frac{7}{18} \\ 0 & 1 & -\frac{2}{3} & 0 & -\frac{1}{3} & -1 & -\frac{8}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{6} & 1 & \frac{17}{6} \\ 0 & 0 & 0 & 0 & \frac{1}{3} & -4 & -\frac{37}{3} \end{array}\right)$$

$$A_{32}\left(\frac{2}{3}\right) \implies \left(\begin{array}{cccccc|c} 1 & 0 & 0 & -2 & -\frac{11}{18} & \frac{1}{3} & -\frac{7}{18} \\ 0 & 1 & 0 & 0 & -\frac{2}{9} & -\frac{1}{3} & -\frac{7}{9} \\ 0 & 0 & 1 & 0 & \frac{1}{6} & 1 & \frac{17}{6} \\ 0 & 0 & 0 & 0 & \frac{1}{3} & -4 & -\frac{37}{3} \end{array}\right)$$

$$M_4(3) \implies \left(\begin{array}{cccccc|c} 1 & 0 & 0 & -2 & -\frac{11}{18} & \frac{1}{3} & -\frac{7}{18} \\ 0 & 1 & 0 & 0 & -\frac{2}{9} & -\frac{1}{3} & -\frac{7}{9} \\ 0 & 0 & 1 & 0 & \frac{1}{6} & 1 & \frac{17}{6} \\ 0 & 0 & 0 & 0 & 1 & -12 & -37 \end{array}\right)$$

$$A_{43}\left(-\frac{1}{6}\right) \implies \left(\begin{array}{cccccc|c} 1 & 0 & 0 & -2 & -\frac{11}{18} & \frac{1}{3} & -\frac{7}{18} \\ 0 & 1 & 0 & 0 & -\frac{2}{9} & -\frac{1}{3} & -\frac{7}{9} \\ 0 & 0 & 1 & 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & 0 & 1 & -12 & -37 \end{array}\right)$$

$$A_{42}\left(\frac{2}{9}\right) \implies \left(\begin{array}{cccccc|c} 1 & 0 & 0 & -2 & -\frac{11}{18} & \frac{1}{3} & -\frac{7}{18} \\ 0 & 1 & 0 & 0 & 0 & -3 & -9 \\ 0 & 0 & 1 & 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & 0 & 1 & -12 & -37 \end{array}\right)$$

$$A_{41}\left(\frac{11}{18}\right) \implies \left(\begin{array}{cccccc|c} 1 & 0 & 0 & -2 & 0 & -7 & -23 \\ 0 & 1 & 0 & 0 & 0 & -3 & -9 \\ 0 & 0 & 1 & 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & 0 & 1 & -12 & -37 \end{array}\right)$$

Q8 (c)

The system is consistent because the row reduced echelon form has no zero rows.

Q8 (d)

Choose parameters $\lambda, \mu \in \mathbb{R}$. Then

$$\begin{aligned} x_1 &= -23 + 2\lambda + 7\mu \\ x_2 &= -9 + 3\mu \\ x_3 &= 9 - 3\mu \\ x_4 &= \lambda \\ x_5 &= -37 + 12\mu \\ x_6 &= \mu \end{aligned}$$