

MA144 Methods of Mathematical Modelling 2, Assignment 2

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Question 1

Q1 (a)

The heat equation is

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

A solution to the heat equation is

$$u(x, t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} = \frac{1}{\sqrt{2\pi}} t^{-\frac{1}{2}} e^{-x^2/2t}$$

Q1 (a) i)

The solution gives

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{\sqrt{2\pi}} \left(t^{-\frac{1}{2}} e^{-x^2/2t} \frac{x^2}{2t^2} - \frac{1}{2} t^{-\frac{3}{2}} e^{-x^2/2t} \right) \\ &= \frac{1}{\sqrt{2\pi}} e^{-x^2/2t} \left(t^{-\frac{1}{2}} \frac{x^2}{2} t^{-2} - \frac{1}{2} t^{-\frac{3}{2}} \right) \\ &= \frac{1}{2\sqrt{2\pi}} e^{-x^2/2t} \left(x^2 t^{-\frac{5}{2}} - t^{-\frac{3}{2}} \right) \\ &= \frac{1}{2\sqrt{2\pi}} e^{-x^2/2t} t^{-\frac{3}{2}} (x^2 t^{-1} - 1) \\ &= \frac{1}{2\sqrt{2\pi t^3}} e^{-x^2/2t} \left(\frac{x^2}{t} - 1 \right) \end{aligned}$$

and

$$\begin{aligned}
 \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} \left(\frac{-x}{t} \right) \right) \\
 &= \frac{\partial}{\partial x} \left(\frac{-1}{\sqrt{2\pi t^3}} x e^{-x^2/2t} \right) \\
 &= \frac{-1}{\sqrt{2\pi t^3}} \left(e^{-x^2/2t} + x e^{-x^2/2t} \left(\frac{-x}{t} \right) \right) \\
 &= \frac{1}{\sqrt{2\pi t^3}} e^{-x^2/2t} \left(\frac{x^2}{t} - 1 \right)
 \end{aligned}$$

Therefore $\alpha = 2$ for this solution.

Q1 (a) ii)

$$u(x, 1) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

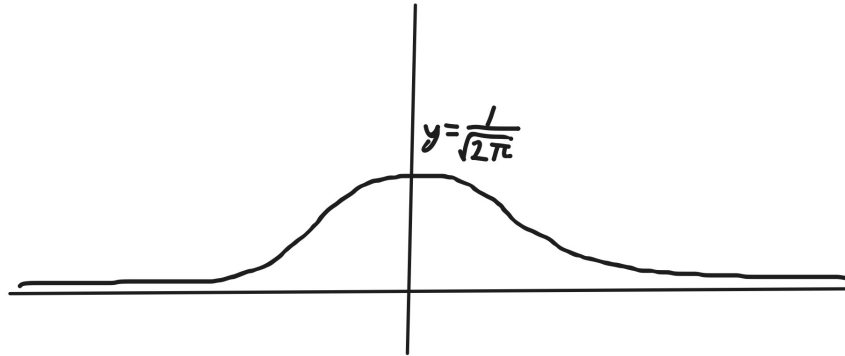


Figure 1: A plot of $y = u(x, 1)$

Q1 (b)

The wave equation is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

A solution to the wave equation is

$$u(x, t) = \sin x \cos \beta t$$

where $\beta > 0$.

Q1 (b) i)

The solution gives

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} (\cos x \cos \beta t) \\ &= -\sin x \cos \beta t \\ &= -u(x, t)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t} (-\beta \sin x \sin \beta t) \\ &= -\beta^2 \sin x \cos \beta t \\ &= -\beta^2 u(x, t)\end{aligned}$$

Plugging this into the wave equation gives

$$-u(x, t) = -\frac{\beta^2}{c^2} u(x, t)$$

Therefore $\frac{\beta^2}{c^2} = 1$ so $\beta = \pm c$. We know $c > 0$ and we want $\beta > 0$, so $\beta = c$.

Q1 (b) ii)

Let $u(x, t) = f(x + ct) + g(x - ct)$. Then by the chain rule,

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} (f'(x + ct) + g'(x - ct)) \\ &= f''(x + ct) + g''(x - ct)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t} (cf'(x + ct) - cg'(x - ct)) \\ &= c^2 f''(x + ct) + c^2 g''(x - ct)\end{aligned}$$

Therefore $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$, so this u satisfies the wave equation.

Question 2

$$f(x, y) = e^x + y^3 - 2x - 3y$$

Q2 (a)

$$\frac{\partial f}{\partial x} = e^x - 2 \quad \frac{\partial f}{\partial y} = 3y^2 - 3$$

Critical points are where $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$. That means $e^x - 2 = 0$ so $x = \log 2$, and $3y^3 - 3 = 0$ so $y = \pm 1$. Therefore the critical points of f are at $(\log 2, 1)$ and $(\log 2, -1)$.

The Hessian matrix of f is $\begin{pmatrix} e^x & 0 \\ 0 & 6y \end{pmatrix}$ and its determinant is $D = 6e^xy$.

At the point $(\log 2, 1)$, $D = 12$ and $f_{xx} = 2$, so this point is a local minimum point.

At the point $(\log 2, -1)$, $D = -12$, so this point is a saddle point.

Q2 (b)

Linear approximation around $P = (0, 2)$ gives

$$\begin{aligned} f(x, y) &\approx f(0, 2) + \nabla f|_{(0,2)} \cdot \begin{pmatrix} x \\ y - 2 \end{pmatrix} \\ &= (1 + 8 - 0 - 6) + \left(\begin{pmatrix} e^x - 2 \\ 3y^2 - 3 \end{pmatrix} \right) \Big|_{(0,2)} \cdot \begin{pmatrix} x \\ y - 2 \end{pmatrix} \\ &= 3 + \begin{pmatrix} -1 \\ 9 \end{pmatrix} \cdot \begin{pmatrix} x \\ y - 2 \end{pmatrix} \\ &= 3 - x + 9y - 18 \\ \therefore f(x, y) &\approx -15 - x + 9y \end{aligned}$$

Therefore

$$\begin{aligned} f(0.01, 1.99) &\approx -15 - 0.01 + 9 \times 1.99 \\ &= -15.01 + 9 + 8.91 \\ &= -15.01 + 17.91 \\ &= 2.9 \end{aligned}$$

For completeness, $f(0.01, 1.99)$ is actually around 2.900649, so this approximation is very good.

Q2 (c)

The direction of steepest descent at point P is $-\nabla f|_P = \begin{pmatrix} 1 \\ -9 \end{pmatrix}$.

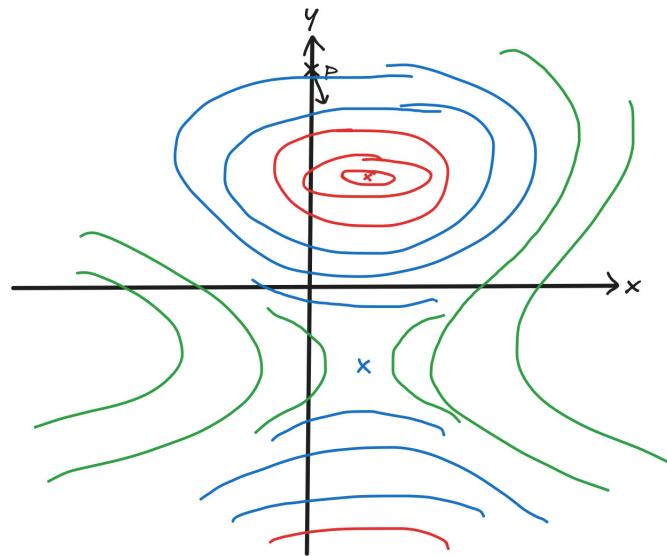


Figure 2: A contour plot of $z = f(x, y)$. Red is negative z , blue is small positive z , and green is large positive z . The red cross is the local minimum point, the blue cross is the saddle point, the black cross is P , and the black arrow is the direction of steepest descent from P .

Q2 (d)

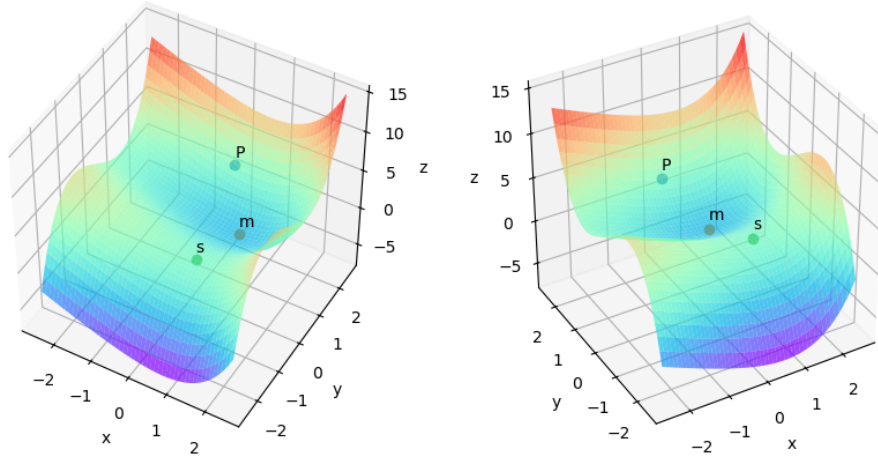


Figure 3: A 3D plot of $z = f(x, y)$ from two angles. m is the local minimum point and s is the saddle point.

```

1  #!/usr/bin/env python3
2
3  from pathlib import Path
4  import numpy as np
5  import matplotlib.pyplot as plt
6
7
8  def f(x, y):
9      return np.exp(x) + y**3 - 2 * x - 3 * y
10
11
12  def plot_point(ax, coords, name):
13      x, y = coords
14      ax.plot(x, y, f(x, y), "o")
15      ax.text(x, y, f(x, y) + 1, name)
16
17
18  def plot_3d_from_angle(filename, *, elev, azim):
19      x = np.linspace(-2.5, 2.5)
20      y = np.linspace(-2.5, 2.5)
21
22      X, Y = np.meshgrid(x, y)
23
24      fig = plt.figure()
25      ax = fig.add_subplot(projection="3d")
26      ax.set_box_aspect([1, 1, 1])
27      ax.view_init(elev=elev, azim=azim)
28
29      ax.plot_surface(X, Y, f(X, Y), alpha=0.7, cmap="rainbow")
30      ax.set_xlabel("x")
31      ax.set_ylabel("y")
32      ax.set_zlabel("z")
33
34      # Plot point P
35      plot_point(ax, (0, 2), "P")
36
37      # Plot critical points
38      plot_point(ax, (np.log(2), 1), "m")
39      plot_point(ax, (np.log(2), -1), "s")
40
41      plt.savefig(
42          Path(_file_).parent.parent / "imgs" / f"{filename}.png",
43          bbox_inches="tight",
44          pad_inches=0.2,
45      )
46      plt.clf()
47
48
49  def main() -> None:
50      plot_3d_from_angle("Q2d-1", elev=45, azim=-60)
51      plot_3d_from_angle("Q2d-2", elev=35, azim=-120)
52
53
54  if __name__ == "__main__":
55      main()

```

Figure 4: The code used to generate Figure 3. The code can also be found on GitHub.