MA268 Algebra 3, Assignment 4

Dyson Dyson

Question 6

Let $\mathbb{R}[X]$ be the $\mathbb{R}[T]$-module where multiplication is given by

$$\begin{aligned}&(a_0 + a_1 T + \cdots + a_n T^n) \cdot f(X) \\ &\qquad = a_0 f(X) + a_1 f'(X) + a_2 f''(X) + \cdots + a_n f^{(n)}(X)\end{aligned}$$

where $f^{(n)}(X)$ denotes the n-th derivative of $f(X)$ with respect to X.

(i) Compute $(1 + T - T^2 - 3T^5) \cdot (X + 3X^2)$.

(ii) Show that $\mathrm{Span}_{\mathbb{R}[T]}(X^n) = \mathrm{Span}_{\mathbb{R}}(1, X, \ldots, X^n)$.

(iii) Show that $\mathbb{R}[X]$ is not free as an $\mathbb{R}[T]$-module.

Q6 (i)

$$\begin{aligned}(1 + T - T^2 - 3T^5) \cdot (X + 3X^2) &= (X + 3X^2) + \frac{\mathrm{d}}{\mathrm{d}X}(X + 3X^2) \\ &\qquad - \frac{\mathrm{d}^2}{\mathrm{d}X^2}(X + 3X^2) - 3\frac{\mathrm{d}^5}{\mathrm{d}X^5}(X + 3X^2) \\ &= (X + 3X^2) + (1 + 6X) - 6 - 3(0) \\ &= -5 + 7X + 3X^2\end{aligned}$$

Q6 (ii)

$\mathrm{Span}_{\mathbb{R}[T]}(X^n)$ is X^n multiplied by a coefficient in $\mathbb{R}[T]$, so

$$\begin{aligned}\mathrm{Span}_{\mathbb{R}[T]}(X^n) &= (a_0 + a_1 T + a_2 T^2 + \cdots + a_n T^n) \cdot X^n \\ &= a_0 X^n + a_1 \frac{\mathrm{d}}{\mathrm{d}X} X^n + a_2 \frac{\mathrm{d}^2}{\mathrm{d}X^2} X^n + \cdots + a_n \frac{\mathrm{d}^n}{\mathrm{d}X^n} X^n \\ &= a_0 X^n + a_1 n X^{n-1} + a_2 n(n-1) X^{n-2} + \cdots + a_n n!\end{aligned}$$

for some $a_i \in \mathbb{R}$.

Likewise,

$$\mathrm{Span}_{\mathbb{R}}(1, X, \ldots, X^n) = b_0 + b_1 X + \cdots + b_n X^n$$

for some $b_i \in \mathbb{R}$.

Both spans contain all powers of X between X^0 and X^n inclusive. By careful inspection, we see that

$$b_i = a_{n-i} \prod_{j=i+1}^{n} j$$

and conclude $\mathrm{Span}_{\mathbb{R}[T]}(X^n) = \mathrm{Span}_{\mathbb{R}}(1, X, \ldots, X^n)$.

Q6 (iii)

If an $\mathbb{R}[T]$-basis existed for $\mathbb{R}[X]$, it would be generated by some set S containing powers of X. There must be some highest power of X in S, call it X^m.

We would expect $\mathrm{Span}_{\mathbb{R}[T]}(S) = \mathbb{R}[X]$ but as seen in part **(b)**, the highest power of X in $\mathrm{Span}_{\mathbb{R}[T]}(S)$ is X^m. But of course $X^{m+1} \in \mathbb{R}[X]$, so $\mathbb{R}[X]$ cannot be spanned by a finite subset, and therefore does not have a free $\mathbb{R}[T]$-basis.