MA265 Methods of Mathematical Modelling 3, Assignment 3

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Question 1

Fourier series: For integers m and n, show that

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\ \mathrm{d}x = \begin{cases} \pi & \text{if } m = n \\ 0 & \text{otherwise,} \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\ \mathrm{d}x = \begin{cases} \pi & \text{if } m = n \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\ \mathrm{d}x = 0 \quad \text{for all } m \text{ and } n.$$

Consider the function $\phi(x) = x$ on $[-\pi, \pi]$. Assuming ϕ can be written in the form

$$\phi(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx), \tag{1}$$

find the values of the coefficients a_k and b_k.

We will first consider the integral of $\sin(mx)\sin(nx)$. If $m = n$, the integral becomes

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\ \mathrm{d}x &= \int_{-\pi}^{\pi} \sin^2(nx)\ \mathrm{d}x \\
&= \frac{1}{2}\int_{-\pi}^{\pi} (1 - \cos(2nx))\ \mathrm{d}x \\
&= \frac{1}{2}\left[x - \frac{1}{2n}\sin(2nx)\right]_{-\pi}^{\pi} \\
&= \frac{1}{2}\left(\pi - \frac{1}{2n}\sin(2n\pi) - \left(-\pi - \frac{1}{2n}\sin(-2n\pi)\right)\right) \\
&= \frac{1}{2}\left(\pi - \frac{1}{2n}\sin(2n\pi) + \pi + \frac{1}{2n}\sin(-2n\pi)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(2\pi - \frac{1}{2n}\sin(2n\pi) - \frac{1}{2n}\sin(2n\pi)\right) \\
&= \pi.
\end{aligned}
$$

If instead $m \neq n$, we get

$$
\begin{aligned}
\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\ \mathrm{d}x &= \int_{-\pi}^{\pi} \frac{-\cos(mx+nx)+\cos(mx-nx)}{2}\ \mathrm{d}x \\
&= \frac{1}{2}\left[\frac{-1}{m+n}\sin(mx+nx) + \frac{1}{m-n}\sin(mx-nx)\right]_{-\pi}^{\pi} \\
&= \frac{1}{2}\Bigg(\frac{-1}{m+n}\sin((m+n)\pi) + \frac{1}{m-n}\sin((m-n)\pi) \\
&\qquad + \frac{1}{m+n}\sin(-(m+n)\pi) - \frac{1}{m-n}\sin(-(m-n)\pi)\Bigg) \\
&= 0
\end{aligned}
$$

since $m+n$ and $m-n$ are always integers, so $\sin((m+n)\pi)$ etc. are all 0.

We will now consider the integral of $\cos(mx)\cos(nx)$. As before, if $m = n$, the integral becomes

$$
\begin{aligned}
\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\ \mathrm{d}x &= \int_{-\pi}^{\pi} \cos^2(nx)\ \mathrm{d}x \\
&= \frac{1}{2}\int_{-\pi}^{\pi} (1+\cos(2nx))\ \mathrm{d}x \\
&= \frac{1}{2}\left[x + \frac{1}{2n}\sin(2nx)\right]_{-\pi}^{\pi} \\
&= \frac{1}{2}\left(\pi + \frac{1}{2n}\sin(2n\pi) - \left(-\pi + \frac{1}{2n}\sin(-2n\pi)\right)\right) \\
&= \frac{1}{2}\left(\pi + \frac{1}{2n}\sin(2n\pi) + \pi - \frac{1}{2n}\sin(-2n\pi)\right) \\
&= \frac{1}{2}\left(2\pi + \frac{1}{2n}\sin(2n\pi) + \frac{1}{2n}\sin(2n\pi)\right) \\
&= \pi + \frac{1}{2n}\sin(2n\pi) \\
&= \pi
\end{aligned}
$$

since $n \in \mathbb{Z}$, so $\sin(2n\pi)$ is always 0.

If instead $m \neq n$, we get

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\ \mathrm{d}x &= \int_{-\pi}^{\pi} \frac{\cos(mx+nx)+\cos(mx-nx)}{2}\ \mathrm{d}x \\
&= \frac{1}{2}\left[\frac{1}{m+n}\sin(mx+nx) + \frac{1}{m-n}\sin(mx-nx)\right]_{-\pi}^{\pi} \\
&= \frac{1}{2}\bigg(\frac{1}{m+n}\sin((m+n)\pi) + \frac{1}{m-n}\sin((m-n)\pi) \\
&\qquad - \frac{1}{m+n}\sin(-(m+n)\pi) - \frac{1}{m-n}\sin(-(m-n)\pi)\bigg) \\
&= 0
\end{aligned}$$

since $m+n$ and $m-n$ are always integers, so $\sin((m+n)\pi)$ etc. are all 0.

We will now consider the integral of $\sin(mx)\cos(nx)$. For any m and n,

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\ \mathrm{d}x &= \int_{-\pi}^{\pi} \frac{\sin(mx+nx)+\sin(mx-nx)}{2}\ \mathrm{d}x \\
&= \frac{1}{2}\left[\frac{-1}{m+n}\cos(mx+nx) - \frac{1}{m-n}\cos(mx-nx)\right]_{-\pi}^{\pi} \\
&= \frac{1}{2}\bigg(\frac{-1}{m+n}\cos((m+n)\pi) - \frac{1}{m-n}\cos((m-n)\pi) \\
&\qquad + \frac{1}{m+n}\cos(-(m+n)\pi) + \frac{1}{m-n}\cos(-(m-n)\pi)\bigg) \\
&= \frac{1}{2}\bigg(\frac{-1}{m+n}\cos((m+n)\pi) - \frac{1}{m-n}\cos((m-n)\pi) \\
&\qquad + \frac{1}{m+n}\cos((m+n)\pi) + \frac{1}{m-n}\cos((m-n)\pi)\bigg) \\
&= 0.
\end{aligned}$$

To find a_k in the Fourier series for ϕ, we can multiply both sides by $\cos(nx)$ and integrate. On the LHS, we get

$$\begin{aligned}
\int_{-\pi}^{\pi} x\cos(nx)\ \mathrm{d}x &= \left[\frac{nx\sin(nx)+\cos(nx)}{n^2}\right]_{-\pi}^{\pi} \\
&= \frac{n\pi\sin(n\pi)+\cos(n\pi) - \big(-n\pi\sin(-n\pi)+\cos(-n\pi)\big)}{n^2} \\
&= \frac{n\pi\sin(n\pi)+n\pi\sin(-n\pi)+\cos(n\pi)-\cos(-n\pi)}{n^2} \\
&= \frac{n\pi\sin(n\pi)-n\pi\sin(n\pi)+\cos(n\pi)-\cos(n\pi)}{n^2} \\
&= 0.
\end{aligned}$$

On the RHS, we get

$$\begin{aligned}
&a_0 \cos(nx) + \sum_{k=1}^{\infty} a_k \cos(kx)\cos(nx) + b_k \sin(kx)\cos(nx) \\
&\quad = \int_{-\pi}^{\pi} \left(a_0 \cos(nx) + \sum_{k=1}^{\infty} a_k \cos(kx)\cos(nx) + b_k \sin(kx)\cos(nx) \right) \,\mathrm{d}x \\
&\quad = a_0 \int_{-\pi}^{\pi} \cos(nx) \,\mathrm{d}x + \sum_{k=1}^{\infty} \left(a_k \int_{-\pi}^{\pi} \cos(kx)\cos(nx) \,\mathrm{d}x \right. \\
&\qquad \left. + \, b_k \int_{-\pi}^{\pi} \sin(kx)\cos(nx) \,\mathrm{d}x \right) \\
&\quad = 0 + \sum_{k=1}^{\infty} a_k \int_{-\pi}^{\pi} \cos(kx)\cos(nx) \,\mathrm{d}x \\
&\quad = 0a_0 + 0a_1 + \cdots + 0a_{n-1} + \pi a_n + 0a_{n+1} \cdots \\
&\quad = \pi a_n.
\end{aligned}$$

So $a_n = 0$.

We can do the same to find b_k, but with $\sin(nx)$. On the LHS, we get

$$\begin{aligned}
\int_{-\pi}^{\pi} x \sin(nx) \,\mathrm{d}x &= \left[\frac{\sin(nx) - nx\cos(nx)}{n^2} \right]_{-\pi}^{\pi} \\
&= \frac{\sin(n\pi) - n\pi\cos(n\pi) - \big(\sin(-n\pi) - n(-\pi)\cos(-n\pi)\big)}{n^2} \\
&= \frac{\sin(n\pi) - \sin(-n\pi) - n\pi\cos(n\pi) - n\pi\cos(-n\pi)}{n^2} \\
&= \frac{\sin(n\pi) + \sin(n\pi) - n\pi\cos(n\pi) - n\pi\cos(n\pi)}{n^2} \\
&= \frac{2\sin(n\pi) - 2n\pi\cos(n\pi)}{n^2}.
\end{aligned}$$

On the RHS, we get

$$\begin{aligned}
&a_0 \sin(nx) + \sum_{k=1}^{\infty} a_k \cos(kx)\sin(nx) + b_k \sin(kx)\sin(nx) \\
&= \int_{-\pi}^{\pi} \left(a_0 \sin(nx) + \sum_{k=1}^{\infty} a_k \cos(kx)\sin(nx) + b_k \sin(kx)\sin(nx) \right) \,\mathrm{d}x \\
&= a_0 \int_{-\pi}^{\pi} \sin(nx) \,\mathrm{d}x + \sum_{k=1}^{\infty} \left(a_k \int_{-\pi}^{\pi} \cos(kx)\sin(nx) \,\mathrm{d}x \right. \\
&\qquad \left. + b_k \int_{-\pi}^{\pi} \sin(kx)\sin(nx) \,\mathrm{d}x \right) \\
&= 0 + \sum_{k=1}^{\infty} b_k \int_{-\pi}^{\pi} \sin(kx)\sin(nx) \,\mathrm{d}x \\
&= 0b_0 + 0b_1 + \cdots + 0b_{n-1} + \pi b_n + 0b_{n+1} \cdots \\
&= \pi b_n.
\end{aligned}$$

So $b_n = \dfrac{2\sin(n\pi) - 2n\pi\cos(n\pi)}{n^2\pi}$.

Question 2

Fourier series: Let $\phi : [-\pi, \pi] \to \mathbb{R}$ be an even function. Prove that $\hat{\phi}(k) = \hat{\phi}(-k)$. Further, show that the Fourier series of ϕ is a cosine series, in the sense that

$$S_n[\phi](x) = \hat{\phi}(0) + 2\sum_{k=1}^{n} \hat{\phi}(k)\cos(kx).$$

With the substitution $y = -x$,

$$\begin{aligned}
\hat{\phi}(k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(x)\mathrm{e}^{-ikx}\ \mathrm{d}x \\
&= \frac{1}{2\pi}\int_{\pi}^{-\pi} \phi(-y)\mathrm{e}^{iky}(-\mathrm{d}y) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(-y)\mathrm{e}^{iky}\ \mathrm{d}y \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(y)\mathrm{e}^{-i(-k)y}\ \mathrm{d}y \\
&= \hat{\phi}(-k).
\end{aligned}$$

Therefore, using de Moivre's theorem,

$$\begin{aligned}
S_n[\phi](x) &= \sum_{k=-n}^{n} \hat{\phi}(k)\mathrm{e}^{ikx} \\
&= \hat{\phi}(0) + \sum_{k=1}^{n}\left(\hat{\phi}(k)\mathrm{e}^{ikx} + \hat{\phi}(-k)\mathrm{e}^{-ikx}\right) \\
&= \hat{\phi}(0) + \sum_{k=1}^{n}\hat{\phi}(k)\left(\mathrm{e}^{ikx} + \mathrm{e}^{-ikx}\right) \\
&= \hat{\phi}(0) + \sum_{k=1}^{n}\hat{\phi}(k)2\cos(kx) \\
&= \hat{\phi}(0) + 2\sum_{k=1}^{n}\hat{\phi}(k)\cos(kx).
\end{aligned}$$

$\square$

Question 3

Wave equation with inhomogeneous boundary conditions: Consider the boundary value problem:

$$\begin{cases} \partial_{tt}u(x,t) = \partial_{xx}u(x,t) & (x,t) \in (0,\pi) \times (0,\infty), \\ u(0,t) = -2 & t \in [0,\infty), \\ u(\pi,t) = 1 & t \in [0,\infty). \end{cases} \tag{2}$$

(i) Verify that

$$\overline{u}(x) = -2 + \frac{3}{\pi}x$$

is a stationary solution of the problem (i.e. one that does not depend on t).

(ii) Assume now the initial conditions

$$u(x,0) = \Phi(x) = \frac{3}{\pi}x - 2 + 5\sin(2x),$$
$$\partial_t u(x,0) = V(x) = \frac{1}{2}\sin(4x) + \frac{1}{3}\sin(11x).$$

Let $w(x,t)$ be the function such that $u(x,t) = \overline{u}(x) + w(x,t)$. State the initial boundary value problem satisfied by w.

(iii) Solve the initial boundary value problem for w, using the general form of the solution

$$w(x,t) = \sum_{j=0}^{\infty} \Big(A_j \cos(jt) + B_j \sin(jt) \Big) \sin(jx).$$

Then, state the solution $u(x,t)$ of (2).

Q3 (i)

Since $\overline{u}$ does not depend on t, $\partial_{tt}\overline{u} = \partial_t\overline{u} = 0$. It is trivial to check that $\overline{u}$ satisfies the initial boundary conditions $\overline{u}(0) = -2$ and $\overline{u}(\pi) = 1$. Also

$$\partial_x\overline{u}(x) = \frac{3}{\pi}, \qquad \partial_{xx}\overline{u}(x) = 0,$$

and so $\partial_{tt}\overline{u} = \partial_{xx}\overline{u}$. Therefore $\overline{u}$ is a stationary solution to (2).

Q3 (ii)

$$\begin{cases} \partial_{tt} w(x,t) = \partial_{xx} w(x,t) & (x,t) \in (0,\pi) \times (0,\infty), \\ w(0,t) = 0 & t \in [0,\infty), \\ w(\pi,t) = 0 & t \in [0,\infty), \\ w(x,0) = 5\sin(2x) & x \in (0,\pi), \\ \partial_t(x,0) = \frac{1}{2}\sin(4x) + \frac{1}{3}\sin(11x) & x \in (0,\pi). \end{cases}$$

Q3 (iii)

Let

$$w(x,t) = \sum_{j=0}^{\infty} \Big(A_j \cos(jt) + B_j \sin(jt) \Big) \sin(jx).$$

This satisfies the conditions $w(0,t) = 0$ and $w(\pi,t) = 0$ because $\sin 0 = \sin \pi = 0$, so the $\sin(jx)$ term annihilates the rest.

For the condition $w(x,0) = 5\sin(2x)$, we have

$$\begin{aligned} w(x,0) &= \sum_{j=0}^{\infty} \Big(A_j \cos(0) + B_j \sin(0) \Big) \sin(jx) \\ &= \sum_{j=0}^{\infty} A_j \sin(jx) \\ &= 5\sin(2x) \end{aligned}$$

which means that $A_2 = 5$ and $A_j = 0$ for all $j \neq 2$.

The derivatives are

$$\partial_t w(x,t) = \sum_{j=0}^{\infty} \Big(-jA_j \sin(jt) + jB_j \cos(jt) \Big) \sin(jx)$$

$$\partial_{tt} w(x,t) = \sum_{j=0}^{\infty} \Big(-j^2 A_j \cos(jt) - j^2 B_j \sin(jt) \Big) \sin(jx)$$

$$\partial_x w(x,t) = \sum_{j=0}^{\infty} \Big(A_j \cos(jt) + B_j \sin(jt) \Big) j \cos(jx)$$

$$\partial_{xx} w(x,t) = \sum_{j=0}^{\infty} \Big(A_j \cos(jt) + B_j \sin(jt) \Big) (-j^2) \sin(jx)$$

So clearly $\partial_{tt} w = \partial_{xx} w$. Also to satisfy the final condition, we need

$$\begin{aligned}
\partial_t w(x,0) &= \sum_{j=0}^{\infty} \Big(-jA_j \sin(0) + jB_j \cos(0) \Big) \sin(jx) \\
&= \sum_{j=0}^{\infty} jB_j \sin(jx) \\
&= \frac{1}{2} \sin(4x) + \frac{1}{3} \sin(11x)
\end{aligned}$$

which means that $4B_4 = \frac{1}{2}$ so $B_4 = 2$, and $11B_{11} = \frac{1}{3}$ so $B_{11} = \frac{11}{3}$, and $B_j = 0$ for all $b \notin \{4, 11\}$.

Therefore

$$w(x,t) = 5\cos(2t)\sin(2x) + 2\sin(4t)\sin(4x) + \frac{11}{3}\sin(11t)\sin(11x)$$

and therefore

$$u(x,t) = -2 + \frac{3}{\pi}x + 5\cos(2t)\sin(2x) + 2\sin(4t)\sin(4x) + \frac{11}{3}\sin(11t)\sin(11x).$$

Question 4

Fourier series: Let $a \in \mathbb{R} \setminus \mathbb{Z}$. Compute the Fourier series of

$$\phi_a(x) = \frac{\pi}{\sin(\pi a)} \mathrm{e}^{-iax}.$$

Then, assuming that

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \left|\phi_a(x) - S_n[\phi_a](x)\right|^2 \, \mathrm{d}x = 0,$$

such that Parseval's identity holds, deduce that

$$\sum_{k\in\mathbb{Z}} \frac{1}{(k+a)^2} = \frac{\pi^2}{\sin^2(\pi a)}.$$

The coefficients are

$$\begin{aligned}
\hat{\phi}_a(k) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_a(x) \mathrm{e}^{-ikx} \, \mathrm{d}x \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\pi}{\sin(\pi a)} \mathrm{e}^{-iax} \mathrm{e}^{-ikx} \, \mathrm{d}x \\
&= \frac{1}{2\sin(\pi a)} \int_{-\pi}^{\pi} \mathrm{e}^{-i(a+k)x} \, \mathrm{d}x \\
&= \frac{1}{2\sin(\pi a)} \left[\frac{\mathrm{e}^{-i(a+k)x}}{-i(a+k)}\right]_{-\pi}^{\pi} \\
&= \frac{i}{2\sin(\pi a)} \frac{\mathrm{e}^{-i(a+k)\pi} - \mathrm{e}^{i(a+k)\pi}}{(a+k)} \\
&= \frac{i\left(\mathrm{e}^{-i(a+k)\pi} - \mathrm{e}^{i(a+k)\pi}\right)}{2(a+k)\sin(\pi a)} \\
&= \frac{i\big(-2i\sinh(i(a+k)\pi)\big)}{2(a+k)\sin(\pi a)} \\
&= \frac{\sinh(i(a+k)\pi)}{(a+k)\sin(\pi a)} \\
&= \frac{\sin((a+k)\pi)}{(a+k)\sin(\pi a)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
S_n[\phi_a](x) &= \sum_{k=-n}^{n} \hat{\phi}(k)\, \mathrm{e}^{-ikx} \\
&= \sum_{k=-n}^{n} \frac{\sin((a+k)\pi)\, \mathrm{e}^{-ikx}}{(a+k)\sin(\pi a)}.
\end{aligned}$$

Parseval's identity says

$$\int_{-\pi}^{\pi} |\phi_a(x)|^2 \, \mathrm{d}x = 2\pi \sum_{k\in\mathbb{Z}} \left|\hat{\phi}_a(k)\right|^2.$$

And so we get

$$\begin{aligned}
\int_{-\pi}^{\pi} \left|\frac{\pi}{\sin(\pi a)}\mathrm{e}^{-iax}\right|^2 \, \mathrm{d}x &= 2\pi \sum_{k\in\mathbb{Z}} \left|\frac{\sin((a+k)\pi)}{(a+k)\sin(\pi a)}\right|^2 \\
\frac{\pi^2}{\sin^2(\pi a)} \int_{-\pi}^{\pi} \left|\mathrm{e}^{-iax}\right|^2 \, \mathrm{d}x &= 2\pi \frac{1}{\sin^2(\pi a)} \sum_{k\in\mathbb{Z}} \frac{\sin^2((a+k)\pi)}{(a+k)^2} \\
\frac{\pi^2}{\sin^2(\pi a)} \int_{-\pi}^{\pi} 1 \, \mathrm{d}x &= \frac{2\pi}{\sin^2(\pi a)} \sum_{k\in\mathbb{Z}} \frac{(\sin(a\pi + k\pi))^2}{(a+k)^2} \\
\pi^2(2\pi) &= 2\pi \sum_{k\in\mathbb{Z}} \frac{\big(\sin(a\pi)\cos(k\pi) + \cos(a\pi)\sin(k\pi)\big)^2}{(a+k)^2} \\
\pi^2 &= \sum_{k\in\mathbb{Z}} \frac{\sin^2(a\pi)\cos^2(k\pi)}{(a+k)^2} \\
\pi^2 &= \sum_{k\in\mathbb{Z}} \frac{\sin^2(a\pi)}{(a+k)^2} \\
\frac{\pi^2}{\sin^2(a\pi)} &= \sum_{k\in\mathbb{Z}} \frac{1}{(a+k)^2}
\end{aligned}$$

□