MA150 Algebra 2, Assignment 4

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Question 4

> Let V be a Euclidean space with inner product $\langle \cdot, \cdot \rangle$. Suppose $w_1, \ldots, w_n$ is an orthonormal basis of V. Of course $\forall\ v \in V,\ \exists\ \lambda_i \in \mathbb{R}$ such that $v = \sum \lambda_i w_i$. Show that in fact $\lambda_i = \langle v, w_i \rangle$.

For any fixed $i \leq n$

$$\begin{aligned}
v &= \sum_{j=1}^{n} \lambda_j w_j \\
\langle v, w_i \rangle &= \left\langle \sum_{j=1}^{n} \lambda_j w_j, w_i \right\rangle \\
&= \sum_{j=1}^{n} \lambda_j \langle w_j, w_i \rangle \\
&= \lambda_1 \langle w_1, w_i \rangle + \cdots + \lambda_i \langle w_i, w_i \rangle + \cdots + \lambda_n \langle w_n, w_i \rangle \\
&= \lambda_i
\end{aligned}$$

Since all $w_1, \ldots, w_n$ are orthonormal, so all but one of the inner products are zero.

Therefore $\langle v, w_i \rangle = \lambda_i$ for all $i = 1, \ldots, n$, as required.

Question 5

Let $V = \mathbb{R}^3$ equipped with the usual inner (dot) product. Let

$$v_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}, \quad v_2 = \begin{pmatrix}2\\0\\1\end{pmatrix}, \quad v_3 = \begin{pmatrix}1\\1\\2\end{pmatrix}$$

Q5 (a)

Apply the Gram-Schmidt orthogonalisation process to v_1, v_2, v_3 to construct an orthonormal basis w_1, w_2, w_3.

$$w_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$\begin{aligned}u_2 &= v_2 - (v_2 \cdot w_1)w_1 \\ &= \begin{pmatrix}2\\0\\1\end{pmatrix} - \frac{2}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\-1\\0\end{pmatrix}\end{aligned}$$

$$w_2 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$\begin{aligned}u_3 &= v_3 - (v_3 \cdot w_1)w_1 - (v_3 \cdot w_2)w_2 \\ &= \begin{pmatrix}1\\1\\2\end{pmatrix} - \frac{2}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} - 0\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\2\end{pmatrix}\end{aligned}$$

$$w_3 = \frac{u_3}{\|u_3\|} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Therefore our orthonormal basis w_1, w_2, w_3 is $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$

Q5 (b)

Consider $v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Find $\lambda_1, \lambda_2, \lambda_3$ such that $v = \lambda_1 w_1 + \lambda_2 w_2 + \lambda_3 w_3$.

Clearly $v = w_3$ so if $v = \lambda_1 w_1 + \lambda_2 w_2 + \lambda_3 w_3$ then, since w_1, w_2, w_3 are linearly independent, $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 1$.

Question 6

Consider the symmetric matrix

$$A = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$$

and define a function

$$\begin{aligned} \varphi : \mathbb{R}^2 &\to \mathbb{R} \\ v = \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto v^T A v = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

Notice that $\varphi(\underline{0}) = 0$. This question determines precise conditions for which $\varphi(v) = 0$ for all $v \neq \underline{0}$.

Notice that $\varphi(\lambda v) = \lambda^2 \varphi(v)$ so we only need to consider $v \in \mathbb{R}^2$ of the form

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v_x = \begin{pmatrix} x \\ 1 \end{pmatrix} \ \forall\, x \in \mathbb{R}$$

Q6 (a)

If $a \leq 0$ find a vector $v \neq \underline{0}$ with $\varphi(v) \leq \underline{0}$.

$\varphi(e_1) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ \frac{b}{2} \end{pmatrix} = a$ so if $a \leq 0$ then $\varphi(e_1) \leq 0$.

Q6 (b)

Express $\varphi(v_x)$ as a polynomial in x.

$$\begin{aligned} \varphi(v_x) &= \begin{pmatrix} x & 1 \end{pmatrix} \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} x & 1 \end{pmatrix} \begin{pmatrix} ax + \frac{b}{2} \\ \frac{b}{2}x + c \end{pmatrix} \\ &= \left(ax + \frac{b}{2} \right) x + \frac{b}{2}x + c \\ &= ax^2 + bx + c \end{aligned}$$

Q6 (c)

Suppose $a > 0$. Prove that $\varphi(v_x) > 0$ for all $x \in \mathbb{R}$ if and only if $b^4 - 4ac < 0$. Recall that $\det A = ac - \frac{b^2}{4}$, so that this condition is exactly the same as $\det A > 0$.

Suppose $a > 0$ and $b^2 - 4ac < 0$. That means $ax^2 + bx + c = 0$ has no roots and since $a > 0$, $ax^2 + bx + c > 0$, so $\varphi(v_x) > 0 \ \forall \ x \in \mathbb{R}$.

Conversely, suppose $\varphi(v_x) > 0$. Therefore $ax^2 + bx + c > 0 \ \forall \ x \in \mathbb{R}$, which means this quadratic has no roots and therefore has a negative discriminant, so $b^2 - 4ac < 0$.

Question 7

Continue with the notation and matrix A from the previous question. For any vectors $v, w \in \mathbb{R}^2$ define $\langle v, w \rangle = v^T A w$.

Q7 (a)

Show that $\langle v, w \rangle = \langle w, v \rangle$ for any $v, w \in \mathbb{R}^2$.

The transpose of a scalar is the same scalar, so

$$\begin{aligned}
\langle v, w \rangle &= \langle v, w \rangle^T \\
&= \left(v^T A w\right)^T \\
&= w^T A^T \left(v^T\right)^T \\
&= w^T A^T v \\
&= w^T A v \qquad \text{since } A^T = A \\
&= \langle w, v \rangle
\end{aligned}$$

Q7 (b)

Show that for any $v_1, v_2, w \in \mathbb{R}^2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$,

$$\langle \lambda_1 v_1 + \lambda_2 v_2, w \rangle = \lambda_1 \langle v_1, w \rangle + \lambda_2 \langle v_2, w \rangle$$

$$\begin{aligned}
\langle \lambda_1 v_1 + \lambda_2 v_2, w \rangle &= (\lambda_1 v_1 + \lambda_2 v_2)^T A w \\
&= \left(\lambda_1 v_1^T + \lambda_2 v_2^T\right) A w \\
&= \lambda_1 v_1^T A w + \lambda_2 v_2^T A w \\
&= \lambda_1 \langle v_1, w \rangle + \lambda_2 \langle v_2, w \rangle
\end{aligned}$$

Q7 (c)

Suppose $a > 0$ and $\det A > 0$. Show that $\langle v, v \rangle \geq 0$ for any $v \in V$, and that $\langle v, v \rangle = 0$ if and only if $v = \underline{0}$.

Since $a > 0$ and $\det A > 0$, $b^4 - 4ac < 0$. We know that $\langle v, v \rangle = v^T A v$ so $\langle v, v \rangle$ is equivalent to $\varphi(v)$ from Question 6, and we know that $\varphi(v) \geq 0 \ \forall \ v \in \mathbb{R}^2$ when $a > 0$ and $\det A > 0$, with equality only when $v = 0$.

Essentially, the desired result follows immediately from Question 6 and the observation that $\langle v, v \rangle = \varphi(v)$.

Q7 (d)

Which of the following matrices A determine an inner product on $V = \mathbb{R}^2$ by the formula $v^T Aw$ above?

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \quad \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix} \quad \begin{pmatrix} 5 & 3 \\ 3 & -2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

For a matrix A to determine an inner product as above, we need $a > 0$ and $\det A > 0$, so only $\begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ determine inner products