MA139 Analysis 2, Assignment 1

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Question 1

Show that the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

converges for every x in the interval satisfying $-1 \leq x < 1$ and that it diverges for all other real values of x.

Let $S = \sum_{n=1}^{\infty} \frac{x^n}{n}$. Using the ratio test, we find

$$\begin{aligned}
r &= \lim_{n\to\infty} \left| \frac{x^{n+1}}{n+1} \middle/ \frac{x^n}{n} \right| \\
&= \lim_{n\to\infty} \left| \frac{x^{n+1} n}{x^n (n+1)} \right| \\
&= \lim_{n\to\infty} \left| x \frac{n}{(n+1)} \right| \\
&= |x| \lim_{n\to\infty} \left| \frac{n}{(n+1)} \right| \\
&= |x|
\end{aligned}$$

S converges when $r < 1$, so S converges when $|x| < 1$.

Note that the ratio test requires the terms to be nonzero, but the terms of S are only 0 when $x = 0$, and S clearly also converges in this case.

Now we check the edges of the radius of convergence. When $x = 1$, S is the harmonic series, which we know diverges.

When $x = -1$, $S = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ and we can use the alternating series test. $\frac{1}{n}$ is decreasing, non-negative, and converges to 0, so S must converge by the alternating series test.

Therefore S converges when $-1 \leq x < 1$.

Question 2

Let $\sum_{n=1}^{\infty} a_n x^n$ be the power series in which

$$a_n = \begin{cases} 1 & \text{if } n \text{ is a prime number} \\ 0 & \text{if } n \text{ is not a prime number} \end{cases}$$

Prove that the series has radius of convergence 1.

Let $S = \sum_{n=1}^{\infty} a_n x^n$. When $0 < x < 1$,

$$\sum_{n=1}^{\infty} a_n x^n < \sum_{n=1}^{\infty} x^n = \frac{1}{1-x},$$

so S converges.

Likewise when $-1 < x < 0$,

$$\sum_{n=1}^{\infty} |a_n x^n| < \sum_{n=1}^{\infty} |x|^n = \frac{1}{1-|x|},$$

so S converges absolutely.

S trivially converges when $x = 0$, so it converges when $-1 < x < 1$. Now we only need to check $x = 1$ and $x = -1$.

We know there are infinitely many primes, so $\sum\limits_{n=1}^{\infty} a_n \to \infty$. And when $x = 1$, $S = \sum\limits_{n=1}^{\infty} a_n$, so S cannot converge.

Note that all prime number except 2 are odd. So when $x = -1$, $S = -1 + 1 + 1 + 1 + \cdots = -1 + \sum\limits_{1}^{\infty} 1$, which clearly diverges to ∞.

Therefore, S converges exactly when $-1 < x < 1$.

Question 3

Let $\sum a_n x^n$ be a power series with radius of convergence R. Show that if the closed interval $[-K, K]$ lies inside the interval $(-R, R)$ then the function f given by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

is bounded on the interval $[-K, K]$.

It was proven in lectures that any convergent power series is continuous in its radius of convergence. Therefore $f(x)$ is continuous in $(-R, R)$ and therefore also in $[-K, K]$.

Since $f(x)$ is continuous in $[-K, K]$, by the extreme value theorem, it is bounded on the interval and attains its bounds.

Question 4

By considering the ratio of successive terms, show that the sequence $\left(\frac{e^n n!}{n^n}\right)$ is increasing. Plot a graph of the first 10 or so terms.

Plot another graph of the squares $\left(\frac{e^n n!}{n^n}\right)^2$. How fast do you think they are growing?

The ratio of successive terms is

$$\begin{aligned}
&\frac{\mathrm{e}^{n+1}\,(n+1)!\,n^n}{\mathrm{e}^n\,n!\,(n+1)^{n+1}} \\
&= \frac{\mathrm{e}\,(n+1)\,n^n}{(n+1)^{n+1}} \\
&= \mathrm{e}\left(\frac{n}{n+1}\right)^n \\
&= \mathrm{e}\left(\frac{n+1}{n}\right)^{-n} \\
&= \mathrm{e}\left(\left(1+\frac{1}{n}\right)^n\right)^{-1} \\
&\to \mathrm{e}\,(\mathrm{e})^{-1} \\
&= 1
\end{aligned}$$

The ratio tends to 1. In particular, $\left(1+\frac{1}{n}\right)^n$ tends to e from below, so $\left(1+\frac{1}{n}\right)^{-n}$ tends to $\frac{1}{\mathrm{e}}$ from above. Therefore the ratio of successive terms tends to 1 from above and the ratio is therefore always ≥ 1, so the terms of the sequence are increasing.

The sequence $\left(\frac{\mathrm{e}^n n!}{n^n}\right)^2$ looks linear. Experimentally it has a gradient of 6.288747, which looks suspiciously like 2π. In fact, the line $y = 2\pi x + 1$ fits it almost exactly.

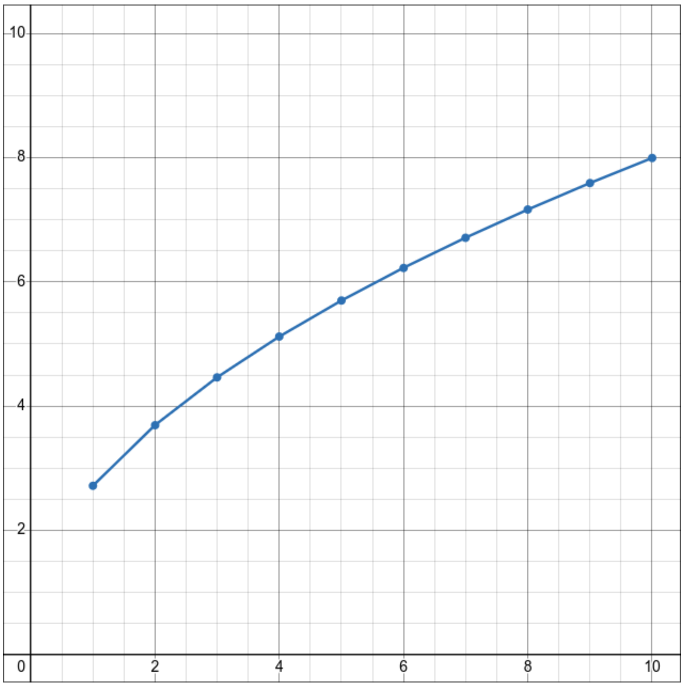

Figure 1: The first 10 terms of $\left(\dfrac{\mathrm{e}^n n!}{n^n}\right)$

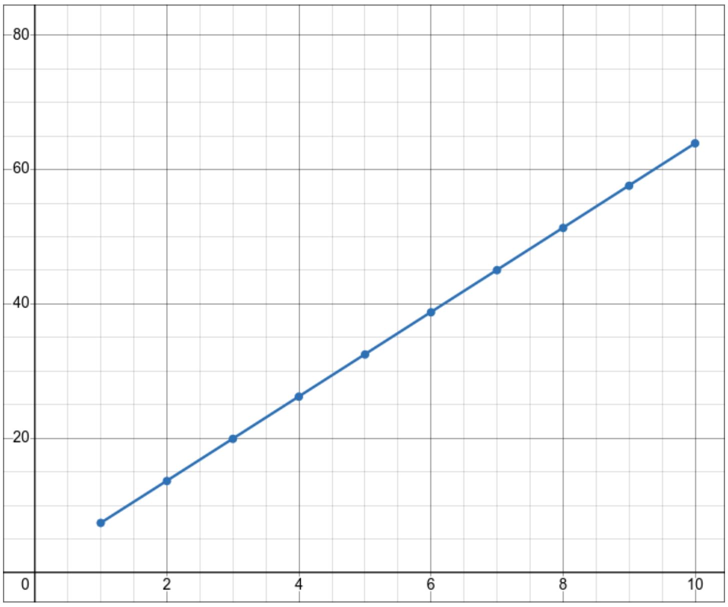

Figure 2: The first 10 terms of $\left(\dfrac{\mathrm{e}^n n!}{n^n}\right)^2$

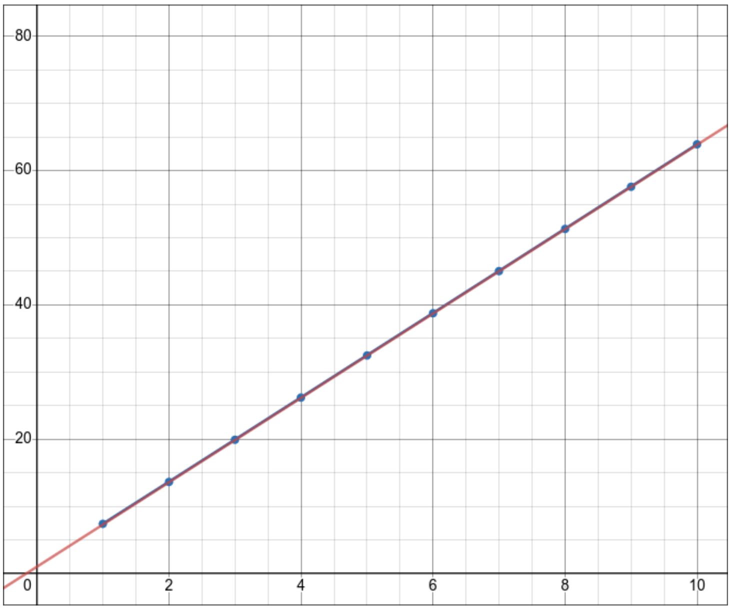

Figure 3: $\left(\frac{\mathrm{e}^n n!}{n^n}\right)^2$ in blue and $y = 2\pi x + 1$ in red