MA243 Geometry, Assignment 4

Dyson Dyson

Question 2

Which of the following sets of points in $\mathbb{P}^2(\mathbb{R})$ are collinear (i.e. lie on a projective line)? In the collinear case, give an equation in the homogeneous coordinates $(X_0 : X_1 : X_2)$ for the projective line.

1. $(2:3:1), (1:3:2), (2:4:2)$
2. $(1:2:3), (3:2:1), (2:4:2)$

For the first set of points to form a plane through the origin, we want to find $a, b, c \in \mathbb{R}$ such that $ax + by + cz = 0$, so

$$\begin{aligned} 2a + 3b + c &= 0 \\ a + 3b + 2c &= 0 \\ 2a + 4b + 2c &= 0 \end{aligned}$$

Solving this system of linear equations gives $a + b = 0$ and $b + c = 0$, so $a = c$ and $b = -a$. Therefore the plane $x - y + z = 0$ contains these three points, so they are collinear in $\mathbb{P}^2(\mathbb{R})$.

The points $(2:3:1)$ and $(1:3:2)$ each correspond to a 1-dimensional subspace of $\mathbb{R}^3$. Their span is a 2-dimensional subspace in $\mathbb{R}^3$ and hence a line in $\mathbb{P}^2(\mathbb{R})$. This line has equation

$$\left\langle \left\{ (2:3:1), (1:3:2) \right\} \right\rangle.$$

For the second set of points to form a plane through the origin, we again want to find $a, b, c \in \mathbb{R}$ such that $ax + by + cz = 0$, so

$$\begin{aligned} a + 2b + 3c &= 0 \\ 3a + 2b + c &= 0 \\ 2a + 4b + 2c &= 0 \end{aligned}$$

That means we need

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 4 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \underline{0} \tag{1}$$

where at least one of a, b, c are non-zero. That means we need the determinant of this matrix to equal 0 but

$$\begin{aligned}\begin{vmatrix}1 & 2 & 3\\ 3 & 2 & 1\\ 2 & 4 & 2\end{vmatrix} &= \begin{vmatrix}2 & 1\\ 4 & 2\end{vmatrix} - 2\begin{vmatrix}3 & 1\\ 2 & 2\end{vmatrix} + 3\begin{vmatrix}3 & 2\\ 2 & 4\end{vmatrix}\\ &= 0 - 2\cdot 4 + 3\cdot 8\\ &= 16\\ &\neq 0\end{aligned}$$

Therefore the only a, b, c that satisfy (1) are $a = b = c = 0$. Therefore the three points $(1:2:3), (3:2:1), (2:4:2)$ are not collinear in $\mathbb{P}^2(\mathbb{R})$.

In contrast, the determinant of the matrix for the first three points would be

$$\begin{aligned}\begin{vmatrix}2 & 3 & 1\\ 1 & 3 & 2\\ 2 & 4 & 2\end{vmatrix} &= 2\begin{vmatrix}3 & 2\\ 4 & 2\end{vmatrix} - 3\begin{vmatrix}1 & 2\\ 2 & 2\end{vmatrix} + \begin{vmatrix}1 & 3\\ 2 & 4\end{vmatrix}\\ &= 2(-2) - 3(-2) + (-2)\\ &= -4 + 6 - 2\\ &= 0\end{aligned}$$

Question 3

Let ℓ_1, ℓ_2 be lines in $\mathbb{R}^2$ defined by $x_0 = -1$ and $x_1 = 3$, respectively. Let $F : \ell_1 \to \ell_2$ be the perspectivity associated with some point $O \in \mathbb{R}^2$ of which we know $O \neq 0, O \notin \ell_1 \cup \ell_2$ (i.e. F is given by projection from O). We may then write $F(-1, x_1) = (f(x_1), 3)$ for some undetermined function $f : \mathbb{R} \to \mathbb{R}$.

(a) Assume we know that $f(1) = 3$ and $f(2) = 0$. Determine O.

(b) Determine $f(3)$.

(c) Consider the (injective) map

$$\begin{aligned} \ell_1 &\to \mathbb{P}^1, \\ (-1, x_1) &\mapsto (x_1 : 1). \end{aligned}$$

Find a point in $\mathbb{P}^1 \setminus \ell_1$. Similarly, in the following we will also use the (injective) map

$$\begin{aligned} \ell_2 &\to \mathbb{P}^1, \\ (x_0, 3) &\mapsto (x_0 : 1). \end{aligned}$$

(d) Let A be an invertible 2×2 matrix. Then multiplication by A sends lines in $\mathbb{R}^2$ to lines in $\mathbb{R}^2$, thus a map $T_A : \mathbb{P}^1 \to \mathbb{P}^1$. Find A such that T_A restricts to $F : \ell_1 \to \ell_2$.

(e) Use the matrix A to determine an expression for f as a ratio of two polynomials.

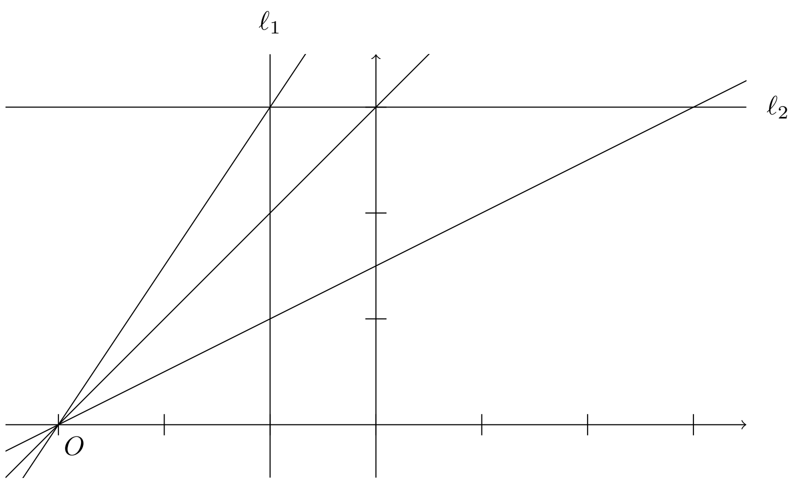

Figure 1: A diagram of the situation

Q3 (a)

Since $f(1) = 3$ and $f(2) = 0$, we have $F(-1, 1) = (3, 3)$ and $F(-1, 2) = (0, 3)$. Therefore we can draw lines through these pairs of points, and O will be their intersection. The lines are

$$x_1 = \frac{1}{2}x_0 + \frac{3}{2}$$
$$x_1 = x_0 + 3$$

or written more nicely,

$$-x_0 + 2x_1 = 3$$
$$-x_0 + x_1 = 3$$

The intersection of these lines is $x_0 = -3$ and $x_1 = 0$, so $O = (-3, 0)$.

Q3 (b)

We know $F(-1, 3) = (f(3), 3)$, so we draw a line through O and $(-1, 3)$ and see where it intersects ℓ_2. The equation of this line is

$$x_1 = \frac{3}{2}x_0 + \frac{9}{2}.$$

We plug in $x_1 = 3$ and find $x_0 = -1$, so $f(3) = -1$.

Q3 (c)

The point $(0 : 1)$ is in $\mathbb{P}^1 \setminus \ell_1$ because it is parallel to ℓ_1 in $\mathbb{R}^2$. Any other line in $\mathbb{R}^2$ would intersect ℓ_1 at one point.

Q3 (d)

The standard frame of reference for $\mathbb{P}^1(\mathbb{R})$ is $(1 : 0), (0 : 1), (1 : 1)$. We want to find matrices B and C such that B sends the standard frame of reference to points on ℓ_1 and C sends the standard frame of reference to points on ℓ_2. Then $A = CB^{-1}$.

Let's say we want B to map $(1 : 0)$ to $(-1 : 2)$ and $(0 : 1)$ to $(-1 : 1)$. Then using Proposition 6.40 in the lecture notes, we get

$$B = \begin{pmatrix} -\lambda & -\mu \\ 2\lambda & \mu \end{pmatrix}$$

for some $\lambda, \mu \in \mathbb{R}$. We also need B to map $(1:1)$ to $(-1:3)$, so we get

$$\begin{aligned} -\lambda - \mu &= -1 \\ 2\lambda + \mu &= 3 \end{aligned}$$

and we conclude that $\lambda = 2$ and $\mu = -1$. Therefore

$$B = \begin{pmatrix} -2 & 1 \\ 4 & -1 \end{pmatrix}.$$

We do the same thing for C, but being careful to map points in the correct pairings. $B(1:0) = (-1:2)$ so we want $C(1:0) = (0:3)$ and $B(0:1) = (-1:1)$ so we want $C(0:1) = (3:3)$, so we get

$$C = \begin{pmatrix} 0 & 3\mu \\ 3\lambda & 3\mu \end{pmatrix}$$

for some $\lambda, \mu \in \mathbb{R}$. We also need $C(1:1) = (-1:3)$, so we get

$$\begin{aligned} 3\mu &= -1 \\ 3\lambda + 3\mu &= 3 \end{aligned}$$

and we conclude that $\lambda = \frac{4}{3}$ and $\mu = -\frac{1}{3}$. Therefore

$$C = \begin{pmatrix} 0 & -1 \\ 4 & -1 \end{pmatrix}.$$

Now we just need to find A.

$$\begin{aligned} \det B &= 2 - 4 \\ &= -2 \\ B^{-1} &= \frac{1}{-2} \begin{pmatrix} -1 & -1 \\ -4 & -2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 2 & 1 \end{pmatrix} \\ A &= CB^{-1} \\ &= \begin{pmatrix} 0 & -1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -2 & -1 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Q3 (e)

We know that $F(-1,x) = (f(x),3)$, we can use A to find f.

$$\begin{pmatrix} -2 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ x \end{pmatrix} = \lambda\begin{pmatrix} 2-x \\ x \end{pmatrix} = \begin{pmatrix} f(x) \\ 3 \end{pmatrix}$$

$$\lambda(2-x) = f(x)$$

$$\lambda x = 3$$

We find $\lambda = \dfrac{x}{3}$ and so

$$f(x) = \frac{6-3x}{x}.$$

Indeed, this satisfies the three values we already know: $f(1) = 3$, $f(2) = 0$, and $f(3) = -1$.