MA141 Analysis 1, Assignment 4

Dyson Dyson

Question 5

> Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are continuous functions, and that we have $f(x) = g(x)$ for every $x \in \mathbb{Q}$. Use sequential continuity to show that $f = g$ everywhere.

For every real number x, we can define a sequence (a_n) as $a_n = \dfrac{\lfloor x \cdot 10^n \rfloor}{10^n}$, starting at $n = 0$. This is a way to generate decimal truncations of x. For example if $x = \pi$, then $a_0 = 3,\ a_1 = 3.1,\ a_2 = 3.14,\ \ldots,\ a_{10} = 3.1415926535,\ \ldots$ It is clear that $a_n \to x$ as $n \to \infty$.

We can use this process of generating a sequence for any real number to fill in the gaps of f and g. Let $x \in \mathbb{R}$ and define (a_n) as above to be the sequence converging to x. All terms of a_n are rational, so $f(a_0) = g(a_0),\ f(a_1) = g(a_1),\ \ldots$ Since f and g are continuous, $f(a_n) \to f(x)$ and $g(a_n) \to g(x)$, and since $f(a_n) = g(a_n)\ \forall\ n$, we must conclude that $f(x) = g(x)\ \forall\ x \in \mathbb{R}$.

Question 7

> Let $f\colon (-\infty, 0] \to \mathbb{R}$ and $g\colon [0, \infty) \to \mathbb{R}$ both be continuous on their entire domain. Show that the function
>
> $$h(x) = \begin{cases} f(x) & x \leq 0 \\ g(x) & x > 0 \end{cases}$$
>
> is continuous at $x = 0$ (and hence on $\mathbb{R}$) if and only if $f(0) = g(0)$.

If h is continuous at 0, then $\forall\ \varepsilon > 0, \exists\ \delta > 0$ such that $|x| < \delta \implies |h(x)| < \varepsilon$.

I just don't know what to do with this question, sorry.

Question 11

> Show that any continuous function $f\colon [a,b] \to [a,b]$ has a fixed point, i.e. there exists an $x^* \in [a,b]$ such that $f(x^*) = x^*$.
>
> Hint: consider the function $g(x) = f(x) - x$ and use the Intermediate Value Theorem.
>
> Give an example to show that the conclusion is not true if $f\colon (a,b) \to (a,b)$.

Let $g(x) = f(x) - x$. Then we have three cases, either $g(a) < g(b)$, $g(a) > g(b)$, or $g(a) = g(b)$. We only get the final case if $f(x) = x$, in which case every point is a fixed point.

In the case of $g(a) < g(b)$, we know $g(a) < 0 < g(b)$ so by the Intermediate Value Theorem, $g(c) = 0$ for some $c \in (a,b)$. Therefore $f(c) = c$, so c is a fixed point of f.

Likewise for the case of $g(a) > g(b)$, we can show $g(a) > 0 > g(b)$ in the same way, so we can find a fixed point using the same logic.

Now let $f : (a,b) \to (a,b)$. The example $f(x) = x^2$ would have fixed points at $x = 0$ and $x = 1$, but these are not in the domain, so $f(0)$ and $f(1)$ are not defined. Therefore f has no fixed point and shows that we can avoid fixed points in this case.

Question 19

> Suppose that $f\colon [0,\infty) \to \mathbb{R}$ is continuous and that $f(x) \to L$ as $x \to \infty$. Show that f is bounded above and below on $[0,\infty)$. Show that f need not attain both its upper and lower bound.

We know that any convergent sequence is bounded above and below, so we can just define the sequence (a_n) as $a_n = f(n)$. Then $a_n \to L$ as $n \to \infty$ and since (a_n) is bounded above and below, f must also be bounded above and below.

The function $f(x) = 1 - \dfrac{1}{1+x}$ is defined on $[0,\infty)$ and its lower bound is 0, which is achieves at $f(0) = 0$, but its upper bound is 1, which it never reaches. $f(x) \to 1$ as $x \to \infty$, but f never actually attains its upper bound.