

# MA265 Methods of Mathematical Modelling 3, Assignment 1

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## Question 1

*Characterisation of second order PDEs:* State the types of the following equations:

(a)  $u_{tt} - 4u_{xx} = 0$

(b)  $u_t = 8u_{xx}$

(c)  $u_{xx} + u_{yy} = 0$

### Q1 (a)

Linear, second order, hyperbolic, homogeneous PDE.

### Q1 (b)

Linear, second order, parabolic, homogeneous PDE.

### Q1 (c)

Linear, second order, elliptic, homogeneous PDE.

## Question 2

*The fundamental solution to the heat equation:* Verify that the solution to the heat equation

$$u_t = ku_{xx} \quad x \in \mathbb{R}, t > 0 \quad (1)$$

for  $k > 0$  is given by

$$u(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}.$$

Set  $k = 0.5$  and sketch  $u$  at different times. What happens as  $t \rightarrow 0$ ?

We will differentiate the  $u$  given in the question.

$$\begin{aligned} u &= (4\pi kt)^{-\frac{1}{2}} e^{-x^2(4kt)^{-1}} \\ \partial_t u &= (4\pi kt)^{-\frac{1}{2}} \left( x^2(4kt^2)^{-1} e^{-x^2(4kt)^{-1}} \right) - \frac{1}{2} (4\pi kt^3)^{-\frac{1}{2}} e^{-x^2(4kt)^{-1}} \\ &= (4\pi kt)^{-\frac{1}{2}} e^{-x^2(4kt)^{-1}} \left( x^2(4kt^2)^{-1} - \frac{1}{2t} \right) \\ \partial_x u &= (4\pi kt)^{-\frac{1}{2}} \left( -2x(4kt)^{-1} \right) e^{-x^2(4kt)^{-1}} \\ &= -2x(4\pi kt)^{-\frac{1}{2}} (4kt)^{-1} e^{-x^2(4kt)^{-1}} \\ \partial_{xx} u &= \partial_x \left( -2x(4\pi kt)^{-\frac{1}{2}} (4kt)^{-1} e^{-x^2(4kt)^{-1}} \right) \\ &= -2(4\pi kt)^{-\frac{1}{2}} (4kt)^{-1} \partial_x \left( x e^{-x^2(4kt)^{-1}} \right) \\ &= -2(4\pi kt)^{-\frac{1}{2}} (4kt)^{-1} \left( x \left( -2x(4kt)^{-1} \right) e^{-x^2(4kt)^{-1}} + e^{-x^2(4kt)^{-1}} \right) \\ &= -2(4\pi kt)^{-\frac{1}{2}} (4kt)^{-1} e^{-x^2(4kt)^{-1}} \left( -2x^2(4kt)^{-1} + 1 \right) \\ &= 4(4\pi kt)^{-\frac{1}{2}} (4kt)^{-1} e^{-x^2(4kt)^{-1}} \left( x^2(4kt)^{-1} - \frac{1}{2} \right) \end{aligned}$$

We want to have  $\partial_t u = k\partial_{xx} u$ , so let's look at each side and manipulate them

to see if they're equal.

$$\begin{aligned}
 \partial_t u &= (4\pi kt)^{-\frac{1}{2}} e^{-x^2(4kt)^{-1}} \left( x^2(4kt^2)^{-1} - \frac{1}{2t} \right) \\
 &= (4\pi kt)^{-\frac{1}{2}} e^{-x^2(4kt)^{-1}} \left( \frac{x^2}{4kt^2} - \frac{1}{2t} \right) \\
 k\partial_{xx} u &= k(4\pi kt)^{-\frac{1}{2}} e^{-x^2(4kt)^{-1}} 4(4kt)^{-1} \left( x^2(4kt)^{-1} - \frac{1}{2} \right) \\
 &= (4\pi kt)^{-\frac{1}{2}} e^{-x^2(4kt)^{-1}} \frac{4k}{4kt} \left( \frac{x^2}{4kt} - \frac{1}{2} \right) \\
 &= (4\pi kt)^{-\frac{1}{2}} e^{-x^2(4kt)^{-1}} \left( \frac{x^2}{4kt^2} - \frac{1}{2t} \right)
 \end{aligned}$$

Therefore this  $u$  satisfies (1).

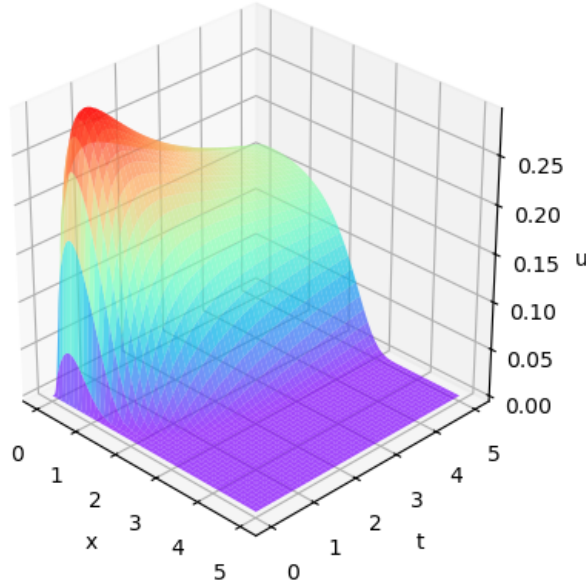


Figure 1: A plot of  $u(x, t)$  with  $k = 0.5$

As  $t \rightarrow 0$ , the model breaks down.

### Question 3

*Well-posedness:* Consider the elliptic problem

$$\begin{aligned} u_{xx}(x) &= f(x) & x \in [0, 1] \\ u_x(0) &= u_x(1) = 0 \end{aligned} \quad (\dagger)$$

Let  $u^*$  denote a solution to  $(\dagger)$ .

- (a) Are there any other solutions to  $(\dagger)$ ? If yes, state them.  
(b) Show that

$$\int_0^1 f(x) \, dx = 0$$

is a necessary condition to have a solution. **Hint:** Integrate  $(\dagger)$  and use the boundary conditions.

#### Q3 (a)

Yes. Any function  $v(x) = u^*(x) + C$ , where  $C \neq 0$  is any constant, is also a solution to  $(\dagger)$ .

#### Q3 (b)

We integrate both sides of  $(\dagger)$  and get

$$\begin{aligned} \int_0^1 u_{xx} \, dx &= \int_0^1 f(x) \, dx \\ u_x(1) - u_x(0) &= \int_0^1 f(x) \, dx \\ 0 &= \int_0^1 f(x) \, dx \end{aligned}$$

## Question 4

*Method of characteristics:* Use the method of characteristics to solve the transport equation

$$u_t + v(x, t)u_x = 0$$

in  $\mathbb{R} \times (0, \infty)$  with initial condition  $u(x, 0) = \Phi_0(x) = 1 - 2x$  for the velocity field

$$v(x, t) = \frac{1 + t^2}{2}.$$

We first have to solve the associated ODE to use the method characteristics.

$$\begin{aligned}\xi'(t) &= v(\xi(t), t) = \frac{1 + t^2}{2} \\ \xi(0) &= x_0\end{aligned}$$

We can solve this with separation of variables.

$$\begin{aligned}\frac{d\xi}{dt} &= \frac{1 + t^2}{2} \\ \int d\xi &= \frac{1}{2} \int 1 + t^2 dt \\ \xi(t) &= \frac{1}{2}t + \frac{1}{6}t^3 + C\end{aligned}$$

We use the initial value to find that

$$\xi(t) = \frac{1}{2}t + \frac{1}{6}t^3 + x_0.$$

We know that  $x = \xi(t)$ , so we can solve for  $x_0$  and find that

$$x_0 = x - \frac{1}{2}t - \frac{1}{6}t^3.$$

Now we apply the fact that  $u(x, t) = \Phi_0(x_0)$  to find that

$$\begin{aligned}u(x, t) &= 1 - 2x_0 \\ &= 1 - 2\left(x - \frac{1}{2}t - \frac{1}{6}t^3\right) \\ &= 1 - 2x + t + \frac{1}{3}t^3.\end{aligned}$$

## Question 5

*Classification of PDEs:* Reduce the elliptic equation

$$u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u = 0$$

to the equation

$$v_{xx} + v_{yy} + cv = 0$$

by changing the dependant variable to  $u(x, y) = v(x, y)e^{\alpha x + \beta y}$  and then using the scaling  $y' = \gamma y$ . What are the values of  $\alpha$ ,  $\beta$ ,  $\gamma$ , and  $c$ ?

We will begin by calculating the derivatives of  $u$ .

$$u(x, y) = v(x, y)e^{\alpha x + \beta y}$$

$$u_x = \alpha v e^{\alpha x + \beta y} + v_x e^{\alpha x + \beta y}$$

$$= e^{\alpha x + \beta y} (\alpha v + v_x)$$

$$u_{xx} = e^{\alpha x + \beta y} (\alpha v_x + v_{xx}) + \alpha e^{\alpha x + \beta y} (\alpha v + v_x)$$

$$= e^{\alpha x + \beta y} (v_{xx} + 2\alpha v_x + \alpha^2 v)$$

$$u_y = e^{\alpha x + \beta y} (\beta v + v_y)$$

$$u_{yy} = e^{\alpha x + \beta y} (v_{yy} + 2\beta v_y + \beta^2 v)$$

We plug these into the equation and get

$$0 = u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u$$

$$\begin{aligned} &= e^{\alpha x + \beta y} \left( v_{xx} + 2\alpha v_x + \alpha^2 v \right. \\ &\quad + 3(v_{yy} + 2\beta v_y + \beta^2 v) \\ &\quad - 2(\alpha v + v_x) \\ &\quad + 24(\beta v + v_y) \\ &\quad \left. + 5v \right) \end{aligned}$$

$$\begin{aligned} &= e^{\alpha x + \beta y} \left( v_{xx} + 2\alpha v_x + \alpha^2 v \right. \\ &\quad + 3v_{yy} + 6\beta v_y + 3\beta^2 v \\ &\quad - 2\alpha v - 2v_x \\ &\quad + 24\beta v + 24v_y \\ &\quad \left. + 5v \right) \end{aligned}$$

$$\begin{aligned} &= e^{\alpha x + \beta y} \left( v_{xx} + 3v_{yy} \right. \\ &\quad + (2\alpha - 2)v_x + (6\beta + 24)v_y \\ &\quad \left. + (\alpha^2 + 3\beta^2 - 2\alpha + 24\beta + 5)v \right) \end{aligned}$$

The coefficients of  $v_x$  and  $v_y$  are 0, so we need  $2\alpha - 2 = 0$  and  $6\beta + 24 = 0$ , which implies  $\alpha = 1$  and  $\beta = -4$ . Therefore the equation becomes

$$\begin{aligned} 0 &= e^{x-4y} (v_{xx} + 3v_{yy} + (1 + 12 - 2 - 96 + 5) v) \\ &= e^{x-4y} (v_{xx} + 3v_{yy} - 78v) \\ \implies 0 &= v_{xx} + 3v_{yy} - 78v \end{aligned}$$

We can divide out the  $e^{x-4y}$  because it is never 0. Then we conclude that  $\alpha = 1$ ,  $\beta = -4$ ,  $\gamma = 3$ ,  $c = -78$ .