MA141 Analysis 1, Assignment 3

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Question 3

> Suppose that (a_n) is an increasing sequence. Assume that (a_n) has a subsequence (a_{n_j}) that converges to some $l \in \mathbb{R}$ as $j \to \infty$. Show that $a_n \to l$ as $n \to \infty$.

Since $\left(a_{n_j}\right) \to \ell$, that means $\exists \varepsilon > 0, N \in \mathbb{N}$ such that $|a_{n_j} - \ell| < \varepsilon \; \forall \; n_j \geq N$.

Since $n_{j+1} > n_j \; \forall \; j \in \mathbb{N}$ and $\ell - \varepsilon < a_{n_j} < \ell + \varepsilon \; \forall \; n_j \geq N$, (a_n) is bounded above by $\ell + \varepsilon$.

Therefore $|a_n - \ell| < \varepsilon \; \forall \; n \geq N$.

Question 10

Use the comparison test to investigate the convergence of $\sum_{n=1}^{\infty} a_n$ in the following cases.

Before you apply the comparison test rigorously, you should use 'heuristics' to work out the answer you expect, so that you know whether to try to bound the terms in the sum above or below.

You can assume that $\sum \frac{1}{n^p}$ diverges if $0 < p \leq 1$ and converges for $p > 1$.

I had absolutely no idea what to do with this one, sorry.

Q10 (a)

$$a_n = \frac{\sqrt{n+1}}{\sqrt{n^3+2}}$$

For large n, $a_n \approx \dfrac{\sqrt{n}}{\sqrt{n^3}} = \dfrac{1}{n^2}$, so we expect $\sum a_n < \infty$.

Q10 (b)

$$a_n = \frac{n-3}{n^3+2}$$

For large n, $a_n \approx \dfrac{1}{n^2}$, so we expect $\sum a_n < \infty$.

Question 15

Use the integral test to find for which $\alpha > 0$ the sum

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^{\alpha}}$$

converges and for which $\alpha > 0$ it diverges.

$$\int_1^n \frac{1}{x(\log x)^{\alpha}} \mathrm{d}x = \left[\frac{(\log x)^{1-\alpha}}{1-\alpha}\right]_1^n = \frac{(\log n)^{1-\alpha}}{1-\alpha} \qquad \text{where } \alpha \neq 1$$

Since $(\log n)^{\beta} \to \infty$ exactly when $b > 0$, we know that the integral is bounded when $1 - \alpha > 0 \implies \alpha > 1$. And the integral is unbounded when $\alpha < 1$ and undefined when $\alpha = 1$.

Therefore $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^{\alpha}}$ converges when $\alpha > 1$ and diverges when $\alpha \leq 1$.

Question 16

Why do the following series converge? Do they converge absolutely?

Q16 (a)

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n+1}$$

The absolute version of this sum is the sum of reciprocals of odd numbers. Much like the Harmonic series, this series diverges to ∞, so the series does not converge absolutely.

It does however converge conditionally to $1-\frac{\pi}{4}$ thanks to the alternating minus signs.

Q16 (b)

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$$

What is the value of this sum?

The absolute version of this sum is

$$\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{1}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

This is the Basel problem, which famously equals $\frac{\pi^2}{6}$. Therefore this series is absolutely convergent, and therefore convergent.