MA263 Multivariable Analysis, Assignment 1

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Question 1

Show directly from the definition that matrix multiplication

$$m : \mathbb{R}^{m,n} \times \mathbb{R}^{n,p} \to \mathbb{R}^{m,p}$$
$$m(A, B) := AB$$

is Fréchet differentiable.

(*Suggestion*: Rather than compute all the partial derivatives, see if you can directly find the linear part of the 'best linear approximation' to m.)

We want to find L such that

$$\lim_{h \to (0,0)} \frac{\|m(x+h) - m(x) - L(h)\|}{\|h\|} = 0,$$

where $x, h \in \mathbb{R}^{m,n} \times \mathbb{R}^{n,p}$.

Question 2

Let $K \subset \mathbb{R}^n$ be compact. Let $\|\cdot\|$ be a norm on $\mathbb{R}^m$. For $f : K \to \mathbb{R}^m$, define

$$\|f\|_\infty := \sup_{x\in K} \|f(x)\|.$$

(a) Show that this is finite and defines a norm on the vector space of continuous functions $f : K \to \mathbb{R}^m$.

(b) Suppose that a sequence of continuous functions $f_i : K \to \mathbb{R}^m$ is such that

$$\forall \varepsilon > 0, \exists I \in \mathbb{N} \text{ such that } i, j \geq I \implies \|f_i - f_j\|_\infty < \varepsilon.$$

Show that there exists a continuous function $f : K \to \mathbb{R}^m$ such that

$$\forall \varepsilon > 0, \exists I \in \mathbb{N} \text{ such that } i \geq I \implies \|f_i - f\|_\infty < \varepsilon.$$

(*Suggestion*: This is a multivariable extension of the uniform convergence material from Analysis 3—see if you can adapt one of the proofs from your notes.)

Q2 (a)

Since K is compact, it is sequentially compact. Since f is continuous, $\|f\|_\infty$ must be finite.

To show that $\|\cdot\|_\infty$ is a norm, we need to show three things. Firstly, since $\|\cdot\|$ is a norm on $\mathbb{R}^m$, it is non-negative. So $\sup_{x\in K} \|f(x)\| \geq 0$, and the only way to get equality is when f is the constant 0 function.

Secondly, we require that for any $\lambda \in \mathbb{R}$, $\|\lambda f\|_\infty = |\lambda| \, \|f\|_\infty$. This is evident from the definition, again since $\|\cdot\|$ is a norm on $\mathbb{R}^m$:

$$\|\lambda f\|_\infty = \sup_{x\in K} \|\lambda f(x)\| = \sup_{x\in K} |\lambda| \, \|f(x)\| = |\lambda| \, \|f\|_\infty.$$

Finally, we require the triangle inequality. Let f and g be functions $K \to \mathbb{R}^m$. Since $\|\cdot\|$ is a norm on $\mathbb{R}^m$,

$$\begin{aligned}
\|f + g\|_\infty &= \sup_{x\in K} \|f(x) + g(x)\| \\
&\leq \sup_{x\in K} \big(\|f(x)\| + \|g(x)\|\big) \\
&\leq \sup_{x\in K} \|f(x)\| + \sup_{x\in K} \|g(x)\| \\
&= \|f\|_\infty + \|g\|_\infty.
\end{aligned}$$

Therefore $\|\cdot\|_\infty$ is a norm.

Q2 (b)

Let $x \in K$ and define $x_i = f_i(x)$. Then (x_i) is a Cauchy sequence and since $\mathbb{R}^m$ is complete, (x_i) converges to some point in $\mathbb{R}^m$. We can do this for any x, so there must exist a function f such that $f_i(x)$ converges to $f(x)$ at least pointwise. We want to show it converges uniformly.

By the hypothesis,

$$\forall \varepsilon > 0, \exists I \in \mathbb{N} \text{ such that } i, j \geq I \implies \|f_i(x) - f_j(x)\|_\infty < \varepsilon.$$

That means that every component of the function is bounded by ε. So for all $k \in \{1, \dots, n\}$,

$$|f_{k,i}(x) - f_{k,j}(x)| < \varepsilon,$$

where $f_{k,i}(x)$ denotes the kth component of $f_i(x)$.

Consider some $k \in \{1, \dots, n\}$. The previous inequality tells us that

$$f_{k,j}(x) - \varepsilon < f_{k,i}(x) < f_{k,j}(x) + \varepsilon.$$

Since this holds for all $j \geq I$, we can take limits as $j \to \infty$. We find

$$f_k(x) - \varepsilon < f_{k,i}(x) < f_k(x) + \varepsilon,$$

from which it follows that

$$|f_k(x) - f_{k,i}(x)| < \varepsilon.$$

which proves the result.

We can apply the previous argument to all k and convert back to the ∞-norm to see that

$$\|f(x) - f_i(x)\|_\infty < \varepsilon,$$

which proves the result.

$\square$

Question 3

Let $A \subset \mathbb{R}^n$, $f : A \to \mathbb{R}^m$. Suppose $\|\cdot\|$ and $\|\cdot\|'$ are norms on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.

We say that f is Lipschitz if there is a constant $L > 0$ so that

$$\|f(x) - f(y)\|' \leq L\|x - y\|.$$

(a) Show that a Lipschitz function is continuous.

(b) Let $U = \{x \in \mathbb{R}^n : \|x\|_\infty < 1\}$ and $f : U \to \mathbb{R}^m$ be differentiable. Suppose also that for some $L > 0$, $\|Df(x)\|_{\infty\to\infty} \leq L$ for every $x \in U$. Show that f is Lipschitz with the ∞-norms on $\mathbb{R}^n$ and $\mathbb{R}^m$.

Q3 (a)

Suppose f is Lipschitz. To show that f is continuous, we want to show that for any $\varepsilon > 0$, we can find a $\delta > 0$ such that $\|x - y\| < \delta \implies \|f(x) - f(y)\|' < \varepsilon$.

We know that $\|f(x) - f(y)\|' \leq L\|x - y\|$ and so if $\|x - y\| < \delta$, then $\|f(x) - f(y)\|' < L\delta$. Now we can just set $L\delta = \varepsilon$ and see that our δ must be $\dfrac{\varepsilon}{L}$. Therefore f is continuous.

Q3 (b)

We know that for some $L > 0$,

$$\|Df(x)\|_{\infty\to\infty} = \sup_{x \in U \setminus \{0\}} \frac{\|Df(x)\|_\infty}{\|x\|_\infty} \leq L$$

Therefore

$$\|f(x) - f(y)\|_\infty \leq L\|x - y\|_\infty.$$

$\square$

Question 4

Use the following strategy to show that all norms on $\mathbb{R}^n$ are equivalent:

(a) Explain why it suffices to prove that an arbitrary norm $\|\cdot\|$ on $\mathbb{R}^n$ is equivalent to $\|\cdot\|_\infty$.

(b) Show that $\exists C > 0$ so that $\|x\| \leq C\|x\|_\infty$.

(*Hint*: Write $x = x_1 e_1 + \cdots + x_n e_n$.)

(c) Show that $\|x\|$ is a continuous function on $\mathbb{R}^n$ with respect to the ∞-norm.

(*Suggestion*: Show that it is Lipschitz using the previous part.)

(d) Show that $S = \{x \in \mathbb{R}^n : \|x\|_\infty = 1\}$ is compact. You may use any of the equivalent formulations of compactness.

(e) Show that $\exists C' > 0$ such that $\|x\| \geq C'$ for $x \in S$.

(f) Conclude that $\|\cdot\|$ and $\|\cdot\|_\infty$ are equivalent.

Q4 (a)

Equivalence of norms is transitive. Suppose $\|\cdot\|$ and $\|\cdot\|'$ are both equivalent to $\|\cdot\|_\infty$. Then for all $x \in \mathbb{R}^n$, there exists $c_1, c_2, c_1', c_2' > 0$ such that

$$c_1\|x\| \leq \|x\|_\infty \leq c_2\|x\|$$

and

$$c_1'\|x\|' \leq \|x\|_\infty \leq c_2'\|x\|'.$$

Therefore

$$c_1\|x\| \leq \|x\|_\infty \leq c_2'\|x\|'$$

and so

$$c_1\|x\| \leq c_2'\|x\|'.$$

Likewise,

$$c_1'\|x\|' \leq c_2\|x\|.$$

Therefore

$$\frac{c_1}{c_2'}\|x\| \leq \|x\|' \leq \frac{c_2}{c_1'}\|x\|,$$

and so $\|\cdot\|$ and $\|\cdot\|'$ are equivalent.

Q4 (b)

$\|x\|_\infty = \max\{x_1, \ldots, x_n\}$, so without loss of generality, assume $\|x\|_\infty = x_1$. Then we just choose $C \geq \dfrac{\|x\|}{x_1}$.

Q4 (c)

Let $x, y \in \mathbb{R}^n$. We want to show that $\|\cdot\|$ is Lipschitz with respect to the ∞-norm, so we want to show that

$$\big\| \|x\| - \|y\| \big\|_\infty \leq L\|x-y\|_\infty$$

for some $L > 0$.

We know there exists $C_x, C_y > 0$ such that

$$\|x\| \leq C_x\|x\|_\infty, \qquad \|y\| \leq C_y\|y\|_\infty.$$

Since $\|\cdot\|$ is Lipschitz, it is continuous, as per **Q3(a)**.

Q4 (d)

Consider a sequence (x_i) in S which converges to a point x. Since all $\|x_i\|_\infty = 1$, $\|x\|_\infty = 1$. The limit of the sequence is in S, so S is closed. Clearly S is bounded since for all $x \in S$, $\|x\|_\infty \leq 1$. Since S is closed and bounded, it is compact.

Q4 (e)

We know $\|x\| = 0$ if and only if $x = 0$ but we require $x \in S$ and we know $0 \neq S$. Therefore we can set $C = \inf_{x \in S} \|x\|$ and we know that $C > 0$.

Q4 (f)

Since $x \in S$ in part **Q4(e)**, we can multiply by $\|x\|_\infty = 1$ and swap sides to get

$$C'\|x\|_\infty \leq \|x\|.$$

We combine this with the result of **Q4(b)** to get

$$C'\|x\|_\infty \leq \|x\| \leq C\|x\|_\infty.$$

Therefore $\|\cdot\|$ is equivalent to $\|\cdot\|_\infty$, so all norms on $\mathbb{R}^n$ are equivalent.

$\square$