MA270 Analysis 3, Assignment 2

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Question 1

(a) For each $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be a function and $|f_n|$ be the function $\mathbb{R} \to \mathbb{R}$ obtained by composing with absolute value (i.e. $|f_n| : x \mapsto |f_n(x)|$). Show that if the series $\sum |f_n|$ converges pointwise then $\sum f_n$ converges pointwise.

(b) Find an example of a sequence (f_n) of functions $\mathbb{R} \to \mathbb{R}$ such that $\sum f_n$ converges pointwise but $\sum |f_n|$ does not converge pointwise.

Q1 (a)

We can apply the theorem from first year analysis for series of real numbers. Let (a_n) be a sequence. If $\sum |a_n|$ converges then $\sum a_n$ converges. This was proved in MA141.

Then we can define a set of sequences $(a_{x,n})$ for all $x \in \mathbb{R}$ where $a_{x,n} = f_n(x)$. Then what we want to show is equivalent to saying that if $\sum |a_{x,n}|$ converges then $\sum a_{x,n}$ converges. This is evident from the theorem above. Therefore if $\sum |f_n|$ converges pointwise then $\sum f_n$ converges pointwise.

$\square$

Q1 (b)

Let $f_n(x) = (-1)^n x$. Then $\sum f_n = -x + x - x + x - \cdots = 0$ so converges pointwise, but $\sum |f_n| = x + x + x + \cdots$ so does not converge pointwise.

Question 2

For each integer $n \geq 1$ and $x \in \mathbb{R}$, let $f_n(x) = \dfrac{x}{x^2 + n^2}$.

(a) Show that the series $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise. (**Hint**: You may use **Q1 (a)**.)

(b) Does the series $\sum_{n=1}^{\infty} f_n(x)$ converge uniformly? Justify your answer. (**Hint**: Try to give a lower bound to $\sum_{k=n+1}^{2n} f_k(x)$ evaluated at $x = 2n$.)

(c) Is the function $\sum_{n=1}^{\infty} f_n(x)$ continuous? (**Hint**: Fix $t > 0$ and restrict the functions f_n to the interval $[-t, t]$ and analyse the convergence of associated series $\sum f_n$.)

Q2 (a)

We shall consider instead $\sum_{n=1}^{\infty} |f_n|(x)$. This converges pointwise by the comparison test, so $\sum_{n=1}^{\infty} f_n(x)$ converges by **Q1 (a)**.

Q2 (b)

We shall consider $\sum_{k=n+1}^{2n} f_k(2n)$. When $n = 1$, this sum is $f_2(2) = \frac{2}{8} = \frac{1}{4}$. As n increases, the sum increases, so $\frac{1}{4}$ is a lower bound.

The series $\sum_{n=1}^{\infty} f_n(x)$ does converge uniformly because $f_n(x)$ converges to 0.

Q2 (c)

Yes it is continuous.

Question 3

Let $f : \mathbb{R}^2 \setminus \{(x, y) \in \mathbb{R}^2 : x = y\} \to \mathbb{R}$ be the function defined by $f(x, y) = \dfrac{\sin x - \sin y}{x - y}$. Show that f extends by continuity to $\mathbb{R}^2$. (**Hint**: Use the Mean Value Theorem.)

We want to extend $f(x, y)$ to points where $x = y$. Fix some arbitrary x and $0 < \varepsilon \ll 1$, then we will consider $y = x \pm \varepsilon$.

We get

$$\cos\left(\frac{\sqrt{2}}{2}\sqrt{x^2 + y^2}\right)$$

on the line $x = y$.

Question 4

Do the following functions defined on $\mathbb{R}^2 \setminus \{(0,0)\}$ admit a limit at $(0,0)$? Justify your claims.

(a) $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$.

(b) $f(x,y) = \dfrac{x^3 + y^3}{x^2 + y^2}$.

(c) $f(x,y) = \dfrac{\sin x^2 + \sin y^2}{\|(x,y)\|}$. (**Hint**: Show that $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$.)

(d) $f(x,y) = (3x + 2y) \sin\left(\dfrac{1}{2x + 3y}\right)$.

Q4 (a)

We can use separate continuity. When we consider the line $y = 0$, we get $f(x,0) = \dfrac{x^2}{x^2} = 1$ for all $x \neq 0$, and it is easy to see the limit as $x \to 0$ is 1. On the line $x = 0$, we get $f(0,y) = \dfrac{-y^2}{y^2} = -1$. Like in the previous case, the limit as $y \to 0$ is now -1. These limits disagree, so f has no limit at $(0,0)$.

Q4 (b)

The limit at $(0,0)$ is 0. We want to show that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\|(x,y) - (0,0)\| < \delta \implies |f(x,y) - 0| < \varepsilon$. That is, we want $\sqrt{x^2 + y^2} < \delta \implies |f(x,y)| < \varepsilon$. Let us assume $\delta < 1$.

$$\begin{aligned}
&\sqrt{x^2 + y^2} < \delta \\
\implies\ & x^2 + y^2 > \delta^2 && \text{sign flips because } 0 < \delta < 1 \\
\implies\ & \frac{1}{x^2 + y^2} < \frac{1}{\delta^2} \\
\implies\ & \frac{|x^3 + y^3|}{x^2 + y^2} < \frac{|x^3 + y^3|}{\delta^2} \\
\implies\ & \left|\frac{x^3 + y^3}{x^2 + y^2}\right| < \frac{|x^3 + y^3|}{\delta^2} \\
& \therefore \varepsilon = \frac{|x^3 + y^3|}{\delta^2} \\
& \therefore \delta = \sqrt{\frac{|x^3 + y^3|}{\varepsilon}}
\end{aligned}$$

Therefore $f(x, y) \to 0$ as $(x, y) \to (0, 0)$.

Q4 (c)

The limit at $(0, 0)$ is 0.

Q4 (d)

The limit at $(0, 0)$ is 0. Clearly $\sin\left(\frac{1}{2x + 3y}\right)$ doesn't have a limit at the origin, but it will oscillate between -1 and 1 with increasing frequency as $(x, y) \to (0, 0)$. However, $3x + 2y$ converges linearly to 0 as $(x, y) \to (0, 0)$, and so this term will dominate f. Therefore $f(x, y) \to 0$ as $(x, y) \to (0, 0)$.