MA265 Methods of Mathematical Modelling 3, Assignment 1

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Question 1

Characterisation of second order PDEs: State the types of the following equations:

(a) $u_{tt} - 4u_{xx} = 0$

(b) $u_t = 8u_{xx}$

(c) $u_{xx} + u_{yy} = 0$

Q1 (a)

Linear, second order, hyperbolic, homogeneous PDE.

Q1 (b)

Linear, second order, parabolic, homogeneous PDE.

Q1 (c)

Linear, second order, elliptic, homogeneous PDE.

Question 2

The fundamental solution to the heat equation: Verify that the solution to the heat equation

$$u_t = ku_{xx} \quad x \in \mathbb{R}, t > 0 \tag{1}$$

for $k > 0$ is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \mathrm{e}^{-\frac{x^2}{4kt}}.$$

Set $k = 0.5$ and sketch u at different times. What happens as $t \to 0$?

We will differentiate the u given in the question.

$$\begin{aligned}
u &= (4\pi kt)^{-\frac{1}{2}} \mathrm{e}^{-x^2(4kt)^{-1}} \\
\partial_t u &= (4\pi kt)^{-\frac{1}{2}} \left(x^2 (4kt^2)^{-1} \mathrm{e}^{-x^2(4kt)^{-1}} \right) - \frac{1}{2} (4\pi kt^3)^{-\frac{1}{2}} \mathrm{e}^{-x^2(4kt)^{-1}} \\
&= (4\pi kt)^{-\frac{1}{2}} \mathrm{e}^{-x^2(4kt)^{-1}} \left(x^2 (4kt^2)^{-1} - \frac{1}{2t} \right) \\
\partial_x u &= (4\pi kt)^{-\frac{1}{2}} \left(-2x(4kt)^{-1} \right) \mathrm{e}^{-x^2(4kt)^{-1}} \\
&= -2x(4\pi kt)^{-\frac{1}{2}} (4kt)^{-1} \mathrm{e}^{-x^2(4kt)^{-1}} \\
\partial_{xx} u &= \partial_x \left(-2x(4\pi kt)^{-\frac{1}{2}} (4kt)^{-1} \mathrm{e}^{-x^2(4kt)^{-1}} \right) \\
&= -2(4\pi kt)^{-\frac{1}{2}} (4kt)^{-1} \partial_x \left(x\mathrm{e}^{-x^2(4kt)^{-1}} \right) \\
&= -2(4\pi kt)^{-\frac{1}{2}} (4kt)^{-1} \left(x \left(-2x(4kt)^{-1} \right) \mathrm{e}^{-x^2(4kt)^{-1}} + \mathrm{e}^{-x^2(4kt)^{-1}} \right) \\
&= -2(4\pi kt)^{-\frac{1}{2}} (4kt)^{-1} \mathrm{e}^{-x^2(4kt)^{-1}} \left(-2x^2(4kt)^{-1} + 1 \right) \\
&= 4(4\pi kt)^{-\frac{1}{2}} (4kt)^{-1} \mathrm{e}^{-x^2(4kt)^{-1}} \left(x^2(4kt)^{-1} - \frac{1}{2} \right)
\end{aligned}$$

We want to have $\partial_t u = k\partial_{xx} u$, so let's look at each side and manipulate them to see if they're equal.

$$\begin{aligned}
\partial_t u &= (4\pi kt)^{-\frac{1}{2}} \mathrm{e}^{-x^2(4kt)^{-1}} \left(x^2 (4kt^2)^{-1} - \frac{1}{2t} \right) \\
&= (4\pi kt)^{-\frac{1}{2}} \mathrm{e}^{-x^2(4kt)^{-1}} \left(\frac{x^2}{4kt^2} - \frac{1}{2t} \right) \\
k\partial_{xx} u &= k(4\pi kt)^{-\frac{1}{2}} \mathrm{e}^{-x^2(4kt)^{-1}} 4(4kt)^{-1} \left(x^2 (4kt)^{-1} - \frac{1}{2} \right) \\
&= (4\pi kt)^{-\frac{1}{2}} \mathrm{e}^{-x^2(4kt)^{-1}} \frac{4k}{4kt} \left(\frac{x^2}{4kt} - \frac{1}{2} \right) \\
&= (4\pi kt)^{-\frac{1}{2}} \mathrm{e}^{-x^2(4kt)^{-1}} \left(\frac{x^2}{4kt^2} - \frac{1}{2t} \right)
\end{aligned}$$

Therefore this u satisfies (1).

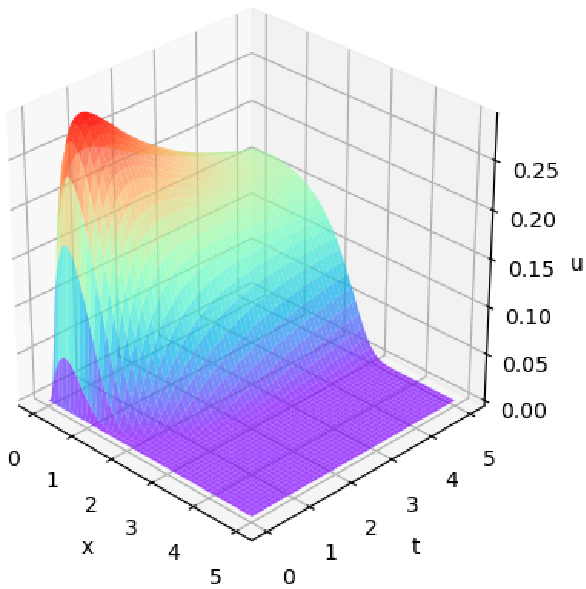

Figure 1: A plot of $u(x, t)$ with $k = 0.5$

As $t \to 0$, the model breaks down.

Question 3

Well-posedness: Consider the elliptic problem

$$\begin{aligned} u_{xx}(x) &= f(x) \qquad\qquad x \in [0,1] \\ u_x(0) &= u_x(1) = 0 \end{aligned} \tag{†}$$

Let u^* denote a solution to (†).

(a) Are there any other solutions to (†)? If yes, state them.

(b) Show that
$$\int_0^1 f(x)\ \mathrm{d}x = 0$$
is a necessary condition to have a solution. **Hint**: Integrate (†) and use the boundary conditions.

Q3 (a)

Yes. Any function $v(x) = u^*(x) + C$, where $C \neq 0$ is any constant, is also a solution to (†).

Q3 (b)

We integrate both sides of (†) and get

$$\begin{aligned} \int_0^1 u_{xx}\ \mathrm{d}x &= \int_0^1 f(x)\ \mathrm{d}x \\ u_x(1) - u_x(0) &= \int_0^1 f(x)\ \mathrm{d}x \\ 0 &= \int_0^1 f(x)\ \mathrm{d}x \end{aligned}$$

Question 4

Method of characteristics: Use the method of characteristics to solve the transport equation

$$u_t + v(x,t)u_x = 0$$

in $\mathbb{R} \times (0, \infty)$ with initial condition $u(x,0) = \Phi_0(x) = 1 - 2x$ for the velocity field

$$v(x,t) = \frac{1+t^2}{2}.$$

We first have to solve the associated ODE to use the method characteristics.

$$\begin{aligned} \xi'(t) &= v(\xi(t), t) = \frac{1+t^2}{2} \\ \xi(0) &= x_0 \end{aligned}$$

We can solve this with separation of variables.

$$\begin{aligned} \frac{\mathrm{d}\xi}{\mathrm{d}t} &= \frac{1+t^2}{2} \\ \int \mathrm{d}\xi &= \frac{1}{2} \int 1 + t^2 \ \mathrm{d}t \\ \xi(t) &= \frac{1}{2}t + \frac{1}{6}t^3 + C \end{aligned}$$

We use the initial value to find that

$$\xi(t) = \frac{1}{2}t + \frac{1}{6}t^3 + x_0.$$

We know that $x = \xi(t)$, so we can solve for x_0 and find that

$$x_0 = x - \frac{1}{2}t - \frac{1}{6}t^3.$$

Now we apply the fact that $u(x,t) = \Phi_0(x_0)$ to find that

$$\begin{aligned} u(x,t) &= 1 - 2x_0 \\ &= 1 - 2\left(x - \frac{1}{2}t - \frac{1}{6}t^3\right) \\ &= 1 - 2x + t + \frac{1}{3}t^3. \end{aligned}$$

Question 5

Classification of PDEs: Reduce the elliptic equation

$$u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u = 0$$

to the equation

$$v_{xx} + v_{y'y'} + cv = 0$$

by changing the dependant variable to $u(x,y) = v(x,y)\mathrm{e}^{\alpha x + \beta y}$ and then using the scaling $y' = \gamma y$. What are the values of α, β, γ, and c?

We will begin by calculating the derivatives of u.

$$\begin{aligned}
u(x,y) &= v(x,y)\mathrm{e}^{\alpha x+\beta y} \\
u_x &= \alpha v \mathrm{e}^{\alpha x+\beta y} + v_x \mathrm{e}^{\alpha x+\beta y} \\
&= \mathrm{e}^{\alpha x+\beta y}\left(\alpha v + v_x\right) \\
u_{xx} &= \mathrm{e}^{\alpha x+\beta y}\left(\alpha v_x + v_{xx}\right) + \alpha \mathrm{e}^{\alpha x+\beta y}\left(\alpha v + v_x\right) \\
&= \mathrm{e}^{\alpha x+\beta y}\left(v_{xx} + 2\alpha v_x + \alpha^2 v\right) \\
u_y &= \mathrm{e}^{\alpha x+\beta y}\left(\beta v + v_y\right) \\
u_{yy} &= \mathrm{e}^{\alpha x+\beta y}\left(v_{yy} + 2\beta v_y + \beta^2 v\right)
\end{aligned}$$

We plug these into the equation and get

$$\begin{aligned}
0 &= u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u \\
&= \mathrm{e}^{\alpha x+\beta y}\Big(v_{xx} + 2\alpha v_x + \alpha^2 v \\
&\quad + 3\left(v_{yy} + 2\beta v_y + \beta^2 v\right) \\
&\quad - 2\left(\alpha v + v_x\right) \\
&\quad + 24\left(\beta v + v_y\right) \\
&\quad + 5v\Big) \\
&= \mathrm{e}^{\alpha x+\beta y}\Big(v_{xx} + 2\alpha v_x + \alpha^2 v \\
&\quad + 3v_{yy} + 6\beta v_y + 3\beta^2 v \\
&\quad - 2\alpha v - 2v_x \\
&\quad + 24\beta v + 24 v_y \\
&\quad + 5v\Big) \\
&= \mathrm{e}^{\alpha x+\beta y}\Big(v_{xx} + 3v_{yy} \\
&\quad + \left(2\alpha - 2\right)v_x + \left(6\beta + 24\right)v_y \\
&\quad + \left(\alpha^2 + 3\beta^2 - 2\alpha + 24\beta + 5\right)v\Big)
\end{aligned}$$

The coefficients of v_x and v_y are 0, so we need $2\alpha - 2 = 0$ and $6\beta + 24 = 0$, which implies $\alpha = 1$ and $\beta = -4$. Therefore the equation becomes

$$\begin{aligned} 0 &= \mathrm{e}^{x-4y}\left(v_{xx} + 3v_{yy} + (1 + 12 - 2 - 96 + 5)\,v\right) \\ &= \mathrm{e}^{x-4y}\left(v_{xx} + 3v_{yy} - 78v\right) \\ \implies 0 &= v_{xx} + 3v_{yy} - 78v \end{aligned}$$

We can divide out the e^{x-4y} because it is never 0. We can also observe that if $y' = \gamma y$, then $\gamma^2 v_{y'y'} = v_{yy}$ so $v_{y'y'} = \frac{1}{\gamma^2} v_{yy}$.

Then we conclude that $\alpha = 1$, $\beta = -4$, $\gamma = \frac{1}{\sqrt{3}}$, $c = -78$.