MA270 Analysis 3, Assignment 1

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Question 1

Let $f : [0, \infty) \to \mathbb{R}$ be a function such that $\lim_{x\to\infty} f(x) = 0$, $f(0) = 0$, and $f(1) = 1$. For each $n \in \mathbb{N}$, let $f_n(x) := f(nx)$ for every $x \in [0, \infty)$.

(a) Show that the sequence (f_n) converges to 0 pointwise but not uniformly.

(b) Show that for any $a > 0$, we have $\sup_{x\in[a,\infty)} |f_n(x)| \to 0$ as $n \to \infty$ (in such case, we say that f_n converges uniformly to 0 on $[a, \infty)$).

Q1 (a)

We will first show that $f_n \to 0$ pointwise. If $x = 0$ then $f_n(x) = 0$ for any n, so consider an arbitrary $x_0 \in (0, \infty)$. For any $\varepsilon > 0$, we want to choose an N such that $\forall n \geq N$, $|f(nx_0)| < \varepsilon$.

Since $\lim_{x\to\infty} f(x) = 0$, we know that $\forall \varepsilon > 0$, $\exists N_0 \in \mathbb{N}$ such that $\forall x \geq N_0$, $|f(x)| < \varepsilon$.

To apply this above, we just want to choose N such that $nx_0 \geq N_0$, so we choose $N = \lceil N_0/x_0 \rceil$. Then $\forall n \geq N$, we have $|f_n(x_0)| = |f(nx_0)| < \varepsilon$ as required. Therefore $f_n \to 0$ pointwise.

However, $f \not\rightrightarrows 0$ because no matter which n we choose, we can find some x such that $f_n(x) \neq 0$. In this case, $x = \frac{1}{n}$, and $f_n(x) = f\left(n\frac{1}{n}\right) = f(1) = 1$. Therefore f does not converge uniformly to 0. $\square$

Q1 (b)

Fix some arbitrary $a > 0$. We want to show that $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall x \in [a, \infty)$, $\forall n \geq N$, $|f(nx)| < \varepsilon$. Since $\lim_{x\to\infty} f(x) = 0$, we know that $\forall \varepsilon > 0$, $\exists N_0 \in \mathbb{N}$ such that $\forall x \geq N_0$, $|f(x)| < \varepsilon$.

So we just choose our N such that when $n > N$, $nx \geq N_0$ for all $x \in [a, \infty)$. So we choose $N = \lceil N_0/a \rceil$. Then the condition is met and therefore f_n converges uniformly on $[a, \infty)$. $\square$

Question 2

Study the pointwise and uniform convergence of the sequence $f_n : (0, \infty) \to \mathbb{R}$ defined by $f_n(x) = \dfrac{\sin(nx)}{n\sqrt{x}}$.

Hint: For uniform convergence, given $n \in \mathbb{N}$, one can analyse the behaviour of $f_n(x)$ separately for $x \in \left[\frac{1}{n}, \infty\right)$ and $x \in \left(0, \frac{1}{n}\right)$. For the analysis in the latter interval, one can consider the function $h : [0, \infty) \to \mathbb{R}$ defined by $h(x) = \dfrac{\sin x}{x}$ for $x \neq 0$ and $h(0) = 1$, show that h is bounded on $[0, \infty)$, and make use of this.

I shall conjecture that f_n converges pointwise to 0. To show this, we need to show that

$$\forall \varepsilon > 0,\ \forall x \in (0, \infty),\ \exists N \in \mathbb{N} \text{ such that } \forall n \geq N,\ |f_n(x)| < \varepsilon.$$

So consider some arbitrary ε and x. Then

$$\begin{aligned} |f_n(x)| &= \left|\frac{\sin(nx)}{n\sqrt{x}}\right| \\ &= \frac{|\sin(nx)|}{|n\sqrt{x}|} \\ &= \frac{|\sin(nx)|}{n\sqrt{x}} \\ |f_n(x)| &< \varepsilon \\ \implies |\sin(nx)| &< \varepsilon n\sqrt{x} \end{aligned}$$

The maximum value of $|\sin(nx)|$ is 1, so as long as $\varepsilon n\sqrt{x} > 1$, we have the desired inequality. That means we can just choose $N = \left\lceil \dfrac{1}{\varepsilon\sqrt{x}} \right\rceil$.

Since this value of N depends on x, f_n does not converge uniformly to 0 in general. We shall consider f_n separately on the intervals $\left(0, \frac{1}{n}\right)$ and $\left[\frac{1}{n}, \infty\right)$.

First, consider f_n only on the interval $\left(0, \frac{1}{n}\right)$. We shall first consider $h : [0, \infty) \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} 1 & \text{if } x = 0 \\ \dfrac{\sin x}{x} & \text{otherwise} \end{cases}$$

The function h is bounded above and below by ± 1. To show this, we want to show that $\forall x \in [0, \infty)$,

$$
\begin{aligned}
|h(x)| &\leq 1 \\
\left|\frac{\sin x}{x}\right| &\leq 1 \\
\frac{|\sin x|}{|x|} &\leq 1 \\
|\sin x| &\leq |x|
\end{aligned}
$$

This is a well-known fact for $x \geq 0$, which is exactly where h is defined. Therefore $-1 \leq h(x) \leq 1$.

This should help with studying the uniform convergence of f_n. We can note that

$$f_n(x) = h(nx)\frac{x}{\sqrt{x}}$$

but unfortunately, I don't know where to go from here.

Question 3

For each $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ be a Riemann integrable function. Suppose that the sequence (f_n) converges uniformly to a function $f : [0,1] \to \mathbb{R}$. Let (g_n) be the sequence defined by $g_n = f_{n+3} - f_n$. Prove that the limit $\lim\limits_{n\to\infty} \int_0^1 g_n$ exists and calculate this limit.

$$\begin{aligned}
\lim_{n\to\infty} \int_0^1 g_n &= \lim_{n\to\infty} \int_0^1 (f_{n+3}(x) - f_n(x))\ \mathrm{d}x \\
&= \lim_{n\to\infty} \int_0^1 f_{n+3}(x)\ \mathrm{d}x - \lim_{n\to\infty} \int_0^1 f_n(x)\ \mathrm{d}x \\
&= \int_0^1 \lim_{n\to\infty} f_{n+3}(x)\ \mathrm{d}x - \int_0^1 \lim_{n\to\infty} f_n(x)\ \mathrm{d}x \qquad (*) \\
&= \int_0^1 f(x)\ \mathrm{d}x - \int_0^1 f(x)\ \mathrm{d}x \\
&= 0
\end{aligned}$$

Note that we're only allowed to move the limits inside the integrals on line (*) because f_n converges to f uniformly. A weaker notion of convergence would not allow for this.

Question 4

Let $g : [0, 1] \to \mathbb{R}$ be a continuous function. For each $n \in \mathbb{N}$, let $g_n : [0, 1] \to \mathbb{R}$ be defined by $g_n(x) = \dfrac{x}{1 + nx^2}\, g(x)$. Show that $\displaystyle\int_0^1 g_n(x)\ \mathrm{d}x \to 0$ as $n \to \infty$.

Hint: One can start by showing that the sequence of functions $h_n : [0, 1] \to \mathbb{R}$ defined by $h_n(x) = \dfrac{x}{1 + nx^2}$ converges uniformly to 0.

Consider the sequence of functions $h_n : [0, 1] \to \mathbb{R}$ defined by $h_n(x) = \dfrac{x}{1 + nx^2}$. For h_n to converge uniformly to 0, we need

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall n \geq N,\ \forall x \in [0, 1],\ |h_n(x)| < \varepsilon.$$

A bit of simple manipulation tells us that

$$\begin{aligned}
|h_n(x)| &< \varepsilon \\
\left|\frac{x}{1 + nx^2}\right| &< \varepsilon \\
\frac{|x|}{|1 + nx^2|} &< \varepsilon \\
\frac{|x|}{1 + nx^2} &< \varepsilon \\
|x| &< \varepsilon(1 + nx^2) \\
|x| &< \varepsilon + \varepsilon nx^2 \\
\frac{|x| - \varepsilon}{\varepsilon x^2} &< n
\end{aligned}$$

needs to hold for all x. Since $x \in [0, 1]$, the left hand side is maximised when $x = 1$, so we need $n > \dfrac{1 - \varepsilon}{\varepsilon}$ and therefore we choose $N = \left\lceil \dfrac{1 - \varepsilon}{\varepsilon} \right\rceil$. Therefore $h_n \rightrightarrows 0$.

Now we want to consider $\displaystyle\lim_{n\to\infty} \int_0^1 g_n(x)\ \mathrm{d}x$. Since h_n converges uniformly and g is continuous, g_n converges uniformly. Therefore we can move the limit inside the integral and instead consider

$$\begin{aligned}
\int_0^1 \lim_{n\to\infty} g_n(x)\ \mathrm{d}x &= \int_0^1 \lim_{n\to\infty} (h_n(x)g(x))\ \mathrm{d}x \\
&= \int_0^1 \left(\lim_{n\to\infty} h_n(x)\right) g(x)\ \mathrm{d}x \\
&= \int_0^1 0g(x)\ \mathrm{d}x \\
&= 0
\end{aligned}$$