MA144 Methods of Mathematical Modelling 2, Assignment 2

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Question 1

Q1 (a)

The heat equation Let $u(x,t)$ be the temperature at a point x on a long solid metal rod lying along the x axis ($x \in \mathbb{R}$) at time $t > 0$.

It can be shown that u satisfies the *heat equation*

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2},$$

where α is a constant called the thermal conductivity of the metal.

A solution to the heat equation is

$$u(x,t) = \frac{1}{\sqrt{2\pi t}}\,\mathrm{e}^{-x^2/2t} = \frac{1}{\sqrt{2\pi}}\,t^{-\frac{1}{2}}\,\mathrm{e}^{-x^2/2t}$$

Q1 (a) i)

Find the value of α.

The solution gives

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{1}{\sqrt{2\pi}}\left(t^{-\frac{1}{2}}\mathrm{e}^{-x^2/2t}\frac{x^2}{2t^2} - \frac{1}{2}t^{-\frac{3}{2}}\mathrm{e}^{-x^2/2t}\right) \\
&= \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2t}\left(t^{-\frac{1}{2}}\frac{x^2}{2}t^{-2} - \frac{1}{2}t^{-\frac{3}{2}}\right) \\
&= \frac{1}{2\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2t}\left(x^2t^{-\frac{5}{2}} - t^{-\frac{3}{2}}\right) \\
&= \frac{1}{2\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2t}t^{-\frac{3}{2}}\left(x^2t^{-1} - 1\right)
\end{aligned}$$

$$= \frac{1}{2\sqrt{2\pi t^3}}\, \mathrm{e}^{-x^2/2t} \left(\frac{x^2}{t} - 1\right)$$

and

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{1}{\sqrt{2\pi t}}\, \mathrm{e}^{-x^2/2t} \left(\frac{-x}{t}\right)\right) \\
&= \frac{\partial}{\partial x} \left(\frac{-1}{\sqrt{2\pi t^3}}\, x\mathrm{e}^{-x^2/2t} \right) \\
&= \frac{-1}{\sqrt{2\pi t^3}} \left(\mathrm{e}^{-x^2/2t} + x\mathrm{e}^{-x^2/2t} \left(\frac{-x}{t}\right)\right) \\
&= \frac{1}{\sqrt{2\pi t^3}} \mathrm{e}^{-x^2/2t} \left(\frac{x^2}{t} - 1\right)
\end{aligned}$$

Therefore $\alpha = 2$ for this solution.

Q1 (a) ii)

> Sketch the solution $u(x, 1)$.

$$u(x, 1) = \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-x^2/2}$$

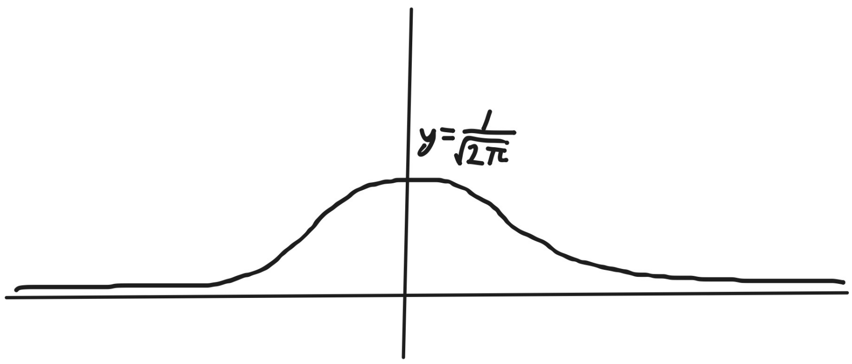

Figure 1: A plot of $y = u(x, 1)$

Q1 (b)

> **The wave equation** A thin elastic string is stretched along the x axis. A point x on the string is free to vibrate along the y direction. Let $u(x, t)$ be the amplitude of the vibration of point x at time t.

It can be shown that u satisfies the *wave equation*

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},$$

where the constant $c > 0$ is the speed of the wave (determined by the material of the string).

Q1 (b) i)

A solution to the wave equation is

$$u(x,t) = \sin x \cos \beta t$$

where $\beta > 0$. Find β.

The solution gives

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}(\cos x \cos \beta t)\\ &= -\sin x \cos \beta t\\ &= -u(x,t)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t}(-\beta \sin x \sin \beta t)\\ &= -\beta^2 \sin x \cos \beta t\\ &= -\beta^2 u(x,t)\end{aligned}$$

Plugging this into the wave equation gives

$$-u(x,t) = -\frac{\beta^2}{c^2}u(x,t)$$

Therefore $\dfrac{\beta^2}{c^2} = 1$ so $\beta = \pm c$. We know $c > 0$ and we want $\beta > 0$, so $\beta = c$.

Q1 (b) ii)

Using the chain rule, show that

$$u(x,t) = f(x+ct) + g(x-ct)$$

satisfies the wave equation. Assume that f and g are twice-differentiable functions defined on $\mathbb{R}$.

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(f'(x+ct) + g'(x-ct)\right) \\ &= f''(x+ct) + g''(x-ct)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t}\left(cf'(x+ct) - cg'(x-ct)\right) \\ &= c^2 f''(x+ct) + c^2 g''(x-ct)\end{aligned}$$

Therefore $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}$, so this u satisfies the wave equation.

Question 2

Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \mathrm{e}^x + y^3 - 2x - 3y.$$

Q2 (a)

Find and classify the critical points of f.

$$\frac{\partial f}{\partial x} = \mathrm{e}^x - 2 \qquad \frac{\partial f}{\partial y} = 3y^2 - 3$$

Critical points are where $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$. That means $\mathrm{e}^x - 2 = 0$ so $x = \log 2$, and $3y^3 - 3 = 0$ so $y = \pm 1$. Therefore the critical points of f are at $(\log 2, 1)$ and $(\log 2, -1)$.

The Hessian matrix of f is $\begin{pmatrix} \mathrm{e}^x & 0 \\ 0 & 6y \end{pmatrix}$ and its determinant is $D = 6\mathrm{e}^x y$.

At the point $(\log 2, 1)$, $D = 12$ and $f_{xx} = 2$, so this point is a local minimum point.

At the point $(\log 2, -1)$, $D = -12$, so this point is a saddle point.

Q2 (b)

Let P be the point $(0, 2)$. Using linear approximation around P, show how we can estimate $f(0.01, 1.99)$ without a calculator.

Linear approximation around P gives

$$\begin{aligned}
f(x,y) &\approx f(0,2) + \nabla f|_{(0,2)} \cdot \begin{pmatrix} x \\ y-2 \end{pmatrix} \\
&= (1 + 8 - 0 - 6) + \begin{pmatrix} \mathrm{e}^x - 2 \\ 3y^2 - 3 \end{pmatrix}\Bigg|_{(0,2)} \cdot \begin{pmatrix} x \\ y-2 \end{pmatrix} \\
&= 3 + \begin{pmatrix} -1 \\ 9 \end{pmatrix} \cdot \begin{pmatrix} x \\ y-2 \end{pmatrix} \\
&= 3 - x + 9y - 18 \\
\therefore \quad f(x,y) &\approx -15 - x + 9y
\end{aligned}$$

Therefore

$$\begin{aligned} f(0.01, 1.99) &\approx -15 - 0.01 + 9 \times 1.99 \\ &= -15.01 + 9 + 8.91 \\ &= -15.01 + 17.91 \\ &= 2.9 \end{aligned}$$

For completeness, $f(0.01, 1.99)$ is actually around 2.900649, so this approximation is very good.

Q2 (c)

Sketch (by hand) the level curves of f. On your sketch, indicate:

- the point P
- the critical points
- a vector representing the direction of *steepest descent* at P

The direction of steepest descent at point P is $-\nabla f|_P = \begin{pmatrix} 1 \\ -9 \end{pmatrix}$.

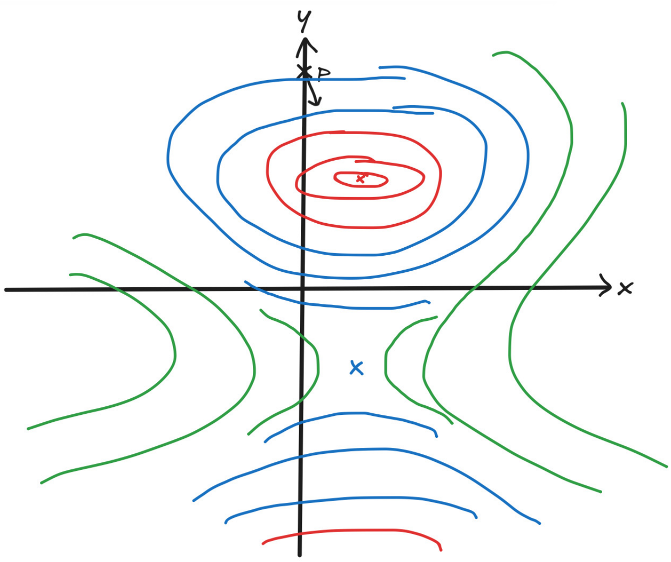

Figure 2: A contour plot of $z = f(x, y)$. Red is negative z, blue is small positive z, and green is large positive z. The red cross is the local minimum point, the blue cross is the saddle point, the black cross is P, and the black arrow is the direction of steepest descent from P.

Q2 (d)

Use Python to produce a 3D plot of the surface $z = f(x, y)$. On your plot, indicate the critical points and point P.

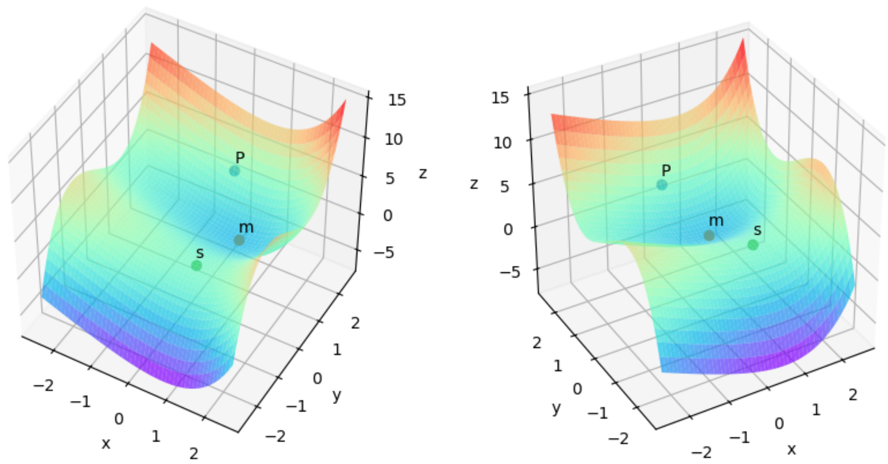

Figure 3: A 3D plot of $z = f(x, y)$ from two angles. m is the local minimum point and s is the saddle point.

```python
#!/usr/bin/env python3

from pathlib import Path

import matplotlib.pyplot as plt
import numpy as np


def f(x, y):
    return np.exp(x) + y**3 - 2 * x - 3 * y


def plot_point(ax, coords, name):
    x, y = coords
    ax.plot(x, y, f(x, y), "o")
    ax.text(x, y, f(x, y) + 1, name)


def plot_3d_from_angle(filename, *, elev, azim):
    x = np.linspace(-2.5, 2.5)
    y = np.linspace(-2.5, 2.5)

    X, Y = np.meshgrid(x, y)

    fig = plt.figure()
    ax = fig.add_subplot(projection="3d")
    ax.set_box_aspect((1, 1, 1))
    ax.view_init(elev=elev, azim=azim)

    ax.plot_surface(X, Y, f(X, Y), alpha=0.7, cmap="rainbow")
    ax.set_xlabel("x")
    ax.set_ylabel("y")
    ax.set_zlabel("z")

    # Plot point P
    plot_point(ax, (0, 2), "P")

    # Plot critical points
    plot_point(ax, (np.log(2), 1), "m")
    plot_point(ax, (np.log(2), -1), "s")

    plt.savefig(
        Path(__file__).parent.parent / "imgs" / f"{filename}.png",
        bbox_inches="tight",
        pad_inches=0.2,
    )
    plt.clf()


def main() -> None:
    plot_3d_from_angle("Q2d-1", elev=45, azim=-60)
    plot_3d_from_angle("Q2d-2", elev=35, azim=-120)


if __name__ == "__main__":
    main()
```

Figure 4: The code used to generate Figure 3. The code can also be found on GitHub.