MA2K4 Numerical Methods and Computing, Assignment 2

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Question 1

Interpolate the function $f(x) = \sqrt{x}$ by a quadratic polynomial $p_2(x)$ with notes at $x_0 = 1$, $x_1 = 2$, and $x_2 = 3$. Isolate the coefficients in the polynomial. Further, compute the error at $x = 6$ to three significant digits.

$$
\begin{aligned}
p_2(x) &= \sum_{k=0}^{2} L_k(x) y_k \\
&= \sum_{k=0}^{2} L_k(x)\sqrt{k} \\
&= L_0(x)\sqrt{1} + L_1(x)\sqrt{2} + L_2(x)\sqrt{3} \\
&= L_0(x) + L_1(x)\sqrt{2} + L_2(x)\sqrt{3} \\
&= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}\sqrt{2} + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}\sqrt{3} \\
&= \frac{(x-2)(x-3)}{(1-2)(1-3)} + \frac{(x-1)(x-3)}{(2-1)(2-3)}\sqrt{2} + \frac{(x-1)(x-2)}{(3-1)(3-2)}\sqrt{3} \\
&= \frac{(x-2)(x-3)}{2} + \frac{(x-1)(x-3)}{-1}\sqrt{2} + \frac{(x-1)(x-2)}{2}\sqrt{3} \\
&= \frac{x^2-5x+6}{2} + \frac{x^2-4x+3}{-1}\sqrt{2} + \frac{x^2-3x+2}{2}\sqrt{3} \\
&= \frac{1}{2}x^2 - \frac{5}{2}x + 3 - \sqrt{2}x^2 + 4\sqrt{2}x - 3\sqrt{2} + \frac{\sqrt{3}}{2}x^2 - \frac{3\sqrt{3}}{2}x + \sqrt{3} \\
&= \left(\frac{1}{2} - \sqrt{2} + \frac{\sqrt{3}}{2}\right)x^2 + \left(-\frac{5}{2} + 4\sqrt{2} - \frac{3\sqrt{3}}{2}\right)x + \left(3 - 3\sqrt{2} + \sqrt{3}\right)
\end{aligned}
$$

The interpolation error at $x = 6$ is

$$
\begin{aligned}
f(6) - p_2(6) &= \sqrt{6} - \left(\frac{1}{2} - \sqrt{2} + \frac{\sqrt{3}}{2}\right) 6^2 \\
&\quad - \left(-\frac{5}{2} + 4\sqrt{2} - \frac{3\sqrt{3}}{2}\right) 6 - \left(3 - 3\sqrt{2} + \sqrt{3}\right) \\
&= \sqrt{6} - \left(18 - 36\sqrt{2} + 18\sqrt{3}\right) \\
&\quad + 15 - 24\sqrt{2} + 9\sqrt{3} - 3 + 3\sqrt{2} - \sqrt{3} \\
&= \sqrt{6} - 18 + 36\sqrt{2} - 18\sqrt{3} \\
&\quad + 15 - 24\sqrt{2} + 9\sqrt{3} - 3 + 3\sqrt{2} - \sqrt{3} \\
&= -18 + 15 - 3 + 36\sqrt{2} - 24\sqrt{2} + 3\sqrt{2} \\
&\quad - 18\sqrt{3} + 9\sqrt{3} - \sqrt{3} + \sqrt{6} \\
&= -6 + 15\sqrt{2} - 10\sqrt{3} + \sqrt{6}
\end{aligned}
$$

This computes to be 0.342 to 3 significant digits.

Question 2

The imaginary error function

$$\operatorname{erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{t^2}\ \mathrm{d}t, \qquad x \in [0,1],$$

is to be linearly interpolated at nodes $x_0, x_1 \in [0,1]$ which are separated by a distance $h = x_1 - x_0$. What node spacing h needs to be chosen to achieve an interpolation error smaller than 10^{-6}?

We know that the error is bounded by

$$|e(x)| \leq \frac{\|\operatorname{erfi}''(x)\|_\infty}{2!} \big|(x - x_0)(x - x_1)\big|$$

and

$$\begin{aligned} \operatorname{erfi}'(x) &= \mathrm{e}^{x^2}, \\ \operatorname{erfi}''(x) &= 2x\mathrm{e}^{x^2}. \end{aligned}$$

Since we're only working within $[0,1]$, we can limit the ∞-norm to this interval so

$$\big\|2x\mathrm{e}^{x^2}\big\|_\infty = 2 \cdot 1 \cdot \mathrm{e}^{1^2} = 2\mathrm{e}$$

Therefore

$$\begin{aligned} |e(x)| &\leq \frac{2\mathrm{e}}{2}\big|(x - x_0)(x - x_1)\big| \\ &= \mathrm{e}\big|(x - x_0)(x - x_1)\big|. \end{aligned}$$

To find the maximum of $(x - x_0)(x - x_1)$, let $\theta \in [0,1]$ and write

$$\begin{aligned} x &= x_0 + \theta h, \\ x_1 &= x_0 + h. \end{aligned}$$

Then

$$\begin{aligned} \big|(x - x_0)(x - x_1)\big| &= \big|(x_0 + \theta h - x_0)(x_0 + \theta h - x_0 - h)\big| \\ &= \big|\theta h(\theta h - h)\big| \\ &= \big|\theta h^2(\theta - 1)\big|. \end{aligned}$$

The maximum is obtained when $\theta = \frac{1}{2}$, giving $\frac{1}{4}h^2$. Therefore

$$|e(x)| \leq \frac{1}{4}h^2\mathrm{e}.$$

We want

$$\begin{aligned} \frac{1}{4}h^2\mathrm{e} &< 10^{-6} \\ h &< \sqrt{4 \cdot 10^{-6}\mathrm{e}} \\ &\approx 0.00329744. \end{aligned}$$

Question 3

Consider $n+2$ distinct nodes $x_i \in \mathbb{R}$, $i \in \{0, 1, \ldots, n+1\}$, and the corresponding data values $y_i \in \mathbb{R}$, $i \in \{0, \ldots, n+1\}$. Let $q(x)$ be the Lagrange interpolating polynomial of degree n for the nodes and values $\{(x_i, y_i) : i = 0, 1, \ldots, n\}$ and let $r(x)$ be the Lagrange interpolating polynomial of degree n for the nodes and values $\{(x_i, y_i) : i = 1, 2, \ldots, n+1\}$.

Define a new polynomial

$$p(x) = \frac{(x - x_0)r(x) - (x - x_{n+1})q(x)}{x_{n+1} - x_0}.$$

Show that this $p(x)$ is the Lagrange interpolation polynomial of degree $n+1$ for the nodes and values $\{(x_i, y_i) : i = 0, 1, \ldots, n+1\}$.

We know

$$\begin{aligned}
q(x) &= \sum_{k=0}^{n} \prod_{j \neq k}^{n} \frac{x - x_j}{x_k - x_j} y_k \\
r(x) &= \sum_{k=1}^{n+1} \prod_{j \neq 0, k}^{n+1} \frac{x - x_j}{x_k - x_j} y_k \\
p(x) &= \sum_{k=0}^{n+1} \prod_{j \neq k}^{n+1} \frac{x - x_j}{x_k - x_j} y_k
\end{aligned}$$

so $q(x)$ is missing the y_{n+1} term

$$\begin{aligned}
p(x) &= \frac{x - x_{n+1}}{x_k - x_{n+1}} q(x) + \prod_{j=1}^{n} \frac{x - x_j}{x_{n+1} - x_j} y_{n+1} \\
&= \frac{x - x_0}{x_k - x_0} r(x) + \prod_{j \neq k}^{n} \frac{x - x_j}{x_0 - x_j} y_0
\end{aligned}$$

Question 4

Given a function $f : [a, b] \to \mathbb{R}$, we make the following recursive definition:

$$\begin{aligned} f[a] &:= f(a), \\ f[a, b] &:= \frac{f[b] - f[a]}{b - a}, \\ f[a, b, c] &:= \frac{f[b, c] - f[a, b]}{c - a}, \\ &\vdots \\ f[a_0, \dots, a_n] &:= \frac{f[a_1, \dots, a_n] - f[a_0, \dots, a_{n-1}]}{a_n - a_0}. \end{aligned}$$

Consider now $m + 1$ distinct nodes $x_j \in \mathbb{R}$ for $j \in \{0, \dots, m\}$. Define the polynomial

$$\begin{aligned} q_0(x) &:= f(x_0), \\ q_{k+1}(x) &:= q_k(x) + f[x_0, \dots, x_{k+1}](x - x_0)(x - x_1) \cdots (x - x_k). \end{aligned}$$

Show that the Lagrange interpolation polynomial of degree m, is in fact given by $p_m(x) = q_m(x)$.

We will do proof by induction. Clearly $p_0(x) = f(x_0) = q_0(x)$ and

$$\begin{aligned} p_1(x) &= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \\ &= q_0(x) + \frac{f[x_1] - f[x_0]}{x_1 - x_0}(x - x_0) \\ &= q_0(x) + f[x_0, x_1](x - x_0), \end{aligned}$$

so we have our base case.

Assume $p_k(x) = q_k(x)$ for some fixed k.

Question 5

For the Newton–Cotes quadrature formula, using equally spaced points $x_j = a + \frac{j}{n}(b-a)$, show that

$$\alpha_j = \alpha_{n-j},$$

where α_j is the quadrature weight of the jth node.

The weights are given by

$$\alpha_j = \int_a^b L_j(x)\ \mathrm{d}x,$$

where L_j is the jth Lagrange polynomial

$$L_j(x) = \prod_{i \neq j}^{m} \frac{x - x_i}{x_j - x_i}.$$

We want to show that

$$\int_a^b \prod_{i \neq j}^{m} \frac{x - x_i}{x_j - x_i}\ \mathrm{d}x = \int_a^b \prod_{i \neq n-j}^{m} \frac{x - x_i}{x_{n-j} - x_i}\ \mathrm{d}x.$$

Both integrands contain a factor of

$$\prod_{i \neq j, n-j}^{k} x - x_i,$$

which we can cancel, giving us

$$\left(\prod_{i \neq j}^{m} \frac{1}{x_j - x_i}\right) \int_a^b (x - x_{n-j})\ \mathrm{d}x = \left(\prod_{i \neq n-j}^{m} \frac{1}{x_{n-j} - x_i}\right) \int_a^b (x - x_j)\ \mathrm{d}x.$$

Then on the LHS, we get

$$\begin{aligned}
\left(\prod_{i \neq j}^{m} \frac{1}{x_j - x_i}\right) \int_a^b (x - x_{n-j})\ \mathrm{d}x &= \left(\prod_{i \neq j}^{m} \frac{1}{x_j - x_i}\right) \left[\frac{1}{2}x^2 - x_{n-j}x\right]_a^b \\
&= \left(\prod_{i \neq j}^{m} \frac{1}{x_j - x_i}\right) \left(\frac{b^2}{2} - x_{n-j}b - \frac{a^2}{2} + x_{n-j}a\right) \\
&= \left(\prod_{i \neq j}^{m} \frac{1}{x_j - x_i}\right) \left(\frac{1}{2}(b^2 - a^2) + x_{n-j}(a - b)\right).
\end{aligned}$$

Similarly on the RHS, we get

$$\left(\prod_{i \neq n-j}^{m} \frac{1}{x_{n-j} - x_i}\right) \left(\frac{1}{2}(b^2 - a^2) + x_j(a - b)\right).$$