MA151 Algebra 1, Assignment 3

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Question 1

> Let ρ and τ be the following permutations:
> $$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}, \qquad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix}$$

Q1 i.

> Express ρ^{-1}, $\rho\tau$, τ^2 in the notation used above.

$$\rho^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 5 & 3 \end{pmatrix}$$
$$\rho\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 4 & 3 & 2 \end{pmatrix}$$
$$\tau^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 1 & 2 \end{pmatrix}$$

Q1 ii.

> Write ρ and τ as products of disjoint cycles.

$$\rho = (1)(2,3,5,4) = (2,3,5,4), \qquad \tau = (1,3,5,4,2)$$

Q1 iii.

> For each of ρ and τ state whether it is an even permutation or an odd permutation.

ρ is an odd permutation (since $\rho = (2,4)(2,5)(2,3)$) and τ is an even permutation (since $\tau = (1,2)(1,4)(1,5)(1,3)$).

Question 2

> Write down the order of each of the following elements of S_6:
>
> i. (1 2) ii. (1 2 3) iii. (1 2 3)(4 6) iv. (1 2 3)(1 2)

Q2 i.

(1 2) has order 2, since it is a transposition.

Q2 ii.

(1 2 3) has order 3.

Q2 iii.

(1 2 3)(4 6) has order 6.

Q2 iv.

$$(1\ 2\ 3)(1\ 2) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (1\ 3)$$

So (1 2 3)(1 2) = (1 3) and has order 2.

Question 3

This question demonstrates a way to argue that two groups cannot be isomorphic.

Q3 (a)

Suppose the group G is isomorphic to the group H. Show that if G contains an element with finite order n then H also contains an element with order n.

Suppose $g \in G$ has order n, so $g^n = 1_G$. Let ϕ be the isomorphic bijection between G and H. We know that $\phi(1_G) = 1_H$ and $\phi(g^n) = \phi(\underbrace{g \cdot g \cdots g}_{n \text{ times}}) = \underbrace{\phi(g) \cdot \phi(g) \cdots \phi(g)}_{n \text{ times}} = \phi(g)^n$.

Therefore $\phi(g^n) = \phi(1_G) \implies \phi(g)^n = 1_H$. Therefore the element $\phi(g) \in H$ has order n.

Q3 (b)

Use **(a)** to deduce that $\mathbb{Z}/6\mathbb{Z}$ cannot be isomorphic to D_6.

$\mathbb{Z}/6\mathbb{Z} \cong C_6$, so every non-identity element of $\mathbb{Z}/6\mathbb{Z}$ has order 6. In D_6, the reflections have order 2, the non-identity rotations have order 3, and the identity has order 1, so no elements of D_6 have order 6. Therefore $\mathbb{Z}/6\mathbb{Z} \ncong D_6$ by **(a)**.

Question 4

> Let G and H be groups and suppose $\phi : G \to H$ is a homomorphism.

Q4 (a)

> Show that $\phi(1_G) = 1_H$.

We know that $1_G 1_G = 1_G$, so $\phi(1_G) = \phi(1_G 1_G) = \phi(1_G)\phi(1_G)$. But $\phi(1_G) \in H$, so it has an inverse in H. Thus, we can say

$$\begin{aligned} \phi(1_G)\phi(1_G)^{-1} &= \phi(1_G)\phi(1_G)\phi(1_G)^{-1} \\ 1_H &= \phi(1_G)1_H \\ &= \phi(1_G) \\ \therefore \phi(1_G) &= 1_H \end{aligned}$$

Q4 (b)

> Show that ϕ is injective if and only if $\ker\phi = \{1_G\}$.

Recall that $\ker\phi = \{g \in G : \phi(g) = 1_H\}$. First we will show that ϕ being injective implies that $\ker\phi = \{1_G\}$.

Suppose ϕ is injective, then $\phi(g_1) = \phi(g_2) \iff g_1 = g_2 \;\forall\; g_1, g_2 \in G$. We already know that $\phi(1_G) = 1_H$ from before. Since ϕ is injective, if $\phi(g) = 1_H$, then $g = 1_G$. Therefore $\ker\phi = \{g \in G : \phi(g) = 1_H\} = \{1_G\}$.

For the converse, now suppose $\ker\phi = \{1_G\}$. That means that $\phi(g) \neq 1_H \;\forall\; g \in G, g \neq 1_G$. Suppose $\phi(g_1) = \phi(g_2)$ for some $g_1 \neq g_2$. Then

$$\begin{aligned} \phi(g_1) &= \phi(g_2) \\ \phi(g_1)^{-1}\phi(g_1) &= \phi(g_1)^{-1}\phi(g_2) \\ 1_H &= \phi\left(g_1^{-1}g_2\right) \\ \implies 1_G &= g_1^{-1}g_2 \\ \implies g_1 &= g_2 \end{aligned}$$

But that's a contradiction, since we assumed $g_1 \neq g_2$. Therefore $\phi(g_1) \neq \phi(g_2)$, so ϕ is injective.

Q4 (c)

Show that if ϕ is surjective and G is abelian then H is abelian.

If ϕ is surjective, then $\forall\ h \in H, \exists g \in G, \phi(g) = h$. If G is Abelian, then $g_1 g_2 = g_2 g_1\ \forall\ g_1, g_2 \in G$.

Then $\forall\ h_1, h_2 \in H$,

$$\begin{aligned} h_1 h_2 &= \phi(g_1)\phi(g_2) \\ &= \phi(g_1 g_2) \\ &= \phi(g_2 g_1) \\ &= \phi(g_2)\phi(g_1) \\ &= h_2 h_1 \end{aligned}$$

Therefore H is also Abelian.

Q4 (d)

Show that if ϕ is injective and H is abelian then G is abelian.

If ϕ is injective, then $\phi(g_1) = \phi(g_2) \iff g_1 = g_2\ \forall\ g_1, g_2 \in G$. If H is Abelian, then $h_1 h_2 = h_2 h_1\ \forall\ h_1, h_2 \in H$.

Then $\forall\ g_1, g_2 \in G$,

$$\phi(g_1)\phi(g_2) = \phi(g_2)\phi(g_1)$$
$$\phi(g_1 g_2) = \phi(g_2 g_1)$$
$$g_1 g_2 = g_2 g_1$$

Therefore G is also Abelian.

Question 5

> Show that, in the ring $M_{2\times 2}(\mathbb{Z})$, there are elements ($2 \times 2$ matrices with integer entries) A, B, C with $A \neq 0$ such that $AB = AC$ but $B \neq C$.

Suppose

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & -1 \\ 10 & 2 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}$$

Clearly $A \neq \mathbf{0}$ and $B \neq C$ but

$$AB = \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad AC = \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}$$

So $AB = AC$.

Question 6

> Suppose R is a ring with the property that whenever $a \neq 0$ and $b \neq 0$ then $ab \neq 0$. Show that if $rs = rt$ where $r, s, t \in R$ with $r \neq 0$ then $s = t$.

Suppose R is a ring where $a \neq 0, b \neq 0 \implies ab \neq 0$ and $rs = rt$. Then either $s = 0$ or $s \neq 0$.

In the case where $s = 0$, we have $r \times 0 = 0 = rt$, therefore $r = 0$ or $t = 0$, but we know $r \neq 0$, so $t = 0$. Therefore $s = t$.

In the case where $s \neq 0$, we have, by distributivity,

$$\begin{aligned} rs &= rt \\ rs - rt &= 0 \\ r(s-t) &= 0 \\ s - t &= 0 \qquad \text{since } r \neq 0 \\ \therefore s &= t \end{aligned}$$

Question 7

Give an example of a non-commutative ring with a finite number of elements. Justify your answer by exhibiting two elements a, b in your ring for which $ab \neq ba$.

Hint: if R is a ring then so is $M_{2\times 2}(R)$.

$M_{2\times 2}(\mathbb{Z}/5\mathbb{Z})$ is a non-commutative ring. We know that $\mathbb{Z}/5\mathbb{Z}$ is a ring, so $M_{2\times 2}(\mathbb{Z}/5\mathbb{Z})$ is also a ring. It has finite elements, since each matrix has 4 numbers, each of which has 5 choices, so there are $5^4 = 625$ elements.

To demonstrate non-commutativity, consider $a = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, b = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$. Then

$$ab = \begin{pmatrix} 4 & 3 \\ 0 & 4 \end{pmatrix}, \qquad ba = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Therefore $ab \neq ba$, so $M_{2\times 2}(\mathbb{Z}/5\mathbb{Z})$ is not commutative.