MA260 Norms Metrics and Topologies, Assignment 1

Dyson Dyson

Question 1

Give a reason why each of the following is **not** a norm on $\mathbb{R}^3$:

(i) $\|(x, y, z)\| = x + y + z$

(ii) $\|(x, y, z)\| = \left(|x|^2 + |y|^2 + |z|^2\right)^{1/3}$

(iii) $\|(x, y, z)\| = \left(|x|^4 - |y|^4 + 2|z|^4\right)^{1/4}$

Q1 (i)

Let $v = (1, -2, 1)$. Then $v \neq \underline{0}$ but $\|v\| = 1 - 2 + 1 = 0$, so this norm does not satisfy non-degeneracy.

Q1 (ii)

Let $v = (x, y, z) \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} \|\lambda v\| &= \left(|\lambda x|^2 + |\lambda y|^2 + |\lambda z|^2\right)^{1/3} \\ &= \left(|\lambda|^2 \left(|x|^2 + |y|^2 + |z|^2\right)\right)^{1/3} \\ &= |\lambda|^{2/3}\|v\|. \end{aligned}$$

It this were a norm, we would expect to get $\|\lambda v\| = |\lambda|\|v\|$, but we don't, so this is not a norm.

Q1 (iii)

If this were a norm, it should satisfy the triangle inequality, so we would have $\|v+w\| \leq \|v\| + \|w\|$. Take $v = (1, 2, 3)$ and $w = (3, 2, 1)$. Then

$$\begin{aligned}
\|v\| &= (1 - 16 + 162)^{1/4} \\
&= 147^{1/4} \\
&\approx 3.482 \\
\|w\| &= (81 - 16 + 2)^{1/4} \\
&= 67^{1/4} \\
&\approx 2.861 \\
\|v\| + \|w\| &\approx 6.343 \\
\|v+w\| &= \|(4, 4, 4)\| \\
&= (256 - 256 + 512)^{1/4} \\
&= 512^{1/4} \\
&\approx 4.757
\end{aligned}$$

This counterexample shows that this 'norm' doesn't satisfy the triangle inequality, so is not a norm.

Question 2

Consider the space ℓ^1 with norm $\|\cdot\|_1$ and define $g : \ell^1 \to \mathbb{R}$ by

$$g\big((x_j)_{j=1}^{\infty}\big) = \sum_{j=1}^{\infty} (-1)^j x_j.$$

Show that $g : \ell^1 \to \mathbb{R}$ is well-defined and continuous.

The space ℓ^1 contains only sequences which are '1-summable', meaning $\sum_{j=1}^{\infty} x_j$ converges to a finite value. Therefore $\sum_{j=1}^{\infty} (-1)^j x_j$ also converges to a finite value. That means $g\big((x_j)\big)$ is in ℓ^1 and is therefore well-defined.

For g to be continuous, we want to show that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$g\big(\mathbb{B}_\delta((x_j))\big) \subset \mathbb{B}_\varepsilon\big(g((x_j))\big).$$

Every sequence in $\mathbb{B}_\delta((x_j))$ is the sequence (x_j) with a perturbation of δ. So if $(x_j) = (x_1, x_2, \dots)$, then an element of $\mathbb{B}_\delta((x_j))$, call it $(x_j) + \delta$, would look something like $(x_1, \dots, x_i + \delta, \dots)$ or $(x_1, \dots, x_i + \delta/3, \dots, x_k + 2\delta/3, \dots)$.

Since ℓ^1 uses the 1-norm, this perturbation of δ will be carried through the norm, so

$$\|(x_j) + \delta\| = \|(x_j)\| + |\delta|.$$

Then this $|\delta|$ will just get included in the sum in g, so

$$g\big((x_j) + \delta\big) = g\big((x_j)\big) + |\delta|.$$

Therefore

$$g\big(\mathbb{B}_\delta((x_j))\big) = \mathbb{B}_\delta\big(g((x_j))\big)$$

Then we just choose $\delta \leq \varepsilon$ and see that g is continuous.