

MA146 Assignment 1

- Q1
- Ⓐ Continuous, deterministic system and an inverse problem.
 - Ⓑ Discrete, stochastic system and a control problem.
 - Ⓒ Discrete, stochastic system and a forward problem.

Q2

Ⓐ $b^2 - 4ac < 0 \quad x_2(t) = e^{-\frac{b}{2a}t} \sin(\omega t)$

$$\begin{aligned} x_2'(t) &= e^{-\frac{b}{2a}t} \omega \cos(\omega t) - \frac{b}{2a} e^{-\frac{b}{2a}t} \sin(\omega t) \\ &= e^{-\frac{b}{2a}t} \left(\omega \cos(\omega t) - \frac{b}{2a} \sin(\omega t) \right) \end{aligned}$$

$$\begin{aligned} x_2''(t) &= e^{-\frac{b}{2a}t} \left(-\omega^2 \sin(\omega t) - \frac{b\omega}{2a} \cos \omega t \right. \\ &\quad \left. - \frac{b}{2a} \left(\omega \cos(\omega t) - \frac{b}{2a} \sin(\omega t) \right) \right) \\ &= e^{-\frac{b}{2a}t} \left(\left(\frac{b^2}{4a^2} - \omega^2 \right) \sin \omega t - \frac{b}{a} \omega \cos \omega t \right) \end{aligned}$$

We plug these into the ODE and get

$$\begin{aligned} e^{-\frac{b}{2a}t} &\left(a \left(\frac{b^2}{4a^2} - \omega^2 \right) \sin \omega t - b\omega \cos \omega t + b\omega \cos \omega t \right. \\ &\quad \left. - \frac{b^2}{2a} \sin \omega t + c \sin \omega t \right) \end{aligned}$$

$$= e^{-\frac{b}{2a}t} \frac{b^2 - 4b^2}{4a} \left(\frac{b^2}{4a} - \frac{b^2}{2a} - a\omega^2 + c \right) \sin \omega t$$

Q2/ (a) cont.

$$= e^{-\frac{b}{2a}t} \left(\frac{-b^2}{4a} - \cancel{a\omega^2} + c \right) \sin \omega t$$

Recall that $\omega = \frac{1}{2a} \sqrt{4ac - b^2} \therefore \omega^2 = \frac{1}{4a^2} (4ac - b^2)$

$$\therefore \text{ODE} = e^{-\frac{b}{2a}t} \left(\frac{-b^2}{4a} - \frac{4ac}{4a} + \frac{b^2}{4a} + c \right) \sin \omega t$$

$$= e^{-\frac{b}{2a}t} \times 0 \times \sin \omega t = 0 \quad \text{as expected.}$$

① $a=2, b=-4, c=-6$

$$b^2 - 4ac = 16 + 8 \times 6 = 64 \quad \lambda_{1,2} = \frac{4 \pm \sqrt{64}}{4} = 1 \pm 2$$

$$\therefore x(t) = l_1 e^{3t} + l_2 e^{-t}$$

② $a=-2, b=3, c=-\frac{5}{4}$

$$b^2 - 4ac = 9 - 5 \times 2 = -1 \quad \lambda_{1,2} = \frac{3}{4} \pm i \frac{1}{4} \sqrt{1}$$

$$\omega = \frac{1}{4} \therefore \lambda_{1,2} = \frac{3}{4} \pm i \frac{1}{4}$$

$$\therefore x(t) = e^{\frac{3}{4}t} \left(l_1 \cos \frac{t}{4} + l_2 \sin \frac{t}{4} \right) = e^{\frac{3}{4}t} \left(l_1 \cos \frac{t}{4} + l_2 \sin \frac{t}{4} \right)$$

③ $b^2 - 4ac = 8 \times 18 = 144$

$$\lambda_{1,2} = \frac{0 \pm 12i}{-4} = \pm 3i$$

$a=-2, b=0, c=-18$

~~$x(t) = l_1 e^{3t} + l_2 e^{-3t}$~~ $\therefore x(t) = e^{0t} (l_1 \sin 3t + l_2 \cos 3t)$

22/2 cont.

$$x(t) = l_1 \sin 3t + l_2 \cos 3t$$

$$x'(t) = 3l_1 \cos 3t - 3l_2 \sin 3t$$

$$x'(2\pi) = 3l_1 \cos 6\pi - 3l_2 \sin 6\pi = 3l_1 - 0 = 12 \therefore l_1 = 4$$

$$x(2\pi) = l_1 \sin 6\pi + l_2 \cos 6\pi = 0 + l_2 = 15 \therefore l_2 = 15$$

$$\therefore x(t) = 4 \sin 3t + 15 \cos 3t$$

② $a=3, b=12, c=12$ Jupyter notebook says

$$x(t) = (-6t - 4)e^{-2t}$$

$$\therefore x'(t) = 2(6t + 4)e^{-2t} - \frac{-(6t + 4)e^{-2t}}{-6e^{-2t}} \\ = (12t + 2)e^{-2t}$$

$$x''(t) = -2(12t + 2)e^{-2t} + 12e^{-2t} = (-24t + 8)e^{-2t}$$

~~ODE = 3(-24t + 8)e^{-2t} + 12(12t + 2)e^{-2t} - 12(-6t - 4)e^{-2t}~~

$$\text{ODE} = 3(-24t + 8)e^{-2t} + 12(12t + 2)e^{-2t} - 12(-6t - 4)e^{-2t} \\ = e^{-2t}(-72t + 24 + 144t + 24 - 72t - 48) \\ = e^{-2t} \times 0 = 0 \quad \text{as expected}$$

```
[2]: a, b, c = 3, 12, 12
```

```
[3]: x = Function("x")
t = Symbol("t")
ode_sc = a * Derivative(x(t), t, t) + b * Derivative(x(t), t) + c * x(t)
# actually, this is no equation but just a term; the subsequent solver will make it an equation by setting it to zero
sol_sc = dsolve(ode_sc)
print(sol_sc)
```

Eq(x(t), (C1 + C2*t)*exp(-2*t))

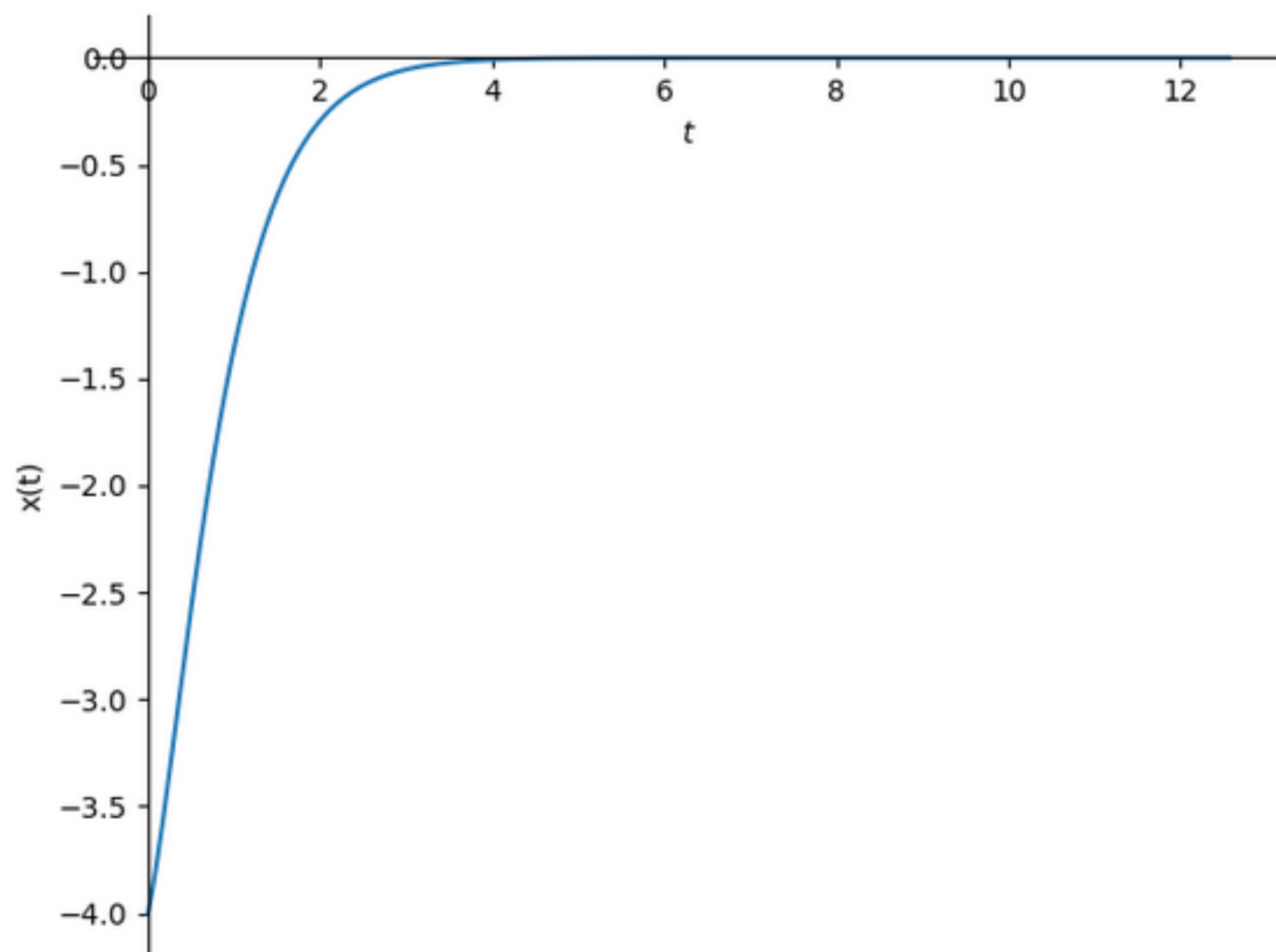
Solving a corresponding initial value problem with the initial condition

$$x(0) = -4, \quad \frac{d}{dt}x(0) = 2.$$

After, we plot the solution for $t \in (0, 4\pi)$ with 500 points to have a sufficient resolution (by default, an adaptive procedure is used but which didn't look great in our tests).

```
[4]: ic_sc = {x(0): -4, x(t).diff(t).subs(t, 0): 2}
sol_sc = dsolve(ode_sc, ics=ic_sc)
print(sol_sc)
plot(sol_sc.rhs, (t, 0, 4 * pi), ylabel="x(t)", adaptive=False, nb_of_points=500)
```

Eq(x(t), (-6*t - 4)*exp(-2*t))




```
[4]: def get_ics(x0):
      return {x(0): x0, x(t).diff(t).subs(t, 0): 0}

initial_x0_list = [1.5, 2, 2.7]

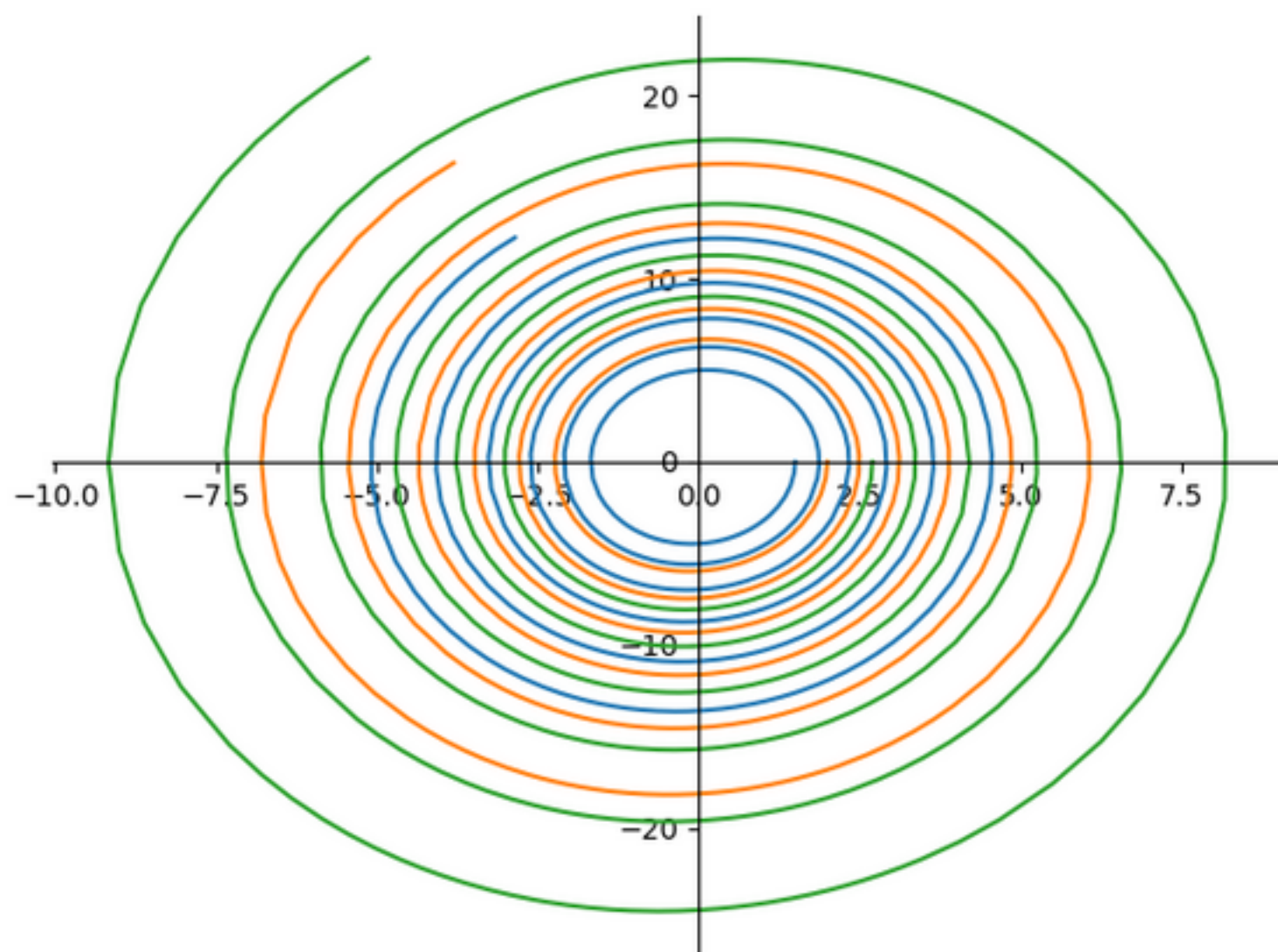
sols_sc = [dsolve(ode_sc, ics=get_ics(x0)) for x0 in initial_x0_list]
print(sols_sc)
# plot(sol_sc.rhs, (t, 0, 4 * pi), ylabel="x(t)", adaptive=False, nb_of_points=500)

[Eq(x(t), (-0.053066185325792*sin(2.82665880502051*t) + 1.5*cos(2.82665880502051*t))*exp(0.35864255*sin(2.82665880502051*t) + 2.7*cos(2.82665880502051*t))*exp(0.1*t))]
```

Retrieve the derivative of the solution and plot it (in orange) together with the solution (in blue as before).

Another popular way to illustrate the solution is via a parametric plot in the phase space. We will look at it

```
[5]: plot_parametric(*[(sol.rhs, sol.rhs.diff(t)) for sol in sols_sc], (t, 0, 4 * pi))
```





Q3 @

During free fall:

$$\boxed{h(0)=0} \quad \boxed{h'(0)=0}$$

$$m \ddot{h}(t) = f(t) = -mg$$

$$\Rightarrow \ddot{h}(t) = -g$$

\therefore In free fall, $\boxed{h(t) = -gt^2}$ when $h(t) \geq -l$.

We don't have to worry about terms in t or constants because of the initial conditions being 0. They are both satisfied by $h(t) = -gt^2$.

During re-bounce: $(h(t) \leq -l)$

$$m \ddot{h}(t) = f(t) = k(-l - h(t)) - mg$$

$$\Rightarrow \ddot{h}(t) = \frac{k}{m}(-l - h(t)) - g$$

$$\Rightarrow \frac{d^2}{dt^2} h(t) + \frac{k}{m} h(t) = -\frac{kl}{m} - g$$

~~$\therefore \lambda^2 + \frac{k}{m} = 0$ is the auxiliary equation, so $\lambda = \pm \sqrt{-\frac{k}{m}}$~~
 ~~$k, m > 0$, so $\lambda = \pm i\sqrt{\frac{k}{m}}$~~

~~$\therefore \lambda^2 + \frac{k}{m} = -\frac{kl}{m} - g$ is the auxiliary equation~~

I was right the first time.

$\lambda^2 + \frac{k}{m} = 0$ is the auxiliary equation for the ~~comp~~ complementary function.

Q3/ (a) $\therefore \lambda = \pm \sqrt{\frac{-k}{m}}$ $k, m > 0$, so $\lambda = \pm i\sqrt{\frac{k}{m}}$

Therefore the complementary function is ~~the~~

$$h(t) = C_1 e^{i\sqrt{\frac{k}{m}}t} + C_2 e^{-i\sqrt{\frac{k}{m}}t}$$

$$= C_1 \left(\cos\sqrt{\frac{k}{m}}t + i \sin\sqrt{\frac{k}{m}}t \right) + C_2 \left(\cos\sqrt{\frac{k}{m}}t - i \sin\sqrt{\frac{k}{m}}t \right)$$

$$= (C_1 + C_2) \cos\sqrt{\frac{k}{m}}t + (C_1 - C_2)i \sin\sqrt{\frac{k}{m}}t$$

I'm not sure why, but we always drop the i here to get

$$h(t) = A \cos\sqrt{\frac{k}{m}}t + B \sin\sqrt{\frac{k}{m}}t$$

The particular solution wants $h(t)$ to be some constant, to reflect the constant on the RHS of the ODE, $-\frac{k\ell}{m} - g$.

Let $h(t) = C$. Then $h'(t) = 0$.

$$\therefore 0 + \frac{k}{m}C = -\frac{k}{m}\ell - g \Rightarrow C = -\ell - \frac{mg}{k}$$

$$\therefore h(t) = A \cos\sqrt{\frac{k}{m}}t + B \sin\sqrt{\frac{k}{m}}t - \ell - \frac{mg}{k}$$

$$= A \cos\left(t\sqrt{\frac{k}{m}}\right) + B \sin\left(t\sqrt{\frac{k}{m}}\right) - \ell - \frac{mg}{k}$$

$$h(0) = A + 0 - \ell - \frac{mg}{k} = 0 \Rightarrow A = \ell + \frac{mg}{k}$$

$$h'(t) = A\sqrt{\frac{k}{m}} \sin\left(t\sqrt{\frac{k}{m}}\right) + B\sqrt{\frac{k}{m}} \cos\left(t\sqrt{\frac{k}{m}}\right)$$

$$h'(0) = 0 + B\sqrt{\frac{k}{m}} = 0 \Rightarrow B = 0 \text{ since } k, m > 0$$

$$\therefore \boxed{h(t) = \left(\ell + \frac{mg}{k}\right) \cos\left(t\sqrt{\frac{k}{m}}\right) - \ell - \frac{mg}{k}}$$

Q3/ (b) $h(t)$ in the re-bounce stage is basically just a cosine function, stretched and ~~scale~~ scaled. That means the minimum of $h(t)$ is $-l - \frac{mg}{k} - (l + \frac{mg}{k})$

$= -2(l + \frac{mg}{k})$ If we want the jumper to just touch the floor in the worst case, then

$h_m \leq -2(l + \frac{mg}{k})$ ~~Then~~ We can then simply rearrange to find l .

$$-\frac{h_m}{2} \geq l + \frac{mg}{k} \quad \therefore \boxed{l \leq -\left(\frac{h_m}{2} + \frac{mg}{k}\right)}$$

$$\therefore \boxed{l \leq -\left(\frac{h_m}{2} + \frac{mg}{k}\right)}$$