CS147 Discrete Maths and its Applications 2, Assignment 2

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Question 1

Let $G = (V, E)$ be a graph, $M \subset E$ be a matching on G, and $Z \subset M$.

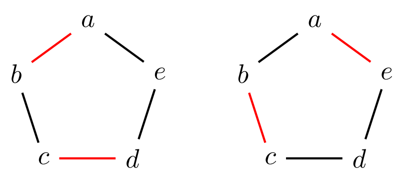
A matching is just a set of edges in G which do not share any common endpoint. For Z to not be a matching, we would need to choose two edges from M which share a node. Since M is a matching, no such pair of edges exists by definition, so Z must also be a matching.

Question 2

Let $G = (V, E)$ be a graph with n nodes, where each node $v \in V$ is incident on exactly 2 edges. The graph is isomorphic to an n-gon. For example, the case of $n = 4$ could be drawn as a square.

In the case of even n, there must exist two maximum matchings of size $\frac{n}{2}$, which are complements of each other in E, so $M_2 = E \setminus M_1$.

In the case of odd n, we still get two complementary matchings, but they are not large enough. Take the case of $n = 5$ for example,



Both of the subsets highlighted in red are matchings, but both are maximum and of size 2. It is clear that in the case of odd n, a maximum matching has size $\left\lfloor \frac{n}{2} \right\rfloor$.

Therefore it is false that there must exist a matching $M \subset E$ with $|M| \geq \frac{n}{2}$.

Question 3

Let $G = (V, E)$ be a graph, and M be a maximal matching on G, so every edge in $E \setminus M$ has at least one endpoint that is matched under M, meaning M cannot be extended. Also let M^* be a matching of maximum size on G.

Is it true that $|M| \geq \frac{1}{2}|M^*|$? Yes.

Suppose we have a situation where $|M| < \frac{1}{2}|M^*|$. Let $|M| = \ell$ and $|M^*| = k$ so that M matches 2ℓ nodes and M^* matches $2k$ nodes. The inequality implies $\ell < \frac{1}{2}k \iff 2\ell < k$.

There are at most 2ℓ edges in M^* which are matched by M. But since $2\ell < k$, there is at least one edge in M^* which is not matched by M. Therefore we can add this edge to M, meaning it is not maximal. That's a contradiction, therefore $|M| < \frac{1}{2}|M^*|$.

Question 4

Let $G = (L \cup R, E)$ be a bipartite graph where every edge connects one node in L to one node in R, and let $N_G(X)$ denote the neighbours of some set X of nodes in G. Also G has the property that for all $A \subset L$, $|N_G(A)| \geq \frac{1}{2}|A|$.

We want to know if there exists a subset $H \subset E$ where $|H| = |L|$, every node in L is incident upon exactly one edge in H, and every node in R is incident upon at most two edges in H.

To satisfy the first two properties, we require that H is constructed by considering each node in L and choosing one of the edges that connects to it.

Is it possible that there exists a $v \in R$ which is incident on three edges in H? Suppose such a v does exist. Then those three edges in H would connect to three distinct nodes in L, call them $S = \{u_1, u_2, u_3\}$. But by the neighbour requirement of G, we have $|N_G(S)| \geq \frac{1}{2}|S|$.

By construction, all three nodes connect to the same $v \in R$ and no other nodes, so $N_G(S) = 1$. Therefore we have $1 \geq \frac{3}{2}$, which is a contradiction.

Therefore we cannot have a $v \in R$ which is incident on three edges in H. Therefore every node in R is incident on at most two edges in H, so the statement is true.