MA265 Methods of Mathematical Modelling 3, Assignment 2

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Question 1

Transport equation: Solve the equation $\partial_t u + v(x,t)\partial_x u = 0$ for $(x,t) \in \mathbb{R} \times (0,\infty)$ with velocity field $v(x,t) = x(t+1)^{-2}$ and initial data $u(x,0) = \cos x$.

We will use the method of characteristics. We first have to solve the associated ODE.

$$\begin{aligned}
\xi'(t) &= v(\xi(t),t) = \xi(t)(t+1)^{-2} \\
\xi(0) &= x_0
\end{aligned}$$

We can solve this with separation of variables.

$$\begin{aligned}
\frac{\mathrm{d}\xi}{\mathrm{d}t} &= \xi(t)(t+1)^{-2} \\
\int \frac{1}{\xi}\,\mathrm{d}\xi &= \int (t+1)^{-2}\,\mathrm{d}t \\
\ln \xi &= \frac{-1}{t+1} + C \\
\xi(t) &= A\mathrm{e}^{-1/(t+1)}
\end{aligned}$$

We use the initial value to find that

$$\begin{aligned}
x_0 &= A\mathrm{e}^{-1} \\
x_0\mathrm{e} &= A \\
\xi(t) &= x_0\mathrm{e}^{1-\frac{1}{t+1}} \\
\xi(t) &= x_0\mathrm{e}^{\frac{t+1-1}{t+1}} \\
\xi(t) &= x_0\mathrm{e}^{t/(t+1)}
\end{aligned}$$

We know that $x = \xi(t)$, so we can solve for x_0 and find that

$$x_0 = x\mathrm{e}^{-t/(t+1)}.$$

Now we apply the fact that $u(x,t) = \cos x_0$ to find that

$$u(x,t) = \cos\left(x\mathrm{e}^{-t/(t+1)}\right).$$

Question 2

Wave equation: Solve the equation $u_{tt}(x,t) = 9u_{xx}(x,t)$ for $(x,t) \in \mathbb{R}\times(0,\infty)$ with initial conditions $u(x,0) = \cos x$ and $u_t(x,0) = 2\cos x$. Verify that if $x_n = \dfrac{2n+1}{2}\pi$ for $n \in \mathbb{Z}$, then $u(x_n,t) = 0$ for all $n \in \mathbb{Z}$ and $t \geq 0$.

We can define

$$\begin{aligned} \phi(x) &= \cos x \\ V(x) &= 2\cos x \\ c &= 3 \end{aligned}$$

and use d'Alembert's Formula

$$u(x,t) = \frac{1}{2}\big(\phi(x+ct) + \phi(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} V(r)\ \mathrm{d}r$$

to find the solution.

$$\begin{aligned} u(x,t) &= \frac{1}{2}\big(\cos(x+3t) + \cos(x-3t)\big) + \frac{1}{6}\int_{x-3t}^{x+3t} 2\cos r\ \mathrm{d}r \\ &= \frac{1}{2}\big(\cos(x+3t) + \cos(x-3t)\big) + \frac{1}{6}\big[2\sin r\big]_{x-3t}^{x+3t} \\ &= \frac{1}{2}\big(\cos x\cos 3t - \sin x \sin 3t + \cos x \cos 3t + \sin x \sin 3t\big) \\ &\quad + \frac{1}{3}\sin(x+3t) - \frac{1}{3}\sin(x-3t) \\ &= \cos x \cos 3t + \frac{1}{3}\big(\sin x\cos 3t + \cos x \sin 3t - (\sin x \cos 3t - \cos x \sin 3t)\big) \\ &= \cos x \cos 3t + \frac{2}{3}\cos x \sin 3t \\ &= \cos x\left(\cos 3t + \frac{2}{3}\sin 3t\right) \end{aligned}$$

So if $x_n = \dfrac{2n+1}{2}\pi$ then $\cos x_n = 0$ and so

$$u(x_n,t) = \cos x_n\left(\cos 3t + \frac{2}{3}\sin 3t\right) = 0$$

as required.

Question 3

Burger's equation: Consider Burger's equation $\partial_t u + u\partial_x u = 0$ for $(x, t) \in \mathbb{R} \times (0, \infty)$ with the initial conditions $u(x, 0) = \Phi(x)$ for all $x \in \mathbb{R}$. Use the method of characteristics to show that the solutions for the following initial data Φ

(i) $\Phi(x) = \begin{cases} 0 & x \leq 0 \\ e^{-1/x} & x > 0 \end{cases}$

(ii) $\Phi(x) = -x$

are given by

(i) $u(x, t) = 0$ for $x \leq 0$ and $u(x, t) = e^{-1/\xi}$ where $\xi > 0$ is such that $x - \xi = te^{-1/\xi}$ for $x > 0$.

(ii) $u(x, t) = \dfrac{-x}{1 - t}$ for $x \in \mathbb{R}$ and times $t < 1$.

Sketch the characteristics in the xt-plane in each case.

Q3 (i)

For the first case of $\Phi(x) = e^{-1/x}$ when $x > 0$ and $\Phi(x) = 0$ otherwise, we consider the ODE

$$\begin{aligned} \xi'(t) &= u(\xi(t), t) = \Phi(x) \\ \xi(0) &= x_0 \end{aligned}$$

We shall consider the cases of $x > 0$ and $x \leq 0$ separately. First the simpler case of $x \leq 0$.

$$\begin{aligned} \zeta'(t) &= u(\zeta(t), t) \\ \frac{\mathrm{d}\zeta}{\mathrm{d}t} &= 0 \\ \int \mathrm{d}\zeta &= 0 \\ \zeta(t) &= C \\ \zeta(0) &= C = x_0 \\ \zeta(t) &= x_0 \\ x &= x_0 \end{aligned}$$

So for $x \leq 0$, $u(x, t) = 0$ as required.

Now for the case of $x > 0$.

$$\begin{aligned}
\xi'(t) &= u(\xi(t), t) \\
\frac{\mathrm{d}\xi}{\mathrm{d}t} &= \mathrm{e}^{-1/\xi} \\
\int \mathrm{d}\xi &= \int \mathrm{e}^{-1/\xi}\ \mathrm{d}t \\
\xi(t) &= t\mathrm{e}^{-1/\xi} + C \\
\xi(0) &= C = x_0 \\
\xi(t) &= t\mathrm{e}^{-1/\xi} + x_0 \\
\xi - x_0 &= t\mathrm{e}^{-1/\xi}
\end{aligned}$$

I'm not sure where to go from here.

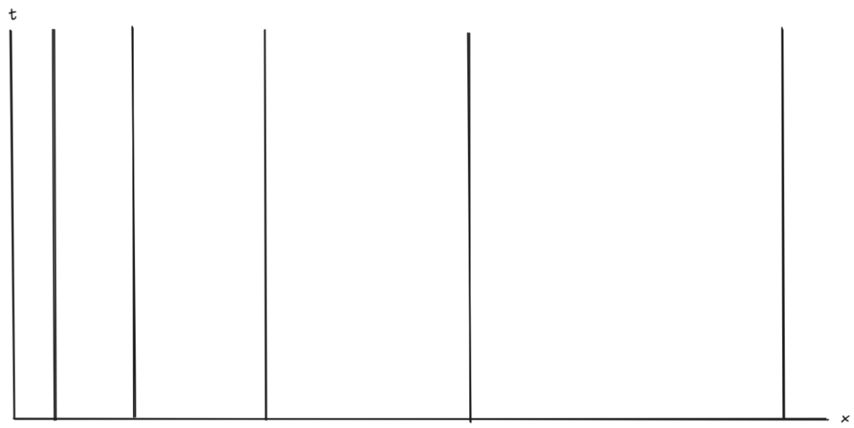

Figure 1: The characteristics for **Q3 (i)**

Q3 (ii)

For the case of $\Phi(x) = -x$, we once again consider the ODE

$$\begin{aligned}
&\xi'(t) = u(\xi(t), t) = \Phi(x_0) \\
&\xi(0) = x_0
\end{aligned}$$

We solve this like so,

$$\begin{aligned}
\xi'(t) &= u(\xi(t), t) \\
\frac{\mathrm{d}\xi}{\mathrm{d}t} &= \frac{-\xi(t)}{1-t} \\
-\frac{1}{\xi}\,\mathrm{d}\xi &= \frac{1}{1-t}\,\mathrm{d}t \\
\int -\frac{1}{\xi}\;\mathrm{d}\xi &= \int \frac{1}{1-t}\;\mathrm{d}t \\
-\ln\xi &= -\ln(1-t) + C \\
\xi(t) &= A\cdot(1-t) \\
\xi(0) &= A = x_0 \\
\xi(t) &= x_0(1-t).
\end{aligned}$$

Or more simply, without involving u,

$$\begin{aligned}
\xi'(t) &= \Phi(x_0) \\
&= -x_0 \\
\xi(t) &= -x_0 t + C \\
\xi(0) &= C = x_0 \\
\xi(t) &= -x_0 t + x_0 \\
x &= x_0(1-t) \\
x_0 &= \frac{x}{1-t}.
\end{aligned}$$

And then $u(x,t) = \Phi(x_0) = \dfrac{-x}{1-t}$ as required.

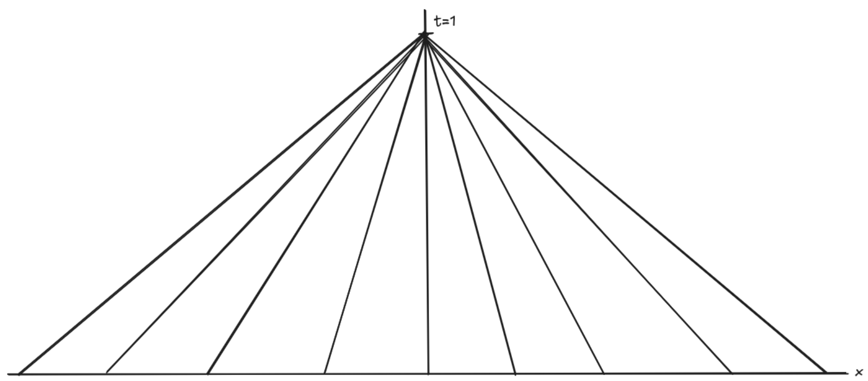

Figure 2: The characteristics for **Q3 (ii)**

Question 4

Initial boundary value problems: Consider the initial boundary value problem (IBVP):

$$\begin{cases} \partial_{tt}u(x,t) = c^2\partial_{xx}u(x,t) & (x,t) \in (0,\infty) \times (0,\times) \\ u(x,0) = \Phi(x) & x \in (0,\infty) \\ \partial_t u(x,0) = V(x) & x \in (0,\infty) \\ u(0,t) = 0 & t \in [0,\infty) \end{cases} \tag{1}$$

for given smooth functions $\Phi(x)$ and $V(x)$. Extend the functions $\Phi(x)$ and $V(x)$ on $(0,\infty)$ to odd functions $\tilde{\Phi}(x)$ and $\tilde{V}(x)$ on $\mathbb{R}$ by

$$\tilde{\Phi}(x) = \begin{cases} \Phi(x) & x > 0 \\ 0 & x = 0 \\ -\Phi(-x) & x > 0 \end{cases} \quad \text{and} \quad \tilde{V}(x) = \begin{cases} V(x) & x > 0 \\ 0 & x = 0 \\ -V(-x) & x > 0 \end{cases}$$

Assume that $\tilde{\Phi}(x)$ and $\tilde{V}(x)$ are smooth functions. Show that d'Alembert's formula with data $\tilde{\Phi}(x)$ and $\tilde{V}(x)$ produces a solution of (1).

D'Alembert's formula with $\tilde{\Phi}$ and $\tilde{V}$ is

$$u(x,t) = \frac{1}{2}\Big(\tilde{\Phi}(x+ct) + \tilde{\Phi}(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{V}(r)\,\mathrm{d}r$$

The wave equation is linear, so we can break this up into three functions and show that each one satisfies the equation.

$$u(x,t) = \frac{1}{2}\Big(\underbrace{\tilde{\Phi}(x+ct)}_{\mathbf{I}} + \underbrace{\tilde{\Phi}(x-ct)}_{\mathbf{II}}\Big) + \underbrace{\frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{V}(r)\,\mathrm{d}r}_{\mathbf{III}}$$

For part **I**, let $v(x,t) = \tilde{\Phi}(x+ct)$. Then

$$\begin{aligned} \partial_t v &= c\,\tilde{\Phi}'(x+ct) & \partial_x v &= \tilde{\Phi}'(x+ct) \\ \partial_{tt} v &= c^2\tilde{\Phi}''(x+ct) & \partial_{xx} v &= \tilde{\Phi}''(x+ct) \end{aligned}$$

And of course by observation, $\partial_{tt}v = c^2\partial_{xx}v$, so v satisfies the wave equation.

Part **II** is shown in the same way. Let $v(x,t) = \tilde{\Phi}(x-ct)$. Then

$$\begin{aligned} \partial_t v &= -c\,\tilde{\Phi}'(x+ct) & \partial_x v &= \tilde{\Phi}'(x+ct) \\ \partial_{tt} v &= c^2\tilde{\Phi}''(x+ct) & \partial_{xx} v &= \tilde{\Phi}''(x+ct) \end{aligned}$$

And so v satisfies the wave equation in this case also.

For part **III**, let $v(x,t) = \displaystyle\int_{x-ct}^{x+ct} \tilde{V}(r)\ \mathrm{d}r$. Then

$$\begin{aligned}
\partial_t v(x,t) &= \frac{1}{2}\left(c\,\tilde{\Phi}'(x+ct) - c\,\tilde{\Phi}'(x-ct)\right) + \partial_t \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{V}(r)\ \mathrm{d}r \\
&= \frac{1}{2}\left(c\,\tilde{\Phi}'(x+ct) - c\,\tilde{\Phi}'(x-ct)\right) + \frac{1}{2c}\left(c\tilde{V}(x+ct) + c\tilde{V}(x-ct)\right) \\
&= \frac{1}{2}\left(c\,\tilde{\Phi}'(x+ct) - c\,\tilde{\Phi}'(x-ct) + \tilde{V}(x+ct) + \tilde{V}(x-ct)\right) \\
\partial_{tt} v(x,t) &= \frac{1}{2}\left(c^2\,\tilde{\Phi}''(x+ct) - c^2\,\tilde{\Phi}''(x-ct) + c\tilde{V}'(x+ct) + c\tilde{V}'(x-ct)\right) \\
\partial_x v(x,t) &= \frac{1}{2}\left(\tilde{\Phi}'(x+ct) - \tilde{\Phi}'(x-ct)\right) + \partial_x \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{V}(r)\ \mathrm{d}r \\
&= \frac{1}{2}\left(\tilde{\Phi}'(x+ct) - \tilde{\Phi}'(x-ct)\right) + \frac{1}{2c}\left(\tilde{V}(x+ct) - \tilde{V}(x-ct)\right) \\
&= \frac{1}{2}\left(\tilde{\Phi}'(x+ct) - \tilde{\Phi}'(x-ct) + \frac{1}{c}\tilde{V}(x+ct) - \frac{1}{c}\tilde{V}(x-ct)\right) \\
\partial_{xx} v(x,t) &= \frac{1}{2}\left(\tilde{\Phi}''(x+ct) - \tilde{\Phi}''(x-ct) + \frac{1}{c}\tilde{V}'(x+ct) - \frac{1}{c}\tilde{V}'(x-ct)\right)
\end{aligned}$$

We can see that $\partial_{tt} v = c^2 \partial_{xx} v$, and so v satisfies the wave equation. Since **I**, **II**, and **III** all satisfy the wave equation, and the wave operator is linear, d'Alembert's formula with $\tilde{\Phi}$ and $\tilde{V}$ satisfies the wave equation.

Now all that remains is to check the initial conditions. We first want $u(x,0) = \tilde{\Phi}(x)$. D'Alembert's formula becomes

$$\begin{aligned}
u(x,0) &= \frac{1}{2}\left(\tilde{\Phi}(x) + \tilde{\Phi}(x)\right) + \frac{1}{2c}\int_{x}^{x} \tilde{V}(r)\ \mathrm{d}r \\
&= \frac{1}{2}\left(2\tilde{\Phi}(x)\right) \\
&= \tilde{\Phi}(x)
\end{aligned}$$

We also want $\partial_t u(x,0) = \tilde{V}(x)$. D'Alembert's formula becomes

$$\begin{aligned}
\partial_t u(x,t) &= \frac{1}{2}\left(c\,\tilde{\Phi}'(x+ct) - c\,\tilde{\Phi}'(x-ct)\right) + \partial_t \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{V}(r)\ \mathrm{d}r \\
&= \frac{1}{2}\left(c\,\tilde{\Phi}'(x+ct) - c\,\tilde{\Phi}'(x-ct)\right) + \frac{1}{2c}\left(c\tilde{V}(x+ct) + c\tilde{V}(x-ct)\right) \\
&= \frac{1}{2}\left(c\,\tilde{\Phi}'(x+ct) - c\,\tilde{\Phi}'(x-ct) + \tilde{V}(x+ct) + \tilde{V}(x-ct)\right) \\
\partial_t u(x,0) &= \frac{1}{2}\left(c\,\tilde{\Phi}'(x) - c\,\tilde{\Phi}'(x) + \tilde{V}(x) + \tilde{V}(x)\right) \\
&= \tilde{V}(x)
\end{aligned}$$

And finally, we want $u(0,t) = 0$.

$$
\begin{aligned}
u(0,t) &= \frac{1}{2}\left(\tilde{\Phi}(ct) + \tilde{\Phi}(-ct)\right) + \frac{1}{2c}\int_{-ct}^{ct} \tilde{V}(r)\ \mathrm{d}r \\
&= \frac{1}{2}\left(\tilde{\Phi}(ct) - \tilde{\Phi}(ct)\right) + \frac{1}{2c}\int_{-ct}^{ct} \tilde{V}(r)\ \mathrm{d}r \\
&= \frac{1}{2} \cdot 0 + \frac{1}{2c} \cdot 0 \\
&= 0
\end{aligned}
$$

To justify the integral becoming 0, we observe that $\tilde{V}$ is an odd function by definition, so integrating it from $-ct$ to ct will always be 0.

And thus, we have a solution to (1).

$\square$