MA146 Methods of Mathematical Modelling 1, Assignment 3

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Question 1

> Objects moving in air experience a drag. Let us specifically consider a ball of radius r that moves with velocity v. We assume that its drag d also depends on the density ρ of the air and its dynamic viscosity μ, and write the problem in the form
> $$d = u(r, v, \rho, \mu)$$
> with some function u that we want to learn more about.
>
> Perform a dimensional analysis using the power of μ to express the solution to the emerging system of linear equations.
>
> Hint: the viscosity has dimension $[\mu] = ML^{-1}T^{-1}$, and the drag (force) $[d] = MLT^{-2}$

Let $a, b, c, d \in \mathbb{Z}$. Then

$$[d] = \left[r^a v^b \rho^c \mu^d\right] = L^a L^b T^{-2b} M^c L^{-3c} M^d L^{-d} T^{-d} = MLT^{-2}$$

Then we get the simultaneous equations

$$\begin{aligned} c + d &= 1 \\ a + b - 3c - d &= 1 \\ -2b - d &= -2 \end{aligned}$$

We don't have enough information to solve the system from here, but we can simplify to get

$$\begin{aligned} a &= 3 - \frac{3}{2}d \\ b &= 1 - \frac{d}{2} \\ c &= 1 - d \end{aligned}$$

Since they're all integers, we know that d is even, so we can just try even values of d.

$d = 2$ gives

$$\begin{aligned} a &= 0 \\ b &= 0 \\ c &= -1 \\ d &= 2 \end{aligned}$$

But we want the radius and velocity to be part of the equation, so we don't want $a = b = 0$.

$d = 4$ gives

$$\begin{aligned} a &= -3 \\ b &= -1 \\ c &= -3 \\ d &= 4 \end{aligned}$$

Therefore $[d] = \left[r^{-3}v^{-1}\rho^{-3}\mu^{4}\right]$.

Question 2

> To model the freezing of a pond at very cold temperatures, assume that the thickness of the ice on it increases at a rate inversely proportional to its thickness (we here ignore the finite depth of the pond).

Q2 (a)

> Denoting the thickness of the ice by $x(t)$ as a function of time t, formulate the problem as a differential equation for x using a proportionality constant denoted by α.

$$\dot{x}(t) = \frac{\alpha}{x(t)}$$

Q2 (b)

> If the ice initially (at midnight) is 2mm thick and at 4am it is 3mm thick, how thick will it be at 9:36am?

Let midnight be time $t = 0$, t be in hours and $x(t)$ be in millimetres. Then $x(0) = 2$ and $x(4) = 3$.

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\alpha}{x} \\
\int x \mathrm{d}x &= \int \alpha \mathrm{d}t \\
\frac{x^2}{2} &= \alpha t + C \\
\therefore x(t) &= \sqrt{2\alpha t + C}
\end{aligned}$$

Then we can use the initial values.

$$\begin{aligned}
x(0) &= \sqrt{2\alpha \times 0 + C} \\
&= \sqrt{C} \\
&= 2 \\
\implies C &= 4
\end{aligned}$$

$$\begin{aligned}
x(2) &= \sqrt{2\alpha \times 4 + 4} \\
&= 3 \\
\implies 8\alpha &= 3^2 - 4 \\
&= 5 \\
\implies \alpha &= \frac{5}{8} \\
\therefore x(t) &= \sqrt{\frac{5}{4}t + 4}
\end{aligned}$$

Now we can just plug in 9:36 am, which is 9.6 hours after midnight, and find that $x(9.6) = \sqrt{12 + 4} = 4$. Therefore the ice will be 4 mm thick at 9:36 am.

Q2 (c)

Assume that the proportionality factor α is replaced by a time dependent function of the form $\alpha(1 + cos(\omega t))$, which aims for modelling temperature changes during the day. Here, t is time measured in hours (denoted h) and $\omega = \frac{2\pi}{24h}$.

Assuming also an initial condition of the form $x(t_0) = x_0$, find the ice thickness as a function of time. (You may keep the parameter α, no need to replace it with the value from part **(b)**.)

Now we have the differential equation

$$\dot{x}(t) = \frac{\alpha\left(1 + \cos\left(\frac{\pi}{12}t\right)\right)}{x(t)}$$

We can solve this like before,

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\alpha\left(1 + \cos\left(\frac{\pi}{12}t\right)\right)}{x} \\
\int x\mathrm{d}x &= \int \alpha\left(1 + \cos\left(\frac{\pi}{12}t\right)\right)\mathrm{d}t \\
\frac{x^2}{2} &= \alpha t + \alpha\frac{12}{\pi}\sin\left(\frac{\pi}{12}t\right) + C \\
\therefore x(t) &= \sqrt{2\alpha t + \frac{24\alpha}{\pi}\sin\left(\frac{\pi}{12}t\right) + C}
\end{aligned}$$

Let $t_0 = 0$. Then $x(t_0) = x(0) = \sqrt{0 + \frac{24\alpha}{\pi}\sin 0 + C} = \sqrt{C} = x_0$, therefore $C = (x_0)^2$.

Therefore,

$$x(t) = \sqrt{2\alpha t + \frac{24\alpha}{\pi}\sin\left(\frac{\pi}{12}t\right) + (x_0)^2}$$

Question 3

A model for the vibrations of a wine glass is given by the differential equation

$$x''(t) + \lambda x'(t) + \omega^2 x(t) = f(t) \tag{1}$$

where x is some measure of the deformation, $\lambda, \omega > 0$ are given numbers and f is a given function called *(acoustic) forcing*. The equation is nondimensional. The glass shatters if $|x(t)| \geq 1$ at any time t.

Please get the Jupyter notebook `MA146_Assignment3.ipynb` for this question. It contains an example on how to solve initial value problems for second order equations with `sympy`.

Q3 (a)

Use the notebook to symbolically solve the initial value problem

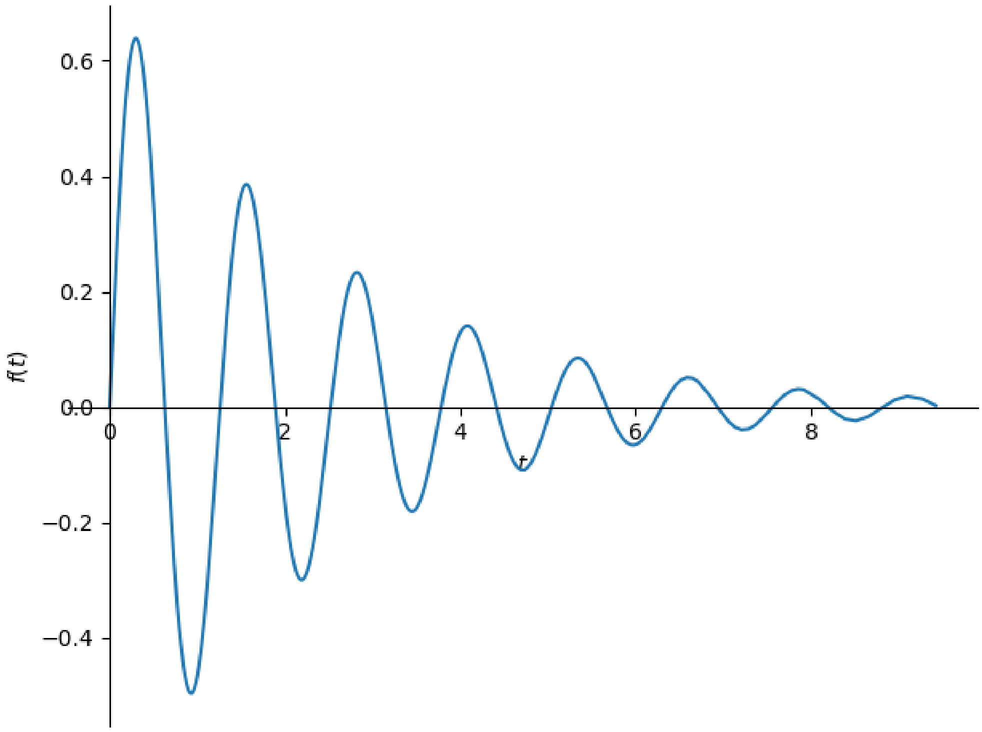
$$x''(t) + \lambda x'(t) + \omega^2 x(t) = 0, \quad x(0) = x_i, x'(0) = d_i$$

with $\lambda = 0.8$, $\omega = 5.0$, $x_i = 0$, and $d = 3.6$.

Provide your code, the algebraic expression for the solution produced by the software, and a plot on the interval $(0, 3\pi)$ for t of the solution.

```
x = Function("x")
t = Symbol('t')

lam = 0.8
omega = 5
ode = Derivative(x(t), t, t) + lam * Derivative(x(t), t) + (omega ** 2) * x(t);

xi = 0
di = 3.6
sol = dsolve(ode, ics={x(0): xi, x(t).diff(t).subs(t, 0): di})

print(sol)
plot(sol.rhs, (t, 0, 3 * pi))
```

```
Eq(x(t), 0.722315118514614*exp(-0.4*t)*sin(4.98397431775085*t))
```



Q3 (b)

Implement now a solver for (1) with

$$f(t) = a\cos(\alpha t)$$

and initial conditions

$$x(0) = 0, x'(0) = 0.$$

Assume that the parameters are $\lambda = 0.0009$, $\omega = 6.415$, $\alpha = 2\pi$.

Use your solver to computationally (for instance, by try and error) find the smallest number $n \in \mathbb{N}$ such that $a = n/10$ is sufficient to ensure that $x(t) \geq 1$ at some time t (i.e., the factor in the forcing is big enough to break the glass).

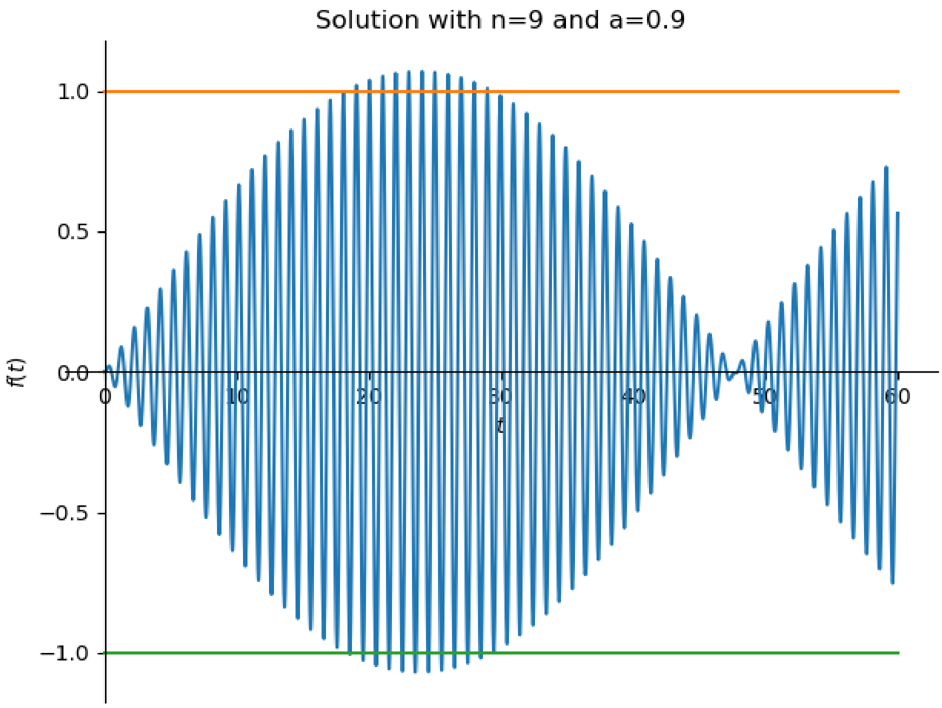
Provide your solution (n and a) and, for evidence, two graphs, one for the minimal n_{min} and one for $n_{\text{min}} - 1$.

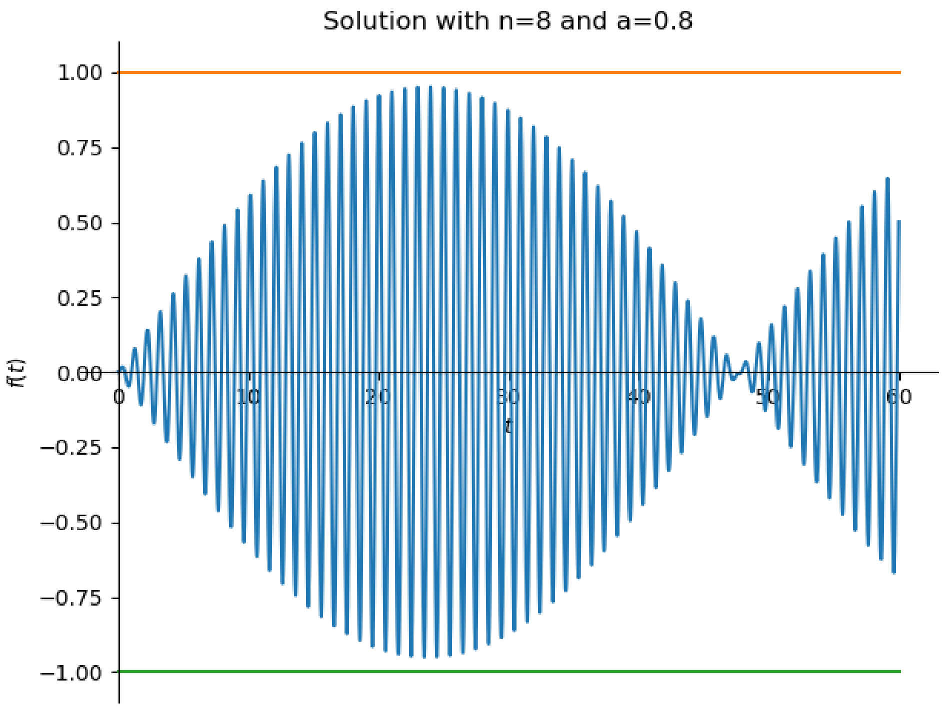
(Hints: Start with $n = 3$. You will need a sufficiently large domain for t to see what is going on, for instance, $t \in (0, 60)$.

You might observe some odd behaviour in the graph but which (hopefully) are visualisation effects only. They should vanish if you increase the number of points in the variable `nb_of_points` that is used in the plotting command.)

My solution is $n_{\text{min}} = 9$.

```
def solve_vibrations_ode(n: int):
    lam = 0.0009
    omega = 6.415
    a = n / 10
    alpha = 2 * pi
    ode = Derivative(x(t), t, t) + lam * Derivative(x(t), t) + (omega ** 2) * x(t) - a * cos(alpha * t);

    xi = 0
    di = 0
    sol = dsolve(ode, ics={x(0): xi, x(t).diff(t).subs(t, 0): di})

    return sol

n = 9
sol = solve_vibrations_ode(n)

from sympy.plotting import plot_parametric

p = plot(sol.rhs, (t, 0, 60), adaptive=False, nb_of_points=8000, show=False, title=f"Solution with n={n} and a={n/10}")

# Use parametric plots to add horizontal lines at y=1 and y=-1
# I'm sure there's a better way to do this, but this method works
p.extend(plot_parametric((t, 1), (t, 0, 60), show=False))
p.extend(plot_parametric((t, -1), (t, 0, 60), show=False))

p.show()
```


Solution with n=9 and a=0.9
1.0
0.5
0.0
−0.5
−1.0
f(t)
0
10
20
30
40
50
60
t


Solution with n=8 and a=0.8
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
f(t)
0
10
20
30
40
50
60
t

Question 4

Find a particular integral for the second order differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}(t) + \eta\frac{\mathrm{d}}{\mathrm{d}t}y(t) + by(t) = f\cos(\theta t)$$

with parameters $b, \eta, \theta, f > 0$.

Since the right hand side is $f\cos(\theta t)$, we will use $y(t) = A\cos(\theta t) + B\sin(\theta t)$ as our particular integral.

$$\begin{aligned} y(t) &= A\cos(\theta t) + B\sin(\theta t) \\ y'(t) &= -\theta A\sin(\theta t) + \theta B\cos(\theta t) \\ y''(t) &= -\theta^2 A\cos(\theta t) - \theta^2 B\sin(\theta t) \end{aligned}$$

Plugging this into the ODE, we get

$$-\theta^2 A\cos(\theta t) - \theta^2 B\sin(\theta t) - \eta\theta A\sin(\theta t) + \eta\theta B\cos(\theta t) + bA\cos(\theta t) + bB\sin(\theta t) = f\cos(\theta t)$$

We can compare coefficients of sin and cos and conclude that

$$\begin{aligned} -\theta^2 B - \eta\theta A + bB &= 0 \\ -\theta^2 A - \eta\theta B + bA &= f \end{aligned}$$

The first equation implies $B(b-\theta^2) = \eta\theta A$. We can use this to get B in terms of A, so $B = \frac{\eta\theta A}{b-\theta^2}$. Then we can plug that into the second equation, which gives

$$A\left(-\theta^2 + \frac{\eta^2\theta^2}{b-\theta^2} + b\right) = f$$

Therefore

$$A = \frac{f(b-\theta^2)}{(b-\theta^2)^2 + \eta^2\theta^2}$$

And therefore

$$B = \frac{\eta\theta f}{(b-\theta^2)^2 + \eta^2\theta^2}$$

Therefore the particular integral is

$$\frac{f(b-\theta^2)}{(b-\theta^2)^2 + \eta^2\theta^2}\cos(\theta t) + \frac{\eta\theta f}{(b-\theta^2)^2 + \eta^2\theta^2}\sin(\theta t)$$

Alternatively written as

$$\frac{f}{(b-\theta^2)^2 + \eta^2\theta^2}\left((b-\theta^2)\cos(\theta t) + \eta\theta\sin(\theta t)\right)$$