MA260 Norms Metrics and Topologies, Assignment 2

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Question 1

> Let (X, d) be a metric space and let $x \neq y$ be two elements of X. Show that $\exists \varepsilon_1, \varepsilon_2 > 0$ such that $\mathbb{B}(x, \varepsilon_1) \cap \mathbb{B}(y, \varepsilon_2) = \emptyset$.

Choose ε_1 and ε_2 such that $\varepsilon_1 + \varepsilon_2 < d(x, y)$ and consider a point $z \in \mathbb{B}(x, \varepsilon_1) \cap \mathbb{B}(y, \varepsilon_2)$. Then $d(x, z) < \varepsilon_1$ and $d(y, z) < \varepsilon_2$. By the triangle inequality, $d(x, y) \leq d(x, z) + d(y, z)$.

But $\varepsilon_1 + \varepsilon_2 < d(x, y)$ and $d(x, z) + d(y, z) < \varepsilon_1 + \varepsilon_2$. Therefore the triangle inequality would require $\varepsilon_1 + \varepsilon_2 < \varepsilon_1 + \varepsilon_2$. This is clearly impossible, and therefore no such z can exist, so $\mathbb{B}(x, \varepsilon_1) \cap \mathbb{B}(y, \varepsilon_2) = \emptyset$.

$\square$

Question 2

> Let (X, d) be a metric space and let $Y \subset X$. Show that U is open in Y if and only if $U = Y \cap V$, where V is open in X.

For U to be open in Y, we require that for all $x \in U$, $x \in Y$ and there exists $\varepsilon > 0$ such that $\mathbb{B}(x, \varepsilon) \subset Y$. The first condition implies that $U \subset Y$. If $V = \bigcup_{x \in U} \mathbb{B}(x, \varepsilon)$ then $U = Y \cap V$.

Conversely, suppose $U = Y \cap V$ for some open set V in X. Then for all $x \in U$, there exists $\varepsilon > 0$ such that $\mathbb{B}(x, \varepsilon) \subset U$, so U is open in Y.

$\square$

Question 3

In this exercise, we will consider $\mathbb{R}^2$ with two different metrics: the standard Euclidean metric d and the *sunflower metric* d_{sf} defined by

$$d_{\text{sf}}(x, y) = \begin{cases} \|x - y\| & \text{if } x \text{ and } y \text{ lie on the same line through the origin,} \\ \|x\| + \|y\| & \text{otherwise.} \end{cases}$$

(i) Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^2$. Show that if (x_n) converges to $x \in \mathbb{R}^2$ with respect to the sunflower metric then (x_n) converges to x with respect to the standard metric.

(ii) By giving an example, show that it is possible for a sequence (x_n) to converge to $x \in \mathbb{R}^2$ with respect to the standard metric but not to converge to x with respect to the sunflower metric.

(iii) Show that any sequence in $\mathbb{R}^2$ with the property described in part **(ii)** does not converge to any limit with respect to the sunflower metric.

Q3 (i)

For (x_n) to converge to x in the sunflower metric, we need $d_{\text{sf}}(x_n, x) \to 0$ as $n \to \infty$. That means either than $x = (0, 0)$ or that x_n converges along the line through x and the origin. In the second case, there is some N such that for all $n > N$, x_n and x are on the same line through the origin.

In both of these cases, we also have $d_2(x_n, x) \to 0$ as $n \to \infty$, so (x_n) also converges to x in the Euclidean metric.

Q3 (ii)

Let $x_n = \left(1, \frac{1}{n}\right)$. In the Euclidean metric, $d_2\left(x_n, (1, 0)\right) \to 0$ as $n \to \infty$. But in the sunflower metric, $d_{\text{sf}}\left(x_n, (1, 0)\right) \to 2$ as $n \to \infty$, so (x_n) converges to $(1, 0)$ in the Euclidean metric but not in the sunflower metric.

Q3 (iii)

Suppose (x_n) converges to x in the Euclidean metric and y in the sunflower metric. That means that for all $\varepsilon > 0$ there exists N_1 such that for all $n > N_1$, $d_{\text{sf}}(x_n, y) < \varepsilon$.

Since $d_2(a, b) \leq d_{\text{sf}}(a, b)$ for any points $a, b \in \mathbb{R}^2$, there also exists N_2 such that for all $n > N_2$, $d_2(x_n, y) < \varepsilon$. Then we choose $N = \max\{N_1, N_2\}$ and see that x must equal y. Therefore (x_n) must converge to the same limit in both the Euclidean metric and the sunflower metric so if it does not converge to x in the sunflower metric, then it has no limit there.

Question 4

Let $\mathcal{T}$ be a topology on $\mathbb{R}$. Suppose that for every pair of real numbers a and b with $a < b$, we have $[a, b] \in \mathcal{T}$. Show that $\mathcal{T}$ must be the discrete topology.

To show that $\mathcal{T}$ is the discrete topology, we want to show that $\mathcal{T} = \mathcal{P}(\mathbb{R})$.

Since $\mathcal{T}$ is a topology, finite intersections of elements of $\mathcal{T}$ are also elements of $\mathcal{T}$. Suppose $a < b < c$, then $[a, b] \cap [b, c] = \{b\} \in \mathcal{T}$. Using this technique, we know that for any $x \in \mathbb{R}$, $\{x\} \in \mathcal{T}$.

Since arbitrary (possibly infinite) unions of elements of $\mathcal{T}$ are also elements of $\mathcal{T}$,

$$\bigcup_{a<x<b} \{x\} = (a, b) \in \mathcal{T}.$$

More generally, any $U \subset \mathcal{P}(\mathbb{R})$ can be constructed as a union of singletons. Since $\mathcal{T}$ contains all singletons, it contains all such U and therefore $\mathcal{T} = \mathcal{P}(\mathbb{R})$ as required.

$\square$