CS147 Discrete Maths and its Applications 2, Assignment 1

Dyson Dyson (ID: 5503449)

Question 1

For each of the following statements, check whether each of them is true or false and provide justifications for your answer.

Q1 (a)

$8^n = \mathcal{O}(7^n)$

This implies that $\exists\, c > 0, N > 0$ such that for all $n > N$,

$$\begin{aligned} 8^n &\leq c7^n \\ n \log 8 &\leq \log c + n \log 7 \\ n(\log 8 - \log 7) &\leq \log c \\ n \log \frac{8}{7} &\leq \log c \end{aligned}$$

$\log \frac{8}{7} > 1$, so whatever value of c we choose, $n \log \frac{8}{7}$ will eventually be larger than $\log c$. Therefore 8^n is not $\mathcal{O}(7^n)$.

Q1 (b)

$n2^{\frac{n}{2}} = \Omega(n2^n)$

This implies that $\exists\, c > 0, N > 0$ such that for all $n > N$,

$$\begin{aligned} n2^{\frac{n}{2}} &\geq cn2^n \\ 2^{\frac{n}{2}} &\geq c2^n \\ \log 2^{\frac{n}{2}} &\geq \log(c2^n) \\ \frac{n}{2} \log 2 &\geq \log c + n \log 2 \\ 0 &\geq \log c + \frac{n}{2} \log 2 \end{aligned}$$

Everything on the right hand side is > 0 and therefore not ≤ 0. Therefore $n2^{\frac{n}{2}}$ is not $\Omega(n2^n)$.

Q1 (c)

$\log(n!) = \mathcal{O}(n \log n)$

This implies that $\exists\, c > 0, N > 0$ such that for all $n > N$,

$$\begin{aligned}
\log(n!) &\leq cn \log n \\
\log(n!) &\leq \log(n^{cn}) \\
0 &\leq \log(n^{cn}) - \log(n!) \\
0 &\leq \log\left(\frac{n^{cn}}{n!}\right)
\end{aligned}$$

We know that $n^n > n!$ for large n, so we see that $\dfrac{(n^n)^c}{n!} > 1$, therefore the logarithm is greater than 0, so $\log(n!)$ is indeed $\mathcal{O}(n \log n)$.

Question 2

> Solve the following recurrences. In each case, $T(c) = \Theta(1)$ for any constant $c > 0$.

Q2 (a)

> $T(n) = 4T\left(\frac{n}{2}\right) + \Theta(\log n)$

We know that $\log n = \Omega(1)$ and $\log n = \mathcal{O}(n)$. So let $T_1(n) = 4T_1\left(\frac{n}{2}\right) + \Theta(1)$ and $T_2(n) = 4T_2\left(\frac{n}{2}\right) + \Theta(n)$ and note that $T_1(n) \leq T(n) \leq T_2(n)$ for large enough n.

We can apply the master theorem to T_1 with $a = 4$, $b = 2$, and $d = 0$. Then $\frac{a}{b^d} > 1$ so $T_1(n) = \Theta\big(n^{\log_2 4}\big) = \Theta(n^2)$.

We can also apply the master theorem to T_2 with $a = 4$, $b = 2$, and $d = 1$. Then $\frac{a}{b^d} > 1$ so $T_2(n) = \Theta\big(n^{\log_2 4}\big) = \Theta(n^2)$.

Therefore $\Theta(n^2) \leq T(n) \leq \Theta(n^2)$ so $T(n) = \Theta(n^2)$.

Q2 (b)

> $T(n) = 8T\left(\frac{n}{2}\right) + \Theta(5^n)$

We know that $5^n = \Omega(n^d)$ for all $d > 0$. We cannot place a polynomial upper bound on an exponential function. Let $T_1(n) = 8T_1\left(\frac{n}{2}\right) + \Theta(n^d)$ for some large d. Then $T_1(n) \leq T(n)$ for large enough n.

We can apply the master theorem to T_1 with $a = 8$, $b = 2$, and large d. Then $\frac{a}{b^d} < 1$ so $T_1(n) = \Theta(n^d)$. Therefore $T(n) \geq \Theta(n^d)$ for large d. Equivalently, $T(n) = \Omega(n^d)$ for large d.

Question 3

Let A, B, C be three events admitting

$$P(A \cap B \cap C) = 0.1, \quad P(A \cap B) = 0.3, \quad P(B \cap C) = 0.4, \quad P(C) = 0.5.$$

If $P(A) = 0.6$, find $P(A \mid B \cap C)$.

$$\begin{aligned}\mathbb{P}(A \mid B \cap C) &= \frac{\mathbb{P}(A \cap (B \cap C))}{\mathbb{P}(B \cap C)} \\ &= \frac{0.1}{0.4} \\ &= \frac{1}{4}\end{aligned}$$

Question 4

Suppose that X and Y are two discrete random variables uniformly distributed over $\{1, 2, \ldots, n\}$. Compute the probability of $X \leq Y$.

Let X and Y be discrete random variables distributed uniformly over $\{1, \ldots, n\}$. Either $X = Y$, $X > Y$, or $X < Y$, and $\mathbb{P}(X > Y \vee X < Y) = 1 - \mathbb{P}(X = Y)$. These are symmetric so $\mathbb{P}(X < Y) = \mathbb{P}(X > Y) = \frac{1}{2} - \frac{1}{2}\mathbb{P}(X = Y)$.

So $\mathbb{P}(X \leq Y) = \mathbb{P}(X < Y) + \mathbb{P}(X = Y) = \frac{1}{2} + \frac{1}{2}\mathbb{P}(X = Y)$ and $\mathbb{P}(X = Y) = \frac{1}{n}$. Therefore

$$\mathbb{P}(X \leq Y) = \frac{1}{2} + \frac{1}{2n} = \frac{n+1}{2n}$$