# MA268 Algebra 3, Assignment 1

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## Question 1

Let

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}, \qquad H = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbb{R} \right\}.$$

Note that $G$ is a subgroup of $\mathrm{GL}_3(\mathbb{R})$.

(i) Let

$$\phi : G \to \mathbb{R}^2, \qquad \phi \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = (x, y).$$

Show that $\phi$ is a homomorphism.

(ii) Show that $H$ is a normal subgroup of $G$.

(iii) Show that the only element of $G$ of finite order is $I_3$, the identity matrix. **Hint**: This is easier if you use $\phi$.

### Q1 (i)

Let

$$A = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \qquad X = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

be elements of $G$. Then $\phi(A) = (a, b)$, $\phi(X) = (x, y)$, and

$$\begin{aligned} \phi(AX) &= \phi\left( \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right) \\ &= \phi \begin{pmatrix} 1 & a+x & z+ay+c \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{pmatrix} \\ &= (a+x, b+y) \end{aligned}$$

And $\phi(A) + \phi(X) = (a + x, b + y)$, so $\phi(AB) = \phi(A)\phi(B)$ and therefore $\phi$ is a homomorphism.

## Q1 (ii)

For $H$ to be a normal subgroup, we need to have $gHg^{-1} = H$ for all $g \in G$, or equivalently, $ghg^{-1} = h$ for all $g \in G, h \in H$.

Let

$$g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in G, \qquad h = \begin{pmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in H$$

Then

$$\begin{aligned}
M_g &= \begin{pmatrix} \begin{vmatrix} 1 & y \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & y \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} x & z \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & z \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & x \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} x & z \\ 1 & y \end{vmatrix} & \begin{vmatrix} 1 & z \\ 0 & y \end{vmatrix} & \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ xy - z & y & 1 \end{pmatrix} \\
C_g &= \begin{pmatrix} 1 & 0 & 0 \\ -x & 1 & 0 \\ xy - z & -y & 1 \end{pmatrix} \\
C_g^T &= \begin{pmatrix} 1 & -x & xy - z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix} \\
\det g &= 1 \begin{vmatrix} 1 & y \\ 0 & 1 \end{vmatrix} + 0 \begin{vmatrix} x & z \\ 0 & 1 \end{vmatrix} + 0 \begin{vmatrix} x & z \\ 1 & y \end{vmatrix} \\
&= 1 \\
\therefore g^{-1} &= \frac{1}{\det g} C_g^T \\
&= \begin{pmatrix} 1 & -x & xy - z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

And then we get

$$
\begin{aligned}
ghg^{-1} &= \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -x & xy-z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & x & w+z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -x & xy-z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & x-x & xy-z-xy+w+z \\ 0 & 1 & y-y \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= h
\end{aligned}
$$

## Q1 (iii)

Let $g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in G$ and suppose $g$ has finite order $n > 0$, so $g^n = I_3$. This means that $\phi(g^n) = \phi(g)^n = \phi(I_3) = (0, 0)$. We can easily see that this requires $x$ and $y$ in $g$ to be 0, so $g$ has the form of an element of $H$.

Let $h = \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in H$ and suppose $h$ has finite order $m > 0$. Trivially, $h^m = \begin{pmatrix} 1 & 0 & mz \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, so to get $h^m = I_3$, we need $mz = 0$. Since $m > 0$, this means the only elements of $H$ that have finite order are those with $z = 0$. That is, the only element of finite order is $I_3$.

Now we return to $g$ and observe further that $z$ must be 0 in $g$. Therefore the only element of $G$ that has finite order is $I_3$.

$\square$

# Question 2

Let

$$V_4 = \{\mathrm{id}, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}.$$

You may assume $V_4$ is a normal subgroup of $S_4$.

(i) Explain why $V_4$ must be a normal subgroup of $A_4$.

(ii) Let $\sigma$ be a 3-cycle. Show that

$$A_4/V_4 = \langle \sigma V_4 \rangle.$$

(iii) Show that $S_4/V_4$ is a non-cyclic group of order 6.

## Q2 (i)

$A_4$ is a subgroup of $S_4$ and $V_4$ is a normal subgroup of $S_4$ which also fits the restrictions of $A_4$ (every element is even). Therefore $V_4$ is a subgroup of $A_4$.

To show that $V_4$ is normal in $A_4$, we can show that $vav^{-1} = a$ for all $v \in V_4$ and $a \in A_4$. This is clearly true for $v = \mathrm{id}$. We can imagine $v^{-1}$ as relabelling all the elements of whatever we're permuting. Then we apply $a$, and then $v$ does the relabelling in reverse, so doing $vav^{-1}$ has the effect of just doing $a$. Therefore $V_4$ is normal in $A_4$.

## Q2 (ii)

Any 3-cycle $\sigma$ has order 3 since $\sigma^3 = \mathrm{id}$, therefore $\langle \sigma V_4 \rangle$ has order 3. That means we should be able to divide $A_4$ into 3 classes, each of which can be mapped to a power of $\sigma$.

Note that every non-identity element of $V_4$ is two disjoint swaps. Every element of $A_4$ is either a product of two disjoint swaps (or the identity), or it is a product of two non-disjoint swaps with two disjoint swaps. The first class are the ones that map to $\sigma^0$, since they don't need to be changed, and the second group can be split into two halves which map to $\sigma$ and $\sigma^2$ respectively.

Therefore $A_4/V_4 = \langle \sigma V_4 \rangle$.

## Q2 (iii)

$S_4/V_4$ is, in a way, 2-cyclic. So we have the cyclic subgroup $A_4/V_4$ from part **(ii)**, and another cyclic almost-subgroup[1] formed of the odd permutations from $S_4$.

We know $\#(A_4/V_4) = \#\langle \sigma V_4 \rangle = 3$, and by a similar argument to that in part **(ii)**, the 'order' of the almost-subgroup is also 3, since it will contain 3 unique elements. Therefore $\#(S_4/V_4) = 6$ and it is not cyclic.

[1] It can't be a proper subgroup because it doesn't contain the identity element.