MA150 Algebra 2, Assignment 4

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Question 4

Let V be a Euclidean space with inner product. Suppose $w_1, \ldots, w_n$ is an orthonormal basis of V. Of course $\forall\, v \in V$, $\exists\, \lambda_i \in \mathbb{R}$ such that $v = \sum \lambda_i w_i$.

We want to show that $\lambda_i = \langle v, w_i \rangle$. For any fixed $i \leq n$

$$\begin{aligned}
v &= \sum_{j=1}^{n} \lambda_j w_j \\
\langle v, w_i \rangle &= \left\langle \sum_{j=1}^{n} \lambda_j w_j, w_i \right\rangle \\
&= \sum_{j=1}^{n} \lambda_j \langle w_j, w_i \rangle \\
&= \lambda_1 \langle w_1, w_i \rangle + \cdots + \lambda_i \langle w_i, w_i \rangle + \cdots + \lambda_n \langle w_n, w_i \rangle \\
&= \lambda_i
\end{aligned}$$

Since all $w_1, \ldots, w_n$ are orthonormal, so all but one of the inner products are zero.

Therefore $\langle v, w_i \rangle = \lambda_i$ for all $i = 1, \ldots, n$, as required.

Question 5

Let $V = \mathbb{R}^3$ and let

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Q5 (a)

We will apply Gram-Schmidt to v_1, v_2, v_3 to get an orthonormal basis w_1, w_2, w_3.

$$\begin{aligned}
w_1 &= \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\
u_2 &= v_2 - (v_2 \cdot w_1) w_1 \\
&= \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \\
w_2 &= \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \\
u_3 &= v_3 - (v_3 \cdot w_1) w_1 - (v_3 \cdot w_2) w_2 \\
&= \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \frac{2}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - 0 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \\
w_3 &= \frac{u_3}{\|u_3\|} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\end{aligned}$$

Therefore our orthonormal basis w_1, w_2, w_3 is $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Q5 (b)

Consider $v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Clearly $v = w_3$ so if $v = \lambda_1 w_1 + \lambda_2 w_2 + \lambda_3 w_3$ then, since w_1, w_2, w_3 are linearly independent, $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 1$.

Question 6

Let

$$A = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$$

And define

$$\varphi : \mathbb{R}^2 \to \mathbb{R}$$
$$v = \begin{pmatrix} x \\ y \end{pmatrix} \mapsto v^T A v = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Note that $\varphi(\underline{0}) = 0$ and $\varphi(\lambda v) = \lambda^2 \varphi(v)$ so we only need to consider $v \in \mathbb{R}^2$ of the form

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v_x = \begin{pmatrix} x \\ 1 \end{pmatrix} \;\forall\; x \in \mathbb{R}$$

Q6 (a)

$\varphi(e_1) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ \frac{b}{2} \end{pmatrix} = a$ so if $a \leq 0$ then $\varphi(e_1) \leq 0$.

Q6 (b)

$$\begin{aligned}
\varphi(v_x) &= \begin{pmatrix} x & 1 \end{pmatrix} \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} x & 1 \end{pmatrix} \begin{pmatrix} ax + \frac{b}{2} \\ \frac{b}{2}x + c \end{pmatrix} \\
&= \left(ax + \frac{b}{2} \right) x + \frac{b}{2} x + c \\
&= ax^2 + bx + c
\end{aligned}$$

Q6 (c)

Suppose $a > 0$. Now suppose $b^2 - 4ac < 0$. That means $ax^2 + bx + c = 0$ has no roots and since $a > 0$, $ax^2 + bx + c > 0$, so $\varphi(v_x) > 0 \;\forall\; x \in \mathbb{R}$.

Conversely, suppose $\varphi(v_x) > 0$. Therefore $ax^2 + bx + c > 0 \;\forall\; x \in \mathbb{R}$, which means this quadratic has no roots and therefore has a negative discriminant, so $b^2 - 4ac < 0$.

Question 7

Let $\langle v, w \rangle = v^T A w$ for any $v, w \in \mathbb{R}^2$.

Q7 (a)

The transpose of a scalar is the same scalar, so

$$\begin{aligned}\langle v, w \rangle &= \langle v, w \rangle^T \\ &= \left(v^T A w\right)^T \\ &= w^T A^T \left(v^T\right)^T \\ &= w^T A^T v \\ &= w^T A v \qquad \text{since } A^T = A \\ &= \langle w, v \rangle\end{aligned}$$

Q7 (b)

$$\begin{aligned}\langle \lambda_1 v_1 + \lambda_2 v_2, w \rangle &= (\lambda_1 v_1 + \lambda_2 v_2)^T A w \\ &= \left(\lambda_1 v_1^T + \lambda_2 v_2^T\right) A w \\ &= \lambda_1 v_1^T A w + \lambda_2 v_2^T A w \\ &= \lambda_1 \langle v_1, w \rangle + \lambda_2 \langle v_2, w \rangle\end{aligned}$$

Q7 (c)

Suppose $a > 0$ and $\det A > 0$, so $b^4 - 4ac < 0$. We know that $\langle v, v \rangle = v^T A v$ so $\langle v, v \rangle$ is equivalent to $\varphi(v)$ from Question 6, and we know that $\varphi(v) \geq 0 \; \forall \, v \in \mathbb{R}^2$ when $a > 0$ and $\det A > 0$, with equality only when $v = 0$.

Essentially, the desired result follows immediately from Question 6 and the observation that $\langle v, v \rangle = \varphi(v)$.

Q7 (d)

For a matrix A to determine an inner product as above, we need $a > 0$ and $\det A > 0$, so only $\begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ determine inner products