CS147 Discrete Maths and its Applications 2, Assignment 2

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Question 1

> Consider any matching $M \subseteq E$ in a graph $G = (V, E)$. Consider any subset of edges $Z \subseteq M$. Is the following statement true or false? Justify your answer.
>
> The set of edges Z must also form a matching in G.

A matching is just a set of edges in G which do not share any common endpoint. For Z to not be a matching, we would need to choose two edges from M which share a node. Since M is a matching, no such pair of edges exists by definition, so Z must also be a matching.

Question 2

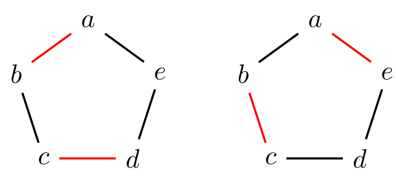
Consider any graph $G = (V, E)$ with $|V| = n$ nodes, where each node $v \in V$ is incident on exactly two edges in E. Is the following statement true or false? Justify your answer.

There must exist a matching $M \subseteq E$ in G with $|M| \geq \frac{n}{2}$.

The graph is isomorphic to an n-gon. For example, the case of $n = 4$ could be drawn as a square.

In the case of even n, there must exist two maximum matchings of size $\frac{n}{2}$, which are complements of each other in E, so $M_2 = E \setminus M_1$.

In the case of odd n, we still get two complementary matchings, but they are not large enough. Take the case of $n = 5$ for example,



Both of the subsets highlighted in red are matchings, but both are maximum and of size 2. It is clear that in the case of odd n, a maximum matching has size $\left\lfloor \frac{n}{2} \right\rfloor$.

Therefore it is false that there must exist a matching $M \subset E$ with $|M| \geq \frac{n}{2}$.

Question 3

Let $M \subseteq E$ be a *maximal* matching in a graph $G = (V, E)$, i.e., M is a matching, and furthermore, every edge $(u, v) \in E \setminus M$ has at least one endpoint that is matched under M. Let $M^* \subseteq E$ be a matching of *maximum* size in G. Is the following statement true or false? Justify your answer.

We must have $|M| \geq \frac{1}{2}|M^*|$.

Suppose we have a situation where $|M| < \frac{1}{2}|M^*|$. Let $|M| = \ell$ and $|M^*| = k$ so that M matches 2ℓ nodes and M^* matches $2k$ nodes. The inequality implies $\ell < \frac{1}{2}k \iff 2\ell < k$.

There are at most 2ℓ edges in M^* which are matched by M. But since $2\ell < k$, there is at least one edge in M^* which is not matched by M. Therefore we can add this edge to M, meaning it is not maximal. That's a contradiction, therefore $|M| < \frac{1}{2}|M^*|$.

Question 4

Consider a bipartite graph $G = (L \cup R, E)$ where each edge $e \in E$ has one endpoint in L and the other endpoint in R. For each set of nodes $X \subseteq L$, let $N_G(X) = \{v \in R\colon \text{there is an edge } (u, v) \in E \text{ for some } u \in X\}$ denote the set of neighbours of X in G.

The graph G has the property that $|N_G(A)| \geq \frac{1}{2}|A|/2$ for all $A \subseteq L$. Is the following statement true or false? Justify your answer.

There is a subset of edges $H \subseteq E$ in G which satisfy the three properties described below:

- $|H| = |L|$.
- Every node $u \in L$ is incident upon exactly one edge from H.
- Every node $v \in R$ is incident upon at most two edges from H.

To satisfy the first two properties, we require that H is constructed by considering each node in L and choosing one of the edges that connects to it.

Is it possible that there exists a $v \in R$ which is incident on three edges in H? Suppose such a v does exist. Then those three edges in H would connect to three distinct nodes in L, call them $S = \{u_1, u_2, u_3\}$. But by the neighbour requirement of G, we have $|N_G(S)| \geq \frac{1}{2}|S|$.

By construction, all three nodes connect to the same $v \in R$ and no other nodes, so $N_G(S) = 1$. Therefore we have $1 \geq \frac{3}{2}$, which is a contradiction.

Therefore we cannot have a $v \in R$ which is incident on three edges in H. Therefore every node in R is incident on at most two edges in H, so the statement is true.