MA150 Algebra 2, Assignment 3

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Question 6

$$W = (x + 2y - 3z = 0) \subset \mathbb{R}^3 \tag{1}$$

Q6 (a)

Show that $W \neq \mathbb{R}^3$, and explain why that implies that $\dim W < 3$.

The vector $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ is not in W since it doesn't satisfy the equation. In particular, $1(1) + 2(1) - 3(-1) = 6 \neq 0$. Therefore $W \neq \mathbb{R}^3$.

We know from lectures that the dimension of a subspace is less than or equal to the dimension of the parent space, and they have the same dimension if and only if they are equal. Since $W \subset \mathbb{R}^3$, $\dim W \leq \dim \mathbb{R}^3$. The dimension of $\mathbb{R}^3$ is 3 (since the standard basis of $\mathbb{R}^3$ has 3 elements). Therefore $\dim W \leq 3$. But $W \neq \mathbb{R}^3$, so $\dim W < 3$.

Q6 (b)

Find a basis of W and find $\dim W$.

We can rearrange equation (1) to get $x = 3z - 2y$. Then we can introduce parameters λ and μ and conclude that any point in W can be written as

$$\begin{pmatrix} 3\mu - 2\lambda \\ \lambda \\ \mu \end{pmatrix} = \lambda \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

Therefore $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis of W.

Call the elements of this basis $\{w_1, w_2\}$ for convenience. Plugging w_1 into equation (1) gives $1(-2) + 2(1) - 3(0) = 0$ as required, and plugging w_2 into equation (1) gives $1(3) + 2(0) - 3(1) = 0$ as required. Therefore $w_1, w_2 \in W$.

For w_1 and w_2 to be independent, we need to show that $\lambda w_1 + \mu w_2 = 0_W$ if and only if $\lambda = \mu = 0$. That linear independence equation expands to

$$\begin{pmatrix} 3\mu - 2\lambda \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The second components of the vectors imply $\lambda = 0$, and the third components imply $\mu = 0$. Therefore w_1 and w_2 are linearly independent.

w_1 and w_2 must span W since any linear combination is of the form $\begin{pmatrix} 3\mu - 2\lambda \\ \lambda \\ \mu \end{pmatrix}$ and we showed before that that is equivalent to equation (1), which is the definition of W.

Since we have a basis of W with 2 elements, we know that $\dim W = 2$.

Question 7

Let $V = \mathbb{R}[x]_{\leq 3}$ be the vector space of polynomials in x of degree at most 3, and let $W = \mathbb{R}^2$. Consider the linear map $\varphi : V \to W$ determined on the basis $1, x, x^2, x^3$ by

$$\varphi(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \varphi(x) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \varphi(x^2) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \varphi(x^3) = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

Q7 (a)

Compute $\varphi(2x^3 - 3x + 2)$.

$$\begin{aligned}
\varphi(2x^3 - 3x + 2) &= \varphi(2x^3) + \varphi(-3x) + \varphi(2) \\
&= 2\varphi(x^3) - 3\varphi(x) + 2\varphi(1) \\
&= 2\begin{pmatrix} -1 \\ 3 \end{pmatrix} - 3\begin{pmatrix} -1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -2 \\ 6 \end{pmatrix} + \begin{pmatrix} 3 \\ -3 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 3 \\ 3 \end{pmatrix}
\end{aligned}$$

Q7 (b)

Consider the linear map $\psi : V \to W$ where

$$\psi = \begin{pmatrix} f(-1) \\ \dfrac{\mathrm{d}f}{\mathrm{d}x}(-1) \end{pmatrix}$$

Show that $\psi = \varphi$.

By proposition 5.17, two linear maps are equal if their domains and codomains are equal and they agree on the elements of a basis of the domain. φ and ψ are both defined on $\varphi, \psi : \mathbb{R}[x]_{\leq 3} \to \mathbb{R}^2$. Then we just have to check that φ and ψ agree on some basis of the domain, and it makes sense to use $\{1, x, x^2, x^3\}$.

$$\psi(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \psi(x) = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$
$$\psi(x^2) = \begin{pmatrix} (-1)^2 \\ 2(-1) \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \psi(x^3) = \begin{pmatrix} (-1)^3 \\ 3(-1)^2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

Since ψ and φ agree on a basis, $\psi = \varphi$.

Q7 (c)

> Compute $\operatorname{Im} \varphi$.

To find the image of a linear transformation, we can write it as a matrix and take the column span of its row reduced echelon form. φ is L_M where

$$M = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \end{pmatrix}$$

Finding $\operatorname{RREF}(M)$ only takes one step, $A_{21}(1)$.

$$\operatorname{RREF}(M) = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 3 \end{pmatrix}$$

Then $\operatorname{Colspan}(\operatorname{RREF}(M)) = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$, so $\operatorname{Im} \varphi = \mathbb{R}^2$.

Q7 (d)

> Compute $\dim \ker \varphi$.

φ is defined on the domain $V = \mathbb{R}[x]_{\leq 3}$, which has dimension 4. Also $\operatorname{Im} \varphi = \mathbb{R}^2$, so $\dim \operatorname{Im} \varphi = 2$. Therefore by the Rank-Nullity Theorem,

$$\dim \ker \varphi = \dim V - \dim \operatorname{Im} \varphi = 4 - 2 = 2$$

Question 8

Let $V = \mathbb{R}[x]_{\leq 2}$ be the vector space of polynomials in x of degree at most 2.

Q8 (a)

For any fixed $a \in \mathbb{R}$, prove that $x \mapsto x + a$ is an isomorphism $\pi : V \to V$. That is, π is the linear map defined by $\pi(x^i) = (x+a)^i$ on the basis $1, x, x^2$ of V.

An isomorphism of vector spaces is just a bijective linear map. We shall first prove that π is a linear map.

We expect $\pi(\lambda x^i) = \lambda \pi(x^i)$.

$$\begin{aligned}
\pi(\lambda x^i) &= \pi\left(\left(\lambda^{\frac{1}{i}} x\right)^i\right) \\
&= \left(\lambda^{\frac{1}{i}} x + a\right)^i
\end{aligned}$$

Q8 (b)

Write the matrix of π with respect to the basis $1, x, x^2$ of V.

$$L_\pi = \begin{pmatrix} 1 & a & a^2 \\ 0 & 1 & 2a \\ 0 & 0 & 1 \end{pmatrix}$$

Question 9

Consider $V = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is differentiable}\}$ which is a (very large) vector space under the usual operations $\lambda f + \mu g$.

Q9 (a)

Let $W = \langle \cos(x), \cos(2x) \rangle$ which is a subspace of V. What is $\dim W$?

$\cos(x)$ and $\cos(2x)$ are linearly independent and span W by definition, so $\{\cos(x), \cos(2x)\}$ is a basis for W. The dimension of a vector space is equal to the number of vectors in a basis, so $\dim W = 2$.

Q9 (b)

Let $\mathcal{U} = \{f \in W : f(10) = 0\}$, which is a subspace of W. What is $\dim \mathcal{U}$?

We want functions of the form $\lambda \cos(x) + \mu \cos(2x)$ for some $\lambda, \mu \in \mathbb{R}$ where $\lambda \cos(10) + \mu \cos(20) = 0$. That means we need

$$\lambda = \frac{-\mu \cos(20)}{\cos(10)}$$

Therefore every element of $\mathcal{U}$ is of the form

$$\mu \left(\frac{-\cos(20)}{\cos(10)} \cos(x) + \cos(2x) \right)$$

and therefore $\left\{ \frac{-\cos(20)}{\cos(10)} \cos(x) + \cos(2x) \right\}$ is a basis of $\mathcal{U}$. Since this basis has 1 element, $\dim \mathcal{U} = 1$.

Q9 (c)

Let $\mathcal{U}_2 = \{f \in W : f(10) = 1\}$. Is $\mathcal{U}_2$ a subspace of W?

We want functions of the form $\lambda \cos(x) + \mu \cos(2x)$ for some $\lambda, \mu \in \mathbb{R}$ where $\lambda \cos(10) + \mu \cos(20) = 1$.

To be a subspace of W, $\mathcal{U}_2$ must be a non-empty subset (this is trivially true), and must be closed under the operations of W. So if we have some $\alpha \cos(x) + \beta \cos(2x) \in \mathcal{U}_2$, then we want $\lambda (\alpha \cos(x) + \beta \cos(2x)) \in \mathcal{U}_2$ for any $\lambda \in \mathbb{R}$.

But $\alpha \cos(10) + \beta \cos(20) = 1$ by definition of $\mathcal{U}_2$, and $\lambda (\alpha \cos(x) + \beta \cos(2x)) = \lambda \neq 1$. Therefore $\mathcal{U}_2$ is not closed under scalar multiplication and therefore is not a subspace of W.

Question 10

Consider $L_A : \mathbb{R}^4 \to \mathbb{R}^3$ (i.e. $\underline{v} \mapsto A\underline{v}$) for the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}$$

where the values for $a, b, c, d \in \mathbb{R}$ are not known.

Q10 (a)

State the Rank–Nullity Theorem.

Let $\varphi : V \to W$ be a linear map. Then $\dim \operatorname{Im} \varphi + \dim \ker \varphi = \dim V$.

Q10 (b)

Provide values for a, b, c, d so that $\dim \operatorname{Colspan} A = 1$. What are $\dim \operatorname{Im} L_A$ and $\dim \ker L_A$ in your example?

$a = b = c = d = 0$ gives $\dim \operatorname{Colspan} A = 1$. Then

$$\dim \operatorname{Im} L_A = \dim \operatorname{Colspan} A = 1$$

and then by the Rank-Nullity Theorem, $\dim \ker L_A = 4 - 1 = 3$.

Q10 (c)

Provide values for a, b, c, d so that $\dim \operatorname{Colspan} A = 2$. What are $\dim \operatorname{Im} L_A$ and $\dim \ker L_A$ in your example?

$a = 1$, $b = c = d = 0$ gives $\dim \operatorname{Colspan} A = 2$. Then $\dim \operatorname{Im} L_A = 2$ and $\dim \ker L_A = 2$.

Q10 (d)

Provide values for a, b, c, d so that $\dim \operatorname{Colspan} A = 3$. What are $\dim \operatorname{Im} L_A$ and $\dim \ker L_A$ in your example?

$a = 1$, $d = 1$, $b = c = 0$ gives $\dim \operatorname{Colspan} A = 3$. Then $\dim \operatorname{Im} L_A = 3$ and $\dim \ker L_A = 1$.