MA266 Multilinear Algebra, Assignment 1

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Question 10

Recall that a matrix $M \in M_n(\mathbb{F})$ is invertible if there exists $N \in M_n(\mathbb{F})$ such that $MN = NM = I_n$, where I_n is the $n \times n$ identity matrix.

(i) Show that matrix multiplication is associative. That is, for $L, M, N \in M_n(\mathbb{F})$, show that $(LM)N = L(MN)$. Deduce that if we set

$$\mathrm{GL}_n(\mathbb{F}) := \{M : M \in M_n(\mathbb{F}) \text{ is invertible}\},$$

then $(\mathrm{GL}_n(\mathbb{F}), \circ)$ is a group where $M \circ N := MN$.

(ii) Suppose that there exists $L, R \in M_n(\mathbb{F})$ with $LM = MR = I_n$. Prove that $L = R$.

(iii) Deduce from **(ii)** that if $M \in M_n(\mathbb{F})$, then M is invertible if and only if M^k is invertible for all $k \in \mathbb{N}$.

Q10 (i)

Let $L, M, N \in M_n(\mathbb{F})$. Then every entry of LMN is a sum of products, all of which are in $\mathbb{F}$. Since addition and multiplication in $\mathbb{F}$ are associative, matrix multiplication in $M_n(\mathbb{F})$ is associative.

The identity element of $\mathrm{GL}_n(\mathbb{F})$ is I_n, whose inverse is I_n. Every element in $\mathrm{GL}_n(\mathbb{F})$ has an inverse by definition. The product of two invertible matrices M and N is MN and it has inverse $N^{-1}M^{-1}$, so $\mathrm{GL}_n(\mathbb{F})$ is closed. We've already proven that matrix multiplication is associative, so $(\mathrm{GL}_n(\mathbb{F}), \circ)$ is a group.

Q10 (ii)

$$\begin{aligned} LM &= I_n \\ LMR &= I_n R \\ &= R \\ MR &= I_n \\ LMR &= LI_n \\ &= L \\ \therefore\ LMR &= R = L \end{aligned}$$

Q10 (iii)

Suppose M is invertible. Then for all $k \in \mathbb{N}$, M^k is invertible and has inverse M^{-k} by induction on k.

For the converse, suppose M^k is invertible for all $k \in \mathbb{N}$. Then we just choose $k = 1$ and get that $MM^{-1} = I_n$, so M is invertible.

Question 11

Let V be an $\mathbb{F}$-vector space and let X be a subset of V.

(i) Prove that $\mathbb{F}\langle X\rangle$ is a subspace of V.

(ii) Prove that $\mathbb{F}\langle X\rangle = \bigcap_{X \subseteq W \leq V} W$. That is, $\mathbb{F}\langle X\rangle$ is the intersection of all subspaces of V containing X.

Q11 (i)

Recall that $\mathbb{F}\langle X\rangle$ is a subspace if and only if $\forall w_1, w_2 \in \mathbb{F}\langle X\rangle$, $\lambda \in \mathbb{F}$, we have $w_1 - \lambda w_2 \in \mathbb{F}\langle X\rangle$. By definition, $\mathbb{F}\langle X\rangle$ contains all linear combinations of elements of X, and therefore all linear combinations of elements of itself. Therefore $\mathbb{F}\langle X\rangle$ is a subspace of V.

Q11 (ii)

Let $x \in \mathbb{F}\langle X\rangle$ and call the intersection I. Then W must contain x, since $X \subseteq W$ implies $\mathbb{F}\langle X\rangle \subseteq W$ because W is a subspace of V. Hence, x is in the intersection of all such W. Therefore $\mathbb{F}\langle X\rangle \subseteq I$.

Conversely, $\mathbb{F}\langle X\rangle$ is a subspace containing X, and so will be one of the W in the intersection. Since the taking repeated intersections can only shrink a set, $I \subseteq \mathbb{F}\langle X\rangle$. Therefore $\mathbb{F}\langle X\rangle = I$.

Question 12

Let r, s be positive integers and let $X \in M_r(\mathbb{F}), Z \in M_s(\mathbb{F})$. Also, let Y be an $r \times s$ matrix over $\mathbb{F}$, and consider the matrix

$$M := \begin{pmatrix} X & Y \\ 0_{s,r} & Z \end{pmatrix}.$$

Let $f \in \mathbb{F}[x]$. Show that there exists an $r \times s$ matrix Y_1 such that

$$f(M) = \begin{pmatrix} f(X) & Y_1 \\ 0_{s,r} & f(Z) \end{pmatrix}.$$

Hint: What does M^d look like for a positive integer d?

$$\begin{aligned}
M^2 &= \begin{pmatrix} X & Y \\ 0_{s,r} & Z \end{pmatrix} \begin{pmatrix} X & Y \\ 0_{s,r} & Z \end{pmatrix} \\
&= \begin{pmatrix} X^2 & XY + YZ \\ 0_{s,r} & Z^2 \end{pmatrix} \\
M^3 &= \begin{pmatrix} X^2 & XY + YZ \\ 0_{s,r} & Z^2 \end{pmatrix} \begin{pmatrix} X & Y \\ 0_{s,r} & Z \end{pmatrix} \\
&= \begin{pmatrix} X^3 & X^2Y + XYZ + YZ^2 \\ 0_{s,r} & Z^3 \end{pmatrix} \\
M^4 &= \begin{pmatrix} X^3 & X^2Y + XYZ + YZ^2 \\ 0_{s,r} & Z^3 \end{pmatrix} \begin{pmatrix} X & Y \\ 0_{s,r} & Z \end{pmatrix} \\
&= \begin{pmatrix} X^4 & X^3Y + X^2YZ + XYZ^2 + YZ^3 \\ 0_{s,r} & Z^4 \end{pmatrix}
\end{aligned}$$

We conclude that

$$M^d = \begin{pmatrix} X^d & \sum_{i=0}^{d-1} X^{d-1-i} Y Z^i \\ 0_{s,r} & Z^d \end{pmatrix},$$

which can be easily proven with induction. The base case of $d = 0$ is just M as given, and for the inductive step, assume the statement holds for d. Then

$$\begin{aligned}
M^{d+1} &= \begin{pmatrix} X^d & \sum_{i=0}^{d-1} X^{d-1-i} Y Z^i \\ 0_{s,r} & Z^d \end{pmatrix} \begin{pmatrix} X & Y \\ 0_{s,r} & Z \end{pmatrix} \\
&= \begin{pmatrix} X^{d+1} & X^d Y + Z \sum_{i=0}^{d-1} X^{d-1-i} Y Z^i \\ 0_{s,r} & Z^{d+1} \end{pmatrix} \\
&= \begin{pmatrix} X^{d+1} & \sum_{i=0}^{d} X^{d-1-i} Y Z^i \\ 0_{s,r} & Z^{d+1} \end{pmatrix}.
\end{aligned}$$

Suppose

$$f(x) = \alpha_n x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_1 x + \alpha_0.$$

Then

$$f(M) = \alpha_n M^n + \alpha_{n-1}M^{n-1} + \cdots + \alpha_1 M + \alpha_0 I.$$

For each term of f, the bottom left entry will always be $0_{s,r}$, so these sum to $0_{s,r}$ as desired. The top left entry of the dth term is $\alpha_d X^d$, so these sum to $f(X)$ as desired. Likewise with the bottom right entry being $f(Z)$.

The top right entry of the dth term is $\sum_{i=0}^{d-1} X^{d-1-i}YZ^i$. These sum to

$$\begin{aligned}\sum_{d=0}^{n}\sum_{i=0}^{d-1} X^{d-1-i}YZ^i &= Y\Big(1 + (X+Z) + (X^2 + XZ + Z^2) + \cdots \\ &\qquad + (X^{n-1} + X^{n-2}Z + \cdots + XZ^{n-2} + Z^{n-1})\Big) \\ &= Y\sum_{t=0}^{n-1}\sum_{\substack{a,b\geq 0 \\ a+b=t}} X^a Z^b.\end{aligned}$$

I'm not sure this is any nicer, but however you want to write it, this is Y_1.