MA243 Geometry, Assignment 3

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Question 1

(a) Define, by analogy with Euclidean geometry, the notions of spherical circle and spherical disc with centre $P \in S^2$ and radius ρ.

(b) Prove that a spherical circle of radius $\rho < \pi$ has (Euclidean) circumference $2\pi \sin \rho$.

(c) Prove that a spherical disc of radius $\rho < \pi$ has area $2\pi(1 - \cos \rho)$.

(d) Let $0 < \rho < \frac{\pi}{2}$ and let C be the spherical circle with centre $P \in S^2$ and radius ρ. Show that (C, d_{S^2}) is not isometric to $(S^1_r, d_{S^1_r})$ for any r.

(e) Show (e.g. using ideas from **(d)**) that there is no isometry from any region in S^2 to any region in $\mathbb{R}^2$.

Q1 (a)

A spherical circle with centre $P \in S^2$ and radius ρ is the set

$$\left\{ x \in S^2 : d_{S^2}(x, P) = \rho \right\}.$$

A (closed) spherical disc with centre $P \in S^2$ and radius ρ is the set

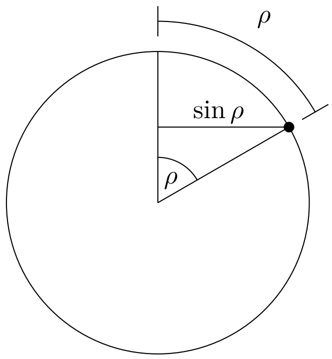
$$\left\{ x \in S^2 : d_{S^2}(x, P) \leq \rho \right\}.$$

And of course an open spherical disc is

$$\left\{ x \in S^2 : d_{S^2}(x, P) < \rho \right\}.$$

Q1 (b)

Consider a cross-section of S^2:



The spherical circle is the black dot rotated about the vertical line. The circumference of this circle is $2\pi r$ where $r = \sin\rho$ as seen in the diagram.

Q1 (c)

The area is of course just the integral of the circumferences of all smaller circles,

$$\begin{aligned}\int_0^\rho 2\pi \sin r \ \mathrm{d}r &= 2\pi \int_0^\rho \sin r \ \mathrm{d}r \\ &= 2\pi[-\cos r]_0^\rho \\ &= 2\pi\left(-\cos\rho - (-\cos 0)\right) \\ &= 2\pi\left(1 - \cos\rho\right)\end{aligned}$$

as required.

Q1 (d)

The shortest path between any two points in S_r^1 is an arc of the circle. But the shortest path between any two points in C is an arc going inside C. This is because the shortest path between two points in S^2 is a great circle, which will always pass through the interior of C since $\rho < \frac{\pi}{2}$. Because distances are measured in completely different ways in the two spaces, they cannot be isometric.

Q1 (e)

If there were such an isometry, then it would preserve the areas of spherical discs. Using the same logic as in part **(d)**, this is not possible.

Question 2

Suppose P and Q are two distinct points in the hyperbolic plane $\mathcal{H}^2$ and let $U := P + Q$ be their vector sum. Show that the mid-point of the hyperbolic line between P and Q is the point

$$R = \frac{U}{\sqrt{-U \cdot_L U}}$$

(where for a positive real number we take the positive square root).

In other words, you need to prove:

(a) $-U \cdot_L U$ is a positive real number.

(b) $R \in \mathcal{H}^2$.

(c) $d_{\mathcal{H}^2}(R, P) = d_{\mathcal{H}^2}(R, Q)$.

Let $P = (x_0, x_1, x_2)$ and $Q = (y_0, y_1, y_2)$. Therefore $U = (x_0 + y_0, x_1 + y_1, x_2 + y_2)$. Since $P, Q \in \mathcal{H}^2$, we know that $x_0 > 0$, $y_0 > 0$, and

$$P \cdot_L P = -x_0^2 + x_1^2 + x_2^2 = -1,$$
$$Q \cdot_L Q = -y_0^2 + y_1^2 + y_2^2 = -1.$$

Q2 (a)

$$\begin{aligned}
-U \cdot_L U &= -(P + Q) \cdot_L (P + Q) \\
&= (-P - Q) \cdot_L (P + Q) \\
&= -P \cdot_L P - P \cdot_L Q - Q \cdot_L P - Q \cdot_L Q \\
&= 1 - P \cdot_L Q - Q \cdot_L P + 1 \\
&= 2 + \cosh\left(\cosh^{-1}(-P \cdot_L Q)\right) + \cosh\left(\cosh^{-1}(-Q \cdot_L P)\right) \\
&= 2 + \cosh(d_{\mathcal{H}^2}(P, Q)) + \cosh(d_{\mathcal{H}^2}(Q, P))
\end{aligned}$$

Since distances are positive real numbers and $\cosh \geq 1$, $-U \cdot U \geq 4$.

Q2 (b)

To have $R \in \mathcal{H}^2$, we need $\|R\|_L = i$. That is, $\|R\|_L^2 = -1$. We have

$$\begin{aligned}
\|R\|_L^2 &= R \cdot_L R \\
&= \frac{1}{-U \cdot_L U} U \cdot_L U \\
&= \frac{U \cdot_L U}{-U \cdot_L U} \\
&= -1
\end{aligned}$$

Therefore $R \in \mathcal{H}^2$.

Q2 (c)

$d_{\mathcal{H}^2}(R, P) = d_{\mathcal{H}^2}(R, Q)$ means $\cosh^{-1}(-R \cdot_L P) = \cosh^{-1}(-R \cdot_L Q)$. Since $\cosh^{-1}$ is a strictly increasing function, this is equivalent to saying that $-R \cdot_L P = -R \cdot_L Q$.

$$\begin{aligned}
-R \cdot_L P &= -\frac{U}{\sqrt{-U \cdot_L U}} \cdot_L P \\
&= \frac{-U \cdot_L P}{\sqrt{-U \cdot_L U}} \\
&= \frac{x_0(x_0 + y_0) - x_1(x_1 + y_1) - x_2(x_2 + y_2)}{\sqrt{-U \cdot_L U}} \\
&= \frac{x_0^2 + x_0 y_0 - x_1^2 - x_1 y_1 - x_2^2 - x_2 y_2}{\sqrt{-U \cdot_L U}} \\
&= \frac{x_0^2 - x_1^2 - x_2^2}{\sqrt{-U \cdot_L U}} + \frac{x_0 y_0 - x_1 y_1 - x_2 y_2}{\sqrt{-U \cdot_L U}} \\
&= \frac{-P \cdot_L P}{\sqrt{-U \cdot_L U}} + \frac{-P \cdot_L Q}{\sqrt{-U \cdot_L U}} \\
&= \frac{1 - P \cdot_L Q}{\sqrt{-U \cdot_L U}}
\end{aligned}$$

$$
\begin{aligned}
-R \cdot_L Q &= -\frac{U}{\sqrt{-U \cdot_L U}} \cdot_L Q \\
&= \frac{-U \cdot_L Q}{\sqrt{-U \cdot_L U}} \\
&= \frac{y_0(x_0 + y_0) - y_1(x_1 + y_1) - y_2(x_2 + y_2)}{\sqrt{-U \cdot_L U}} \\
&= \frac{y_0^2 + x_0 y_0 - y_1^2 - x_1 y_1 - y_2^2 - x_2 y_2}{\sqrt{-U \cdot_L U}} \\
&= \frac{y_0^2 - y_1^2 - y_2^2}{\sqrt{-U \cdot_L U}} + \frac{x_0 y_0 - x_1 y_1 - x_2 y_2}{\sqrt{-U \cdot_L U}} \\
&= \frac{-Q \cdot_L Q}{\sqrt{-U \cdot_L U}} + \frac{-P \cdot_L Q}{\sqrt{-U \cdot_L U}} \\
&= \frac{1 - P \cdot_L Q}{\sqrt{-U \cdot_L U}}
\end{aligned}
$$

Therefore $-R \cdot_L P = -R \cdot_L Q$ and therefore $d_{\mathcal{H}^2}(R, P) = d_{\mathcal{H}^2}(R, Q)$.

Therefore R is the midpoint of P and Q.

$\square$

Question 3

A hyperbolic line $L \subset \mathcal{H}^2$ is the non-empty intersection $\mathcal{H}^2 \cap V$ with a plane V through the origin in $\mathbb{R}^3$. Prove that any hyperbolic line on $\mathcal{H}^2$ that passes through the point $(1, 0, 0)$ projects to a straight line in x_1x_2-plane. (Here, the projection is given by $(x_0, x_1, x_2) \mapsto (x_1, x_2)$).

Any such line contains the point $(1, 0, 0)$ and some other point (y_0, y_1, y_2). The plane V defining this line must therefore contain the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}.$$

Or we can instead use the orthogonal vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ y_1 \\ y_2 \end{pmatrix}.$$

Therefore any point on the hyperbolic line must be of the form

$$\lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ y_1 \\ y_2 \end{pmatrix}$$

for some $\lambda, \mu \in \mathbb{R}$, and so the projection of any point is $(\mu y_1, \mu y_2)$. The set of projected points is therefore a straight line in $\mathbb{R}^2$.

$\square$