

MA270 Analysis 3, Assignment 3

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Question 1

Let $n, k \geq 1$ be two integers. In the following, the vector space $L(\mathbb{R}^n, \mathbb{R}^k)$ of linear maps from \mathbb{R}^n to \mathbb{R}^k is endowed with the operator norm $\|\cdot\|_{\text{op}}$. Let (A_m) be a Cauchy sequence in the normed vector space $L(\mathbb{R}^n, \mathbb{R}^k)$.

- (a) Show that (A_m) is a bounded sequence, i.e. $\exists M > 0$ such that $\forall m \geq 1$, $\|A_m\|_{\text{op}} \leq M$.
- (b) Show that $\forall v \in \mathbb{R}^n$, the sequence $(A_m v)$ in \mathbb{R}^k converges.
- (c) Define a function $A : \mathbb{R}^n \rightarrow \mathbb{R}^k$ by setting $Av = \lim_{m \rightarrow \infty} A_m v$. Show that $A \in L(\mathbb{R}^n, \mathbb{R}^k)$.
- (d) Show that $\|A_m - A\|_{\text{op}} \rightarrow 0$ as $m \rightarrow \infty$.

Q1 (a)

By the definition of a Cauchy sequence, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n, m \geq N$, $\|A_m - A_n\|_{\text{op}} < \varepsilon$.

Take $\varepsilon = 1$ and $n = N$ in this definition. Then we get that $\exists N \in \mathbb{N}$ such that $\forall m \geq N$, $\|A_m - A_N\|_{\text{op}} < 1$.

Since the operator norm satisfies the triangle inequality, we have that for $m \geq N$,

$$\begin{aligned}\|A_m\|_{\text{op}} &\leq \|A_m - A_N\|_{\text{op}} + \|A_N\|_{\text{op}} \\ &< 1 + \|A_N\|_{\text{op}}.\end{aligned}$$

Therefore $\forall m \in \mathbb{N}$, $\|A_m\|_{\text{op}} \leq \max\{\|A_1\|_{\text{op}}, \dots, \|A_{N-1}\|_{\text{op}}, \|A_N\|_{\text{op}} + 1\}$, so (A_m) is bounded.

□

Q1 (b)

Consider an arbitrary $v \in \mathbb{R}^n$. Then $(A_m v)_m$ is a Cauchy sequence because $\forall m, n \in \mathbb{N}$,

$$\|A_m v - A_n v\| = \|(A_m - A_n)v\| \leq \|A_m - A_n\|_{\text{op}} \|v\|.$$

Of course $A_m v \in \mathbb{R}^k$ and we know that \mathbb{R}^k is complete for any k . That means that any Cauchy sequence in \mathbb{R}^k converges to an element of \mathbb{R}^k . Since $(A_m v)_m$ is a Cauchy sequence in \mathbb{R}^k , it converges.

□

Q1 (c)

Clearly A must be a map $\mathbb{R}^n \rightarrow \mathbb{R}^k$, since no other domain and codomain would make sense for the limit, so we just have to show that it's linear. Since all A_m are linear,

$$\begin{aligned} A(\lambda v + u) &= \lim_{m \rightarrow \infty} A_m(\lambda v + u) \\ &= \lim_{m \rightarrow \infty} (\lambda A_m v + A_m u) \\ &= \lim_{m \rightarrow \infty} \lambda A_m v + \lim_{m \rightarrow \infty} A_m u \\ &= \lambda \lim_{m \rightarrow \infty} A_m v + \lim_{m \rightarrow \infty} A_m u \\ &= \lambda A v + A u \end{aligned}$$

Therefore $A \in L(\mathbb{R}^n, \mathbb{R}^k)$.

□

Q1 (d)

Suppose $\|A_m - A\|_{\text{op}} \not\rightarrow 0$ as $m \rightarrow \infty$. That means that $\exists \varepsilon_0 > 0$ such that $\forall m, \|A_m - A\|_{\text{op}} \geq \varepsilon_0$. Using the equivalent definition of the operator norm as $\|A\|_{\text{op}} := \sup_{\|x\|=1} \|Ax\|$, we can deduce that there exists a sequence m_k in \mathbb{N} and a sequence x_k of elements of the unit sphere $\{x \in \mathbb{R}^n : \|x\| = 1\}$ such that $\forall k, \|(A_{m_k} - A)x_k\| \geq \varepsilon_0$. Let x be the limit of (x_k) .

By part (a), we know that (A_m) is bounded, so $\exists M > 0$ such that $\|A_m - A\|_{\text{op}} \leq M$. Then

$$\begin{aligned} \varepsilon_0 &\leq \|(A_{m_k} - A)x_k\| \\ &\leq \|(A_{m_k} - A)(x_k - x)\| + \|(A_{m_k} - A)x\| \\ &\leq M\|x - x_k\| + \|(A_{m_k} - A)x\|. \end{aligned}$$

As $k \rightarrow \infty$, $\|x - x_k\| \rightarrow 0$ and $A_{m_k} - A \rightarrow 0$. Therefore $\exists \varepsilon > 0$ such that $\varepsilon_0 \leq \varepsilon$. But ε_0 is chosen first, so we can always choose a k that makes $\varepsilon < \varepsilon_0$, which is a contradiction. Therefore $\|A_m - A\|_{\text{op}} \rightarrow 0$ as $m \rightarrow \infty$.

□

Question 2

Let $K \subset \mathbb{R}^n$ be a sequentially compact subset and $f : K \rightarrow K$ a continuous function such that $\|f(x) - f(y)\| < \|x - y\|$ for every $x, y \in K$ such that $x \neq y$.

- (a) Show that the function $K \rightarrow \mathbb{R}$, defined by $x \mapsto \|f(x) - x\|$ attains a minimum in K (i.e. $\exists x_* \in K$ such that $\|f(x_*) - x_*\| \leq \|f(x) - x\|$ for every $x \in K$).
- (b) Show that f admits a unique fixed point in K (i.e. a point $y_0 \in K$ such that $f(y_0) = y_0$).

Q2 (a)

Since K is sequentially compact, it is bounded. Since f is continuous, the function $g : x \mapsto \|f(x) - x\|$ is continuous. By the Extreme Value Theorem, g is bounded and attains its bounds, so $\exists x_* \in K$ such that $g(x_*) = \inf_{x \in K} g(x)$. Therefore $\forall x \in K$, $\|f(x_*) - x_*\| \leq \|f(x) - x\|$.

□

Q2 (b)

Clearly y_0 minimises g . We have to prove that $g(y_0) = 0$ and y_0 is unique. Unfortunately I don't know how to do this.

Question 3

Let A and B be two subsets of \mathbb{R}^n such that A is closed and B is sequentially compact.

- (a) Show that the set $A + B := \{a + b : a \in A \text{ and } b \in B\}$ is closed.
- (b) Find an example of $n \geq 1$, closed sets A and B in \mathbb{R}^n such that $A + B$ is not closed.

Q3 (a)

For $A + B$ to be closed, we need any sequence (x_n) in $A + B$ to converge to some element $x \in A + B$.

The sequence (x_n) must be the sum of two sequences (a_n) in A and (b_n) in B . Since B is sequentially compact, there exists a convergent subsequence $b_{n_k} \rightarrow b \in B$. Note that since A is only known to be closed, we can't assume that (a_n) converges.

Since $b_{n_k} \rightarrow b$, we have $x_{n_k} \rightarrow x$, which means a_{n_k} converges to $x - b$. Since A is closed, this limit $a = x - b \in A$. Therefore $x = a + b \in A + B$, so $A + B$ is closed.

□

Q3 (b)

Let $A = \mathbb{N}$ and $B = \left\{ \frac{1}{n+1} - n : n \in \mathbb{N} \right\}$. They are both closed but $A + B$ contains the sequence $\left(\frac{1}{n+1} \right)_{n \in \mathbb{N}}$, whose limit is zero, but $0 \notin A + B$.

Question 4

Let $n \geq 1$ be an integer and $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ be a linear map. Suppose that $\|A\|_{\text{op}} < 1$.

- (a) Show that the sequence $\left(\sum_{k=0}^m A^k\right)_m$ converges in the normed vector space $(L(\mathbb{R}^n, \mathbb{R}^n), \|\cdot\|_{\text{op}})$ (i.e. $\exists B \in L(\mathbb{R}^n, \mathbb{R}^n)$ such that $\left\|\sum_{k=0}^m A^k - B\right\|_{\text{op}} \rightarrow 0$ as $m \rightarrow \infty$).
- (b) Show that if (C_m) is a sequence in $L(\mathbb{R}^n, \mathbb{R}^n)$ converging to $D \in L(\mathbb{R}^n, \mathbb{R}^n)$, then $\forall E \in L(\mathbb{R}^n, \mathbb{R}^n)$, we have $\|C_m E - D E\|_{\text{op}} \rightarrow 0$ as $m \rightarrow \infty$.
- (c) Let $\text{Id} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ denote the identity linear operator ($x \mapsto x$) and let $B \in L(\mathbb{R}^n, \mathbb{R}^n)$ denote the limit of the sequence $\left(\sum_{k=0}^m A^k\right)_m$. Show that $(\text{Id} - A)B = \text{Id}$.

Q4 (a)

Since the operator norm is a norm, $\|A^{n+1}\|_{\text{op}} = \|A\|_{\text{op}}\|A^n\|_{\text{op}}$ for any $n \in \mathbb{N}$. Therefore $\|A^n\|_{\text{op}} = \|A\|_{\text{op}}^n$. We know that $\|A\|_{\text{op}} < 1$, so $\|A^n\|_{\text{op}} \rightarrow 0$ as $n \rightarrow \infty$. We define B as $\sum_{k=0}^{\infty} A^k$ so that

$$\begin{aligned} \left\|\sum_{k=0}^m A^k - B\right\|_{\text{op}} &= \left\|\sum_{k=0}^m A^k - \sum_{k=0}^{\infty} A^k\right\|_{\text{op}} \\ &= \left\|-\sum_{k=m+1}^{\infty} A^k\right\|_{\text{op}} \\ &= |-1| \left\|\sum_{k=m+1}^{\infty} A^k\right\|_{\text{op}} \\ &= \left\|\sum_{k=m+1}^{\infty} A^k\right\|_{\text{op}} \end{aligned}$$

Then by the triangle inequality,

$$\begin{aligned}
 &\leq \|A^{m+1}\|_{\text{op}} + \left\| \sum_{k=m+2}^{\infty} A^k \right\|_{\text{op}} \\
 &\leq \|A^{m+1}\|_{\text{op}} + \|A^{m+2}\|_{\text{op}} + \left\| \sum_{k=m+3}^{\infty} A^k \right\|_{\text{op}} \\
 &\leq \sum_{k=m+1}^{\infty} \|A^k\|_{\text{op}}
 \end{aligned}$$

And we know that $\|A^k\|_{\text{op}} \rightarrow 0$ as $k \rightarrow \infty$, so the sum above goes to 0 as $m \rightarrow \infty$. Therefore $\left\| \sum_{k=0}^m A^k - B \right\|_{\text{op}} \rightarrow 0$ as $m \rightarrow \infty$.

□

Q4 (b)

Since $C_m \rightarrow D$, we have $\|C_m - D\|_{\text{op}} \rightarrow 0$ by definition. Since $L(\mathbb{R}^n, \mathbb{R}^n)$ can be represented by matrices, these linear maps are distributive, so $C_m E - DE = (C_m - D)E$. Therefore we have

$$\begin{aligned}
 \|C_m E - DE\|_{\text{op}} &= \|(C_m - D)E\|_{\text{op}} \\
 &= \|C_m - D\|_{\text{op}} \|E\|_{\text{op}} \\
 &= 0 \|E\|_{\text{op}} \quad \text{as } m \rightarrow \infty \\
 &= 0
 \end{aligned}$$

Q4 (c)

$$\begin{aligned}
 (\text{Id} - A)B &= (\text{Id} - A) \sum_{k=0}^{\infty} A^k \\
 &= \sum_{k=0}^{\infty} A^k - A \sum_{k=0}^{\infty} A^k \\
 &= \sum_{k=0}^{\infty} A^k - \sum_{k=1}^{\infty} A^k \\
 &= A^0 \\
 &= \text{Id}
 \end{aligned}$$