MA150 Algebra 2, Assignment 2

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Question 5

Let $P = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $Q = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} \in \mathbb{R}^3$ and let L be the (unique, infinite) line that passes through them both.

Describe L in two ways: first, parametrically as

$$L = \{\underline{v} + \lambda \underline{w} \colon \lambda \in \mathbb{R}\}$$

where $\underline{v}$ is a point of L and $\underline{w}$ is a vector parallel to L, and second implicitly by two equations of the form $ax + by + cz = d$ (for suitable values $a, b, c, d \in \mathbb{R}$).

We can parametrise L as $\overrightarrow{P} + \lambda\overrightarrow{PQ}$ and $\overrightarrow{PQ} = \overrightarrow{Q} - \overrightarrow{P} = \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix}$ so L can be parametrised as

$$L = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} : \lambda \in \mathbb{R} \right\}$$

To describe L in terms of two implicit equations, we need two planes. We will start with the plane Π_1 through P, Q, and the origin. The vectors $\overrightarrow{P}$ and $\overrightarrow{Q}$ will both be in Π_1 so a normal vector is

$$\underline{n} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ -11 \\ 5 \end{pmatrix}$$

Therefore any point on Π_1 will satisfy

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 7 \\ -11 \\ 5 \end{pmatrix} = 0$$

So we get the equation $7x - 11y + 5z = 0$.

Note that the point $R = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ does not satisfy this equation and thus is not on Π_1. So we can define Π_2 as the plane containing P, Q, and R.

Two vectors in Π_2 are $\overrightarrow{RP} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$ and $\overrightarrow{RQ} = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix}$. Then we can find a normal vector

$$\underline{n} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ -9 \\ 6 \end{pmatrix}$$

R is a point on Π_2 and $\overrightarrow{R} \cdot \underline{n} = 7$. Therefore any point on Π_2 will satisfy

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 7 \\ -9 \\ 6 \end{pmatrix} = 7$$

So we get the equation $7x - 9y + 6z = 7$.

Therefore the line L can be described by the pair of equations

$$\begin{aligned} 7x - 11y + 5z &= 0 \\ 7x - 9y - 6z &= 7 \end{aligned}$$

We can check and indeed, P and Q both satisfy both of these equations.

Question 6

Compute the RREF of

$$A = \begin{pmatrix} 1 & -3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ -2 & 6 & 1 & 0 \end{pmatrix}$$

$$A_{13}(2) \implies \begin{pmatrix} 1 & -3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$A_{23}(-1) \implies \begin{pmatrix} 1 & -3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Question 7

For each of the following matrices A, determine whether A in invertible, and if it is, compute A^{-1}.

Q7 (a)

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Only square matrices can be invertible, so A is not invertible.

Q7 (b)

$$A = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$$

$$\begin{aligned}
(A \,|\, I) &= \left(\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 6 & 8 & 0 & 1 \end{array}\right) \\
M_1\left(\frac{1}{2}\right) \quad &\Longrightarrow \left(\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 6 & 8 & 0 & 1 \end{array}\right) \\
A_{12}\left(-6\right) \quad &\Longrightarrow \left(\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & -4 & -3 & 1 \end{array}\right) \\
M_2\left(-\frac{1}{4}\right) \quad &\Longrightarrow \left(\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{4} & -\frac{1}{4} \end{array}\right) \\
A_{21}\left(-2\right) \quad &\Longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & -1 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} & -\frac{1}{4} \end{array}\right)
\end{aligned}$$

Therefore $A^{-1} = A_{21}(-2)\, M_2\left(-\frac{1}{4}\right)\, A_{12}(-6)\, M_1\left(\frac{1}{2}\right) = \begin{pmatrix} -1 & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} \end{pmatrix}$.

Q7 (c)

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(A\,|\,I) = \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 2 & -3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$A_{12}(-2) \implies \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$A_{21}(2) \implies \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Therefore $A^{-1} = A_{21}(2)\,A_{12}(-2) = \begin{pmatrix} -3 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Question 8

Consider the system of equations

$$\begin{array}{rcrcrcrcrcrcr} & & 6x_2 & + & 2x_3 & & & - & x_5 & & & = & 1 \\ & & & & 4x_3 & & & + & x_5 & & & = & -1 \\ -2x_1 & + & x_2 & & & + & 4x_4 & + & x_5 & - & x_6 & = & 0 \\ -3x_1 & & & + & x_3 & + & 6x_4 & + & 2x_5 & & & = & 4 \end{array}$$

Q8 (a)

Write down the augmented matrix for this system.

$$\left(\begin{array}{cccccc|c} 0 & 6 & 2 & 0 & -1 & 0 & 1 \\ 0 & 0 & 4 & 0 & 1 & 0 & -1 \\ -2 & 1 & 0 & 4 & 1 & -1 & 0 \\ -3 & 0 & 1 & 6 & 2 & 0 & 4 \end{array}\right)$$

Q8 (b)

Compute the RREF of the augmented matrix.

$$\left(\begin{array}{cccccc|c} 0 & 6 & 2 & 0 & -1 & 0 & 1 \\ 0 & 0 & 4 & 0 & 1 & 0 & -1 \\ -2 & 1 & 0 & 4 & 1 & -1 & 0 \\ -3 & 0 & 1 & 6 & 2 & 0 & 4 \end{array}\right)$$

$$S_{13} \implies \left(\begin{array}{cccccc|c} -2 & 1 & 0 & 4 & 1 & -1 & 0 \\ 0 & 0 & 4 & 0 & 1 & 0 & -1 \\ 0 & 6 & 2 & 0 & -1 & 0 & 1 \\ -3 & 0 & 1 & 6 & 2 & 0 & 4 \end{array}\right)$$

$$M_1\left(-\frac{1}{2}\right) \implies \left(\begin{array}{cccccc|c} 1 & -\frac{1}{2} & 0 & -2 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 4 & 0 & 1 & 0 & -1 \\ 0 & 6 & 2 & 0 & -1 & 0 & 1 \\ -3 & 0 & 1 & 6 & 2 & 0 & 4 \end{array}\right)$$

$$A_{14}(3) \implies \left(\begin{array}{cccccc|c} 1 & -\frac{1}{2} & 0 & -2 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 4 & 0 & 1 & 0 & -1 \\ 0 & 6 & 2 & 0 & -1 & 0 & 1 \\ 0 & -\frac{3}{2} & 1 & 0 & \frac{1}{2} & \frac{3}{2} & 4 \end{array}\right)$$

$$S_{24} \implies \left(\begin{array}{cccccc|c} 1 & -\frac{1}{2} & 0 & -2 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{3}{2} & 1 & 0 & \frac{1}{2} & \frac{3}{2} & 4 \\ 0 & 6 & 2 & 0 & -1 & 0 & 1 \\ 0 & 0 & 4 & 0 & 1 & 0 & -1 \end{array}\right)$$

$$M_2\left(-\frac{2}{3}\right) \implies \left(\begin{array}{cccccc|c} 1 & -\frac{1}{2} & 0 & -2 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} & 0 & -\frac{1}{3} & -1 & -\frac{8}{3} \\ 0 & 6 & 2 & 0 & -1 & 0 & 1 \\ 0 & 0 & 4 & 0 & 1 & 0 & -1 \end{array}\right)$$

$$A_{21}\left(\frac{1}{2}\right) \implies \left(\begin{array}{cccccc|c} 1 & 0 & -\frac{1}{3} & -2 & -\frac{2}{3} & 0 & -\frac{4}{3} \\ 0 & 1 & -\frac{2}{3} & 0 & -\frac{1}{3} & -1 & -\frac{8}{3} \\ 0 & 6 & 2 & 0 & -1 & 0 & 1 \\ 0 & 0 & 4 & 0 & 1 & 0 & -1 \end{array}\right)$$

$$A_{23}\left(-6\right) \implies \left(\begin{array}{cccccc|c} 1 & 0 & -\frac{1}{3} & -2 & -\frac{2}{3} & 0 & -\frac{4}{3} \\ 0 & 1 & -\frac{2}{3} & 0 & -\frac{1}{3} & -1 & -\frac{8}{3} \\ 0 & 0 & 6 & 0 & 1 & 6 & 17 \\ 0 & 0 & 4 & 0 & 1 & 0 & -1 \end{array}\right)$$

$$M_3\left(\frac{1}{6}\right) \implies \left(\begin{array}{cccccc|c} 1 & 0 & -\frac{1}{3} & -2 & -\frac{2}{3} & 0 & -\frac{4}{3} \\ 0 & 1 & -\frac{2}{3} & 0 & -\frac{1}{3} & -1 & -\frac{8}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{6} & 1 & \frac{17}{6} \\ 0 & 0 & 4 & 0 & 1 & 0 & -1 \end{array}\right)$$

$$A_{34}\left(-4\right) \implies \left(\begin{array}{cccccc|c} 1 & 0 & -\frac{1}{3} & -2 & -\frac{2}{3} & 0 & -\frac{4}{3} \\ 0 & 1 & -\frac{2}{3} & 0 & -\frac{1}{3} & -1 & -\frac{8}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{6} & 1 & \frac{17}{6} \\ 0 & 0 & 0 & 0 & \frac{1}{3} & -4 & -\frac{37}{3} \end{array}\right)$$

$$A_{31}\left(\frac{1}{3}\right) \implies \left(\begin{array}{cccccc|c} 1 & 0 & 0 & -2 & -\frac{11}{18} & \frac{1}{3} & -\frac{7}{18} \\ 0 & 1 & -\frac{2}{3} & 0 & -\frac{1}{3} & -1 & -\frac{8}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{6} & 1 & \frac{17}{6} \\ 0 & 0 & 0 & 0 & \frac{1}{3} & -4 & -\frac{37}{3} \end{array}\right)$$

$$A_{32}\left(\frac{2}{3}\right) \implies \left(\begin{array}{cccccc|c} 1 & 0 & 0 & -2 & -\frac{11}{18} & \frac{1}{3} & -\frac{7}{18} \\ 0 & 1 & 0 & 0 & -\frac{2}{9} & -\frac{1}{3} & -\frac{7}{9} \\ 0 & 0 & 1 & 0 & \frac{1}{6} & 1 & \frac{17}{6} \\ 0 & 0 & 0 & 0 & \frac{1}{3} & -4 & -\frac{37}{3} \end{array}\right)$$

$$M_4(3) \implies \left(\begin{array}{cccccc|c} 1 & 0 & 0 & -2 & -\frac{11}{18} & \frac{1}{3} & -\frac{7}{18} \\ 0 & 1 & 0 & 0 & -\frac{2}{9} & -\frac{1}{3} & -\frac{7}{9} \\ 0 & 0 & 1 & 0 & \frac{1}{6} & 1 & \frac{17}{6} \\ 0 & 0 & 0 & 0 & 1 & -12 & -37 \end{array}\right)$$

$$A_{43}\left(-\frac{1}{6}\right) \implies \left(\begin{array}{cccccc|c} 1 & 0 & 0 & -2 & -\frac{11}{18} & \frac{1}{3} & -\frac{7}{18} \\ 0 & 1 & 0 & 0 & -\frac{2}{9} & -\frac{1}{3} & -\frac{7}{9} \\ 0 & 0 & 1 & 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & 0 & 1 & -12 & -37 \end{array}\right)$$

$$A_{42}\left(\frac{2}{9}\right) \implies \left(\begin{array}{cccccc|c} 1 & 0 & 0 & -2 & -\frac{11}{18} & \frac{1}{3} & -\frac{7}{18} \\ 0 & 1 & 0 & 0 & 0 & -3 & -9 \\ 0 & 0 & 1 & 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & 0 & 1 & -12 & -37 \end{array}\right)$$

$$A_{41}\left(\frac{11}{18}\right) \implies \left(\begin{array}{cccccc|c} 1 & 0 & 0 & -2 & 0 & -7 & -23 \\ 0 & 1 & 0 & 0 & 0 & -3 & -9 \\ 0 & 0 & 1 & 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & 0 & 1 & -12 & -37 \end{array}\right)$$

Q8 (c)

In terms of this RREF (without actually identifying a solution) explain why the system is consistent.

The system is consistent because the row reduced echelon form has no zero rows.

Q8 (d)

Give the general solution of this system of equations.

Choose parameters $\lambda, \mu \in \mathbb{R}$. Then

$$\begin{aligned}
x_1 &= -23 + 2\lambda + 7\mu \\
x_2 &= -9 + 3\mu \\
x_3 &= 9 - 3\mu \\
x_4 &= \lambda \\
x_5 &= -37 + 12\mu \\
x_6 &= \mu
\end{aligned}$$