# MA139 Analysis 2, Assignment 4

Dyson Dyson

## Question 1

Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = \left(1 - \frac{1}{2}x + \frac{1}{12}x^2\right) \mathrm{e}^x.$$

### Q1 (a)

Use Taylor's theorem with remainder to show that

$$F(x) \geq 1 + \frac{1}{2}x + \frac{1}{12}x^2 \qquad \text{for } x \geq 0.$$

To apply Taylor's theorem with remainder to $F$, we need the first few derivatives of $F$.

$$\begin{aligned}
F'(x) &= \left(1 - \frac{1}{2}x + \frac{1}{12}x^2\right) \mathrm{e}^x + \left(-\frac{1}{2} + \frac{1}{6}x\right) \mathrm{e}^x \\
&= F(x) + \left(-\frac{1}{2} + \frac{1}{6}x\right) \mathrm{e}^x \\
F''(x) &= F'(x) + \left(-\frac{1}{2} + \frac{1}{6}x\right) \mathrm{e}^x + \frac{1}{6}\mathrm{e}^x \\
&= F'(x) + \left(-\frac{1}{3} + \frac{1}{6}x\right) \mathrm{e}^x \\
F^{(3)}(x) &= F''(x) + \left(-\frac{1}{3} + \frac{1}{6}x\right) \mathrm{e}^x + \frac{1}{6}\mathrm{e}^x \\
&= F''(x) + \left(-\frac{1}{6} + \frac{1}{6}x\right) \mathrm{e}^x \\
F^{(4)}(x) &= F^{(3)}(x) + \left(-\frac{1}{6} + \frac{1}{6}x\right) \mathrm{e}^x + \frac{1}{6}\mathrm{e}^x \\
&= F^{(3)}(x) + \frac{1}{6}x\mathrm{e}^x
\end{aligned}$$

$$
\begin{aligned}
F^{(5)} &= F^{(4)}(x) + \frac{1}{6}x\mathrm{e}^x + \frac{1}{6}\mathrm{e}^x \\
&= F^{(4)}(x) + \left(\frac{1}{6} + \frac{1}{6}x\right)\mathrm{e}^x \\
F^{(n)}(x) &= F^{(n-1)}(x) + \frac{n-4+x}{6}\mathrm{e}^x
\end{aligned}
$$

We want to prove that $F^{(4)}(x) \geq 0$ for all $x \geq 0$. We have

$$
\begin{aligned}
F^{(4)}(x) &= F^{(3)} + \frac{1}{6}x\mathrm{e}^x \\
&= F''(x) + \left(-\frac{1}{6} + \frac{1}{6}x\right)\mathrm{e}^x + \frac{1}{6}x\mathrm{e}^x \\
&= F''(x) + \left(-\frac{1}{6} + \frac{1}{3}x\right)\mathrm{e}^x \\
&= F'(x) + \left(-\frac{1}{3} + \frac{1}{6}x - \frac{1}{6} + \frac{1}{3}x\right)\mathrm{e}^x \\
&= F'(x) + \left(-\frac{1}{2} + \frac{1}{2}x\right)\mathrm{e}^x \\
&= F(x) + \left(-\frac{1}{2} + \frac{1}{6}x - \frac{1}{2} + \frac{1}{2}x\right)\mathrm{e}^x \\
&= F(x) + \left(-1 + \frac{2}{3}x\right)\mathrm{e}^x \\
&= \left(1 - \frac{1}{2}x + \frac{1}{12}x^2 - 1 + \frac{2}{3}x\right)\mathrm{e}^x \\
&= \frac{1}{12}\mathrm{e}^x\left(2x^2 + x\right)
\end{aligned}
$$

Clearly $\frac{1}{12}\mathrm{e}^x > 0$ for all $x$ and $2x^2 + x \geq 0$ for all $x \geq 0$, therefore $F^{(4)}(x) \geq 0$ for all $x \geq 0$.

Now applying Taylor's theorem with Lagrange remainder around 0, we get

$$
\begin{aligned}
F(x) &= F(0) + F'(0)x + \frac{F''(0)x^2}{2} + \frac{F^{(3)}(0)x^3}{6} + \frac{F^{(4)}(t)x^4}{24} \\
&= 1 + \frac{1}{2}x + \frac{1}{6}\frac{x^2}{2} + 0x^3 + \frac{F^{(4)}(t)x^4}{24} \\
&= 1 + \frac{1}{2}x + \frac{1}{12}x^2 + \frac{1}{24}x^4F^{(4)}(t)
\end{aligned}
$$

for some $t$ between 0 and $x$. Assuming $x \geq 0$, then $0 \leq t \leq x$.

Since $F^{(4)}(t) \geq 0$ for all $t \geq 0$, we get $F(x) \geq 1 + \frac{1}{2}x + \frac{1}{12}x^2$ for all $x \geq 0$, as required.

## Q1 (b)

Show that for all $x$,

$$1 - \frac{1}{2}x + \frac{1}{12}x^2 \geq 0.$$

We shall treat $\frac{1}{12}x^2 - \frac{1}{2}x + 1$ as a quadratic equation in $x$. Then we can see its discriminant '$b^2 - 4ac$' to be $\frac{1}{4} - \frac{4}{12} = -\frac{1}{12} < 0$. The discriminant is negative, which means the quadratic has no real roots.

Since the quadratic equation has no real roots, and the $x^2$ coefficient is positive, we can conclude that $1 - \frac{1}{2}x + \frac{1}{12}x^2 > 0 \; \forall \; x \in \mathbb{R}$.

## Q1 (c)

Deduce that for $x \geq 0$,

$$e^x \geq \frac{1 + \frac{1}{2}x + \frac{1}{12}x^2}{1 - \frac{1}{2}x + \frac{1}{12}x^2}$$

and hence that $e \geq \frac{19}{7}$.

In part **(a)**, we showed that for all $x \geq 0$,

$$e^x \left(1 - \frac{1}{2}x + \frac{1}{12}x^2\right) \geq 1 + \frac{1}{2}x + \frac{1}{12}x^2$$

The quadratic in brackets on the LHS has discriminant $\frac{1}{4} - \frac{1}{3} < 0$, and positive coefficient of $x^2$, so that term in brackets is always strictly positive, so we can divide by it.

Therefore, for $x \geq 0$,

$$e^x \geq \frac{1 + \frac{1}{2}x + \frac{1}{12}x^2}{1 - \frac{1}{2}x + \frac{1}{12}x^2}$$

Plugging in $x = 1$ gives

$$\begin{aligned} e &\geq \frac{1 + \frac{1}{2} + \frac{1}{12}}{1 - \frac{1}{2} + \frac{1}{12}} \\ &= \frac{\frac{18}{12} + \frac{1}{12}}{\frac{6}{12} + \frac{1}{12}} \\ &= \frac{19}{7} \end{aligned}$$

Therefore $e \geq \frac{19}{7}$ as required.

# Question 2

Suppose $f, g\colon [a,b] \to \mathbb{R}$ are integrable and agree except at finitely many points in the interval. Show that

$$\int_a^b f = \int_a^b g.$$

We know that $f(x) = g(x)\ \forall\ x \in [a,b] \setminus \{c_1,\ \ldots,\ c_n\}$.

Since $f$ and $g$ are integrable, there exist partitions $P$ and $Q$ of $[a,b]$ such that for any $\varepsilon > 0$,

$$\begin{aligned} U(f,P) - L(f,P) &< \varepsilon \\ U(g,Q) - L(g,Q) &< \varepsilon \end{aligned}$$

Since $f$ and $g$ only disagree at finitely many points $c_1,\ \ldots,\ c_n$, for each $c_i$, either $f$ or $g$ (or both) must be discontinuous at $c_i$. Therefore either $P$ or $Q$ must "cut out the bad bit" at $c_i$. Therefore we can take a common refinement $R$ of $P$ and $Q$, which will "cut out the bad bits" at all $c_1,\ \ldots,\ c_n$.

Say for each $c_i$ we choose $\delta_i$ such that $[c_i - \delta_i, c_i + \delta_i]$ is one of the intervals in $R$, evidently the one containing $c_i$. Then for any $\varepsilon$, we can choose all $\delta_i$ small enough such that the rectangles in the upper and lower sums will have arbitrarily small area.

Let's say the sum of all the rectangles containing each $c_i$ for the upper sum of $f$ is $\Gamma_f$ and the sum of all the rectangles containing each $c_i$ for the lower sum of $f$ is $\gamma_f$. Let's also define $\Gamma_g$ and $\gamma_g$ similarly for $g$.

Let $S$ be the partition $R$ but excluding each of the intervals containing $c_1,\ \ldots,\ c_n$. Then

$$\begin{aligned} U(f,R) &= U(f,S) + \Gamma_f \\ U(g,R) &= U(g,S) + \Gamma_g \\ L(f,R) &= L(f,S) + \gamma_f \\ L(g,R) &= L(g,S) + \gamma_g \end{aligned}$$

Since $f$ and $g$ agree at all points in $S$, $U(f,S) = U(g,S)$.

We can choose all the $\delta_i$ accordingly to make $\Gamma_f = \Gamma_g = \Gamma$ and $\gamma_f = \gamma_g = \gamma$ and make both arbitrarily small, so

$$\begin{aligned} U(f,R) &= U(f,S) + \Gamma = U(g,R) \\ L(f,R) &= L(f,S) + \gamma = L(g,R) \end{aligned}$$

Since the upper sums and lower sums for $f$ and $g$ both agree on the partition $S$, we can conclude that $\displaystyle\int_a^b f = \int_a^b g$.

# Question 3

Find

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \log\left(1 + \frac{k}{n}\right).$$

Be careful to explain what facts you use from the course.

This can be viewed as $\lim\limits_{n\to\infty} U(f, P_n)$ where $f(x) = \log(1+x)$ and $P_n$ is the partition of $[0,1]$ into $n$ equal intervals, since the area of each rectangle is the width $\frac{1}{n}$ times the height $f(x_i)$, and since $f$ is increasing and the sum always takes $f(x_i)$ on the right hand side of the interval, we get the upper sum.

Note that $f$ is continuous and so by Homework 8 Question 4, this upper sum converges to the integral

$$\begin{aligned}\int_0^1 \log(1+x)\ \mathrm{d}x &= \big[(1+x)\log(1+x) - x\big]_0^1 \\ &= 2\log 2 - 1 - \log 1 - 0 \\ &= 2\log 2 - 1\end{aligned}$$