

# MA146 Assignment 1

- Q1
- ① Continuous, deterministic system and an inverse problem.
  - ② Discrete, stochastic system and a control problem.
  - ③ Discrete, stochastic system and a forward problem.

Q2

①  $b^2 - 4ac < 0$   $x_2(t) = e^{-\frac{b}{2a}t} \sin(\omega t)$

$$\begin{aligned} x_2'(t) &= e^{-\frac{b}{2a}t} \omega \cos(\omega t) - \frac{b}{2a} e^{-\frac{b}{2a}t} \sin(\omega t) \\ &= e^{-\frac{b}{2a}t} \left( \omega \cos(\omega t) - \frac{b}{2a} \sin(\omega t) \right) \end{aligned}$$

$$\begin{aligned} x_2''(t) &= e^{-\frac{b}{2a}t} \left( -\omega^2 \sin(\omega t) - \frac{b\omega}{2a} \cos \omega t \right. \\ &\quad \left. - \frac{b}{2a} \left( \omega \cos(\omega t) - \frac{b}{2a} \sin(\omega t) \right) \right) \\ &= e^{-\frac{b}{2a}t} \left( \left( \frac{b^2}{4a^2} - \omega^2 \right) \sin \omega t - \frac{b}{a} \omega \cos \omega t \right) \end{aligned}$$

We plug these into the ODE and get

$$\begin{aligned} e^{-\frac{b}{2a}t} &\left( a \left( \frac{b^2}{4a^2} - \omega^2 \right) \sin \omega t - b\omega \cos \omega t + b\omega \cos \omega t \right. \\ &\quad \left. - \frac{b^2}{2a} \sin \omega t + c \sin \omega t \right) \end{aligned}$$

$$= e^{-\frac{b}{2a}t} \frac{b^2 - 4b^2}{4a} \left( \frac{b^2}{4a} - \frac{b^2}{2a} - a\omega^2 + c \right) \sin \omega t$$

Q2/ (a) cont.

$$= e^{-\frac{b}{2a}t} \left( \frac{-b^2}{4a} - \cancel{a\omega^2} + c \right) \sin \omega t$$

Recall that  $\omega = \frac{1}{2a} \sqrt{4ac - b^2} \therefore \omega^2 = \frac{1}{4a^2} (4ac - b^2)$

$$\therefore \text{ODE} = e^{-\frac{b}{2a}t} \left( \frac{-b^2}{4a} - \frac{4ac}{4a} + \frac{b^2}{4a} + c \right) \sin \omega t$$

$$= e^{-\frac{b}{2a}t} \times 0 \times \sin \omega t = 0 \quad \text{as expected.}$$

①  $a=2, b=-4, c=-6$

$$b^2 - 4ac = 16 + 8 \times 6 = 64 \quad \lambda_{1,2} = \frac{4 \pm \sqrt{64}}{4} = 1 \pm 2$$

$$\therefore x(t) = l_1 e^{3t} + l_2 e^{-t}$$

②  $a=-2, b=3, c=-\frac{5}{4}$

$$b^2 - 4ac = 9 - 5 \times 2 = -1 \quad \lambda_{1,2} = \frac{3}{4} \pm i \frac{1}{4} \sqrt{1}$$

$$\omega = \frac{1}{4} \therefore \lambda_{1,2} = \frac{3}{4} \pm i \frac{1}{4}$$

$$\therefore x(t) = e^{\frac{3}{4}t} \left( l_1 \cos \frac{t}{4} + l_2 \sin \frac{t}{4} \right) = e^{\frac{3}{4}t} \left( l_1 \cos \frac{t}{4} + l_2 \sin \frac{t}{4} \right)$$

③  $b^2 - 4ac = 8 \times 18 = 144$

$$\lambda_{1,2} = \frac{0 \pm 12i}{-4} = \pm 3i$$

$a=-2, b=0, c=-18$

~~$x(t) = l_1 e^{3t} + l_2 e^{-3t}$~~   $\therefore x(t) = e^{0t} (l_1 \sin 3t + l_2 \cos 3t)$

22/2 cont.

$$x(t) = l_1 \sin 3t + l_2 \cos 3t$$

$$x'(t) = 3l_1 \cos 3t - 3l_2 \sin 3t$$

$$x'(2\pi) = 3l_1 \cos 6\pi - 3l_2 \sin 6\pi = 3l_1 - 0 = 12 \therefore l_1 = 4$$

$$x(2\pi) = l_1 \sin 6\pi + l_2 \cos 6\pi = 0 + l_2 = 15 \therefore l_2 = 15$$

$$\therefore x(t) = 4 \sin 3t + 15 \cos 3t$$

②  $a=3, b=12, c=12$       Jupyter notebook says

$$x(t) = (-6t - 4)e^{-2t}$$

$$\therefore x'(t) = 2(6t + 4)e^{-2t} - \frac{-(6t + 4)e^{-2t}}{-6e^{-2t}} \\ = (12t + 2)e^{-2t}$$

$$x''(t) = -2(12t + 2)e^{-2t} + 12e^{-2t} = (-24t + 8)e^{-2t}$$

~~ODE = 3(-24t + 8)e^{-2t} + 12(12t + 2)e^{-2t} - 12(-6t - 4)e^{-2t}~~

$$\begin{aligned} \text{ODE} &= 3(-24t + 8)e^{-2t} + 12(12t + 2)e^{-2t} - 12(-6t - 4)e^{-2t} \\ &= e^{-2t}(-72t + 24 + 144t + 24 - 72t - 48) \\ &= e^{-2t} \times 0 = 0 \quad \text{as expected} \end{aligned}$$





Q3 @

During free fall:

$$\boxed{h(0)=0} \quad \boxed{h'(0)=0}$$

$$m \ddot{h}(t) = f(t) = -mg$$

$$\Rightarrow \ddot{h}(t) = -g$$

$\therefore$  In free fall,  $\boxed{h(t) = -gt^2}$  when  $h(t) \geq -l$ .

We don't have to worry about terms in  $t$  or constants because of the initial conditions being 0. They are both satisfied by  $h(t) = -gt^2$ .

During re-bounce:  $(h(t) \leq -l)$

$$m \ddot{h}(t) = f(t) = k(-l - h(t)) - mg$$

$$\Rightarrow \ddot{h}(t) = \frac{k}{m}(-l - h(t)) - g$$

$$\Rightarrow \frac{d^2}{dt^2} h(t) + \frac{k}{m} h(t) = -\frac{kl}{m} - g$$

~~$\therefore \lambda^2 + \frac{k}{m} = 0$  is the auxiliary equation, so  $\lambda = \pm \sqrt{-\frac{k}{m}}$~~   
 ~~$k, m > 0$ , so  $\lambda = \pm i\sqrt{\frac{k}{m}}$~~

~~$\therefore \lambda^2 + \frac{k}{m} = -\frac{kl}{m} - g$  is the auxiliary equation~~

I was right the first time.

$\lambda^2 + \frac{k}{m} = 0$  is the auxiliary equation for the ~~comp~~ complementary function.

Q3/ (a)  $\therefore \lambda = \pm \sqrt{\frac{-k}{m}}$   $k, m > 0$ , so  $\lambda = \pm i\sqrt{\frac{k}{m}}$

Therefore the complementary function is ~~the~~

$$h(t) = C_1 e^{i\sqrt{\frac{k}{m}}t} + C_2 e^{-i\sqrt{\frac{k}{m}}t}$$

$$= C_1 \left( \cos\sqrt{\frac{k}{m}}t + i \sin\sqrt{\frac{k}{m}}t \right) + C_2 \left( \cos\sqrt{\frac{k}{m}}t - i \sin\sqrt{\frac{k}{m}}t \right)$$

$$= (C_1 + C_2) \cos\sqrt{\frac{k}{m}}t + (C_1 - C_2)i \sin\sqrt{\frac{k}{m}}t$$

I'm not sure why, but we always drop the  $i$  here to get

$$h(t) = A \cos\sqrt{\frac{k}{m}}t + B \sin\sqrt{\frac{k}{m}}t$$

The particular solution wants  $h(t)$  to be some constant, to reflect the constant on the RHS of the ODE,  $-\frac{k\ell}{m} - g$ .

Let  $h(t) = C$ . Then  $h'(t) = 0$ .

$$\therefore 0 + \frac{k}{m}C = -\frac{k}{m}\ell - g \Rightarrow C = -\ell - \frac{mg}{k}$$

$$\therefore h(t) = A \cos\sqrt{\frac{k}{m}}t + B \sin\sqrt{\frac{k}{m}}t - \ell - \frac{mg}{k}$$

$$= A \cos\left(t\sqrt{\frac{k}{m}}\right) + B \sin\left(t\sqrt{\frac{k}{m}}\right) - \ell - \frac{mg}{k}$$

$$h(0) = A + 0 - \ell - \frac{mg}{k} = 0 \Rightarrow A = \ell + \frac{mg}{k}$$

$$h'(t) = A\sqrt{\frac{k}{m}} \sin\left(t\sqrt{\frac{k}{m}}\right) + B\sqrt{\frac{k}{m}} \cos\left(t\sqrt{\frac{k}{m}}\right)$$

$$h'(0) = 0 + B\sqrt{\frac{k}{m}} = 0 \Rightarrow B = 0 \text{ since } k, m > 0$$

$$\therefore \boxed{h(t) = \left(\ell + \frac{mg}{k}\right) \cos\left(t\sqrt{\frac{k}{m}}\right) - \ell - \frac{mg}{k}}$$

Q3/ (b)  $h(t)$  in the re-bounce stage is basically just a cosine function, stretched and ~~scale~~ scaled. That means the minimum of  $h(t)$  is  $-l - \frac{mg}{k} - (l + \frac{mg}{k})$

$= -2(l + \frac{mg}{k})$  If we want the jumper to just touch the floor in the worst case, then

$h_m \leq -2(l + \frac{mg}{k})$  ~~Then~~ We can then simply rearrange to find  $l$ .

$$-\frac{h_m}{2} \geq l + \frac{mg}{k} \quad \therefore \boxed{l \leq -\left(\frac{h_m}{2} + \frac{mg}{k}\right)}$$

$$\therefore \boxed{l \leq -\left(\frac{h_m}{2} + \frac{mg}{k}\right)}$$