MA139 Analysis 2, Assignment 4

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Question 1

> Let $F : \mathbb{R} \to \mathbb{R}$ be defined by
>
> $$F(x) = \left(1 - \frac{1}{2}x + \frac{1}{12}x^2\right) \mathrm{e}^x.$$

Q1 (a)

> Use Taylor's theorem with remainder to show that
>
> $$F(x) \geq 1 + \frac{1}{2}x + \frac{1}{12}x^2 \quad \text{for } x \geq 0.$$

To apply Taylor's theorem with remainder to F, we need the first few derivatives of F.

$$\begin{aligned}
F'(x) &= \left(1 - \frac{1}{2}x + \frac{1}{12}x^2\right) \mathrm{e}^x + \left(-\frac{1}{2} + \frac{1}{6}x\right) \mathrm{e}^x \\
&= F(x) + \left(-\frac{1}{2} + \frac{1}{6}x\right) \mathrm{e}^x \\
F''(x) &= F'(x) + \left(-\frac{1}{2} + \frac{1}{6}x\right) \mathrm{e}^x + \frac{1}{6}\mathrm{e}^x \\
&= F'(x) + \left(-\frac{1}{3} + \frac{1}{6}x\right) \mathrm{e}^x \\
F^{(3)}(x) &= F''(x) + \left(-\frac{1}{3} + \frac{1}{6}x\right) \mathrm{e}^x + \frac{1}{6}\mathrm{e}^x \\
&= F''(x) + \left(-\frac{1}{6} + \frac{1}{6}x\right) \mathrm{e}^x \\
F^{(4)}(x) &= F^{(3)}(x) + \left(-\frac{1}{6} + \frac{1}{6}x\right) \mathrm{e}^x + \frac{1}{6}\mathrm{e}^x \\
&= F^{(3)}(x) + \frac{1}{6}x\mathrm{e}^x
\end{aligned}$$

$$\begin{aligned}
F^{(5)} &= F^{(4)}(x) + \frac{1}{6}x\mathrm{e}^x + \frac{1}{6}\mathrm{e}^x \\
&= F^{(4)}(x) + \left(\frac{1}{6} + \frac{1}{6}x\right)\mathrm{e}^x \\
F^{(n)}(x) &= F^{(n-1)}(x) + \frac{n-4+x}{6}\mathrm{e}^x
\end{aligned}$$

We want to prove that $F^{(4)}(x) \geq 0$ for all $x \geq 0$. We have

$$\begin{aligned}
F^{(4)}(x) &= F^{(3)} + \frac{1}{6}x\mathrm{e}^x \\
&= F''(x) + \left(-\frac{1}{6} + \frac{1}{6}x\right)\mathrm{e}^x + \frac{1}{6}x\mathrm{e}^x \\
&= F''(x) + \left(-\frac{1}{6} + \frac{1}{3}x\right)\mathrm{e}^x \\
&= F'(x) + \left(-\frac{1}{3} + \frac{1}{6}x - \frac{1}{6} + \frac{1}{3}x\right)\mathrm{e}^x \\
&= F'(x) + \left(-\frac{1}{2} + \frac{1}{2}x\right)\mathrm{e}^x \\
&= F(x) + \left(-\frac{1}{2} + \frac{1}{6}x - \frac{1}{2} + \frac{1}{2}x\right)\mathrm{e}^x \\
&= F(x) + \left(-1 + \frac{2}{3}x\right)\mathrm{e}^x \\
&= \left(1 - \frac{1}{2}x + \frac{1}{12}x^2 - 1 + \frac{2}{3}x\right)\mathrm{e}^x \\
&= \frac{1}{12}\mathrm{e}^x\left(2x^2 + x\right)
\end{aligned}$$

Clearly $\frac{1}{12}\mathrm{e}^x > 0$ for all x and $2x^2 + x \geq 0$ for all $x \geq 0$, therefore $F^{(4)}(x) \geq 0$ for all $x \geq 0$.

Now applying Taylor's theorem with Lagrange remainder around 0, we get

$$\begin{aligned}
F(x) &= F(0) + F'(0)x + \frac{F''(0)x^2}{2} + \frac{F^{(3)}(0)x^3}{6} + \frac{F^{(4)}(t)x^4}{24} \\
&= 1 + \frac{1}{2}x + \frac{1}{6}\frac{x^2}{2} + 0x^3 + \frac{F^{(4)}(t)x^4}{24} \\
&= 1 + \frac{1}{2}x + \frac{1}{12}x^2 + \frac{1}{24}x^4F^{(4)}(t)
\end{aligned}$$

for some t between 0 and x. Assuming $x \geq 0$, then $0 \leq t \leq x$.

Since $F^{(4)}(t) \geq 0$ for all $t \geq 0$, we get $F(x) \geq 1 + \frac{1}{2}x + \frac{1}{12}x^2$ for all $x \geq 0$, as required.

Q1 (b)

Show that for all x,
$$1 - \frac{1}{2}x + \frac{1}{12}x^2 \geq 0.$$

We shall treat $\dfrac{1}{12}x^2 - \dfrac{1}{2}x + 1$ as a quadratic equation in x. Then we can see its discriminant '$b^2 - 4ac$' to be $\dfrac{1}{4} - \dfrac{4}{12} = -\dfrac{1}{12} < 0$. The discriminant is negative, which means the quadratic has no real roots.

Since the quadratic equation has no real roots, and the x^2 coefficient is positive, we can conclude that $1 - \dfrac{1}{2}x + \dfrac{1}{12}x^2 > 0 \ \forall \ x \in \mathbb{R}$.

Q1 (c)

Deduce that for $x \geq 0$,
$$\mathrm{e}^x \geq \frac{1 + \frac{1}{2}x + \frac{1}{12}x^2}{1 - \frac{1}{2}x + \frac{1}{12}x^2}$$
and hence that $\mathrm{e} \geq \dfrac{19}{7}$.

In part **(a)**, we showed that for all $x \geq 0$,
$$\mathrm{e}^x \left(1 - \frac{1}{2}x + \frac{1}{12}x^2\right) \geq 1 + \frac{1}{2}x + \frac{1}{12}x^2$$

The quadratic in brackets on the LHS has discriminant $\frac{1}{4} - \frac{1}{3} < 0$, and positive coefficient of x^2, so that term in brackets is always strictly positive, so we can divide by it.

Therefore, for $x \geq 0$,
$$\mathrm{e}^x \geq \frac{1 + \frac{1}{2}x + \frac{1}{12}x^2}{1 - \frac{1}{2}x + \frac{1}{12}x^2}$$

Plugging in $x = 1$ gives
$$\begin{aligned}
\mathrm{e} &\geq \frac{1 + \frac{1}{2} + \frac{1}{12}}{1 - \frac{1}{2} + \frac{1}{12}} \\
&= \frac{\frac{18}{12} + \frac{1}{12}}{\frac{6}{12} + \frac{1}{12}} \\
&= \frac{19}{7}
\end{aligned}$$

Therefore $\mathrm{e} \geq \dfrac{19}{7}$ as required.

Question 2

Suppose $f, g\colon [a,b] \to \mathbb{R}$ are integrable and agree except at finitely many points in the interval. Show that

$$\int_a^b f = \int_a^b g.$$

We know that $f(x) = g(x)\ \forall\ x \in [a,b] \setminus \{c_1,\ \ldots,\ c_n\}$.

Since f and g are integrable, there exist partitions P and Q of $[a,b]$ such that for any $\varepsilon > 0$,

$$U(f,P) - L(f,P) < \varepsilon$$
$$U(g,Q) - L(g,Q) < \varepsilon$$

Since f and g only disagree at finitely many points $c_1,\ \ldots,\ c_n$, for each c_i, either f or g (or both) must be discontinuous at c_i. Therefore either P or Q must "cut out the bad bit" at c_i. Therefore we can take a common refinement R of P and Q, which will "cut out the bad bits" at all $c_1,\ \ldots,\ c_n$.

Say for each c_i we choose δ_i such that $[c_i - \delta_i, c_i + \delta_i]$ is one of the intervals in R, evidently the one containing c_i. Then for any ε, we can choose all δ_i small enough such that the rectangles in the upper and lower sums will have arbitrarily small area.

Let's say the sum of all the rectangles containing each c_i for the upper sum of f is Γ_f and the sum of all the rectangles containing each c_i for the lower sum of f is γ_f. Let's also define Γ_g and γ_g similarly for g.

Let S be the partition R but excluding each of the intervals containing $c_1,\ \ldots,\ c_n$. Then

$$U(f,R) = U(f,S) + \Gamma_f$$
$$U(g,R) = U(g,S) + \Gamma_g$$
$$L(f,R) = L(f,S) + \gamma_f$$
$$L(g,R) = L(g,S) + \gamma_g$$

Since f and g agree at all points in S, $U(f,S) = U(g,S)$.

We can choose all the δ_i accordingly to make $\Gamma_f = \Gamma_g = \Gamma$ and $\gamma_f = \gamma_g = \gamma$ and make both arbitrarily small, so

$$U(f,R) = U(f,S) + \Gamma = U(g,R)$$
$$L(f,R) = L(f,S) + \gamma = L(g,R)$$

Since the upper sums and lower sums for f and g both agree on the partition S, we can conclude that $\displaystyle\int_a^b f = \int_a^b g$.

Question 3

Find

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \log\left(1 + \frac{k}{n}\right).$$

Be careful to explain what facts you use from the course.

This can be viewed as $\lim_{n\to\infty} U(f, P_n)$ where $f(x) = \log(1+x)$ and P_n is the partition of $[0,1]$ into n equal intervals, since the area of each rectangle is the width $\frac{1}{n}$ times the height $f(x_i)$, and since f is increasing and the sum always takes $f(x_i)$ on the right hand side of the interval, we get the upper sum.

Note that f is continuous and so by Homework 8 Question 4, this upper sum converges to the integral

$$\begin{aligned}\int_0^1 \log(1+x)\,\mathrm{d}x &= \big[(1+x)\log(1+x) - x\big]_0^1 \\ &= 2\log 2 - 1 - \log 1 - 0 \\ &= 2\log 2 - 1\end{aligned}$$