

# CS147 Discrete Maths and its Applications 2, Assignment 2

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## Question 1

Let  $G = (V, E)$  be a graph,  $M \subset E$  be a matching on  $G$ , and  $Z \subset M$ .

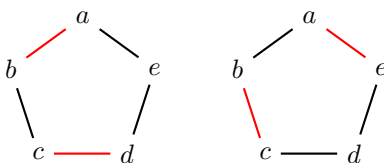
A matching is just a set of edges in  $G$  which do not share any common endpoint. For  $Z$  to not be a matching, we would need to choose two edges from  $M$  which share a node. Since  $M$  is a matching, no such pair of edges exists by definition, so  $Z$  must also be a matching.

## Question 2

Let  $G = (V, E)$  be a graph with  $n$  nodes, where each node  $v \in V$  is incident on exactly 2 edges. The graph is isomorphic to an  $n$ -gon. For example, the case of  $n = 4$  could be drawn as a square.

In the case of even  $n$ , there must exist two maximum matchings of size  $\frac{n}{2}$ , which are complements of each other in  $E$ , so  $M_2 = E \setminus M_1$ .

In the case of odd  $n$ , we still get two complementary matchings, but they are not large enough. Take the case of  $n = 5$  for example,



Both of the subsets highlighted in red are matchings, but both are maximum and of size 2. It is clear that in the case of odd  $n$ , a maximum matching has size  $\left\lfloor \frac{n}{2} \right\rfloor$ .

Therefore it is false that there must exist a matching  $M \subset E$  with  $|M| \geq \frac{n}{2}$ .

### Question 3

Let  $G = (V, E)$  be a graph, and  $M$  be a maximal matching on  $G$ , so every edge in  $E \setminus M$  has at least one endpoint that is matched under  $M$ , meaning  $M$  cannot be extended. Also let  $M^*$  be a matching of maximum size on  $G$ .

Is it true that  $|M| \geq \frac{1}{2}|M^*|$ ? Yes.

Suppose we have a situation where  $|M| < \frac{1}{2}|M^*|$ . Let  $|M| = \ell$  and  $|M^*| = k$  so that  $M$  matches  $2\ell$  nodes and  $M^*$  matches  $2k$  nodes. The inequality implies  $\ell < \frac{1}{2}k \iff 2\ell < k$ .

There are at most  $2\ell$  edges in  $M^*$  which are matched by  $M$ . But since  $2\ell < k$ , there is at least one edge in  $M^*$  which is not matched by  $M$ . Therefore we can add this edge to  $M$ , meaning it is not maximal. That's a contradiction, therefore  $|M| \geq \frac{1}{2}|M^*|$ .

## Question 4

Let  $G = (L \cup R, E)$  be a bipartite graph where every edge connects one node in  $L$  to one node in  $R$ , and let  $N_G(X)$  denote the neighbours of some set  $X$  of nodes in  $G$ . Also  $G$  has the property that for all  $A \subset L$ ,  $|N_G(A)| \geq \frac{1}{2}|A|$ .

We want to know if there exists a subset  $H \subset E$  where  $|H| = |L|$ , every node in  $L$  is incident upon exactly one edge in  $H$ , and every node in  $R$  is incident upon at most two edges in  $H$ .

To satisfy the first two properties, we require that  $H$  is constructed by considering each node in  $L$  and choosing one of the edges that connects to it.

Is it possible that there exists a  $v \in R$  which is incident on three edges in  $H$ ? Suppose such a  $v$  does exist. Then those three edges in  $H$  would connect to three distinct nodes in  $L$ , call them  $S = \{u_1, u_2, u_3\}$ . But by the neighbour requirement of  $G$ , we have  $|N_G(S)| \geq \frac{1}{2}|S|$ .

By construction, all three nodes connect to the same  $v \in R$  and no other nodes, so  $N_G(S) = 1$ . Therefore we have  $1 \geq \frac{3}{2}$ , which is a contradiction.

Therefore we cannot have a  $v \in R$  which is incident on three edges in  $H$ . Therefore every node in  $R$  is incident on at most two edges in  $H$ , so the statement is true.