MA139 Analysis 2, Assignment 3

Dyson Dyson

Question 1

$$f(t) = \log\left(\frac{1+t}{1-t}\right) - 2t = \log(1+t) - \log(1-t) - 2t$$

Therefore

$$\begin{aligned} f'(t) &= \frac{1}{1+t} - (-1)\frac{1}{1-t} - 2 \\ &= \frac{1-t+1+t}{1-t^2} - 2 \\ &= \frac{2-2(1-t^2)}{1-t^2} \\ &= \frac{2t^2}{1-t^2} \end{aligned}$$

In the range $t \in (-1,1)$, $t^2 \in (0,1)$. Therefore $2t^2 > 0$ and $1-t^2 > 0$, so $f'(t) > 0$. Therefore $f(t)$ is increasing for $t \in (-1,1)$.

$f(0) = \log(1) - 0 = 0$ and since $f(t)$ is increasing, $f(t) \geq 0$ for $t \in [0,1)$. Therefore $\log\left(\frac{1+t}{1-t}\right) - 2t \geq 0 \implies \log\left(\frac{1+t}{1-t}\right) \geq 2t$ for $0 \leq t < 1$ as required.

Let $t = \dfrac{1}{2x+1}$. Then

$$\begin{aligned} \frac{1+t}{1-t} &= \frac{1+\frac{1}{2x+1}}{1-\frac{1}{2x+1}} \\ &= \frac{2x+1+1}{2x+1-1} \\ &= \frac{2x+2}{2x} \\ &= 1+\frac{1}{x} \end{aligned}$$

Therefore

$$\log\left(\frac{1+t}{1-t}\right) \geq 2t \implies \log\left(1+\frac{1}{x}\right) \geq \frac{2}{2x+1}$$

for the condition

$$0 \le t < 1$$

$$0 \le \frac{1}{2x+1} < 1$$

$$0 \le 1 < 2x+1$$

$$0 < 2x$$

$$0 < x$$

Then

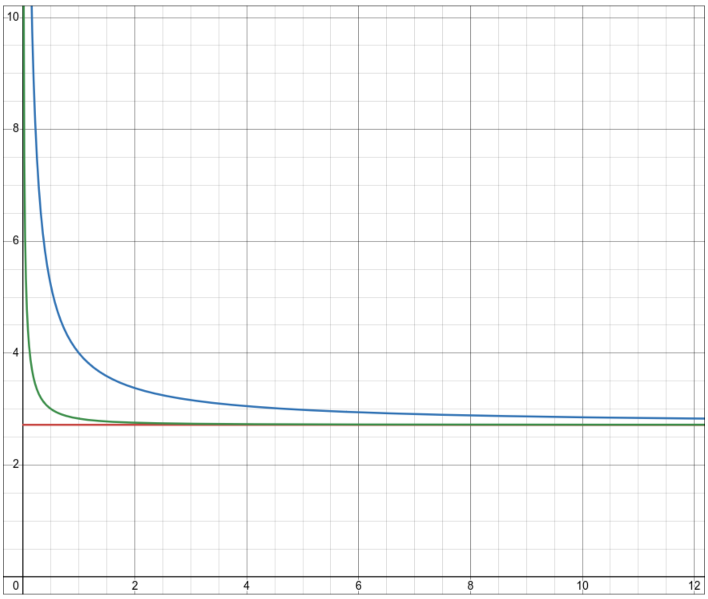
$$\log\left(1+\frac{1}{x}\right) \ge \frac{2}{2x+1}$$

$$(2x+1)\log\left(1+\frac{1}{x}\right) \ge 2$$

$$(x+\frac{1}{2})\log\left(1+\frac{1}{x}\right) \ge 1$$

$$\log\left(\left(1+\frac{1}{x}\right)^{x+\frac{1}{2}}\right) \ge 1$$

$$\left(1+\frac{1}{x}\right)^{x+\frac{1}{2}} \ge \mathrm{e}$$



Question 2

$$y = \frac{1}{\sqrt{x}} - \frac{1}{x} \qquad x \in (0, \infty)$$

For $x \in (0,1)$, $\sqrt{x} > x$, so $y < 0$. And for $x > 1$, $x > \sqrt{x}$, so $y > 0$. Clearly the maximum will be when $y > 0$, so $x > 1$.

The derivative is

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{1}{2}x^{-\frac{3}{2}} + x^{-2} \\ &= -\frac{1}{2\sqrt{x^3}} + \frac{1}{x^2}\end{aligned}$$

This equals 0 when $x^2 = 2\sqrt{x^3} \implies x^4 = 4x^3 \implies x = 4$. Therefore $x = 4$ is the only extremum point of the function with $x > 1$. The value at this point is

$$\frac{1}{\sqrt{4}} - \frac{1}{4} = \frac{1}{4}$$

We can evaluate the derivative at either side of $x = 4$ to show that y is increasing on the left and decreasing on the right, therefore $x = 4$ is the maximum.

$$\begin{aligned}\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=3} &= -\frac{1}{2\sqrt{27}} + \frac{1}{9} \\ &= -\frac{1}{6\sqrt{3}} + \frac{1}{9} \\ &= \frac{6\sqrt{3} - 9}{54\sqrt{3}} \\ &= \frac{2\sqrt{3} - 3}{18\sqrt{3}} \\ &= \frac{2 - \sqrt{3}}{18}\end{aligned}$$

$$3 < 4 \implies \sqrt{3} < \sqrt{4} = 2$$

$$\therefore \frac{2 - \sqrt{3}}{18} > 0$$

$$
\begin{aligned}
\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=5} &= -\frac{1}{2\sqrt{125}} + \frac{1}{25} \\
&= -\frac{1}{10\sqrt{5}} + \frac{1}{25} \\
&= \frac{10\sqrt{5} - 25}{250\sqrt{5}} \\
&= \frac{2\sqrt{5} - 5}{50\sqrt{5}} \\
&= \frac{2 - \sqrt{5}}{50} \\
5 > 4 &\implies \sqrt{5} > \sqrt{4} = 2 \\
\therefore\ \frac{2 - \sqrt{5}}{50} &< 0
\end{aligned}
$$

Therefore $x = 4$, $y = \dfrac{1}{4}$ is the maximum of this function.

Question 3

Let

$$f(n) = \frac{n}{t} \log\left(1 + \frac{t}{n}\right)$$

And therefore

$$
\begin{aligned}
f'(n) &= \frac{1}{t} \log\left(1 + \frac{t}{n}\right) + \frac{n}{t} \frac{1}{1 + \frac{t}{n}} \left(-\frac{t}{n^2}\right) \\
&= \frac{1}{t} \log\left(1 + \frac{t}{n}\right) + \frac{-nt}{t\left(1 + \frac{t}{n}\right) n^2} \\
&= \frac{1}{t} \log\left(1 + \frac{t}{n}\right) - \frac{nt}{tn^2 + t^2 n} \\
&= \frac{1}{t} \log\left(1 + \frac{t}{n}\right) - \frac{1}{n + t}
\end{aligned}
$$

Since we only care about what happens as $n \to \infty$, we can choose to only consider $n > 0$. We will split t into two cases, $t > 0$ and $t < 0$.

Since when $t > 0$, $f(n) < 1$ and $f(n)$ is increasing, we must have $\lim_{n \to \infty} f(n) = 1$, as required.

Since when $t < 0$, $f(n) > 1$ and $f(n)$ is decreasing, we must have $\lim_{n\to\infty} f(n) = 1$, as required.

From the previous result, we get

$$\log\left(\left(1+\frac{t}{n}\right)^n\right) \to t$$

We want to "apply exp to both sides" to get the desired result. We are allowed to do this because exp is a monotonic function, so it preserves limits.