MA139 Analysis 2, Assignment 3

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Question 1

Let $f\colon (-1,1) \to \mathbb{R}$ be the function defined by

$$f(t) = \log\left(\frac{1+t}{1-t}\right) - 2t = \log(1+t) - \log(1-t) - 2t.$$

Assuming knowledge of the derivative of log show that f is increasing on $(-1,1)$.

Deduce that $\log\left(\frac{1+t}{1-t}\right) \geq 2t$ for $0 \leq t < 1$.

Prove that if $x > 0$ then $\log\left(1 + \frac{1}{x}\right) \geq \frac{2}{2x+1}$.

Deduce that for each positive x, $\left(1 + \frac{1}{x}\right)^{x+\frac{1}{2}} \geq \mathrm{e}$.

You already saw that $\left(1 + \frac{1}{x}\right)^{x+1} \geq \mathrm{e}$.

Draw a graph of the two functions $x \mapsto \left(1 + \frac{1}{x}\right)^{x+\frac{1}{2}}$ and $x \mapsto \left(1 + \frac{1}{x}\right)^{x+1}$ for $x > 0$ and the horizontal line $y = \mathrm{e}$ to see how much more accurate the new inequality is. (This gives you some idea of the power of the derivative.)

$$\begin{aligned} f'(t) &= \frac{1}{1+t} - (-1)\frac{1}{1-t} - 2 \\ &= \frac{1-t+1+t}{1-t^2} - 2 \\ &= \frac{2 - 2(1-t^2)}{1-t^2} \\ &= \frac{2t^2}{1-t^2} \end{aligned}$$

In the range $t \in (-1, 1)$, $t^2 \in (0, 1)$. Therefore $2t^2 > 0$ and $1 - t^2 > 0$, so $f'(t) > 0$. Therefore $f(t)$ is increasing for $t \in (-1, 1)$.

$f(0) = \log(1) - 0 = 0$ and since $f(t)$ is increasing, $f(t) \geq 0$ for $t \in [0, 1)$. Therefore $\log\left(\frac{1+t}{1-t}\right) - 2t \geq 0 \implies \log\left(\frac{1+t}{1-t}\right) \geq 2t$ for $0 \leq t < 1$ as required.

Let $t = \dfrac{1}{2x+1}$. Then

$$
\begin{aligned}
\frac{1+t}{1-t} &= \frac{1 + \frac{1}{2x+1}}{1 - \frac{1}{2x+1}} \\
&= \frac{2x+1+1}{2x+1-1} \\
&= \frac{2x+2}{2x} \\
&= 1 + \frac{1}{x}
\end{aligned}
$$

Therefore

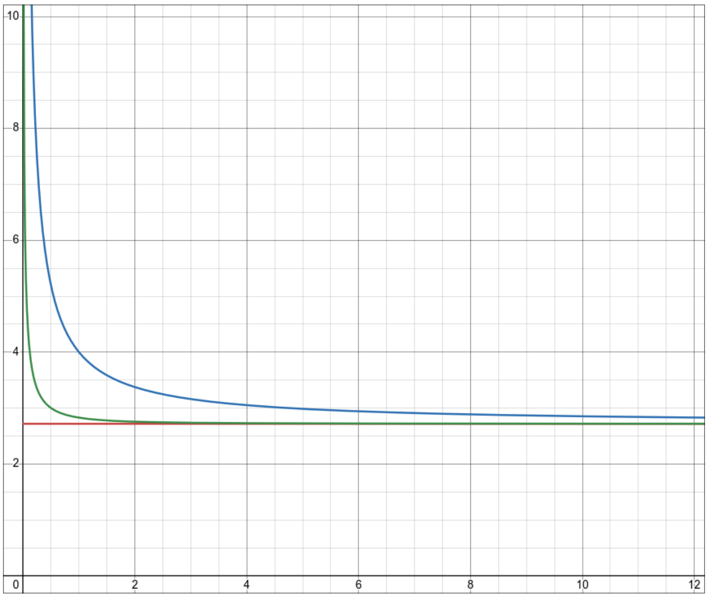
$$\log\left(\frac{1+t}{1-t}\right) \geq 2t \implies \log\left(1 + \frac{1}{x}\right) \geq \frac{2}{2x+1}$$

for the condition

$$
\begin{aligned}
&0 \leq t < 1 \\
&0 \leq \frac{1}{2x+1} < 1 \\
&0 \leq 1 < 2x+1 \\
&0 < 2x \\
&0 < x
\end{aligned}
$$

Then

$$
\begin{aligned}
\log\left(1 + \frac{1}{x}\right) &\geq \frac{2}{2x+1} \\
(2x+1)\log\left(1 + \frac{1}{x}\right) &\geq 2 \\
(x + \frac{1}{2})\log\left(1 + \frac{1}{x}\right) &\geq 1 \\
\log\left(\left(1 + \frac{1}{x}\right)^{x+\frac{1}{2}}\right) &\geq 1 \\
\left(1 + \frac{1}{x}\right)^{x+\frac{1}{2}} &\geq \mathrm{e}
\end{aligned}
$$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Question 2

> Find the maximum value of $y = \dfrac{1}{\sqrt{x}} - \dfrac{1}{x}$ on $(0, \infty)$.

For $x \in (0,1)$, $\sqrt{x} > x$, so $y < 0$. And for $x > 1$, $x > \sqrt{x}$, so $y > 0$. Clearly the maximum will be when $y > 0$, so $x > 1$.

The derivative is

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{1}{2}x^{-\frac{3}{2}} + x^{-2} \\ &= -\frac{1}{2\sqrt{x^3}} + \frac{1}{x^2}\end{aligned}$$

This equals 0 when $x^2 = 2\sqrt{x^3} \implies x^4 = 4x^3 \implies x = 4$. Therefore $x = 4$ is the only extremum point of the function with $x > 1$. The value at this point is

$$\frac{1}{\sqrt{4}} - \frac{1}{4} = \frac{1}{4}$$

We can evaluate the derivative at either side of $x = 4$ to show that y is increasing on the left and decreasing on the right, therefore $x = 4$ is the maximum.

$$\begin{aligned}\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=3} &= -\frac{1}{2\sqrt{27}} + \frac{1}{9} \\ &= -\frac{1}{6\sqrt{3}} + \frac{1}{9} \\ &= \frac{6\sqrt{3} - 9}{54\sqrt{3}} \\ &= \frac{2\sqrt{3} - 3}{18\sqrt{3}} \\ &= \frac{2 - \sqrt{3}}{18} \\ 3 < 4 &\implies \sqrt{3} < \sqrt{4} = 2 \\ \therefore \frac{2 - \sqrt{3}}{18} &> 0\end{aligned}$$

$$\begin{aligned}
\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=5} &= -\frac{1}{2\sqrt{125}} + \frac{1}{25} \\
&= -\frac{1}{10\sqrt{5}} + \frac{1}{25} \\
&= \frac{10\sqrt{5} - 25}{250\sqrt{5}} \\
&= \frac{2\sqrt{5} - 5}{50\sqrt{5}} \\
&= \frac{2 - \sqrt{5}}{50}
\end{aligned}$$

$$5 > 4 \implies \sqrt{5} > \sqrt{4} = 2$$

$$\therefore \ \frac{2 - \sqrt{5}}{50} < 0$$

Therefore $x = 4$, $y = \dfrac{1}{4}$ is the maximum of this function.

Question 3

Use the differentiability of log at 1 to show that for each t

$$\frac{n}{t}\log\left(1+\frac{t}{n}\right)\to 1 \quad \text{as } n\to\infty$$

What property of the exponential function do you need (in addition to the fact that it is the inverse of the logarithm) to deduce that $\left(1+\frac{t}{n}\right)^n \to \mathrm{e}^t$?

Let

$$f(n)=\frac{n}{t}\log\left(1+\frac{t}{n}\right)$$

And therefore

$$\begin{aligned}
f'(n) &= \frac{1}{t}\log\left(1+\frac{t}{n}\right)+\frac{n}{t}\frac{1}{1+\frac{t}{n}}\left(-\frac{t}{n^2}\right)\\
&= \frac{1}{t}\log\left(1+\frac{t}{n}\right)+\frac{-nt}{t\left(1+\frac{t}{n}\right)n^2}\\
&= \frac{1}{t}\log\left(1+\frac{t}{n}\right)-\frac{nt}{tn^2+t^2n}\\
&= \frac{1}{t}\log\left(1+\frac{t}{n}\right)-\frac{1}{n+t}
\end{aligned}$$

Since we only care about what happens as $n\to\infty$, we can choose to only consider $n>0$. We will split t into two cases, $t>0$ and $t<0$.

Since when $t>0$, $f(n)<1$ and $f(n)$ is increasing, we must have $\lim_{n\to\infty} f(n)=1$, as required.

Since when $t<0$, $f(n)>1$ and $f(n)$ is decreasing, we must have $\lim_{n\to\infty} f(n)=1$, as required.

From the previous result, we get

$$\log\left(\left(1+\frac{t}{n}\right)^n\right)\to t$$

We want to "apply exp to both sides" to get the desired result. We are allowed to do this because exp is a monotonic function, so it preserves limits.