MA243 Geometry, Assignment 2

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Question 1

> Let R_L denote the reflection in the line L. Let H_P be the rotation through P by 180°. Prove that
>
> $$R_L \circ H_P = H_P \circ R_L \iff P \in L$$

For the $\Rightarrow$ direction, assume $R_L \circ H_P = H_P \circ R_L$ but suppose $P \notin L$. We can define P' as the point in L which minimises $d(P, P')$, so P' is the perpendicular projection of P onto L. Then consider a point Q where Q is closer to L than P is, but on the same side as P.

Then $Q_{HR} := (H_P \circ R_L)(Q)$ will be on the same side of L as P, but $Q_{RH} := (R_L \circ H_P)(Q)$ will be on the opposite side from P.

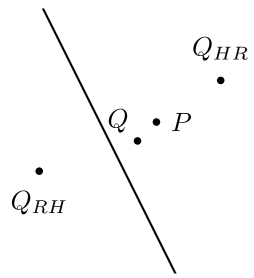

Figure 1: A sketch of the situtation

Conversely, for the $\Leftarrow$ direction, assume $P \in L$. Without loss of generality, we can work in $\mathbb{R}^2$ and assume $P = (0, 0)$, using an isometry to get to and from any Euclidean space. Since we're in $\mathbb{R}^2$, we can choose a unit vector $\underline{v}$ in the direction of L and an orthogonal unit vector $\underline{u}$ with $\underline{v} \cdot \underline{u} = 0$. Then $\underline{v}, \underline{u}$ is an orthonormal basis, and

$$R_L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad H_P = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

An arbitrary point Q can be written as $\lambda\underline{v} + \mu\underline{u}$. Then we can just compute

$$\begin{aligned}
R_L \circ H_P &= R_L(H_P(Q)) \\
&= R_L(H_P(\lambda\underline{v} + \mu\underline{u})) \\
&= R_L(-\lambda\underline{v} - \mu\underline{u}) \\
&= -\lambda\underline{v} + \mu\underline{u} \\
H_P \circ R_L &= H_P(R_L(Q)) \\
&= H_P(R_L(\lambda\underline{v} + \mu\underline{u})) \\
&= H_P(\lambda\underline{v} - \mu\underline{u}) \\
&= -\lambda\underline{v} + \mu\underline{u}
\end{aligned}$$

These are the same, so $R_L \circ H_P = H_P \circ R_L$.

$\square$

Question 2

Define the flat torus to be the metric space given by the set $T := [0,1) \times [0,1)$ together with the distance

$$d((x_1,y_1),(x_2,y_2)) = \min\left\{d_{\mathbb{R}^2}((x_1,y_1),(x_2+i,y_2+j)) : i,j \in \mathbb{Z}\right\}.$$

You can think of this as a square with the top and bottom edges identified, and the left and right sides identified. Here $d_{\mathbb{R}^2}$ is the usual Euclidean metric on $\mathbb{R}^2$. For example, on the flat torus, $(0.9,0)$ is distance 0.2 from the point $(0.1,0)$.

For $z = (x,y) \in \mathbb{R}^2$, define

$$\{z\} = (\{x\},\{y\})$$

where $\{a\} \in [0,1)$ is the fractional part of the real number a. This is the unique value in $[0,1)$ such that $a - \{a\}$ is an integer.

(a) For any $(a,b) \in \mathbb{R}^2$, define a translation function

$$\begin{aligned} f &: T \to T \\ (x,y) &\mapsto \{(x+a, y+b)\}. \end{aligned}$$

Prove that f is an isometry of T.

(b) Define a "rotation" function

$$\begin{aligned} g &: T \to T \\ \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto \left\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right\}. \end{aligned}$$

Prove that g is an isometry of T.

(c) Define a "rotation" function

$$\begin{aligned} h &: T \to T \\ \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto \left\{\begin{pmatrix} \cos(45°) & -\sin(45°) \\ \sin(45°) & \cos(45°) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right\}. \end{aligned}$$

Prove that h is *not* an isometry of T.

Isometries preserve distance, so in each case we want to show $d(\underline{u},\underline{v}) = d(f(\underline{u}), f(\underline{v}))$ for all $\underline{u},\underline{v} \in T$, or likewise with g or h for the other parts. Let $\underline{u} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$.

Q2 (a)

$$
\begin{aligned}
d(\underline{u}, \underline{v}) &= \min\left\{ d_{\mathbb{R}^2}\left(\underline{u}, \underline{v} + \begin{pmatrix} i \\ j \end{pmatrix}\right) : i, j \in \mathbb{Z} \right\} \\
d(f(\underline{u}), f(\underline{v})) &= d\left(\underline{u} + \begin{pmatrix} a \\ b \end{pmatrix}, \underline{v} + \begin{pmatrix} a \\ b \end{pmatrix}\right) \\
&= \min\left\{ d_{\mathbb{R}^2}\left(\underline{u} + \begin{pmatrix} a \\ b \end{pmatrix}, \underline{v} + \begin{pmatrix} i \\ j \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}\right) : i, j \in \mathbb{Z} \right\}
\end{aligned}
$$

Since $d_{\mathbb{R}^2}(\underline{x} + \underline{z}, \underline{y} + \underline{z}) = d_{\mathbb{R}^2}(\underline{x}, \underline{y})$, we get the desired result that $d(\underline{u}, \underline{v}) = d(f(\underline{u}), f(\underline{v}))$.

$\square$

Q2 (b)

$$
\begin{aligned}
d(\underline{u}, \underline{v}) &= \min\left\{ d_{\mathbb{R}^2}\left(\underline{u}, \underline{v} + \begin{pmatrix} i \\ j \end{pmatrix}\right) : i, j \in \mathbb{Z} \right\} \\
d(g(\underline{u}), g(\underline{v})) &= d\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \underline{u}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \underline{v}\right) \\
&= d\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) \\
&= d\left(\begin{pmatrix} -y_1 \\ x_1 \end{pmatrix}, \begin{pmatrix} -y_2 \\ x_2 \end{pmatrix}\right) \\
&= \min\left\{ d_{\mathbb{R}^2}\left(\begin{pmatrix} -y_1 \\ x_1 \end{pmatrix}, \begin{pmatrix} -y_2 \\ x_2 \end{pmatrix} + \begin{pmatrix} i \\ j \end{pmatrix}\right) : i, j \in \mathbb{Z} \right\}
\end{aligned}
$$

Since the components are kept separate, we can choose i and j independently for x and y, which allows this to be an isometry.

$\square$

Q2 (c)

$$
\begin{aligned}
d(\underline{u}, \underline{v}) &= \min \left\{ d_{\mathbb{R}^2} \left(\underline{u}, \underline{v} + \begin{pmatrix} i \\ j \end{pmatrix} \right) : i, j \in \mathbb{Z} \right\} \\
d(h(\underline{u}), h(\underline{v})) &= d \left(\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \underline{u}, \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \underline{v} \right) \\
&= d \left(\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right) \\
&= d \left(\frac{\sqrt{2}}{2} \begin{pmatrix} x_1 - y_1 \\ x_1 + y_1 \end{pmatrix}, \frac{\sqrt{2}}{2} \begin{pmatrix} x_2 - y_2 \\ x_2 + y_2 \end{pmatrix} \right) \\
&= \min \left\{ d_{\mathbb{R}^2} \left(\frac{\sqrt{2}}{2} \begin{pmatrix} x_1 - y_1 \\ x_1 + y_1 \end{pmatrix}, \frac{\sqrt{2}}{2} \begin{pmatrix} x_2 - y_2 \\ x_2 + y_2 \end{pmatrix} + \begin{pmatrix} i \\ j \end{pmatrix} \right) : i, j \in \mathbb{Z} \right\}
\end{aligned}
$$

Since the components of these vectors are combinations of the original components, we cannot choose i and j independently for x and y, so we can get stuck with some points which won't permit a good choice. Therefore there exist points $\underline{r}$ and $\underline{s}$ for which $d(\underline{r}, \underline{s}) \neq d(h(\underline{r}), h(\underline{s}))$, so h is not an isometry.

$\square$

Question 3

For a metric space (X, d), the group of isometries is defined to be

$$\mathrm{Isom}(X, d) = \{f : f \text{ is an isometry of } (X, d)\}$$

with group operation being composition of functions.

Prove that $\mathrm{Isom}(X, d)$ is a group.

For $\mathrm{Isom}(X, d)$ to be a group, it needs to satisfy the group rules:

1. Closure
2. Associativity
3. Existence of the identity
4. Existence of inverses

An isometry is, by definition, a distance-preserving bijective function. The composition of any two distance-preserving functions will of course also preserve distance, and thus be an isometry. So this group is closed.

Function composition is always associative, so this group is associative.

The identity in the group is the isometry that maps every point to itself.

Isometries are bijective by definition, and so all have inverses.

Since $\mathrm{Isom}(X, d)$ satisfies all the group rules, it is a group.

$\square$

Question 4

For any field k, we define

$$O(n,k) = \{A : A \text{ is an invertible } n \times n \text{ matrix with coefficients in } k, \text{ and } A^T A = I_n\}.$$

Prove that $O(n,k)$ is a group.

For $O(n,k)$ to be a group, it needs to satisfy the group rules:

1. Closure
2. Associativity
3. Existence of the identity
4. Existence of inverses

Let $A, B \in O(n,k)$. To have closure, we need $AB \in O(n,k)$. Since A and B are both $n \times n$ matrices, so is their product. The inverse is $(AB)^{-1} = B^{-1}A^{-1}$. And for the orthogonality condition:

$$\begin{aligned}(AB)^T(AB) &= B^T A^T AB \\ &= B^T(A^T A)B \\ &= B^T I_n B \\ &= B^T B \\ &= I_n\end{aligned}$$

Therefore AB satisfies all the requirements to be in $O(n,k)$, so $O(n,k)$ is closed.

Matrix multiplication is associate, so $O(n,k)$ is associative.

The identity in $O(n,k)$ is

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

where 1 is the additive identity in k and 0 is the multiplicative identity in k.

The restriction on the set requires that every matrix is invertible, so inverses are given by the definition.

Since $O(n,k)$ satisfies all the group rules, it is a group.

$\square$

Question 5

Let $T = T_{I_2,(4,0)}$ be the translation map

$$T(\underline{v}) = \underline{v} + \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

Find two rotations, R_1 and R_2, so that

$$T = R_1 \circ R_2.$$

What are the centres of both these rotations? Write out both rotations in the form $\underline{v} \mapsto A\underline{v} + \underline{b}$.

Is your solution unique?

We will rotate 180° about $(0, 0)$ and then 180° about $(2, 0)$. So

$$R_2(\underline{v}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \underline{v}$$

$$R_1(\underline{v}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \underline{v} + \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

As long as the second centre is offset by the first one by $(2, 0)$, and both rotations are by 180°, this will work, so this solution is not unique. Consider instead some arbitrary point (a, b):

$$R_2(\underline{v}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \underline{v} + 2 \begin{pmatrix} a \\ b \end{pmatrix}$$

$$R_1(\underline{v}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \underline{v} + 2 \begin{pmatrix} a+2 \\ b \end{pmatrix}$$

Then we get

$$\begin{aligned} T(\underline{v}) &= T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) \\ &= R_1\left(R_2\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)\right) \\ &= R_1\left(\begin{pmatrix} -x \\ -y \end{pmatrix} + 2\begin{pmatrix} a \\ b \end{pmatrix}\right) \\ &= R_1\left(\begin{pmatrix} -x+2a \\ -y+2b \end{pmatrix}\right) \\ &= \begin{pmatrix} x-2a \\ y-2b \end{pmatrix} + 2\begin{pmatrix} a+2 \\ b \end{pmatrix} \\ &= \begin{pmatrix} x+4 \\ y \end{pmatrix} \end{aligned}$$

which is exactly what we'd expect from the translation.