MA141 Analysis 1, Assignment 3

Dyson Dyson

Question 3

(a_n) is an increasing sequence and the subsequence $\left(a_{n_j}\right)$ converges to some $\ell \in \mathbb{R}$.

Since $\left(a_{n_j}\right) \to \ell$, that means $\exists \varepsilon > 0, N \in \mathbb{N}$ such that $|a_{n_j} - \ell| < \varepsilon \ \forall \ n_j \geq N$.

Since $n_{j+1} > n_j \ \forall \ j \in \mathbb{N}$ and $\ell - \varepsilon < a_{n_j} < \ell + \varepsilon \ \forall \ n_j \geq N$, (a_n) is bounded above by $\ell + \varepsilon$.

Therefore $|a_n - \ell| < \varepsilon \ \forall \ n \geq N$.

Question 10

I had absolutely no idea what to do with this one, sorry.

Q10 (a) $a_n = \dfrac{\sqrt{n+1}}{\sqrt{n^3+2}}$

For large n, $a_n \approx \dfrac{\sqrt{n}}{\sqrt{n^3}} = \dfrac{1}{n^2}$, so we expect $\sum a_n < \infty$.

Q10 (b) $a_n = \dfrac{n-3}{n^3+2}$

For large n, $a_n \approx \dfrac{1}{n^2}$, so we expect $\sum a_n < \infty$.

Question 15

We care about the convergence of $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^{\alpha}}$, so we will use the integral test.

$$\int_1^n \frac{1}{x(\log x)^{\alpha}} \mathrm{d}x = \left[\frac{(\log x)^{1-\alpha}}{1-\alpha} \right]_1^n = \frac{(\log n)^{1-\alpha}}{1-\alpha} \qquad \text{where } \alpha \neq 1$$

Since $(\log n)^{\beta} \to \infty$ exactly when $b > 0$, we know that the integral is bounded when $1 - \alpha > 0 \implies \alpha > 1$. And the integral is unbounded when $\alpha < 1$ and undefined when $\alpha = 1$.

Therefore $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^{\alpha}}$ converges when $\alpha > 1$ and diverges when $\alpha \leq 1$.

Question 16

Q16 (a) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n+1}$

The absolute version of this sum is the sum of reciprocals of odd numbers. Much like the Harmonic series, this series diverges to ∞, so the series does not converge absolutely.

It does however converge conditionally to $1-\frac{\pi}{4}$ thanks to the alternating minus signs.

Q16 (b) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$

The absolute version of this sum is

$$\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{1}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

This is the Basel problem, which famously equals $\frac{\pi^2}{6}$. Therefore this series is absolutely convergent, and therefore convergent.