MA144 Methods of Mathematical Modelling 2, Assignment 3

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Question 1

Consider the volume of the solid S bounded between the plane $z = 1$ and the surface $z = x^2 + 4y^2$.

Q1 (a)

Express the volume as a triple integral in Cartesian coordinates. Do this in *three* ways with different orders of integration.

$$S = \int_0^1 \int_{-\frac{\sqrt{z}}{2}}^{\frac{\sqrt{z}}{2}} \int_{-\sqrt{z-4y^2}}^{\sqrt{z-4y^2}} \mathrm{d}x \ \mathrm{d}y \ \mathrm{d}z$$

$$S = \int_0^1 \int_{\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} \mathrm{d}y \ \mathrm{d}x \ \mathrm{d}z$$

$$S = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}} \int_{x^2+4y^2}^{1} \mathrm{d}z \ \mathrm{d}x \ \mathrm{d}y$$

Q1 (b)

Choose one of the triple integrals you wrote down in part **(a)**. Calculate the volume of the solid.

$$
\begin{aligned}
S &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}} \int_{x^2+4y^2}^{1} \mathrm{d}z \ \mathrm{d}x \ \mathrm{d}y \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}} \left(1 - x^2 - 4y^2\right) \mathrm{d}x \ \mathrm{d}y \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[x - \frac{1}{3}x^3 - 4xy^2\right]_{x=-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}} \mathrm{d}y \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(k - \frac{1}{3}k(1-4y^2) - 4ky^2 - \left(-k + \frac{1}{3}k(1-4y^2) + 4ky^2\right)\right) \mathrm{d}y \\
&\quad \left(\text{where } k = \sqrt{1-4y^2}\right) \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{1-4y^2} \left(1 - \frac{1}{3}(1-4y^2) - 4y^2 + 1 - \frac{1}{3}(1-4y^2) - 4y^2\right) \mathrm{d}y \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{1-4y^2} \left(2 - \frac{2}{3}(1-4y^2) - 8y^2\right) \mathrm{d}y \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{1-4y^2} \left(\frac{4}{3} + \frac{8}{3}y^2 - \frac{24}{3}y^2\right) \mathrm{d}y \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{1-4y^2} \left(\frac{4}{3} - \frac{16}{3}y^2\right) \mathrm{d}y \\
&= \frac{4}{3} \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{1-4y^2} \left(1-4y^2\right) \mathrm{d}y \\
&= \frac{4}{3} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(1-4y^2\right)^{\frac{3}{2}} \mathrm{d}y
\end{aligned}
$$

I don't know how to continue, but SageMath says this integral is $\frac{\pi}{4}$.

Question 2

A thin metal plate has mass M and uniform density ρ (where density in this case equals mass per unit area). The boundary of the plate is the polar curve $r = 1 + \sin\theta$.

Q2 (a)

Show that $M = \dfrac{3\pi\rho}{2}$.

$M = \displaystyle\iint_\Omega \rho \,\mathrm{d}A$ where Ω is the region defined by the curve $r = 1+\sin\theta$. Therefore

$$
\begin{aligned}
M &= \int_0^{2\pi}\int_0^{1+\sin\theta} \rho r \ \mathrm{d}r \ \mathrm{d}\theta \\
&= \frac{\rho}{2}\int_0^{2\pi} \left[r^2\right]_0^{1+\sin\theta} \ \mathrm{d}\theta \\
&= \frac{\rho}{2}\int_0^{2\pi} \left(1 + 2\sin\theta + \sin^2\theta\right) \ \mathrm{d}\theta \\
&= \frac{\rho}{2}\int_0^{2\pi} \left(1 + 2\sin\theta + \frac{1}{2} - \frac{1}{2}\cos 2\theta\right) \ \mathrm{d}\theta \\
&= \frac{\rho}{4}\int_0^{2\pi} \left(3 + 4\sin\theta - \cos 2\theta\right) \ \mathrm{d}\theta \\
&= \frac{\rho}{4}\left[3\theta - 4\cos\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi} \\
&= \frac{\rho}{4}\left(6\pi - 4 + 0 - (0 - 4 - 0)\right) \\
&= \frac{6\pi\rho}{4} \\
&= \frac{3\pi\rho}{2}
\end{aligned}
$$

Q2 (b)

Locate the centre of mass $(\overline{x}, \overline{y})$ of the plate.

$\overline{x} = \dfrac{1}{M}\displaystyle\iint_\Omega x\rho \,\mathrm{d}A$ and $\overline{y} = \dfrac{1}{M}\displaystyle\iint_\Omega y\rho \,\mathrm{d}A$, where $x = r\cos\theta$ and $y = r\sin\theta$.

Therefore

$$
\begin{aligned}
\overline{x} &= \frac{2}{3\pi\rho}\int_0^{2\pi}\int_0^{1+\sin\theta}\rho r^2\cos\theta\ \mathrm{d}r\ \mathrm{d}\theta \\
&= \frac{2}{9\pi}\int_0^{2\pi}\cos\theta\big[r^3\big]_0^{1+\sin\theta}\ \mathrm{d}\theta \\
&= \frac{2}{9\pi}\int_0^{2\pi}\cos\theta\left(1+3\sin\theta+3\sin^2\theta+\sin^3\theta\right)\ \mathrm{d}\theta \\
&= \frac{2}{9\pi}\int_0^{2\pi}\left(\cos\theta+3\sin\theta\cos\theta+3\sin^2\theta\cos\theta+\sin^3\theta\cos\theta\right)\ \mathrm{d}\theta \\
&= \frac{2}{9\pi}\left[\sin\theta+\frac{3}{2}\sin^2\theta+\sin^3\theta+\frac{1}{4}\sin^4\theta\right]_0^{2\pi} \\
&= \frac{2}{9\pi}\times 0 \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{y} &= \frac{2}{3\pi\rho}\int_0^{2\pi}\int_0^{1+\sin\theta} \rho r^2 \sin\theta \ \mathrm{d}r \ \mathrm{d}\theta \\
&= \frac{2}{9\pi}\int_0^{2\pi} \sin\theta \big[r^3\big]_0^{1+\sin\theta} \ \mathrm{d}\theta \\
&= \frac{2}{9\pi}\int_0^{2\pi} \sin\theta \left(1 + 3\sin\theta + 3\sin^2\theta + \sin^3\theta\right) \ \mathrm{d}\theta \\
&= \frac{2}{9\pi}\int_0^{2\pi} \left(\sin\theta + 3\sin^2\theta + 3\sin^3\theta + \sin^4\theta\right) \ \mathrm{d}\theta \\
&= \frac{2}{9\pi}\int_0^{2\pi} \left(\sin\theta + \frac{3}{2} - \frac{3}{2}\cos 2\theta + 3\sin\theta\left(1-\cos^2\theta\right) + \left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)^2\right) \ \mathrm{d}\theta \\
&= \frac{2}{9\pi}\int_0^{2\pi} \left(4\sin\theta + \frac{3}{2} - \frac{3}{2}\cos 2\theta - 3\sin\theta\cos^2\theta + \frac{1}{4} - \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos^2 2\theta\right) \ \mathrm{d}\theta \\
&= \frac{2}{9\pi}\int_0^{2\pi} \left(4\sin\theta + \frac{7}{4} - 2\cos 2\theta - 3\sin\theta\cos^2\theta + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos 4\theta\right)\right) \ \mathrm{d}\theta \\
&= \frac{2}{9\pi}\int_0^{2\pi} \left(4\sin\theta + \frac{7}{4} - 2\cos 2\theta - 3\sin\theta\cos^2\theta + \frac{1}{8} + \frac{1}{8}\cos 4\theta\right) \ \mathrm{d}\theta \\
&= \frac{2}{9\pi}\int_0^{2\pi} \left(4\sin\theta + \frac{15}{8} - 2\cos 2\theta - 3\sin\theta\cos^2\theta + \frac{1}{8}\cos 4\theta\right) \ \mathrm{d}\theta \\
&= \frac{2}{9\pi}\left[-4\cos\theta + \frac{15}{8}\theta - \sin^3\theta + \frac{1}{32}\sin 4\theta\right]_0^{2\pi} \\
&= \frac{2}{9\pi}\left(-4 + \frac{15\pi}{4} + 0 - (-4+0)\right) \\
&= \frac{2}{9\pi}\,\frac{15\pi}{4} \\
&= \frac{5}{6}
\end{aligned}
$$

Question 3

Q3 (a)

The triple integral below represents the volume of a solid in $\mathbb{R}^3$.

$$\int_0^{2\pi} \int_0^1 \int_{r^2}^{3-2r} r \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta.$$

Sketch the solid and evaluate the triple integral.

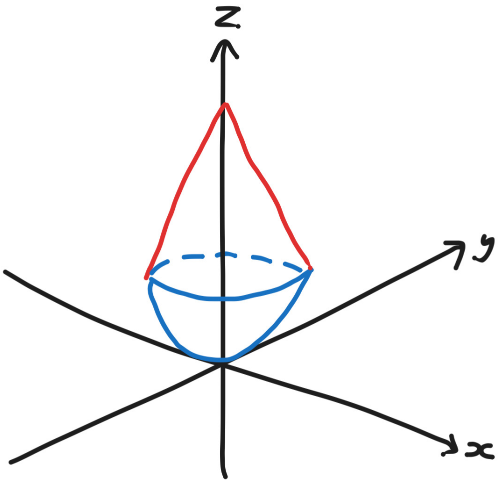

Figure 1: A sketch of the solid

$$\begin{aligned}
V &= \int_0^{2\pi} \int_0^1 \int_{r^2}^{3-2r} r \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta \\
&= \int_0^{2\pi} \int_0^1 [rz]_{z=r^2}^{3-2r} \, \mathrm{d}r \, \mathrm{d}\theta \\
&= \int_0^{2\pi} \int_0^1 r\left(3 - 2r - r^2\right) \mathrm{d}r \, \mathrm{d}\theta \\
&= \int_0^{2\pi} \int_0^1 \left(3r - 2r^2 - r^3\right) \mathrm{d}r \, \mathrm{d}\theta \\
&= \int_0^{2\pi} \left[\frac{3}{2}r^2 - \frac{2}{3}r^3 - \frac{1}{4}r^4\right]_0^1 \mathrm{d}\theta
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{2\pi} \left(\frac{3}{2} - \frac{2}{3} - \frac{1}{4} \right) \mathrm{d}\theta \\
&= \int_0^{2\pi} \left(\frac{18}{12} - \frac{8}{12} - \frac{3}{12} \right) \mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{7}{12} \,\mathrm{d}\theta \\
&= \frac{7\pi}{6}
\end{aligned}
$$

Q3 (b)

> Evaluate the integral numerically using SciPy's `tplquad` function. How accurate is Python's answer? (i.e. to how many decimal places?)

SciPy gives 3.6651914291880914 with an estimated error of approximately 4.07×10^{-14}. The actual answer is $\frac{7\pi}{6} = 3.665191429188092$, so SciPy is correct to 14 decimal places, which also matches the estimated error.

```python
#!/usr/bin/env python3

from math import pi

from scipy.integrate import tplquad

print(
    "SciPy gives:",
    tplquad(
        lambda _z, r, _t: r,
        0,
        2 * pi,
        0,
        1,
        lambda _t, r: r * r,
        lambda _t, r: 3 - 2 * r,
    ),
)

print("Actual answer:", 7 * pi / 6)
```

Figure 2: The Python code used to generate this answer