MA150 Algebra 2, Assignment 1

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Question 1

By any means you like, solve the following system of simultaneous linear equations

$$\begin{aligned} 2x - 5y &= b_1 \\ x - 3y &= b_2 \end{aligned}$$

in each of the following three cases.

Q1 (a) $b_1 = 0$, $b_2 = 0$

$$\begin{aligned} 2x - 5y &= 0 && (1) \\ x - 3y &= 0 && (2) \end{aligned}$$

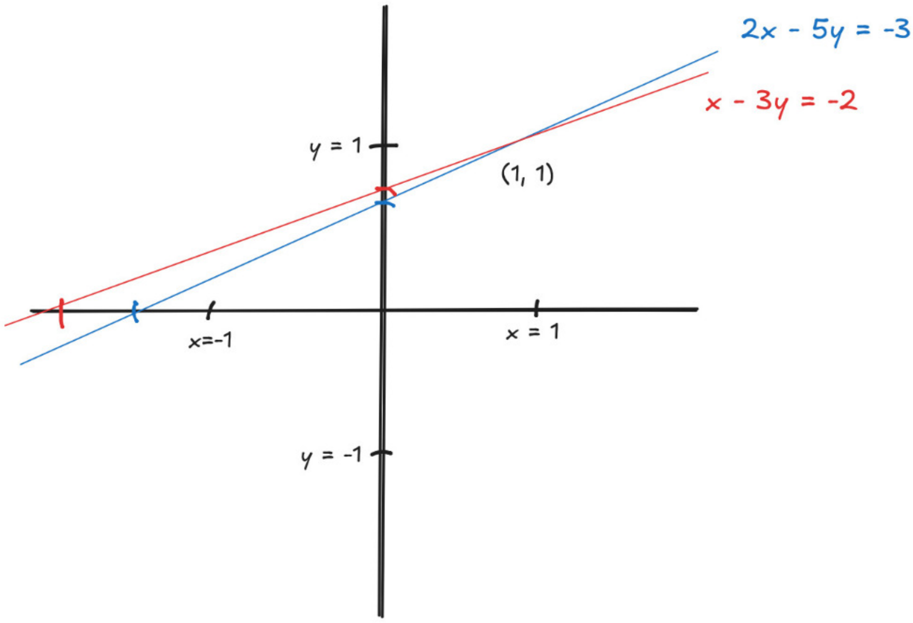
If a solution exists, equation (2) implies $x = 3y$. Plugging this into (1) gives $6x - 5y = 0$, so $y = 0$. Plugging this into either equation gives $x = 0$. So if a solution exists, it must be $x = 0$, $y = 0$. Indeed, this solution satisfies both simultaneous equations.

Q1 (b) $b_1 = -3$, $b_2 = -2$

$$\begin{aligned} 2x - 5y &= -3 && (1) \\ x - 3y &= -2 && (2) \end{aligned}$$

If a solution exists, $(1) - 2 \times (2)$ implies $0x + y = 1$, so $y = 1$. Plugging this into either equation gives $x = 1$. So if a solution exists, it must be $x = 1$, $y = 1$. Indeed, this solution satisfies both simultaneous equations.

For your solution – to **(b)**, sketch (reasonably accurately) the lines $2x-5y = -3$ and $x - 3y = -2$ in the plane $\mathbb{R}^2$ (labelled so as to be instantly unambiguous) and confirm that your solution is the point where they intersect.



Q1 (c) $b_1 = \lambda$, $b_2 = \mu$ $\quad \lambda, \mu \in \mathbb{R}$

$$2x - 5y = \lambda \tag{1}$$
$$x - 3y = \mu \tag{2}$$

If a solution exists, $(1) - 2 \times (2)$ implies $0x + y = \lambda - 2\mu$, so $y = \lambda - 2\mu$. Plugging this into (2) gives $x = \mu + 3(\lambda - 2\mu) = 3\lambda - 5\mu$. So if a solution exists, it must be $x = 3\lambda - 5\mu$, $y = \lambda - 2\mu$. Indeed, this solution satisfies both simultaneous equations:

$$2(3\lambda - 5\mu) - 5(\lambda - 2\mu) = 6\lambda - 10\mu - 5\lambda + 10\mu = \lambda$$
$$(3\lambda - 5\mu) - 3(\lambda - 2\mu) = 3\lambda - 5\mu - 3\lambda + 6\mu = \mu$$

Question 2

Solve the following linear equations

$$\begin{aligned} x + y + z &= b_1 \\ x - 2y + 3z &= b_2 \end{aligned}$$

in each of the following cases.

In each case there will be infinitely many solutions, with 1 degree of freedom, and you should find **all** solutions.

Q2 (a) $b_1 = 0$, $b_2 = 0$

$$x + y + z = 0 \tag{1}$$
$$x - 2y + 3z = 0 \tag{2}$$

The pair of simultaneous equations describes a straight line in $\mathbb{R}^3$. Trivially $x = 0$, $y = 0$, $z = 0$ is a solution, so the line must pass through the origin.

To find another solution, $(1) - (2)$ implies $3y - 2z = 0$, so for any solution, we require $z = \frac{3}{2}y$. Plugging this into either equation (1) or (2) gives $x + \frac{5}{2}y = 0$. By observation, a solution for this is $x = 5$, $y = -2$. Since $z = \frac{3}{2}y$, this potential solution would also have $z = -3$. Indeed, this solution satisfies both simultaneous equations, so the line passes through $(0, 0, 0)$ and $(5, -2, -3)$.

We can parametrise the line as $\lambda \begin{pmatrix} 5 \\ -2 \\ -3 \end{pmatrix}$ for $\lambda \in \mathbb{R}$.

Q2 (b) $b_1 = 2$, $b_2 = 3$

$$x + y + z = 2 \tag{1}$$
$$x - 2y + 3z = 3 \tag{2}$$

The pair of simultaneous equations describes a straight line in $\mathbb{R}^3$. $(1) - (2)$ implies $3y - 2z = -1$, so any solution must have $z = \frac{3}{2}y + \frac{1}{2}$. Plugging this into either equation (1) or (2) gives $x + \frac{5}{2}y = \frac{3}{2}$. By observation, one solution of this equation is $x = -1$, $y = 1$. This solution would require $z = 2$. Indeed, this solution satisfies both simultaneous equations, so the point $(-1, 1, 2)$ is on the line.

To find more points on the line, we can rearrange the previous equation in x and y to get $2x + 5y = 3$. So when $x = 0$, $y = \frac{3}{5}$ and this would imply $z = \frac{14}{10}$. Likewise, when $y = 0$, $x = \frac{3}{2}$ and this would imply $z = \frac{1}{2}$. We can check these and see that both $\left(0, \frac{3}{5}, \frac{14}{10}\right)$ and $\left(\frac{3}{2}, 0, \frac{1}{2}\right)$ are on the line.

I will choose $(-1,1,2)$ as the fixed point for my parametrisation. The vector from $(-1,1,2)$ to $\left(\frac{3}{2},0,\frac{1}{2}\right)$ is

$$\begin{pmatrix} \frac{3}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ -1 \\ -\frac{3}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 5 \\ -2 \\ -3 \end{pmatrix}$$

Therefore we can parametrise the line as $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -2 \\ -3 \end{pmatrix}$ for $\lambda \in \mathbb{R}$.

Question 3

Q3 (a)

Let $\underline{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\underline{w} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \in \mathbb{R}^2$. Find the component (a real number) of $\underline{v}$ in the direction of $\underline{w}$. Compute the (vector) orthogonal projection of $\underline{v}$ in the direction of $\underline{w}$.

The component of $\underline{v}$ in the direction of $\underline{w}$ is $\underline{v} \cdot \hat{\underline{w}}$. First, $\|\underline{w}\| = \sqrt{5}$, so $\hat{\underline{w}} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Therefore the component of $\underline{v}$ in the direction of $\underline{w}$ is $\frac{2}{\sqrt{5}} + \frac{1}{\sqrt{5}} = \frac{3}{\sqrt{5}}$.

The orthogonal projection of $\underline{v}$ in the direction of $\underline{w}$ is

$$(\underline{v} \cdot \hat{\underline{w}})\,\hat{\underline{w}} = \frac{3}{\sqrt{5}}\,\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{3}{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Q3 (b)

Let $\underline{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\underline{w} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \in \mathbb{R}^3$. Compute the orthogonal projection of $\underline{v}$ in the direction of $\underline{w}$.

The orthogonal projection of $\underline{v}$ in the direction of $\underline{w}$ is $(\underline{v} \cdot \hat{\underline{w}})\,\hat{\underline{w}}$.

Firstly, $\|\underline{w}\| = \sqrt{1+4+4} = 3$, so $\hat{\underline{w}} = \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$. Then $\underline{v} \cdot \hat{\underline{w}} = \frac{1}{3} - \frac{2}{3} - \frac{2}{3} = -1$.

Finally, the orthogonal projection of $\underline{v}$ in the direction of $\underline{w}$ is

$$(\underline{v} \cdot \hat{\underline{w}})\,\hat{\underline{w}} = -\hat{\underline{w}} = \frac{1}{3}\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

Question 4

> Let $P = \begin{pmatrix}1\\1\\1\end{pmatrix}$ and $Q = \begin{pmatrix}1\\-1\\3\end{pmatrix} \in \mathbb{R}^3$, on which we have coordinates x, yz.

Q4 (a)

> Give the equation of the plane $\Pi \in \mathbb{R}^3$ that passes through P, Q, and the origin $\underline{0} \in \mathbb{R}^3$.

The plane will have equation $\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \hat{\underline{n}} = 0$, where $\hat{\underline{n}}$ is a unit normal vector to the plane. We can find $\underline{n}$ with the cross product:

$$\underline{n} = \overrightarrow{P} \times \overrightarrow{Q} = \begin{pmatrix}1\\1\\1\end{pmatrix} \times \begin{pmatrix}1\\-1\\3\end{pmatrix} = \begin{pmatrix}4\\-2\\-2\end{pmatrix}$$

Then $\|\underline{n}\| = \sqrt{16+4+4} = 2\sqrt{6}$, so $\hat{\underline{n}} = \dfrac{1}{2\sqrt{6}} \begin{pmatrix}4\\-2\\-2\end{pmatrix} = \dfrac{1}{\sqrt{6}} \begin{pmatrix}2\\-1\\-1\end{pmatrix}$

Therefore Π has equation $\frac{1}{\sqrt{6}}(2x - y - z) = 0$. Or equivalently, $2x - y - z = 0$.

Q4 (b)

> Give two equations that together define the line L_{PQ} through P and Q.

We need two equations that define planes containing P and Q. We've already got the plane through the origin.

The point $(1, 0, 0)$ does not satisfy the equation of the plane from part **(a)**, so it is not on the plane Π. It also therefore not collinear with P and Q, so we can use it to find a different plane, Π_2.

For this plane, we need the normal vector from before, which we find slightly differently, since Π_2 doesn't include the origin, so we can't just use the position vectors of P and Q. Let's call $(1, 0, 0)$ the point X.

$$\begin{aligned}
\underline{n} &= \overrightarrow{XP} \times \overrightarrow{XQ} \\
&= \left(\overrightarrow{P} - \overrightarrow{X}\right) \times \left(\overrightarrow{Q} - \overrightarrow{X}\right) \\
&= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \\
\therefore \ \hat{\underline{n}} &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}$$

Then Π_2 is defined by $x+0y+0z = k$ for some constant k. Solving this equation with any of the points P, Q, or X gives $k = 1$. Therefore Π_2 is defined by $x = 1$.

Therefore the line through P and Q is defined as all the points that satisfy both equations:

$$\begin{aligned}
2x - y - z &= 0 \\
x &= 1
\end{aligned}$$

Equivalently, Π_2 is defined as all the points that satisfy both equations

$$\begin{aligned}
x &= 1 \\
y + z &= 2
\end{aligned}$$

Q4 (c)

> Giv the line L_{PQ} in parametrised form, for as example as $\{\underline{v} + \mu\underline{w} : \mu \in \mathbb{R}\}$ for specified $\underline{v}$ and $\underline{w}$.

The line L_{PQ} can be parametrised as $\overrightarrow{P} + \lambda\overrightarrow{PQ}$ for $\lambda \in \mathbb{R}$.

$$\overrightarrow{PQ} = \overrightarrow{Q} - \overrightarrow{P} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}$$

Therefore L_{PQ} can be parametrised as $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ for $\lambda \in \mathbb{R}$.