MA266 Multilinear Algebra, Assignment 2

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Question 11

Let V be a finite dimensional $\mathbb{F}$-vector space and let B be a basis for V. Let

$$M = (V \otimes V)_{-} := \mathbb{F}\left\langle \{v \otimes w - w \otimes v : v, w \in V\} \right\rangle.$$

Show that the set

$$\{b \otimes c + M : \{b, c\} \subset B \text{ of size at most } 2\} = \{b \otimes c + M : b, c \in B\}$$

is a basis for the symmetric square $S^2(V) = (V \otimes V)/M$.

Set $X := \{b_i \otimes b_j + M : 1 \leq i \leq j \leq n\}$. We need to prove that X is linearly independent and that X spans $S^2(V)$.

First, we prove linear independence. Assume $\exists \lambda_{i,j} \in \mathbb{F}$ such that

$$\sum_{1 \leq i < j \leq n} \lambda_{i,j}(b_i \otimes b_j + M) = 0_{S^2(V)}.$$

By the distributive laws for tensor products, we have

$$\sum_{1 \leq i < j \leq n} \lambda_{i,j}(b_i \otimes b_j) + M = 0_{S^2(V)}.$$

Equivalently,

$$\sum_{1 \leq i < j \leq n} \lambda_{i,j}(b_i \otimes b_j) \in M.$$

Thus, by the definition of M, $\exists \mu_{i,j}, \alpha_i \in \mathbb{F}$ such that

$$\sum_{i \leq i < j \leq n} \lambda_{i,j}(b_i \otimes b_j) = \sum_{i=1}^{n} \alpha_i(b_i \otimes b_i) + \sum_{i \neq j} \mu_{i,j}(b_i \otimes b_j - b_j \otimes b_i).$$

Hence,

$$\sum_{1 \leq i < j \leq n} \lambda_{i,j}\mu_{i,j}(b_i \otimes b_j) + \sum_{1 \leq i < j \leq n} \mu_{i,j}(b_j \otimes b_i) - \sum_{1=1}^{n} \alpha_i(b_i \otimes b_i) = 0_{V \otimes V}.$$

By linear independence of the basis $\{b_i \otimes b_j : 1 \leq i, j \leq n\}$ in $V \otimes V$, it follows that $\alpha_i = 0_{\mathbb{F}}$ for all i and $\mu_{i,j} = 0_{\mathbb{F}}$ for all $i > j$, and so $\lambda_{i,j} = 0_{\mathbb{F}}$ for all $1 \leq i < j \leq n$. Thus, X is linearly independent.

Secondly, we prove that X spans $S^2(V)$. So let $w + M \in S^2(V) = (V \otimes V)/M$. Then since $\{b_i \otimes b_j : 1 \leq i, j \leq n\}$ is a basis for $V \otimes V$ and $w \in V \otimes V$, $\exists \lambda_{i,j} \in \mathbb{F}$ such that

$$w = \sum_{1 \leq i,j \leq n} \lambda_{i,j}(b_i \otimes b_j).$$

Thus,

$$w = \sum_{i<j} \big(\lambda_{i,j}(b_i \otimes b_j) + \lambda_{i,j}(b_j \otimes b_i)\big) + \sum_{1 \leq i \leq n} \lambda_{i,i}(b_i \otimes b_i).$$

Clearly the first term on the RHS is in M and the second term is in $\mathrm{Diag}(V \otimes V)$. From lectures, we know $b_j \otimes b_i + D = -b_i \otimes b_j + D$. Thus, by definition of addition and scalar multiplication in a quotient vector space, we have

$$w + D = \sum_{1 \leq i < j \leq n} (\lambda_{i,j}\mu_{i,j})(b_i \otimes b_j) + D.$$

Therefore X spans V.

$\square$

Question 12

Let V be an $\mathbb{F}$-vector space. Define

$$(V \otimes V)_+ := \mathbb{F}\left\langle \{v \otimes w + w \otimes v : v, w \in V\} \right\rangle .$$

(i) Prove that $(V \otimes V)_+ \subset \mathrm{Diag}(V \otimes V)$.

(ii) Suppose that $1_\mathbb{F} \neq -1_\mathbb{F}$. Prove that $(V \otimes V)_+ = \mathrm{Diag}(V \otimes V)$.

(iii) Let $V = \mathbb{F}_2^2$ be the space of 2-dimensional column vectors over the Galois field $\mathbb{F}_2 = \{0, 1\}$. Show that $(V \otimes V)_+ \neq \mathrm{Diag}(V \otimes V)$.

Q12 (i)

Let $D := \mathrm{Diag}(V \otimes V)$.

We want to show that every element $v \otimes w + w \otimes v$ is in D. Every other element of $(V \otimes V)_+$ will follow as linear combinations.

Clearly $(v + w) \otimes (v + w)$ is an element of D and we can expand it with the distributive laws to see that

$$v \otimes v + v \otimes w + w \otimes v + w \otimes w$$

is an element of D. Since the first and last terms of this expression are clearly elements of D, the part in the middle, $v \otimes w + w \otimes v$ must also be an element of D by closure.

□

Q12 (ii)

Now we want to show that $D \subset (V \otimes V)_+$, so we want to show that every element $v \otimes v$ is in $(V \otimes V)_+$. Every other element of D will follow as linear combinations.

We can choose $v = w$ and see that $v \otimes v + v \otimes v$ is an element of $(V \otimes V)_+$. Since we're taking the span, we can just multiply this by $\frac{1}{2}$ and see that $v \otimes v$ is an element of $(V \otimes V)_+$ as required.

Note that if $1_\mathbb{F} = -1_\mathbb{F}$ then $v \otimes v + v \otimes v = 0_{V \otimes V}$ so this trick wouldn't work.

Since $(V \otimes V) \subset D$ and $D \subset (V \otimes V)_+$, we have $(V \otimes V)_+ = D$ as required.

□

Q12 (iii)

In this V, $1_{\mathbb{F}_2^2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $-1_{\mathbb{F}_2^2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus $1_{\mathbb{F}_2^2} = -1_{\mathbb{F}_2^2}$ and so we can't do our trick from **Q12 (ii)**. The logic in **Q12 (i)** still works, so we still have $(V \otimes V)_+ \subset D$.

The only way to get any element of the form $\lambda v \otimes v$ from $(V \otimes V)_+$ is to add an element to itself. Since $1_{\mathbb{F}_2^2} = -1_{\mathbb{F}_2^2}$, adding any element to itself will give zero, and so we cannot make those required elements from $(V \otimes V)_+$. Therefore $D \not\subset (V \otimes V)_+$ and so $(V \otimes V)_+ \neq D$.

$\square$