MA139 Analysis 2, Assignment 2

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Question 1

Let $(x_i)_1^n$ be a finite sequence of positive numbers whose mean is

$$m = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

Use the fact that for each positive t we have $\log t \geq 1 - \dfrac{1}{t}$ to show that

$$\frac{1}{n}\sum_{i=1}^{n} x_i \log x_i \geq m \log m.$$

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} x_i \log x_i &\geq \frac{1}{n}\sum_{i=1}^{n} x_i \left(1 - \frac{1}{x_i}\right) \\
&= \frac{1}{n}\sum_{i=1}^{n} (x_i - 1) \\
&= \frac{1}{n}\sum_{i=1}^{n} x_i - \frac{1}{n}\sum_{i=1}^{n} 1 \\
&= m - \frac{n}{n} \\
&= m - 1 \\
&= m\left(1 - \frac{1}{m}\right) \\
&\leq m \log m
\end{aligned}$$

This is obviously not correct, so I'll try a special case of $m = 1$. Then

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} x_i \log x_i &\geq \cdots \\ &= m - \frac{n}{n} \\ &= 1 - 1 \\ \therefore\ \frac{1}{n}\sum_{i=1}^{n} x_i \log x_i &\geq 0\end{aligned}$$

Also $m \log m = 1 \log 1 = 0$. Therefore $\frac{1}{n}\sum_{i=1}^{n} x_i \log x_i \geq m \log m$ when $m = 1$, as required.

Question 2

Prove that for each positive integer m,

$$\lim_{u\to\infty} \frac{u^m}{\mathrm{e}^u} = 0.$$

Let $f(u) = \frac{u^m}{\mathrm{e}^u}$ for some positive integer m. Then $\frac{\mathrm{d}f}{\mathrm{d}u} = \frac{mu^{m-1}\mathrm{e}^u - u^m\mathrm{e}^u}{\mathrm{e}^{2u}} = \frac{u^{m-1}(m-u)}{\mathrm{e}^u}$.

Consider $u > m > 0$. Then $f(u)$ is positive, since u^m and e^u are both positive. And $f'(u)$ is negative, since u^{m-1} and e^m are positive, but $m-u$ is negative. So for sufficiently large u, the function is always positive but its derivative is always negative. Therefore when $u > m$, f is a strictly decreasing function bounded below by 0, so $\lim_{u\to\infty} \frac{u^m}{\mathrm{e}^u} = 0$ for all m.

Question 3

Prove that

$$\lim_{x \to 0^+} \log x = -\infty.$$

Hint: How small does x have to be to guarantee that $\log x < -M$?

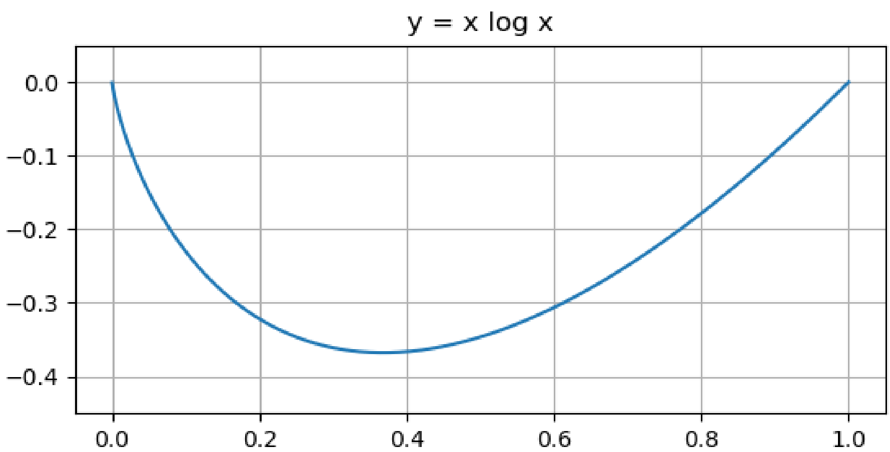
Plot a graph of the function $x \mapsto x \log x$ on the interval $(0, 1]$.

By taking $u = -\log x$ and using the previous question prove that (as the graph suggests)

$$\lim_{x \to 0^+} (x \log x) = 0.$$

What is $\lim_{x \to 0^+} x^x$?

We want to show that for all $M > 0$, there exists x such that $\log x < -M$. We can just choose any positive $x < \mathrm{e}^{-M}$. Since exp is a positive, strictly increasing function, e^{-M} will always be positive and will approach 0 as M grows. So $x \to 0^+$ as $M \to \infty$. Therefore $\lim_{x \to 0^+} \log x = -\infty$ as required.



Let $u = -\log x$. Then as $x \to 0^+$, $u \to \infty$.

Also $\dfrac{1}{e^u} = e^{\log x} = x$ so $\dfrac{u}{e^u} = -x \log x$. Therefore

$$\lim_{x \to 0^+} (x \log x) = -\lim_{u \to \infty} \frac{u}{e^u} = -0$$

by Question 2. Therefore $\lim_{x \to 0^+} (x \log x) = 0$ as required.

Note that $x^x = e^{x \log x}$, so

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{x \log x} = e^{\lim_{x \to 0^+} (x \log x)} = e^0 = 1$$