MA268 Algebra 3, Assignment 2

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Question 1

Let $n \geq 3$. Recall that $D_{2n} = \left\langle r, s \,\middle|\, r^n = s^2 = \mathrm{id}, srs = r^{-1} \right\rangle$. We follow the convention of writing elements of D_{2n} as r^k or sr^k where k only matters modulo n.

(i) Show that $r^k \cdot s = sr^{-k}$.

(ii) Complete the following multiplication rules for D_{2n}:

	r^k	sr^k
r^ℓ		
sr^ℓ		

Q1 (i)

We know that $srs = r^{-1}$, so we can derive that $rs = s^{-1}r^{-1} = sr^{-1}$, since s is self-inverse. Then we have

$$\begin{aligned} r^k s &= r^{k-1} rs \\ &= r^{k-1} sr^{-1} \\ &= r^{k-2} rsr^{-1} \\ &= r^{k-2} sr^{-2} \\ &\vdots \\ &= sr^{-k} \end{aligned}$$

Q1 (ii)

	r^k	sr^k
r^ℓ	$r^{\ell+k}$	$sr^{k-\ell}$
sr^ℓ	$sr^{\ell+k}$	$r^{k-\ell}$

Question 2

> Determine all homomorphisms $D_{2n} \to \mathbb{C}^*$.

To preserve the cyclic nature of rotations, we need to have $\phi(r) = \mathrm{e}^{2i\pi k/n}$ for some $k \in \{1, \dots, n\}$.

However, we cannot preserve the relation that $srs = r^{-1}$, since it requires some sort of reflection. Since every element of $\mathbb{C}^*$ geometrically represents a rotation, we cannot preserve reflections. Thus, no homomorphisms exists from D_{2n} to $\mathbb{C}^*$.

Question 3

Recall the rotation matrix $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. This represents anti-clockwise rotation through angle θ. It is obvious geometrically, and easy to check using trig identities that $R_\theta R_\phi = R_{\theta+\phi}$. Let $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(i) What does multiplying a vector by S do geometrically?

(ii) Show that there is a unique homomorphism

$$\rho : D_{2n} \to \mathrm{GL}_2(\mathbb{R}), \qquad \rho(r) = R_{2\pi/n}, \qquad \rho(s) = S.$$

Q3 (i)

Multiplying a vector by S reflects it in the x-axis.

Q3 (ii)

Clearly ρ is a homomorphism since it preserves the relations that define D_{2n}:

$$r^n = s^2 = \mathrm{id}, srs = r^{-1}$$

$$\begin{aligned} \rho(r^n) &= \rho(\mathrm{id}) \\ &= I \\ &= (R_{2\pi/n})^n \\ &= \rho(r)^n \\ \rho(s^2) &= \rho(\mathrm{id}) \\ &= I \\ &= S^2 \\ &= \rho(s)^2 \\ \rho(srs) &= \rho(r^{-1}) \\ &= (R_{2\pi/n})^{-1} \\ &= R_{-2\pi/n} \\ &= SR_{2\pi/n}S \\ &= \rho(s)\rho(r)\rho(s) \end{aligned}$$

By the Fundamental Theorem of Group Presentations, this homomorphism is unique.

Question 4

Let
$$G = \langle x, y \mid x^4 = y^5 = 1, xy = y^2x \rangle .$$
It can be shown (you don't have to) that $\#G = 20$ and that every element of G can be written uniquely as $y^b x^a$ where $b \in \{0,1,2,3,4\}$ and $a \in \{0,1,2,3\}$. Complete the following table of multiplication rules for G. **Hint**: Start by proving that $xy^b = y^{2b}x$.

	y^k	y^kx	y^kx^2	y^kx^3
y^ℓ				
$y^\ell x$				
$y^\ell x^2$				
$y^\ell x^3$				

We will start by proving that $xy^b = y^{2b}x$.

$$\begin{aligned} xy^b &= xyy^{b-1} \\ &= y^2xy^{b-1} \\ &= y^2xyy^{b-2} \\ &= y^4xy^{b-2} \\ &\vdots \\ &= y^{2b}x \end{aligned}$$

□

Thus the multiplication table is

	y^k	y^kx	y^kx^2	y^kx^3
y^ℓ	$y^{\ell+k}$	$y^{\ell+k}x$	$y^{\ell+k}x^2$	$y^{\ell+k}x^3$
$y^\ell x$	$y^{\ell+2k}x$	$y^{\ell+2k}x^2$	$y^{\ell+2k}x^3$	$y^{\ell+2k}$
$y^\ell x^2$	$y^{\ell+4k}x^2$	$y^{\ell+4k}x^3$	$y^{\ell+4k}$	$y^{\ell+4k}x$
$y^\ell x^3$	$y^{\ell+8k}x^3$	$y^{\ell+8k}$	$y^{\ell+8k}x$	$y^{\ell+8k}x^2$

Question 5

(i) Let $n \geq 3$. Determine the elements of order 2 in D_{2n}.

(ii) Determine the elements of order 2 in Q_8.

(iii) Show that $Q_8 \not\cong D_{2n}$ for all $n \geq 3$.

Q5 (i)

Clearly all reflections are order 2, and there are n reflections. If n is even, then $r^{n/2}$ is also of order 2. Therefore there are n or $n+1$ elements of order 2 in D_{2n}.

Q5 (ii)

We know that

$$Q_8 = \left\langle a, b \,\middle|\, a^4 = \mathrm{id}, a^2 = b^2, bab^{-1} = a^{-1} \right\rangle$$

and we know from Lemma V.5.2 in the lectures that every element of Q_8 can be written uniquely as $a^i b^j$ for some $0 \leq i \leq 3$ and $0 \leq j \leq 1$.

Since $bab^{-1} = a^{-1}$ and $a^4 = \mathrm{id}$, we get $ba = a^3 b$.

Since there are only 8 elements of Q_8, we can just square all of them and see which ones have order 2. We will skip $a^0 b^0$ however, since we know this has order 1.

$$\begin{aligned}
\left(a^1 b^0\right)^2 &= a^2 \\
\left(a^2 b^0\right)^2 &= a^4 = \mathrm{id} \\
\left(a^3 b^0\right)^2 &= a^6 = a^2 \\
\left(a^0 b^1\right)^2 &= b^2 = a^2 \\
\left(a^1 b^1\right)^2 &= abab \\
&= aa^3bb \\
&= a^2 \\
\left(a^2 b^1\right)^2 &= a^2ba^2b \\
&= a^2(ba)ab \\
&= a^2(a^3b)ab \\
&= a(ba)b \\
&= a(a^3b)b \\
&= a^2
\end{aligned}$$

$$\begin{aligned}
(a^3b^1)^2 &= a^3ba^3b \\
&= a^3(ba)a^2b \\
&= a^3(a^3b)a^2b \\
&= a^2(ba)ab \\
&= a^2(a^3b)ab \\
&= a(ba)b \\
&= a(a^3b)b \\
&= a^2
\end{aligned}$$

Thus the only element of order 2 is a^2.

Q5 (iii)

Since $n \geq 3$, the number of elements in D_{2n} of order two must be at least 3. But Q_8 only has one element of order 2, so Q_8 cannot be isomorphic to D_{2n}.