

# MA270 Analysis 3, Assignment 3

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## Question 1

Let  $n, k \geq 1$  be two integers. In the following, the vector space  $L(\mathbb{R}^n, \mathbb{R}^k)$  of linear maps from  $\mathbb{R}^n$  to  $\mathbb{R}^k$  is endowed with the operator norm  $\|\cdot\|_{\text{op}}$ . Let  $(A_m)$  be a Cauchy sequence in the normed vector space  $L(\mathbb{R}^n, \mathbb{R}^k)$ .

- (a) Show that  $(A_m)$  is a bounded sequence, i.e.  $\exists M > 0$  such that  $\forall m \geq 1$ ,  $\|A_m\|_{\text{op}} \leq M$ .
- (b) Show that  $\forall v \in \mathbb{R}^n$ , the sequence  $(A_m v)$  in  $\mathbb{R}^k$  converges.
- (c) Define a function  $A : \mathbb{R}^n \rightarrow \mathbb{R}^k$  by setting  $Av = \lim_{m \rightarrow \infty} A_m v$ . Show that  $A \in L(\mathbb{R}^n, \mathbb{R}^k)$ .
- (d) Show that  $\|A_m - A\|_{\text{op}} \rightarrow 0$  as  $m \rightarrow \infty$ .

### Q1 (a)

By the definition of a Cauchy sequence,  $\forall \varepsilon > 0$ ,  $\exists N \in \mathbb{N}$  such that  $\forall n, m \geq N$ ,  $\|A_m - A_n\|_{\text{op}} < \varepsilon$ .

Take  $\varepsilon = 1$  and  $n = N$  in this definition. Then we get that  $\exists N \in \mathbb{N}$  such that  $\forall m \geq N$ ,  $\|A_m - A_N\|_{\text{op}} < 1$ .

Since the operator norm satisfies the triangle inequality, we have that for  $m \geq N$ ,

$$\begin{aligned}\|A_m\|_{\text{op}} &\leq \|A_m - A_N\|_{\text{op}} + \|A_N\|_{\text{op}} \\ &< 1 + \|A_N\|_{\text{op}}.\end{aligned}$$

Therefore  $\forall m \in \mathbb{N}$ ,  $\|A_m\|_{\text{op}} \leq \max\{\|A_1\|_{\text{op}}, \dots, \|A_{N-1}\|_{\text{op}}, \|A_N\|_{\text{op}} + 1\}$ , so  $(A_m)$  is bounded.

□

**Q1 (b)**

Consider an arbitrary  $v \in \mathbb{R}^n$ . Then  $(A_m v)_m$  is a Cauchy sequence because  $\forall m, n \in \mathbb{N}$ ,

$$\|A_m v - A_n v\| = \|(A_m - A_n)v\| \leq \|A_m - A_n\|_{\text{op}} \|v\|.$$

Of course  $A_m v \in \mathbb{R}^k$  and we know that  $\mathbb{R}^k$  is complete for any  $k$ . That means that any Cauchy sequence in  $\mathbb{R}^k$  converges to an element of  $\mathbb{R}^k$ . Since  $(A_m v)_m$  is a Cauchy sequence in  $\mathbb{R}^k$ , it converges.

□

**Q1 (c)**

Clearly  $A$  must be a map  $\mathbb{R}^n \rightarrow \mathbb{R}^k$ , since no other domain and codomain would make sense for the limit, so we just have to show that it's linear. Since all  $A_m$  are linear,

$$\begin{aligned} A(\lambda v + u) &= \lim_{m \rightarrow \infty} A_m(\lambda v + u) \\ &= \lim_{m \rightarrow \infty} (\lambda A_m v + A_m u) \\ &= \lim_{m \rightarrow \infty} \lambda A_m v + \lim_{m \rightarrow \infty} A_m u \\ &= \lambda \lim_{m \rightarrow \infty} A_m v + \lim_{m \rightarrow \infty} A_m u \\ &= \lambda Av + Au \end{aligned}$$

Therefore  $A \in L(\mathbb{R}^n, \mathbb{R}^k)$ .

□

**Q1 (d)**

Suppose  $\|A_m - A\|_{\text{op}} \not\rightarrow 0$  as  $m \rightarrow \infty$ . That means that  $\exists \varepsilon_0 > 0$  such that  $\forall m$ ,  $\|A_m - A\|_{\text{op}} \geq \varepsilon_0$ . Using the equivalent definition of the operator norm as  $\|A\|_{\text{op}} := \sup_{x=\|1\|} \|Ax\|$ , we can deduce that there exists a sequence  $m_k$  in  $\mathbb{N}$  and a sequence  $x_k$  of elements of the unit sphere  $\{x \in \mathbb{R}^n : \|x\| = 1\}$  such that  $\forall k$ ,  $\|(A_{m_k} - A)x_k\| \geq \varepsilon_0$ . Let  $x$  be the limit of  $(x_k)$ .

By part (a), we know that  $(A_m)$  is bounded, so  $\exists M > 0$  such that  $\|A_m - A\|_{\text{op}} \leq M$ . Then

$$\begin{aligned} \varepsilon_0 &\leq \|(A_{m_k} - A)x_k\| \\ &\leq \|(A_{m_k} - A)(x_k - x)\| + \|(A_{m_k} - A)x\| \\ &\leq M\|x_k - x\| + \|(A_{m_k} - A)x\|. \end{aligned}$$

As  $k \rightarrow \infty$ ,  $\|x - x_k\| \rightarrow 0$  and  $A_{m_k} - A \rightarrow 0$ . Therefore  $\exists \varepsilon > 0$  such that  $\varepsilon_0 \leq \varepsilon$ . But  $\varepsilon_0$  is chosen first, so we can always choose a  $k$  that makes  $\varepsilon < \varepsilon_0$ , which is a contradiction. Therefore  $\|A_m - A\|_{\text{op}} \rightarrow 0$  as  $m \rightarrow \infty$ .

□

## Question 2

Let  $K \subset \mathbb{R}^n$  be a sequentially compact subset and  $f : K \rightarrow K$  a continuous function such that  $\|f(x) - f(y)\| < \|x - y\|$  for every  $x, y \in K$  such that  $x \neq y$ .

- (a) Show that the function  $K \rightarrow \mathbb{R}$ , defined by  $x \mapsto \|f(x) - x\|$  attains a minimum in  $K$  (i.e.  $\exists x_* \in K$  such that  $\|f(x_*) - x_*\| \leq \|f(x) - x\|$  for every  $x \in K$ ).
- (b) Show that  $f$  admits a unique fixed point in  $K$  (i.e. a point  $y_0 \in K$  such that  $f(y_0) = y_0$ ).

### Q2 (a)

Since  $K$  is sequentially compact, it is bounded. Since  $f$  is continuous, the function  $g : x \mapsto \|f(x) - x\|$  is continuous. By the Extreme Value Theorem,  $g$  is bounded and attains its bounds, so  $\exists x_* \in K$  such that  $g(x_*) = \inf_{x \in K} g(x)$ . Therefore  $\forall x \in K$ ,  $\|f(x_*) - x_*\| \leq \|f(x) - x\|$ .

□

### Q2 (b)

Clearly  $y_0$  minimises  $g$ . We have to prove that  $g(y_0) = 0$  and  $y_0$  is unique. Unfortunately I don't know how to do this.

## Question 3

Let  $A$  and  $B$  be two subsets of  $\mathbb{R}^n$  such that  $A$  is closed and  $B$  is sequentially compact.

- (a) Show that the set  $A + B := \{a + b : a \in A \text{ and } b \in B\}$  is closed.
- (b) Find an example of  $n \geq 1$ , closed sets  $A$  and  $B$  in  $\mathbb{R}^n$  such that  $A + B$  is not closed.

### Q3 (a)

For  $A + B$  to be closed, we need any sequence  $(x_n)$  in  $A + B$  to converge to some element  $x \in A + B$ .

The sequence  $(x_n)$  must be the sum of two sequences  $(a_n)$  in  $A$  and  $(b_n)$  in  $B$ . Since  $B$  is sequentially compact, there exists a convergent subsequence  $b_{n_k} \rightarrow b \in B$ . Note that since  $A$  is only known to be closed, we can't assume that  $(a_n)$  converges.

Since  $b_{n_k} \rightarrow b$ , we have  $x_{n_k} \rightarrow x$ , which means  $a_{n_k}$  converges to  $x - b$ . Since  $A$  is closed, this limit  $a = x - b \in A$ . Therefore  $x = a + b \in A + B$ , so  $A + B$  is closed.

□

### Q3 (b)

Let  $A = \mathbb{N}$  and  $B = \left\{ \frac{1}{n+1} - n : n \in \mathbb{N} \right\}$ . They are both closed but  $A + B$  contains the sequence  $\left( \frac{1}{n+1} \right)_{n \in \mathbb{N}}$ , whose limit is zero, but  $0 \notin A + B$ .

## Question 4

Let  $n \geq 1$  be an integer and  $A \in L(\mathbb{R}^n, \mathbb{R}^n)$  be a linear map. Suppose that  $\|A\|_{\text{op}} < 1$ .

- (a) Show that the sequence  $\left( \sum_{k=0}^m A^k \right)_m$  converges in the normed vector space  $(L(\mathbb{R}^n, \mathbb{R}^n), \|\cdot\|_{\text{op}})$  (i.e.  $\exists B \in L(\mathbb{R}^n, \mathbb{R}^n)$  such that  $\left\| \sum_{k=0}^m A^k - B \right\|_{\text{op}} \rightarrow 0$  as  $m \rightarrow \infty$ ).
- (b) Show that if  $(C_m)$  is a sequence in  $L(\mathbb{R}^n, \mathbb{R}^n)$  converging to  $D \in L(\mathbb{R}^n, \mathbb{R}^n)$ , then  $\forall E \in L(\mathbb{R}^n, \mathbb{R}^n)$ , we have  $\|C_m E - D E\|_{\text{op}} \rightarrow 0$  as  $m \rightarrow \infty$ .
- (c) Let  $\text{Id} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  denote the identity linear operator ( $x \mapsto x$ ) and let  $B \in L(\mathbb{R}^n, \mathbb{R}^n)$  denote the limit of the sequence  $\left( \sum_{k=0}^m A^k \right)_m$ . Show that  $(\text{Id} - A)B = \text{Id}$ .

### Q4 (a)

Since the operator norm is a norm,  $\|A^{n+1}\|_{\text{op}} = \|A\|_{\text{op}} \|A^n\|_{\text{op}}$  for any  $n \in \mathbb{N}$ . Therefore  $\|A^n\|_{\text{op}} = \|A\|_{\text{op}}^n$ . We know that  $\|A\|_{\text{op}} < 1$ , so  $\|A^n\|_{\text{op}} \rightarrow 0$  as  $n \rightarrow \infty$ . We define  $B$  as  $\sum_{k=0}^{\infty} A^k$  so that

$$\begin{aligned} \left\| \sum_{k=0}^m A^k - B \right\|_{\text{op}} &= \left\| \sum_{k=0}^m A^k - \sum_{k=0}^{\infty} A^k \right\|_{\text{op}} \\ &= \left\| - \sum_{k=m+1}^{\infty} A^k \right\|_{\text{op}} \\ &= |-1| \left\| \sum_{k=m+1}^{\infty} A^k \right\|_{\text{op}} \\ &= \left\| \sum_{k=m+1}^{\infty} A^k \right\|_{\text{op}} \end{aligned}$$

Then by the triangle inequality,

$$\begin{aligned}
&\leq \|A^{m+1}\|_{\text{op}} + \left\| \sum_{k=m+2}^{\infty} A^k \right\|_{\text{op}} \\
&\leq \|A^{m+1}\|_{\text{op}} + \|A^{m+2}\|_{\text{op}} + \left\| \sum_{k=m+3}^{\infty} A^k \right\|_{\text{op}} \\
&\leq \sum_{k=m+1}^{\infty} \|A^k\|_{\text{op}}
\end{aligned}$$

And we know that  $\|A^k\|_{\text{op}} \rightarrow 0$  as  $k \rightarrow \infty$ , so the sum above goes to 0 as  $m \rightarrow \infty$ . Therefore  $\left\| \sum_{k=0}^m A^k - B \right\|_{\text{op}} \rightarrow 0$  as  $m \rightarrow \infty$ .

□

#### Q4 (b)

Since  $C_m \rightarrow D$ , we have  $\|C_m - D\|_{\text{op}} \rightarrow 0$  by definition. Since  $L(\mathbb{R}^n, \mathbb{R}^n)$  can be represented by matrices, these linear maps are distributive, so  $C_m E - DE = (C_m - D)E$ . Therefore we have

$$\begin{aligned}
\|C_m E - DE\|_{\text{op}} &= \|(C_m - D)E\|_{\text{op}} \\
&= \|C_m - D\|_{\text{op}} \|E\|_{\text{op}} \\
&= 0 \|E\|_{\text{op}} \quad \text{as } m \rightarrow \infty \\
&= 0
\end{aligned}$$

#### Q4 (c)

$$\begin{aligned}
(\text{Id} - A)B &= (\text{Id} - A) \sum_{k=0}^{\infty} A^k \\
&= \sum_{k=0}^{\infty} A^k - A \sum_{k=0}^{\infty} A^k \\
&= \sum_{k=0}^{\infty} A^k - \sum_{k=1}^{\infty} A^k \\
&= A^0 \\
&= \text{Id}
\end{aligned}$$