MA139 Analysis 2, Assignment 2

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Question 1

Let $(x_i)_1^n$ be a finite sequence of positive numbers with mean m. Recall that $\log t \ge 1 - \frac{1}{t}$. Then

$$\frac{1}{n} \sum_{i=1}^{n} x_i \log x_i \ge \frac{1}{n} \sum_{i=1}^{n} x_i \left(1 - \frac{1}{x_i}\right)$$

$$= \frac{1}{n} \sum_{i=1}^{n} (x_i - 1)$$

$$= \frac{1}{n} \sum_{i=1}^{n} x_i - \frac{1}{n} \sum_{i=1}^{n} 1$$

$$= m - \frac{n}{n}$$

$$= m - 1$$

$$= m \left(1 - \frac{1}{m}\right)$$

$$\le m \log m$$

This is obviously not correct, so I'll try a special case of m=1. Then

$$\frac{1}{n} \sum_{i=1}^{n} x_i \log x_i \ge \cdots$$

$$= m - \frac{n}{n}$$

$$= 1 - 1$$

$$\therefore \frac{1}{n} \sum_{i=1}^{n} x_i \log x_i \ge 0$$

Also $m \log m = 1 \log 1 = 0$. Therefore $\frac{1}{n} \sum_{i=1}^{n} x_i \log x_i \ge m \log m$ when m = 1, as required.

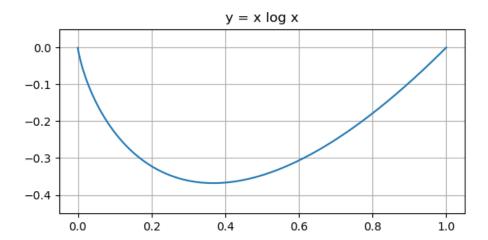
Question 2

Let
$$f(u) = \frac{u^m}{e^u}$$
 for some positive integer m . Then $\frac{\mathrm{d}f}{\mathrm{d}u} = \frac{mu^{m-1}\mathrm{e}^u - u^m\mathrm{e}^u}{\mathrm{e}^{2u}} = \frac{u^{m-1}(m-u)}{\mathrm{e}^u}$.

Consider u>m>0. Then f(u) is positive, since u^m and e^u are both positive. And f'(u) is negative, since u^{m-1} and e^m are positive, but m-u is negative. So for sufficiently large u, the function is always positive but its derivative is always negative. Therefore when u>m, f is a strictly decreasing function bounded below by 0, so $\lim_{u\to\infty}\frac{u^m}{e^u}=0$ for all m.

Question 3

We want to prove that $\lim_{x\to 0^+}\log x=-\infty$. That means for all M>0, there exists x such that $\log x<-M$. We can just choose any positive $x<\mathrm{e}^{-M}$. Since exp is a positive, strictly increasing function, e^{-M} will always be positive and will approach 0 as M grows. So $x\to 0^+$ as $M\to\infty$. Therefore $\lim_{x\to 0^+}\log x=-\infty$ as required.



Let $u = -\log x$. Then as $x \to 0^+$, $u \to \infty$.

Also
$$\frac{1}{e^u} = e^{\log x} = x$$
 so $\frac{u}{e^u} = -x \log x$. Therefore

$$\lim_{x \to 0^+} (x \log x) = -\lim_{u \to \infty} \frac{u}{e^u} = -0$$

by Question 2. Therefore $\lim_{x\to 0^+} (x \log x) = 0$ as required.

Note that $x^x = e^{x \log x}$, so

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{x \log x} = e^{\lim_{x \to 0^+} (x \log x)} = e^0 = 1$$