DATA 252/DATA 551
Modeling and Simulation

Lecture 5: Simulating from random distributions
Acceptance/rejection method

February 17, 2020

Review and set-up

We can make random draws from common distributions (e.g. beta) using software directly (e.g. `rbeta`).

Generating from common distributions can usually be done using transformation, which usually only requires drawing from uniform (0,1). See table below from *Computational Statistics* (Givens and Hoeting, 2013).

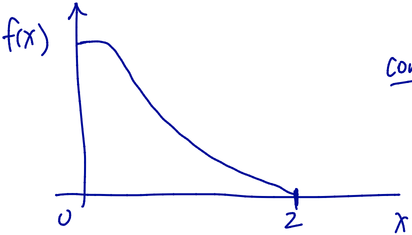
/ not required

TABLE 6.1 Some methods for generating a random variable X from familiar distributions.

Distribution	Method
Uniform	See [195, 227, 383, 538, 539, 557]. For $X \sim \text{Unif}(a, b)$; draw $U \sim \text{Unif}(0, 1)$; then let $X = a + (b - a)U$.
Normal(μ, σ^2) and Lognormal(μ, σ^2)	Draw $U_1, U_2 \sim$ i.i.d. Unif(0, 1); then $X_1 = \mu + \sigma\sqrt{-2\log U_1}\cos\{2\pi U_2\}$ and $X_2 = \mu + \sigma\sqrt{-2\log U_1}\sin\{2\pi U_2\}$ are independent $N(\mu, \sigma^2)$. If $X \sim N(\mu, \sigma^2)$ then $\exp\{X\} \sim \text{Lognormal}(\mu, \sigma^2)$.
Multivariate $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$	Generate standard multivariate normal vector, $\mathbf{Y}$, coordinatewise; then $\mathbf{X} = \boldsymbol{\Sigma}^{-1/2}\mathbf{Y} + \boldsymbol{\mu}$.
Cauchy(α, β)	Draw $U \sim \text{Unif}(0, 1)$; then $X = \alpha + \beta\tan\{\pi(U - \frac{1}{2})\}$.
Exponential(λ)	Draw $U \sim \text{Unif}(0, 1)$; then $X = -(\log U)/\lambda$.
Poisson(λ)	Draw $U_1, U_2, \ldots \sim$ i.i.d. Unif(0, 1); then $X = j - 1$, where j is the lowest index for which $\prod_{i=1}^{j} U_i < e^{-\lambda}$.
Gamma(r, λ)	See Example 6.1, references, or for integer r, $X = -(1/\lambda)\sum_{i=1}^{r} \log U_i$ for $U_1, \ldots, U_r \sim$ i.i.d. Unif(0, 1).
Chi-square (df $= k$)	Draw $Y_1, \ldots, Y_k \sim$ i.i.d. $N(0, 1)$, then $X = \sum_{i=1}^{k} Y_i^2$; or draw $X \sim \text{Gamma}(k/2, \frac{1}{2})$.
Student's t (df $= k$) and $F_{k,m}$ distribution	Draw $Y \sim N(0, 1)$, $Z \sim \chi^2_k$, $W \sim \chi^2_m$ independently, then $X = Y/\sqrt{Z/k}$ has the t distribution and $F = (Z/k)/(W/m)$ has the F distribution.
Beta(a, b)	Draw $Y \sim \text{Gamma}(a, 1)$ and $Z \sim \text{Gamma}(b, 1)$ independently; then $X = Y/(Y + Z)$.
Bernoulli(p) and Binomial(n, p)	Draw $U \sim \text{Unif}(0, 1)$; then $X = 1_{\{U<p\}}$ is Bernoulli(p). The sum of n independent Bernoulli(p) draws has a Binomial(n, p) distribution.
Negative Binomial(r, p)	Draw $U_1, \ldots, U_r \sim$ i.i.d. Unif(0, 1); then $X = \sum_{i=1}^{r} \lfloor(\log U_i)/\log\{1 - p\}\rfloor$, and $\lfloor\cdot\rfloor$ means greatest integer.
Multinomial$(1, (p_1, \ldots, p_k))$	Partition [0, 1] into k segments so the ith segment has length p_i. Draw $U \sim \text{Unif}(0, 1)$; then let X equal the index of the segment into which U falls. Tally such draws for Multinomial$(n, (p_1, \ldots, p_k))$.
Dirichlet$(\alpha_1, \ldots, \alpha_k)$	Draw independent $Y_i \sim \text{Gamma}(\alpha_i, 1)$ for $i = 1, \ldots, k$; then $\mathbf{X}^{\mathrm{T}} = \left(Y_1/\sum_{i=1}^{k} Y_i, \ldots, Y_k/\sum_{i=1}^{k} Y_i\right)$.

Review and set-up

What if X is not a common distribution? In general, the distribution of a *continuous* random variable X can be specified by its probability density function $f(x)$.



$\underline{\text{constant}} \cdot e^{-x^2}$ $\quad x \in (0, 2)$

We'll introduce a general **acceptance-rejection method** that allows us to simulate from the distribution of X, only knowing its the density function $f(x)$.

"Accept with probability ..."

Many simulation methods involve an acceptance-rejection step. Suppose your goal is to simulate some data $\{x_1, x_2, \ldots x_{nsim}\}$. Key idea of acceptance-rejection:

- Make a **proposal** x'
- Calculate the **acceptance probability** $p(x')$
- **Accept** x' **with probability (w.p.)** $p(x')$; reject it with probability $1 - p(x')$

say 0.38

Q: How do you write code to "accept with probability $\underline{p(x')}$"?

→ draw $u \sim \text{Unif}(0,1)$

→ accept if $u < 0.38$

Simulating from a random distribution

Ingredients:

i Target density: $f(x)$

ii Proposal density that is easy to sample from: $g(x)$, satisfying $f(x) \leq C \cdot g(x)$ for all x and some positive constant C. You can think of $C \cdot g(X)$ as an *envelope*.

Algorithm:

Step 1 Sample x' from $g(x)$

Step 2 Sample u from uniform $(0, 1)$

Step 3 If $u < f(x')/(Cg(x'))$, accept x'; otherwise reject x'.

accept x' w.p. $\frac{f(x')}{Cg(x')}$

Exercise

Our goal is to sample from the density function:
$f(x) = 6x(1-x), 0 < x < 1.$

- Group discussion: sketch this function (on paper or on the computer) and come up with a proposal density.
- Implement the acceptance-rejection algorithm for a total of $nsim = 10000$ updates.
- Calculate the mean of the resulting random draws (should be close to 0.5).
- Plot a histogram of the resulting random draws (should be close to the functional form of $f(x)$, which is actually the density of a beta distribution with shape parameters 2, 2).
- Calculate the *acceptance rate*.