homework 4

Q1

give the linear programming:

$$\begin{gathered}\min z = -x_1 - 2x_2 \\ s.t. -2x_1 + x_2 \leq 2 \\ -x_1 + 2x_2 \leq 7 \\ x_1 + 2x_2 \leq 3 \\ x_1, x_2 \geq 0\end{gathered}$$

use the prime-dual interior-point method to iterate once. Suppose $\mu = 10$ in the beginning, and the **infeasible** starting guess is:

$$x = (1,1,1,1,1)^T, y = (0,0,0)^T, s = (1,1,1,1,1)^T$$

answer

$$A = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{pmatrix}$$

$$b = \begin{pmatrix} 2 & 7 & 3 \end{pmatrix}^T, c = \begin{pmatrix} -1 & -2 & 0 & 0 & 0 \end{pmatrix}^T$$

$$X = diag(x) = I_{5\times5}, S = diag(s) = I_{5\times5}$$

$$\nu(\mu) = \mu e - XSe = \begin{pmatrix} 9 & 9 & 9 & 9 & 9 \end{pmatrix}^T$$

$$r_p = b - Ax = \begin{pmatrix} 2 & 5 & -1 \end{pmatrix}^T$$

$$r_D = c - A^T y - s = \begin{pmatrix} -2 & -3 & -1 & -1 & -1 \end{pmatrix}^T$$

$$D = S^{-1}X = I_{5\times5}$$

$$\Delta y = -(ADA^T)^{-1}[AS^{-1}\nu(\mu) - ADr_D - r_p] = \begin{pmatrix} -1 & 2 & -\frac{26}{3} \end{pmatrix}^T$$

$$\Delta s = -A^T \Delta y = \begin{pmatrix} 20/3 & 34/3 & 0 & -3 & 23/3 \end{pmatrix}^T$$

$$\Delta x = S^{-1}\nu(\mu) - D\Delta s = \begin{pmatrix} 7/3 & -7/3 & 9 & 12 & 4/3 \end{pmatrix}^T$$

$$x = x + \Delta x = \begin{pmatrix} 10/3 & -4/3 & 10 & 13 & 7/3 \end{pmatrix}^T$$

$$y = y + \Delta y = \begin{pmatrix} -1 & 2 & -26/3 \end{pmatrix}^T$$

$$s = s + \Delta s = \begin{pmatrix} 23/3 & 37/3 & 1 & -2 & 26/3 \end{pmatrix}^T$$

Q2

think the optimal problem:

$$\begin{gathered} \min \frac{1}{2}||x||^2 \\ s.t. a^T x = b \\ x \geq 0 \end{gathered}$$

for $a \in R^n$,$a \geq 0$ and $b \in R, b > 0$. Try to explain if this problem has the optimal solution, then the optimal solution is uniform and use a and b to express the optimal solution.

answer

define $q(x) = \frac{1}{2}x^T x$, then $\nabla_x q(x) = \frac{x}{2}$ and $\nabla_{xx} q(x) = \frac{1}{2} \succ 0$. So it's a convex QP.

define $L(x) = \frac{1}{2}x^T x - \lambda(a^T x - b)$, then $\nabla L = \frac{x}{2} - \lambda a$. With $\nabla L = 0$, we get $x^* = 2\lambda a$. With $a^T x^* = b$, then $2\lambda a^T a = b$ which infer $\lambda = \frac{b}{2a^T a}$.

So $x^* = \frac{ba}{a^T a}$

Q3

Solve the equality-constrained quadratic problem:

$$\begin{gathered} \min \frac{3}{2}x_1^2 - x_1 x_2 + x_2^2 - x_2 x_3 + \frac{1}{2}x_3^2 + x_1 + x_2 + x_3 \\ s.t. x_1 + 2x_2 + x_3 = 4 \end{gathered}$$

answer

Define $x = (x_1, x_2, x_3)^T$ and $q(x) = \frac{1}{2}x^T G x + x^T c$ with $G = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ and $c = (1, 1, 1)^T$. From equality constrain, we get $A = (1, 2, 1)$ and $b = 4$.

So we get the Null space of A which is $Z = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $Z^TGZ \succ 0$, is nonsingular, hence there is an unique global solution.

With KKT condition and schur-complement method, we get:

$$\begin{aligned} C &= G^{-1} - G^{-1}A^T(AG^{-1}A^T)^{-1}AG^{-1} \\ &= \begin{pmatrix} 5/18 & -1/18 & -1/6 \\ -1/18 & 1/9 & -1/6 \\ -1/6 & -1/6 & 1/2 \end{pmatrix} \end{aligned}$$

$$E = G^{-1}A^T(AG^{-1}A^T)^{-1} = (1/9, 5/18, 1/3)^T$$

$$F = -(AG^{-1}A^T)^{-1} = -1/18$$

At last

$$x^* = -Cd + Eb = (7/18, 11/9, 7/6)^T, \lambda^* = E^Td - Fb = 17/18$$

Q4

Solve the inequality-constrained problem:

$$\begin{gathered} \min 3x_1^2 + 3x_2^2 - 10x_1 - 24x_2 \\ s.t. -2x_1 - x_2 \geq -4 \\ x_1, x_2 \geq 0 \end{gathered}$$

with the starting feasible point $x^{(0)} = (0, 0)^T$

answer

Define $q(x) = \frac{1}{2}x^TGx + x^Tb$ with $G = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$ and $d = (-10, -24)^T$.

Label the constrains as 1,2,3 respectively. With the starting point, we get starting working set $W_0 = \{2, 3\}$.

So we get suboptimal problem as:

$$\begin{aligned} &\min \frac{1}{2}p^TGp + g_0^Tp \\ &s.t. \quad a_i^Tp = 0 \quad i = 2, 3 \end{aligned}$$

With $g_0 = Gx_0 + d = (-10, -24)^T$, the suboptimal problem can be solved and we get $p = (0, 0)^T$.

Then we compute λ_2, λ_3 from $\lambda_2 a_2^T + \lambda_3 a_3^T = g_0$ and get $\lambda_2 = -10, \lambda_3 = -24$.

So we remove constrain 3 from the working set, because it has the most negative multiplier, and set $W_1 = \{2\}$.

With $x_1 = x_0 + p$, we get $x_1 = (0,0)^T$

Iteration 2 starts by solving the suboptimal problem with working set $W_1 = \{2\}$. Then we get $p = (0,4)^T$. The step-length formula yields $\alpha_2 = 1$ with the blocking constrain 1. So $W_2 = \{1,2\}$ and $x_2 = x_1 + \alpha_2 p = (0,4)^T$.

Iteration 3 starts by solving the suboptimal problem with working set $W_2 = \{1,2\}$. Then we get $p = (0,0)^T$. We deduce that the Lagrange multiplier for the working constraint is $\lambda_1 = 0, \lambda_2 = -10$, so we drop 2 from working set to obtain $W_3 = \{1\}$. And $x_3 = (0,4)^T$

Iteration 4 starts by solving the suboptimal problem with working set $W_3 = \{1\}$. Then we get $p = (1/3, -2/3)^T$. The step-length formula yields $\alpha_4 = 1$. So $W_4 = \{1\}$ and $x_4 = x_3 + p = (1/3, 10/3)^T$.

Iteration 5 starts by solving the suboptimal problem with working set $W_4 = \{1\}$. Then we get $p = (0,0)^T$. We deduce that the Lagrange multiplier for the working constraint is $\lambda_1 = 0$, so we get the final solution which is $x^* = (1/3, 10/3)^T$