SI231 - Matrix Computations, Fall 2020-21

Homework Set #1

Name: Sujinning **Major:** Master in IE

Student No.: 2020232125 **E-mail:** sujn@shanghaitech.edu.cn

I. Orthogonality

Problem1

1) **Solution:**

   Firstly for any vector $x \in \mathcal{N}$(A), we have $Ax = 0$, which means for every row in A , the dot product of the row and x is 0. Obviously , vectors $x \in \mathcal{N}$ (A) and linear conbinations of vectors in $\mathcal{R}$(A^T) are orthogonal

   From the hint $dim(\mathcal{N}(A)) + dim(\mathcal{R}(A^T)) = n$

   We have the union of their basis are still linear independent and they can span a $\mathbb{R}$ space

   Then $\mathbb{R} = \mathcal{N}(A) \oplus \mathcal{R}(A^T)$ is proved

2) **Solution:**

   let $rank(A) = p, rank(B) = q$, then we have a set of basis [$\alpha_1, ...\alpha_p, \beta_{p+1}, \beta_{p+2}...\beta_{p+q}$]

   Then we have $A + B$ is the linear combination of the basis from the set

   Resultly $rank(A + B) = dim(A + B) \leq p + q = rank(A) + rank(B)$

   proved

3) **Solution:**

   a) We know AB can be regarded as the linear combination of matrix A 's columns. So $rank(AB) \leq rank(A)$ the same, AB can also be regarded as the linear combination of matrix B 's rows. So $rank(AB) \leq rank(B^T) = rank(B)$

   b) We know AB can be regarded as the linear combination of matrix A 's columns. So $rank(AB) \leq rank(A)$ and AB can also be regarded as the linear combination of matrix B 's rows.

      Assume A has no full-column rank,then we know the linear combination of A 's column should have a dimension lower than n, which means $rank(AB)$ can not be n. The same, if B has no full-row rank, $rank(AB)$ can not have rank n. Therefore,only when A has full-column rank and B has full-row rank can we have $rank(AB) = n$

4) **Solution:**

   $\mathcal{R}(A|B)$ can be written as the linear combination of columns of matrix A and B, which means vector x in $\mathcal{R}(A|B)$ can be written as the linear combination of $[\alpha_1, \alpha_2......\alpha_n, \beta_1, ...\beta_p]$. Similarly, vectors a in A can be writen as the linear combination $[\alpha_1......\alpha_n]$ and vectors b in B can be writen as the linear combination $[\beta_1......\beta_p]$, which means x can be writen as the linear combination of a and b. Then x are in $\mathcal{R}(A) + \mathcal{R}(B)$. which means $\mathcal{R}(A|B) \subseteq (\mathcal{R}(A) + \mathcal{R}(B))$

   Obversiouly, vectors in $\mathcal{R}(A)$ and $\mathcal{R}(B)$ can be writen as the linear combination of $[\alpha_1, \alpha_2......\alpha_n, \beta_1, ...\beta_p]$, which means $(\mathcal{R}(A)+\mathcal{R}(B)) \subseteq \mathcal{R}(A|B)$

Finally, from $\mathcal{R}(A|B) \subseteq (\mathcal{R}(A)+\mathcal{R}(B))$ and $(\mathcal{R}(A)+\mathcal{R}(B)) \subseteq \mathcal{R}(A|B)$, $\mathcal{R}(A|B) = \mathcal{R}(A)+\mathcal{R}(B)$ is proved

5) **Solution:**

We have $dim(\mathcal{R}(A)+\mathcal{R}(B)) = dim(\mathcal{R}(A)) + dim(\mathcal{R}(B)) - dim(\mathcal{R}(A) \cap \mathcal{R}(B)))$

From (4), we have $dim(\mathcal{R}(A)+\mathcal{R}(B)) = dim(\mathcal{R}(A|B))$

Then, $rank(A|B) = dim(\mathcal{R}(A|B)) = dim(\mathcal{R}(A)) + dim(\mathcal{R}(B)) - dim(\mathcal{R}(A) \cap \mathcal{R}(B))$

$= rank(A) + rank(B) - dim(\mathcal{R}(A) \cap \mathcal{R}(B))$

II. UNDERSTAND SPAN,SUBSPACE

Problem1

1) **Solution:**

a) For $span(\mathcal{S}) \subseteq \mathcal{M}$

for vector x in $span(\mathcal{S})$, it can be writen as the linear combination of [$v_1 ... v_n$].

Consider $\mathcal{M} = \cap_{s \subseteq \mathcal{V}} \mathcal{V}$, we have $S \subseteq \mathcal{M}$.Resultly,all the vectors in $span(\mathcal{S})$ can be writen as the linear combination of [$v_1 ... v_n$],which are in S , belonging to $\mathcal{M}$

b) For $\mathcal{M} \subseteq span(\mathcal{S})$

Obviously, $span(\mathcal{S})$ is one of the subspace that contain vectors set $\mathcal{S}$, so $span(\mathcal{S})$ is a special $\mathcal{V}$. It is proved that $\mathcal{M} \subseteq span(\mathcal{S})$ because $\mathcal{M} = \cap_{s \subseteq \mathcal{V}} \mathcal{V}$

III. BASIS,DIMENSION AND PROJECTION

problem1

1) **Solution:**

The dimension is $n+1$

2) **Solution:**

The dimension is $\frac{n(n+1)}{2}$

problem2

1) **Solution:**

a) Assume $R = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, From $RR^T = I$ and $\det(R) = 1$ we have:

$$a^2 + b^2 = 1$$

$$ac - bd = 0$$

$$c^2 + d^2 = 1$$

$$ad - bc = 1$$

After solve the equation we have

$$R = \begin{bmatrix} sin(\theta) & cos(\theta) \\ -cos(\theta) & sin(\theta) \end{bmatrix}, \theta \in [0, 2\pi)$$

b) $\mathbf{R}x = \begin{bmatrix} cos(\frac{5\pi}{6}) \\ sin(\frac{5\pi}{6}) \end{bmatrix}$

2) **Solution**

$$\begin{aligned} QHx &= (I - uu^T)(I - 2uu^T)x \\ &= (I - 3uu^T + 2u(u^Tu)^Tu)x \\ &= (I - uu^T)x = Qx \end{aligned}$$

Then

$$\begin{aligned} ||Hx - QHx||_2 &= ||(I - Q)Hx||_2 \\ &= ||uu^T(I - 2uu)^Tx||_2 \\ &= ||uu^Tx||_2 \\ &= ||I - Qx||_2 \\ &= ||x - Qx||_2 \end{aligned}$$

Resultly,Hx is a reflection of x with respect to $\mathcal{H}_u$

IV. DIRECT SUM

Problem1

Solution

Assume that $\mathcal{V}$ is a n-dimension space, then $\mathcal{B}$ can be written as $[v_1, v_2......v_n]$ in which vectors are linear independent.subsets $\mathcal{B}_1$ and $\mathcal{B}_2$ can be writen as $\mathcal{B}_1 = [v_1......v_p]$ and $\mathcal{B}_2 = [v_{p+1}......v_n]$ .Then we know vectors in $span(\mathcal{B}_1)$ are linear combinations of vectors in $\mathcal{B}_1$, while vectors in $span(\mathcal{B}_1)$ can not be written by linear combination of vectors in $\mathcal{B}_2$. So $span(\mathcal{B}_1) \cap span(\mathcal{B}_2) = \emptyset$ Besides,dimension of $\mathcal{B}_1$ should be p and dimension of $\mathcal{B}_2$ should be $n - p$ which are the same as $span(\mathcal{B}_1)$ and $span(\mathcal{B}_2)$. So $dim(span(\mathcal{B}_1)) + dim(span(\mathcal{B}_2)) = n$ and $span(\mathcal{B}_1) \cap span(\mathcal{B}_2) = \emptyset$, which prove $\mathcal{V} = span(\mathcal{B}_1) \oplus span(\mathcal{B}_2)$

Problem2

Solution

we have d-dimension subspace $\mathcal{S}$ Assume $v = [v_1, ...v_n]$ is a set of basis of $\mathcal{V}$. Then there must be d vectors in the set and we can span the d vectors to construct $\mathcal{S}$. Assume $[v_1, ...v_d]$ is the a set of basis for $\mathcal{S}$,then the rest vectors in $v = [v_1, ...v_n]$ are $v_{rest} = [v_{d+1}, ...v_n]$ Then from Problem1 we have $\mathcal{S} \cap span(v_{rest}) = \emptyset$ and $dim(S) + dim(span(v_{rest})) = n$ Finally we have $\mathcal{V} = \mathcal{S} \oplus span(v_{rest})$,which means $\mathcal{V} = \mathcal{S} \oplus \mathcal{T}$ is proved

V. UNDERSTANDING THE MATRIX NORM

Problem1

1) **Solution**

   The result of Ax is the linear combinaton of column vectors $[\alpha_1, ...\alpha_n]$ in A , if we add a 1-norm to the result,$||Ax||_1 = ||x_1a_1 + ... + x_na_n||_1 \leq max(||a_1||_1, ..., ||a_n||_1)$, equality holds when
   $||a_i||_1 = max(||a_1||_1, ..., ||a_n||_1)$ and $x_i = 1$
   so
   $\max_{||x||_1=1} ||Ax||_1 = \max(||a_1||_1, ..., ||a_n||_1) = \max_j \sum_i^m |a_{ij}|$
2) **Solution**

   like (1) , if we add a $\infty - norm$ to the result

$$\begin{aligned}&||Ax||_\infty\\ =&||x_1a_1 + \cdots + x_na_n||\\ =&||||a_1||_1 + \cdots + ||a_n||_1||_\infty\\ =&\max_{1\leq i<m} \sum_{j=1}^{n} |a_ij|\end{aligned}$$

   Equality holds when we firstly choose the a largest absolute sum row and secondly for every element in row vector $a_{kj}, j = 1, ..., n, x_ja_{kj} = |a_{kj}|$
   so

$$\begin{aligned}||A||_\infty &= \max_{||x||_\infty=1} ||Ax||_\infty\\ &= \max_i \sum_j^m |a_{ij}|\end{aligned}$$

VI. UNDERSTANDING THE HOLDER INEQUALITY

problem1

1) **Solution**

$f^{'}(t) = \lambda - \lambda t^{\lambda-1}, 0 < \lambda < 1.$ when $0 < t < 1, f^{'}(t) < 0$ and when $t > 1, f^{'}(t) > 0$ So $f_{min}(t) = f(0) = 0$

Let $t = \frac{\alpha}{\beta}$, then $f(\frac{\alpha}{\beta}) = (1-\lambda) + \lambda(\frac{\alpha}{\beta}) - (\frac{\alpha}{\beta})^{\lambda} \geq 0$, when we mutiple β on both side , we get: $(1-\lambda)\beta + \lambda\alpha - \alpha^{\lambda}\beta^{1-\lambda} \geq 0 \Rightarrow \alpha^{\lambda}\beta^{\lambda} \leq \lambda\alpha + (1-\lambda)\beta$

2) **Solution**

Let $\alpha = |\hat{x_i}|^p$, $\beta = |\hat{y_i}|^q$, $\lambda = \frac{1}{p}$, then we have:

$$|\hat{x_i}\hat{y_i}| \leq \frac{1}{p}|\hat{x_i}|^p + \frac{1}{q}|\hat{y_i}|^q$$

So

$$\sum_{i=1}^{n} |\hat{x_i}\hat{y_i}| \leq \sum_{i=1}^{n} \frac{1}{p}|\hat{x_i}|^p + \sum_{i=1}^{n} \frac{1}{q}|\hat{y_i}|^q = \frac{1}{p} + \frac{1}{q} = 1$$

3) **Solution**

$\sum_{i=1}^{n} |\hat{x_i}\hat{y_i}| \leq 1 \Rightarrow \sum_{i=1}^{n} |\hat{x_i}\hat{y_i}| \leq ||x||_p||y||_q \Rightarrow |x^T y| \leq ||x||_p||y||_q$

proved