

# SI231 - Matrix Computations, Fall 2020-21

## Homework Set #1

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### Acknowledgements:

- 1) Deadline: **2020-09-27 23:59:59**
- 2) No handwritten is accepted. You need to use L<sup>A</sup>T<sub>E</sub>X. (If you have difficulty in using L<sup>A</sup>T<sub>E</sub>X, you are allowed to use **Word** for the first and the second homework to accommodate yourself.)
- 3) Do use the given template.

### I. UNDERSTANDING RANK, RANGE SPACE AND NULL SPACE

#### Problem 1. (4 points × 5)

- 1) For matrix  $\mathbf{A} \in \mathbb{R}^{m \times n}$ , prove that  $\mathbb{R}^n = \mathcal{N}(\mathbf{A}) \oplus \mathcal{R}(\mathbf{A}^T)$ <sup>1</sup>.
- Hint:**  $\dim(\mathcal{N}(\mathbf{A})) + \dim(\mathcal{R}(\mathbf{A}^T)) = n$ .
- 2) For matrices  $\mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{B} \in \mathbb{R}^{m \times n}$ , prove that  $\text{rank}(\mathbf{A} + \mathbf{B}) \leq \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B})$ .
- 3) For matrices  $\mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{B} \in \mathbb{R}^{n \times p}$ , prove that  $\text{rank}(\mathbf{AB}) \leq \min\{\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})\}$  and  $\text{rank}(\mathbf{AB}) = n$  only when  $\mathbf{A}$  has full-column rank and  $\mathbf{B}$  has full-row rank.
- 4) For matrix  $\mathbf{A} \in \mathbb{R}^{m \times n}$  and  $\mathbf{B} \in \mathbb{R}^{m \times p}$ , prove that  $\mathcal{R}(\mathbf{A|B}) = \mathcal{R}(\mathbf{A}) + \mathcal{R}(\mathbf{B})$ <sup>2</sup>
- 5) For matrix  $\mathbf{A} \in \mathbb{R}^{m \times n}$  and  $\mathbf{B} \in \mathbb{R}^{m \times p}$ , prove that

$$\text{rank}(\mathbf{A|B}) = \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B}) - \dim(\mathcal{R}(\mathbf{A}) \cap \mathcal{R}(\mathbf{B})).$$

**Hint:** Recall the result in 4).

<sup>1</sup>Let  $\mathcal{S}_1$  and  $\mathcal{S}_2$  be two subspaces of  $\mathbb{R}^n$ , if  $\mathcal{S}_1 \cap \mathcal{S}_2 = \{\mathbf{0}\}$  and  $\mathcal{S}_1 + \mathcal{S}_2 = \mathbb{R}^n$ , we define the **direct sum**  $\mathbb{R}^n = \mathcal{S}_1 \oplus \mathcal{S}_2$ .

<sup>2</sup>Here  $\mathbf{A|B}$  denotes a new matrix combined by  $\mathbf{A}$  and  $\mathbf{B}$ . For example,  $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ ,  $\mathbf{B} = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}$ , then  $\mathbf{A|B} = \begin{bmatrix} a_{11} & a_{12} & b_{11} \\ a_{21} & a_{22} & b_{21} \end{bmatrix}$ .

## II. UNDERSTANDING SPAN, SUBSPACE

**Problem 1. (10 points)** For a set of vectors  $\mathcal{S} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ , prove that  $\text{span}(\mathcal{S})$  is the intersection of all subspaces that contain  $\mathcal{S}$ , i.e., prove that  $\text{span}(\mathcal{S}) = \mathcal{M}$  where  $\mathcal{M} := \bigcap_{\mathcal{S} \subseteq \mathcal{V}} \mathcal{V}$  is the intersection of all subspaces that contain  $\mathcal{S}$  and  $\mathcal{V}$  denotes the subspace containing  $\mathcal{S}$ .

**Hint:** Prove that  $\text{span}(\mathcal{S}) \subseteq \mathcal{M}$  and  $\mathcal{M} \subseteq \text{span}(\mathcal{S})$ .

### III. BASIS, DIMENSION AND PROJECTION

**Problem 1. (2 points  $\times$  2)** Determine the dimension of each of the following vector spaces:

- 1) The space of polynomials having degree  $n$  or less;
- 2) The space of  $n \times n$  symmetric matrices.

**Problem 2. Some Important linear transformations**

1) **Rotations. (6 points)** A rotation matrix  $\mathbf{R} \in \mathbb{R}^{n \times n}$  is an orthogonal matrix ( $\mathbf{R}\mathbf{R}^T = \mathbf{I}$ ) such that  $\det(\mathbf{R}) = 1$ .

- According to above definition, find all rotation matrix in  $\mathbb{R}^{2 \times 2}$ .
- Geometrically, if  $\mathbf{R} \in \mathbb{R}^{2 \times 2}$ , then  $\mathbf{R}\mathbf{x}$  means we rotate the vector  $\mathbf{x} \in \mathbb{R}^2$  from some angle  $\theta \in [0, 2\pi]$  in anti-clockwise direction. For  $\mathbf{x} = [\cos(\pi/4), \sin(\pi/4)]^T$ , compute  $\mathbf{R}\mathbf{x}$ , where  $\mathbf{R}$  represents the matrix that rotating  $\mathbf{x}$  by  $7/12\pi$  in anti-clockwise direction.

**Hint:** draw a plot of  $\mathbf{x}$  and  $\mathbf{R}\mathbf{x}$ .

2) **Reflections. (8 points)** Let  $\mathbf{u} \in \mathbb{R}^n$  be a unit vector,  $\|\mathbf{u}\|_2 = 1$ . For a given vector  $\mathbf{x} \in \mathbb{R}^n$  and a hyperplane  $\mathcal{H}_u = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{u}^T \mathbf{x} = 0\}$ . Let  $\mathbf{Q} = \mathbf{I} - \mathbf{u}\mathbf{u}^T$ . Then a vector  $\mathbf{y} \in \mathbb{R}^n$  is said to be a *reflection* of  $\mathbf{x}$  with respect to  $\mathcal{H}$  if their projections onto the hyperplane  $\mathcal{H}$  (denoted as  $\mathbf{Q}\mathbf{x}$  and  $\mathbf{Q}\mathbf{y}$  respectively) satisfy

$$\mathbf{Q}\mathbf{x} = \mathbf{Q}\mathbf{y}, \quad \|\mathbf{x} - \mathbf{Q}\mathbf{x}\|_2 = \|\mathbf{y} - \mathbf{Q}\mathbf{y}\|_2.$$

See Figure.1 for visualization.

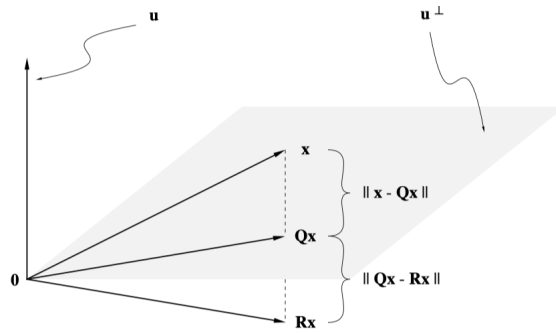


Figure 1. Reflection of  $\mathbf{x}$

A Householder matrix has the form  $\mathbf{H} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T$ . Prove that  $\mathbf{H}\mathbf{x}$  is a reflection of  $\mathbf{x}$  with respect to  $\mathcal{H}_u$ .

## IV. DIRECT SUM

**Problem 1. (10 points)** Let  $\mathcal{V}$  be a vector space, and  $\mathcal{B}$  be a basis for  $\mathcal{V}$ . Suppose that there exist subsets  $\mathcal{B}_1, \mathcal{B}_2$  of  $\mathcal{B}$ , such that  $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$  and  $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$ . Then show that  $\mathcal{V} = \text{span}(\mathcal{B}_1) \oplus \text{span}(\mathcal{B}_2)$ .

**Problem 2. (10 points)** Let  $\mathcal{V}$  be a real vector space of dimension  $n$ . Let  $\mathcal{S}$  be a subspace of  $\mathcal{V}$  of dimension  $d \leq n$ . Prove that there exists a subspace  $\mathcal{T}$  of  $\mathcal{V}$  such that  $\mathcal{V} = \mathcal{S} \oplus \mathcal{T}$ .

## V. UNDERSTANDING THE MATRIX NORM

**Problem 1.** (7 points  $\times$  2) Matrix norm is induced by vector norm,

$$\|\mathbf{A}\|_p = \max_{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{Ax}\|_p}{\|\mathbf{x}\|_p} = \max_{\|\mathbf{x}\|_p=1} \|\mathbf{Ax}\|_p, \quad \mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{x} \in \mathbb{R}^{n \times 1},$$

prove that

1) the matrix 1-norm

$$\begin{aligned} \|\mathbf{A}\|_1 &= \max_{\|\mathbf{x}\|_1=1} \|\mathbf{Ax}\|_1 = \max_j \sum_i^m |a_{ij}| \\ &= \text{the largest absolute column sum.} \end{aligned}$$

2) the matrix  $\infty$ -norm

$$\begin{aligned} \|\mathbf{A}\|_\infty &= \max_{\|\mathbf{x}\|_\infty=1} \|\mathbf{Ax}\|_\infty = \max_i \sum_j^n |a_{ij}| \\ &= \text{the largest absolute row sum.} \end{aligned}$$

## VI. UNDERSTANDING THE HÖLDER INEQUALITY

**Problem 1.** (6 points × 3) Hölder inequality:

$$|\mathbf{x}^T \mathbf{y}| \leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q,$$

for any  $p, q$  such that  $1/p + 1/q = 1$ ,  $p \geq 1$ . Derive this inequality by executing the following steps:

- 1) Consider the function  $f(t) = (1 - \lambda) + \lambda t - t^\lambda$  for  $0 < \lambda < 1$ , establish the inequality

$$\alpha^\lambda \beta^{1-\lambda} \leq \lambda \alpha + (1 - \lambda) \beta,$$

for nonnegative real numbers  $\alpha$  and  $\beta$ .

- 2) Let  $\hat{\mathbf{x}} = \mathbf{x}/\|\mathbf{x}\|_p$  and  $\hat{\mathbf{y}} = \mathbf{y}/\|\mathbf{y}\|_q$ , and apply the inequality of part (a) to obtain

$$\sum_{i=1}^n |\hat{x}_i \hat{y}_i| \leq \frac{1}{p} \sum_{i=1}^n |\hat{x}_i|^p + \frac{1}{q} \sum_{i=1}^n |\hat{y}_i|^q = 1.$$

- 3) Deduce the Hölder inequality with the above results.  
 4) (Bouns question: 10 points) Prove the general form of triangle inequality

$$\|\mathbf{x} + \mathbf{y}\|_p \leq \|\mathbf{x}\|_p + \|\mathbf{y}\|_p.$$

**Hint:** For  $p > 1$ , let  $q$  be the number such that  $1/q = 1 - 1/p$ . Verify that for scalars  $\alpha$  and  $\beta$ ,

$$|\alpha + \beta|^p = |\alpha + \beta| |\alpha + \beta|^{p/q} \leq |\alpha| |\alpha + \beta|^{p/q} + |\beta| |\alpha + \beta|^{p/q}$$

and make use of Hölder's inequality.