SI231 - Matrix Computations, Fall 2020-21

Homework Set #1

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I. UNDERSTANDING RANK, RANGE SPACE AND NULL SPACE

Solution 1:

1) $\because \mathcal{R}(\mathrm{A}^T) = \mathcal{N}(\mathrm{A})^{\perp}$ and $\because N(\mathrm{A}) \oplus N(\mathrm{A})^{\perp} = \mathbb{R}^n$

   $\therefore \mathcal{N}(\mathrm{A}) \oplus \mathcal{R}(\mathrm{A}^T) = \mathbb{R}^n$

2) $\because \mathcal{R}(\mathrm{A}+\mathrm{B}) \in \mathcal{R}(\mathrm{A}) + \mathcal{R}(\mathrm{B})$

   $\therefore \mathrm{rank}(\mathrm{A}+\mathrm{B}) = \mathrm{dim}\mathcal{R}(\mathrm{A}+\mathrm{B}) \leq \mathrm{dim}\Big(\mathcal{R}(\mathrm{A})+\mathcal{R}(\mathrm{B})\Big) = \mathrm{dim}\mathcal{R}(\mathrm{A})+\mathrm{dim}\mathcal{R}(\mathrm{B})-\mathrm{dim}\Big(\mathcal{R}(\mathrm{A})\cap\mathcal{R}(\mathrm{B})\Big) \leq \mathrm{dim}\mathcal{R}(\mathrm{A}) + \mathrm{dim}\mathcal{R}(\mathrm{B}) = \mathrm{rank}(\mathrm{A}) + \mathrm{rank}(\mathrm{B})$

3) $\because \mathrm{AB} = (a_1, a_2, \cdots a_n)(b_1, b_2, \cdots b_n)^T$

   $\therefore$ AB is the linear combination of matrix A 's columns and matrix B 's rows.

   $\therefore \mathrm{rank}(\mathrm{AB}) \leq \mathrm{rank}(\mathrm{A})$ and $\mathrm{rank}(\mathrm{AB}) \leq \mathrm{rank}(\mathrm{B}^T) = \mathrm{rank}(\mathrm{B})$

   if A has full-column rank and B has full-row rank

   $then$ $\mathrm{rank}(\mathrm{AB}) = \mathrm{rank}(\mathrm{A}) = \mathrm{rank}(\mathrm{B}) = n$

   if A has no full-column rank or B has no full-row rank

   $then$ $\mathrm{rank}(\mathrm{AB}) \leq n$

4) $\mathcal{R}(A|B)$ means all the linear combination of $(a_1, a_2, \cdots a_n, b_1, b_2, \cdots b_p)$

   $\mathcal{R}(A) + \mathcal{R}(B)$ means all the linear combination of $(a_1, a_2, \cdots a_n)$ and $(b_1, b_2, \cdots b_p)$

   Every vectors x in $\mathcal{R}(A|B)$ and in $\mathcal{R}(A)+\mathcal{R}(B)$ both can be written in the same form like $\sum_{i=1}^n x_i a_i + \sum_{i=1}^p y_i b_i$

   Finally, $\mathcal{R}(A|B) = \mathcal{R}(A) + \mathcal{R}(B)$ is proved

5) $\because \mathrm{dim}(\mathcal{R}(A)+\mathcal{R}(B)) = \mathrm{dim}(\mathcal{R}(A))+\mathrm{dim}(\mathcal{R}(B))-\mathrm{dim}(\mathcal{R}(A)\cap\mathcal{R}(B)))$ and $\because \mathrm{dim}(\mathcal{R}(A)+\mathcal{R}(B)) = \mathrm{dim}(\mathcal{R}(A|B))$

   $\therefore \mathrm{rank}(A|B) = \mathrm{dim}(\mathcal{R}(A|B)) = \mathrm{dim}(\mathcal{R}(A))+\mathrm{dim}(\mathcal{R}(B))-\mathrm{dim}(\mathcal{R}(A)\cap\mathcal{R}(B)) = \mathrm{rank}(A)+\mathrm{rank}(B)-\mathrm{dim}(\mathcal{R}(A)\cap\mathcal{R}(B))$

II. UNDERSTAND SPAN,SUBSPACE

Solution 1:

1) To prove $span(\mathcal{S}) \subseteq \mathcal{M}$

   $\because \mathcal{M} = \cap_{s\subseteq\mathcal{V}}\mathcal{V}$

   $\therefore S = \{v_1, \cdots v_n\} \subseteq \mathcal{M}$

$\because \mathcal{M}$ is a vactor space

$\therefore \sum_{i=1}^{n} x_i v_i \subseteq \mathcal{M}$

$\therefore span(\mathcal{S}) \subseteq \mathcal{M}$

2) To prove $\mathcal{M} \subseteq span(\mathcal{S})$

$\because span(\mathcal{S})$ is one of the subspace which contain $\mathcal{S}$

$\therefore \mathcal{S}$ is one of the $\mathcal{V}$ which contain $\mathcal{M}$

$\therefore \mathcal{M} \subseteq span(\mathcal{S})$

3) $\because span(\mathcal{S}) \subseteq \mathcal{M}$ and $\mathcal{M} \subseteq span(\mathcal{S})$

$\therefore \mathcal{M} = span(\mathcal{S})$

III. BASIS,DIMENSION AND PROJECTION

Solution 1:

1) The dimension is $n+1$
2) The dimension is $\frac{n(n+1)}{2}$

Solution 2:

1) **Rotations**

a) Assume $R = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\because RR^T = I$ and $\det(R) = 1$

$$\begin{cases} a^2 + b^2 = 1 \\ ac - bd = 0 \\ c^2 + d^2 = 1 \\ ad - bc = 1 \end{cases}$$

Therefore

$$R = \begin{bmatrix} sin(\theta) & cos(\theta) \\ -cos(\theta) & sin(\theta) \end{bmatrix}, \theta \in [0, 2\pi)$$

b) $\mathbf{R}x = (cos(\frac{5\pi}{6}), sin(\frac{5\pi}{6}))^T$

2) **Reflections**

$$\begin{aligned} QHx &= (I - uu^T)(I - 2uu^T)x \\ &= (I - 3uu^T + 2u(u^Tu)^Tu)x \\ &= (I - uu^T)x = Qx \end{aligned}$$

Therefore

$$\begin{aligned} ||Hx - QHx||_2 &= ||(I - Q)Hx||_2 \\ &= ||uu^T(I - 2uu)^Tx||_2 \\ &= ||uu^Tx||_2 \\ &= ||I - Qx||_2 \\ &= ||x - Qx||_2 \end{aligned}$$

$\therefore Hx$ is a reflection of x with respect to $\mathcal{H}_u$

IV. DIRECT SUM

Solution 1:

1) According to the question, assume $dim(\mathcal{V}) = n$, $\mathcal{B} = (v_1, v_2, \cdots v_n)$ in which vectors are linear independent, $\mathcal{B}_1 = (v_1, v_2, \cdots v_m)$, $\mathcal{B}_2 = (v_{m+1}, v_{m+2}, \cdots v_n)$
$\therefore span(\mathcal{B}_1) = \sum_{i=1}^{m} x_i v_i$ and $span(\mathcal{B}_2) = \sum_{i=m+1}^{n} y_i v_i$
$\therefore span(\mathcal{B}_1) \cap span(\mathcal{B}_2) = \emptyset$ and $dim(span(\mathcal{B}_1)) + dim(span(\mathcal{B}_2)) = m + (n - m) = n$
$\therefore \mathcal{V} = span(\mathcal{B}_1) \oplus span(\mathcal{B}_2)$

Solution 2:

1) According to the question, assume $dim(\mathcal{V}) = n$, the basis is $(v_1, v_2, \cdots v_n)$ in which vectors are linear independent, $\mathcal{S} = span(v_1, v_2, \cdots v_d)$, $\mathcal{T} = span(v_{rest})$
$\therefore \mathcal{S} \cap \mathcal{T} = \emptyset$ and $dim(\mathcal{S}) + dim(\mathcal{T}) = d + (n - d) = n$
$\therefore \mathcal{V} = \mathcal{S} \oplus \mathcal{T}$

V. UNDERSTANDING THE MATRIX NORM

Solution 1:

1) $\because$ The result of Ax is the linear combinaton of column vectors $[\alpha_1, ...\alpha_n]$ in A

   $\therefore if$ we add a 1-norm to the result,$||Ax||_1 = ||x_1 a_1 + ... + x_n a_n||_1 \leq max(||a_1||_1, ..., ||a_n||_1)$, equality holds when $||a_i||_1 = max(||a_1||_1, ..., ||a_n||_1)$ and $x_i = 1$

   $\therefore \max_{||x||_1=1} ||Ax||_1 = \max(||a_1||_1, ..., ||a_n||_1) = \max\limits_j \sum\limits_i^m |a_{ij}|$

2) if we add a $\infty - norm$ to the result

$$\begin{aligned} ||Ax||_\infty =& ||x_1 a_1 + \cdots + x_n a_n|| \\ =& ||||a_1||_1 + \cdots + ||a_n||_1||_\infty \\ =& \max_{1 \leq i < m} \sum_{j=1}^{n} |a_i j| \end{aligned}$$

   Equality holds when we firstly choose the a largest absolute sum row and secondly for every element in row vector $a_{kj}, j = 1, ..., n, x_j a_{kj} = |a_{kj}|$

   $\therefore ||A||_\infty = \max_{||x||_\infty=1} ||Ax||_\infty = \max_i \sum_j^m |a_{ij}|$

VI. UNDERSTANDING THE HOLDER INEQUALITY

Solution 1:

1) $f^{'}(t) = \lambda - \lambda t^{\lambda-1}, 0 < \lambda < 1.$ when $\begin{cases} 0 < t < 1, f^{'}(t) < 0 \\ t > 1, f^{'}(t) > 0 \end{cases}$

   $\therefore f_{min}(t) = f(0) = 0$

   Let $t = \frac{\alpha}{\beta}$, then $f(\frac{\alpha}{\beta}) = (1-\lambda) + \lambda(\frac{\alpha}{\beta}) - (\frac{\alpha}{\beta})^\lambda \geq 0$, when we mutiple β on both side , we get: $(1-\lambda)\beta + \lambda\alpha - \alpha^\lambda \beta^{1-\lambda} \geq 0 \Rightarrow \alpha^\lambda \beta^\lambda \leq \lambda\alpha + (1-\lambda)\beta$

2) Let $\alpha = |\hat{x_i}|^p$, $\beta = |\hat{y_i}|^q$, $\lambda = \frac{1}{p}$

   $\therefore |\hat{x_i}\hat{y_i}| \leq \frac{1}{p}|\hat{x_i}|^p + \frac{1}{q}|\hat{y_i}|^q$

   $\therefore \sum_{i=1}^n |\hat{x_i}\hat{y_i}| \leq \sum_{i=1}^n \frac{1}{p}|\hat{x_i}|^p + \sum_{i=1}^n \frac{1}{q}|\hat{y_i}|^q = \frac{1}{p} + \frac{1}{q} = 1$

3) $\sum_{i=1}^n |\hat{x_i}\hat{y_i}| \leq 1 \Rightarrow \sum_{i=1}^n |\hat{x_i}\hat{y_i}| \leq ||x||_p ||y||_q \Rightarrow |x^T y| \leq ||x||_p ||y||_q$

   Proved.