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Numerical methods for partial differential equations

Exercise 1 – Calender week 11 - 2020

Example 1.1

Let V be a Hilbert space, and let $A : V \times V \to \mathbb{R}$ be a continuous, elliptic and symmetric bilinear form. Further, let $f : V \to \mathbb{R}$ be a continuous linear form. We define the functional

$$J : V \to \mathbb{R}, \quad J(v) := \frac{1}{2} A(v, v) - f(v).$$

Let $V_0 \subset V$ be a linear subspace and define for $g \in V$ the subspace $V_g := \{v + g : v \in V_0\}$. Prove that the following two statements are equivalent:

i.) The function u minimizes the functional J, thus

$$J(u) = \inf_{v \in V_g} J(v)$$

ii.) The function $u \in V_g$ solves the problem

$$A(u, v) = f(v) \; \forall v \in V_0.$$

Does V_0 has to be a closed subspace?

Example 1.2

Let V be a Hilbert space, and let $A : V \to V$ be a bounded, linear and self-adjoint operator. Show (without using general spectral theory) that

$$\|A\| := \sup_{\substack{v \in V \\ v \neq 0}} \frac{\|Av\|_V}{\|v\|_V} = \sup_{\substack{v \in V \\ v \neq 0}} \frac{|(Av, v)_V|}{\|v\|_V^2},$$

where $(\cdot, \cdot)_V$ and $\| \cdot \|_V$ is the inner product and the norm on V, respectively.

Example 1.3 (C^1 is not complete)

Let $X := \{v \in C^1([0, 1]) : v(0) = 0, v'(1) = 1\}$. We define the functional

$$J(v) := \int_0^1 (v'(x))^2 \, \mathrm{d}x \, .$$

Prove that there is no function $u \in X$ such that $J(u) = \inf_{v \in X} J(v)$.

Example 1.4 (First problem in NGSolve)

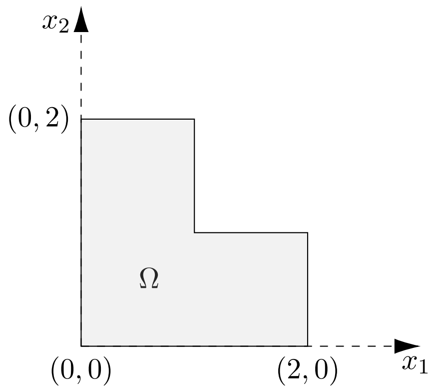
Solve the Poisson equation with homogeneous boundary conditions

$$\begin{aligned} -\Delta u &= 1 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

on the domain (called "L-shape") $\Omega := (0,2)^2 \setminus [1,2]^2$ in NGSolve. Plot the partial derivatives of u. Calculate the mean value of u on Ω. Estimate the error measured in the L^2-norm and the H^1-norm given by

$$\|u\|^2_{H^1(\Omega)} := \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)},$$

for different polynomial orders $k = 1, \dots, 8$ by comparing your solution with a finite element solution of higher order (for example $k + 2$).
For the definition of the geometry look at the introduction of Netgen (the mesh generator of NGSolve): `https://ngsolve.org/docu/latest/netgen_tutorials/define_2d_geometries.html`



Remark 1 *The H^1-norm above is the norm of the Sobolev space $W^{k,p}$ with $k = 1$ and $p = 2$. It will be of main interest for the analysis of partial differential equations (and their approximation by finite elements).*