

# A Relatively Small Turing Machine Whose Behavior Is Independent of Set Theory

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## Abstract

Since the definition of the Busy Beaver function by Radó in 1962, an interesting open question has been what the smallest value of  $n$  for which  $BB(n)$  is independent of ZFC set theory. Is this  $n$  approximately 10, or closer to 1,000,000, or is it even larger? In this paper, we show that it is at most 8,013 by presenting an explicit description of a 8,013-state Turing machine  $Z$  with 1 tape and a 2-symbol alphabet that cannot be proved to run forever in ZFC (even though it presumably does), assuming ZFC is consistent. The machine is based on work of Harvey Friedman on independent statements involving order-invariant graphs. In doing so, we give the first known upper bound on the highest provable Busy Beaver number in ZFC. We also present an explicit description of a 4,888-state Turing machine  $G$  that halts if and only if there is a counterexample to Goldbach's conjecture, and an explicit description of a 5,372-state Turing machine  $R$  that halts if and only if the Riemann hypothesis is false. To create  $G$ ,  $R$ , and  $Z$ , we develop and use a higher-level language, Laconic, which is much more convenient than direct state manipulation.

## 1 Introduction

### 1.1 Background and Motivation

*Zermelo-Fraenkel set theory with the axiom of choice*, more commonly known as ZFC, is an axiomatic system invented in the twentieth which has since been used as the foundation of most of modern mathematics. It encodes arithmetic by describing natural numbers as increasing sets of sets.

Like any axiomatic system capable of encoding arithmetic, ZFC is constrained by Gödel's two incompleteness theorems. The first incompleteness theorem states that if ZFC is *consistent* (it never proves both a statement and its opposite), then ZFC cannot also be *complete* (able to prove every true statement). The second incompleteness theorem states that if ZFC is consistent, then ZFC cannot prove its own consistency. Because we have built modern mathematics on top of ZFC, we can reasonably be said to have assumed ZFC's consistency. This means that we must also believe that ZFC cannot prove its own consistency. This fact carries with it certain surprising conclusions.

In particular, consider a Turing machine  $Z$  that enumerates, one after the other, each of the provable statements in ZFC. To describe how such a machine might be constructed,  $Z$  could iterate over the axioms and inference rules of ZFC, applying each in every possible way to each conclusion

or pair of conclusions that had been reached so far. We might ask  $Z$  to halt if it ever reaches a contradiction; in other words,  $Z$  will halt if and only if it ever finds a proof of  $0 = 1$ . Because we know that this machine will enumerate *every* provable statement in ZFC, we know that it will run forever if and only if ZFC is consistent.

It follows that  $Z$  is a Turing machine for which the question of its behavior (whether or not it halts when run indefinitely) is equivalent to the consistency of ZFC. While we will talk about ZFC throughout this paper, rather than simple Zermelo-Fraenkel set theory, this is simply convention brought about by the fact that ZFC is a more powerful and more commonly-used set of axioms. In more detail, for the purposes of this paper, the Axiom of Choice is irrelevant: the consistency of ZFC is equivalent to the consistency of simple ZF set theory, [8] and ZFC and ZF prove exactly the same arithmetical statements (which include, among other things, statements about whether Turing machines halt). [17] Therefore, just as ZFC cannot prove its own consistency (assuming ZFC is consistent), ZFC also cannot prove that  $Z$  will run forever.

This is interesting because, while the undecidability of the halting problem tells us that there cannot exist an algorithmic method for determining whether an *arbitrary* Turing machine loops or halts,  $Z$  is an example of a *specific* Turing machine whose behavior cannot be proven one way or the other using the foundation of modern mathematics. Mathematicians and computer scientists think of themselves as being able to determine how a given algorithm will behave if we are given enough time to stare at it; despite this intuition,  $Z$  is a machine whose behavior we can never prove without assuming axioms more powerful than those generally assumed in most of modern mathematics.

## 1.2 Turing Machines

There are many definitions for Turing machines, each differing slightly from the other. For example, some definitions allow the machine to have multiple tapes; others only allow it to have one. Some definitions allow an arbitrarily large alphabet, while others allow only two symbols. Some definitions allow the tape head to remain in place, while others require it to move at every time-step. In most research regarding Turing machines, mathematicians don't concern themselves with which of these models to use, because any one of them can simulate the others. However, because this work is concerned with upper-bounding the exact number of states required to perform certain tasks, it is important to define precisely what model of Turing machine is being used.

Formally, a  $k$ -state Turing machine is a 7-tuple  $M = (Q, \Gamma, b, \Sigma, \delta, q_0, F)$ , where:

$Q$  is the set of  $k$  states  $\{q_0, q_1, \dots, q_{k-2}, q_{k-1}\}$

$\Gamma = \{a, b\}$  is the set of *tape alphabet symbols*

$a$  is the *blank symbol*

$\Sigma$  is the set of *input symbols*

$\delta = Q \times \Gamma \rightarrow (Q \cup F) \times \Gamma \times \{L, R\}$  is the *transition function*

$q_0$  is the *start state*

$F = \{\text{ACCEPT}, \text{REJECT}, \text{ERROR}\}$  is the set of *halting transitions*.

A Turing machine's *states* make up the Turing machine's easily-accessible, finite memory. The Turing machine's state is initialized to  $q_0$ .

The *tape alphabet symbols* correspond to the symbols that can be written on the Turing machine's infinite tape.

In this work, all Turing machines discussed are run on the all-**a** input.

The *transition function* encodes the Turing machine's behavior. It takes two inputs: the current state of the Turing machine (an element of  $Q$ ) and the symbol read off the tape (an element of  $\Gamma$ ). It outputs three separate instructions: what state to enter (an element of  $Q$ ), what symbol to write onto the tape (an element of  $\Gamma$ ) and what direction to move the head in (an element of  $\{L, R\}$ ). A transition function specifies the entire behavior of the Turing machine in all cases.

The *start state* is the state that the Turing machine is in at initialization.

A *halting transition* is a transition that causes the Turing machine to halt. While having three possible halting transitions is not necessary for our purposes, being able to differentiate between three different types of halting (ACCEPT, REJECT, and ERROR) is useful for testing.

### 1.3 The Busy Beaver Function

Consider the set of all Turing machines with  $k$  states, for some positive integer  $k$ . We call a Turing machine  $B$  a *k-state Busy Beaver* if when run on the empty tape as input,  $B$  halts, and also runs for at least as many steps before halting as all other halting  $k$ -state Turing machines. [16]

In other words, a Busy Beaver is a Turing machine that runs for at least as long as all other halting Turing machines with as many states as it. Another common definition for a Busy Beaver is a Turing machine that writes as many 1's on the tape as possible; because the number of 1's written is a somewhat arbitrary measure, it is not used in this work.

The *Busy Beaver function*, written  $BB(k)$ , equals the number of steps it takes for a  $k$ -state Busy Beaver to halt. The Busy Beaver function has many striking properties. To begin with, it is not *computable*; in other words, there does not exist an algorithm that takes  $k$  as input and returns  $BB(k)$ , for arbitrary values of  $k$ . This follows directly from the undecidability of the halting problem. Suppose an algorithm existed to compute the Busy Beaver function; then given a  $k$ -state Turing machine  $M$  as input, we could compute  $BB(k)$  and run  $M$  for  $BB(k)$  steps. If, after  $BB(k)$  steps,  $M$  had not yet halted, we could safely conclude that  $M$  would never halt. Thus, we could solve the halting problem, which we know is impossible.

By the same argument,  $BB(k)$  must grow faster than any computable function. (To check this, assume that some computable function  $f(k)$  grows faster than  $BB(k)$ , and substitute  $f(k)$  for  $BB(k)$  in the rest of the proof.) In particular, the Busy Beaver grows even faster than (for instance) the Ackermann function, a well-known fast-growing function.

Because finding the value of  $BB(k)$  for a given  $k$  requires so much work (one must fully explore the behavior of all  $k$ -state Turing machines), few explicit values of the Busy Beaver function are known. The known values are [10] [3]:

$$BB(1) = 1$$

$$BB(2) = 6$$

$$BB(3) = 21$$

$$BB(4) = 107$$

For  $BB(5)$  and  $BB(6)$ , only lower bounds are known:  $BB(5) \geq 47,176,870$ , and  $BB(6) \geq 7.4 \times 10^{36,534}$ . Researchers are currently working on pinning down the value of  $BB(5)$  exactly, and some consider it to be possibly within reach. A summary of the current state of human knowledge about Busy Beaver values can be found at [13].

Another way to discuss the Busy Beaver sequence is to say that modern mathematics has established a *lower bound* of 4 on the highest provable Busy Beaver value. In this paper, we prove the first known *upper bound* on the highest provable Busy Beaver value in ZFC; that is, we give a value of  $k$ , namely 8,013, such that the value of  $BB(k)$  cannot be proven in ZFC.

Intuitively, one might expect that while no algorithm may exist to compute  $BB(k)$  for *all* values of  $k$ , we could find the value of  $BB(k)$  for any *specific*  $k$  using a procedure similar to the one we used to find the value of  $BB(k)$  for  $k \leq 4$ . The reason this is not so is closely tied to the existence of a machine like the Gödelian machine  $Z$ , as described in Section 1.1. Suppose that  $Z$  has  $k$  states. Because  $Z$ 's behavior (whether it halts or loops) cannot be proven in ZFC, it follows that the value of  $BB(k)$  also cannot be proven in ZFC; if it could, then a proof would exist of  $Z$ 's behavior in ZFC. Such a proof would consist of a *computation history* for  $Z$ , which is an explicit step-by-step description of  $Z$ 's behavior for a certain number of steps. If  $Z$  halts, a computation history leading up to  $Z$ 's halting would be the entire proof; if  $Z$  loops, then a computation history that takes  $BB(k)$  steps, combined with a proof of the value of  $BB(k)$ , would constitute a proof that  $Z$  will run forever.

In this paper we construct a machine like  $Z$ , for which a proof that  $Z$  runs forever would imply that ZFC was consistent. In doing so, we give an explicit upper bound on the highest Busy Beaver value provable in ZFC assuming the consistency of a slightly stronger set theory. Our machine, which we shall refer to as  $Z$  hereafter, contains 8,013 states. Therefore, we will never be able to prove the value of  $BB(8,013)$  without assuming more powerful axioms than those of ZFC. This upper bound is presumably very far from tight, but it is a first step.

Nontrivial ideas are needed to achieve such a small state count, without which the size of  $Z$  might range in the hundreds of thousands or even millions of states. We briefly introduce these ideas in the subsection following this one, and explore them in much greater detail later in the paper. These ideas constitute this paper's main technical contribution.

## 1.4 Parsimony

In most algorithmic study, efficiency is the primary concern. In designing  $Z$ , however, parsimony is the only thing that matters. One historical analogue is the practice of “code-golfing”: a recreational pursuit adopted by some programmers in which the goal is to produce a piece of code in a given programming language, using as few characters as possible. Many examples of code-golfing can be found at [19]. The goal of designing a Turing machine with as few states as possible to accomplish a certain task, without concern for the machine's efficiency or space usage, can be thought of as code-golfing with a particularly low-level programming language.

Part of the charm of Turing machines is that they give us a “standard reference point” for measuring complexity, unencumbered by the details of more sophisticated programming languages. Also, with Turing machines, there can be no suspicion that we engineered a programming formalism just for the purpose of code-golfing, or for making the concepts we want expressed artificially simple to describe. This is why we prefer Turing machines as a tool for measuring complexity; not because they are particularly special, but simply because they are so primitive and so minimal that their specifics will interfere minimally with what we mean by an algorithm being “complicated.”

In this paper, we use three ideas for generating parsimonious Turing machines: Harvey Friedman's mathematical statements, *on-tape processing*, and *introspective* Turing machines. The last of these ideas was proposed, under a different name and with some variations, by Ben-Amram and Petersen in 2002. [2] These three ideas are explained in more detail in Subsections 3.1, 8.1, and 8.3,

respectively, but we will summarize them very briefly here.

The first idea, that of using Friedman’s mathematical statements, is simply to use the research done by Friedman into finding simple-to-express statements that are equivalent to the consistency of various axiomatic systems. In particular, we make use of a statement discovered by Friedman to be equivalent to the consistency of a set theory known to be stronger than ZFC (and whose consistency, therefore, would imply the consistency of ZFC). [6]

The second idea, on-tape processing, is a way to encode high-level commands into a Turing machine parsimoniously. Instead of converting commands to groups of states directly, which incurs a multiplicative overhead based on how large these groups need to be, on-tape processing begins by writing the commands onto the tape, using as efficient an encoding as possible. Then, once the commands are on the tape, the commands are processed by a single group of states that understands how to interpret them.

The third idea, introspective Turing machines, is a way to write long strings onto the tape using as few states as possible. The idea is to encode information one of each state’s transitions, instead of encoding information in each state’s write field. This is advantageous because there are many choices for which state to point a transition to, but only two choices for what bit to write. Therefore, more information can be encoded in each state using this method.

## 1.5 Implementation Overview

To generate descriptions of Turing machines with nice mathematical properties entirely by hand is a daunting task. Rather than approach the problem directly, we created tools for generating parsimonious Turing machines while presenting an interface that is comfortably familiar to most programmers (and to us!)

We created two tools. At the top level is the Laconic programming language, whose syntax and capabilities are similar to those of most programming languages, such as Java or Python. Beneath it we created a lower-level language called Turing Machine Descriptor (TMD). TMD is quite unlike most programming languages, and is better thought of as a convenient way to describe a multi-tape, 3-symbol Turing machine plus a function stack. The style of multi-tape Turing machine used in TMD is the commonly used “one-tape-at-a-time” abstraction: only one tape at a time can be interacted with, for reading, writing, and moving the head. Laconic compiles down to a TMD program, and TMD compiles down to a description of a single-tape, 2-symbol Turing machine. This process is illustrated in Figure 1.

We recommend that programmers hoping to use our tools to generate their own encodings of mathematical statements or algorithms as Turing machines use Laconic. Laconic’s interface is perfect for somebody hoping to write in a “traditional” language. On the other hand, if the programmer wishes to improve upon Laconic’s compilation process, writing code directly in TMD is likely to be the better option.

## 2 Related Work

This paper is not the first to attempt to quantify the complexity of arithmetical statements. Calude and Calude [5] define a register machine of their own design, and provide quantifications of the complexity of Legendre’s conjecture, Fermat’s last theorem, Goldbach’s conjecture, Dyson’s conjecture,

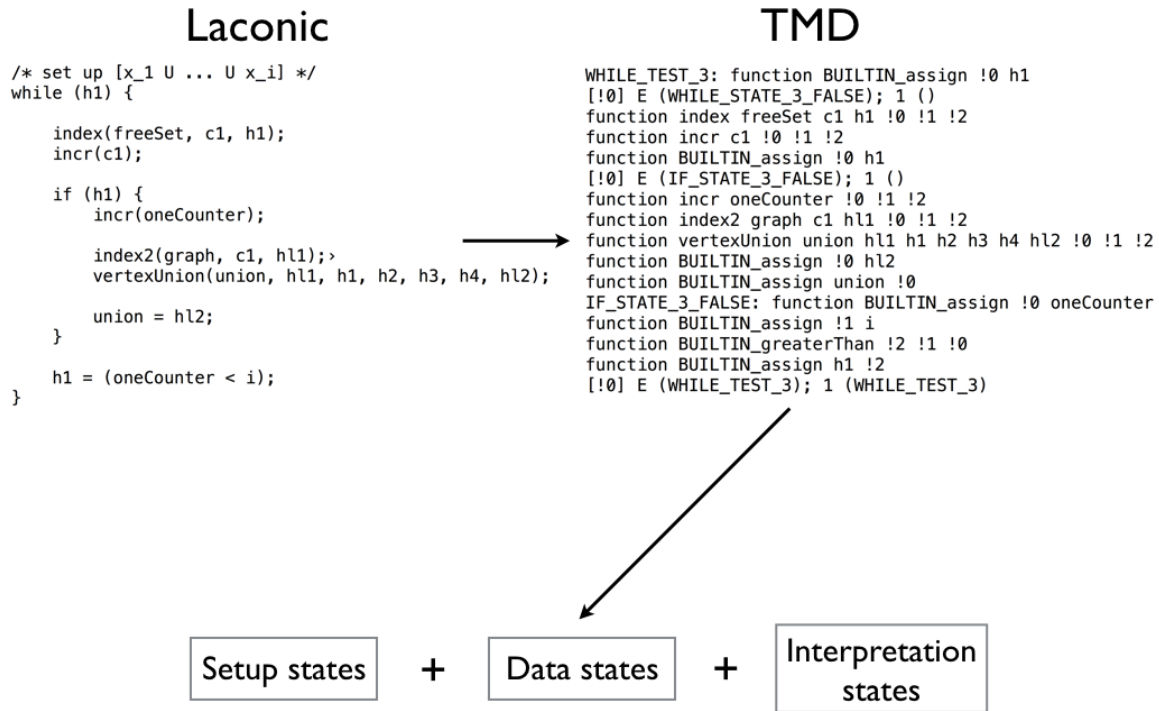


Figure 1: A visual overview of the compilation process.

the Riemann hypothesis, and the four color theorem.<sup>1</sup> In addition, Koza [9] and Pargellis [15] each invent instruction sets that are particularly well-suited to representing self-reproducing programs simply, and show that starting from a “primordial soup” of such instructions distributed about a large memory, along with an increasing number of program threads, a rich ecosystem of increasingly efficient self-reproducing programs start to dominate the “landscape.”

This paper differs from the previous work in two ways: firstly, it is the first to give explicit, relatively small machines whose behavior is provably independent of the standard axioms of modern mathematics. Secondly, to our knowledge, this paper is the first concrete study of parsimony to use Turing machines as the model of computation—rather than (for example) a new programming language proposed by the authors! We consider it important to use the weakest and most common model of computation for complexity comparisons across different mathematical statements. This is because the more powerful and complex the model of computation used, the more of the complexity of the algorithm can be “shunted” onto the model of computation, and the greater the potential distortion created by the choice of model. As a *reductio ad absurdum*, we could imagine a programming language that included “test the Riemann hypothesis” and “test the consistency of ZFC” as primitive operations. By using the “weakest” model of computation that is commonly known, and one which is generally accepted as the mathematical basis of algorithms, we hope to avoid this pitfall and make it easier to interpret our results in a model-independent way.

### 3 A Turing Machine that Cannot Be Shown to Run Forever Using ZFC

We present a 8,013-state Turing machine whose behavior is *independent of ZFC*; it would not be possible to prove that this machine would halt or wouldn’t halt using the axioms of ZFC, assuming a slightly stronger set theory is consistent. It is therefore impossible to prove the value of  $BB(8,013)$  to be any given value without assuming axioms more powerful than ZFC, assuming that ZFC is consistent.

For an explicit listing of this machine, see Appendix C.

We call this machine  $Z$ . One way to build this machine would be to start with the axioms of ZFC and apply the inference rules of first-order logic repeatedly in each possible way so as to enumerate every statement ZFC could prove, and to halt if ever a contradiction was found. While this method is conceptually simple, to actually construct such a machine would lead to a huge number of states, because it would require writing a program to manipulate the axioms of ZFC and the inference rules of first-order logic, and then compiling that program all the way down to Turing machine states.

#### 3.1 Friedman’s Mathematical Statement

Thankfully, a simpler method exists for creating  $Z$ . Friedman [6] was able to derive a graph-theoretic statement whose truth implies the consistency of ZFC, and which will be false if ZFC is inconsistent.<sup>2</sup> Here is Friedman’s statement (the notation will be explained in the rest of this

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<sup>1</sup>Because Fermat’s last theorem and the four color theorem have been proved, their complexity is now known to be 0.

<sup>2</sup>In fact, Friedman’s statement is equivalent to the consistency of SRP (“stationary Ramsey property”), which is a system of axioms more powerful than ZFC. Because SRP is strictly more powerful than ZFC (it in fact consists of

section):

**Statement 1.** *For all  $k, n, r > 0$ , every order invariant graph on  $[\mathbb{Q}]^{\leq k}$  has a free  $\{x_1, \dots, x_r, \text{ush}(x_1), \dots, \text{ush}(x_r)\}$  of complexity  $\leq (8knr)!$ , each  $\{x_1, \dots, x_{(8kni)!}\}$  reducing  $[x_1 \cup \dots \cup x_i \cup \{0, \dots, n\}]^{\leq k}$ . [6]*

A number of *complexity* at most  $c$  refers to a number that can be written as a fraction  $a/b$ , where  $a$  and  $b$  are both integers less than or equal to  $c$ . A set has complexity at most  $c$  if all the numbers it contains have complexity at most  $c$ .

An *order invariant graph* is a graph containing a countably infinite number of nodes. In particular, it has one node for each finite set of rational numbers. The only numbers relevant to the statement are numbers of complexity  $(8knr)!$  or smaller. In every description of nodes that follows, the term *node* refers both to the object in the order invariant graph and to the set of numbers that it represents.

In an order invariant graph, two nodes  $(a, b)$  have an edge between them if and only if each other pair of nodes  $(c, d)$  that is *order equivalent* with  $(a, b)$  has an edge between them. Two pairs of nodes  $(a, b)$  and  $(c, d)$  are *order equivalent* if  $a$  and  $c$  are the same size and  $b$  and  $d$  are the same size and if for all  $1 \leq i \leq |a|$  and  $1 \leq j \leq |b|$ , the  $i$ -th element of  $a$  is less than the  $j$ -th element of  $b$  if and only if the  $i$ -th element of  $c$  is less than the  $j$ -th element of  $d$ .

To give some trivial examples of order invariant graphs: the graph with no edges is order invariant, as is the complete graph. A less trivial example is a graph on  $[\mathbb{Q}]^2$ , in which each node corresponds to a set of two rational numbers of a given complexity, and there is an edge between two nodes if and only if their corresponding sets  $a$  and  $b$  satisfy  $a_1 < b_1 < a_2 < b_2$ . (Because edges are undirected in order invariant graphs, such an edge will exist if *either* assignment of the vertices to  $a$  and  $b$  satisfies the inequality above).

The  $\text{ush}()$  function takes as input a set and returns a copy of that set with all non-negative numbers in that set incremented by 1.

For vertices  $x$  and  $y$ ,  $x \leq_{\text{rlex}} y$  if and only if  $x = y$  or  $x_i < y_i$  where  $i$  is least such that  $x_i \neq y_i$ .

Finally, a set of vertices  $X$  *reduces* a set of vertices  $Y$  if and only if for all  $y \in Y$ , there exists  $x \in X$  such that  $x \leq_{\text{rlex}} y$  and an edge exists between  $x$  and  $y$ .

## 3.2 Implementation Methods

To create  $Z$ , we needed to design a Turing machine that halts if Statement 1 is false, and loops if Statement 1 is true. Such a Turing Machine's behavior would necessarily be independent of ZFC, because the truth or falsehood of Statement 1 is independent of ZFC, assuming the consistency of SRP, a slightly more powerful set theory. [6]

To design such a Turing machine, we wrote a Laconic program which encoded Friedman's statement, then compiled the program down to a description of a single-tape, 2-symbol Turing machine. What follows is an extremely brief description of the design of the Laconic program; for the documented Laconic code itself, along with a detailed explanation of the full compilation process, please see [18].

Our Laconic program begins by looping over all non-negative values for  $k$ ,  $n$ , and  $r$ . For each trio  $(k, n, r)$ , our program generates a list  $N$  of all numbers of complexity at most  $(8knr)!$ . These

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ZFC plus some additional axioms), the consistency of SRP implies the consistency of ZFC, and the inconsistency of ZFC implies the inconsistency of SRP.



numbers represent the vertices in our putative order invariant graph. Because Laconic does not support floating-point numbers, the list is entirely composed of integers; it is a list of all numbers that can be written in the form  $((8knr)!)((8kni)!)/((8knj)!)$ , where  $i$  and  $j$  are integers satisfying  $-(8knr)! \leq i \leq (8knr)!$  and  $1 \leq j \leq (8knr)!$ . (Note that any number that can be expressed in this form is necessarily an integer, because of the large scaling factor in front.)

After we generate  $N$ , we generate the nodes in a potential order invariant graph by adding to  $N$  all possible lists of  $k$  or fewer numbers from  $N$ . We call this list of lists  $V$ .

We iterate over all binary lists of length  $|V|^2$ . Any such list  $E$  represents a possible set of edges in the graph. To be more precise, we say that an edge exists between node  $i$  and node  $j$  (represented by  $V_i$  and  $V_j$  respectively) if and only if  $E_{i|V|+j}$  is 1.

For any graph  $(V, E)$ , we say that it is “valid” if the following three conditions hold:

1. No node has an edge to itself.
2. If an edge exists between node  $i$  and node  $j$ , an edge also exists between node  $j$  and node  $i$ .
3. The graph has a free  $\{x_1, \dots, x_r, \text{ush}(x_1), \dots, \text{ush}(x_r)\}$ , each  $\{x_1, \dots, x_{(8kni)!}\}$  reducing  $[x_1 \cup \dots \cup x_i \cup \{0, \dots, n\}]^{\leq k}$ .

For each list of nodes  $V$ , we loop over every possible binary list  $E$ , and if no pair  $(V, E)$  yields a valid graph, we halt.

When verifying the validity of a graph, checking the first two conditions is trivial, but the third merits further explanation. In order to verify that a given graph  $(V, E)$  has a free  $\{x_1, \dots, x_r, \text{ush}(x_1), \dots, \text{ush}(x_r)\}$ , each  $\{x_1, \dots, x_{(8kni)!}\}$  reducing  $[x_1 \cup \dots \cup x_i \cup \{0, \dots, n\}]^{\leq k}$ , we look at every possible subset of the nodes in  $V$ . For each subset, we verify that it has length  $r$ , that  $\text{ush}(x_1), \dots, \text{ush}(x_r)$  all exist in  $V$ , and for each  $i$  such that  $(8kni)! \leq r$ , that  $\{x_1, \dots, x_{(8kni)!}\}$  reduces  $[x_1 \cup \dots \cup x_i \cup \{0, \dots, n\}]^{\leq k}$ . Once we have found such a subset, we know that the third condition is satisfied.

## 4 A Turing Machine that Encodes Goldbach’s Conjecture

We present a 4,888-state Turing machine that *encodes Goldbach’s conjecture*; in other words, to know whether this machine halts is to know whether Goldbach’s conjecture is true. It is therefore impossible to prove the value of  $BB(4,888)$  without simultaneously proving or disproving Goldbach’s conjecture.

Recall that Goldbach’s conjecture is as follows:

**Statement 2.** Every even integer greater than 2 can be expressed as the sum of two primes.

Because Goldbach’s conjecture is so simple to state, the Laconic program encoding the statement is also quite simple. It can be found in Appendix A. A detailed explanation of the compilation process, documentation for the Laconic language, and an explicit description of this Turing machine are available at [18].

## 5 A Turing Machine that Encodes Riemann's Hypothesis

We present a 5,372-state Turing machine that *encodes Riemann's hypothesis*; in other words, to know whether this machine halts is to know whether Riemann's hypothesis is true. An explicit description of this machine can be found at [18]

Riemann's hypothesis is traditionally stated as follows:

**Statement 3.** The Riemann zeta function has its zeros only at the negative even integers and the complex numbers with real part  $1/2$ .

### 5.1 Equivalent Statement

Instead of encoding the Riemann zeta function into a Laconic program, it is simpler to use the following statement, which was shown by Lagarias to be equivalent to the Riemann hypothesis [4]:

**Statement 4.** For all integers  $n \geq 1$ ,

$$\left( \left( \sum_{k \leq \delta(n)} \frac{1}{k} \right) - \frac{n^2}{2} \right)^2 < 36n^3$$

The function  $\delta(n)$  used in Statement 4 is defined as follows:

$$\begin{aligned} \eta(j) &= p \text{ if } j = p^k, p \text{ is prime, } k \text{ is a positive integer} \\ \eta(j) &= 1 \text{ otherwise} \\ \delta(x) &= \prod_{n < x} \prod_{j \leq n} \eta(j) \end{aligned}$$

### 5.2 Implementation Methods

This statement is equivalent to the following statement, which contains only positive integers<sup>3</sup>:

$$l(n) < r(n)$$

for all positive integers  $n$ , where

$$\begin{aligned} l(n) &= (a(n))^2 + (b(n))^2 \\ r(n) &= 36n^3(\delta(n)!)^2 + 2a(n)b(n) \end{aligned}$$

$$\begin{aligned} a(n) &= \sum_{k \leq \delta(n)} \frac{\delta(n)!}{k} \\ b(n) &= n^2 \frac{\delta(n)!}{2} \end{aligned}$$

To check the Riemann hypothesis, our program computes  $a(n)$ ,  $b(n)$ ,  $l(n)$ , and  $r(n)$ , in that order, for each possible value of  $n$ . If  $l(n) \geq r(n)$ , our program halts.

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<sup>3</sup>Although it is not immediately obvious at first glance,  $\frac{\delta(n)!}{k}$  is necessarily an integer for all  $k \leq \delta(n)$ , and  $\frac{\delta(n)!}{2}$  is an integer for all  $n > 1$ .

## 6 Laconic

Laconic is a programming language designed to be both user-friendly and easy to compile down to parsimonious Turing machine descriptions.

Laconic is a strongly-typed language that supports recursive functions. Laconic compiles to an intermediate language called TMD. TMD programs are spread across multiple files and grouped into directories. TMD directories are meant to represent a sequence of commands that could be given to a multi-tape, 3-symbol Turing machine, using the Turing machine abstraction that allows the machine to read and write from one head at a time.

For an example of a Laconic program, see Appendix A. For a visual illustration of the compilation process, see Figure 1.

## 7 TMD

TMD is a programming language designed to help the user describe the behavior of a multi-tape, 3-symbol Turing machine with a function stack. Each tape is infinite in one direction and supports three symbols:  $\_$ , 1, and E. The blank symbol is  $\_$ : that is,  $\_$  is the only symbol that can appear on the tape an infinite number of times. The tape must always have the form  $\_?(1|E)^+\_$ ; in other words, each tape must always contain a string of 1's and E's of size at least 1, possibly preceded by a  $\_$  symbol, and necessarily followed by an infinite number of copies of the  $\_$  symbol.

What is the purpose of having a language like TMD as an intermediary between Laconic and a description of a single-tape machine? The concept of tapes in a multi-tape Turing machine and the concept of variables in standard imperative programming languages map to one another very nicely. The idea of the Laconic-to-TMD compiler is to encode the value of each variable on one tape. Then, each Laconic command that manipulates the value of one or more variables compiles down to a TMD function call that manipulates the tapes that correspond to those variables appropriately.

As an example, consider the following Laconic command:

```
a=b*c;
```

This Laconic command assigns the value of **a** to the value of **b\*c**. It compiles down to the following TMD function call:

```
function BUILTIN_multiply a b c
```

This function call will result in **BUILTIN\_multiply** being run on the three tapes **a**, **b**, and **c**. This will cause the symbols on tape **a** to take on a representation of an integer whose value is equal to  $bc$ .

In turn, the TMD code compiles directly to a string of bits that are written onto the tape at the start of the Turing machine's execution.

A TMD directory consists of three types of files:

1. The **functions** file. This file contains a list of the names of all the functions used by the TMD program. The top function in the file is pushed onto the stack at initialization. Moreover, when this top function returns, the Turing machine halts.

2. The `initvar` file. This file contains the non-`_` symbols that start in each register at initialization.
3. Any files used to describe TMD functions. These files all end in a `.tfn` extension and only have any relevance to the compiled program if they show up in the functions file.

## 8 Compilation and Processing

When discussing the layout of the tape symbols and patterns, there are two ways to think about it: one is with a 4-symbol alphabet (`{_, 1, H, E}`, blank symbol `_`), and one is with a 2-symbol alphabet (`{a, b}`, blank symbol `a`). The 2-symbol alphabet version is the one that's ultimately used for the results in this paper, since we advertised a Turing machine that used only two symbols. However, in nearly all parts of the Turing machine, the 2-symbol version of the machine is a direct translation of the 4-symbol version, according to the following mapping:

- `_`  $\leftrightarrow$  `aa`
- `1`  $\leftrightarrow$  `ab`
- `H`  $\leftrightarrow$  `ba`
- `E`  $\leftrightarrow$  `bb`

Additionally, the sections that follow may make reference to the `ERROR` state. Transitions to the `ERROR` state are stand-ins for transitions that should never be taken under any circumstances and are useful for debugging purposes.

### 8.1 Concept

A directory of TMD functions is converted at compilation time to a string of bits to be written onto the tape, along with other states designed to interpret these bits. The resulting Turing machine has three main components, or *submachines*:

1. The *initializer* sets up the basic structure of the variable registers and the function stack.
2. The *printer* writes down the binary string that corresponds to the compiled TMD code.
3. The *processor* interprets the compiled binary, modifying the variable registers and the function stack as necessary.

The Turing machine's control flow proceeds from the initializer to the printer to the interpreter. In other words, initializer states point only to initializer states or to printer states, printer states point only to printer states or to interpreter states, and interpreter states point only to interpreter states or the `HALT` state.

This division of labor, while seemingly straightforward, actually constitutes a very important and non-obvious idea. The problem of the compiler is to convert a higher-level representation—a machine with many tapes, a larger alphabet, and a function stack—to the lower-level representation of a machine with a single tape, a 2-symbol alphabet and no function stack. The immediately

obvious solution, and the one taught in every computability theory class as a proof of the equivalence of different kinds of Turing machines, is to have every “state” in the higher-level machine compile down to many states in the lower-level machine.

While simple, this approach is suboptimal in terms of the number of states. As is nearly always true when designing systems to be parsimonious, the clue that improvement is possible lies in the presence of repetition. Each state transition in the higher-level machine is converted to a group of lower-level states with the same basic structure. Why not instead explain how to perform this conversion exactly once, and then apply the conversion many times?

This idea is at the core of the division of labor described previously. We begin by writing a description of the higher-level machine onto the tape, and then “run” the higher-level machine by reading what is on the tape with a set of states that understands how to interpret the encoded higher-level machine. We refer to this idea as *on-tape processing*.

In this paper, we use TMD as the representation of the higher-level machine.<sup>4</sup> The printer writes the TMD program onto the tape, and the processor executes it. As a result of using this scheme, we incur a constant *additive* overhead—we have to include the processor in our final Turing machine—but we avoid the constant *multiplicative* overhead required for the naïve scheme.

Is this additive overhead small enough to be worth it? We found that it is. Our implementation of the processor requires 3,860 states. (See Section 8.5 for a detailed breakdown of the state cost by submachine.) In contrast to this additive overhead of 3,860, the naïve approach incurs a large multiplicative overhead that depends in part on how many states must be used to represent each higher-level state transition, and in part on how efficient an encoding scheme can be devised for the on-tape approach. The following table compares the performance of on-tape processing to the performance of an implementation that used the naïve approach. The comparison is shown for three kinds of machines: a machine that halts if and only if Goldbach’s conjecture is false, a machine that halts if and only if the Riemann hypothesis is false, and a machine whose behavior is independent of ZFC.

Program	States (Naïve)	States (On-Tape Processing)
Goldbach	7,902	4,888
Riemann	36,146	5,372
ZFC	340,943	8,013

As can be seen from this table, on-tape interpretation results in huge gains, particularly in large and complex programs.

The subsections that follow describe each of the three submachines—the initializer, the printer, and the processor—in greater detail.

## 8.2 The Initializer

The initializer starts by writing a counter onto the tape which encodes how many registers there will be in the program. Using the value in that counter, it creates each register, with demarcation

---

<sup>4</sup>Note that instead of TMD, the on-tape processing scheme could be used for any language, assuming the designer provides both a processor and an encoding for that language. We chose TMD because it made the interpreter easy to write, but other minimalist languages, like Unlambda [11], Brainf\*ck [14], or Iota and Jot [1], might be good candidates for parsimonious designs, with the additional advantage of being already known to some programmers! Thanks to Luke Schaeffer for this point.

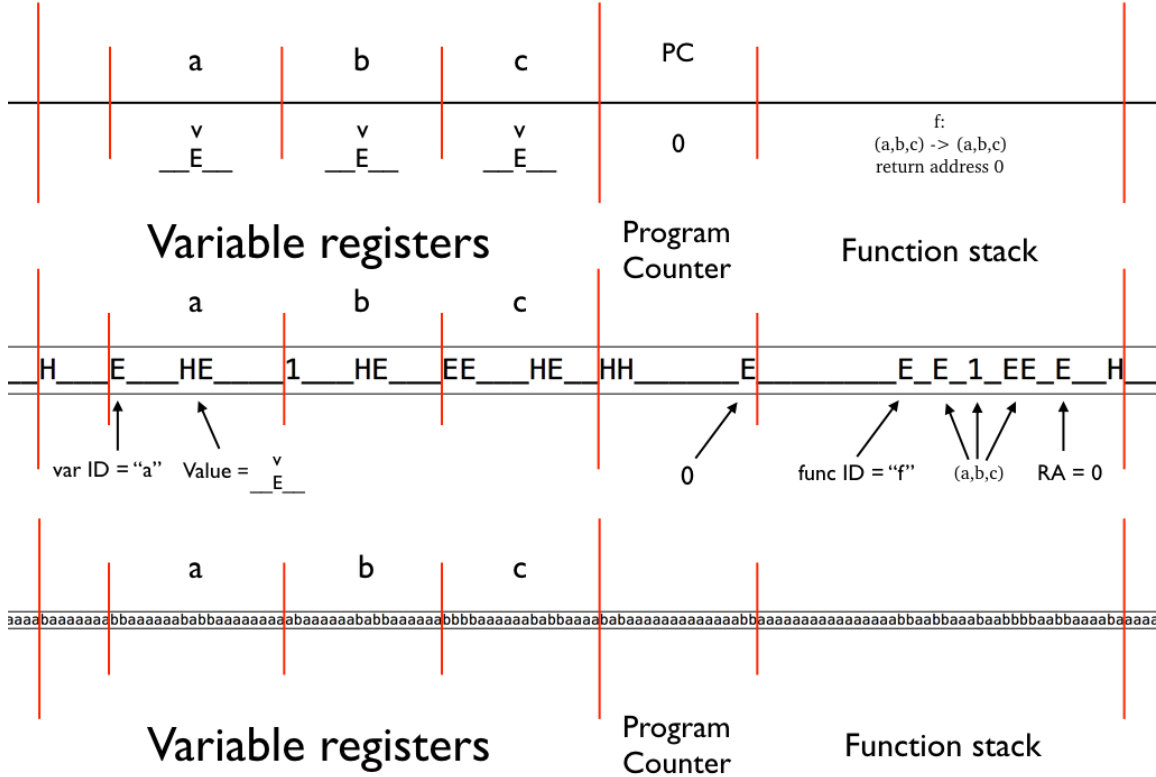


Figure 2: The state of the Turing machine tape after the initializer completes. The TMD program being expressed in Turing machine form is described in full in Appendix B. The top bar is a high-level description of what each part of the Turing machine tape represents. The middle bar is an encoding of the tape in the standard 4-symbol alphabet; the bottom bar is simply the translation of that tape into the 2-symbol alphabet. For a more detailed explanation of how to interpret the tape patterns, see [18].

patterns between registers, and unique identifiers for each register. Each register’s value begins with the pattern of non-`_` symbols laid out in the `initvar` file. The initializer also creates the program counter, which starts at 0, and the function stack, which starts out with only a single function call to the top function in the `functions` file.

Figure 2 is a detailed diagram describing the tape’s state when the initializer passes control to the printer.

## 8.3 The Printer

### 8.3.1 Specification

The printer writes down a long binary string which encodes the entirety of the TMD program onto the tape.

Figure 3 is a detailed diagram describing the tape’s state when the printer passes control to the processor.

### 8.3.2 Introspection

Writing down a long binary string onto a Turing machine tape in a parsimonious fashion is not as straightforward as it might initially appear. The first idea that comes to mind is simply to use one state per symbol, with each state pointing to the next, as shown in Figure 4.

Upon closer examination, however, it is apparent that this approach is quite wasteful for all but the smallest binary files. Every **a** transition points to the next state in the sequence, and none of the **b** transitions are used at all! Indeed, the only information-bearing part of the state is the single bit contained in the choice of which symbol to write. But in theory, far more information than that could be encoded with each state. In a machine that contains  $n$  states, each state could contain  $2(\log(n) + 1)$  bits of information, because each of its two transitions could point to any of the  $n$  states, and write either an **a** or a **b** onto the tape. Of course, this is only in theory; in practice, to extract the information contained in the Turing machine’s states and translate it into bits on the tape is nontrivial.

What we propose here is a scheme originally conceived by Ben-Amram and Petersen [2] and refined further and suggested to us by Luke Schaeffer. It does not achieve the optimal theoretical encoding described above, but is relatively simple to implement and understand, and is within a factor of 2 of optimal for large binary strings. Schaeffer named Turing machines that use this idea *introspective*.

Introspection works as follows. If the binary string contains  $k$  bits, then let  $w$  be the *word size*.  $w$  takes the largest value it can such that  $w2^w \leq k$ . We can split the binary string into  $n_w = \lceil \frac{k}{w} \rceil$  different *words* of size  $w$  bits each (we can pad the last word with copies of the blank symbol). In our scheme, each word in the bit-string will be represented by a *data state*. Each data state points to the state representing the next word in the sequence for its **a** transition, but which state the **b** transition points to will encode the next word. Every **b** transition points to one of the last  $2^w$  data states, thereby encoding  $w$  bits of information.

Of course, the encoding is useless until we specify how to extract the encoded bit-string from the data states. The extraction scheme works as follows. To query the  $i^{\text{th}}$  data state for the bits it encodes, we run the data states on the string  $\mathbf{a}^{i-1}\mathbf{b}\mathbf{a}^\infty$  (a string of  $i - 1$  **a**’s followed by a **b** in the  $i^{\text{th}}$  position). After running the data states on that string, what remains on the tape is the string  $\mathbf{b}^{i-1}\mathbf{a}\mathbf{b}^r\mathbf{a}^\infty$ , assuming that the  $i^{\text{th}}$  data state pointed to the  $r^{\text{th}}$ -to-last data state. Thus, what we are left with is essentially a unary encoding of the “value” of the word in binary. Thus, the job of the extractor is to set up a binary counter which removes one **b** at a time and increments the counter appropriately. Then, afterward, the extractor reverts the tape back to the form  $\mathbf{a}^i\mathbf{b}\mathbf{a}^\infty$ , shifts all symbols on the tape over by  $w$  bits, and repeats the process. Finally, when the state beyond the last data state sees a **b** on the tape, we know that the process has completed, and we can pass control to the processor. Figure 5 depicts the introspection algorithm.

How much have we gained by using an introspective technique for encoding the program binary, instead of the naïve approach? It depends on how large the program binary is. By using introspection, we incur an  $O(\log k)$  *additive* overhead, because we have to include the extractor in our machine. (Our implementation of the extractor takes  $10w + 17$  states.) But in return, we save a *multiplicative* factor of  $w$  (which scales with  $\log k$ ) on the number of data states needed.

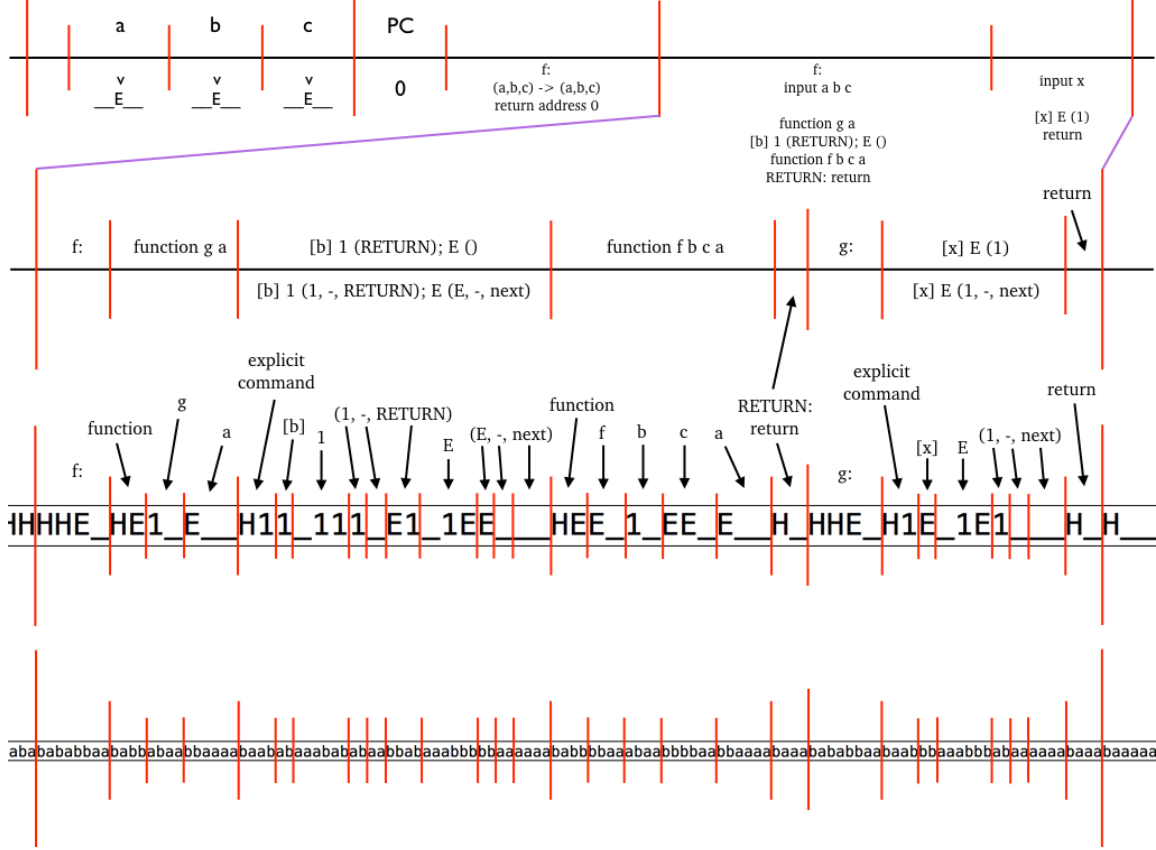


Figure 3: The state of the Turing machine tape after the printer completes. The TMD program being expressed in Turing machine form is described in full in Appendix B. The top bar is a high-level description of the entire tape; unfortunately, at this point there are so many symbols on the tape that it is impossible to see everything at once. For a detailed view of the first two-thirds of the tape (registers, program counter, and stack), see Figure 2. The bottom three bars show a zoomed-in view of the program binary. From the top, the second bar gives a high-level description of what each part of the program binary means; the third bar gives the direct correspondence between 4-symbol alphabet symbols on the tape and their meaning in TMD; the fourth and final bar gives the translation of the third bar into the 2-symbol alphabet. For a more detailed explanation of the encoding of TMD into tape symbols, see [18].



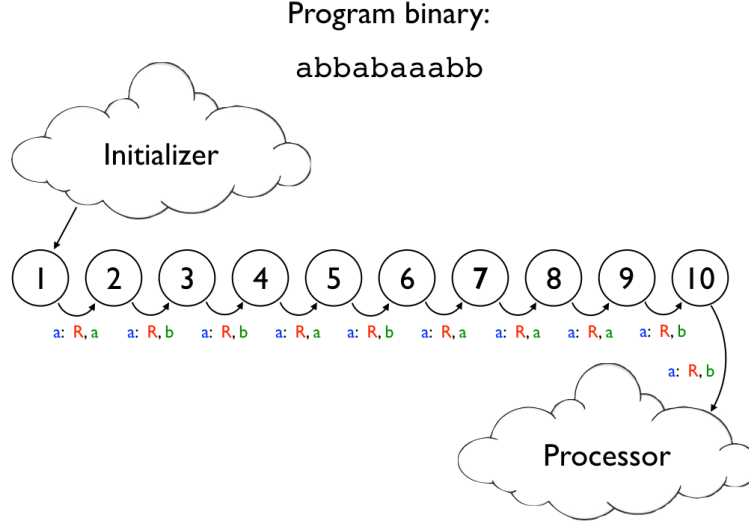


Figure 4: A naïve implementation of the printer. In this example, the hypothetical program is ten bits long, and the printer uses ten states, one for each bit. In the diagram, the blue symbol is the symbol that is read on a transition, the red letter indicates the direction the head moves, and the green symbol indicates the symbol that it written. Note the lack of transitions on reading a **b**; this is because in this implementation, the printer will only ever read the blank symbol, which is **a**, since the head is always proceeding to untouched parts of the tape. It therefore makes no difference what behavior the Turing machine adopts upon reading a **b** in states 1-10 (and therefore **b** transitions are presumed to lead to the **ERROR** state)

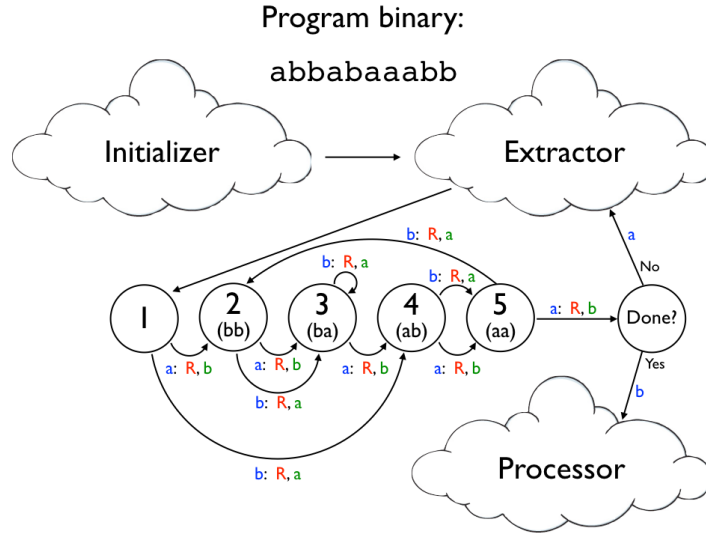


Figure 5: An introspective implementation of the printer. In this example, the hypothetical program is  $k = 10$  bits long, and so the word size must be 2 (since  $w = 2$  is the largest  $w$  such that  $w2^w \leq 10$ ). There are therefore  $n_w = \left\lceil \frac{k}{w} \right\rceil = 5$  data states, each encoding two bits. The **b** transitions carry the information about the encoding; note that each one only points to one of the last four data states. The last four data states have in parentheses what word we mean to encode if we point to them.

This is plainly not worth it for the 10-bit example binary shown in Figs. 4 and 5. For that binary, we require 69 additional states for the extractor in order to save 5 states on the data states. For real programs, however, it is worth it, as can be seen from the following table.

Program	Binary Size	$w$	$n_w$	Extractor Size	States (Naïve)	States (Introspective)
Example TMD	116	4	29	57	116	86
Goldbach	4,964	9	552	107	4,964	659
Riemann	9,532	10	1,024	117	9,532	1,141
ZFC	35,906	11	3,265	127	35,906	3,392

One minor detail concerns the numbers presented for the Riemann program. Ordinarily, with a binary of size 9,532, we would opt to split the program into 1,060 words of 9 bits each plus a 107-state extractor, since 9 is the greatest  $w$  such that  $w2^w < 9,532$ . But because 9,532 is so close to the “magic number” 10,240, it’s actually more parsimonious to pad the program with copies of the blank symbol until it’s 10,240 bits long, and split it into 1,024 words of 10 bits each plus a 117-state extractor.

## 8.4 The Processor

The processor’s job is to interpret the code written onto the tape and modify the variable registers and function stack accordingly. The processor does this by the following sequence of steps:

START:

1. Find the function call at the top of the stack. Mark the function  $f$  in the code whose ID matches that of the top function call.
2. Read the current program counter. Mark the line of code  $l$  in  $f$  whose line number matches the program counter.
3. Read  $l$ . Depending on what type of command  $l$  is, carry out one of the following three lists of tasks.

IF  $l$  IS AN EXPLICIT TAPE COMMAND:

1. Read the variable name off  $l$ . Index the variable name into the list of variables in the top function on the stack. This list of variables corresponds to the mapping between the function’s local variables and the register names.
2. Match the indexed variable to its corresponding register  $r$ . Mark  $r$ . Read the symbol  $s_r$  to the right of the head marker in that register.
3. Travel back to  $l$ , remembering the value of  $s_r$  using states. Find and mark the reaction  $x$  corresponding to the symbol. See what symbol  $s_w$  should be written in response to reading  $s_r$ .
4. Travel back to  $r$ , remembering the value of  $s_w$  using states. Replace  $s_r$  with  $s_w$ .

5. Travel back to  $x$ . See which direction  $d$  the head should move in response to reading  $s_r$ .
6. Travel back to  $r$ , remembering the value of  $d$  using states. Move the head marker accordingly.
7. Travel back to  $x$ . See if a jump is specified. If a jump is specified, copy the jump address onto the program counter. Otherwise, increment the program counter by 1.
8. Go back to START.

IF  $l$  IS A FUNCTION CALL:

1. Write the function's name to the top of the stack.
2. For each variable in the function call, index the variable name into the list of variables in the top function on the stack. This list of variables corresponds to the mapping between the function's local variables and the register names. Push the corresponding register names in the order that they correspond to the variables in the function call.
3. Copy the current program counter to the return address of the newborn function call at the top of the stack.
4. Replace the current program counter with 0 (meaning "read the first line of code").
5. Go back to START.

IF  $l$  IS A RETURN STATEMENT:

1. Replace the current program counter with  $f$ 's return address.
2. Increment the program counter by 1.
3. Erase the call to  $f$  from the top of the stack.
4. Check if the stack is now empty. If so, halt.
5. Go back to START.

## 8.5 Cost Analysis

It is worthwhile to analyze the relative contributions of the initializer, the printer, and the processor to the machine's final state count. The following table lists the number of states in each submachine for each of the four different TMD programs under discussion.

Program	Initializer	Printer	Processor	Total
Example TMD	349	86	3,860	4,295
Goldbach	369	659	3,860	4,888
Riemann	371	1,141	3,860	5,372
ZFC	389	3,392	3,860	8,013

As can be seen from this table, the processor makes the largest contribution to all four programs. Improving the processor, therefore, is probably the best approach for improving upon the bounds we present. Equally clear, however, is that for programs more complicated than the ones presented here, the cost of the printer will grow almost linearly but the cost of the processor will stay the same. The cost of the initializer grows very slightly with the complexity of programs because of the need to initialize additional registers.

Improving the printer, and with it the TMD and Laconic languages, is probably the best approach for reducing state count for very large and complex programs.

## 9 Future Work

How much further can  $Z$ 's state count be reduced? Without a significant new insight, further order-of-magnitude reductions seem unlikely via tweaks to our existing system: note that such tweaks would need to reduce the sizes of both the printer and the processor by an order of magnitude. However, improvement is possible by (for example) a redesigned processor, combined with a simpler independent mathematical statement than Friedman's.

Other future work might involve further use of our Laconic language to upper-bound the 'complexities' of mathematical statements and algorithms, in as standardized and model-independent a way as possible. Perhaps Laconic could be used to measure the complexity of other well-known conjectures, or even to compare different algorithms for solving the same problem to each other (e.g. to try to quantify the notion that an insertion sort is simpler than a merge sort)!

## 10 Acknowledgements

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We thank Alex Arkhipov for introducing us to the term "code golfing."

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# Appendices

## A Example Laconic Program: Goldbach's Conjecture

The following is an example Laconic program, which compiles down to the Turing machine  $G$  mentioned in Section 4 (which halts if and only Goldbach's Conjecture is false).

```
func zero(x) {
    x = 0;
    return;
}

func one(x) {
    x = 1;
    return;
}

func incr(x) {
    x = x + 1;
    return;
}

/* Computes x modulo y */
func modulus(x, y, out) {
    out = x;

    while (out >= y) {
        out = out - y;
    }

    return;
}

func assignXtoYminusX(x, y) {
    x = y - x;
    return;
}

/* Figures out if x is prime, and puts the output in y */
/* Does not modify x, modifies y */
func isPrime(x, h, y) {
    if (x == 1) {
        zero(y);
        return;
    }

    y = 2;

    while (x > y) {
        modulus(x, y, h);

        if (h == 0) {
            zero(y);
            return;
        }
        incr(y);
    }

    return;
}

int evenNumber;
```

```

int primeCounter;
int isThisOnePrime;
int foundSum;
int h;

evenNumber = 2;
one(foundSum);

while (foundSum) {
    zero(foundSum);
    evenNumber = evenNumber + 2;
    one(primeCounter);

    while (primeCounter < evenNumber) {
        isPrime(primeCounter, h, isThisOnePrime);

        if (isThisOnePrime) {
            assignXtoYminusX(primeCounter, evenNumber);
            isPrime(primeCounter, h, isThisOnePrime);
            assignXtoYminusX(primeCounter, evenNumber);

            if (isThisOnePrime) {
                print evenNumber;
                print primeCounter;

                one(foundSum);
            }
        }

        incr(primeCounter);
    }
}

halt;

```

For detailed documentation of the Laconic programming language, see [18]. To find this file specifically, navigate to `parsimony/src/laconic/laconic_files/goldbach.lac` at [18].

## B Example TMD Program

The following is an example TMD directory, which compiles down to a binary string to be written on a Turing machine tape. It is the example that is used in illustrations throughout this paper, most notably in the example compilation shown in Figs. 2 and 3. The program calls itself recursively three times until the starting symbol on each tape, E, is replaced with a 1, at which point the program halts.

This TMD directory is called `example_tmd_dir`, and contains four files: `f.tmd`, `g.tmd`, `initvar`, and `functions`.

```

f.tmd:

input a b c

// Recursively writes a 1 on every tape.

function g a
[b] 1 (RETURN); E ()
function f b c a
RETURN: return

```

```

g.tmd:
input x
// Writes a 1 on the input tape.

[x] E (1)
return
  functions:
f
g
  initvar:
E

```

For detailed documentation of the TMD programming language, see [18]. To find this directory specifically, navigate to `parsimony/src/tmd/tmd_dirs/example_tmd_dir/` at [18].

## C Explicit Listing of $Z$

### A description of a single state in $Z$

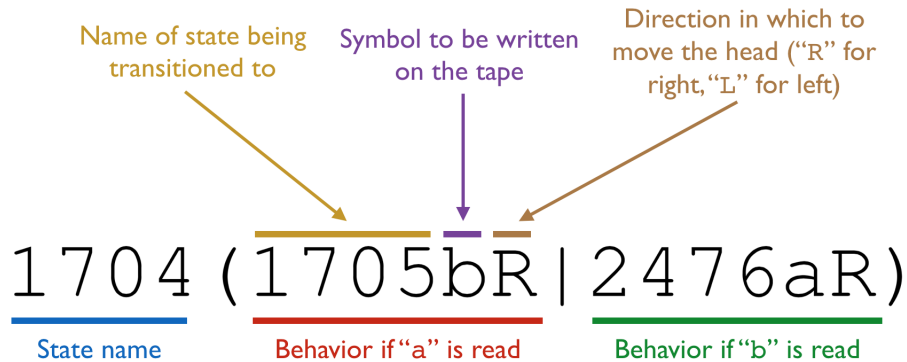


Figure 6: This figure explains how to read a description of a single state. Note that “**ERROR-**” or “**HALT--**” denote transitions to the **ERROR** or **HALT** states, respectively (no further information is provided because what symbol is written and which direction the head moves are at that point irrelevant).

We present below an explicit listing of  $Z$ . For a more easily readable version of  $Z$ , complete with descriptive state names, see [18].

We ran this Turing Machine for 10,000,000,000 steps (more than half a day on our simulators) and within that time it did not halt. We note, however, that it was designed for parsimony and not for efficiency, and that this “experiment” is of little consequence! We similarly designed and ran a Turing machine designed to test the conjecture that all perfect squares are less than 5, and it



ran for 2,895,083,899 steps (a couple hours on our simulators) before it found the counterexample 9 and halted.

Figure 6 presents useful information for how to interpret the description shown below. In addition, note the following:

1. The tape has a 2-symbol alphabet, with tape symbols  $\{a, b\}$  and blank symbol  $a$  (in other words,  $a$  is the only symbol that can appear an infinite times on the tape).
2. The start state of  $Z$  is 0000.
3.  $Z$  will never transition to the ERROR state. Any transition to the ERROR state could be replaced by a transition to any other state (including HALT) and the Turing machine's behavior would remain identical.
4.  $Z$  contains only one transition to the HALT state, out of state 7593.

0000(0001bR ERROR-)	0001(0004bR ERROR-)	0002(0003bR ERROR-)	0003(0012aR 0012bR)	0004(0005bR ERROR-)	0005(0006bR ERROR-)	0006(0007aR ERROR-)	0007(0008bR ERROR-)	0008(0009aR ERROR-)	0009(0010bR ERROR-)
0010(0011aR ERROR-)	0011(0020bR ERROR-)	0012(0013aR ERROR-)	0013(0014aR 0014bR)	0014(0015aR ERROR-)	0015(0057aR 0057bR)	0016(0017bR ERROR-)	0017(0018bR ERROR-)	0018(0019aR ERROR-)	0019(0020aR 0020bR)
0020(0021aR 0021bR)	0021(0022aR 0022bR)	0022(0023aR ERROR-)	0023(0024aR 0024bR)	0024(0025aR ERROR-)	0025(0067aR 0067bR)	0026(0027aR 0027bR)	0027(0028aR 0030bR)	0028(0029aR 0029bR)	0029(0029aR 0029bR)
0030(0031aR 0031bR)	0031(0026aR 0026bR)	0032(0033aR 0033bR)	0033(0034aR 0034bR)	0034(0037aR 0037bR)	0035(0037aR 0037bR)	0036(0038aR 0038bR)	0037(0038aR 0041bR)	0038(0040aR 0039bR)	0039(0040aR 0040bR)
0040(0049aR 0049bR)	0041(0049aR 0042bR)	0042(0043aR 0043bR)	0043(0037aR 0037bR)	0044(0045aR 0048bR)	0045(0049aR 0046aR)	0046(0047aR 0047bR)	0047(0049aR 0049bR)	0048(0071aR ERROR-)	0049(0050aR 0051bR)
0050(0049aR 0049bR)	0051(0052aR 0049bR)	0052(0053aR 0054bR)	0053(0054aR 0052bR)	0054(0055aR 0052bR)	0055(0056aR 0056aR)	0056(0012aR 0012bR)	0057(0058aR ERROR-)	0058(0059aR 0059bR)	0059(0060aR ERROR-)
0060(0061aR 0061bR)	0061(0062aR 0063bR)	0062(0063aR 0063bR)	0063(0064aR 0063bR)	0064(0065aR 0065bR)	0065(0066aR 0066bR)	0066(0016aR 0016bR)	0067(0068aR 0067bR)	0068(0069aR 0069bR)	0069(0070bR ERROR-)
0070(0026aR 0026bR)	0071(0072aR 0075bR)	0072(0071aR 0073bR)	0073(0074aR 0074bR)	0074(0075aR 0078bR)	0075(0076aR 0076bR)	0076(0077aR 0077bR)	0077(0078aR 0078bR)	0078(0079aR 0082bR)	0079(0080aR 0079bR)
0080(0081aR 0081bR)	0081(0083aR 0083bR)	0082(0079aR 0079bR)	0083(0084aR 0085bR)	0084(0085aR 0083bR)	0085(0086aR 0083bR)	0086(0087aR 0087bR)	0087(0088aR 0088bR)	0088(0089aR 0094bR)	0089(0090aR 0092bR)
0090(0091aR 0091bR)	0091(0099aR 0099bR)	0092(0091aR 0093bR)	0093(0088aR 0088bR)	0094(0099aR 0097bR)	0095(0096aR 0096bR)	0096(0098aR 0098bR)	0097(0098aR 0098bR)	0098(0098aR 0098bR)	0099(0100aR 0105bR)
0100(0101aR 0103bR)	0101(0102aR 0102bR)	0102(0099aR 0099bR)	0103(0104aR 0104bR)	0104(0101aR 0110bR)	0105(0106aR 0108bR)	0106(0107aR 0107bR)	0107(0139aR 0139bR)	0108(0109aR 0109bR)	0109(0110aR 0110bR)
0110(0111aR 0116bR)	0111(0112aR 0114bR)	0112(0113bR 0113bR)	0113(0113aR 0130bR)	0114(0115bR 0115bR)	0115(0110aR 0110bR)	0116(0117aR 0117bR)	0117(0118aR 0118bR)	0118(0119aR 0119bR)	0119(0120aR 0125bR)
0120(0121aR 0123bR)	0121(0122aR 0122bR)	0122(0130aR 0130bR)	0123(0124aR 0124bR)	0124(0101aR 0119bR)	0125(0126aR 0128bR)	0126(0127aR 0127bR)	0127(0119aR 0119bR)	0128(0129aR 0129bR)	0129(0119aR 0119bR)
0130(0131aR 0136bR)	0131(0132aR 0134bR)	0132(0133aR 0133bR)	0133(0130aR 0130bR)	0134(0135aR 0135bR)	0135(0088aR 0088bR)	0136(0088aR 0088bR)	0137(0138aR 0138bR)	0138(0088aR 0088bR)	0139(0140aR 0143bR)
0140(0139aR 0141bR)	0141(0142aR 0142bR)	0142(0146aR 0146bR)	0143(0144aR 0144bR)	0144(0145aR 0145bR)	0145(0146aR 0146bR)	0146(0147aR 0150bR)	0147(0148aR 0149bR)	0148(0149aR 0149bR)	0149(0071aR 0149bR)
0150(0151aR 0146bR)	0151(0152aR 0152bR)	0152(0153aR 0153bR)	0153(0154aR 0153bR)	0154(0155aR 0155bR)	0155(0156aR 0156bR)	0156(0157aR 0157bR)	0157(0158aR 0163bR)	0158(0159aR 0161bR)	0159(0160aR 0160bR)
0160(0157aR 0157bR)	0161(0162aR 0162bR)	0162(0166aR 0166bR)	0163(0197aR 0164bR)	0164(0165aR 0165bR)	0165(0166aR 0166bR)	0166(0167aR 0172bR)	0167(0168aR 0170bR)	0168(0169bR 0169bR)	0169(0177aR 0177bR)
0170(0171aR 0171bR)	0171(0166aR 0166bR)	0172(0173aR 0173bR)	0173(0174aR 0174bR)	0174(0166aR 0166bR)	0175(0176aR 0176bR)	0176(0166aR 0166bR)	0177(0178aR 0183bR)	0178(0179aR 0181bR)	0179(0180aR 0180bR)
0180(0177aR 0177bR)	0181(0182aR 0182bR)	0182(0186aR 0186bR)	0183(0187aR 0184bR)	0184(0185aR 0185bR)	0185(0186aR 0186bR)	0186(0187aR 0192bR)	0187(0188aR 0190bR)	0188(0189aR 0189bR)	0189(0157aR 0157bR)
0190(0191aR 0191bR)	0191(0186aR 0186bR)	0192(0193aR 0195bR)	0193(0194aR 0194bR)	0194(0186aR 0186bR)	0195(0196aR 0196bR)	0196(0186aR 0186bR)	0197(0198aR 0199bR)	0198(0197aR 0197bR)	0199(0200aR 0197bR)
0200(0201aR 0204bR)	0201(0202aR 0197bR)	0202(0203bR 0203bR)	0203(0205aR 0205bR)	0204(0206aR 0197bR)	0205(0206aR 0197bR)	0206(0207aR 0207bR)	0207(0208aR ERROR-)	0208(0209aR 0209bR)	0209(0210aR ERROR-)
0210(0219aR 0219bR)	0211(0213bR ERROR-)	0212(0213bR ERROR-)	0213(0214aR ERROR-)	0214(0233aR 0233bR)	0215(0216aR ERROR-)	0216(0217bR ERROR-)	0217(0218bR ERROR-)	0218(0263aR ERROR-)	0219(0220aR ERROR-)
0220(0221aR 0221bR)	0221(0222aR ERROR-)	0222(0223aR 0223bR)	0223(0224aR ERROR-)	0224(0225aR 0225bR)	0225(0226aR ERROR-)	0226(0227aR 0227bR)	0227(0228aR ERROR-)	0228(0229aR 0229bR)	0229(0230aR ERROR-)
0230(0231aR 0231bR)	0231(0232aR ERROR-)	0232(0211aR 0211bR)	0233(0234aR ERROR-)	0234(0237bR ERROR-)	0235(0236aR ERROR-)	0236(0245aR 0245bR)	0237(0238aR ERROR-)	0238(0239aR ERROR-)	0239(0240aR ERROR-)
0240(0241aR ERROR-)	0241(0242bR ERROR-)	0242(0243bR ERROR-)	0243(0244bR ERROR-)	0244(0245bR ERROR-)	0245(0246bR ERROR-)	0246(0247bR ERROR-)	0247(0248bR ERROR-)	0248(0249bR ERROR-)	0249(0250aR ERROR-)
0250(0251aR 0251bR)	0251(0252aR ERROR-)	0252(0253aR 0253bR)	0253(0254aR 0254bR)	0254(0255aR 0255bR)	0255(0256aR ERROR-)	0256(0257aR 0257bR)	0257(0258aR ERROR-)	0258(0259aR 0259bR)	0259(0260aR ERROR-)
0260(0261aR 0261bR)	0261(0262aR ERROR-)	0262(0215aR 0215bR)	0263(0264aR 0269bR)	0264(0265aR 0267bR)	0265(0266aR 0266bR)	0266(0267aR 0267bR)	0267(0268aR 0268bR)	0268(0269aR 0269bR)	0269(0270aR 0272bR)
0270(0271aR 0271bR)	0271(0272aR 0273bR)	0272(0273aR 0273bR)	0273(0313aR 0313bR)	0274(0308aR 0308bR)	0275(0308aR 0308bR)	0276(0309aR 0309bR)	0277(0309aR 0309bR)	0278(0309aR 0309bR)	0279(0309aR 0309bR)
0280(0274aR 0274bR)	0281(0282aR 0285bR)	0282(0283aR 0283bR)	0283(0284aR 0284bR)	0284(0285aR 0288bR)	0285(0286aR ERROR-)	0286(0287aR 0287bR)	0287(0300aR 0300bR)	0288(0289aR 0290bR)	0289(0288aR 0288bR)
0290(0291aR 0291bR)	0291(0292aR 0292bR)	0292(0293aR 0293bR)	0293(0294aR 0291bR)	0294(0295aR 0295bR)	0295(0296aR 0296bR)	0296(0297aR ERROR-)	0297(0298aR ERROR-)	0298(0299aR 0317aR)	0299(0300aR 0305bR)
0300(0301aR 0302bR)	0301(0302aR 0302bR)	0302(0303aR 0303bR)	0303(0304aR 0305bR)	0304(0305aR 0306bR)	0305(0306aR 0306bR)	0306(0307aR 0307bR)	0307(0308aR 0308bR)	0308(0309aR 0310bR)	0309(0308aR 0308bR)
0310(0311aR 0306bR)	0311(0312aR 0312bR)	0312(0313aR 0313bR)	0313(0314aR 0317bR)	0314(0315aR 0315bR)	0315(0316aR 0316bR)	0316(0317aR 0317bR)	0317(0318aR 0318bR)	0318(0319aR 0319bR)	0319(0320aR 0306bR)
0320(0321aR 0324bR)	0321(0320aR 0322aR)	0322(0323aR 0323bR)	0323(0324aR 0324bR)	0324(0325aR 0327aR)	0325(0326aR 0326bR)	0326(0327aR 0327bR)	0327(0328aR 0328bR)	0328(0329aR 0329bR)	0329(0330aR 0333bR)
0330(0331aR 0334bR)	0331(0332aR 0332bR)	0332(0333aR 0333bR)	0333(0334aR 0336bR)	0334(0335aR 0335bR)	0335(0336aR 0336bR)	0336(0337aR 0337bR)	0337(0338aR 0338bR)	0338(0339aR 0344bR)	0339(0340aR 0342bR)
0340(0341aR 0341bR)	0341(0320aR 0320bR)	0342(0343aR 0343bR)	0343(0329aR 0329bR)	0344(0345aR 0338bR)	0345(0346aR 0346bR)	0346(0347aR 0347bR)	0347(0348aR 0348bR)	0348(0349aR 0349bR)	0349(0350aR 0355bR)
0350(0351aR 0354bR)	0351(0352aR 0352bR)	0352(0353aR 0353bR)	0353(0354aR 0354bR)	0354(0355aR 0355bR)	0355(0356aR 0356bR)	0356(0357aR 0357bR)	0357(0358aR 0358bR)	0358(0359aR 0359bR)	0359(0360aR 0357bR)
0360(0361aR 0366bR)	0361(0362aR 0364bR)	0362(0363aR 0363bR)	0363(0313aR 0313bR)	0364(0365aR 0365bR)	0365(0366aR 0366bR)	0366(0367aR 0367bR)	0367(0368aR 0368bR)	0368(0371aR 0371bR)	0369(0370aR 0370bR)
0370(0360aR 0360bR)	0371(0372aR 0375bR)	0372(0373aR 0371bR)	0373(0374aR 0374bR)	0374(0308aR 0308bR)	0375(0376aR 0371bR)	0376(0377aR 0377bR)	0377(0378aR 0378bR)	0378(0379aR 0379bR)	0379(0380aR ERROR-)
0380(0381aR ERROR-)	0381(0382aR 0382bR)	0382(0383aR 0383bR)	0383(0384aR 0384bR)	0384(0385aR 0385bR)	0385(0386aR 0386bR)	0386(0387aR 0387bR)	0387(0388aR 0388bR)	0388(0389aR 0389bR)	0389(0390aR 0390bR)
0390(0391bR ERROR-)	0391(0395aR ERROR-)	0392(0393aR 0392aR)	0393(0394aR 0394bR)	0394(0415aR 0394bR)	0395(0396aR 0396aR)	0396(0397bR 0318aR)	0397(0398aR 0357aR)	0398(0399aR 0396aR)	0399(0400bR 0310aR)
0400(0401aR 0216aR)	0401(0402bR 0392aR)	0402(0403bR 0252aR)	0403(0404bR 0208aR)	0404(0405bR 0198aR)	0405(0406bR 0390aR)	0406(0407bR 0319aR)	0407(0408bR 0237aR)	0408(0409bR 0214aR)	0409(0410bR 0252aR)
0410(0411bR 0212bR)	0411(0412bR 0327aR)	0412(0413bR 0385bR)	0413(0414bR 0354bR)	0414(0415bR 0276aR)	0415(0416bR 0261aR)	0416(0417bR 0305aR)	0417(0418bR 0354aR)	0418(0419bR 0354aR)	0419(0420bR 0211aR)
0420(0421bR 0215aR)	0421(0422bR 0307aR)	0422(0423bR 0263aR)	0423(0424bR 0339aR)	0424(0425bR 0318aR)	0425(0426bR 0207aR)	0426(0427bR 0254aR)	0427(0428bR 0251aR)	0428(0429bR 0304aR)	0429(0430bR 0352aR)
0430(0431bR 0296aR)	0431(0432bR 0310aR)	0432(0433bR 0244aR)	0433(0434bR 0394aR)	0434(0435bR 0217aR)	0435(0436bR 0361aR)	0436(0437bR 0254aR)	0437(0438bR 0310aR)	0438(0439bR 0353aR)	0439(0440bR 0347aR)
0440(0441bR 0254aR)	0441(0442bR 0210aR)	0442(0443bR 0324aR)	0443(0444bR 0324aR)	0444(0445bR 0296aR)	0445(0446bR 0303aR)	0446(0447bR 0303aR)	0447(0448bR 0303aR)	0448(0449bR 0303aR)	0449(0450bR 0205aR)
0450(0451bR 0217aR)	0451(0452bR 0210aR)	0452(0453bR 0250aR)	0453(0454bR 0236aR)	0454(0455bR 0217aR)	0455(0456bR 0305aR)	0456(0457bR 0387aR)	0457(0458bR 0293aR)	0458(0459bR 0226aR)	0459(0460bR 0312aR)
0460(0461bR 0299aR)	0461(0462bR 0191aR)	0462(0463bR 0254aR)	0463(0464bR 0206aR)	0464(0465bR 0207aR)	0465(0466bR 0385aR)	0466(0467bR 0379aR)	0467(0468bR 0289aR)	0468(0469bR 0353aR)	0469(0470bR 0301aR)
0470(0471bR 0267aR)	0471(0472bR 0210aR)	0472(0473bR 0324aR)	0473(0474bR 0324aR)	0474(0475bR 0276aR)	0475(0476bR 0397aR)	0476(0477bR 0397aR)	0477(0478bR 0396aR)	0478(0479bR 0396aR)	0479(0480bR 0243aR)
0480(0481bR 0370aR)	0481(0482bR 0339aR)	0482(0483bR 0269aR)	0483(0484bR 0310aR)	0484(0485bR					

1050 (1051br|2923ar) 1051 (1052br|3962ar) 1052 (1053br|3218ar) 1053 (1054br|2399ar) 1054 (1055br|3146ar) 1055 (1056br|3639ar) 1056 (1057br|3304ar) 1057 (1058br|2398ar) 1058 (1059br|3859ar) 1059 (1060br|2845ar) 1060 (1061br|2552ar) 1061 (1062br|2375ar) 1062 (1063br|3406ar) 1063 (1064br|2263ar) 1064 (1065br|2808ar) 1065 (1066br|3668ar) 1066 (1067br|2223ar) 1067 (1068br|2911ar) 1068 (1069br|3476ar) 1069 (1070br|2078ar) 1070 (1071br|3575ar) 1071 (1072br|2511ar) 1072 (1073br|3513ar) 1073 (1074br|2068ar) 1074 (1075br|3413ar) 1075 (1076br|2951ar) 1076 (1077br|3451ar) 1077 (1078br|2908ar) 1078 (1079br|3478ar) 1079 (1080br|3170ar) 1080 (1081br|2730ar) 1081 (1082br|3548ar) 1082 (1083br|2237ar) 1083 (1084br|2325ar) 1084 (1085br|3264ar) 1085 (1086br|2101ar) 1086 (1087br|2894ar) 1087 (1088br|3126ar) 1088 (1089br|3176ar) 1089 (1090br|2674ar) 1090 (1091br|3341ar) 1091 (1092br|3124ar) 1092 (1093br|2210ar) 1093 (1094br|3614ar) 1094 (1095br|3517ar) 1095 (1096br|3131ar) 1096 (1097br|2557ar) 1097 (1098br|3161ar) 1098 (1099br|2927ar) 1099 (1100br|2781ar) 1100 (1101br|2557ar) 1101 (1102br|3159ar) 1102 (1103br|2740ar) 1103 (1104br|3127ar) 1104 (1105br|2973ar) 1105 (1106br|3195ar) 1106 (1107br|2973ar) 1107 (1108br|3126ar) 1108 (1109br|3164ar) 1109 (1110br|2927ar) 1110 (1111br|2740ar) 1111 (1112br|2265ar) 1112 (1113br|2941ar) 1113 (1114br|2055ar) 1114 (1115br|3179ar) 1115 (1116br|2565ar) 1116 (1117br|3704ar) 1117 (1118br|3614ar) 1118 (1119br|3629ar) 1119 (1120br|2097ar) 1120 (1121br|2923ar) 1121 (1122br|3936ar) 1122 (1123br|2158ar) 1123 (1124br|3612ar) 1124 (1125br|2927ar) 1125 (1126br|2513ar) 1126 (1127br|3184ar) 1127 (1128br|3937ar) 1128 (1129br|2211ar) 1129 (1130br|3497ar) 1130 (1131br|1999ar) 1131 (1132br|3974ar) 1132 (1133br|2764ar) 1133 (1134br|3127ar) 1134 (1135br|2959ar) 1135 (1136br|3127ar) 1136 (1137br|3127ar) 1137 (1138br|3127ar) 1138 (1139br|3127ar) 1139 (1140br|3127ar) 1140 (1141br|2124ar) 1141 (1142br|3550ar) 1142 (1143br|3195ar) 1143 (1144br|3012ar) 1144 (1145br|3291ar) 1145 (1146br|3291ar) 1146 (1147br|3127ar) 1147 (1148br|3127ar) 1148 (1149br|3127ar) 1149 (1150br|3127ar) 1150 (1151br|2220ar) 1151 (1152br|3936ar) 1152 (1153br|2767ar) 1153 (1154br|2800ar) 1154 (1155br|2171ar) 1155 (1156br|3399ar) 1156 (1157br|3127ar) 1157 (1158br|3614ar) 1158 (1159br|3859ar) 1159 (1160br|2401ar) 1160 (1161br|2813ar) 1161 (1162br|3937ar) 1162 (1163br|2754ar) 1163 (1164br|3131ar) 1164 (1165br|3893ar) 1165 (1166br|3831ar) 1166 (1167br|2046ar) 1167 (1168br|3390ar) 1168 (1169br|3534ar) 1169 (1170br|3364ar) 1170 (1171br|2880ar) 1171 (1172br|3952ar) 1172 (1173br|2942ar) 1173 (1174br|3783ar) 1174 (1175br|3695ar) 1175 (1176br|3649ar) 1176 (1177br|3127ar) 1177 (1178br|3661ar) 1178 (1179br|2702ar) 1179 (1180br|3953ar) 1180 (1181br|3534ar) 1181 (1182br|3386ar) 1182 (1183br|2024ar) 1183 (1184br|3873ar) 1184 (1185br|3883ar) 1185 (1186br|3339ar) 1186 (1187br|2042ar) 1187 (1188br|3496ar) 1188 (1189br|2123ar) 1189 (1190br|2473ar) 1190 (1191br|2156ar) 1191 (1192br|3899ar) 1192 (1193br|2208ar) 1193 (1194br|3908ar) 1194 (1195br|3948ar) 1195 (1196br|3366ar) 1196 (1197br|2230ar) 1197 (1198br|2433ar) 1198 (1199br|3179ar) 1199 (1200br|3173ar) 1200 (1201br|3692ar) 1201 (1202br|2096ar) 1202 (1203br|2230ar) 1203 (1204br|2103ar) 1204 (1205br|3174ar) 1205 (1206br|3989ar) 1206 (1207br|3984ar) 1207 (1208br|2131ar) 1208 (1209br|2726ar) 1209 (1210br|2845ar) 1210 (1211br|2722ar) 1211 (1212br|3785ar) 1212 (1213br|3840ar) 1213 (1214br|3097ar) 1214 (1215br|3791ar) 1215 (1216br|3782ar) 1216 (1217br|2740ar) 1217 (1218br|3286ar) 1218 (1219br|2928ar) 1219 (1220br|3962ar) 1220 (1221br|3224ar) 1221 (1222br|3317ar) 1222 (1223br|3875ar) 1223 (1224br|3495ar) 1224 (1225br|3476ar) 1225 (1226br|3193ar) 1226 (1227br|2638ar) 1227 (1228br|2058ar) 1228 (1229br|3883ar) 1229 (1230br|3329ar) 1230 (1231br|3553ar) 1231 (1232br|3719ar) 1232 (1233br|2670ar) 1233 (1234br|2078ar) 1234 (1235br|3859ar) 1235 (1236br|2513ar) 1236 (1237br|3247ar) 1237 (1238br|2581ar) 1238 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