McNaughton's Theorem (1966)

4.4 Theorem: **If L is Büchi recognisable then L is recognisable by a deterministic Muller automaton.**

This is the main theorem of the theory of ω-automata.

Given Büchi automaton $\mathcal{B} = (Q, \Sigma, q_0, \Delta, F)$.

First try: Powerset construction

Determine after each input-prefix w the set of states reachable via w,

and declare as final states the sets which contain an F-state.

Safra Trees

We present the Safra construction (S. Safra 1988)

Idea: Branch off a separate computation thread starting from final states

To record these different computation branches we use a tree structure.

A Safra tree over Q is an ordered finite tree

- with node names from $\{1, \ldots, 2|Q|\}$,
- where each node is labelled by a nonempty set $R \subseteq Q$, possibly with an extra marker "!"
- where labels of brother nodes are disjoint
- where the union of brother nodes is a proper subset of the parent node

Definition of the Muller Automaton

Remark: There are only finitely many possible Safra trees over Q.

For the given Büchi automaton $\mathcal{B} = (Q, \Sigma, q_0, \Delta, F)$

define the Muller automaton $\mathcal{M} = (Q', \Sigma, q_0', \delta, \mathcal{F})$:

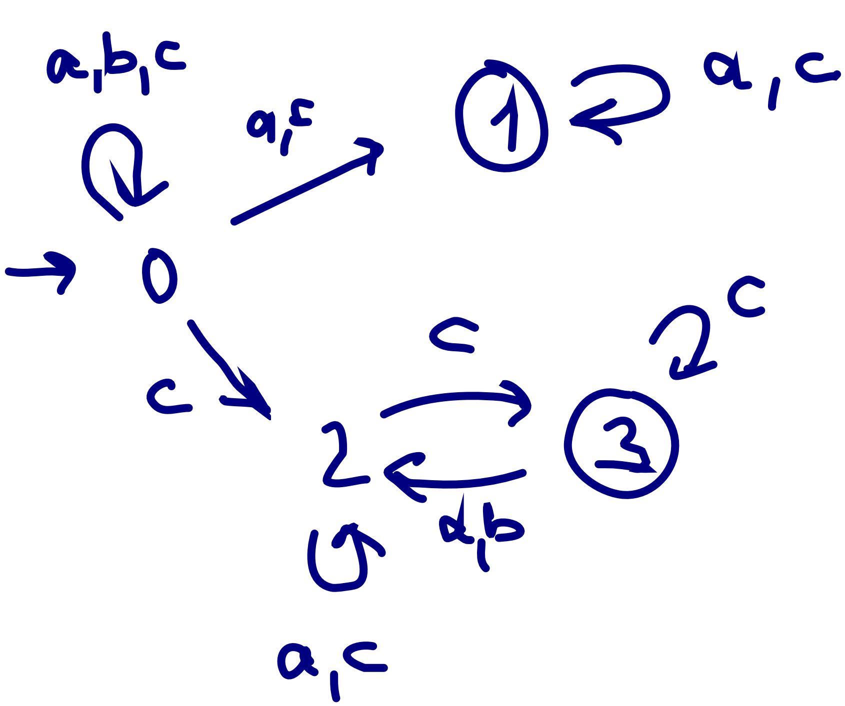
- Q' := set of Safra trees over Q.
- q_0' := Safra tree consisting just of root labelled $\{q_0\}$
- Define $\delta(s, a)$ (for Safra tree s, input letter a) in four stages as described below
- Declare a set S of Safra trees to be in $\mathcal{F}$ if some node name appears in each tree $s \in S$, and in some tree $s \in S$ the label of this node name carries the marker "!"

A Transition of $\mathcal{M}$

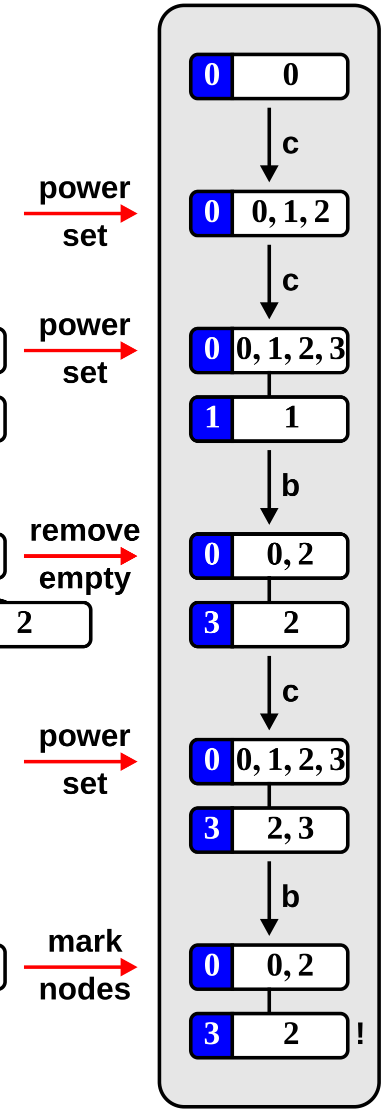
Define $\delta(s, a)$ (for Safra tree s, input letter a) in four stages as follows:

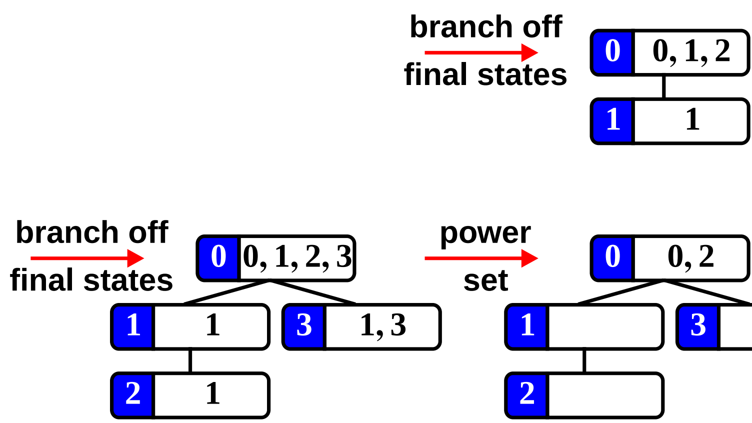
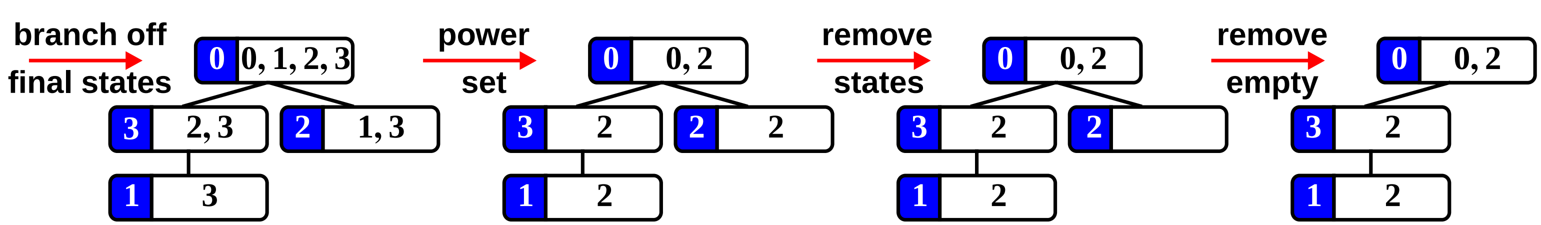
1. For each node whose label contains final states, branch off a new son containing these final states. (Take as node name a free number $\leq 2|Q|$)
2. To each node label apply the powerset construction via input letter a: $R \rightarrow \{r' \mid \exists r \in R : (r, a, r') \in \Delta\}$
3. Cancel state q if it occurs also in an older brother node. Cancel a node if it carries label $\emptyset$ (unless it is the root)
4. Cancel all sons (and their descendants) if the union of their labels is the parent label, and in this case mark the parent by "!"

Example



INPUT:
c c b c b



McNaughton's Theorem (1966)

4.5 Theorem: If L is Büchi recognisable then L is recognisable by a deterministic Muller automaton.

For the given Büchi automaton $\mathcal{B} = (Q, \Sigma, q_0, \Delta, F)$

define the Muller automaton $\mathcal{M} = (Q', \Sigma, q_0', \delta, \mathcal{F})$ by

- Q' := set of Safra trees over Q.
- q_0' := Safra tree consisting just of root labelled $\{q_0\}$
- $S \in \mathcal{F} \iff$ if some node name appears in each tree $s \in S$, and in some tree $s \in S$ the label of this node name carries the marker "!"

Definition of δ

Define $\delta(s, a)$ (for Safra tree s, input letter a) in four stages as follows:

1. For each node whose label contains final states, branch off a new son containing these final states. (Take as node name a free number $\leq 2|Q|$)
2. To each node label apply the powerset construction via input letter a: $R \to \{r' \mid \exists r \in R : (r, a, r') \in \Delta\}$
3. Cancel state q if it occurs also in an older brother node. Cancel a node if it carries label $\emptyset$ (unless it is the root)
4. Cancel all sons (and their descendants) if the union of their labels is the parent label, and in this case mark the parent by "!"

Towards the Correctness Proof

Notation: $P \overset{w}{\rightsquigarrow} R$ for $P, R \subseteq Q$ means

$\forall r \in R\, \exists p \in P$ s.t. $\mathcal{B} : p \xrightarrow{w} r$.

We analyse the case where a node name stays alive and has "!" again and again:

$$\begin{array}{ccccccccccccc}
R_0 & \overset{u_1}{\rightsquigarrow} & P_1 & \overset{v_1}{\rightsquigarrow} & R_1! & \overset{u_2}{\rightsquigarrow} & P_2 & \overset{v_2}{\rightsquigarrow} & R_2! & \ldots & P_i & \overset{v_i}{\rightsquigarrow} & R_i! \\
 & & \cup\!\mid & & \parallel & & \cup\!\mid & & \parallel & & \cup\!\mid & & \parallel \\
 & & F_1 & \overset{v_1}{\rightsquigarrow} & Q_1 & & F_2 & \overset{v_2}{\rightsquigarrow} & Q_2 & & F_i & \overset{v_i}{\rightsquigarrow} & Q_i
\end{array}$$

where F_i = set of final states from P_i.

Then $\forall r \in R_i\, \exists p \in R_0$:

$\mathcal{A}$ reaches from p via input $u_1 v_1 u_2 v_2 \ldots u_i v_i$ the state r with $\geq i$ visits of final states.

König's Lemma

4.6 Lemma: An infinite finitely branching tree has an infinite path.

Proof:

Let t be an infinite finitely branching tree.

Define a path π such that each node v on π has the following property: the subtree at v is infinite.

The root has the property by assumption.

If v has the property then we can pick a son v' with the same property (because the tree is finitely branching!)

Iterating this we obtain an infinite path.

Run Lemma

4.7 Lemma: Let $R_0 \overset{u_1 v_1}{\rightsquigarrow} R_1! \overset{u_2 v_2}{\rightsquigarrow} R_2! \ldots R_i! \overset{u_{i+1} v_{i+1}}{\rightsquigarrow} \ldots$ be as before.

Then on the input $u_1 v_1 u_2 v_2 \ldots$ there is a successful run of the Büchi automaton $\mathcal{B}$, starting in a state of R_0.

Proof:

Consider for each state from $r \in R_i$ a run from R_0 to r via $u_1 v_1 \ldots u_i v_i$.

These runs form a run tree which is infinite and finitely branching.

By König's Lemma there is an infinite run in this tree.

By construction, a final state is visited after each prefix $u_1 v_1 \ldots u_i v_i$.

Correctness of Safra Construction

Claim: $L(\mathcal{B}) = L(\mathcal{M})$

Show first $L(\mathcal{M}) \subseteq L(\mathcal{B})$:

Let $\alpha \in L(\mathcal{M})$

Consider the successful run of Safra trees on α.

Pick a node k which from some point onwards occurs in each Safra tree and has marker "!" infinitely often.

Consider the labels where "!" occurs at k, call them $R_1, R_2, \ldots$.

The Run Lemma applies and yields an infinite run of $\mathcal{B}$ on α.

Show $L(\mathcal{B}) \subseteq L(\mathcal{M})$:

Let $\alpha \in L(\mathcal{B})$, consider a successful run of $\mathcal{B}$ on α, visiting say the final state q again and again.

Consider the $\mathcal{M}$-run of Safra trees on α

If root is marked "!" infinitely often, $\mathcal{M}$ accepts.

Otherwise look at first occurrence of q afterwards:

Here q is put into a son of the root, and the Büchi run finally stays in a fixed son k_1 of the root.

If k_1 is marked "!" infinitely often, $\mathcal{M}$ accepts.

Otherwise we continue analogously and get a son k_2 of k_1 where the Büchi run finally stays.

At some stage the marker "!" occurs infinitely often, otherwise height of the Safra trees would be unbounded.

Rabin Automata: Motivation

We may define the sets S which form the acceptance component $\mathcal{F}$ by two conditions:

For some node name j

- S should not contain a tree without node name j
- S should contain a tree where node name j appears with marker "!"

Define

- E_j = set of Safra trees without node name j
- F_j = tree where node name j appears with marker "!"

Then: ρ is successful if for some j,
$\mathrm{Inf}(\rho) \cap E_j = \emptyset$ and $\mathrm{Inf}(\rho) \cap F_j \neq \emptyset$

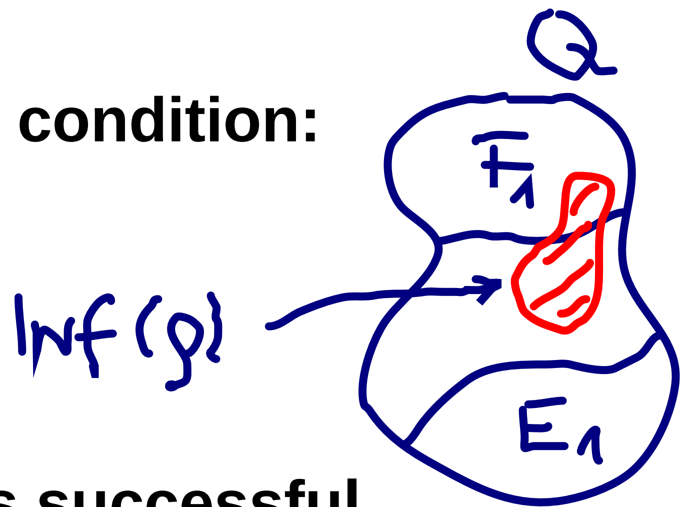
Rabin Automata

A (deterministic) **Rabin automaton** is of the form
$\mathcal{A} = (Q, \Sigma, q_0, \delta, \Omega)$ where $\Omega = ((E_1, F_1), \ldots, (E_k, F_k))$ is a list of "accepting pairs" with $E_i, F_i \subseteq Q$,

used with the following **Rabin acceptance** condition:

A run ρ is **successful** if for some $j \leq k$

$$\mathrm{Inf}(\rho) \cap E_j = \emptyset \text{ and } \mathrm{Inf}(\rho) \cap F_j \neq \emptyset$$



$\mathcal{A}$ accepts α if the unique run of $\mathcal{A}$ on α is successful.

$$L(\mathcal{A}) = \{\alpha \mid \mathcal{A} \text{ accepts } \alpha\}$$

L is called **Rabin recognisable** if $L = L(\mathcal{A})$ for a Rabin automaton $\mathcal{A}$

Equivalence Theorem

The Safra construction transforms a Büchi automaton into a deterministic Rabin automaton

A Rabin automaton $\mathcal{R} = (Q, \Sigma, q_0, \delta, \Omega)$ with $\Omega = ((E_1, F_1), \ldots, (E_k, F_k))$ is equivalent to the Muller automaton $\mathcal{M} = (Q, \Sigma, q_0, \delta, \mathcal{F})$ with

$$P \in \mathcal{F} \iff \bigvee_{j=1}^{k} (P \cap E_j = \emptyset \wedge P \cap F_j \neq \emptyset)$$

So we have proved

4.8 Theorem: For an ω-language, the following are equivalent:

1. L is recognised by a nondeterministic Büchi automaton.
2. L is recognised by a deterministic Rabin automaton.
3. L is recognised by a deterministic Muller automaton.