Exercise 09

$\mathbf{P} \subsetneq \mathbf{E}$

Proof. It is obvious that $\mathbf{P} \subseteq \mathbf{E}$ holds. Now construct a language $D \in \mathbf{E}$, but $D \notin \mathbf{P}$.

$$D = \{\alpha \mid M_\alpha \text{ outputs } O \text{ after } \alpha \text{ steps}\}$$

This is obviously decidable in $\mathbf{E}$ since M_α can be simulated for each hier fehlt noch was

Towards a contradiction assume there is a machine M_x that decides D in $\ln^c, c \in \mathbb{N}$. Choose a $x' > x^c$ such that $M_{x'} = M_x$ this is possible since each TM can be represented by infinitely many strings. Now construct a TM M_y that cleans the output, writes x on the onto the output tape and behave like M afterwards, $y > x$. □

Exercise 10

running times	algorithm
$\log\log n$	For $o \leq i < \log\log n$
$\log n$	For $0 \leq j < \log n$
$T(j) = T(\log n)$	$h = H(j)$
$2^{\log n} = n$	For $x \in \{0,1\}^\star$ with $\lvert x\rvert = j$
$(i\lvert x\rvert^i)^{1,5} < \psi < n^3$	Run U on i for $i\lvert x\rvert^i$ steps. Save output to o
$H(\lvert x\rvert) \cdot 2^{\lvert x\rvert} = n$	Run $M_{\mathbf{SAT}_H}$ on x save output to o'
$O(1)$	if $o == O'$
$O(1)$	return i

Let $T(n)$ be the running to of $H(n)$. Proof by induction.

Base Case:

$n \leq 2:$ The outer loop is never executed an thus $T(n) \in O(1) \subseteq O(n^3)$.

Induction Step:

$$\begin{aligned}
&T(n) \in O(\log\log(n)\log(n)\,(T(\log(n)) + n(log(n)\cdot n + n)) \\
&\in O\left(/log^2(n)T(\log(n)) + n^2 \cdot \log^3(n)\right) \\
&\in O\left(\log^2(n)\log\log^3(n) + n^3 \cdot \log^3(n)\right) \\
&\in O\left(n^2 \cdot \log^3(n))\right) \\
&\in O(n^3)
\end{aligned}$$

Exercise 11

Since $\mathbf{SAT}_H$ is $\mathbf{NP}$-complete, there is a reduction f in polynomial time, say $O(n^i)$ from $\mathbf{SAT}$ to $\mathbf{SAT}_H$.

We now construct a polynomial time algorithm A that decides $\mathbf{SAT}$ on φ in $O(n^j)$, $j \geq i$:

Let $N \in \mathbb{N}$ be the number such that $H(n) > i$ for $n > N$. If $|\varphi| \leq N$ solve $\mathbf{SAT}(\varphi)$ using a brute force assignment. This can be done in constant time since the length of φ is bounded. If $|\varphi| > N$, compute $\eta = f(\varphi)$. By construction of $f, \eta \in \mathbf{SAT}_H \Leftrightarrow \varphi \in \mathbf{SAT}$. If η is not of the form $\psi 01^{n^{H(n)}}$ $\eta\neg \in \mathbf{SAT}_H$ and thus $\varphi\neg \in \mathbf{SAT}$, so return false. This check can be done in polynomial time since the length of φ up to the marker can be counted in polynomial time an H can be computed in polynomial time by definition.

Otherwise run $A(\psi)$ and forward its output. Proof by induction.

Base Case:
: As described above.

Induction Step:
: f runs in polynomial time of $|\varphi|$ so eta has to be in $O(n^j)$. So $|\psi| < |\varphi|$ must be sub linear in size of $|\varphi|$. Otherwise
$|\psi|^{H(|\psi|)} \geq |\varphi|^{H(|\varphi|)} > |\varphi|^j < |\varphi|^i$ and f hasn't got enough time to print sufficiently man 1's. On input ψ A runs at most $O(|\psi|)$ steps by the induction step. The reduction can be performed in $O(|\varphi|^i)$ steps and since $i \leq j$ A terminates on ϖ in $O(|\psi|^j + |\varphi|^j) \in O(|\varphi|^j + |\varphi|^j) \in O(|\varphi|^j)$