Exercise 30

The idea is to use a language $\mathcal{L}$ that is difficult to compute. Therefore let $\mathcal{L}$ not be in $E = \bigcup_{c=1}^{\infty} \mathsf{TIME}(2^{cn})$. Furthermore let be $\mathcal{L}' = \{1^m : m \in \mathcal{L}\}$. Obviously $\mathcal{L} \in \mathsf{P}_{/\mathsf{poly}}$, but $\mathcal{L}' \notin \mathsf{P}$. If $\mathcal{L}' \in \mathsf{P}$ a TM $\mathcal{M}$ would compute $\mathcal{L}'$ in $O(m^k)$. So $\mathcal{M}$ could decide $\mathcal{L}$ in $O((2^n)^k)$ and thus $\mathcal{L} \in E$.

Show that $\mathcal{L}' \in \mathsf{P}_{/\mathsf{poly}}$. Let $\mathcal{M}'$ be a TM with the advice $a(n)$. $a(n) = 1$ if and only if $n \in \mathcal{L}$, so for every input length exists exactly one advice. Thus $\mathcal{M}'$ can recognize $\mathcal{L}'$.

Therefore $\mathcal{M}'$ rejects if the input has not the form 1^m or it has the form but $n \in \mathcal{L}$, otherwise accepts. If $\mathcal{L}$ is decidable $\mathcal{L}'$ is trivially decidable too. Because of the time hierarchy theorem such an $\mathcal{L}$ must exist.

Exercise 31

A language $\mathcal{L}$ is in $\mathcal{NC}^d$ if $\mathcal{L}$ can be decided by a family of circuits $\{C_n\}$, where C_n has poly(n) size and depth $O(\log^d n)$. So $\mathcal{NC}^0$ has a depth of $\log^0 n = 1$.

a)

b)

c)

Because of the depth of the graph the languages $\mathcal{L} \in \mathcal{NC}^0$ work on inputs of the form $x = x_1 x_2$ or $x = x_1$. In case an input has the form $x = x_1 x_2 x_3 ...$ there exists a path from x_i to the output node that is longer then 1. All languages that can be constructed by a single $\wedge, \vee$ or $\neg$ are not infinite, so the union of them is not infinite, so $\mathcal{NC}^0$ does not contain any infinite language.

$|\mathsf{PARITY}|$ obviously is infinite, so $\mathsf{PARITY} \notin \mathcal{NC}^0$.