

Exercise 4.4 - Solution

Are the following statements correct or not? Prove each of your claims either by means of resolution or by specifying a suitable interpretation as a counterexample.

The numbers in the transformations below stand for the respective step (c.f. Resolution slide 14/37):

1. Eliminate \supset and \equiv .
2. Push \neg inside.
3. Rename variables to make them syntactically distinct.
4. Eliminate \exists 's by Skolemization.
5. Move \forall 's to the left.
6. Distribute \vee over \wedge ($(\alpha \wedge \beta) \vee \gamma \equiv (\alpha \vee \gamma) \wedge (\beta \vee \gamma)$)
7. Simplify.

Conversions into CNF:

$$\begin{aligned}
 \text{(a) } \{ \exists x P(x), \exists x Q(x) \} &\models \exists x [P(x) \wedge Q(x)] \\
 &\{ \exists x P(x), \exists x Q(x) \} \cup \{ \neg \exists x [P(x) \wedge Q(x)] \} \\
 &\stackrel{\wedge}{=} \exists x P(x) \wedge \exists x Q(x) \wedge \neg \exists x [P(x) \wedge Q(x)] \\
 &\stackrel{1.}{\equiv} \text{---} \text{''} \text{---} \\
 &\stackrel{2.}{\equiv} \exists x P(x) \wedge \exists x Q(x) \wedge \forall x [\neg P(x) \vee \neg Q(x)] \\
 &\stackrel{3.}{\equiv} \exists x P(x) \wedge \exists y Q(y) \wedge \forall z [\neg P(z) \vee \neg Q(z)] \\
 &\stackrel{4.}{\rightsquigarrow} P(\mathbf{f}) \wedge Q(\mathbf{f}') \wedge \forall z [\neg P(z) \vee \neg Q(z)] \\
 &\stackrel{5.}{\equiv} \forall z [P(\mathbf{f}) \wedge Q(\mathbf{f}') \wedge [\neg P(z) \vee \neg Q(z)]] \\
 &\stackrel{6.}{\equiv} \text{---} \text{''} \text{---} \\
 &\stackrel{7.}{\equiv} \text{---} \text{''} \text{---} \\
 &\stackrel{\wedge}{=} \{ [P(\mathbf{f})], [Q(\mathbf{f}')], [\sim P(z), \sim Q(z)] \}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b) } \{ \forall x P(x) \vee \forall x Q(x) \} &\models \forall x [P(x) \vee Q(x)] \\
 &\{ \forall x P(x) \vee \forall x Q(x) \} \cup \{ \neg \forall x [P(x) \vee Q(x)] \} \\
 &\stackrel{\wedge}{=} [\forall x P(x) \vee \forall x Q(x)] \wedge \neg \forall x [P(x) \vee Q(x)] \\
 &\stackrel{1.}{\equiv} \text{---} \text{''} \text{---} \\
 &\stackrel{2.}{\equiv} [\forall x P(x) \vee \forall x Q(x)] \wedge \exists x [\neg P(x) \wedge \neg Q(x)] \\
 &\stackrel{3.}{\equiv} [\forall x P(x) \vee \forall y Q(y)] \wedge \exists z [\neg P(z) \wedge \neg Q(z)] \\
 &\stackrel{4.}{\rightsquigarrow} [\forall x P(x) \vee \forall y Q(y)] \wedge [\neg P(\mathbf{f}) \wedge \neg Q(\mathbf{f})] \\
 &\stackrel{5.}{\equiv} \forall x \forall y [P(x) \vee Q(y)] \wedge \neg P(\mathbf{f}) \wedge \neg Q(\mathbf{f}) \\
 &\stackrel{6.}{\equiv} \text{---} \text{''} \text{---} \\
 &\stackrel{7.}{\equiv} \text{---} \text{''} \text{---} \\
 &\stackrel{\wedge}{=} \{ [P(x), Q(y)], [\sim P(\mathbf{f})], [\sim Q(\mathbf{f})] \}
 \end{aligned}$$

$$(c) \models (\forall x P(x) \wedge \forall x Q(x)) \equiv \forall x [P(x) \wedge Q(x)]$$

$$\begin{aligned}
& \{ \} \cup \{ \neg[(\forall x P(x) \wedge \forall x Q(x)) \equiv \forall x [P(x) \wedge Q(x)]] \} \\
& \triangleq \neg[(\forall x P(x) \wedge \forall x Q(x)) \equiv \forall x [P(x) \wedge Q(x)]] \\
& \stackrel{1.}{\equiv} \neg[\neg[(\forall x P(x) \wedge \forall x Q(x)) \vee \forall x [P(x) \wedge Q(x)]] \wedge [\neg \forall x [P(x) \wedge Q(x)] \vee (\forall x P(x) \wedge \forall x Q(x))]] \\
& \stackrel{2.}{\equiv} [\forall x P(x) \wedge \forall x Q(x) \wedge \exists x [\neg P(x) \vee \neg Q(x)]] \vee [\forall x [P(x) \wedge Q(x)] \wedge (\exists x \neg P(x) \vee \exists x \neg Q(x))] \\
& \stackrel{3.}{\equiv} [\forall u P(u) \wedge \forall v Q(v) \wedge \exists w [\neg P(w) \vee \neg Q(w)]] \vee [\forall x [P(x) \wedge Q(x)] \wedge (\exists y \neg P(y) \vee \exists z \neg Q(z))] \\
& \stackrel{4.}{\rightsquigarrow} [[\forall u P(u) \wedge \forall v Q(v) \wedge [\neg P(\mathbf{f}) \vee \neg Q(\mathbf{f})]] \vee [\forall x [P(x) \wedge Q(x)] \wedge (\neg P(\mathbf{f}') \vee \neg Q(\mathbf{f}''))]] \\
& \stackrel{5.}{\equiv} \forall u \forall v \forall x [[P(u) \wedge Q(v) \wedge [\neg P(\mathbf{f}) \vee \neg Q(\mathbf{f})]] \vee [P(x) \wedge Q(x) \wedge (\neg P(\mathbf{f}') \vee \neg Q(\mathbf{f}''))]] \\
& \stackrel{6.}{\equiv} \forall u \forall v \forall x [\begin{array}{ccc} [P(u) \vee P(x)] \wedge & [P(u) \vee Q(x)] \wedge & [P(u) \vee \neg P(\mathbf{f}') \vee \neg Q(\mathbf{f}'')] \wedge \\ [Q(v) \vee P(x)] \wedge & [Q(v) \vee Q(x)] \wedge & [Q(v) \vee \neg P(\mathbf{f}') \vee \neg Q(\mathbf{f}'')] \wedge \\ [\neg P(\mathbf{f}) \vee \neg Q(\mathbf{f}) \vee P(x)] \wedge & [\neg P(\mathbf{f}) \vee \neg Q(\mathbf{f}) \vee Q(x)] \wedge & [\neg P(\mathbf{f}) \vee \neg Q(\mathbf{f}) \vee \neg P(\mathbf{f}') \vee \neg Q(\mathbf{f}'')] \end{array}] \\
& \stackrel{7.}{\equiv} \text{---} " \text{---} \\
& \triangleq \{ \begin{array}{ccc} [P(u), P(x)], & [P(u), Q(x)], & [P(u), \sim P(\mathbf{f}'), \sim Q(\mathbf{f}'')] , \\ [Q(v), P(x)], & [Q(v), Q(x)], & [Q(v), \sim P(\mathbf{f}'), \sim Q(\mathbf{f}'')] , \\ [\sim P(\mathbf{f}), \sim Q(\mathbf{f}), P(x)], & [\sim P(\mathbf{f}), \sim Q(\mathbf{f}), Q(x)], & [\sim P(\mathbf{f}), \sim Q(\mathbf{f}), \sim P(\mathbf{f}'), \sim Q(\mathbf{f}'')] \end{array} \}
\end{aligned}$$

Alternatively, show validity of the \supset and \subset directions of \equiv separately:

$$“\supset” \models (\forall x P(x) \wedge \forall x Q(x)) \supset \forall x [P(x) \wedge Q(x)]$$

$$\begin{aligned}
& \{ \} \cup \{ \neg[(\forall x P(x) \wedge \forall x Q(x)) \supset \forall x [P(x) \wedge Q(x)]] \} \\
& \triangleq \neg[(\forall x P(x) \wedge \forall x Q(x)) \supset \forall x [P(x) \wedge Q(x)]] \\
& \stackrel{1.}{\equiv} \neg[\neg[(\forall x P(x) \wedge \forall x Q(x)) \vee \forall x [P(x) \wedge Q(x)]] \\
& \stackrel{2.}{\equiv} \forall x P(x) \wedge \forall x Q(x) \wedge \exists x [\neg P(x) \vee \neg Q(x)] \\
& \stackrel{3.}{\equiv} \forall x P(x) \wedge \forall y Q(y) \wedge \exists z [\neg P(z) \vee \neg Q(z)] \\
& \stackrel{4.}{\rightsquigarrow} \forall x P(x) \wedge \forall y Q(y) \wedge [\neg P(\mathbf{f}) \vee \neg Q(\mathbf{f})] \\
& \stackrel{5.}{\equiv} \forall x \forall y [P(x) \wedge Q(y) \wedge [\neg P(\mathbf{f}) \vee \neg Q(\mathbf{f})]] \\
& \stackrel{6.}{\equiv} \text{---} " \text{---} \\
& \stackrel{7.}{\equiv} \text{---} " \text{---} \\
& \triangleq \{ [P(x)], [Q(y)], [\sim P(\mathbf{f}), \sim Q(\mathbf{f})] \}
\end{aligned}$$

$$“\subset” \models \forall x [P(x) \wedge Q(x)] \supset (\forall x P(x) \wedge \forall x Q(x))$$

$$\begin{aligned}
& \{ \} \cup \{ \neg[\forall x [P(x) \wedge Q(x)] \supset (\forall x P(x) \wedge \forall x Q(x))] \} \\
& \triangleq \neg[\forall x [P(x) \wedge Q(x)] \supset (\forall x P(x) \wedge \forall x Q(x))] \\
& \stackrel{1.}{\equiv} \neg[\neg \forall x [P(x) \wedge Q(x)] \vee (\forall x P(x) \wedge \forall x Q(x))] \\
& \stackrel{2.}{\equiv} \forall x [P(x) \wedge Q(x)] \wedge (\exists x \neg P(x) \vee \exists x \neg Q(x)) \\
& \stackrel{3.}{\equiv} \forall x [P(x) \wedge Q(x)] \wedge (\exists y \neg P(y) \vee \exists z \neg Q(z)) \\
& \stackrel{4.}{\rightsquigarrow} \forall x [P(x) \wedge Q(x)] \wedge (\neg P(\mathbf{f}) \vee \neg Q(\mathbf{f}')) \\
& \stackrel{5.}{\equiv} \forall x [P(x) \wedge Q(x) \wedge (\neg P(\mathbf{f}) \vee \neg Q(\mathbf{f}'))] \\
& \stackrel{6.}{\equiv} \text{---} " \text{---} \\
& \stackrel{7.}{\equiv} \text{---} " \text{---} \\
& \triangleq \{ [P(x)], [Q(x)], [\sim P(\mathbf{f}), \sim Q(\mathbf{f}')] \}
\end{aligned}$$

(d) $\models \neg\phi$ where ϕ is $\forall x[P(x) \supset Q(x, g(x))] \wedge \exists x[P(g(x)) \wedge \neg Q(g(x), g(g(x)))]$

$$\{\} \cup \{\neg\neg\phi\}$$

$$\begin{aligned} &\triangleq \neg\neg[\forall x[P(x) \supset Q(x, g(x))] \wedge \exists x[P(g(x)) \wedge \neg Q(g(x), g(g(x)))] \\ &\stackrel{1.}{\equiv} \neg\neg[\forall x[\neg P(x) \vee Q(x, g(x))] \wedge \exists x[P(g(x)) \wedge \neg Q(g(x), g(g(x)))] \\ &\stackrel{2.}{\equiv} \forall x[\neg P(x) \vee Q(x, g(x))] \wedge \exists x[P(g(x)) \wedge \neg Q(g(x), g(g(x)))] \\ &\stackrel{3.}{\equiv} \forall x[\neg P(x) \vee Q(x, g(x))] \wedge \exists y[P(g(y)) \wedge \neg Q(g(y), g(g(y)))] \\ &\stackrel{4.}{\rightsquigarrow} \forall x[\neg P(x) \vee Q(x, g(x))] \wedge P(g(\mathbf{f})) \wedge \neg Q(g(\mathbf{f}), g(g(\mathbf{f}))) \\ &\stackrel{5.}{\equiv} \forall x[\neg P(x) \vee Q(x, g(x))] \wedge P(g(\mathbf{f})) \wedge \neg Q(g(\mathbf{f}), g(g(\mathbf{f}))) \\ &\stackrel{6.}{\equiv} \text{---} \text{''} \text{---} \\ &\stackrel{7.}{\equiv} \text{---} \text{''} \text{---} \\ &\triangleq \{ [\sim P(x), Q(x, g(x))], [P(g(\mathbf{f}))], [\sim Q(g(\mathbf{f}), g(g(\mathbf{f})))] \} \end{aligned}$$

(e) $\{ \forall x \exists y [(P(y) \supset P(f(x))) \wedge (P(f(x)) \supset Q(x, f(y))) \wedge (Q(x, f(x)) \vee P(y))] \} \models \forall x \exists y Q(x, y)$

$$\begin{aligned} &\{ \forall x \exists y [(P(y) \supset P(f(x))) \wedge (P(f(x)) \supset Q(x, f(y))) \wedge (Q(x, f(x)) \vee P(y))] \} \cup \{ \neg \forall x \exists y Q(x, y) \} \\ &\triangleq \forall x \exists y [(P(y) \supset P(f(x))) \wedge (P(f(x)) \supset Q(x, f(y))) \wedge (Q(x, f(x)) \vee P(y))] \wedge \neg \forall x \exists y Q(x, y) \\ &\stackrel{1.}{\equiv} \forall x \exists y [(\neg P(y) \vee P(f(x))) \wedge (\neg P(f(x)) \vee Q(x, f(y))) \wedge (Q(x, f(x)) \vee P(y))] \wedge \neg \forall x \exists y Q(x, y) \\ &\stackrel{2.}{\equiv} \forall x \exists y [(\neg P(y) \vee P(f(x))) \wedge (\neg P(f(x)) \vee Q(x, f(y))) \wedge (Q(x, f(x)) \vee P(y))] \wedge \exists x \forall y \neg Q(x, y) \\ &\stackrel{3.}{\equiv} \forall x \exists y [(\neg P(y) \vee P(f(x))) \wedge (\neg P(f(x)) \vee Q(x, f(y))) \wedge (Q(x, f(x)) \vee P(y))] \wedge \exists v \forall z \neg Q(v, z) \\ &\stackrel{4.}{\rightsquigarrow} \forall x [(\neg P(\mathbf{f}'(x)) \vee P(f(x))) \wedge (\neg P(f(x)) \vee Q(x, f(\mathbf{f}'(x)))) \wedge (Q(x, f(x)) \vee P(\mathbf{f}'(x)))] \wedge \forall z \neg Q(\mathbf{f}'', z) \\ &\stackrel{5.}{\equiv} \forall x \forall z [(\neg P(\mathbf{f}'(x)) \vee P(f(x))) \wedge (\neg P(f(x)) \vee Q(x, f(\mathbf{f}'(x)))) \wedge (Q(x, f(x)) \vee P(\mathbf{f}'(x))) \wedge \neg Q(\mathbf{f}'', z)] \\ &\stackrel{6.}{\equiv} \text{---} \text{''} \text{---} \\ &\stackrel{7.}{\equiv} \text{---} \text{''} \text{---} \\ &\triangleq \{ [\sim P(\mathbf{f}'(x)), P(f(x))], [\sim P(f(x)), Q(x, f(\mathbf{f}'(x)))] , [Q(x, f(x)), P(\mathbf{f}'(x))], [\sim Q(\mathbf{f}'', z)] \} \end{aligned}$$

(f) $\{ \forall x \exists y P(x, y), \neg \exists z P(z, a) \} \models \exists y \forall x P(x, y)$

$$\begin{aligned} &\{ \forall x \exists y P(x, y), \neg \exists z P(z, a) \} \cup \{ \neg \exists y \forall x P(x, y) \} \\ &\triangleq \forall x \exists y P(x, y) \wedge \neg \exists z P(z, a) \wedge \neg \exists y \forall x P(x, y) \\ &\stackrel{1.}{\equiv} \text{---} \text{''} \text{---} \\ &\stackrel{2.}{\equiv} \forall x \exists y P(x, y) \wedge \forall z \neg P(z, a) \wedge \forall y \exists x \neg P(x, y) \\ &\stackrel{3.}{\equiv} \forall x \exists y P(x, y) \wedge \forall z \neg P(z, a) \wedge \forall v \exists w \neg P(w, v) \\ &\stackrel{4.}{\rightsquigarrow} \forall x P(x, \mathbf{f}(x)) \wedge \forall z \neg P(z, a) \wedge \forall v \neg P(\mathbf{f}'(v), v) \\ &\stackrel{5.}{\equiv} \forall x \forall z \forall v [P(x, \mathbf{f}(x)) \wedge \neg P(z, a) \wedge \neg P(\mathbf{f}'(v), v)] \\ &\stackrel{6.}{\equiv} \text{---} \text{''} \text{---} \\ &\stackrel{7.}{\equiv} \text{---} \text{''} \text{---} \\ &\triangleq \{ [P(x, \mathbf{f}(x))], [\sim P(z, a)], [\sim P(\mathbf{f}'(v), v)] \} \end{aligned}$$

Note that these transformations into CNF are only necessary for proving the statements with resolution. If a statement is not true, this is shown by specifying an interpretation as counterexample. A CNF transformation is not required in this case, but only shown here for the sake of completeness.