

Landau level formulation (v2)

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Contents

1	Useful formulas	1
2	Peierls substitution, ladder operators, guiding centers	2
2.1	Landau gauge	2
2.1.1	First Landau gauge	2
2.1.2	Second Landau gauge	3
2.1.3	Bloch Landau levels	3
2.2	Symmetric gauge	5
3	LLs of simple systems	5
3.1	Free Schrodinger electrons	5
3.2	Free Dirac electrons	6

1 Useful formulas

The Hermite polynomial is

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \quad (1)$$

The generalized Laguerre polynomial is (α can take any integers satisfying $\alpha \geq -n$)

$$L_n^\alpha(x) = \sum_{l=\max(0, -\alpha)}^n (-1)^l \frac{(n+\alpha)!}{(n-l)!(\alpha+l)!} \frac{x^l}{l!} = \sum_{l=0}^{\min(n, n+\alpha)} (-1)^{n-l} \frac{(n+\alpha)!}{l!(\alpha+n-l)!} \frac{x^{n-l}}{(n-l)!}, \quad (2)$$

satisfying

$$\frac{(-x)^m}{m!} L_n^{m-n}(x) = \frac{(-x)^n}{n!} L_m^{n-m}(x). \quad (3)$$

The BCH formula is

$$e^A e^B = e^{A+B} e^{[A,B]/2} = e^B e^A e^{[A,B]}, \quad \text{for } [A, [A, B]] = [B, [A, B]] = 0. \quad (4)$$

The Leibniz rule of derivatives reads

$$\frac{d^n}{dx^n} (fg) = \sum_{l=0}^n \frac{n!}{l!(n-l)!} \frac{d^l f}{dx^l} \frac{d^{n-l} g}{dx^{n-l}}. \quad (5)$$

2 Peierls substitution, ladder operators, guiding centers

If a magnetic field is exerted on an electron moving in the 2D plane, its momentum will be replaced by $\mathbf{p} \rightarrow \boldsymbol{\pi}$, where

$$(\pi_x, \pi_y) = \boldsymbol{\pi} = \mathbf{p} + e\mathbf{A}, \quad (6)$$

$e > 0$ is the electron charge, \mathbf{A} is the gauge potential so that $\nabla \times \mathbf{A} = \mathbf{B}$. Here we assume $\mathbf{B} = -B\mathbf{e}_z$ is along the negative z -axis with $B > 0$. Another set of useful operators are

$$(\pi'_x, \pi'_y) = \boldsymbol{\pi}' = \boldsymbol{\pi} - e\mathbf{B} \times \mathbf{r} = (\pi_x - eBy, \pi_y + eBx). \quad (7)$$

It can be verified that

$$[\pi_x, \pi_y] = i\hbar eB, \quad [\pi'_x, \pi'_y] = -i\hbar eB, \quad [\pi_\mu, \pi'_\nu] = 0. \quad (8)$$

The kinetic energy term contains only $\boldsymbol{\pi}$, while $\boldsymbol{\pi}'$ is related to guiding centers and plays the role of generating operators of magnetic translations. We define the ladder operator as

$$a = i \frac{l_B}{\sqrt{2}\hbar} (\pi_x + i\pi_y), \quad a^\dagger = -i \frac{l_B}{\sqrt{2}\hbar} (\pi_x - i\pi_y), \quad (9)$$

satisfying $[a, a^\dagger] = 1$, where $l_B = \sqrt{\hbar/(eB)}$ is the magnetic length. The guiding center operators are

$$X = \frac{\pi'_y}{eB} = x + \frac{\pi_y}{eB}, \quad Y = -\frac{\pi'_x}{eB} = y - \frac{\pi_x}{eB}, \quad (10)$$

satisfying $[X, Y] = -il_B^2$. The magnetic translation operators are

$$t(\mathbf{R}) = \exp\left(-\frac{i}{\hbar} \boldsymbol{\pi}' \cdot \mathbf{R}\right) = \exp\left[-\frac{i}{l_B^2} (XR_y - YR_x)\right]. \quad (11)$$

The magnetic translation operators satisfy

$$t(\mathbf{R})t(\mathbf{R}') = t(\mathbf{R}')t(\mathbf{R}) \exp\left[\frac{i}{l_B^2} \mathbf{e}_z \cdot (\mathbf{R} \times \mathbf{R}')\right] = t(\mathbf{R}')t(\mathbf{R}) \exp\left[\frac{i}{l_B^2} (R_x R'_y - R_y R'_x)\right]. \quad (12)$$

2.1 Landau gauge

2.1.1 First Landau gauge

If the Landau gauge $\mathbf{A} = -Bxe_y$ is applied, the ladder operators (9) become

$$a = i \frac{l_B}{\sqrt{2}\hbar} [p_x + i(p_y - eBx)] = \frac{1}{\sqrt{2}} (l_B \partial_x + il_B \partial_y + x/l_B), \quad (13)$$

$$a^\dagger = -i \frac{l_B}{\sqrt{2}\hbar} [p_x - i(p_y - eBx)] = -\frac{1}{\sqrt{2}} (l_B \partial_x - il_B \partial_y - x/l_B), \quad (14)$$

satisfying $[a, p_y] = [a^\dagger, p_y] = 0$ so that $[a^\dagger a, p_y] = 0$, meaning that we can use the common eigenstates $|n, k_y\rangle$ of $a^\dagger a$ and p_y as the basis to expand any 2D wavefunctions, so that ($n \geq 0$)

$$a^\dagger a |n, k_y\rangle = n |n, k_y\rangle \quad p_y |n, k_y\rangle = \hbar k_y |n, k_y\rangle. \quad (15)$$

The Landau level (LL) state $|n, k_y\rangle$ has the real-space wave function

$$\varphi_{n, k_y}(\mathbf{r}) = \langle \mathbf{r} | n, k_y \rangle = \frac{1}{\sqrt{2^n n! \sqrt{\pi} l_B}} e^{-\frac{1}{2}(x/l_B - k_y l_B)^2} H_n(x/l_B - k_y l_B) \frac{e^{ik_y y}}{\sqrt{L_y}}, \quad (16)$$

where L_y is the length of the system along y direction. The above formula can be obtained by first solving $a\varphi_{0, k_y} = 0 \Rightarrow (l_B \partial_x + x/l_B - k_y l_B)\varphi_{0, k_y} = 0$, which gives $\varphi_{0, k_y} = (\sqrt{\pi} l_B L_y)^{-1/2} \exp[-(x/l_B - k_y l_B)^2/2] e^{ik_y y}$.

Then the higher-order Landau levels are set as (using $a^\dagger|n, k_y\rangle = \sqrt{n+1}|n+1, k_y\rangle$ and $\partial_z - z = e^{z^2/2} \frac{d}{dz} e^{-z^2/2}$, $(\partial_z - z)^n = e^{z^2/2} \frac{d^n}{dz^n} e^{-z^2/2}$)

$$\begin{aligned}\varphi_{n, k_y}(\mathbf{r}) &= \frac{1}{\sqrt{n!}} (a^\dagger)^n \psi_{0, k_y}(\mathbf{r}) = \frac{(-1)^n}{\sqrt{2^n n!} \sqrt{\pi} l_B} \frac{e^{i k_y y}}{\sqrt{L_y}} \left[\frac{\partial}{\partial(x/l_B)} - \left(\frac{x}{l_B} - k_y l_B \right) \right]^n e^{-\frac{1}{2}(x/l_B - k_y l_B)^2} \\ &= \frac{(-1)^n}{\sqrt{2^n n!} \sqrt{\pi} l_B} \frac{e^{i k_y y}}{\sqrt{L_y}} e^{\frac{1}{2}(x/l_B - k_y l_B)^2} \frac{d^n}{d(x/l_B - k_y l_B)^n} e^{-(x/l_B - k_y l_B)^2}.\end{aligned}$$

The LL degeneracy is (A is the system area, $\Phi_0 = h/e$ is the flux quantum)

$$N_\phi = \frac{L_x}{(2\pi/L_y)l_B^2} = \frac{A}{2\pi l_B^2} = \frac{BA}{h/e} = \frac{BA}{\Phi_0}. \quad (17)$$

We calculate the plane wave matrix elements in LL basis. In this gauge we have $p_x = \pi_x = i\hbar(a^\dagger - a)/(\sqrt{2}l_B)$, $p_y - eBx = \pi_y = -\hbar(a^\dagger + a)/(\sqrt{2}l_B)$. Using $|n, k_y\rangle = e^{i k_y y} e^{-i p_x k_y l_B^2/\hbar} |n, 0\rangle$, and Eq. (4), we have

$$\begin{aligned}\langle n', k'_y | e^{i\mathbf{q}\cdot\mathbf{r}} | n, k_y \rangle &= \langle n', 0 | e^{\frac{i}{\hbar} p_x k'_y l_B^2} e^{i q_x x + i(q_y + k_y - k'_y)y} e^{-\frac{i}{\hbar} p_x k_y l_B^2} | n, 0 \rangle \\ &= \delta_{k'_y, k_y + q_y} e^{i q_x k_y l_B^2} \langle n', 0 | e^{\frac{i}{\hbar} p_x q_y l_B^2} e^{i q_x x} | n, 0 \rangle \\ &= \delta_{k'_y, k_y + q_y} e^{i q_x k_y l_B^2} e^{\frac{i}{2} q_x q_y l_B^2} \langle n', 0 | e^{\frac{i}{\hbar} p_x q_y l_B^2 + i q_x x} | n, 0 \rangle \\ &= \delta_{k'_y, k_y + q_y} e^{i q_x k_y l_B^2} e^{\frac{i}{2} q_x q_y l_B^2} \langle n', 0 | e^{i Q a^\dagger + i \bar{Q} a} | n, 0 \rangle \\ &= \delta_{k'_y, k_y + q_y} e^{i q_x (k_y + \frac{q_y}{2}) l_B^2} e^{-\frac{|Q|^2}{2}} \langle n', 0 | e^{i Q a^\dagger} e^{i \bar{Q} a} | n, 0 \rangle \\ &= \delta_{k'_y, k_y + q_y} e^{i q_x (k_y + \frac{q_y}{2}) l_B^2} e^{-\frac{|Q|^2}{2}} \sum_{l=0}^{\min(n', n)} \frac{(iQ)^{n'-l} (i\bar{Q})^{n-l}}{(n'-l)!(n-l)!} \langle n', 0 | (a^\dagger)^{n'-l} a^{n-l} | n, 0 \rangle \\ &= \delta_{k'_y, k_y + q_y} e^{i q_x (k_y + \frac{q_y}{2}) l_B^2} F_{n'n}(Q),\end{aligned} \quad (18)$$

where $Q = \frac{(q_x + i q_y) l_B}{\sqrt{2}}$, $\bar{Q} = \frac{(q_x - i q_y) l_B}{\sqrt{2}}$, and the form factor

$$F_{n'n}(Q) = \begin{cases} e^{-\frac{|Q|^2}{2}} \sqrt{\frac{n!}{n'!}} (i\bar{Q})^{n-n'} L_{n'}^{n-n'}(|Q|^2) & (n' \leq n) \\ e^{-\frac{|Q|^2}{2}} \sqrt{\frac{n!}{n'!}} (iQ)^{n'-n} L_n^{n'-n}(|Q|^2) & (n' \geq n) \end{cases}. \quad (19)$$

2.1.2 Second Landau gauge

Alternatively, if the Landau gauge $\mathbf{A} = B y \mathbf{e}_x$ is applied, then the ladder operators (9) will satisfy $[a, p_x] = [a^\dagger, p_x] = 0$ so that $[a^\dagger a, p_x] = 0$, meaning that we can use the common eigenstates of $a^\dagger a$ and p_x as the basis to expand any 2D wavefunctions. We denote the eigenfunction $|n, k_x\rangle$, $n \geq 0$ so that $a^\dagger a |n, k_x\rangle = n |n, k_x\rangle$ and $p_x |n, k_x\rangle = \hbar k_x |n, k_x\rangle$. The state $|n, k_x\rangle$ has the wave function

$$\varphi_{n, k_x}(\mathbf{r}) = \langle \mathbf{r} | n, k_x \rangle = \frac{(-i)^n}{\sqrt{2^n n!} \sqrt{\pi} l_B} e^{-\frac{1}{2}(y/l_B + k_x l_B)^2} H_n(y/l_B + k_x l_B) \frac{e^{i k_x x}}{\sqrt{L_x}}, \quad (20)$$

where L_x is the length of the system along x direction. Similar to the first Landau gauge case, the plane wave matrix element reads

$$\langle n', k'_x | e^{i\mathbf{q}\cdot\mathbf{r}} | n, k_x \rangle = \delta_{k'_x, k_x + q_x} e^{-i q_y (k_x + \frac{q_x}{2}) l_B^2} F_{n'n}(Q). \quad (21)$$

2.1.3 Bloch Landau levels

We can even make the LLs to have similar form as Bloch wavefunctions. Notice that for two general vectors $\mathbf{m}_1, \mathbf{m}_2$ the magnetic translations $t(\mathbf{m}_1), t(\mathbf{m}_2)$ do not commute with each other, due to Eq. (12). However, if the area closed by $\mathbf{m}_1, \mathbf{m}_2$ (i.e., $(\mathbf{m}_1 \times \mathbf{m}_2)_z$) is integer times of $2\pi l_B^2$, they commute.

So we can define the magnetic unit cell as the smallest area to make the magnetic translations commute. Explicitly we define it as spanned by $\mathbf{m}_1, \mathbf{m}_2 \parallel \mathbf{e}_y$, and $(\mathbf{m}_1 \times \mathbf{m}_2)_z = m_{1x} m_2 = 2\pi l_B^2$. Then $t(m\mathbf{m}_1 + n\mathbf{m}_2)$ are

all commutative with each other and the Hamiltonian. Correspondingly, we define the magnetic Brillouin zone (MBZ), which is spanned by $\mathbf{g}_1, \mathbf{g}_2$ so that $\mathbf{m}_i \cdot \mathbf{g}_j = 2\pi\delta_{ij}$. Explicitly, $\mathbf{g}_1 = \frac{2\pi}{m_{1x}}\mathbf{e}_x$, $\mathbf{g}_2 = \frac{2\pi}{m_2}(-\frac{m_{1y}}{m_{1x}}\mathbf{e}_x + \mathbf{e}_y)$.

We can impose the periodic boundary condition: $t(N_1\mathbf{m}_1) = t(N_2\mathbf{m}_2) = 1$, where N_1, N_2 are total number of unit cells along the $\mathbf{m}_1, \mathbf{m}_2$ direction, and $N_1N_2 = N_\phi$. Then the eigenvalue of $t(\mathbf{m}_i)$ is $e^{-i\mathbf{k} \cdot \mathbf{m}_i}$, where $\mathbf{k} = k_x\mathbf{e}_x + k_y\mathbf{e}_y = k_1\mathbf{g}_1 + k_2\mathbf{g}_2 \in \text{MBZ}$. One may expect the following basis from Eq. (16)

$$\psi_{n,\mathbf{k}}(\mathbf{r}) = \frac{e^{-i\pi k_2^2 m_{1y}/m_2}}{\sqrt{N_1}} \sum_s e^{is\mathbf{k} \cdot \mathbf{m}_1} t^s(\mathbf{m}_1) \varphi_{n,k_y}(\mathbf{r}) = \frac{1}{\sqrt{N_1}} \sum_s e^{is2\pi k_1} e^{-i\pi \frac{m_{1y}}{m_2} (k_2+s)^2} \varphi_{n, \frac{2\pi}{m_2} (k_2+s)}(\mathbf{r}).$$

However, it can be checked that it doesn't give the correct eigenvalues strictly due to $t_1^N(\mathbf{m}_1) \neq 1$. Instead, we construct such basis in the following way,

$$\begin{aligned} \psi_{n,\mathbf{k}}(\mathbf{r}) &= \frac{e^{-i\pi k_2^2 m_{1y}/m_2}}{\sqrt{N_1 N_t}} \sum_{s=0}^{N_1-1} \sum_{l \in \mathbb{Z}} e^{i(s+lN_1)\mathbf{k} \cdot \mathbf{m}_1} t^{s+lN_1}(\mathbf{m}_1) \varphi_{n,k_y}(\mathbf{r}) \\ &= \frac{1}{\sqrt{N_1 N_t}} \sum_{sl} e^{is2\pi k_1} e^{-i\pi \frac{m_{1y}}{m_2} (k_2+s+lN_1)^2} \varphi_{n, \frac{2\pi}{m_2} (k_2+s+lN_1)}(\mathbf{r}) \\ &= \frac{1}{\sqrt{N_1 N_t}} \sum_{s \in \mathbb{Z}} e^{is2\pi k_1} e^{-i\pi \frac{m_{1y}}{m_2} (k_2+s)^2} \varphi_{n, \frac{2\pi}{m_2} (k_2+s)}(\mathbf{r}), \end{aligned} \quad (22)$$

i.e., the summation over s is rendered into infinity, and N_t is simply a normalization factor. Actually this wavefunction is nothing but the Bloch LLs proposed in PRL 110, 106802 (2013). In that paper the Bloch LLs are defined as

$$\psi_{n,\mathbf{k}}(\mathbf{r}) = \frac{1}{\sqrt{N_1}} \sum_{s=0}^{N_x-1} e^{i2\pi s k_1} \tilde{\psi}_{n, \frac{2\pi}{m_2} (k_2+s)}(\mathbf{r}), \quad (23)$$

where $\tilde{\psi}_{n, \frac{2\pi}{m_2} k_2}(\mathbf{r})$ is the LL on the torus

$$\tilde{\psi}_{n, \frac{2\pi}{m_2} k_2}(\mathbf{r}) = \frac{e^{-i\pi k_2^2 m_{1y}/m_2}}{\sqrt{N_t}} \sum_{l \in \mathbb{Z}} t^{lN_1}(\mathbf{m}_1) \varphi_{n, \frac{2\pi}{m_2} k_2}(\mathbf{r}), \quad (24)$$

satisfying $t(\frac{\mathbf{m}_1}{N_2})\tilde{\psi}_{n, \frac{2\pi}{m_2} k_2} = \tilde{\psi}_{n, \frac{2\pi}{m_2} (k_2 + \frac{1}{N_2})}$ and $t(\frac{\mathbf{m}_2}{N_1})\tilde{\psi}_{n, \frac{2\pi}{m_2} k_2} = e^{-i2\pi k_2/N_1} \tilde{\psi}_{n, \frac{2\pi}{m_2} k_2}$. As a result, $\tilde{\psi}_{n, \frac{2\pi}{m_2} (k_2+s)} = t^{sN_2}(\frac{\mathbf{m}_1}{N_2})\tilde{\psi}_{n, \frac{2\pi}{m_2} k_2} = t^s(\mathbf{m}_1)\tilde{\psi}_{n, \frac{2\pi}{m_2} k_2}$. Putting all together, we find

$$\psi_{n,\mathbf{k}}(\mathbf{r}) = \frac{1}{\sqrt{N_1}} \sum_{s=0}^{N_x-1} e^{i2\pi s k_1} t^s(\mathbf{m}_1) \tilde{\psi}_{n, \frac{2\pi}{m_2} k_2}(\mathbf{r}) = \frac{e^{-i\pi k_2^2 m_{1y}/m_2}}{\sqrt{N_1 N_t}} \sum_{s=0}^{N_x-1} \sum_l e^{i2\pi s k_1} t^{s+lN_1}(\mathbf{m}_1) \varphi_{n, \frac{2\pi}{m_2} k_2}(\mathbf{r}),$$

which is same to Eq. (22).

We can check $t(\mathbf{m}_i)\psi_{n,\mathbf{k}} = e^{-i\mathbf{k} \cdot \mathbf{m}_i} \psi_{n,\mathbf{k}}$, and the periodic boundary condition is automatically satisfied. The Bloch LLs satisfy $\psi_{n,\mathbf{k}+\mathbf{g}_1} = \psi_{n,\mathbf{k}}$, and $\psi_{n,\mathbf{k}+\mathbf{g}_2} = e^{-i\mathbf{k} \cdot \mathbf{m}_1} \psi_{n,\mathbf{k}}$. We may also prove the orthogonality

$$\begin{aligned} \langle n', \mathbf{k}' | n, \mathbf{k} \rangle &= \frac{1}{N_1 N_t} \sum_{s's'} e^{i2\pi (s k_1 - s' k'_1)} e^{i\pi \frac{m_{1y}}{m_2} (s+s'+k_2+k'_2)(s'-s+k'_2-k_2)} \langle n', \frac{2\pi}{m_2} (k'_2 + s') | n, \frac{2\pi}{m_2} (k_2 + s) \rangle \\ &= \delta_{n'n} \delta_{\mathbf{k}'\mathbf{k}}. \end{aligned} \quad (25)$$

The plane wave matrix element then reads ($\mathbf{q} = q_1\mathbf{g}_1 + q_2\mathbf{g}_2$)

$$\begin{aligned} &\langle n', \mathbf{k}' | e^{i\mathbf{q} \cdot \mathbf{r}} | n, \mathbf{k} \rangle \\ &= \frac{1}{N_1 N_t} \sum_{s's'} e^{i2\pi (s k_1 - s' k'_1)} e^{i\pi \frac{m_{1y}}{m_2} (s+s'+k_2+k'_2)(s'-s+k'_2-k_2)} \langle n', \frac{2\pi}{m_2} (k'_2 + s') | e^{i\mathbf{q} \cdot \mathbf{r}} | n, \frac{2\pi}{m_2} (k_2 + s) \rangle \\ &= \frac{1}{N_1 N_t} \sum_{s's'} e^{i2\pi (s k_1 - s' k'_1)} e^{i\frac{2\pi}{m_2} (k_2+s+\frac{q_2}{2})(m_{1y}q_2+q_x l_B^2)} F_{n'n} \left(\frac{q_x + i q_y}{\sqrt{2}} l_B \right) \delta_{k'_2, k_2+q_2+s-s'} \\ &= \delta_{k'_2, [k_2+q_2]} \frac{1}{N_1 N_t} \sum_s e^{i2\pi (s k_1 - (k_2+q_2-k'_2+s)k'_1)} e^{i2\pi (k_2+s+\frac{q_2}{2})q_1} F_{n'n} \left(\frac{q_x + i q_y}{\sqrt{2}} l_B \right) \\ &= \delta_{k'_1, [k_1+q_1]} \delta_{k'_2, [k_2+q_2]} e^{i2\pi k'_1 (k'_2 - k_2 - q_2)} e^{i2\pi q_1 (k_2 + \frac{q_2}{2})} F_{n'n} \left(\frac{q_x + i q_y}{\sqrt{2}} l_B \right), \end{aligned} \quad (26)$$

where $[k_i]$ represents the fractional part of k_i . We see the matrix element is nonzero only when $\mathbf{k}' = \mathbf{k} + \mathbf{q} \bmod(\mathbf{g}_1, \mathbf{g}_2)$. Such kind of LL formulation will be quite useful in diagonalizing continuum Hamiltonian with plane wave potentials.

2.2 Symmetric gauge

If the symmetric gauge $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r} = \left(\frac{By}{2}, -\frac{Bx}{2}, 0\right)$ is applied, we define another set of ladder operators

$$b = i\frac{l_B}{\sqrt{2}\hbar}(\pi'_x - i\pi'_y), \quad b^\dagger = -i\frac{l_B}{\sqrt{2}\hbar}(\pi'_x + i\pi'_y), \quad (27)$$

which satisfies $[a, b] = [a, b^\dagger] = [a^\dagger, b] = [a^\dagger, b^\dagger] = 0$ so that $[a^\dagger a, b^\dagger b] = 0$, meaning that we can use the common eigenstates of $a^\dagger a$ and $b^\dagger b$ to expand any 2D wavefunctions. We denote the eigenfunction $|n, m\rangle, n, m \geq 0$ so that $a^\dagger a|n, m\rangle = n|n, m\rangle$ and $b^\dagger b|n, m\rangle = m|n, m\rangle$. We introduce the complex coordinates $z = (x + iy)/l_B$ and $\bar{z} = (x - iy)/l_B$. Then the operators will have the differential form

$$a = \frac{1}{\sqrt{2}}\left(\frac{z}{2} + 2\frac{\partial}{\partial \bar{z}}\right), \quad a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{\bar{z}}{2} - 2\frac{\partial}{\partial z}\right), \quad b = \frac{1}{\sqrt{2}}\left(\frac{\bar{z}}{2} + 2\frac{\partial}{\partial z}\right), \quad b^\dagger = \frac{1}{\sqrt{2}}\left(\frac{z}{2} - 2\frac{\partial}{\partial \bar{z}}\right), \quad (28)$$

and the wavefunction $\phi_{n,m}(\mathbf{r})$ is

$$\phi_{n,m}(\mathbf{r}) = \langle \mathbf{r} | n, m \rangle = \frac{(-1)^n}{\sqrt{2\pi l_B^2}} \sqrt{\frac{n!}{m!}} \left(\frac{z}{\sqrt{2}}\right)^{m-n} e^{-\frac{z\bar{z}}{4}} L_n^{m-n}\left(\frac{z\bar{z}}{2}\right). \quad (29)$$

To obtain it, we first solve $a\phi_{0,0} = b\phi_{0,0} = 0$, which gives $\phi_{0,0} = e^{-z\bar{z}/4}/\sqrt{2\pi l_B^2}$. Then higher order LLs are

$$\begin{aligned} \phi_{n,m} &= \frac{1}{\sqrt{n!m!}} (b^\dagger)^m (a^\dagger)^n \phi_{0,0} = \frac{(-1)^{n+m}}{\sqrt{2\pi l_B^2 n!m! 2^{n+m}}} \left(e^{\frac{z\bar{z}}{4}} \frac{\partial}{\partial z} e^{-\frac{z\bar{z}}{4}}\right)^n \left(e^{\frac{z\bar{z}}{4}} \frac{\partial}{\partial \bar{z}} e^{-\frac{z\bar{z}}{4}}\right)^m e^{-\frac{z\bar{z}}{4}} \\ &= \frac{(-2)^{n+m}}{\sqrt{2\pi l_B^2 n!m! 2^{n+m}}} e^{\frac{z\bar{z}}{4}} \frac{\partial^n}{\partial z^n} \frac{\partial^m}{\partial \bar{z}^m} e^{-\frac{z\bar{z}}{2}} = \frac{(-2)^n}{\sqrt{2\pi l_B^2 n!m! 2^{n+m}}} e^{\frac{z\bar{z}}{4}} \frac{\partial^n}{\partial z^n} z^m e^{-\frac{z\bar{z}}{2}} \\ &= \frac{(-2)^n}{\sqrt{2\pi l_B^2 n!m! 2^{n+m}}} e^{-\frac{z\bar{z}}{4}} \sum_{l=0}^{\min(n,m)} \frac{n!m!}{l!(n-l)!(m-l)!} z^{m-l} \left(-\frac{\bar{z}}{2}\right)^{n-l} \\ &= \frac{(-1)^n}{\sqrt{2\pi l_B^2}} \sqrt{\frac{n!}{m!}} \left(\frac{z}{\sqrt{2}}\right)^{m-n} e^{-\frac{z\bar{z}}{4}} \sum_{l=0}^{\min(n,m)} \frac{(-1)^{n-l} m!}{l!(n-l)!(m-l)!} \left(\frac{z\bar{z}}{2}\right)^{n-l} \\ &= \frac{(-1)^n}{\sqrt{2\pi l_B^2}} \sqrt{\frac{n!}{m!}} \left(\frac{z}{\sqrt{2}}\right)^{m-n} e^{-\frac{z\bar{z}}{4}} L_n^{m-n}\left(\frac{z\bar{z}}{2}\right) = \frac{(-1)^n}{\sqrt{2\pi l_B^2}} F_{n,m}\left(-i\frac{z}{\sqrt{2}}\right), \end{aligned} \quad (30)$$

where $\partial_{\bar{z}} = z/4 + e^{z\bar{z}/4} \partial_z e^{-z\bar{z}/4}$ and Eq. (5) have been used. It is easy to verify that $\phi_{n,m}(\mathbf{r}) = \phi_{m,n}^*(\mathbf{r})$ since z and z^* are symmetric, or explicitly proved using Eq. (3).

The plane wave matrix element under symmetric gauge Landau basis is ($Q = \frac{(q_x + iq_y)l_B}{\sqrt{2}}, \bar{Q} = \frac{(q_x - iq_y)l_B}{\sqrt{2}}$)

$$\begin{aligned} \langle n', m' | e^{i\mathbf{q} \cdot \mathbf{r}} | n, m \rangle &= \langle n', m' | e^{i(Qa^\dagger + \bar{Q}a) + i(\bar{Q}b^\dagger + Qb)} | n, m \rangle = \langle n' | e^{i(Qa^\dagger + \bar{Q}a)} | n \rangle \langle m' | e^{i(\bar{Q}b^\dagger + Qb)} | m \rangle \\ &= F_{n'n}(Q) F_{m'm}(\bar{Q}). \end{aligned} \quad (31)$$

3 LLs of simple systems

3.1 Free Schrodinger electrons

The free electron moving in 2D under an external magnetic field has the Hamiltonian

$$H = \frac{(\mathbf{p} + e\mathbf{A})^2}{2m} = \frac{\pi_x^2 + \pi_y^2}{2m} = \hbar\omega \left(a^\dagger a + \frac{1}{2}\right), \quad (32)$$

where $\omega = \frac{eB}{m}$ is the cyclotron frequency. The eigenenergy is $E_n = \hbar\omega \left(n + \frac{1}{2}\right), n \geq 0$.

3.2 Free Dirac electrons

The free Dirac electrons in 2D under an external field has the Hamiltonian (near the Dirac point)

$$H = v_F(\mathbf{p} + e\mathbf{A}) \cdot (\eta\sigma_x, \sigma_y). \quad (33)$$

The eigenenergy is $E_n = \text{sgn}(n)\hbar\omega\sqrt{|n|}$, where $\omega = \sqrt{2}v_F/l_B$ and n takes all integers.