Landau level formulation (v2)

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May 21, 2025

Contents

1 Useful formulas

The Hermite polynomial is

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \tag{1}$$

The generalized Laguerre polynomial is (α can take any integers satisfying $\alpha \geq -n$)

$$L_n^\alpha(x) = \sum_{l=\max(0,-\alpha)}^{n} (-1)^l \frac{(n+\alpha)!}{(n-l)!(\alpha+l)!} \frac{x^l}{l!} = \sum_{l=0}^{\min(n,n+\alpha)} (-1)^{n-l} \frac{(n+\alpha)!}{l!(\alpha+n-l)!} \frac{x^{n-l}}{(n-l)!}, \tag{2}$$

satisfying

$$\frac{(-x)^m}{m!} L_n^{m-n}(x) = \frac{(-x)^n}{n!} L_m^{n-m}(x). \tag{3}$$

The BCH formula is

$$e^A e^B = e^{A+B} e^{[A,B]/2} = e^B e^A e^{[A,B]}, \quad \text{for } [A,[A,B]] = [B,[A,B]] = 0. \tag{4}$$

The Leibniz rule of derivatives reads

$$\frac{d^n}{dx^n}(fg) = \sum_{l=0}^{n} \frac{n!}{l!(n-l)!} \frac{d^l f}{dx^l} \frac{d^{n-l} g}{dx^{n-l}}. \tag{5}$$

2 Peierls substitution, ladder operators, guiding centers

If a magnetic field is exerted on an electron moving in the 2D plane, its momentum will be replaced by $\boldsymbol{p} \to \boldsymbol{\pi}$, where

$$(\pi_x, \pi_y) = \boldsymbol{\pi} = \boldsymbol{p} + e\boldsymbol{B}, \tag{6}$$

$e > 0$ is the electron charge, $\boldsymbol{A}$ is the gauge potential so that $\nabla \times \boldsymbol{A} = \boldsymbol{B}$. Here we assume $\boldsymbol{B} = -B\boldsymbol{e}_z$ is along the negative z-axis with $B > 0$. Another set of useful operators are

$$(\pi'_x, \pi'_y) = \boldsymbol{\pi}' = \boldsymbol{\pi} - e\boldsymbol{B} \times \boldsymbol{r} = (\pi_x - eBy, \pi_y + eBx). \tag{7}$$

It can be verified that

$$[\pi_x, \pi_y] = i\hbar eB, \quad [\pi'_x, \pi'_y] = -i\hbar eB, \quad [\pi_\mu, \pi'_\nu] = 0. \tag{8}$$

The kinetic energy term contains only $\boldsymbol{\pi}$, while $\boldsymbol{\pi}'$ is related to guiding centers and plays the role of generating operators of magnetic translations. We define the ladder operator as

$$a = i\frac{l_B}{\sqrt{2}\hbar}(\pi_x + i\pi_y), \quad a^\dagger = -i\frac{l_B}{\sqrt{2}\hbar}(\pi_x - i\pi_y), \tag{9}$$

satisfying $[a, a^\dagger] = 1$, where $l_B = \sqrt{\hbar/(eB)}$ is the magnetic length. The guiding center operators are

$$X = \frac{\pi'_y}{eB} = x + \frac{\pi_y}{eB}, \quad Y = -\frac{\pi'_x}{eB} = y - \frac{\pi_x}{eB}, \tag{10}$$

satisfying $[X, Y] = -il_B^2$. The magnetic translation operators are

$$t(\boldsymbol{R}) = \exp\left(-\frac{i}{\hbar}\boldsymbol{\pi}' \cdot \boldsymbol{R}\right) = \exp\left[-\frac{i}{l_B^2}(XR_y - YR_x)\right]. \tag{11}$$

The magnetic translation operators satisfy

$$t(\boldsymbol{R})t(\boldsymbol{R}') = t(\boldsymbol{R}')t(\boldsymbol{R})\exp\left[\frac{i}{l_B^2}\boldsymbol{e}_z \cdot (\boldsymbol{R} \times \boldsymbol{R}')\right] = t(\boldsymbol{R}')t(\boldsymbol{R})\exp\left[\frac{i}{l_B^2}(R_xR'_y - R_yR'_x)\right]. \tag{12}$$

2.1 Landau gauge

2.1.1 First Landau gauge

If the Landau gauge $\boldsymbol{m} = -Bx\boldsymbol{e}_y$ is applied, the ladder operators (9) become

$$a = i\frac{l_B}{\sqrt{2}\hbar}[p_x + i(p_y - eBx)] = \frac{1}{\sqrt{2}}(l_B\partial_x + il_B\partial_y + x/l_B), \tag{13}$$

$$a^\dagger = -i\frac{l_B}{\sqrt{2}\hbar}[p_x - i(p_y - eBx)] = -\frac{1}{\sqrt{2}}(l_B\partial_x - il_B\partial_y - x/l_B), \tag{14}$$

satisfying $[a, p_y] = [a^\dagger, p_y] = 0$ so that $[a^\dagger a, p_y] = 0$, meaning that we can use the common eigenstates $|n, k_y\rangle$ of $a^\dagger a$ and p_y as the basis to expand any 2D wavefunctions, so that $(n \geq 0)$

$$a^\dagger a|n, k_y\rangle = n|n, k_y\rangle \quad p_y|n, k_y\rangle = \hbar k_y|n, k_y\rangle. \tag{15}$$

The Landau level (LL) state $|n, k_y\rangle$ has the real-space wave function

$$\varphi_{n,k_y}(\boldsymbol{r}) = \langle \boldsymbol{r}|n, k_y\rangle = \frac{1}{\sqrt{2^n n!\sqrt{\pi}l_B}}e^{-\frac{1}{2}(x/l_B - k_yl_B)^2}H_n\left(x/l_B - k_yl_B\right)\frac{e^{ik_yy}}{\sqrt{L_y}}, \tag{16}$$

where L_y is the length of the system along y direction. The above formula can be obtained by first solving $a\varphi_{0,k_y} = 0 \Rightarrow (l_B\partial_x + x/l_B - k_yl_B)\varphi_{0,k_y} = 0$, which gives $\varphi_{0,k_y} = (\sqrt{\pi}l_BL_y)^{-1/2}\exp[-(x/l_B - k_yl_B)^2/2]e^{ik_yy}$.

Then the higher-order Landau levels are set as (using $a^\dagger|n,k_y\rangle = \sqrt{n+1}|n+1,k_y\rangle$ and $\partial_z - z = e^{z^2/2}\frac{d}{dz}e^{-z^2/2}$, $(\partial_z - z)^n = e^{z^2/2}\frac{d^n}{dz^n}e^{-z^2/2}$)

$$\begin{aligned}\varphi_{n,k_y}(\boldsymbol{r}) =& \frac{1}{\sqrt{n!}}(a^\dagger)^n\psi_{0,k_y}(\boldsymbol{r}) = \frac{(-1)^n}{\sqrt{2^n n!\sqrt{\pi}l_B}}\frac{e^{ik_y y}}{\sqrt{L_y}}\left[\frac{\partial}{\partial(x/l_B)} - \left(\frac{x}{l_B} - k_y l_B\right)\right]^n e^{-\frac{1}{2}(x/l_B - k_y l_B)^2} \\ =& \frac{(-1)^n}{\sqrt{2^n n!\sqrt{\pi}l_B}}\frac{e^{ik_y y}}{\sqrt{L_y}}e^{\frac{1}{2}(x/l_B - k_y l_B)^2}\frac{d^n}{d(x/l_B - k_y l_B)^n}e^{-(x/l_B - k_y l_B)^2}.\end{aligned}$$

The LL degeneracy is (A is the system area, $\Phi_0 = h/e$ is the flux quantum)

$$N_\phi = \frac{L_x}{(2\pi/L_y)l_B^2} = \frac{A}{2\pi l_B^2} = \frac{BA}{h/e} = \frac{BA}{\Phi_0}. \tag{17}$$

We calculate the plane wave matrix elements in LL basis. In this gauge we have $p_x = \pi_x = i\hbar(a^\dagger - a)/(\sqrt{2}l_B)$, $p_y - eBx = \pi_y = -\hbar(a^\dagger + a)/(\sqrt{2}l_B)$. Using $|n,k_y\rangle = e^{ik_y y}e^{-ip_x k_y l_B^2/\hbar}|n,0\rangle$, and Eq. (4), we have

$$\begin{aligned}\langle n',k_y'|e^{i\boldsymbol{q}\cdot\boldsymbol{r}}|n,k_y\rangle &= \langle n',0|e^{\frac{i}{\hbar}p_x k_y' l_B^2}e^{iq_x x + i(q_y + k_y - k_y')y}e^{-\frac{i}{\hbar}p_x k_y l_B^2}|n,0\rangle \\ &= \delta_{k_y',k_y+q_y}e^{iq_x k_y l_B^2}\langle n',0|e^{\frac{i}{\hbar}p_x q_y l_B^2}e^{iq_x x}|n,0\rangle \\ &= \delta_{k_y',k_y+q_y}e^{iq_x k_y l_B^2}e^{\frac{i}{2}q_x q_y l_B^2}\langle n',0|e^{\frac{i}{\hbar}p_x q_y l_B^2 + iq_x x}|n,0\rangle \\ &= \delta_{k_y',k_y+q_y}e^{iq_x k_y l_B^2}e^{\frac{i}{2}q_x q_y l_B^2}\langle n',0|e^{iQa^\dagger + i\bar{Q}a}|n,0\rangle \\ &= \delta_{k_y',k_y+q_y}e^{iq_x(k_y + \frac{q_y}{2})l_B^2}e^{-\frac{|Q|^2}{2}}\langle n',0|e^{iQa^\dagger}e^{i\bar{Q}a}|n,0\rangle \\ &= \delta_{k_y',k_y+q_y}e^{iq_x(k_y + \frac{q_y}{2})l_B^2}e^{-\frac{|Q|^2}{2}}\sum_{l=0}^{\min(n',n)}\frac{(iQ)^{n'-l}(i\bar{Q})^{n-l}}{(n'-l)!(n-l)!}\langle n',0|(a^\dagger)^{n'-l}a^{n-l}|n,0\rangle \\ &= \delta_{k_y',k_y+q_y}e^{iq_x(k_y + \frac{q_y}{2})l_B^2}F_{n'n}(Q),\end{aligned} \tag{18}$$

where $Q = \frac{(q_x + iq_y)l_B}{\sqrt{2}}$, $\bar{Q} = \frac{(q_x - iq_y)l_B}{\sqrt{2}}$, and the form factor

$$F_{n'n}(Q) = \begin{cases} e^{-\frac{|Q|^2}{2}}\sqrt{\frac{n'!}{n!}}(i\bar{Q})^{n-n'}L_{n'}^{n-n'}(|Q|^2) & (n' \le n) \\ e^{-\frac{|Q|^2}{2}}\sqrt{\frac{n!}{n'!}}(iQ)^{n'-n}L_{n}^{n'-n}(|Q|^2) & (n' \ge n)\end{cases}. \tag{19}$$

2.1.2 Second Landau gauge

Alternatively, if the Landau gauge $\boldsymbol{m} = By\boldsymbol{e}_x$ is applied, then the ladder operators (9) will satisfy $[a,p_x] = [a^\dagger,p_x] = 0$ so that $[a^\dagger a, p_x] = 0$, meaning that we can use the common eigenstates of $a^\dagger a$ and p_x as the basis to expand any 2D wavefunctions. We denote the eigenfunction $|n,k_x\rangle, n \geqslant 0$ so that $a^\dagger a|n,k_x\rangle = n|n,k_x\rangle$ and $p_x|n,k_x\rangle = \hbar k_x|n,k_x\rangle$. The state $|n,k_x\rangle$ has the wave function

$$\varphi_{n,k_x}(\boldsymbol{r}) = \langle\boldsymbol{r}|n,k_x\rangle = \frac{(-i)^n}{\sqrt{2^n n!\sqrt{\pi}l_B}}e^{-\frac{1}{2}(y/l_B + k_x l_B)^2}H_n(y/l_B + k_x l_B)\frac{e^{ik_x x}}{\sqrt{L_x}}, \tag{20}$$

where L_x is the length of the system along x direction. Similar to the first Landau gauge case, the plane wave matrix element reads

$$\langle n',k_x'|e^{i\boldsymbol{q}\cdot\boldsymbol{r}}|n,k_x\rangle = \delta_{k_x',k_x+q_x}e^{-iq_y(k_x + \frac{q_x}{2})l_B^2}F_{n'n}(Q). \tag{21}$$

2.1.3 Bloch Landau levels

We can even make the LLs to have similar form as Bloch wavefunctions. Notice that for two general vectors $\boldsymbol{m}_1$, $\boldsymbol{m}_2$ the magnetic translations $t(\boldsymbol{m}_1)$, $t(\boldsymbol{m}_2)$ do not commute with each other, due to Eq. (12). However, if the area closed by $\boldsymbol{m}_1$, $\boldsymbol{m}_2$ (i.e., $(\boldsymbol{m}_1 \times \boldsymbol{m}_2)_z$) is integer times of $2\pi l_B^2$, they commute.

So we can define the magnetic unit cell as the smallest area to make the magnetic translations commute. Explicitly we define it as spanned by $\boldsymbol{m}_1$, $\boldsymbol{m}_2 \parallel \boldsymbol{e}_y$, and $(\boldsymbol{m}_1 \times \boldsymbol{m}_2)_z = m_{1x}m_2 = 2\pi l_B^2$. Then $t(m\boldsymbol{m}_1 + n\boldsymbol{m}_2)$ are all commutative with each other and the Hamiltonian. Correspondingly, we define the magnetic Brillouin zone (MBZ), which is spanned by $\boldsymbol{g}_1$, $\boldsymbol{g}_2$ so that $\boldsymbol{m}_i\cdot\boldsymbol{g}_j=2\pi\delta_{ij}$. Explicitly, $\boldsymbol{g}_1=\frac{2\pi}{m_{1x}}\boldsymbol{e}_x$, $\boldsymbol{g}_2=\frac{2\pi}{m_2}(-\frac{m_{1y}}{m_{1x}}\boldsymbol{e}_x+\boldsymbol{e}_y)$.

We can impose the periodic boundary condition: $t(N_1\boldsymbol{m}_1)=t(N_2\boldsymbol{m}_2)=1$, where N_1, N_2 are total number of unit cells along the $\boldsymbol{m}_1$, $\boldsymbol{m}_2$ direction, and $N_1N_2=N_\phi$. Then the eigenvalue of $t(\boldsymbol{m}_i)$ is $e^{-i\boldsymbol{k}\cdot\boldsymbol{m}_i}$, where $\boldsymbol{k}=k_x\boldsymbol{e}_x+k_y\boldsymbol{e}_y=k_1\boldsymbol{g}_1+k_2\boldsymbol{g}_2\in$ MBZ. One may expect the following basis from Eq. (16)

$$\psi_{n,\boldsymbol{k}}(\boldsymbol{r})=\frac{e^{-i\pi k_2^2m_{1y}/m_2}}{\sqrt{N_1}}\sum_s e^{is\boldsymbol{k}\cdot\boldsymbol{m}_1}t^s(\boldsymbol{m}_1)\varphi_{n,k_y}(\boldsymbol{r})=\frac{1}{\sqrt{N_1}}\sum_s e^{is2\pi k_1}e^{-i\pi\frac{m_{1y}}{m_2}(k_2+s)^2}\varphi_{n,\frac{2\pi}{m_2}(k_2+s)}(\boldsymbol{r}).$$

However, it can be checked that it doesn't give the correct eigenvalues strictly due to $t_1^N(\boldsymbol{m}_1)\neq 1$. Instead, we construct such basis in the following way,

$$\begin{aligned}\psi_{n,\boldsymbol{k}}(\boldsymbol{r})=&\frac{e^{-i\pi k_2^2m_{1y}/m_2}}{\sqrt{N_1N_t}}\sum_{s=0}^{N_1-1}\sum_{l\in\mathbb{Z}}e^{i(s+lN_1)\boldsymbol{k}\cdot\boldsymbol{m}_1}t^{s+lN_1}(\boldsymbol{m}_1)\varphi_{n,k_y}(\boldsymbol{r})\\=&\frac{1}{\sqrt{N_1N_t}}\sum_{sl}e^{is2\pi k_1}e^{-i\pi\frac{m_{1y}}{m_2}(k_2+s+lN_1)^2}\varphi_{n,\frac{2\pi}{m_2}(k_2+s+lN_1)}(\boldsymbol{r})\\=&\frac{1}{\sqrt{N_1N_t}}\sum_{s\in\mathbb{Z}}e^{is2\pi k_1}e^{-i\pi\frac{m_{1y}}{m_2}(k_2+s)^2}\varphi_{n,\frac{2\pi}{m_2}(k_2+s)}(\boldsymbol{r}),\end{aligned}\tag{22}$$

i.e., the summation over s is rendered into infinity, and N_t is simply a normalization factor. Actually this wavefunction is nothing but the Bloch LLs proposed in PRL 110, 106802 (2013). In that paper the Bloch LLs are defined as

$$\psi_{n,\boldsymbol{k}}(\boldsymbol{r})=\frac{1}{\sqrt{N_1}}\sum_{s=0}^{N_x-1}e^{i2\pi sk_1}\tilde{\psi}_{n,\frac{2\pi}{m_2}(k_2+s)}(\boldsymbol{r}),\tag{23}$$

where $\tilde{\psi}_{n,\frac{2\pi}{m_2}k_2}(\boldsymbol{r})$ is the LL on the torus

$$\tilde{\psi}_{n,\frac{2\pi}{m_2}k_2}(\boldsymbol{r})=\frac{e^{-i\pi k_2^2m_{1y}/m_2}}{\sqrt{N_t}}\sum_{l\in\mathbb{Z}}t^{lN_1}(\boldsymbol{m}_1)\varphi_{n,\frac{2\pi}{m_2}k_2}(\boldsymbol{r}),\tag{24}$$

satisfying $t(\frac{\boldsymbol{m}_1}{N_2})\tilde{\psi}_{n,\frac{2\pi}{m_2}k_2}=\tilde{\psi}_{n,\frac{2\pi}{m_2}(k_2+\frac{1}{N_2})}$ and $t(\frac{\boldsymbol{m}_2}{N_1})\tilde{\psi}_{n,\frac{2\pi}{m_2}k_2}=e^{-i2\pi k_2/N_1}\tilde{\psi}_{n,\frac{2\pi}{m_2}k_2}$. As a result, $\tilde{\psi}_{n,\frac{2\pi}{m_2}(k_2+s)}=t^{sN_2}(\frac{\boldsymbol{m}_1}{N_2})\tilde{\psi}_{n,\frac{2\pi}{m_2}k_2}=t^s(\boldsymbol{m}_1)\tilde{\psi}_{n,\frac{2\pi}{m_2}k_2}$. Putting all together, we find

$$\psi_{n,\boldsymbol{k}}(\boldsymbol{r})=\frac{1}{\sqrt{N_1}}\sum_{s=0}^{N_x-1}e^{i2\pi sk_1}t^s(\boldsymbol{m}_1)\tilde{\psi}_{n,\frac{2\pi}{m_2}k_2}(\boldsymbol{r})=\frac{e^{-i\pi k_2^2m_{1y}/m_2}}{\sqrt{N_1N_t}}\sum_{s=0}^{N_x-1}\sum_l e^{i2\pi sk_1}t^{s+lN_1}(\boldsymbol{m}_1)\varphi_{n,\frac{2\pi}{m_2}k_2}(\boldsymbol{r}),$$

which is same to Eq. (22).

We can check $t(\boldsymbol{m}_i)\psi_{n,\boldsymbol{k}}=e^{-i\boldsymbol{k}\cdot\boldsymbol{m}_i}\psi_{n,\boldsymbol{k}}$, and the periodic boundary condition is automatically satisfied. The Bloch LLs satisfy $\psi_{n,\boldsymbol{k}+\boldsymbol{g}_1}=\psi_{n,\boldsymbol{k}}$, and $\psi_{n,\boldsymbol{k}+\boldsymbol{g}_2}=e^{-i\boldsymbol{k}\cdot\boldsymbol{m}_1}\psi_{n,\boldsymbol{k}}$. We may also prove the orthogonality

$$\begin{aligned}\langle n',\boldsymbol{k}'|n,\boldsymbol{k}\rangle=&\frac{1}{N_1N_t}\sum_{s's}e^{i2\pi(sk_1-s'k_1')}e^{i\pi\frac{m_{1y}}{m_2}\left(s+s'+k_2+k_2'\right)\left(s'-s+k_2'-k_2\right)}\langle n',\frac{2\pi}{m_2}(k_2'+s')|n,\frac{2\pi}{m_2}(k_2+s)\rangle\\=&\delta_{n'n}\delta_{\boldsymbol{k}'\boldsymbol{k}}.\end{aligned}\tag{25}$$

The plane wave matrix element then reads ($\boldsymbol{q}=q_1\boldsymbol{g}_1+q_2\boldsymbol{g}_2$)

$$\begin{aligned}&\langle n',\boldsymbol{k}'|e^{i\boldsymbol{q}\cdot\boldsymbol{r}}|n,\boldsymbol{k}\rangle\\=&\frac{1}{N_1N_t}\sum_{s's}e^{i2\pi(sk_1-s'k_1')}e^{i\pi\frac{m_{1y}}{m_2}\left(s+s'+k_2+k_2'\right)\left(s'-s+k_2'-k_2\right)}\langle n',\frac{2\pi}{m_2}(k_2'+s')|e^{i\boldsymbol{q}\cdot\boldsymbol{r}}|n,\frac{2\pi}{m_2}(k_2+s)\rangle\\=&\frac{1}{N_1N_t}\sum_{s's}e^{i2\pi(sk_1-s'k_1')}e^{i\frac{2\pi}{m_2}\left(k_2+s+\frac{q_2}{2}\right)(m_{1y}q_2+q_xl_B^2)}F_{n'n}\left(\frac{q_x+iq_y}{\sqrt{2}}l_B\right)\delta_{k_2',k_2+q_2+s-s'}\\=&\delta_{k_2',[k_2+q_2]}\frac{1}{N_1N_t}\sum_s e^{i2\pi(sk_1-(k_2+q_2-k_2'+s)k_1')}e^{i2\pi\left(k_2+s+\frac{q_2}{2}\right)q_1}F_{n'n}\left(\frac{q_x+iq_y}{\sqrt{2}}l_B\right)\\=&\delta_{k_1',[k_1+q_1]}\delta_{k_2',[k_2+q_2]}e^{i2\pi k_1'(k_2'-k_2-q_2)}e^{i2\pi q_1\left(k_2+\frac{q_2}{2}\right)}F_{n'n}\left(\frac{q_x+iq_y}{\sqrt{2}}l_B\right),\end{aligned}\tag{26}$$

where $[k_i]$ represents the fractional part of k_i. We see the matrix element is nonzero only when $\boldsymbol{k}' = \boldsymbol{k} + \boldsymbol{q} \mod(\boldsymbol{g}_1, \boldsymbol{g}_2)$. Such kind of LL formulation will be quite useful in diagonalizing continuum Hamiltonian with plane wave potentials.

2.2 Symmetric gauge

If the symmetric gauge $\boldsymbol{A} = \frac{1}{2}\boldsymbol{B} \times \boldsymbol{r} = \left(\frac{By}{2}, -\frac{Bx}{2}, 0\right)$ is applied, we define another set of ladder operators

$$b = i\frac{l_B}{\sqrt{2}\hbar}(\pi'_x - i\pi'_y), \quad b^\dagger = -i\frac{l_B}{\sqrt{2}\hbar}(\pi'_x + i\pi'_y), \tag{27}$$

which satisfies $[a, b] = [a, b^\dagger] = [a^\dagger, b] = [a^\dagger, b^\dagger] = 0$ so that $[a^\dagger a, b^\dagger b] = 0$, meaning that we can use the common eigenstates of $a^\dagger a$ and $b^\dagger b$ to expand any 2D wavefunctions. We denote the eigenfunction $|n, m\rangle, n, m \geqslant 0$ so that $a^\dagger a|n, m\rangle = n|n, m\rangle$ and $b^\dagger b|n, m\rangle = m|n, m\rangle$. We introduce the complex coordinates $z = (x + iy)/l_B$ and $\bar{z} = (x - iy)/l_B$. Then the operators will have the differential form

$$a = \frac{1}{\sqrt{2}}\left(\frac{z}{2} + 2\frac{\partial}{\partial \bar{z}}\right), \quad a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{\bar{z}}{2} - 2\frac{\partial}{\partial z}\right), \quad b = \frac{1}{\sqrt{2}}\left(\frac{\bar{z}}{2} + 2\frac{\partial}{\partial z}\right), \quad b^\dagger = \frac{1}{\sqrt{2}}\left(\frac{z}{2} - 2\frac{\partial}{\partial \bar{z}}\right), \tag{28}$$

and the wavefunction $\phi_{n,m}(\boldsymbol{r})$ is

$$\phi_{n,m}(\boldsymbol{r}) = \langle \boldsymbol{r}|n, m\rangle = \frac{(-1)^n}{\sqrt{2\pi l_B^2}}\sqrt{\frac{n!}{m!}}\left(\frac{z}{\sqrt{2}}\right)^{m-n} e^{-\frac{z\bar{z}}{4}} L_n^{m-n}\left(\frac{z\bar{z}}{2}\right). \tag{29}$$

To obtain it, we first solve $a\phi_{0,0} = b\phi_{0,0} = 0$, which gives $\phi_{0,0} = e^{-z\bar{z}/4}/\sqrt{2\pi l_B^2}$. Then higher order LLs are

$$\begin{aligned}
\phi_{n,m} =& \frac{1}{\sqrt{n!m!}}(b^\dagger)^m(a^\dagger)^n\phi_{0,0} = \frac{(-1)^{n+m}}{\sqrt{2\pi l_B^2 n!m!2^{n+m}}}\left(e^{\frac{z\bar{z}}{4}}\frac{\partial}{\partial z}e^{-\frac{z\bar{z}}{4}}\right)^n\left(e^{\frac{z\bar{z}}{4}}\frac{\partial}{\partial \bar{z}}e^{-\frac{z\bar{z}}{4}}\right)^m e^{-\frac{z\bar{z}}{4}} \\
=& \frac{(-2)^{n+m}}{\sqrt{2\pi l_B^2 n!m!2^{n+m}}}e^{\frac{z\bar{z}}{4}}\frac{\partial^n}{\partial z^n}\frac{\partial^m}{\partial \bar{z}^m}e^{-\frac{z\bar{z}}{2}} = \frac{(-2)^n}{\sqrt{2\pi l_B^2 n!m!2^{n+m}}}e^{\frac{z\bar{z}}{4}}\frac{\partial^n}{\partial z^n}z^m e^{-\frac{z\bar{z}}{2}} \\
=& \frac{(-2)^n}{\sqrt{2\pi l_B^2 n!m!2^{n+m}}}e^{-\frac{z\bar{z}}{4}}\sum_{l=0}^{\min(n,m)}\frac{n!m!}{l!(n-l)!(m-l)!}z^{m-l}\left(-\frac{\bar{z}}{2}\right)^{n-l} \\
=& \frac{(-1)^n}{\sqrt{2\pi l_B^2}}\sqrt{\frac{n!}{m!}}\left(\frac{z}{\sqrt{2}}\right)^{m-n}e^{-\frac{z\bar{z}}{4}}\sum_{l=0}^{\min(n,m)}\frac{(-1)^{n-l}m!}{l!(n-l)!(m-l)!}\left(\frac{z\bar{z}}{2}\right)^{n-l} \\
=& \frac{(-1)^n}{\sqrt{2\pi l_B^2}}\sqrt{\frac{n!}{m!}}\left(\frac{z}{\sqrt{2}}\right)^{m-n}e^{-\frac{z\bar{z}}{4}}L_n^{m-n}\left(\frac{z\bar{z}}{2}\right) = \frac{(-1)^n}{\sqrt{2\pi l_B^2}}F_{n,m}\left(-i\frac{z}{\sqrt{2}}\right),
\end{aligned} \tag{30}$$

where $\partial_{\bar{z}} - z/4 = e^{z\bar{z}/4}\partial_{\bar{z}}e^{-z\bar{z}/4}$ and Eq. (5) have been used. It is easy to verify that $\phi_{n,m}(\boldsymbol{r}) = \phi^*_{m,n}(\boldsymbol{r})$ since z and z^* are symmetric, or explicitly proved using Eq. (3).

The plane wave matrix element under symmetric gauge Landau basis is ($Q = \frac{(q_x+iq_y)l_B}{\sqrt{2}}$, $\bar{Q} = \frac{(q_x-iq_y)l_B}{\sqrt{2}}$)

$$\begin{aligned}
\langle n', m'|e^{i\boldsymbol{q}\cdot\boldsymbol{r}}|n, m\rangle &= \langle n', m'|e^{i(Qa^\dagger+\bar{Q}a)+i(\bar{Q}b^\dagger+Qb)}|n, m\rangle = \langle n'|e^{i(Qa^\dagger+\bar{Q}a)}|n\rangle\langle m'|e^{i(\bar{Q}b^\dagger+Qb)}|m\rangle \\
&= F_{n'n}(Q)\,F_{m'm}(\bar{Q}).
\end{aligned} \tag{31}$$

3 LLs of simple systems

3.1 Free Schrodinger electrons

The free electron moving in 2D under an external magnetic field has the Hamiltonian

$$H = \frac{(\boldsymbol{p} + e\boldsymbol{B})^2}{2m} = \frac{\pi_x^2 + \pi_y^2}{2m} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right), \tag{32}$$

where $\omega = \frac{eB}{m}$ is the cyclotron frequency. So the its eigenenergy is $E_n = \hbar\omega\left(n + \frac{1}{2}\right), n \geqslant 0$, the eigenfunctions can take any gauge.

3.2 Free Dirac electrons

The free Dirac electrons in 2D under an external field has the Hamiltonian (near the Dirac point)

$$H = v_F(\boldsymbol{p} + e\boldsymbol{B}) \cdot \boldsymbol{\sigma}^\eta = v_F(\boldsymbol{p} + e\boldsymbol{B}) \cdot (\eta\sigma_x, \sigma_y). \tag{33}$$

The eigenenergy is $E_n = \mathrm{sgn}(n)\hbar\omega\sqrt{|n|}$, where $\omega = \sqrt{2}v_F/l_B$ and n takes all integers.