Landau level formulation

SHI Hao

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1 Peierls substitution, ladder operator

In quantum mechanics, if a field is exerted on an electron moving in the two dimensional plane, its momentum operator will be replaced by $\boldsymbol{p} \to \boldsymbol{\pi} = \boldsymbol{p} + e\boldsymbol{A}$, where e is the electron charge, $\boldsymbol{A}$ is the gauge potential so that $\nabla \times \boldsymbol{A} = \boldsymbol{B} = B\boldsymbol{e}_z$. First we can define the ladder operator (the prefactor i added here is just a convention to make the formulation more symmetric)

$$a = i\frac{l_B}{\sqrt{2}\hbar}(\pi_x - i\pi_y), \quad a^\dagger = -i\frac{l_B}{\sqrt{2}\hbar}(\pi_x + i\pi_y), \tag{1}$$

where $l_B = \sqrt{\hbar/(eB)}$ is the magnetic length. The above operators obey $[a, a^\dagger] = 1$.

1.1 Landau gauge

1.1.1 First Landau gauge

If the Landau gauge $\boldsymbol{A} = Bx\boldsymbol{e}_y$ is applied, then the ladder operators (1) will satisfy $[a, p_y] = [a^\dagger, p_y] = 0$ so that $[a^\dagger a, p_y] = 0$, meaning that we can use the common eigenstates of $a^\dagger a$ and p_y as the basis to expand any 2D wavefunctions. We denote the eigenfunction $|n, k_y\rangle, n \geqslant 0$ so that $a^\dagger a|n, k_y\rangle = n|n, k_y\rangle$ and $p_y|n, k_y\rangle = \hbar k_y|n, k_y\rangle$. The state $|n, k_y\rangle$ has the wave function

$$\varphi_{n,k_y}(\boldsymbol{r}) = \langle \boldsymbol{r}|n, k_y\rangle = \frac{1}{\sqrt{2^n n! \sqrt{\pi} l_B}} e^{-\frac{1}{2}\left(\frac{x}{l_B} + k_y l_B\right)^2} H_n\left(\frac{x}{l_B} + k_y l_B\right) \frac{e^{ik_y y}}{\sqrt{L_y}}, \tag{2}$$

where L_y is the length of the system along y direction, and $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$ is the Hermite polynomial. Note that in this gauge we have

$$|n, k_y\rangle = e^{ik_y y} e^{ip_x k_y l_B^2/\hbar}|n, 0\rangle, \tag{3}$$

$$p_x = \frac{i\hbar}{\sqrt{2}l_B}(a^\dagger - a), \quad p_y + eBx = \frac{\hbar}{\sqrt{2}l_B}(a^\dagger + a), \tag{4}$$

Usually we will calculate the matrix elements of plane waves under the Landau level basis, which are

$$\begin{aligned}
\langle n', k_y'|e^{i\boldsymbol{q}\cdot\boldsymbol{r}}|n, k_y\rangle &= \langle n', 0|e^{-\frac{i}{\hbar}p_x k_y' l_B^2} e^{iq_x x + i(q_y + k_y - k_y')y} e^{\frac{i}{\hbar}p_x k_y l_B^2}|n, 0\rangle \\
&= \delta_{k_y', k_y + q_y} \langle n', 0|e^{\frac{k_y' l_B}{\sqrt{2}}(a^\dagger - a)} e^{i\frac{q_x l_B}{\sqrt{2}}(a^\dagger + a)} e^{-\frac{k_y l_B}{\sqrt{2}}(a^\dagger - a)}|n, 0\rangle \\
&= \delta_{k_y', k_y + q_y} e^{-\left(iq_x k_y + i\frac{q_x q_y}{2} + \frac{q\bar{q}}{4}\right)l_B^2} \langle n', 0|e^{i\frac{l_B}{\sqrt{2}}a^\dagger \bar{q}} e^{i\frac{l_B}{\sqrt{2}}aq}|n, 0\rangle \\
(n \geqslant n') &= \delta_{k_y', k_y + q_y} e^{-\left(iq_x k_y + i\frac{q_x q_y}{2} + \frac{q\bar{q}}{4}\right)l_B^2} \sqrt{\frac{n'!}{n!}} \left(\frac{iql_B}{\sqrt{2}}\right)^{n-n'} L_{n'}^{n-n'}\left(\frac{q\bar{q}}{2}l_B^2\right) \\
(n' \geqslant n) &= \delta_{k_y', k_y + q_y} e^{-\left(iq_x k_y + i\frac{q_x q_y}{2} + \frac{q\bar{q}}{4}\right)l_B^2} \sqrt{\frac{n!}{n'!}} \left(\frac{i\bar{q}l_B}{\sqrt{2}}\right)^{n'-n} L_n^{n'-n}\left(\frac{q\bar{q}}{2}l_B^2\right),
\end{aligned} \tag{5}$$

where $q = q_x + iq_y, \bar{q} = q^*$, $L_n^\alpha(x) = \sum_{k=0}^n (-1)^k \begin{pmatrix} n+\alpha \\ n-k \end{pmatrix} \frac{x^k}{k!}$ is the Laguerre polynomial. The first step uses (3), the second step uses (4) and p_y commutes with $a, a^\dagger$, and $p_y|n, 0\rangle = 0$, the third step uses the BCH formula $e^A e^B = e^{A+B} e^{[A,B]/2}$.

1.1.2 Second Landau gauge

Alternatively, if the Landau gauge $\boldsymbol{A} = -By\boldsymbol{e}_x$ is applied, then the ladder operators (1) will satisfy $[a, p_x] = [a^\dagger, p_x] = 0$ so that $[a^\dagger a, p_x] = 0$, meaning that we can use the common eigenstates of $a^\dagger a$ and p_x as the basis to expand any 2D wavefunctions. We denote the eigenfunction $|n, k_x\rangle, n \geqslant 0$ so that $a^\dagger a|n, k_x\rangle = n|n, k_x\rangle$ and $p_x|n, k_x\rangle = \hbar k_x|n, k_x\rangle$. The state $|n, k_x\rangle$ has the wave function

$$\varphi_{n,k_x}(\boldsymbol{r}) = \langle \boldsymbol{r}|n, k_x\rangle = \frac{i^n}{\sqrt{2^n n!\sqrt{\pi} l_B}} e^{-\frac{1}{2}\left(\frac{y}{l_B} - k_x l_B\right)^2} H_n\left(\frac{y}{l_B} - k_x l_B\right) \frac{e^{ik_x x}}{\sqrt{L_x}}, \tag{6}$$

where L_x is the length of the system along x direction. Similar to the first Landau gauge case, the plane wave matrix element reads

$$\begin{aligned}
\langle n', k_x'|e^{i\boldsymbol{q}\cdot\boldsymbol{r}}|n, k_x\rangle &= \langle n', 0|e^{\frac{i}{\hbar}p_y k_x' l_B^2} e^{i(q_x + k_x - k_x')x + iq_y y} e^{-\frac{i}{\hbar}p_y k_x l_B^2}|n, 0\rangle \\
&= \delta_{k_x', k_x + q_x} \langle n', 0|e^{i\frac{k_x' l_B}{\sqrt{2}}(a^\dagger + a)} e^{\frac{q_y l_B}{\sqrt{2}}(a^\dagger - a)} e^{-i\frac{k_x l_B}{\sqrt{2}}(a^\dagger + a)}|n, 0\rangle \\
&= \delta_{k_x', k_x + q_x} e^{\left(iq_y k_x + i\frac{q_x q_y}{2} - \frac{q\bar{q}}{4}\right) l_B^2} \langle n', 0|e^{i\frac{l_B}{\sqrt{2}}a^\dagger \bar{q}} e^{i\frac{l_B}{\sqrt{2}}aq}|n, 0\rangle \\
(n \geqslant n') &= \delta_{k_x', k_x + q_x} e^{\left(iq_y k_x + i\frac{q_x q_y}{2} - \frac{q\bar{q}}{4}\right) l_B^2} \sqrt{\frac{n'!}{n!}} \left(\frac{iql_B}{\sqrt{2}}\right)^{n-n'} L_{n'}^{n-n'}\left(\frac{q\bar{q}}{2} l_B^2\right) \\
(n' \geqslant n) &= \delta_{k_x', k_x + q_x} e^{\left(iq_y k_x + i\frac{q_x q_y}{2} - \frac{q\bar{q}}{4}\right) l_B^2} \sqrt{\frac{n!}{n'!}} \left(\frac{i\bar{q}l_B}{\sqrt{2}}\right)^{n'-n} L_n^{n'-n}\left(\frac{q\bar{q}}{2} l_B^2\right).
\end{aligned} \tag{7}$$

1.2 Symmetric gauge

If the symmetric gauge $\boldsymbol{A} = -\frac{1}{2}\boldsymbol{r} \times \boldsymbol{B} = \left(-\frac{By}{2}, \frac{Bx}{2}, 0\right)$ is applied, then we can define another set of operators

$$b = i\frac{l_B}{\sqrt{2}\hbar}(\pi_x' + i\pi_y'), \quad b^\dagger = -i\frac{l_B}{\sqrt{2}\hbar}(\pi_x' - i\pi_y'), \tag{8}$$

where $\boldsymbol{\pi}' = \boldsymbol{p} - e\boldsymbol{A} = (p_x + eBy/2, p_y - eBx/2)$, and the symmetric gauge guarantees that $[a, b] = [a, b^\dagger] = [a^\dagger, b] = [a^\dagger, b^\dagger] = 0$ so that $[a^\dagger a, b^\dagger b] = 0$, meaning that we can use the common eigenstates of $a^\dagger a$ and $b^\dagger b$ as the basis to expand any 2D wavefunctions. We denote the eigenfunction $|n, m\rangle, n, m \geqslant 0$ so that $a^\dagger a|n, m\rangle = n|n, k_y\rangle$ and $b^\dagger b|n, m\rangle = m|n, m\rangle$. The state $|n, m\rangle$ also has an analytical wavefunction $\phi_{n,m}(\boldsymbol{r})$, which is most easily expressed after introducing the complex coordinates $z = (x + iy)/l_B$ and $\bar{z} = (x - iy)/l_B$. Then the operators will have the differential form

$$a = \frac{1}{\sqrt{2}}\left(\frac{\bar{z}}{2} + 2\frac{\partial}{\partial z}\right), \quad a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{z}{2} - 2\frac{\partial}{\partial \bar{z}}\right), \tag{9}$$

$$b = \frac{1}{\sqrt{2}}\left(\frac{z}{2} + 2\frac{\partial}{\partial \bar{z}}\right), \quad b^\dagger = \frac{1}{\sqrt{2}}\left(\frac{\bar{z}}{2} - 2\frac{\partial}{\partial z}\right), \tag{10}$$

and the wavefunction $\phi_{n,m}(\boldsymbol{r})$ is

$$\phi_{n,m}(\boldsymbol{r}) = \langle \boldsymbol{r}|n, m\rangle = \frac{(-1)^m}{\sqrt{2\pi l_B^2}} e^{-\frac{z\bar{z}}{4}} \sqrt{\frac{m!}{n!}} \left(\frac{z}{\sqrt{2}}\right)^{n-m} L_m^{n-m}\left(\frac{z\bar{z}}{2}\right), \tag{11}$$

where $L_n^\alpha(x) = \frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}(e^{-x}x^{n+\alpha}) = \sum_{k=0}^n (-1)^k \begin{pmatrix} n+\alpha \\ n-k \end{pmatrix} \frac{x^k}{k!}$ is the generalized Laguerre polynomial. It is easy to verify that $\phi_{n,m}(\boldsymbol{r}) = \phi_{m,n}^*(\boldsymbol{r})$ since z and z^* are symmetric, or using the identity $\frac{(-x)^m}{m!}L_n^{m-n}(x) = \frac{(-x)^n}{n!}L_m^{n-m}(x)$.

The plane wave matrix element under symmetric gauge Landau basis is

$$\begin{aligned}
\langle n', m'|e^{i\boldsymbol{q}\cdot\boldsymbol{r}}|n, m\rangle &= \langle n', m'|e^{i\frac{\bar{q}a^\dagger + qa}{\sqrt{2}} l_B} e^{i\frac{qb^\dagger + \bar{q}b}{\sqrt{2}} l_B}|n, m\rangle \\
&= e^{-\frac{q\bar{q}}{2} l_B^2} \langle n'|e^{i\frac{\bar{q}l_B}{\sqrt{2}}a^\dagger} e^{i\frac{ql_B}{\sqrt{2}}a}|n\rangle \langle m'|e^{i\frac{ql_B}{\sqrt{2}}b^\dagger} e^{i\frac{\bar{q}l_B}{\sqrt{2}}b}|m\rangle \\
&= e^{-\frac{q\bar{q}}{2} l_B^2} \sqrt{\frac{n!}{n'!}} \left(\frac{i\bar{q}l_B}{\sqrt{2}}\right)^{n'-n} L_n^{n'-n}\left(\frac{q\bar{q}}{2} l_B^2\right) \sqrt{\frac{m!}{m'!}} \left(\frac{iql_B}{\sqrt{2}}\right)^{m'-m} L_m^{m'-m}\left(\frac{q\bar{q}}{2} l_B^2\right).
\end{aligned} \tag{12}$$

Eqs. (5), (7) and (12) have been checked numerically.

2 LLs of simple systems

2.1 Free Schrodinger electrons

The free electron moving in 2D under an external magnetic field has the Hamiltonian

$$H = \frac{(\boldsymbol{p} + e\boldsymbol{A})^2}{2m} = \frac{\pi_x^2 + \pi_y^2}{2m} = \hbar\omega_c \left(a^\dagger a + \frac{1}{2} \right), \tag{13}$$

where $\omega_c = \frac{eB}{m}$ is the cyclotron frequency. So the its eigenenergy is $E_n = \hbar\omega \left(n + \frac{1}{2}\right), n \geqslant 0$, the eigenfunctions can take any gauge.

2.2 Free Dirac electrons

The free Dirac electrons in 2D under an external field has the Hamiltonian (near the Dirac point)

$$H = v_F(\boldsymbol{p} + e\boldsymbol{A}) \cdot \boldsymbol{\sigma}^\eta = v_F(\boldsymbol{p} + e\boldsymbol{A}) \cdot (\eta\sigma_x, \sigma_y). \tag{14}$$

The eigenenergy is $E_n = \mathrm{sgn}(n)\hbar\omega_D\sqrt{|n|}$, where $\omega_D = \sqrt{2}v_F/l_B$ and n takes all integers.