ANALYTICAL SOLUTIONS AND PROPERTIES OF A FUNCTIONAL EQUATION

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ANALYTICAL SOLUTIONS AND PROPERTIES OF A FUNCTIONAL EQUATION

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Let Z be the set of integers. Determine all functions $\boldsymbol{f}: \boldsymbol{Z} \rightarrow \boldsymbol{Z}$, such that for all integers a and b,

$$f(2a) + 2f(b) = f\big(f(a+b)\big)$$

International Math Olympiad - 2019

Introduction

This problem was a part of the International Math Olympiad for 2019. The participants of this competition are high-school students from all around the world, and they are generally considered to be at the top of their game for their age group. The competition is a 2-day event. Each day the participants are given 3 problems to solve, and about 4.5 hours to solve it. Each problem is worth 7 points, which brings the total up to 42 points.

This problem looks difficult to approach for the average person. However, looking at the statistics for the 2019 IMO, this problem has more than 60% of the participants getting a perfect score, i.e., 7 points for it. This makes it a relatively easy problem for an elite mathematician. However, solving this equation requires some complex number theory, which an average person would not understand

This solution has been laid out in such a way that it uses basic logic and high school math so that everyone can understand.

Approach

When dealing with problems that involve functions of different variables, in this case a and b, what one tries to do is use special/trivial values of the variables. In case of an arithmetic problem, the values of the functions at 0 are taken into account, because of the identity property of 0. This states that:

$$x + 0 = x, \qquad \forall x$$

Similarly, in a multiplicative function, the value 1 is used;

$$x \cdot 1 = x, \qquad \forall x$$

As we move on with solving, our best aim is to try and reduce the function to its basic form, i.e., getting f(x) on one side. We try to build upon whatever information is given to us through the question. This allows us to use the function directly and input values to see what results they give us.

$f(x)$ = some stuff

It is also good to make clever guesses for the solution based on evidence (as we will see later). When we assume a given form, it makes it easier for us to develop upon ideas as we can now use the rules of the "given form" to proceed with the solving. For example, if we see that the function is some sort of a quadratic function, we can guess the value of it to be:

$$f(x) = ax^2 + bx + c$$

Later on, we can use other properties that we have already established to find out values for a, b and c, and thus we can complete our solution.

Solution

Consider the original equation

$$f(2a) + 2f(b) = f\big(f(a+b)\big)$$

Consider the trivial values $a = 0$, as this will help give us a good starting point.
Then,
For $a = 0$

$$f(2 \cdot 0) + 2f(b) = f\big(f(0+b)\big)$$
$$f(0) + 2f(b) = f\big(f(b)\big) \quad \text{....(1)}$$

But $b \in Z$

Thus, from equation (1), we get:

$$f(0) + 2f(x) = f\big(f(x)\big), \forall\, x \in Z \quad \text{....(2)}$$

At $a = 1$ in the original equation:

$$f(2 \cdot 1) + 2f(b) = f\big(f(b+1)\big)$$
$$f(2) + 2f(b) = f\big(f(b+1)\big) \quad \text{....(3)}$$

Consider value $x = b + 1$ in Equation (2). We get:

$$f(0) + 2f(b+1) = f\big(f(b+1)\big) \quad \text{....(4)}$$

Comparing equation (4) with (3), both the right-hand sides have $f\big(f(b+1)\big)$.

$$f(2) + 2f(b) = f(0) + 2f(b+1)$$

$$f(2) - f(0) = 2f(b+1) - 2f(b)$$

Rearranging the terms, we get:

$$f(b+1) - f(b) = \frac{f(2) - f(0)}{2} \quad \text{....(5)}$$

Notice that $b \in Z$. Thus, the left-hand side is just a difference of 2 consecutive terms. No matter what b value we take, the right-hand side remains constant.

Because the difference of any 2 consecutive terms is a constant, this has to imply that $f(x)$ has to be a linear function. If it was any other type of function, this would mean that the difference would not be constant.

We can now assume $f(x)$ to be a linear function, as this will allow us to move on and use the properties of liner functions to continue our solving process.

Therefore, let:

$$f(x) = mx + n \quad \text{....(6)}$$

Where $m, n \in Z$. Normally we would have assumed $f(x) = ax + b$ but we are already using a and b in our equations. Anyway, it is irrelevant because m, n are just any arbitrary variables and their names do not matter.

Applying equation (5) in our original equation, we get:

$$f(2a) + 2f(b) = f(f(a + b)$$

$$m(2a) + n + 2[m(b) + n] = f[m(a + b) + n]$$

$$2am + n + 2mb + 2n = m^2(a + b) + nm + n$$

Rearranging, we get:

$$2m(a + b) + 3n = m[m(a + b) + n] + n$$

$$2m(a + b) + 2n = m[m(a + b) + n]$$

$$2[m(a + b) + n] = m[m(a + b) + n]$$

Notice, at $x = a + b$,

$f(x) = m(a + b) + n$

$$2f(a + b) = mf(a + b)$$

We can conclude that

$$\boldsymbol{m = 2}$$

Now we know that our function is of the form

$$f(x) = 2x + n$$

To find the value of n, we can plug it in various equations.

In our original equation:

$$f(2a) + 2f(b) = f\big(f(a + b)\big)$$

$$4a + n + 4b + 2n = f(2a + 2b + n)$$

$$4a + 4b + 3n = 4a + 4b + 3n$$

The equations on both end match up, thus we need to check for another equation. Let us check for equation (5)

$$f(x+1) - f(x) = \frac{f(2) - f(0)}{2}$$

$$2x + 2 + n - 2x - n = \frac{4 + n - n}{2}$$

$$= 2$$

Both sides give a 2, which just proves that the difference of any 2 consecutive terms**.** No matter what value of n we take, in each equation it will either get cancelled out, or it will be equal on both sides.

Using this information, we can establish that it does not matter which value of n we take.

Thus, our final answer is the equation

$$f(x) = 2x + n, \qquad \forall n \in Z$$

There is also the **trivial case** of

$$m = 0$$

In this case, the value of n also has to be 0

Then our equation becomes:

$$f(x) = 0$$

The functions that satisfy the equation $f(2a) + 2f(b) = f\big(f(a+b)\big)$, are:

a. $\boldsymbol{f(x) = 2x + n,\ \forall n \in R}$
b. $\boldsymbol{f(x) = 0}$ **(trivial solution)**

We can try to generalise the solution for this form of equation. Let us try and solve the equation:

$$f(3a) + 3f(b) = f\Big(f\big(f(a+b)\big)\Big)$$

Using a similar approach:

At $a = 0$:

$$f(0) + f(3b) = f^3(b) \qquad \ldots(7)$$

Here we define the function such that:

$$f^n(x) = f(f(f(\ldots n\ times\ (x))$$

Thus $f^3(x) = f\Big(f\big(f(x)\big)\Big)$

At $a = 1$:

$$f(3) + 3f(b) = f^3(b+1) \qquad \ldots(8)$$

If we replace b with $b + 1$ in equation 7:

$$f(0) + 3f(b+1) = f^3(b+1) \qquad \ldots(9)$$

We can now equate the equations (8) and (9) as they have the same right-hand side.

$$f(3) + 3f(b) = f(0) + 3f(b+1)$$
$$3[f(b+1) - f(b)] = f(3) - f(0)$$
$$f(b+1) - f(b) = \frac{1}{3}[f(3) - f(0)]$$

The difference of 2 consecutive terms is a difference of 2 constants; hence this function is also linear.

We can generalise this for any functional equation of this form:

$$f(pa) + pf(b) = f^p(a+b)$$

Using similar substitutions, we can prove that the difference of 2 consecutive terms is the difference of 2 constant terms:

$$f(b+1) - f(b) = \frac{1}{p}[f(p) - f(0)]$$

This is true $\forall\, p$. We will define what values p can take later. For now, assume that p takes any positive integer.

Now we know that the functional equation of the form $f(pa) + pf(b) = f^p(a+b)$ has a linear function as a solution.

Assume $f(x) = mx + n$.

For $p = 3$:

$$f(3a) + 3f(b) = f^3(a+b)$$

Let us try and define the function for any value of p;

At $p = 2$

$$f^2(x) \to f(mx+n) \to m(mx+n) + n$$
$$= m^2x + mn + n$$

Here the x term is connected with m^2 with $p = 2$

At $p = 3$

$$f^3(x) \to f^2(mx+n)$$

We can try and use a recursive approach by directly substituting $mx + n$ in $f^2(x)$. We get:

$$f^2(mx+n) = m^2(mx+n) + mn + n$$
$$= m^3x + m^2n + mn + n$$

We can see that the x term is only connected with m^3 term when $p = 3$. We can generalise this for any p:

$$f^p(x) = \ m^p x + m^{p-1}n + \cdots m^2 n + mn + n$$
$$f^p(x) = m^p x + \sum_{i=0}^{p-1} m^i n$$
$$= m^p x + n\sum_{i=0}^{p-1} m^i$$

This function is also linear with respect to x as x is present in the power of 1 and only the constant terms m and n that change with relation to p

Let us try and solve the general equation : $f(pa) + pf(b) = f^p(a+b)$.

LHS:

$$f(pa) + pf(b)$$
$$mpa + n + mpb + pn$$
$$mp(a+b) + n(p+1)$$

RHS:

$$f^p(a+b)$$
$$= m^p(a+b) + \sum_{i=0}^{p-1} m^i n$$

Comparing $(a+b)$ coefficients:

$$mp = m^p$$
$$m^p - mp = 0$$
$$m(m^{p-1} - p) = 0$$

Using Zero Product property:

$$m = 0;\ m^{p-1} - p = 0$$

We will ignore $m = 0$ as it will lead to a trivial solution.

$$m^{p-1} - p = 0$$
$$m^{p-1} = p$$
$$m = \sqrt[p-1]{p}$$

If we consider $n = 0$; (n can take any real value), this can simplify our equation to:

$$f(x) = mx$$
$$f(x) = \sqrt[p-1]{p}\, x$$

Thus, $f(x) = \sqrt[p-1]{p}\, x$ is always a solution for the functional equation $f(pa) + pf(b) = f^p(a+b)$

We still need to find what values p can take. Let us define a function $g(x)$ such that

$$g(x) = \sqrt[x-1]{x}$$

So that we can map the slope of x in our original functional equation. We can write $g(x)$ as:

$$g(x) = x^{\frac{1}{x-1}}$$

There is a discontinuity at $x = 1$, as we cannot have the denominator of the power as 0.

We will try and take the limit of this function as it approaches $x \to 1$

$$\lim_{x \to 1} x^{\frac{1}{x-1}}$$

Let $t = x - 1$. Then:

$$x = t + 1 \text{ and as } x \to 1; t \to 0$$

Thus, our limit changes to:

$$\lim_{t \to 0} (t+1)^{\frac{1}{t}}$$

This of course is a very common limit, and its value is $e = 2.718..$

Thus at $p = 1 \to m = e$

As p approaches infinity:

$$g(x) = \lim_{x \to \infty} x^{\frac{1}{x-1}}$$

Take natural on both sides:

$$\ln\big(g(x)\big) = \lim_{x \to \infty} \frac{1}{x-1} \ln(x)$$

This is $\frac{\infty}{\infty}$ form. We can apply L'Hôpital's rule by taking the derivative on numerator and denominator

$$\ln(g(x)) = \lim_{x \to \infty} \frac{1}{x} \to 0$$
$$g(x) = e^0 = 1$$

Thus as p approaches infinity, $g(x) = 1$

Thus, we know that p can take any positive integer values.

We have found that the function $\boldsymbol{f(x) = \sqrt[p-1]{p}\, x}$ is always a solution to the functional equation: $\boldsymbol{f(pa) + pf(b) = f^p(a+b)}$, with $\boldsymbol{p \in \mathbf{Z}^+, and\ a, b \in Z}$

Bibliography and links:

This paper was written by using references from the IMO official problems: https://www.imo-official.org/problems/IMO2019SL.pdf