MODELLING AND SIMULATION - SHEET 4

① Malthusian Model

a) $\dot{p}(t) = \lambda \cdot p(t) \Rightarrow \frac{dp}{dt} = \lambda p \Rightarrow \frac{dp}{p} = \lambda dt \Rightarrow \int \frac{dp}{p} = \int \lambda dt$

$\Rightarrow \ln p = \lambda t + C \Rightarrow p = e^{\lambda t + C} \Rightarrow p = e^{\lambda t} \cdot e^{C}$

Let $C' = e^{C}$ as new constant:

$\Rightarrow p = C' \cdot e^{\lambda t}$

For given $p(0) = p_0$;

$p_0 = C' \cdot \cancel{e^{\lambda \cdot 0}} \Rightarrow C' = p_0$

when we rewrite C' in the equation again;

$$\boxed{p = p_0 \cdot e^{\lambda t}}$$

b) $p = p_0 \cdot e^{\lambda t}$

At $t = 0 \Rightarrow p(0) = 232$ million

At $t = 1000 \Rightarrow p(1000) = 323$ million

$\Rightarrow$

$232 = p_0 \cdot \cancel{e^{\lambda \cdot 0}} \Rightarrow \boxed{p_0 = 232}$

$323 = 232 \cdot e^{\lambda \cdot 1000} \Rightarrow \boxed{\lambda = \frac{\ln(323/232)}{1000}}$

c) For $t = 2000 \rightarrow p(2000) = 232 \cdot e^{\frac{\ln(323/232)}{1000} \cdot 2000} \cong \underline{449.7}$ million

Reality $p(2000) = 6140$ million.

? Our model underestimates the population at $t = 2000$.

- Flaws in the model;

  → The model does not account for technological advancements, health improvements, changes in social norms and other factors that can significantly influence population grow rates.

- Improvements:

  → Adjusting λ over time: Defining a dynamic λ can reflect the changes due to significant historical events or changes.

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② Modelling Population Dynamics

$$\dot{p} = \lambda p^2 \left(1 - \frac{p}{k}\right)$$

a) Scenarios we want to use positive p^2 term:

? This factor is useful when interactions within the ~~related~~ population significantly enhance growth.

→ Useful for the populations benefit from larger group sizes. Particularly noticable in social and cooperative ~~groups~~ species.

→ This term indicates that growth accelerates more quickly than linear models. This might be typical in species with high fecundity.

b) The purpose of $\left(1 - \frac{p}{k}\right)$ term:

→ Regularization of Growth: When the population (p) approaches to k, the growing process is getting slower.

→ Limiting Population Size: When the population (p) reaches carrying capacity (k) the terms become 0. This zeros out the growth rate ($\dot{p} = 0$) and indicates that the population is stabilized (no longer growing).

? This term is fundamental part of logistic growth models and reflects the reality that no environment can support an infinite population due to limited resources.

c) Stationary points: $\boxed{p = 0}$, $\boxed{p = K}$ $\Rightarrow \dot{p} = 0$

Interpretations:

$p = 0 \Rightarrow$ Represents extinct or non-existing population.

$p = k \Rightarrow$ Represents an equilibrium state where the population is stable and does not change. Also, the maximum capacity the population can reach for this model.

d) $\frac{d\dot{p}}{dp} = \lambda . \left(2p - \frac{3p^2}{k}\right)$

e)

For $p = 0 \Rightarrow \frac{d\dot{p}}{dp} = 0$ → unstable equilibrium point

For $p = k \Rightarrow \frac{d\dot{p}}{dp} = -\lambda K < 0$ → stable equilibrium point

For $0 < p < k \Rightarrow$

$p < \frac{2k}{3} \Rightarrow \frac{d\dot{p}}{dp} > 0$ → population grows

$p > \frac{2k}{3} \Rightarrow \frac{d\dot{p}}{dp} < 0$ → population decays

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③ Predator - Prey Models

$$\left.\begin{aligned} \frac{du}{dt} &= u(1-u-av) \\ \frac{dv}{dt} &= sv(1+bu-v) \end{aligned}\right\} \text{ where } a,b,s>0$$

a) For given case $\underline{v}$ represent the <u>predator</u> population, whereas $\underline{u}$ represents the <u>prey</u> population.

→ When we increase v, we can see from the first equation growth in u (du/dt) decreases, indicating v hunts u.

→ When we increase u, as we can see from equation 2, the growth in v (dv/dt) also increases. This indicates v is fed by u.

b) Finding steady states;

For $\frac{du}{dt}=0$ → $u=0$ or $u=1-av$

For $\frac{dv}{dt}=0$ → $v=0$ or $v=1+bu$

Consider all of the combinations;

$u=0, v=0 \Rightarrow (0,0)$ → Both populations are extinct.

$u=0, v=1+bu \Rightarrow (0,1)$ → Prey population at carrying capacity. no predators.

$u=1-av, v=0 \Rightarrow (1,0)$ → Predator capacity at carrying capacity. no prey. NOT REALISTIC !!

$u=1-av, v=1+bu \Rightarrow \left(\frac{1-a}{1+ab}, \frac{1+b}{1+ab}\right)$

↳ Coexistance equilibrium. Prey and predator have stable population

c) Jacobian and its eigenvalues where u is extinct.

$$J = \begin{bmatrix} 1-2u-av & -au \\ sbv & s(1+bu-2v) \end{bmatrix} \Rightarrow \text{Evaluate at } (u,v)=(0,1).$$

$$J = \begin{bmatrix} 1-a & 0 \\ 0 & -s \end{bmatrix} \rightarrow \text{Eigenvalues}$$

$\det(\underbrace{J-\lambda I}) = 0$

$\begin{pmatrix} 1-a-\lambda & 0 \\ 0 & -s-\lambda \end{pmatrix}$

$\Rightarrow (1-a-\lambda)(-s-\lambda)=0 \Rightarrow \boxed{\lambda_1 = 1-a}$ $\boxed{\lambda_2 = -s}$

d) Stability check;

- $\lambda_1 = 1-a$ is negative if $a>1$.
- $\lambda_2 = -s$ is always negative since $s>0$

! $a>1$, state <u>stable</u> since both eigenvalues have negative real part.

$a\leq 1$, state <u>unstable</u>.

state $\Rightarrow (u,v)=(0,1)$.

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