ENPH270: Mechanics II
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Engineering Mechanics: Dynamics, 14th Edition; R. C. Hibbeler

16 Planar Kinematics of a Rigid Body

16.1 Planar Rigid-Body Motion

planar motion: motion such that all particles move along paths equidistant from a fixed plane

16.2 Translation

translation: planar motion such that all lines remain parallel to their original orientations
rectilinear translation: translation such that all points move along parallel paths
curvilinear translation: translation such that all points move along curved paths
base point: the origin of a relative coordinate system
The kinematic equations of point B and base point A on a translating rigid body are:

$$\begin{aligned} \mathbf{r}_B &= \mathbf{r}_A + \mathbf{r}_{B/A} \\ \mathbf{v}_B &= \mathbf{v}_A \\ \mathbf{a}_B &= \mathbf{a}_A \end{aligned} \tag{1}$$

16.3 Rotation about a Fixed Axis

rotation: planar motion such that all points travel in circular paths about a fixed axis
The kinematic equations for rotation with constant angular acceleration are:

$$\begin{aligned} \omega &= \omega_0 + \alpha t \\ \theta &= \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \\ \omega^2 &= \omega_0^2 + 2\alpha(\theta - \theta_0) \end{aligned} \tag{2}$$

Let $\mathbf{r}_P$ be the vector from any point on the axis of rotation to the position of the rotated point, $\mathbf{a}_t$ be the tangential acceleration, and $\mathbf{a}_n$ be the normal acceleration. Then the kinematic equations for general rotation about a fixed axis are:

$$\begin{aligned} \alpha \ \mathrm{d}\theta &= \omega \ \mathrm{d}\omega \\ \mathbf{v} &= \boldsymbol{\omega} \times \mathbf{r}_P \\ \mathbf{a} &= \mathbf{a}_t + \mathbf{a}_n = \boldsymbol{\alpha} \times \mathbf{r}_P - \omega^2 \mathbf{r}_P \end{aligned} \tag{3}$$

16.4 Absolute Motion Analysis

general plane motion: planar motion that combines translation and rotation
Analyze general plane motion by relating a rectilinear position s to an angular position θ by geometry and trigonometry, then differentiate to relate velocity v to angular velocity ω.
centre of mass (G): the one point in a body such that all external forces whose lines of action pass through it create no rotation of the body
At the centre of mass of a circular object that rolls without slipping:

$$s_G = \theta r \qquad v_G = \omega r \qquad a_G = \alpha r \tag{4}$$

16.5 Relative-Motion Analysis: Velocity

Let A be the base point of a translating coordinate system and ω be the angular velocity of B about A. Then the velocity of point B is:

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A} \tag{5}$$

16.6 Instantaneous Center of Zero Velocity

instantaneous centre of zero velocity: a base point with zero velocity
On a rigid body, the velocity of every point is perpendicular to their displacement from the instantaneous centre of zero velocity, and every point purely rotates about it:

$$\mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_{B/IC} \tag{6}$$

centrode: the path traced by the instantaneous centre of zero velocity as it moves over time

16.7 Relative-Motion Analysis: Acceleration

Let A be the base point of a translating coordinate system, ω be the angular velocity, and α be the angular acceleration of B about A. Then the acceleration of point B is:

$$\mathbf{a}_B = \mathbf{a}_A + (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2 \mathbf{r}_{B/A} \tag{7}$$

16.8 Relative-Motion Analysis using Rotating Axes

Let A be the base point of a translating and rotating coordinate system, $(v_{B/A})_{rel}$ and $(a_{B/A})_{rel}$ be the velocity and acceleration of B inside the coordinate system, and ω and α be the angular velocity and acceleration of the coordinate system. Then the kinematic equations of point B are:

$$\begin{aligned}
\mathbf{r}_B &= \mathbf{r}_A + \mathbf{r}_{B/A} \\
\mathbf{v}_B &= \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A} + (v_{B/A})_{rel} \\
\mathbf{a}_B &= \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2 \mathbf{r}_{B/A} + 2\boldsymbol{\omega} \times (v_{B/A})_{rel} + (a_{B/A})_{rel}
\end{aligned} \tag{8}$$

Coriolis acceleration: the $2\boldsymbol{\omega} \times (v_{B/A})_{rel}$ term of acceleration in a rotating reference frame

17 Planar Kinetics of a Rigid Body: Force and Acceleration

17.1 Mass Moment of Inertia

mass moment of inertia (I): a measure of the resistance of a body to angular acceleration
Let r be the moment arm and ρ be the density of a body. Then the mass moment of inertia is the integral of the second moment of mass about some axis:

$$I = \int_m r^2 \, \mathrm{d}m = \int_V r^2 \rho \, \mathrm{d}V \tag{9}$$

The usual mass moment of inertia I_G is about the axis normal to the plane of motion that passes through the centre of mass.
parallel-axis theorem: the moment of inertia about a parallel axis d away is:

$$I = I_G + md^2 \tag{10}$$

radius of gyration (k): a measure of the resistance of a shape to angular acceleration

$$I = mk^2 \tag{11}$$

17.2 Planar Kinetic Equations of Motion

The kinetic equations of a rigid body in general plane motion are:

$$\begin{aligned} \mathbf{F}_R &= m\mathbf{a}_G \\ (M_R)_G &= I_G \boldsymbol{\alpha} \end{aligned} \tag{12}$$

17.3 Equations of Motion: Translation

The kinetic equations of a translating rigid body reduce to:

$$\begin{aligned} &(F_R)_x = m(a_G)_x \quad (F_R)_n = m(a_G)_n \\ &(F_R)_y = m(a_G)_y \quad (F_R)_t = m(a_G)_t \\ &(M_R)_G = 0 \end{aligned} \tag{13}$$

17.4 Equations of Motion: Rotation about a Fixed Axis

The kinetic equations of a rigid body rotating about a fixed axis O are:

$$\begin{aligned} (F_R)_n &= m\omega^2 r_G \\ (F_R)_t &= m\alpha r_G \\ (M_R)_G &= I_G \alpha \\ (M_R)_O &= I_O \alpha \end{aligned} \tag{14}$$

17.5 Equations of Motion: General Plane Motion

The kinetic equations of a rigid body in general plane motion at a point A are:

$$\begin{aligned} \mathbf{F}_R &= m\mathbf{a}_G \\ (M_R)_A &= I_G\alpha + \mathbf{r}_{G/A} \times (m\mathbf{a}_G) \end{aligned} \tag{15}$$

At the instantaneous centre of zero velocity of a circular object that rolls without slipping:

$$(M_R)_{IC} = I_{IC}\boldsymbol{\alpha} \tag{16}$$

18 Planar Kinetics of a Rigid Body: Work and Energy

18.1 Kinetic Energy

The kinetic energy of a rigid body rotating about a fixed axis O is:

$$T = \frac{1}{2} I_O \omega^2 \tag{17}$$

The kinetic energy of a rigid body in general plane motion is:

$$T = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2 = \frac{1}{2} I_{IC} \omega^2 \tag{18}$$

18.2 The Work of a Force

The work done by an external force $\mathbf{F}$ is:

$$U_F = \int \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{s_1}^{s_2} F \cos\theta \, \mathrm{d}s \tag{19}$$

The work done by a constant force with magnitude F and direction θ is:

$$U_F = (F \cos\theta)\, s \tag{20}$$

The work done by a weight W over a vertical displacement Δy is:

$$U_W = -W \Delta y \tag{21}$$

The work done by a spring force with spring constant k over a displacement from s_1 to s_2 is:

$$U_s = -\left(\frac{1}{2} k s_2^2 - \frac{1}{2} k s_1^2 \right) \tag{22}$$

Forces that act at fixed points on the body or perpendicular to their displacement do no work.

18.3 The Work of a Couple Moment

The work done by an external couple moment M is:

$$U_M = \int_{\theta_1}^{\theta_2} M \, \mathrm{d}\theta \tag{23}$$

The work done by a constant couple moment is:

$$U_M = M\,(\theta_2 - \theta_1) \tag{24}$$

18.4 Principle of Work and Energy

principle of work and energy: the work done on a rigid body is its change in energy

$$T_1 + \Sigma U = T_2 \tag{25}$$

18.5 Conservation of Energy

The gravitational potential energy of a weight W at a height y_G above the horizontal datum is:

$$V_g = W y_g \tag{26}$$

The elastic potential energy imparted by a spring with spring constant k and displacement s is:

$$V_e = \frac{1}{2} k s^2 \tag{27}$$

conservation of energy: if only conservative forces do work, then energy is conserved

$$T_1 + V_1 = T_2 + V_2 \tag{28}$$