Vertices of Uniform Polyhedra

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Notation List

I use the following notations in this essay, which I have adapted according to IB's standard system (International Baccalaureate Organization, 73–77):

$P\langle x, y, z\rangle$	the point *P* with Cartesian coordinates x, y, z
$P(r, \theta)$	the point *P* with polar coordinates r, θ
$P(r, \theta, \phi)$	the point *P* with spherical coordinates r, θ, ϕ
(AB)	the line or arc containing points *A* and *B*
AB	the length of the line or arc segment with end points *A* and *B*
$\overrightarrow{AB}$	the vector represented in magnitude and direction by the directed line segment from *A* to *B*
$\mathbf{a}$	the position vector $\overrightarrow{OA}$
$\begin{bmatrix} x_{\mathbf{a}} \\ y_{\mathbf{a}} \\ z_{\mathbf{a}} \end{bmatrix}$	the column vector representation of $\mathbf{a}$
$\lvert\overrightarrow{AB}\rvert$	the magnitude of $\overrightarrow{AB}$
$\lvert\mathbf{a}\rvert$	the magnitude of $\mathbf{a}$
$\mathbf{v} \cdot \mathbf{w}$	the scalar product of $\mathbf{v}$ and $\mathbf{w}$
$\mathbf{v} \times \mathbf{w}$	the vector product of $\mathbf{v}$ and $\mathbf{w}$
$\in$	is an element of
$\{1,2, \ldots, p\}$	the set of integers between and including 1 and p
$\mathbb{Z}$	the set of integers, $\{0, \pm 1, \pm 2, \pm 3, \ldots\}$
$\{p\}$	the regular polygon with Schläfli symbol $\{p\}$
$\{p, q\}$	the regular polyhedron with Schläfli symbol $\{p, q\}$
$(p\ q\ s)$	the uniform polyhedron with Wythoff symbol $(p\ q\ s)$

Introduction

Polytopes are, informally, collections of points and lines connecting those points. I spent much time finding the coordinates of these points for my extended essay, which dealt with rotating and displaying them on a computer program. However, I could not add in my extended essay any personal narrative nor discussion of the coordinates of these points, for want of space. Therefore, the aim of this internal assessment is to find the coordinates of 3D polytope points and discuss the long process I went through to find my final results. Its only connection with my extended essay is that both use the same code.

I first really became interested in polytopes after stumbling on Hedrondude's Home Page at http://www.polytope.net/hedrondude/home.htm. His renderings of four- and five-dimensional polytopes mesmerized me, but I struggled to understand their properties and how he discovered them. It seemed to me that he simply sat down and found new polytopes out of thin air! I read many books, but I eventually realized that an essay on, for example, the use of symmetry groups to find polytope vertices, would require too much background explanation and rely too much on rehashing others' ideas. It would have been a mathematical research paper, not a mathematical investigation.

Later, I wanted to use some visualization software to help me see different polytopes, but I could find no free four-dimensional visualization software. The one free software program I did find, vZome, did not work on my computer! I spent several hours working with its creator to debug the program, to no avail. After spending so much time on finding appropriate software, I eventually decided to create my own instead, from scratch. This eventually evolved into my extended essay and this internal assessment.

Although I created all my code from scratch, there is some mathematical context. I use Schwarz triangles to tile the sphere and spherical trigonometry to find the Wythoff points on those triangles. Wikipedia gave me the inspiration to use the Wythoff construction to find polytope vertices, and I later referenced Eric Weisstein's Wolfram MathWorld to check Wikipedia's validity and to find the spherical trigonometric laws.

The Two-Argument Inverse Tangent

The programming language that I used to program my polyhedron display system has a function $\mathrm{atan2}$ that takes two inputs y and x and outputs a value θ such that $\tan\theta = y/x$ with the output between $-\pi$ and π (Python Software Foundation sec. 9.2.3). I use $\mathrm{atan2}$ instead of $\tan^{-1}$ as the inverse tangent function and omit all quadrant conversions in this essay because unlike the arctangent function, whose range is between $-\pi/2$ and $\pi/2$, $\mathrm{atan2}$ outputs angles in the correct quadrant.

Regular Polygon Vertices

I began my investigation by considering *polygons*. A p-*gon* is "a circuit of p line-segments $(A_1A_2), (A_2A_3), \dots, (A_pA_1)$ joining consecutive pairs of p points $A_1, A_2, \dots, A_p$", and these line-segments and points are respectively *sides* and *vertices* (singular: *vertex*), so that a p-gon is a polygon with p vertices (Coxeter 1). A *regular* polygon is a polygon whose sides are of equal length and whose angles, between each consecutive pair of sides, are of equal size. Coxeter notes that all regular polygons have a *centre*, such that all of the polygon's vertices' distance to the centre is the same *circumradius* (2).

For simplicity, I set this centre to be the origin O, and represented the circumradius as some quantity r_c. Then every vertex of a regular polygon is r_c from O, and *polar coordinates* are more suited to describe regular polygons than Cartesian coordinates are. If point P has the polar coordinates $P(r, \theta)$, then $r = \left|\overrightarrow{OP}\right|$ and θ is the angle that $\overrightarrow{OP}$ makes with the x-axis (Martin 487). But my program uses Cartesian coordinates to rotate objects, so I must somehow convert from polar coordinates to Cartesian coordinates. To do this, I solved for x and y on Figure 1.

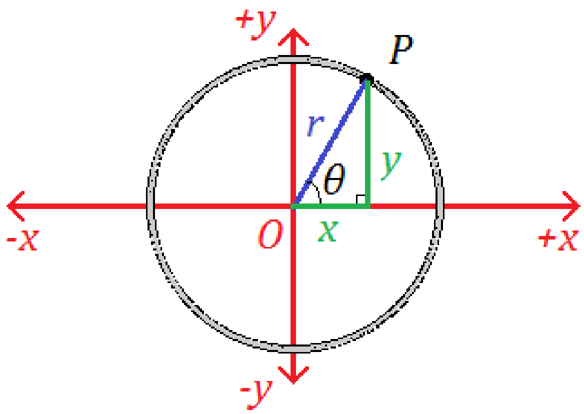

Figure 1. Conversion from polar to Cartesian coordinates.

$$y = x \tan\theta$$

$$r = \sqrt{x^2 + (x \tan\theta)^2} = \sqrt{x^2(1 + \tan^2\theta)} = \sqrt{x^2 \sec^2\theta} = x \sec\theta$$

$$x = r\cos\theta$$

$$y = x\tan\theta = r\cos\theta\tan\theta = r\sin\theta$$

So the conversion formula from polar to Cartesian coordinates is:

$$P(r, \theta) \overset{C}{\rightarrow} P\langle r\cos\theta\,, r\sin\theta\rangle \qquad [5.1]$$

My first idea for finding the vertices of a regular polygon was to express each side as a vector, then start from $(r_c, 0)$ and add each vector in order around the polygon to create a list of points at the tip of each vertex. This idea was ridiculously inefficient, but I implemented it anyway, in version 0.05 of my program.

I used a simpler idea for version 0.06. I extended lines from O to every vertex, which partitioned a p-gon into p triangles, which are congruent and isosceles because the sides of a regular polygon are equal, as are the distances $OA_n = r_c$. The tips of these triangles form a circle around O, so the sum of all p triangles' vertex angles is 2π. Because the triangles are congruent, their vertex angles must then all be the same: $2\pi/p$.

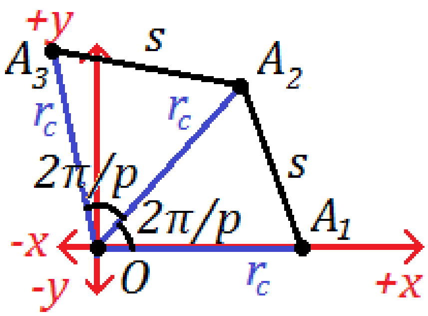

Figure 2. Calculation of the θ-coordinates of a regular p-gon.

Each vertex of the p-gon is r_c away from O and makes an angle with the x-axis starting at 0 which increases by $2\pi/p$ every vertex. So the vertices have polar coordinates:

$$A_n = \left(r_c, \frac{2n\pi}{p}\right), n \in \{1,2,\dots,p\} \qquad [5.2]$$

A polygon contains both vertices and sides. To find the sides of this polygon, I could simply connect the vertices A_nA_{n+1} in order. If the vertices are not connected in order, some sides may intersect other sides at points that are not vertices, creating a *star* polygon. For example, in Figure 3, the heptagram in the middle is created by connecting A_nA_{n+2} and the great heptagram in the right is created by connecting A_nA_{n+3}. Coxeter defines the *Schläfli symbol* of a regular polygon whose p edges connect the vertices A_nA_{n+k} as $\{p/k\}$ (3). When $k = 1$, we obtain the regular non-star p-gon $\{p\}$.

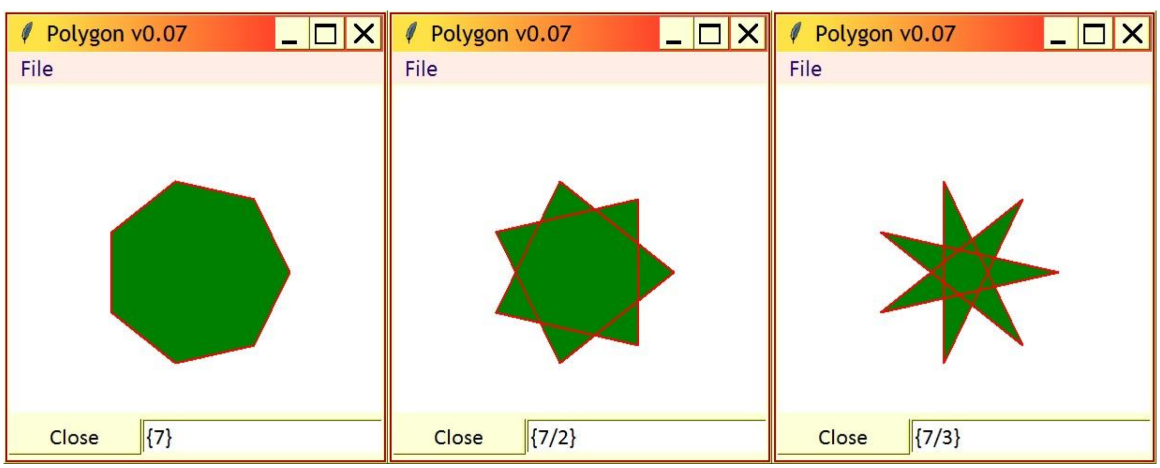

Figure 3. The three regular star heptagons.

My code at this time needed me to find a list of vertices, which the program would then connect in the order given, so I could not simply run through n from 0 to $p-1$ and connect all the A_nA_{n+k} edges, because this would not be in order. Instead, I had to run through kn with n going from 0 to $p-1$ so that the vertices would be in the correct order. Since there were only p vertices and going up to kn would go through $k(p-1)$ points, I had to concatenate k copies of the p vertices and then take every kn point to get the correct order. I did this in version 0.07.

But in version 0.10, I realized that the points could be obtained in the proper order simply by increasing the angle by $2k\pi/p$ between points. Then I would not have to worry about concatenating copies of lists because the trigonometric functions are periodic in 2π. So, I found this final algorithm for generating vertices of all regular polygons:

$$A_n = \left(r_c, \frac{2nk\pi}{p}\right), n \in \{1,2,\dots,p\} \qquad [5.3]$$

Spherical Coordinates

Coxeter defines a *polyhedron* (plural: *polyhedra*) as a finite, connected set of polygons, each of whose sides are shared with exactly one other polygon (4). These polygons and their sides are the polyhedron's *faces* and their *edges*. *Regular* polyhedra have a circumradius r_c (5). A regular polyhedron's faces are regular and congruent, and its vertices are surrounded by the same number of faces (6). Coxeter defines the Schläfli symbol of a regular polyhedron with q $\{p\}$ faces around each vertex as $\{p,q\}$ (3, 5).

If I set the centre of all regular polyhedra to be at O, then *spherical coordinates* are perfectly suited to describing them, since all their vertices' distances from O are r_c. If point P has the spherical coordinates $P(r,\theta,\phi)$, then $r = \left|\overrightarrow{OP}\right|$, θ is the angle that the orthogonal projection of $\overrightarrow{OP}$ on the xy-plane makes with the x-axis, and ϕ is the angle that $\overrightarrow{OP}$ makes with the z-axis (Weisstein, “Spherical Coordinates”). Weisstein restricts ϕ to $0 \leq \phi \leq \pi$ so that different coordinates, modulo 2π, do not describe the same point, except when $r_c = 0$. If $\pi < \phi < 2\pi$, then $P(r, \theta, \phi) = P(r, \theta + \pi, 2\pi - \phi)$.

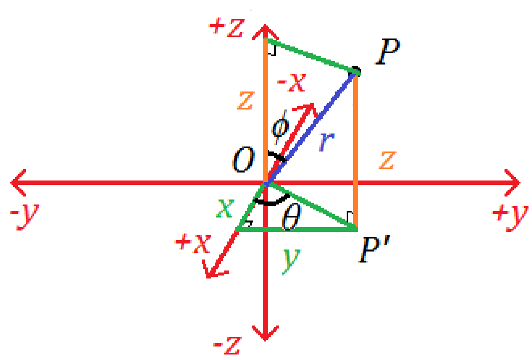

Figure 4. Conversion between spherical and Cartesian coordinates.

To convert from spherical coordinates to Cartesian coordinates, I use $\overrightarrow{OP'}$ in Figure 4 to represent the orthogonal projection of $\overrightarrow{OP}$ on the xy-plane. The normal equation of the xy-plane is $z = 0$, so the z-component of $\overrightarrow{OP'}$ is 0:

$$\overrightarrow{OP} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \overrightarrow{OP'} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

I solved for x, y, and z on Figure 4:

$$z = r \sin\left(\frac{\pi}{2} - \phi\right) = r \cos \phi$$

$$\overrightarrow{OP'} = r \cos\left(\frac{\pi}{2} - \phi\right) = r \sin \phi$$

$$x = \overrightarrow{OP'} \cos \theta = r \sin \phi \cos \theta$$

$$y = \overrightarrow{OP'} \sin \theta = r \sin \phi \sin \theta$$

So the conversion formula from spherical to Cartesian coordinates is:

$$P(r, \theta, \phi) \xrightarrow{C} P\langle r\sin\phi\cos\theta\,, r\sin\phi\sin\theta\,, r\cos\phi\rangle \qquad [6.1]$$

In version 0.56, I decided that using spherical coordinates would help random sampling. So, I came up with methods to convert Cartesian coordinates into spherical coordinates. The most straightforward method is by inspecting Figure 4 to get:

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \operatorname{atan2}(y, x)$$

$$\phi = \arccos(z/r)$$

Another method I found is more mathematically rigorous and comes only from the definition. Given the vector $\overrightarrow{OP}$ and its orthogonal projection on the xy-plane $\overrightarrow{OP'}$:

$$\overrightarrow{OP} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \overrightarrow{OP'} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

Since $r = \left|\overrightarrow{OP}\right|$, $r = \sqrt{x^2 + y^2 + z^2}$.

Since θ is the angle between $\overrightarrow{OP'}$ and the x-axis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$:

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$$\cos\theta = \frac{1x + 0y + 0}{\sqrt{1^2}\sqrt{x^2 + y^2}}$$

$$\theta = \arccos\left(\frac{x}{\sqrt{x^2 + y^2}}\right)$$

Finally, since ϕ is the angle between $\overrightarrow{OP}$ and the z-axis $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$:

$$\cos\phi = \frac{0x + 0y + 1z}{\sqrt{1^2}\sqrt{x^2 + y^2 + z^2}}$$

$$\phi = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

These two methods give the same result, since:

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \operatorname{atan2}(y, x) = \arctan\left(\frac{opp}{adj}\right) = \arccos\left(\frac{adj}{hyp}\right) = \arccos\left(\frac{x}{\sqrt{x^2 + y^2}}\right)$$

$$\phi = \arccos(z/r) = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

So the conversion formula from Cartesian to spherical coordinates is:

$$P\langle x, y, z\rangle \xrightarrow{S} P\left(\sqrt{x^2 + y^2 + z^2}, \operatorname{atan2}(y, x), \arccos(z/r)\right) \qquad [6.2]$$

The programming language that I used ensures that ϕ satisfies Weisstein’s constraint $0 \leq \phi \leq \pi$ because its $\arccos$ function returns an angle between 0 and π.

Regular Polyhedron Vertices

I only started to find regular polyhedron vertices in version 0.18; I used version 0.01 to 0.04 to set up my display program and versions 0.08 to 0.17 to implement 2D rotation and fiddle around unsuccessfully with 3D rotation algorithms. These details are irrelevant to this investigation, since they did not help me find any vertices.

I wanted to derive all these vertices only from the Schläfli symbols $\{p, q\}$. I first set one vertex to be the north pole $N(r_c, 0, 0)$. There are q p-gons around all vertices, so if I extend q edges down from N, then I would have q new vertices. The sides of all regular polygons all have the same length s, so the edges of all regular polyhedra also have the same length s and the distance between each new vertex and N must be s. Therefore, these new vertices all lie in the same plane, perpendicular to the z-axis, which I call the *first vertex plane*, as shown in Figure 5a. Let the radius of the intersection of the first vertex plane with the *polyhedron sphere* $r = r_c$ be r_1.

I sliced a p-gon into $p - 2$ triangles, such as the one on Figure 5c, by joining lines from one vertex to every other vertex. The vertices of each triangle coincide with some vertex of the p-gon, so adding the angle sums of each triangle gives the angle sum of the whole p-gon, and the vertex angle α of the p-gon is that angle sum divided by the number of vertices p. All triangles have an angle sum of π, so for all p-gons:

$$\alpha = (p - 2)\pi/p \qquad [7.1]$$

I then extended lines from every vertex on the first vertex plane to the centre C of the first vertex plane. There are q vertices on the first vertex plane, so the vertex angle of the isosceles triangle in Figure 5d is:

$$\beta = 2\pi/q \qquad [7.2]$$

This angle β corresponds exactly with the $2\pi/p$ of [5.2], and from the same reasoning, the vertices on the first vertex plane have θ-coordinates:

$$\theta = \beta n, n \in \{1,2, \dots, q\} \qquad [7.3]$$

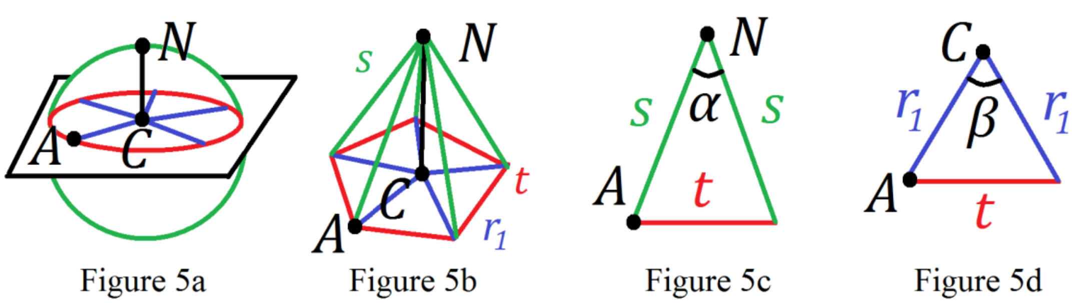

Figure 5. Calculation of triangles on the first vertex plane.

I applied the cosine law on Figures 5c and 5d to solve for r_1/s:

$$t = s\sqrt{2(1-\cos\alpha)} = r_1\sqrt{2(1-\cos\beta)}$$

$$r_1/s = \sqrt{(1-\cos\alpha)/(1-\cos\beta)} \quad [7.4]$$

Consider one arbitrary vertex A on the first vertex plane. I connected (AN), (OA), and (ON) as shown in Figure 6b. $AN = s$ because (AN) is a side and $OA = ON = r_c$ because A and N are vertices. So, (AN), (OA), and (ON) are sides of an isosceles triangle. I found that if ϕ_1 is the vertex angle of this triangle and δ are its base angles, then from inspecting Figure 6c and subtracting equations:

$$\gamma = \cos^{-1}(r_1/s) \quad [7.5]$$

$$\gamma + \delta = \pi/2$$

$$\delta - \gamma + \phi_1 = \pi/2$$

$$\phi_1 = 2\gamma \quad [7.6]$$

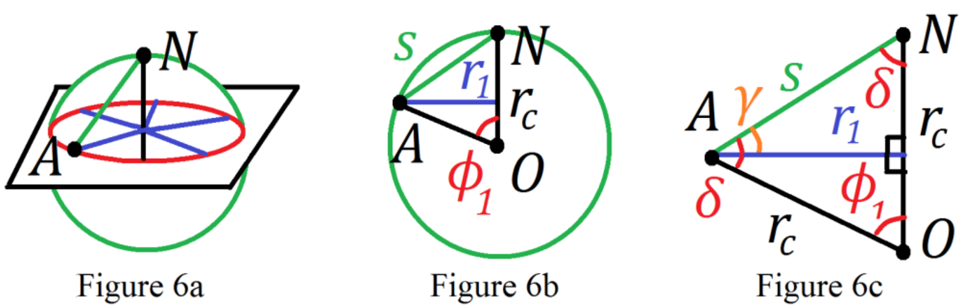

Figure 6. Calculation of the ϕ-coordinate of the first vertex plane.

Since the first vertex plane is perpendicular to the z-axis, all its points must have the same z=coordinate and thus ϕ-coordinate. From [7.4], [7.5] and [7.6], I found that $\phi_1 = 2\cos^{-1}\left(\sqrt{\frac{1-\cos\alpha}{1-\cos\beta}}\right)$. Combining this with the θ-coordinates I found in [7.3], I found the coordinates of all vertices on the first vertex plane:

$$(r_c, \beta n, \phi_1), n \in \{1,2,\ldots,q\} \quad [7.7]$$

From pictures of the regular polyhedra, I noticed that if $\phi_1 \leq \pi/2$, then there is also a vertex on the south pole $S(r_c, 0, \pi)$. This makes sense because of symmetry, but I do not know how to formally prove it. I expected that I could flip polyhedra upside-down and apply [7.7] to S instead of N to make a *last vertex plane*. However, this failed.

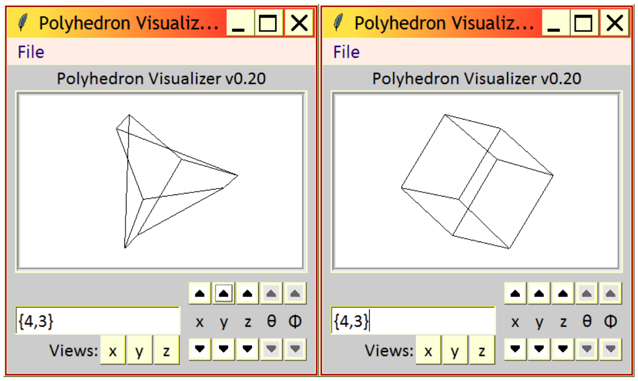

Figure 7. The cube, with an aligned and a shifted last vertex plane.

The tetrahedron neither has a south pole nor a last vertex plane because its $\phi_1 > \pi$, so its last vertex plane's ϕ-coordinates $\pi - \phi_1$ would be negative! The octahedron does have a south pole, but its last vertex plane coincides with its first vertex plane since $\phi_1 = \pi/2$ so $\phi_1 = \pi - \phi_1$. However, all other regular polyhedra should have a last vertex plane, and its vertices' θ-coordinates are $\beta/2$ more than those of the first vertex plane. I cannot formally prove why, but if the vertices in both planes were aligned and there were only two vertex planes, then there must be rectangles connecting the two planes. But p is not always 4, so there must be some shift. The first pictures in Figures 7 and 8 show how strange polyhedra become when the vertex planes are aligned.

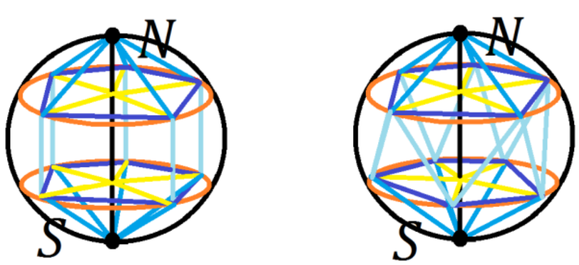

Figure 8. The icosahedron, with an aligned and a shifted last vertex plane.

Side lengths could only be equal if every vertex in the last vertex plane was put halfway between two vertices in the first vertex plane, a shift of $\beta/2$. Then its vertices are:

$$\left(r_c, \beta\left(n+\frac{1}{2}\right), \pi-\phi_1\right), n \in \{1,2, \dots, q\} \tag{7.8}$$

Dodecahedron Vertices

N, [7.7], S, and [7.8] specify the vertices of all regular polyhedra that have two or fewer vertex planes. However, the dodecahedron has four vertex planes. I struggled for two whole weeks on this problem, starting with version 0.21 and going around and around in circles until I had a flash of inspiration in version 0.31 that greatly simplified calculations. Most of the following equations use the cosine law.

Firstly, I solved for the side length s in Figure 6c:

$$s=\sqrt{2{r_c}^2(1-\cos\phi_1)} \tag{8.1}$$

Secondly, I solved for the diagonal length d of a pentagon in Figure 5c and 9b:

$$d=t=\sqrt{2s^2(1-\cos\alpha)} \tag{8.2}$$

Thirdly, I solved for the ϕ-coordinate of the second vertex plane in Figure 9c:

$$\phi_2=\cos^{-1}\left(1-\frac{d^2}{2{r_c}^2}\right) \tag{8.3}$$

Finally, I solved for the radius r_2 of the middle plane in Figure 9c:

$$r_2=r_c\sin\phi_2 \tag{8.4}$$

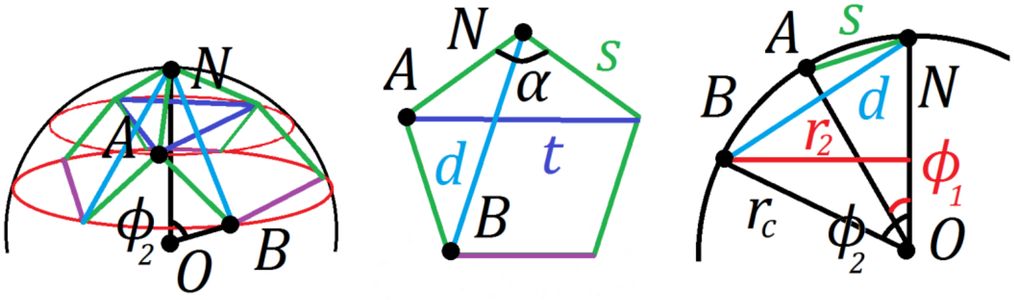

Figure 9a Figure 9b Figure 9c

Figure 9. Calculation of the ϕ-coordinate of the second vertex plane.

The vertices on the dodecahedron's second vertex plane are alternately separated by s and t by inspection of Figure 9a. This is because the pentagonal faces alternate between having sides and diagonals on the vertex plane. I cannot formally prove that this is the only possible pattern, but there definitely is a pattern, as the vertices on the cube's first vertex plane are also alternately separated by s and t.

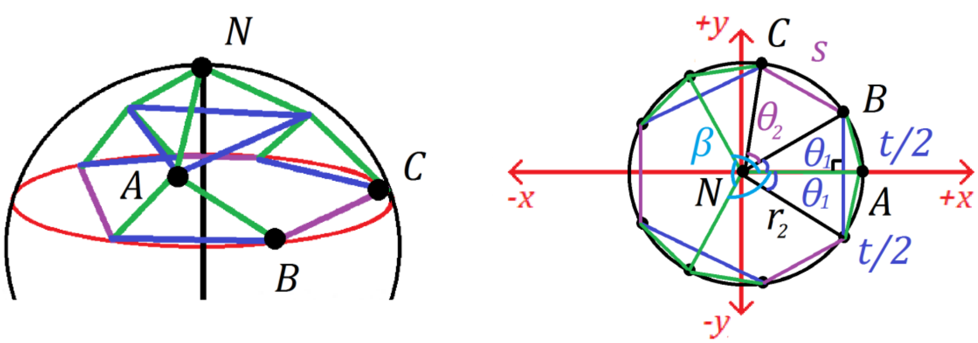

Figure 10. Calculation of the θ-coordinates of the second vertex plane.

I used the cosine law to find θ_1 on Figure 10:

$$\theta_1 = \frac{\arccos\left(1 - \frac{t^2}{2{r_2}^2}\right)}{2} \quad [8.5]$$

By inspection, I used θ_1 to find all six θ-coordinates on the second vertex plane. Each vertex has an angle either θ_1 greater or θ_1 smaller than the angle of the nearest green line, and these green lines separated by $\beta = \pi/3$, so the θ-coordinates are:

$$(\beta n \pm \theta_1), n \in \{1,2,3\} \quad [8.6]$$

The θ-coordinates of the third vertex plane are shifted by $\beta/2$, so they are:

$$\left(\beta\left(n + \frac{1}{2}\right) \pm \theta_1\right), n \in \{1,2,3\} \quad [8.7]$$

To confirm this, Figure 10 shows that $2\theta_1 + \theta_2 = \pi/3$. By the cosine law:

$$\theta_2 = \arccos\left(1 - \frac{s^2}{2{r_2}^2}\right)$$

Plugging this into my computer, I get $\theta_1 = 0.6590580358264087$ and $\theta_2 = 0.7762790307403772$. But $2\pi/3 = 2.0943951023931954$ and:

$$\begin{aligned} 2\theta_1 + \theta_2 &= 2 \times 0.6590580358264087 + 0.7762790307403772 \\ &= 2.0943951023931946 \end{aligned}$$

So my calculations are correct, at least up to 10^{-15}. The difference likely comes from floating-point computation errors in all the square roots and trigonometric functions.

This was the most challenging part of my investigation, mainly because I had no focus during my calculations, until the end. I defined ϕ_2 in terms of θ_1 and then tried to solve for θ_1 knowing only ϕ_2, but I did not realize what I was doing because the reality was buried under piles of other variables: α, β, d, t, s, r_2, r_c… I lost track and gave up.

Then, after I completely shifted my focus to 3D rotations between versions 0.22 and 24 and foreshortening between versions 0.25 and 0.30, I came back to dodecahedron vertices with a fresh mind in 0.31 and solved the entire thing within half an hour. From then on, I always gave myself a week to clear my mind before tackling difficult problems. My failure to find the dodecahedron coordinates until after the peer review influenced my decision to remove all mention of vertex coordinates from my extended essay.

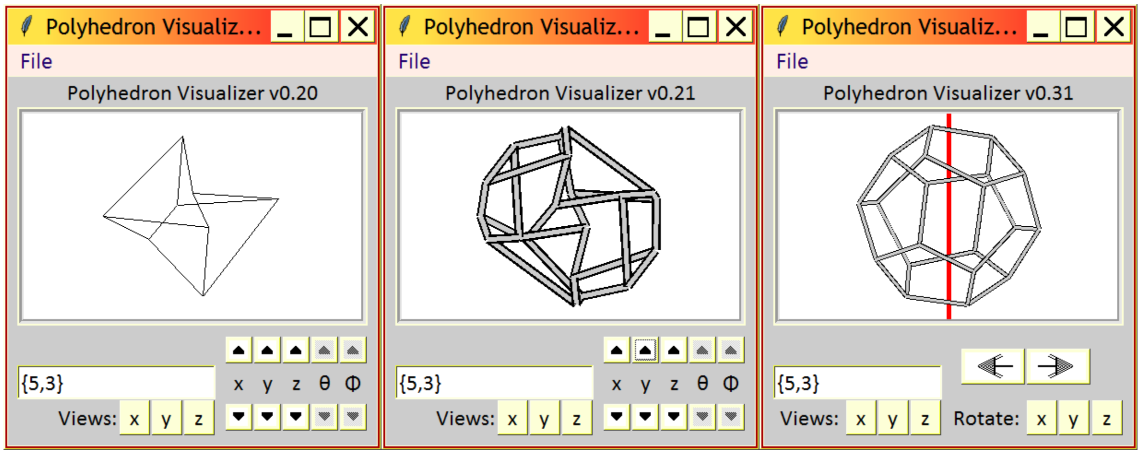

Figure 11. The dodecahedron, with incorrect and correct middle vertex planes.

Schwarz Triangle Tilings

I spent versions 0.32 to 0.40 working on 4D algorithms. I could not derive coordinates of 4D regular polytopes by myself, so I copied their coordinates off of Wolfram MathWorld in my code and still cannot discuss them in this investigation. Nevertheless, after I finished my extended essay, I still only had the coordinates of five polyhedra. So, I tried to find the coordinates of all uniform polyhedra.

Uniform polyhedra have identical vertices (Weisstein, “Uniform Polyhedron”). Since these vertices are identical, they must all be the same distance from the centre O, and so uniform polyhedra also have a circumradius r_c. All edge lengths must be the same—if one edge were different from the others, its vertices would no longer be identical.

One way to find the coordinates of uniform polyhedra is through *Wythoff’s kaleidoscopic method of construction* (Weisstein, “Uniform Polyhedron”). I can choose a spherical triangle and imagine its sides to be mirrors. If it has suitable angles, then it is a *Schwarz triangle* that I can continuously reflect until it tiles the sphere (Weisstein, “Uniform Polyhedron”). Then, I can choose suitable points in the triangle to be reflected, with the triangle, such that its reflections form the vertices of a uniform polyhedron.

I first found the suitable Schwarz triangles. Each of their angles must be a rational multiple of 2π; otherwise, there would be gaps at some triangle vertices. If I let the three vertices of a generic Schwarz triangle be P, Q, and S, and the angle at each vertex be respectively π/p, π/q, and π/s, then p, q, and s each represent half the number of triangles that fit around each vertex, and p, q, and s must all be rational. At the beginning, in version 0.41, I only considered integer values of p, q, and s, so that no triangle reflections would overlap each other.

On a sphere, the angle sum of a triangle is always greater than π, and is only equal to π in the limit as curvature goes to 0, that is, when the sphere becomes a plane (Weisstein, "Spherical Triangle"). For the generic Schwarz triangle:

$$\frac{\pi}{p}+\frac{\pi}{q}+\frac{\pi}{s}>\pi$$

I ignored values of 1 because those would make degenerate lunes instead of triangles. Otherwise, to avoid permutations, if $p \leq q \leq s$, then if $p = 2$ and $q = 2$:

$$\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{s}>\pi$$
$$\frac{\pi}{s}>0$$

Then s can be any integer. If $p = 2$ and $q = 3$:

$$\frac{\pi}{2}+\frac{\pi}{3}+\frac{\pi}{s}>\pi$$
$$\frac{\pi}{s}>\frac{\pi}{6}$$

Then $s < 6$, but $3 = q \leq s$, so s can be 3, 4, or 5.

If $p = 2$ and $q \geq 4$:

$$\frac{\pi}{2}+\frac{\pi}{4}+\frac{\pi}{s}\geq\frac{\pi}{2}+\frac{\pi}{q}+\frac{\pi}{s}>\pi$$
$$\frac{\pi}{s}>\frac{\pi}{4}$$

Then $s<4$ and $s\geq 4$, so there is no valid value of s. If $p\geq 3$, then $q\geq 3$:

$$\frac{\pi}{3}+\frac{\pi}{3}+\frac{\pi}{s}\geq\frac{\pi}{p}+\frac{\pi}{q}+\frac{\pi}{s}>\pi$$
$$\frac{\pi}{s}\geq\frac{\pi}{3}$$

Then $s<3$ and $s\geq 3$, so there is no valid value of s. Therefore, the only valid integer combinations $p\ q\ s$ are $2\ 2\ n$, $2\ 3\ 3$, $2\ 3\ 4$, and $2\ 3\ 5$, with any $n\geq 2, n\in\mathbb{Z}$.

I first tried to use the surface area formula, $A=(\text{angle sum}-\pi)r_c^2$ to find the first vertex plane of my generic triangle (Weisstein, "Spherical Triangle"). I put S at the north pole $N(r_c,0,0)$ and tried to integrate to find the area of the circular cap between the first vertex plane and S. Although I spent many hours drawing diagrams and calculating coordinates on the assumption that these vertex planes existed, I eventually realized that P and Q generally do not lie on the same plane, unless $p\ q\ s=2\ 2\ n$ because of symmetry, so I calculated the coordinates of the triangles generated by angles of $\pi/2$, $\pi/2$, and $\pi/$s in version 0.42. The surface area of a $2\ 2\ n$ Schwarz triangle is:

$$A_t=\left(\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{s}-\pi\right)r_c^2=\pi r_c^2/s \qquad [9.1]$$

But there are $2s$ Schwarz triangles arranged around the north pole, so the area of the spherical cap between the first vertex plane and the north pole is:

$$A_s=2sA_t=2\pi r_c^2 \qquad [9.2]$$

But the surface area of a sphere is $4\pi r_c^2$, so this first vertex plane must bisect the sphere, that is, $\phi_1=\pi/2$. Then, as in [5.2] and [7.3], the coordinates of its vertices are:

$$A_n=\left(r_c,\frac{\pi n}{s},\frac{\pi}{2}\right),n\in\{1,2,\ldots,2s\} \qquad [9.3]$$

I connected these vertices with each other and with the north and south poles to render these $2\ 2\ n$ triangle tilings. This method did not work for any other $p\ q\ s$.

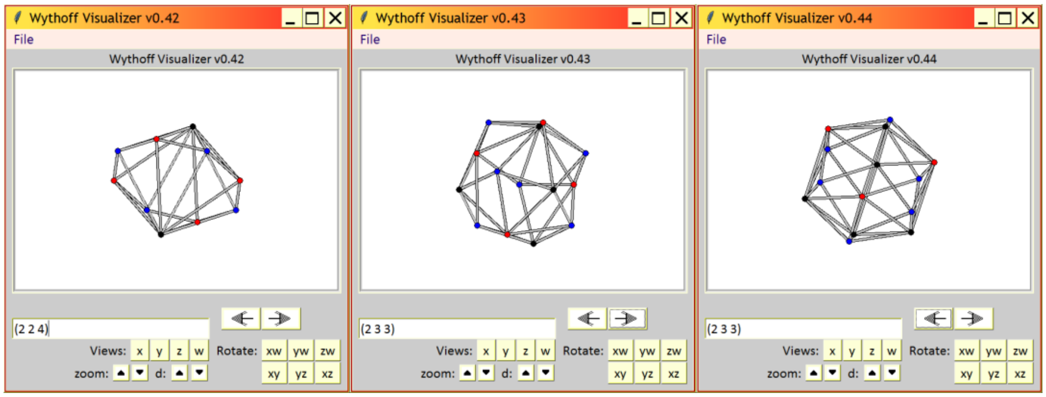

Figure 12. The correct $2\ 2\ n$, incorrect $2\ 3\ 3$, and correct $2\ 3\ 3$ Schwarz triangles.

Thankfully, I know the angles of a generic triangle, so I can use the spherical angle cosine law to find its coordinates (Weisstein, “Spherical Trigonometry”). Given a triangle with sides a, b, and c respectively opposite to angles A, B, C:

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a \qquad [9.4]$$

So, solving for a, I found a formula I can apply to find the sides of the triangle:

$$a = \cos^{-1} \frac{(\cos A + \cos B \cos C)}{\sin B \sin C} \qquad [9.5]$$

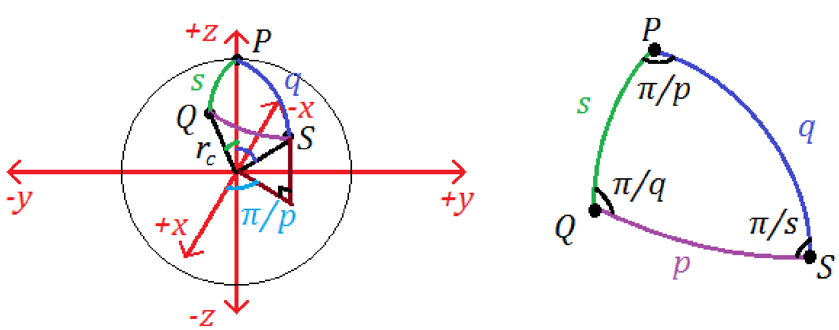

Figure 13. Calculation of the vertices of the first Schwarz triangle.

I set P at the north pole, so by $\theta = lr$ the ϕ-coordinates of Q and S are just respectively s/r_c and q/r_c. If I let the θ-coordinate of Q be 0, then the θ-coordinate of S is the angle at P, π/p. Since these formulae work for all p, q, and s, I was not restricted to integral values anymore! Thus, the coordinates of all possible triangles are:

$$P = (r_c, 0,0) \qquad [9.6]$$

$$Q = \left(r_c, 0, \cos^{-1}\left(\frac{\left(\cos\frac{\pi}{s} + \cos\frac{\pi}{p}\cos\frac{\pi}{q}\right)}{\sin\frac{\pi}{p}\sin\frac{\pi}{q}}\right)\right) \qquad [9.7]$$

$$S = \left(r_c, \frac{\pi}{p}, \cos^{-1}\left(\frac{\left(\cos\frac{\pi}{q} + \cos\frac{\pi}{s}\cos\frac{\pi}{p}\right)}{\sin\frac{\pi}{s}\sin\frac{\pi}{p}}\right)\right) \qquad [9.8]$$

In version 0.44, I tried to tile the sphere with a similar method as in my derivation of the coordinates of regular polyhedra: by duplicating the first vertex 'plane' while replacing ϕ with $\pi - \phi$ and shifting all θ-coordinates forward by π/p.

But now I was stumped again, by the dodecahedron problem! If finding the coordinates of the 20 dodecahedron vertices was hard, despite its equal side lengths and angles, then how could I ever find the coordinates of the 62 vertices of the disdyakis triacontahedron, whose angles were different at every vertex?

This investigation involves a computer program. So, of course I would develop a recursive reflection algorithm, instead of calculating every point analytically! Given points P, Q, and S, I had to find their reflected points P', Q', and S', respectively reflected about the arcs (QS), (SP), and (PQ) of length p, q, and s.

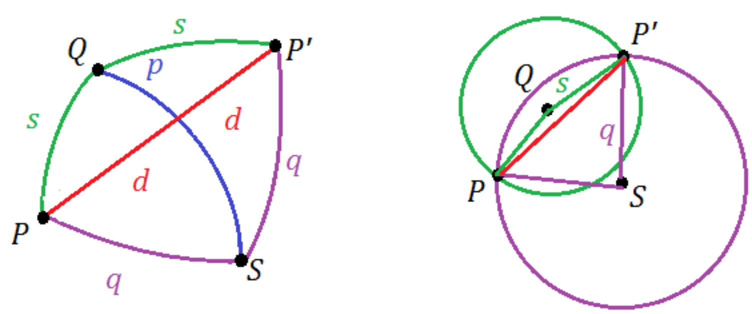

Figure 14. Calculation of P' by intersecting three spheres.

Consider the reflection of P across (QS) to P', with Cartesian coordinates $P'\langle x, y, z\rangle$. My first instinct was to find the intersection of the spheres with radii s and q respectively centered at Q and S. If Q has Cartesian coordinates $Q\langle x_Q, y_Q, z_Q\rangle$ and S has Cartesian coordinates $S\langle x_S, y_S, z_S\rangle$, then the equations of their respective spheres are:

$$(x - x_Q)^2 + (y - y_Q)^2 + (z - z_Q)^2 = s^2$$
$$(x - x_S)^2 + (y - y_S)^2 + (z - z_S)^2 = q^2$$

And of course, P' is a vertex, so it must be on the polyhedron sphere:

$$x^2 + y^2 + z^2 = r_C^2$$

This system is probably solvable but after several fruitless hours, I gave up. On closer reflection, no pun intended, I realized that the radii of the spheres could not be q and s because radii measure straight-line distance, and q and s were distances on a sphere! I would have saved much time had I realized this before I began.

Later, it hit me that points on a sphere remain on a sphere after reflection across any plane that passes through the centre. This is because reflecting a point across any plane preserves its distance from the plane, and if the plane passes through the origin, then the point, the origin, and the orthogonal projection of the point on the plane make a right-angled triangle whose sides remain the same after reflection, and whose hypotenuse is thus the radius of the sphere. However, if the plane does not pass through the origin, then this triangle is not right, and its hypotenuse will change depending on its angles. Knowing this, to reflect a point across a line on a sphere, all I had to do was perform a Euclidean reflection about the plane passing through the origin containing that line.

This plane has Q, S, and O, so its unit normal $\mathbf{n}$ is perpendicular to OQ and OS:

$$\mathbf{n} = \frac{OQ \times OS}{|OQ \times OS|} \qquad [9.9]$$

Then the equation of this plane is:

$$\mathbf{n} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{n} \cdot O = 0 \qquad [9.10]$$

I set the distance between P and this plane as d, so that:

$$\mathbf{n} \cdot (P + d\mathbf{n}) = 0$$

$$d = \frac{\mathbf{n} \cdot P}{|\mathbf{n}|} \qquad [9.11]$$

The reflection of P about this plane is the point P' that is d away from the plane and on its other side, so the distance between P and P' is $2d$:

$$P' = P + 2d\mathbf{n} \qquad [9.12]$$

The simplicity of this result shocked me immensely and I felt somewhat bitter that I had worked so long just to find such a simple answer. However, my main feeling was of relief: I could finally display all the integral Schwarz triangle tilings.

In version 0.46, I reflected every vertex of the first triangle, added the vertices of the newly-created triangles to a list, reflected every vertex of every triangle in the list, added the vertices of the newly-created triangles to the list, and so forth. It took seconds to process at the beginning, and though I ended up with over a thousand triangles at times, this method could not even tile the whole sphere, since so many triangles were repeated. In version 0.47, I kept track of ordered concatenated rounded vertex coordinates so that previously added triangles would not be readded to the list. This was much faster and I tiled the sphere, but because I made three edges and three points for each triangle and each edge was shared by two triangles and each point by even more, I ended up with doubled edges and multiple copies of the same point, which limited my speed. I still do not know how to only process each point and edge once.

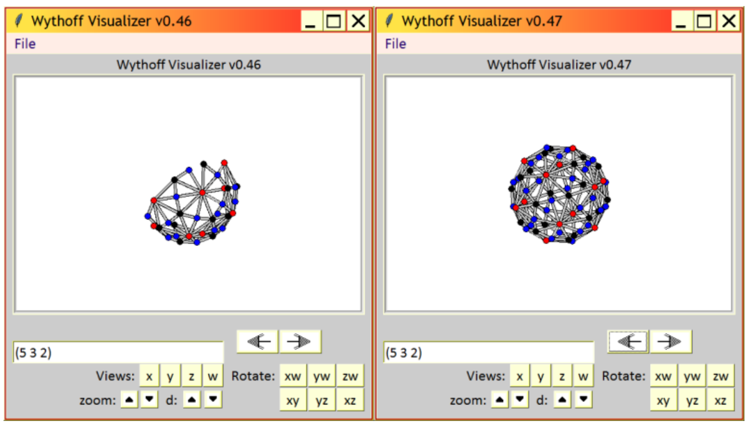

Figure 15. The disdyakis triacontahedron, incompletely and completely tiled.

Uniform Polyhedron Vertices

None of the Schwarz triangle polyhedra were uniform because their faces were isosceles or scalene triangles. If their faces were equilateral triangles, they would not be able to fold down into a sphere! I mentioned that I must choose a point in the Schwarz triangle to truly follow Wythoff's method. Now, in this final section, I created all 74 Wythoff-constructable uniform polyhedra by finding these points.

Since uniform polyhedra have equal edge length, I found the edge length between a point and its reflection. Edges are straight, so this straight distance must be $2d$, except in the snub case, which I will discuss later. To connect vertices in edges, I went through every pair of vertices, checked if their distance was $2d$, and connected them in an edge if they were. Although this is not analytic, I had already left that realm when I had to reflect

Schwarz triangles recursively. I added this in version 0.49, so Figure 16 has weird edges.

The first three possible points are P, Q, and S (Weisstein, “Wythoff Symbol”). Polyhedra created by reflecting point P in a $p\ q\ s$ Schwarz triangle have a Wythoff symbol $(p \mid q\ s)$. I already found all the coordinates of the Schwarz triangles’ vertices when tiling them; to display these polyhedra, all I had to do was select only the P vertices of the Schwarz triangles that tiled the sphere.

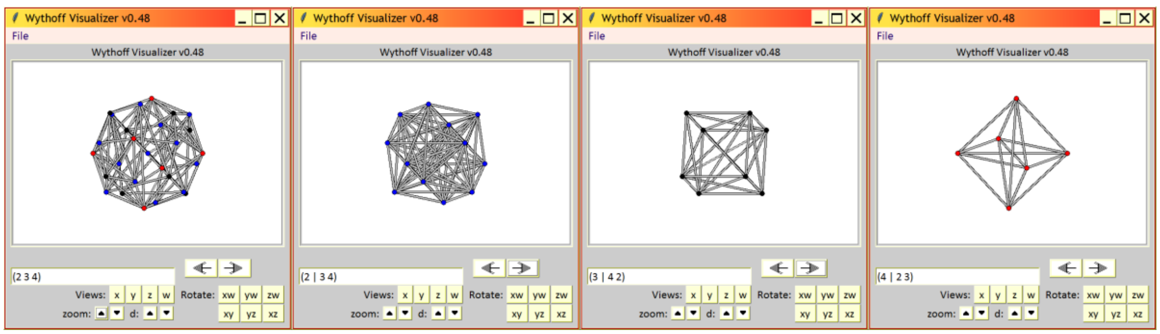

Figure 16. The $(p \mid q\ s)$ polyhedra of the $2\ 3\ 4$ Schwarz triangle.
(The disdyakis dodecahedron, the cuboctahedron, the cube, and the octahedron.)

The next three possible points are the feet of the angle bisector of P, Q, or S on its opposite arc (Weisstein, “Wythoff Symbol”). Polyhedra created by reflecting a point on side (PQ) of a $p\ q\ s$ Schwarz triangle have a Wythoff symbol $(p\ q \mid s)$. In version 0.49, I thought this point would be the projection on the sphere of the midpoint of the arc’s endpoints, so I made some disproportioned non-uniform polyhedra, such as the first picture in Figure 18. But midpoints are not feet of angle bisectors!

The spherical sine and cosine laws are (Weisstein, “Spherical Trigonometry”):

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \qquad [10.1]$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \qquad [10.2]$$

I replaced a with c in [10.2]:

$$\cos c = \cos b \cos a + \sin b \sin a \cos C$$

Then I substituted this into [10.2] and used [10.1]:

$$\cos a = \cos b\,(\cos b \cos a + \sin b \sin a \cos C) + \sin b \sin c \cos A \left(\frac{\sin C}{\sin c} \div \frac{\sin A}{\sin a}\right)$$

I rearranged this to get:

$$\cos a - \cos a \cos^2 b = \cos b \sin b \sin a \cos C + \sin b \sin C \frac{\cos A}{\sin A} \sin a$$

$$\cos a\,(1 - \cos^2 b) = \sin a \sin b\,(\cos b \cos C + \cot A \sin C)$$

$$\cot a \sin b = \cos b \cos C + \cot A \sin C$$

Now I solved this for a:

$$a = \operatorname{atan2}\left(\frac{\sin b}{\cos b \cos C + \cot A \sin C}\right) \qquad [10.3]$$

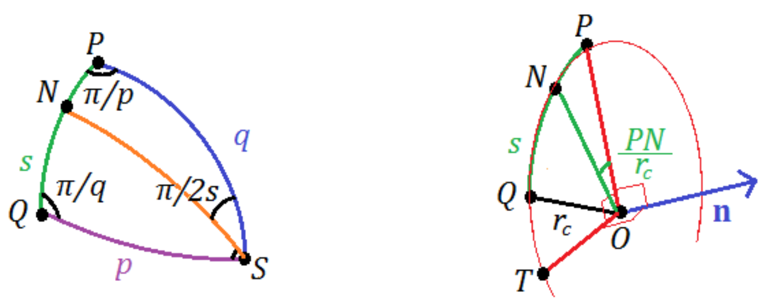

Figure 17. Calculation of the foot on (PQ) of the angle bisector of S.

Looking at Figure 17, I replaced a in [10.3] with PN, C with π/p, b with q, and A with $\pi/2s$ to find the distance PN along the arc (PQ). N is PN away from P. Since $l = \theta r$, this means N has an angle of PN/r_c more than P on (PQ).
To find N given this angle PN/r_c, I parametrized PQ as a circle by finding the normal vector $\mathbf{n} = \overrightarrow{OP} \times \overrightarrow{OQ}$ to the plane. Then, I found another vector $\overrightarrow{OT}$ perpendicular to $\overrightarrow{OP}$ on this circle by $\overrightarrow{OT} = \mathbf{n} \times \overrightarrow{OP}$. Finally, I normalized both $\overrightarrow{OP}$ and $\overrightarrow{OT}$.

This parametrized a point C on this circle as:

$$C = r_c \cos t\, \overrightarrow{OP} + r_c \sin t\, \overrightarrow{OT} \qquad [10.4]$$

When $t = 0$, $C = P$, and when $t = PN/r_c$, $C = N$.

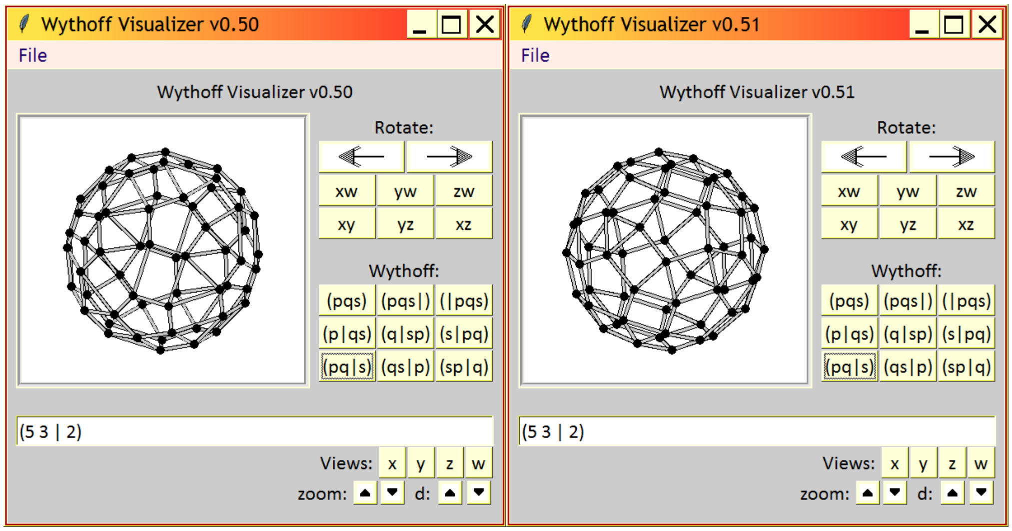

Figure 18. The rhomboicosidodecahedron, with rectangular and square faces.

The seventh possible point is the *incentre* C of the $p\ q\ s$ Schwarz triangle, the point at which all three angle bisectors intersect, which has the Wythoff symbol $(p\ q\ s\ |)$ (Weisstein, "Wythoff Symbol"). To find this intersection, I first found the feet N and M of the angle bisectors of S and Q respectively, using [10.4]. Then, I found the normals $\mathbf{n}$ of the plane containing $\overrightarrow{ON}$ and $\overrightarrow{OS}$ and $\mathbf{m}$ of the plane containing $\overrightarrow{OM}$ and $\overrightarrow{OQ}$:

$$\mathbf{n} = \overrightarrow{ON} \times \overrightarrow{OS} \qquad [10.5]$$

$$\mathbf{m} = \overrightarrow{OM} \times \overrightarrow{OQ} \qquad [10.6]$$

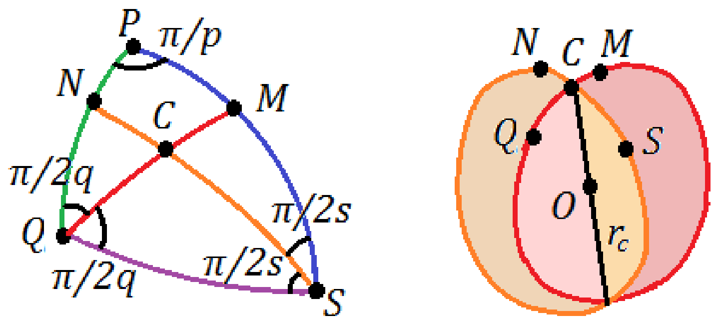

Figure 19. Calculation of the incentre by intersecting two planes.

$\overrightarrow{OC}$ is the unique intersection of both planes and thus perpendicular to both $\mathbf{n}$ and $\mathbf{m}$. Since S is lower down than N, Q is lower down than M, and S is more clockwise than Q, so by the right-hand rule, the shorter turn from $\mathbf{n}$ to $\mathbf{m}$ is clockwise. Then by the right-hand rule again, OC points upwards only if its direction is $\mathbf{m} \times \mathbf{n}$. This must hold because Schwarz triangles are in the northern hemisphere of the polyhedron sphere. Now C is a vertex of a uniform polyhedron, so $\left|\overrightarrow{OC}\right| = r_c$ and:

$$\overrightarrow{OC} = r_c \frac{\mathbf{m} \times \mathbf{n}}{|\mathbf{m} \times \mathbf{n}|} \qquad [10.7]$$

These may seem like all the possible points. Either the point sits on two sides, one side, or none. Since each side of the Schwarz triangle is a mirror, then if the point sits on two sides, it is only reflected by one mirror. If the point sits on one side, its perpendicular distance to the other two sides must be the same for the edge length to be the same, and the only possible point on that side would then be the foot of the opposite angle bisector. If the point sits on no sides, then its perpendicular distance to all three sides must be the same for the edge length to be the same, and so it would then be the incentre.

But there is one final possible point, which has the Wythoff symbol $(|\, p\ q\ s)$ (Weisstein, “Wythoff Symbol”). This point, when reflected once about each triangle side, creates reflected points that form an equilateral triangle. So, the edge length is equal. These reflected points are then reflected twice about each side to create new points, so only even reflections are used. The original point is not included as a vertex of the polyhedron, only the points that join together in equilateral triangles. This creates many *snub* polyhedra that may only have rotational symmetry, and the side length is not $2d$.

I had no idea how to solve this analytically. But once again, I had a computer, and so I decided to find the point through random sampling. My goal was to optimize for minimum side length variance. I would choose a point N, reflect it across the triangle sides (QS), (SP), and (PQ) using [9.12] to image points N_P, N_Q, and N_S, find the side lengths $N_P N_Q$, $N_Q N_S$, and $N_S N_P$, and finally calculate the variance. If variance is 0, then all three sides are the same length, and I will know I have found the correct point N.

My sampling used spherical coordinates for simplicity, which is why I had to derive [6.2]. The shaded area of Figure 20 gives a θ-range within the triangle of $0 \leq \theta_{low} \leq \theta_{random} \leq \theta_{high} \leq \pi/p$ and a corresponding ϕ-range within the triangle of $0 \leq \phi_{low} \leq \phi_{random} \leq \phi_{high} \leq \max(q,s)$, with $\max(q,s)$ being the maximum ϕ-coordinate as derived in [9.5]. I then generated the angles θ_{random} and ϕ_{random} between each range in a uniform distribution to find a point $N(r_c, \theta_{random}, \phi_{random})$.

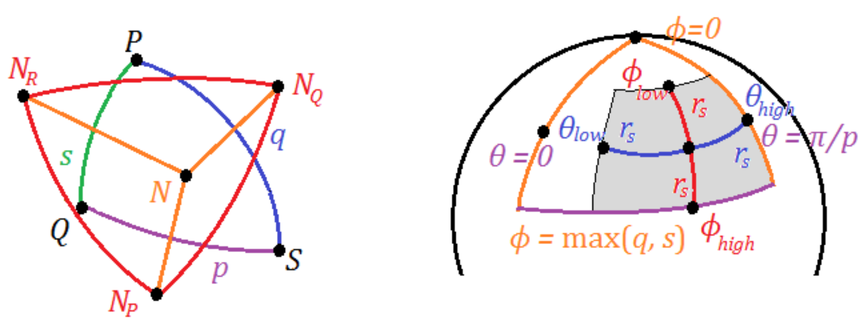

Figure 20. Calculation of the final Wythoff point by random sampling.

But this sampling is not really uniform. Since the circumference of the circle $r = r_c, \phi = \phi_{low}$ is larger than the circumference of the circle $r = r_c, \phi = \phi_{high}$, assuming

$\pi/p \leq \pi/2$, but the probability of choosing within some small value ϵ of ϕ_{low} is the same as the probability of choosing within ϵ of ϕ_{high} because my distribution was uniform, so the probability of choosing some point in an arc of length l within ϵ of ϕ_{low} is greater than the probability of choosing some point in an arc of the same length, but within ϵ of ϕ_{high}!

To minimize this effect, I decided to systematically measure the variance of points on the triangle. I started at $\phi = \max(q, s)$ and iterated down, decreasing ϕ by 0.1 every loop. Inside that loop, I started at $\theta = \pi/p$ and iterated down, decreasing θ by $0.1/\phi$ every loop. In this way, as ϕ increases, the number of θs considered increases too, so as to counterbalance the effect of the previous paragraph. The first picture of Figure 21 shows possible resulting sampled points.

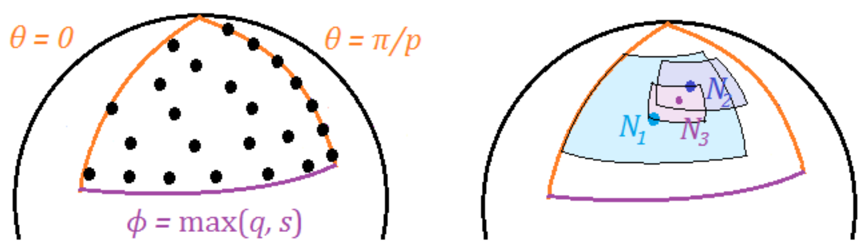

Figure 21. Systematic sampling and recursion methods.

After this systematic sampling, I had a point $N_1(r_c, \theta_1, \phi_1)$ with minimal side length variance. To find the actual point N, I had to find a better approximation, through recursion. Given constants r_s and r_d, every loop, I set θ_{high}, ϕ_{high}, θ_{low}, and ϕ_{low} as $\theta_n + r_s/r_d^n$, $\phi_n + r_s/r_d^n$, $\theta_n - r_s/r_d^n$, and $\phi_n - r_s/r_d^n$, within their restrictions. Then I randomly sampled f points inside that range, set the point with the lowest edge variance as N_{n+1}, and repeated the process. Through this method, as n increases, the edge variance decreases and the sampling range decreases because r_s/r_d^n goes to zero. After d recursions, N_d should be so precise that no further recursions are necessary.

Although this method, being random, sometimes finds undesirable local minima, I found after much testing in version 0.56 that $d = 4$, $f = 1024$, $r_s = \pi/8$, and $r_d = 8$ give adequate results within a second. I am still trying to find a method that always works.

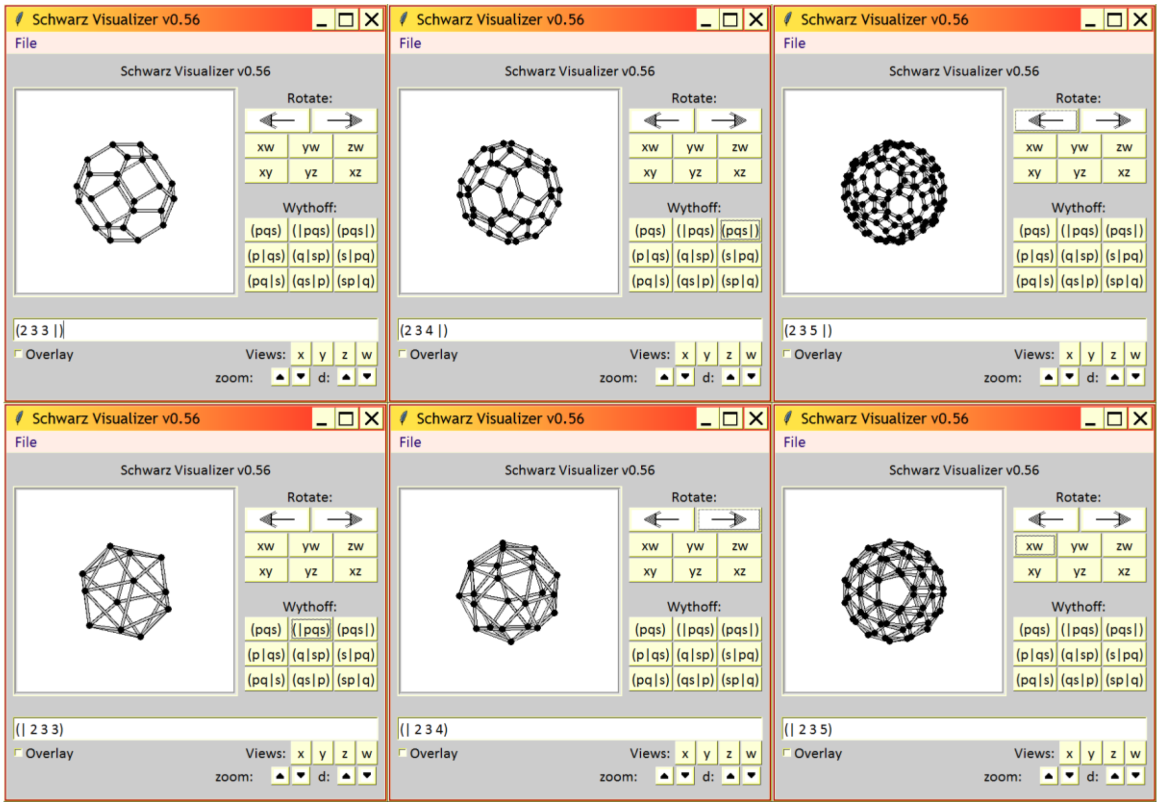

Figure 22. The $(p\ q\ s\,|)$ and $(|\,p\ q\ s)$ polyhedra of the integral Schwarz triangles. (The truncated octahedron, the truncated cuboctahedron, the truncated icosidodecahedron, the snub tetrahedron or icosahedron, the snub cube, and the snub dodecahedron.)

Conclusion

In this exploration, I was able to find the vertices of all regular polygons, all regular polyhedra, and all Wythoff-constructable—that is, all but one—uniform polyhedra. It was definitely an adventure, and I did start to understand a bit about how geometers discover new objects. You can imagine the shock I had a late night last week, when I accidentally typed $(2\ 3/2\ 2)$ instead of $(2\ 3\ 2)$ and my program actually worked! It did not fully render the triangles, but it was marvellous how I could create these objects without even meaning to. This month was certainly a very different experience from writing my extended essay, when I pored for hours and hours over the same five boring polyhedra (and later, the same three polychora). Now, I was discovering new, strange creatures with every new permutation of numbers and point location. I wondered how Hedrondude discovered all those polytopes and now here I am, with the coordinates of all but one uniform polyhedra! Granted, four-dimensional polytopes are more conceptually demanding than three-dimensional polyhedra, but discovery is a process.

My program, now version 0.56, is of course, not being version 1.00, incomplete. The main issue is that my algorithms are slow, especially all the recursion and reflection algorithms, and even the edge connection algorithm runs in $O(V^2)$ where V is the number of vertices. This is because my algorithms are optimized for conceptual clarity, and it takes deep mathematical insight to optimize for speed instead, an insight I cannot comprehend. Also, my final method for constructing snub polyhedra sometimes renders improperly, and $(|\, p\ q\ s)$ and $(p\ q\ s\,|)$ polyhedra sometimes only fill up half the sphere when p, q, or s are not integers. But throughout this entire investigation, I have never had to actively ponder what new thread to pursue, because after every single version, I always discover some bug, whose resolution naturally leads me to implement the next extension. And so, I foresee myself pursuing this mathematical investigation for a while.

I believe this investigation was an adequate application of my current mathematical ability. From basic planar trigonometry to spherical calculations, from Cartesian equations of curves to coordinate transformations and parametrization, from discrete casework to vector constructions, I was surprised at how many different areas of mathematics my investigation led me to use. I was also surprised at how useful approximation was when analytic methods failed. I cannot understand how Coxeter diagrams represent vertex coordinates so easily, but at least my recursive algorithms found them. Using randomness to optimize for variance was certainly unexpected when I first started this investigation.

If I could redo this entire investigation without all the twists and turns, if some magical mathematical being infused my mind with the essence of every algorithm I found, I would not. Even though I wasted so much time pursuing silly ideas and suffered horribly from a lack of formalism, I honestly believe my mathematical investigative skills are now nonpareil. More importantly, I discovered how incredibly euphoric the emotional thrill of mathematical discovery could be after wading through equations after equations, diagrams after diagrams for hours and hours, a far cry from the everyday humdrum of carefully pruned, five-minute test questions. I had actually made my own mathematical discoveries.

Bibliography

Coxeter, Harold Scott MacDonald. *Regular Polytopes.* 3rd ed. New York: Dover Publications, 1973. Print.

International Baccalaureate Organization. *Diploma Programme: Mathematics HL Guide.* Cardiff: International Baccalaureate Organization (UK), 2012. Print.

Martin, David et al. *Mathematics for the International Student: Mathematics HL (Core).* 3rd ed. Adelaide: Haese Mathematics, 2012. Print.

The Python Standard Library, Version 3.4.2. Python Software Foundation, 2014. Web. 2 Oct. 2014.

Weisstein, Eric. "Spherical Coordinates." *Wolfram MathWorld*. Wolfram Research, n.d. Web. 15 Sept. 2014.

Weisstein, Eric. "Spherical Triangle." *Wolfram MathWorld*. Wolfram Research, n.d. Web. 14 Dec. 2014.

Weisstein, Eric. "Spherical Trigonometry." *Wolfram MathWorld*. Wolfram Research, n.d. Web. 15 Dec. 2014.

Weisstein, Eric. "Uniform Polyhedron." *Wolfram MathWorld*. Wolfram Research, n.d. Web. 14 Dec. 2014.

Weisstein, Eric. "Wythoff Symbol." *Wolfram MathWorld*. Wolfram Research, n.d. Web. 14 Dec. 2014.