Deriving the Geodesic Equation from the Euler-Lagrange Equations

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1 Preliminaries and Definitions

We use the Einstein summation convention, where indices that are repeated twice, once lower, once upper, are implicitly summed over. For example for $x, y \in \mathbb{R}^3$, we can write their dot product:

$$\boldsymbol{z} = \boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^{3} x^i y_i = x^i y_i = x_i y^i.$$

Given an N-dimensional manifold we label our cartesian basis $x^a = (x^1, \ldots, x^N)$ and our general basis $u^a = (u^1, \ldots, u^N)$. We define the *natural basis* $\boldsymbol{e_a}$ and our *metric* g_{ab} by

$$\boldsymbol{e_a} = \frac{\partial \boldsymbol{r}}{\partial u^a}, \quad g_{ab} = \boldsymbol{e_a} \cdot \boldsymbol{e_b}.$$

Note that due to commutativity of the dot product we obtain that $g_{ab} = g_{ba}$. We also introduce the inverse of the metric g^{bc} such that

$$g_{ab} g^{bc} = g^{bc} g_{ab} = \delta_a^c$$

Here we label the Kronecker delta tensor δ_{ij} in tensor notation as δ^i_j.

2 Arc Length

Given a curve parametrised by $t_1 \leq t \leq t_2$, we can calculate the length of this curve by

$$l = \int_{r(t_1)}^{r(t_2)} dr = \int_{t_1}^{t_2} \frac{dr}{dt} dt = \int_{t_1}^{t_2} \sqrt{\frac{d\boldsymbol{r}}{dt} \cdot \frac{d\boldsymbol{r}}{dt}} dt.$$

Using the chain rule and the definition of the metric we can simplify the integrand as follows:

$$\frac{d\boldsymbol{r}}{dt} = \frac{\partial \boldsymbol{r}}{\partial u^b} \frac{du^b}{dt} = \boldsymbol{e_b} \dot{u}^b$$

$$\Rightarrow \sqrt{\frac{d\boldsymbol{r}}{dt} \cdot \frac{d\boldsymbol{r}}{dt}} = \sqrt{(\boldsymbol{e_b} \dot{u}^b) \cdot (\boldsymbol{e_c} \dot{u}^c)} = \sqrt{g_{bc} \dot{u}^b \dot{u}^c}$$

3 Euler-Lagrange Equations for the Arc Length

Given a functional $S[x^a]$ (sometimes called an *action*), with fixed end points t_1 and t_2, defined by

$$S[x^a] = \int_{t_1}^{t_2} L(t, x^a(t), \dot{x}^a)dt,$$

(where L is the Lagrangian), we can find the *path* $x^a(t)$ that extremises this value using the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = \frac{\partial L}{\partial x^a},$$

one equation for each coordinate x^a. A *geodesic* is a path along the manifold that *minimises* the length between two fixed points. Equating the functional with our arc length, we find that the geodesics are given by the Euler-Lagrange equations of $L = \sqrt{g_{bc}\dot{u}^b\dot{u}^c}$. We introduce the quantity K defined by

$$K = \frac{1}{2}L^2 = \frac{1}{2}g_{bc}\dot{u}^b\dot{u}^c$$

By considering the derivatives of K and using the Euler-Lagrange equations we obtain

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{u}^a}\right) = \frac{\partial K}{\partial u^a} + \dot{L}\frac{\partial L}{\partial \dot{u}^a}.$$

Since $L = \frac{dr}{dt} = \dot{r}$, we can rewrite $\dot{L} = \ddot{r}$. We're free to choose our parametrisation t to relate to the displacement r along the curve as we wish - we can choose it to be related by $t = \alpha r + \beta$ ($\alpha \neq 0, b$ constant) such that $\ddot{r} = 0$. Then t is called an *affine parameter*. Choosing this parametrisation we obtain

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{u}^a}\right) = \frac{\partial K}{\partial u^a},$$

i.e. K itself obeys the Euler-Lagrange equations.

4 Deriving the Geodesic Equation

We begin by taking our Euler-Lagrange equation with respect to a coordinate u^d, i.e.

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{u}^d}\right) = \frac{\partial K}{\partial u^d}. \tag{1}$$

We introduce the notation ∂_d to denote the partial derivative with respect to u^d. Then we can calculate the RHS as

$$\frac{\partial K}{\partial u^d} = \frac{1}{2}\partial_d g_{bc}\dot{u}^b\dot{u}^c. \tag{2}$$

Since the metric $g_{bc} = g_{bc}(u^a(t))$ (i.e. the metric does not depend on the time derivative of our coordinates), it is held constant when differentiating with respect to $\dot{u}^d$. We note that

$$\frac{\partial \dot{u}^b}{\partial \dot{u}^c} = \delta^b_c,$$

which, along with the product rule, allows us to write the partial derivate on the LHS as

$$\frac{\partial K}{\partial \dot{u}^d} = \frac{1}{2}g_{bc}\left(\dot{u}^b\delta^c_d + \dot{u}^c\delta^b_d\right).$$

We notice that since c is repeated (and thus summed over), we can simplify $g_{bc}\delta^c_d$ by writing

$$g_{bc}\delta^c_d = \sum_c g_{bc}\delta^c_d = g_{bd},$$

and similarly for $g_{bc}\delta^b_d$. Our expression for the partial derivate then becomes

$$\frac{\partial K}{\partial \dot{u}^d} = \frac{1}{2}g_{bd}\dot{u}^b + \frac{1}{2}g_{cd}\dot{u}^c. \tag{3}$$

We'll now take the time derivative of equation (3). Focusing on the first term on the RHS:

$$\frac{d}{dt}\left(\frac{1}{2}g_{bd}\dot{u}^b\right) = \frac{1}{2}g_{bd}\ddot{u}^b + \frac{1}{2}\cdot\frac{d}{dt}(g_{bd})\dot{u}^b.$$

Using the chain rule we obtain

$$\frac{d}{dt}(g_{bd}) = \frac{\partial}{\partial u^c}(g_{bd})\frac{du^c}{dt} = \partial_c g_{bd}\dot{u}^c,$$

which simplifies our previous equation to

$$\frac{d}{dt}\left(\frac{1}{2}g_{bd}\dot{u}^b\right) = \frac{1}{2}g_{bd}\ddot{u}^b + \frac{1}{2}\partial_c g_{bd}\dot{u}^b\dot{u}^c$$

Doing the same for the second term in equation (3), and mirroring the use of u^c in the chain rule step, we get that equation (3) becomes

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{u}^d}\right) = \frac{1}{2}g_{bd}\ddot{u}^b + \frac{1}{2}g_{cd}\ddot{u}^c + \frac{1}{2}\partial_c g_{bd}\dot{u}^b\dot{u}^c + \frac{1}{2}\partial_b g_{cd}\dot{u}^b\dot{u}^c.$$

We can manipulate the first two terms by swapping both their indices as following:

$$\frac{1}{2}g_{bd}\ddot{u}^b+\frac{1}{2}g_{cd}\ddot{u}^c = \frac{1}{2}\sum_b g_{bd}\ddot{u}^b+\frac{1}{2}\sum_c g_{cd}\ddot{u}^c = \frac{1}{2}\sum_e g_{ed}\ddot{u}^e+\frac{1}{2}\sum_e g_{ed}\ddot{u}^e = g_{ed}\ddot{u}^e,$$

therefore obtaining our LHS of equation (1) as

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{u}^d}\right) = g_{ed}\ddot{u}^e + \frac{1}{2}\partial_c g_{bd}\dot{u}^b\dot{u}^c + \frac{1}{2}\partial_b g_{cd}\dot{u}^b\dot{u}^c \tag{4}$$

Now by using our original Euler-Lagrange equation, equation (1), and plugging in equations (2) and (4) we obtain:

$$g_{ed}\ddot{u}^e + \frac{1}{2}\partial_c g_{bd}\dot{u}^b\dot{u}^c + \frac{1}{2}\partial_b g_{cd}\dot{u}^b\dot{u}^c - \frac{1}{2}\partial_d g_{bc}\dot{u}^b\dot{u}^c = 0.$$

By multiplying the left-most term by g^{ad} and noting the definition of the inverse of the metric we see that

$$g^{ad}g_{ed}\ddot{u}^e = \delta^a_e\ddot{u}^e = \ddot{u}^a.$$

Hence we multiply the entire equation by g^{ad}, and factor out $\dot{u}^b\dot{u}^c$ from the 3 right-most terms to get it in the following form:

$$\ddot{u}^a + \Gamma^a_{bc}\dot{u}^b\dot{u}^c = 0, \tag{5}$$

where Γ^a_{bc} is the Christoffel symbol (of the second kind) defined by

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(\partial_c g_{bd} + \partial_b g_{cd} - \partial_d g_{bc} \right).$$

Note that the Christoffel symbol is symmetric ($\Gamma^a_{bc} = \Gamma^a_{cb}$). Equation (5) is the geodesic equation.

5 References

Sections 1-3 were based on Dr. Ryan Barnett's notes from his course (module) in Tensor Calculus and General Relativity from Spring 2017 at Imperial College London. His page can be found at
`http://wwwf.imperial.ac.uk/~rlbarnet/`
and the lecture notes used are available at
`http://wwwf.imperial.ac.uk/~rlbarnet/grnotes2017.pdf`.