An Introduction to Quivers in 3d $\mathcal{N} = 4$ Superconformal Field Theories

M3R

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Abstract

This paper is meant as an introduction to quivers, a class of objects particularly useful in representing gauge theories. In this paper begin with the definition of quivers, both in a graph theoretic approach and in a linear algebra approach. We define the moduli space of vacua, an algebraic object that arises from quivers, and we use Hilbert series in order to classify global symmetries. We then use tools in linear algebra to observe nilpotent orbits and their relation to quivers through [5]. We introduce Hasse diagrams as an easy way to visualise Coulomb branch dimensions and symplectic leaves [1]. Next we turn to the recent quiver subtraction algorithm and its use in calculating Hasse diagrams for some nilpotent orbits. Finally we look to applications in string theory using work from [11].

Contents

1 Introduction

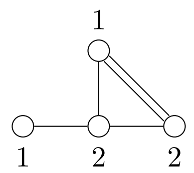
In this paper we attempt to introduce quivers and their utility from the ground up in an accessible way to both mathematicians and physicists, highlighting crucial ideas which can be studied further. This paper is meant as an introduction to the topic for readers with knowledge in linear algebra and representation theory. Quivers play an important role, both in the theory of representations and in their use in supersymmetry. In relation to common quantum field theories, supersymmetry and conformality lead to a simplification in which the masses, charges and coupling constants are naturally fixed [4]. Throughout this paper, we use work from many sources, mainly [2], however almost all the work has been rephrased for the purpose of accessibility to those not currently in the field. The originality of this project comes mostly from presentation and the collection of these ideas, along with some explicit examples that are often excluded in expert texts.

The ideas of this paper come mostly from conversations with my supervisor, Dr Antoine Bourget. I would like to give him a special thank you for his help over the last 6 months.

2 Quivers

2.1 Basic definitions

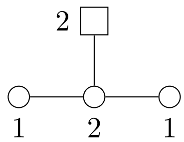
Definition 2.1. Quiver
A quiver is a finite, connected, labeled graph with no loops from a node to itself. We illustrate a simple example below:



Quivers are used to describe a $3d$ $\mathcal{N} = 4$ supersymmetric quantum field theories as follows: to every node we associate a complex Lie group G_i, and to every link we associate a finite-dimensional complex bifundamental representation of the Lie groups on either end, known as the *matter content.* Here the $\mathcal{N} = 4$ tells us our theory has 8 supercharges - a quantity describing how much supersymmetry is present. In this paper we will mostly discuss Lie groups $U(n)$, where n is determined by the label of each node, known as *unitary* quivers. In this case, a link between nodes with labels n_i and n_j corresponds to the bifundamental representation of $U(n_i) \times U(n_j)$.

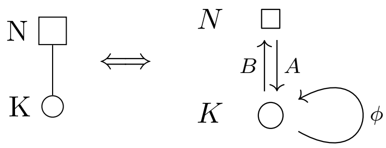
2.2 Flavor nodes

Along with the usual gauge nodes appearing in quivers, we may also come across *flavor nodes*, represented by a box $\square$. When present in quivers, we label them as *framed* quivers, and we can convert these to *unframed* quivers where all nodes are gauge nodes. For details on framing/unframing quivers, see [2]. Flavor nodes appear naturally in the construction of type IIB string theories from quivers and will be used in later chapters. Flavor nodes generally represent $SU(n)$ groups. Here we show an example:



2.3 Quivers as a shorthand for linear maps

For each quiver we can draw it as an extensive directed graph between $\mathbb{C}^n$. We will outline how we achieve this and its importance in finding geometric invariants. For example, translating the following quiver into linear maps:



where $A : \mathbb{C}^N \to \mathbb{C}^K$, $B : \mathbb{C}^K \to \mathbb{C}^N$, $\phi : \mathbb{C}^K \to \mathbb{C}^K$. In general we replace every link l_i from nodes labelled n_i to m_i with a pair of maps A_i, B_i where say, $A_i : \mathbb{C}^{n_i} \to \mathbb{C}^{m_i}$, $B_i : \mathbb{C}^{m_i} \to \mathbb{C}^{n_i}$, and for every gauge node of $U(n)$ we introduce an extra map $\phi_j : \mathbb{C}^n \to \mathbb{C}^n$.

Our condition of gauge invariances tells us that under a transformation $g \in U(K)$, such that $A \mapsto gA$, $B \mapsto Bg^{-1}$, $\phi \mapsto g\phi g^{-1}$, we only care about composed maps that are invariant under this transformation. In this case, we can see that the *meson matrix* $M = BA$ is invariant: $BA \mapsto Bg^{-1}gA = BA$, and similarly maps of the form $B\phi^m A$, where m is an integer. In general, we consider this separately for each gauge node.

Now we use the condition of supersymmetry to fix some relations between our maps. We define the *superpotential* $W = \mathrm{Tr}(B\phi A)$, and the *F-terms* as $\frac{\partial W}{\partial \phi} = 0 = AB$. In general this gives us relations $\sum A_i B_i = 0$ for each gauge node in our quiver.

Finally we define our first object of important:

Definition 2.2. The Higgs Branch of a quiver Q is defined as polynomial ring of maps A_i, B_i, quotiented by our F-terms, such that the resulting space is gauge invariant, i.e.

$$\text{Higgs}(Q) = \left(\frac{\mathbb{C}[A_1, B_1, \cdots A_n, B_n]}{F\text{-}terms} \right)_{gauge\ invariant}$$

2.4 Moduli spaces of vacua

To each quiver we can associate a moduli space of vacua - that is, a singular algebraic varieties representing the configuration of fields with 0 energy. This moduli space can be split into two important spaces; the Higgs branch and the Coulomb branch.

The Higgs branch that we've just discussed is easy to formulate mathematically, however the Coulomb branch is more difficult to give an exact expression for. A rigorous description is still hard to define, and we redirect readers to [3]. Instead of constructing the Coulomb branch explicitly, we will study it through a variety of related tools and objects in the next few sections. Here we add a useful definition which will be used later.

Definition 2.3. The quaternionic dimension of the Coulomb branch of a quiver is given by the summing the labels of each gauge node in its framed representation. This represents the rank of the global symmetry $G = U(n_1) \times U(n_2) \times \cdots \times U(n_m)$. In the unframed representation of a quiver, we must uncouple an extra $U(1)$ group from our global symmetry, hence reducing the dimension to the sum of the labels of the gauge nodes minus 1.

2.5 Hilbert series

Given a graded coordinate ring over a symplectic singularity X, we define the Hilbert series of the space as follows:

$$H_X(t) = \sum_{d=0}^{\infty} dim(\mathbb{C}[X]_d) t^d.$$

This is a useful tool in the study of quivers as we can study the Hilbert series of the Higgs and Coulomb branches of a quiver to find out more about their global properties. It has been shown [8] that the coefficient of the t^2 term of the Hilbert series of the Coulomb branch is exactly the dimension of the global symmetry.

3 Nilpotent Orbits

Here we take a turn into linear algebra to study nilpotent orbits, a powerful tool in representations of Lie groups and Lie algebras, and have been studied by mathematicians for a long time [7]. Note that here all our matrices are over $\mathbb{C}$. Given a matrix $M \in \mathbb{C}^{n\times n}$, we say it is *nilpotent* if there exists a positive integer N such that $M^N = 0$. The orbit of a matrix M is defined as

$$\mathcal{O}(M) := \left\{P^{-1}MP | P \in GL(n)\right\}.$$

It is easy to see that if M is nilpotent, then every element of its orbit is also nilpotent. Therefore we call the orbits of such matrices *nilpotent orbits*. We would like to study what the matrices of these nilpotent orbits look like, up to similarity.

3.1 Nilpotent orbits in one dimension

Here we'll consider the most trivial example, $\mathbb{C}^{1\times 1}$. Here our nilpotent orbit is made up of just the 0 matrix as a scalar in $\mathbb{C}$. We simply obtain a one-element set, whose dimension is 0.

3.2 Nilpotent orbits in two dimensions

Now for a non-trivial example, $\mathbb{C}^{2\times 2}$. The first thing we notice is that our matrices must not have non-zero eigenvalues λ, as raising these matrices to a power N gives us eigenvalue $\lambda^N \neq 0$, therefore meaning we do not end up with the zero matrix. Therefore all eigenvalues must be zero. Now we can use a fact of all matrices over $\mathbb{C}$ - Jordan decomposition. Up to similarity, any matrix can be written in terms of Jordan blocks by selecting the right P, in this case with all eigenvalues equal to 0. Therefore we see that our orbit can be generated with elements dependent on the Jordan blocks possible for $n \times n$ matrices. For $n = 2$ this gives us:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

corresponding to Jordan decompositions of blocks size 1 and 1, and a block of size 2. Therefore we have a bijection between nilpotent orbits of $\mathbb{C}^{n\times n}$ and the integer partitions of n.

First of all we see that the space generated by the Jordan blocks of size 1, which we'll call $\mathcal{O}_{[1^2]}$, has dimension 0 similar to the one-dimensional case. Next we look in detail at the orbit generated by the Jordan block of size 2, lets call this block M. We denote the orbit of M as $\mathcal{O}_{[2]}$. Let's first define what the P matrix must be:

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}^{2\times 2}, \text{ such that } ad - bc = 1,$$

from the requirement that $P \in GL(2)$. Then $P^{-1}MP$ gives us

$$\mathcal{O}_{[2]} = \left\{ \begin{pmatrix} cd & d^2 \\ -c^2 & -cd \end{pmatrix} \,\middle|\, (c,d) \in \mathbb{C}^2 - \{(0,0)\} \right\}$$

Notice that the map $(c,d) \mapsto (-c,-d)$ does not change our final matrix, hence quotienting our space by this relation leaves us

$$\mathcal{O}_{[2]} \cong \frac{\mathbb{C}^2 - \{(0,0)\}}{\mathbb{Z}_2}.$$

We now turn to looking at the closure of this orbit - we end up including the origin of $\mathbb{C}^2$, which is exactly the singularity in the Coulomb branch that was described in the introduction. Including the origin, we obtain $\mathbb{C}^2/\mathbb{Z}_2$, which is a cone. For this reason, the union of all nilpotent orbits is often called the nilpotent cone, or *nilcone* for short. This is also important as we see that $\mathcal{O}_{[1^2]} \subset \overline{\mathcal{O}}_{[2]}$. We will see later how this defines a partial order on the nilcones which we can visualise with Hasse diagrams.
Writing the coordinates instead in terms of $(x,y,z) \in \mathbb{C}^3, x = cd, y = d^2, z = -c^2$, we obtain the relation

$$x^2 + yz = 0.$$

This gives us a complex homogeneous algebraic variety of degree 4 defining our orbit (since (x,y,z) are degree 2 in (c,d)). Finally we turn to look at the coordinate ring of this set to compute its Hilbert series:

$$\mathbb{C}[\overline{\mathcal{O}}_{[2]}] = \frac{\mathbb{C}[x,y,z]}{(x^2+yz)}.$$

Then we obtain

$$\begin{aligned}\mathbb{C}[\overline{\mathcal{O}}_{[2]}] &= \mathbb{C} \oplus (x\mathbb{C} \oplus y\mathbb{C} \oplus z\mathbb{C}) \oplus (x^2\mathbb{C} \oplus y^2\mathbb{C} \oplus z^2\mathbb{C} \oplus xy\mathbb{C} \oplus xz\mathbb{C}) \oplus \cdots \\ \Rightarrow H_{\overline{\mathcal{O}}_{[2]}}(t) &= 1 + 3t^2 + 5t^4 + \cdots = \frac{1-t^4}{(1-t^2)^3}\end{aligned}$$

3.3 Partial order and Hasse diagrams

For n-dimension orbits we can impose a partial order given by the integer partitions as follows:

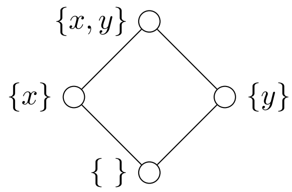
Given two partitions $[r] = [r_1 + r_2 + \cdots + r_l]$ and $[q] = [q_1 + q_2 + \cdots + q_m]$ of n, we define the partial order:

$$[r] \leq [q] \iff \sum_{i=1}^{k} r_i \leq \sum_{i=1}^{k} q_i, \quad \forall k. \tag{1}$$

This translates into

$$[r] \leq [q] \iff \overline{\mathcal{O}}_{[r]} \subset \overline{\mathcal{O}}_{[q]} \tag{2}$$

We use this partial order to construct a *Hasse diagram*, showing subspaces of our orbits. As an example, we can look at the two element set $\{x, y\}$, and draw a hasse diagram based on the subsets as follows [1]:

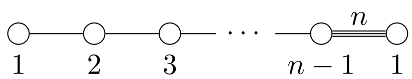


In the next section we use this as a way to visualise the subspaces of orbits, however we usually write the dimension of the orbits alongside the nodes on our diagram.

3.4 Quivers to nilpotent orbits

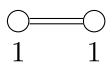
Here we state a theorem which we will use alongside quiver subtraction to study Hasse diagrams of orbits.

Theorem 3.1. The Kraft-Procesi transition [5]
For any value of n, the following quiver:



has a Coulomb branch of the form $\overline{\mathcal{O}}_{[n]}$. We will use this to study various nilpotent orbits using quiver subtraction and compute their Hasse diagrams. In terms of our dimension two example, we obtain:

Coulomb (



) $= \overline{\mathcal{O}}_{[2]}$

4 The Quiver Subtraction Algorithm

Here we will showcase the quiver subtraction algorithm as outlined in by Hanany and Cabrera [6]. Quiver subtraction is a way to view subspaces (known as *transverse slices*) of Higgs branches and Coulomb branches of quivers for unframed quivers.

We will first discuss balancing, a condition that must be satisfied on each node when using the algorithm, and we also need to define some basic quiver which we will use in the algorithm, known as *elementary slices.*

4.1 Balancing

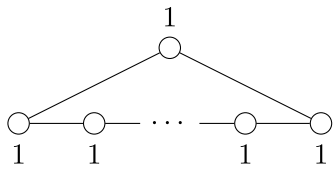
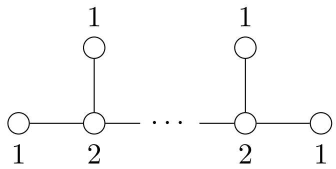
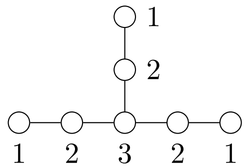
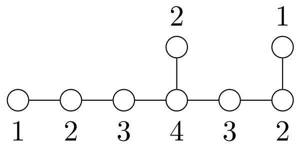
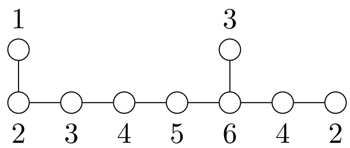
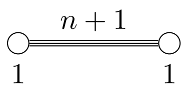
Given a quiver, for all nodes v_i labelled by n_i, we define their *balance* b_i as:

$$b_i = \sum_j n_j - 2n_i \geq 0,$$

where v_j are all nodes linked to v_i. When equality is reached we call the node *balanced.*
Note that for multiple links between nodes we count n_j in the balance formula accordingly.

4.2 Elementary slices and minimal nilpotent orbits

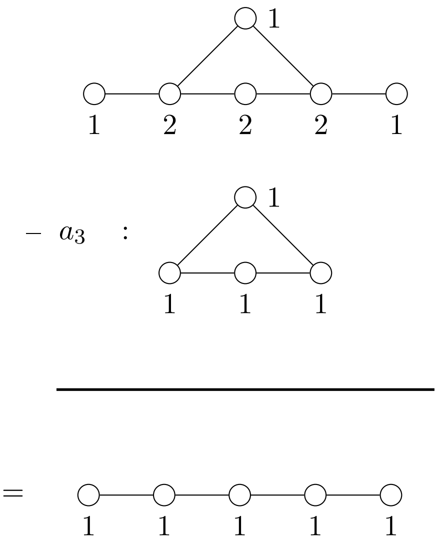
To subtract quivers we firstly need to define some basic quivers. From [2] we view some unitary quivers useful in our algorithm. We will look closer at a few of these which occur most often on the next page. The coulomb branches shown are from [10] , the missing coulomb branches do not have a simple form, and as such have been omitted. Note that in the diagram below, a_n and d_n quivers have $n + 1$ gauge nodes.

Slice	Quiver	Coulomb Branch
a_n		$\overline{\mathcal{O}_{min}(sl(n+1,\mathbb{C}))}$
d_n		$\overline{\mathcal{O}_{min}(so(2n,\mathbb{C}))}$
e_6		
e_7		
e_8		
A_n		

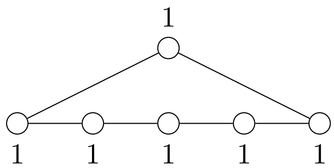
4.3 The quiver subtraction algorithm

We now describe the quiver subtraction algorithm. We begin with an unframed quiver $\mathcal{Q}$. We then identify a *subquiver* of $\mathcal{Q}$ as follows; we look for a connected subgraph of our quiver matching the elementary slices such that each of the nodes included have weights equal to or less than the weight on the corresponding nodes in $\mathcal{Q}$. We then subtract the weights of the nodes from our elementary slices from the weights on $\mathcal{Q}$, such that any node whose weight becomes 0 disappears. Finally we add on a $U(1)$ node to our diagram and create links to each node so that they remain balanced. Below is an illustration of those steps:

Firstly in the following quiver we notice an a_3 subquiver in the bottom three and top nodes.



We see that the top node on the quiver has now gone. Calculating the balance of all the other nodes in the original quiver we see that they are all balanced. Now we add another node to our quiver, adding links to make sure that the new quiver is balanced. Since only the outermost nodes are unbalanced now, we add links to those.



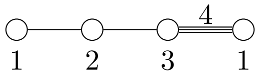
This is our final quiver after subtracting a_3! We will illustrate other examples in the next section when calculating the Hasse diagram for 5-dimensional orbits.

5 Hasse Diagrams of Nilpotent Orbits

In this section we will use the quiver diagrams for some n-dimensional orbits, describing the quiver subtractions used in each step.

5.1 Hasse Diagram for 4 dimensional nilpotent orbits

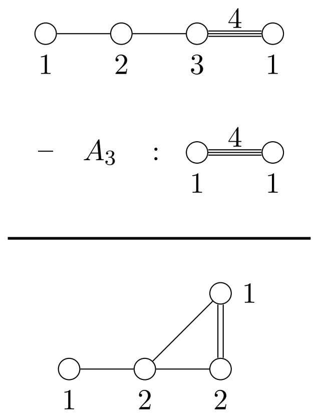
Firstly we start with the 4-dimensional case. By theorem 3.1, we can describe the nilpotent orbits in 4 dimensions as follows:

$Coulomb($ 

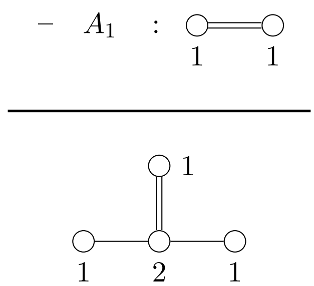
$) \quad = \quad \overline{\mathcal{O}}_{[4]}$

As we go along we will calculate the dimension of each Coulomb branch - summing the gauge nodes and subtracting one we obtain 6 as the first dimension. Now we start performing quiver subtraction until we reach the trivial quiver (i.e. one gauge node).

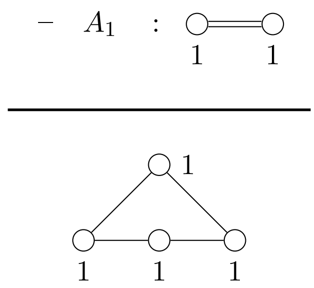
Our first subtraction comes from noticing an A_3 subquiver on the right of our quiver. Subtracting this gives:



This has a Coulomb branch of dimension 5. Once again we find another subquiver, A_1, on the right side of this quiver.

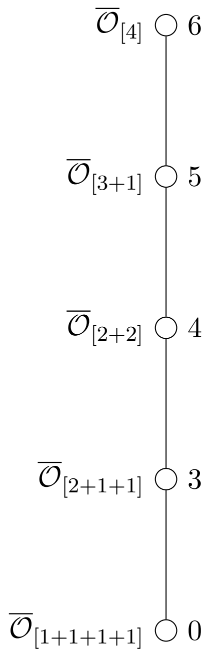


This quiver Coulomb branch of dimension 4. Another A_1 subquiver can be seen in the middle two nodes:



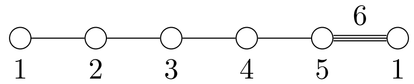
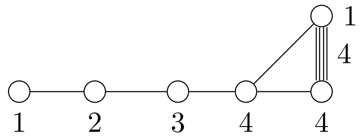
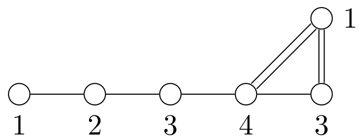
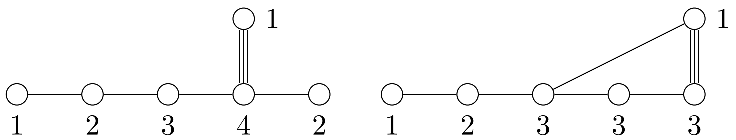
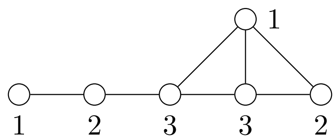
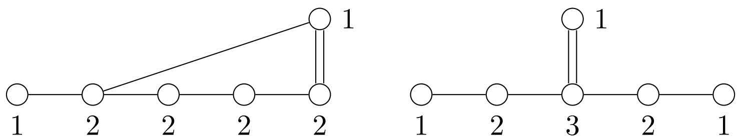
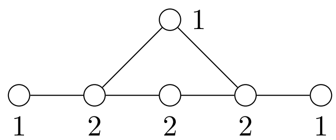
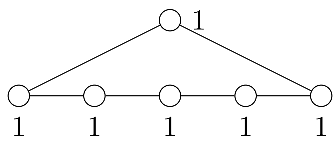
We see this as exactly the a_3 quiver, with a Coulomb branch of dimension 3. Hence subtracting it leaves us with the trivial one node quiver, which has dimension 0.

We can now represent all of these subtractions with a Hasse diagram. Each node represents a quiver's Coulomb branch, labelled by its dimension. On the left of these nodes we'll also include the relevant integer partition for this series of quiver subtractions.

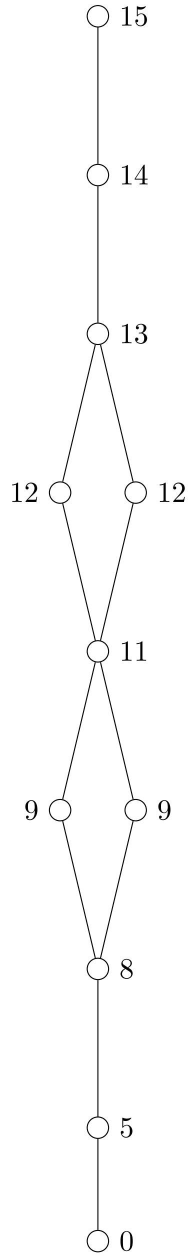


Next we show the $n = 6$ case where a bifurcation appears due to multiple integer partitions where no ordering is possible.

5.2 Hasse Diagram for 6 dimensional nilpotent orbits

Quivers	Coulomb Branches
	$\overline{\mathcal{O}}_{[6]}$
	$\overline{\mathcal{O}}_{[5+1]}$
	$\overline{\mathcal{O}}_{[4+2]}$
	$\overline{\mathcal{O}}_{[3+3]}$, $\overline{\mathcal{O}}_{[4+1+1]}$
	$\overline{\mathcal{O}}_{[3+2+1]}$
	$\overline{\mathcal{O}}_{[3+1+1+1]}$, $\overline{\mathcal{O}}_{[2+2+2]}$
	$\overline{\mathcal{O}}_{[2+2+1+1]}$
	$\overline{\mathcal{O}}_{[2+1+1+1+1]}$
	$\overline{\mathcal{O}}_{[1+1+1+1+1+1]}$

Hasse Diagram



6 Brane Systems and Quivers

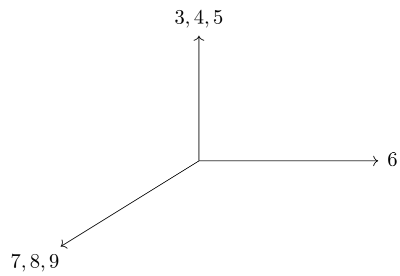
Brane systems play an important role in string theory - they allow us to visualise the structure of branes in a simple way. Here we will outline the Hanany-Witten setup [11], and show its relation to quivers. In the end our goal is to form a connection between the Higgs branch and Coulomb branch of a quiver by going through intermediate brane diagrams. We focus here on type IIB string theories. Starting with a 10-dimensional Minkowski space, labelled $\mathbb{R}^{1,9}$, we define various different objects. We label the one-dimensional strings of our system by $F1$ (F here standing for *fundamental*, and 1 describing the spatial dimensions). More importantly, we look at *D-branes* (named after Dirichlet); these are n-dimensional (in space) surfaces on which the boundaries of our strings end. Due to boundary conditions in type IIB string theories, n must be odd here, so we may have D1, D3 ... D9 branes in our system. One other important object is the *NS5-brane*, named after Neveu and Schwarz. This is another type of 5-dimensional brane. In our system all D3-branes must end on NS5-branes. We will first describe how to build diagrams from these objects, and later relate them to quivers.

6.1 The Hanany-Witten Setup

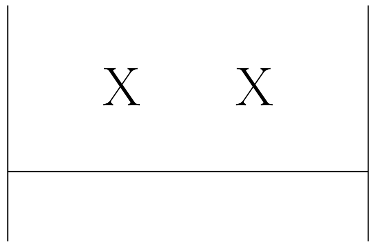
Here we will set out a method with which to draw brane systems consisting of D3, D5 and NS5 branes. In the following diagram we set out to show the possible dimensions that each of these branes occupy:

Branes	Dimension									
	0	1	2	3	4	5	6	7	8	9
D3	x	x	x				x			
D5	x	x	x					x	x	x
NS5	x	x	x	x	x	x				

We see that there is a 3-dimensional intersection - this is where the field theory occurs. We look to the other dimensions, 3-9, to look at the geometry. Of course a 7-dimensional diagram may be hard to visualise, so we group together the associated dimensions (3-5, 6, 7-9) to create our diagrams in 3 dimensions. We draw these axes as shown below.



With this orientation NS5 branes become vertical lines, D3 branes become horizontal lines, and D5 branes become lines in/out of the page, which we'll represent with a single x for each D5 brane. Below we illustrate a simple example of a brane diagram.

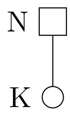


In this example we have 2 NS5-branes, bounding the D3 brane on either side, and 2 D5-branes. In the next section we describe how quivers can be represented with brane systems and introduce the Hanany-Witten transition.

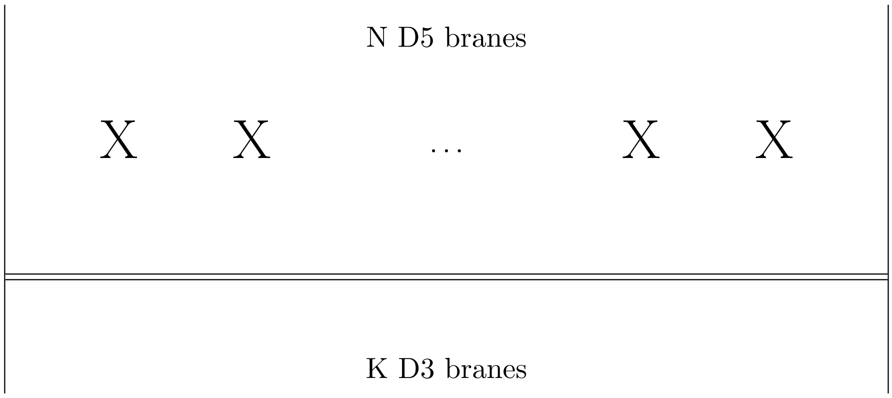
6.2 From Quivers to Branes

Firstly we describe how we can represent framed quivers by brane systems. To do this, we replace flavor nodes with D5 branes and gauge nodes with D3 branes. Between each link in our quiver from a gauge node to another gauge node we insert an NS5 brane splitting the to D3 branes. As mentioned before, we also insert NS5 branes on the edges of remaining D3 branes.

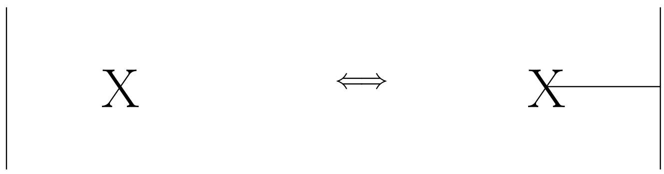
For example, taking the following quiver from section 2:



we can represent it as a brane diagram as follows:



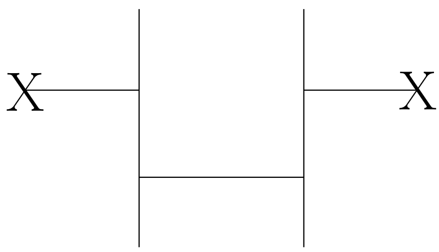
We next describe the "Hanany-Witten" transition [11], where we can move a D5 brane across an NS5 brane, bringing an D3 brane along with it. This is a useful tool when working with brane systems. We show this in the diagram below:



6.3 S-duality and mirror quivers

S-duality is an important phenomenon in string theory and quantum field theories in general that has been getting attention from mathematicians. It is a way to connect equivalent string theories where one of the theories may be easier to performs calculations on than the other.

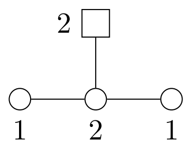
In our simple brane system this is shown by swapping NS5 and D5 branes. Any D3 branes that now intersect the new NS5 branes will be split. Illustrating this with our original example of a brane diagram we obtain:



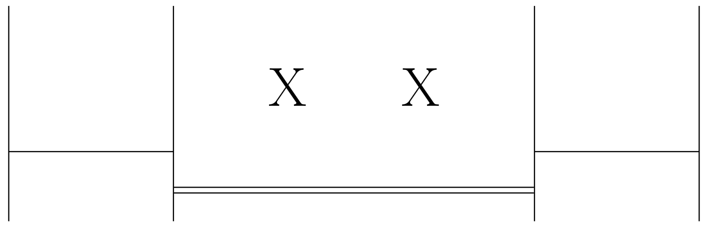
It turns out that in this example, using the Hanany-Witten transition, we can move both D5 branes back between the NS5 branes giving us our original quiver back! This isn't a general result however - we usually expect to get different brane diagrams after this step.

We can now work towards defining *mirror quivers.* Given a framed quiver our goal is to transform it into a brane diagram, use S-duality to obtain a new brane, and using the Hanany-Witten transition to get it into the right form before transforming it back into a quiver. The resulting quiver is the *mirror* of the original.

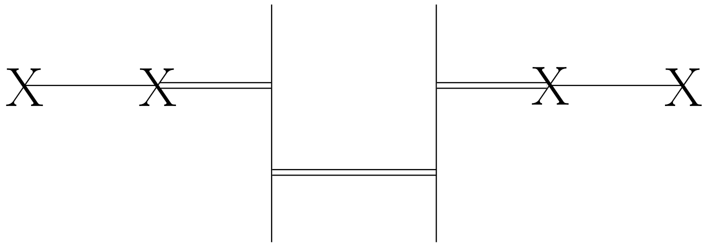
Lets consider our original framed quiver from section 1:



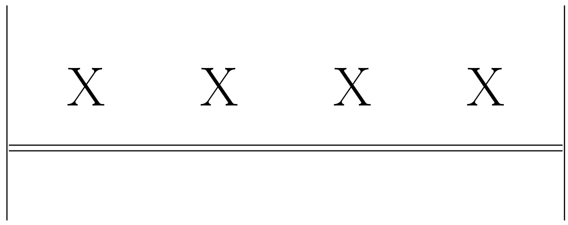
Transforming this into a brane system we obtain:



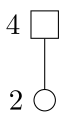
Now we apply S-duality to this diagram - we obtain the following:



Applying the Hanany-Witten transition to each of the 4 D5 branes, bringing them inside the NS5 branes, we get the following brane diagram:



which is exactly the brane diagram for the following quiver:



Finally we state a useful result that ties this section to the rest of the paper.

Theorem 6.1. Given a quiver $\mathcal{Q}$ and its mirror $\mathcal{P}$, the following relation holds: [9]

$$\begin{aligned} Higgs(\mathcal{Q}) &= Coulomb(\mathcal{P}) \\ Coulomb(\mathcal{Q}) &= Higgs(\mathcal{P}) \end{aligned}$$

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