

A First Guide of RSA Public Key Encryption By Example

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This document as well as its tex file can be obtained from

<https://github.com/dogtordoggo/rsa-guide>

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The purpose of this writeup is to serve as simple as possible an explanation of RSA public-key encryption as well as a demo showing how it works.

Prerequisites

Basic understanding of **modulo arithmetic** as well as **Euler's Theorem** and **Fermat's Little Theorem** are assumed prerequisites before heading into this document. Their proof isn't too hard though if anyone is interested in figuring them out, but here they are (from wikipedia):

Theorem 1 (Euler's Theorem): If n and a are coprime positive integers, $\phi(n)$ is Euler's totient function, then

$$a^{\phi(n)} = 1 \bmod n$$

Corollary 1.1 (Corollary of Euler's Theorem): For any integer $M < n = p \cdot q$, where p and q are two prime numbers, and any integer r , there is

$$M^{r \cdot \phi(n) + 1} = M \bmod n$$

See Appendix for the proof of this corollary.

Theorem 2 (Fermat's Little Theorem): If p is a prime number, then for any integer a

$$a^p = a \bmod p$$

or

$$a^{p-1} = 1 \bmod p$$

What we need:

1. Two large prime numbers p and q and $n = p \cdot q$, for the sake of demonstration and simplicity we pick $p = 41, q = 67$. We can then calculate

$$\begin{aligned} n &= p \cdot q = 41 \cdot 67 = 2747 \\ \phi(n) &= p \cdot q \cdot \left(1 - \frac{1}{p}\right) \cdot \left(1 - \frac{1}{q}\right) = (p-1) \cdot (q-1) = 40 \cdot 66 = 2640 \end{aligned}$$

Note that $\phi(n)$ is Euler's totient function of n .

2. A small prime number $s = 13$.

From here we can calculate other variables needed for our RSA algorithm. First of all, according to Corollary of Euler's Theorem, for any integer $M < n$, and any integer r , there is

$$M^{r \cdot \phi(n) + 1} = M \mod n$$

s is a (small) prime number, therefore $\gcd(s, \phi(n)) = 1$, so the inverse of $s \mod \phi(n)$ exists (See Lemma 1 in Appendix), note it as $t = s^{-1} \mod \phi(n)$. t could be calculated using *Euclid's algorithm*. See Appendix for a source code doing Euclid's algorithm calculating those coefficients. What this also means is that there exists some integer r such that $s \times t = r \times \phi(n) + 1$.

tl;dr: We need two large prime numbers p and q , and a smaller prime number s , then $(s, p \cdot q)$ would be public key pair. Another integer $t = s^{-1} \mod \phi(p \cdot q)$ is calculated, and $(t, p \cdot q)$ would be private key pair.

RSA

Now we can deploy our public-key encryption algorithm. Make (s, n) the public key pair, and (t, n) private key pair. Denote M as plaintext message, E as encrypted message.

Encryption:

$$E = M^s \mod n$$

Decryption:

$$M = E^t \mod n$$

Proof:

$$\begin{aligned} E^t \mod n &= (M^s)^t \mod n \\ &= M^{st} \mod n \\ &= M^{r \cdot \phi(n) + 1} \mod n \\ &= M \mod n \\ &= M \end{aligned}$$

Why is this secure?

Without the knowledge of p and q , we can't find $\phi(n) = (p-1) \cdot (q-1)$, therefore we can't calculate $t = s^{-1} \bmod \phi(n)$ based on the information of (s, n) . This means we can't calculate private key pair's t from the public key pair (s, n) .

Example:

Let's continue with the example where we chose $p = 41, q = 67$ then we have $n = 2747, \phi(n) = 2640$, then we choose $s = 13$. Then by Euclid's Algorithm, we have

$$\begin{aligned} st + r\phi(n) &= 13 \times -203 + 1 \times 2640 = 1 \\ t &= -203 \bmod 2640 = 2437 \end{aligned}$$

Therefore in this case public key is $(s, n) = (13, 2747)$, the private key pair is $(t, n) = (2437, 2747)$. Suppose our original message is an integer $M = 2017$, the year of this writeup. We can then go ahead and calculate encrypted message

$$E = M^s \bmod n = 2017^{13} \bmod 2747 = 800$$

There is a binary-like efficient algorithm to calculate the above modulo due to the simple fact that

$$a \cdot b = (a \bmod n) \cdot (b \bmod n) \bmod n$$

A matlab command of `mod(2017^13, 2747)` would produce *wrong* result. An efficient algorithm calculating modulus of large exponential modulus is given in Appendix. To recover original message from the encrypted message, we can do

$$M = E^t \bmod n = 800^{2437} \bmod 2747 = 2017$$

And we recovered the original message of 2017 from the encrypted message of 800.

Appendix

Lemma 1 For two positive integers s and n , $n > 1$, there exists s 's inverse mod n , noted as $s^{-1} \bmod n$, if and only if $\gcd(s, n) = 1$.

Proof

Necessity: If $\gcd(s, n) = 1$, by Euclid's algorithm, there exist integers t and r such that

$$t \cdot s + r \cdot n = 1$$

Therefore $t = s^{-1} \bmod n$.

Sufficiency: If there exists an integer $t = s^{-1} \bmod n$, it means for some integer r there is

$$t \cdot s = r \cdot n + 1$$

If $\gcd(s, n) = k > 1$, we can divide both sides of above equation by k and get

$$t \cdot \frac{s}{k} = r \cdot \frac{n}{k} + \frac{1}{k}$$

Which is impossible. ■

Proof of Corollary of Euler's Theorem:

When M is coprime of n , the above equation is direct result of Euler's Theorem.

When not, M must be multiple of p , $M = kp$ where $1 \leq k < q$ or multiple of q , $M = kq$ where $1 \leq k < p$. Suppose the first case, $M = kp$, $1 \leq k < q$, M is coprime of q obviously, according to Fermat's little theorem, we have

$$M^{q-1} = 1 \bmod q$$

Therefore

$$M^{r \cdot (p-1) \cdot (q-1)} = M^{r \cdot \phi(n)} = 1 \bmod q$$

This can be re-written as

$$M^{r \cdot \phi(n)} = zq + 1$$

for some integer z . Multiple both sides of the equation by $M = k \cdot p$, we have

$$M^{r \cdot \phi(n) + 1} = z \cdot k \cdot p \cdot q + M = z \cdot k \cdot n + M$$

Which is equivalent to

$$M^{r \cdot \phi(n) + 1} = M \bmod n$$

The $M = k \cdot q$ case is symmetrical thus proof is omitted. ■

A piece of code doing Euler's Algorithm, calculating gcd and the coefficients s and t of two integers a and b such that

$$sa + tb = \gcd(a, b)$$

can be found at <https://github.com/dogtordoggo/rsa-guide/EuclidsAlgorithm.kt>.

A piece of code calculating modulus of large exponentials like

$$800^{2437} = 2017 \bmod 2747$$

can be found at <https://github.com/dogtordoggo/rsa-guide/BigNumberModulo.kt>.