9.4 Parabolic PDEs: Heat Equations

Modify the program "solve_heat.m" (in Section 9.2.3) so that it can solve the following PDEs by using the explicit forward Euler method, the implicit backward Euler method, and the Crank–Nicholson method.

(a) $$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \quad \text{for } 0 \le x \le 1, 0 \le t \le 0.1 \qquad \text{(P9.4.1)}$$

with the initial/boundary conditions

$$u(x,0) = x^4, \qquad u(0,t) = 0, \qquad u(1,t) = 1 \qquad \text{(P9.4.2)}$$

(i) With the solution region divided into $M \times N = 10 \times 20$ sections, does the explicit forward Euler method converge? What is the value of $r = A\Delta t/(\Delta x)^2$? We use $r = a(\Delta t)/(\Delta x)^2$

(ii) If you increase M and N to make $M \times N = 20 \times 40$ for better accuracy, does the explicit forward Euler method still converge? What is the value of $r = A\Delta t/(\Delta x)^2$?

(iii) What is the number N of subintervals along the t axis that we should choose in order to keep the same value of r for $M = 20$? With that value of r, does the explicit forward Euler method converge?

$$a \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

small letter!

Stability Condition: $r = a\dfrac{\Delta t}{(\Delta x)^2} \le \dfrac{1}{2}$ (9.2.6)

(b) $$10^{-5}\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \quad \text{for } 0 \le x \le 1, 0 \le t \le 6000 \tag{P9.4.3}$$

with the initial/boundary conditions

$$u(x,0) = 2x + \sin(2\pi x), \quad u(0,t) = 0, \quad u(1,t) = 2 \tag{P9.4.4}$$

(i) With the solution region divided into $M \times N = 20 \times 40$ sections, does the explicit forward Euler method converge? What is the value of $r = A\Delta t/(\Delta x)^2$? Does the numerical stability condition (9.2.6) seem to be so demanding?

(ii) If you increase M and N to make $M \times N = 40 \times 160$ for better accuracy, does the explicit forward Euler method still converge? What is the value of $r = A\Delta t/(\Delta x)^2$? Does the numerical stability condition (9.2.6) seem to be so demanding?

(iii) With the solution region divided into $M \times N = 40 \times 200$ sections, does the explicit forward Euler method converge? What is the value of $r = A\Delta t/(\Delta x)^2$?

(c) $$2\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \quad \text{for } 0 \le x \le \pi, 0 \le t \le 0.2 \tag{P9.4.5}$$

with the initial/boundary conditions

$$u(x,0) = \sin(2x), \quad u(0,t) = 0, \quad u(\pi,t) = 0 \tag{P9.4.6}$$

(i) By substituting

$$u(x,t) = \sin(2x)e^{-8t} \tag{P9.4.7}$$

into the above equation (P9.4.5), verify that this is a solution to the PDE.

(ii) With the solution region divided into $M \times N = 40 \times 100$ sections, does the explicit forward Euler method converge? What is the value of $r = A\Delta t/(\Delta x)^2$?

(iii) If you increase N (the number of subintervals along the t-axis) to 125 for improving the numerical stability, does the explicit forward Euler method converge? What is the value of $r = A\Delta t/(\Delta x)^2$? Use the MATLAB statements in the following box to find the maximum absolute errors of the numerical solutions obtained by the three methods. Which method yields the smallest error?

```
uo = inline('sin(2*x)*exp(-8*t)','x','t'); %true analytical solution
Uo = uo(x,t);
err = max(max(abs(u1 - Uo)))
```

(iv) If you increase N to 200, what is the value of $r = A\Delta t/(\Delta x)^2$? Find the maximum absolute errors of the numerical solutions obtained by the three methods as in (iii). Which method yields the smallest error?

(d) $$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \quad \text{for } 0 \le x \le 1, 0 \le t \le 0.1 \qquad \text{(P9.4.8)}$$

with the initial/boundary conditions

$$u(x,0) = \sin(\pi x) + \sin(3\pi x), \quad u(0,t) = 0, \quad u(1,t) = 0 \quad \text{(P9.4.9)}$$

(i) By substituting

$$u(x,t) = \sin(\pi x)e^{-\pi^2 t} + \sin(3\pi x)e^{-(3\pi)^2 t} \qquad \text{(P9.4.10)}$$

into Eq. (P9.4.5), verify that this is a solution to the PDE.

(ii) With the solution region divided into $M \times N = 25 \times 80$ sections, does the explicit forward Euler method converge? What is the value of $r = A\Delta t/(\Delta x)^2$?

(iii) If you increase N (the number of subintervals along the t axis) to 100 for improving the numerical stability, does the explicit forward Euler method converge? What is the value of $r = A\Delta t/(\Delta x)^2$? Find the maximum absolute errors of the numerical solutions obtained by the three methods as in (c)(iii).

(iv) If you increase N to 200, what is the value of $r = A\Delta t/(\Delta x)^2$? Find the maximum absolute errors of the numerical solutions obtained by the three methods as in (c)(iii). Which one gained the accuracy the most of the three methods through increasing N?