MAT488E

Hwk-1

Due: October 20^{th}, 2017
17:00

1. Consider the boundary-initial value problem (in nondimensional form)

(20 pts.)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad u = u(x,t)$$

$$u(0,t) = u(1,t) = 0$$

$$u(x,0) = f(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2(1-x), & \frac{1}{2} \le x \le 1 \end{cases}$$

The physical problem is to compute the temperature history $u(x,t)$ for a bar with a prescribed initial temperature distribution $f(x)$, no internal heat sources, and zero temperature prescribed at both ends. Solve this problem using the explicit finite difference algorithm such as forward-time central-space with $h = \Delta x = 0.1$, and $k = \Delta t = r(h^2) = r(\Delta x)^2$ for two different values of r: a) $r = 0.48$ and b) $r = 0.52$. Plot the results. For both cases calculate the L^2-norm error at $T_{final} = 0.192$ for $r = 0.48$ and $T_{final} = 0.208$ for $r = 0.52$ by

$$\|Error\|_2 = \left(h \sum_{j=1}^{J} \Big| \underset{\text{approx. sol}^n}{w_j^{T_{final}}} - \underbrace{u(x_j, T_{final})}_{\text{exact sol}^n} \Big|^2 \right)^{1/2}.$$

You already know that the analytic solution can be obtained by the technique of separation of variables as

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8}{(n\pi)^2} \sin\frac{n\pi}{2} (\sin n\pi x)\, e^{-(n\pi)^2 t}.$$

You do NOT need to show this.

2. (Calculation of scheme order) (10 pts.)

The 1D linear advection equation is

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \quad c>0 \text{ (constant)}$$

Apply the forward-time central-space finite difference scheme and determine scheme order, i.e. $O((\Delta t)^?)$, $O((\Delta x)^?)$.

3. Consider an iron bar and the heat equation
(10 pts.)

$$\rho c \frac{\partial u}{\partial t} - K\frac{\partial^2 u}{\partial x^2} = 0 \qquad (*)$$

where $\rho :=$ density, $c :=$ specific heat, $K :=$ thermal conductivity
$l :=$ length of the bar. The following table gives us their dimensions:

parameter	units
ρ	g/cm^3
c	$J/(g\,K)$
K	$J/(sn\ cm\ K)$
l	cm

J: joule
K: Kelvin
sn: seconds
g: gram

Then the ratio

$$\frac{\rho c l^2}{K}$$

is seen to have units of seconds, so we define a dimensionless time variable s by

$$s = \frac{t}{\rho c l^2/K} = \frac{Kt}{\rho c l^2}$$

We then define $v(y,s) = u(x,t)$, where (y,s) and (x,t) are related by $y = \frac{x}{l}$, $s = \frac{Kt}{\rho c l^2}$. Show that the eqn. $(*)$ is equivalent to $\frac{\partial v}{\partial s} - \frac{\partial^2 v}{\partial y^2} = 0$ (in nondimensionalized form).