Exo 1 : $I = \displaystyle\int_{(\mathbb{R}_+)^2} \frac{dx\,dy}{(1+y)(1+x^2y)}$.

1/ On a $\forall (x,y) \in \mathbb{R}_+^2$, $\dfrac{1}{(1+y)(1+x^2y)} \geq 0$.

et $(x,y) \mapsto \dfrac{1}{(1+y)(1+x^2y)}$ est més sur $\mathbb{R}_+^2$.

Donc d'après le Th de Fubini-Tonelli, on a :

$$I = \int_0^{+\infty} \left(\int_0^{+\infty} \frac{1}{\sqrt{y}(1+y)} \frac{\sqrt{y}}{1+(x\sqrt{y})^2} dx \right) dy.$$

$$= \int_0^{+\infty} \frac{1}{\sqrt{y}(1+y)} \cdot \underbrace{\left[\arctan(x\sqrt{y}) \right]_0^{+\infty}}_{\pi/2} dy. = \pi \cdot \int_0^{+\infty} \frac{1}{1+y} \cdot \frac{dy}{2\sqrt{y}}.$$

$t = \sqrt{y}$. $\quad = \pi \displaystyle\int_0^{+\infty} \frac{1}{1+t^2} dt = \pi^2/2.$

$$* \quad \frac{1}{(1+y)(1+x^2y)} = \frac{a}{1+y} + \frac{b}{1+x^2y}, \quad \forall x>0 \text{ tq } x \neq 1.$$

$$a = \frac{1}{1-x^2}, \quad b = \frac{1}{1-\frac{1}{x^2}} = \frac{x^2}{x^2-1} = \frac{-x^2}{1-x^2}$$

$$\int_0^\alpha \frac{1}{(1+y)(1+x^2y)}\,dy = \frac{1}{1-x^2}\int_0^\alpha \left(\frac{1}{1+y} - \frac{x^2}{1+x^2y}\right)dy.$$

$$= \frac{1}{1-x^2}\Big(\ln(1+\alpha) - \ln(1+x^2\alpha)\Big).$$

$$= \frac{1}{1-x^2}\ln\left(\frac{1+\alpha}{1+x^2\alpha}\right) \xrightarrow[\alpha\to+\infty]{} \frac{2\ln(x)}{x^2-1}.$$

i.e $$\int_0^{+\infty} \frac{1}{(1+y)(1+x^2y)}\,dy = \frac{2\ln(x)}{x^2-1}.$$

Donc $I = \int_0^{+\infty} \frac{2\ln(x)}{x^2-1}\,dx = \frac{\pi^2}{2}.$

1/. $\int_0^{+\infty} \frac{\ln(x)}{x^2-1}\,dx = \frac{\pi^2}{4}.$

2/ – $\int_0^{+\infty} \frac{\ln(x)}{x^2-1}\,dx = \int_0^1 \frac{\ln(x)}{x^2-1}\,dx + \int_1^{+\infty} \frac{\ln(x)}{x^2-1}\,dx$

$$= \int_0^1 \frac{\ln(x)}{x^2-1}\,dx + \int_1^0 \frac{+\ln(t)}{\frac{1}{t^2}-1}\cdot\frac{+1}{t^2}\,dt \qquad t = 1/x$$

$$= 2\int_0^1 \frac{\ln(x)}{x^2-1}\,dx = \pi^2/4$$

D'où $\int_0^1 \frac{\ln(x)}{x^2-1}\,dx = \frac{\pi^2}{8}.$

On a $\forall x \in [0,1[$, $\dfrac{1}{x^2-1} = -\sum_{n=0}^{+\infty} x^{2n}$.

On a $x \overset{f_n}{\longmapsto} \sum_{k=0}^{n} x^{2k} \ln(x)$ est monotone.

par CV monotone on a $\displaystyle\int_0^1 \left(-\sum_{n=0}^{+\infty} x^{2n} \ln(x)\right) dx = -\sum_{n=0}^{+\infty} \underbrace{\int_0^1 x^{2n} \ln(x)\, dx}_{J}$

$$J = \left[\frac{1}{2n+1} x^{2n+1} \ln(x)\right]_0^1 - \frac{1}{2n+1}\int_0^1 x^{2n+1} \cdot \frac{1}{x}\, dx = \frac{-1}{(2n+1)^2}.$$

D'où $\displaystyle\int_0^1 \frac{\ln(x)}{x^2-1}\, dx = \sum_{n \geq 0} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$.

Exo 9 : $f(t) = \int_0^{+\infty} \frac{\sin(x)}{x} e^{-tx}\, dx$

$$g(t) = \int_0^{+\infty} \left(\frac{\sin(x)}{x}\right)^2 e^{-tx}\, dx.$$

1/- Par Th de continuité sous-signe intégral.

2/- $\frac{\sin(x)}{x} = \int_0^1 \cos(xy)\, dy.$

On a $f(t) = \int_0^{+\infty} \left(\int_0^1 \cos(xy)\, dy\right) e^{-tx}\, dx.$

$(x,y) \overset{\varphi_t(x,y)}{\longmapsto} \cos(xy) e^{-tx}$ est intégrable sur $\mathbb{R}_+ \times [0,1]$?

On a $\forall t>0, \forall (x,y) \in \mathbb{R}_+ \times [0,1], |\varphi_t(x,y)| \leq e^{-tx}$,

avec $(x,y) \mapsto e^{-tx}$ est intégrable sur $\mathbb{R}_+ \times [0,1]$.

Donc par le Th. de Fubini-Lebesgue, on a :

$$f(t) = \int_0^1 \left(\int_0^{+\infty} \cos(xy) e^{-tx} \, dx \right) dy.$$

$\underline{\to M1}$: $\cos(xy) = \mathrm{Re}(e^{ixy})$.

$\underline{\to M2}$: Double IPP.

$\underline{\to M3}$: $\cos(xy) = \displaystyle\sum_{n=0}^{+\infty} \frac{(xy)^{2n}}{(2n)!}$ $\qquad u = tx.$

$$\int_0^{+\infty} \cos(xy) e^{-tx} \, dx = \sum_{n=0}^{+\infty} y^{2n}/(2n)! \int_0^{+\infty} x^{2n} e^{-tx} \, dx.$$

$$= \sum_{n=0}^{+\infty} \frac{y^{2n}}{(2n)!} \cdot \frac{1}{t^{2n}} \int_0^{+\infty} u^{2n} e^{-u} \cdot \frac{1}{t} \, du.$$

$$= \sum_{n=0}^{+\infty} \frac{y^{2n}}{\cancel{(2n)!}} \frac{1}{t^{2n+1}} \cdot \cancel{\Gamma(2n+1)}$$

Dc $f(t) = \sum_{n=0}^{+\infty} \left(\frac{1}{t}\right)^{2n+1} \cdot \int_0^1 y^{2n} \, dy = \sum_{n=0}^{+\infty} \frac{1}{2n+1} \cdot \left(\frac{1}{t}\right)^{2n+1}$.

Cl : $\forall t > 0$, $f(t) = \operatorname{arctg}(1/t)$.

3/- $\left(\frac{\sin(x)}{x}\right)^2 = \int_0^1 \frac{\sin(2xy)}{x} \, dy$.

On a $g(t) = \int_0^{+\infty} \left(\int_0^1 \frac{\sin(2xy)}{x} \, dy \right) e^{-tx} \, dx$,

et par les mêmes arguments de Q.2, on a :

$$g(t) = \int_0^1 \left(\int_0^{+\infty} \frac{\sin(2xy)}{2xy} e^{-tx} \, 2y \, dx \right) dy. \qquad u = 2xy.$$

$$= \int_0^1 \left(\int_0^{+\infty} \frac{\sin(u)}{u} e^{-\frac{t}{2y} \cdot u} \, du \right) dy.$$

$$= \int_0^1 f\left(\frac{t}{2y}\right) dy = \int_0^1 \operatorname{arctg}\left(\frac{2y}{t}\right) dy.$$

$s = \dfrac{2y}{t}$ $\quad \hookrightarrow \; = \dfrac{t}{2} \displaystyle\int_0^{\frac{2}{t}} \operatorname{arctg}(s) \cdot ds.$

$$= \frac{t}{2}\left[s \operatorname{arctg}(s) - \frac{1}{2}\ln(1+s^2)\right]_0^{2/t}.$$

$$g(t) = \operatorname{arctg}\left(\frac{2}{t}\right) - \frac{t}{4}\ln\left(1+\frac{4}{t^2}\right), \quad \forall t>0,$$

4/- On a g est cont sur $\mathbb{R}^+$.

(CVD + caractérisation séquentielle de la limite)

$$\Rightarrow \lim_{t\to 0^+} g(t) = \int_0^{+\infty} \left(\frac{\sin(x)}{x}\right)^2 dx.$$

$g(t) \xrightarrow[t\to 0^+]{} \pi/2.$ $\quad$ C/C : $\displaystyle\int_0^{+\infty} \left(\frac{\sin(x)}{x}\right)^2 dx = \pi/2.$

<u>Exo3</u> : $\Gamma(a) = \int_0^{+\infty} e^{-t}.t^{a-1}\,dt.$

$$B(a,b) = \int_0^1 t^{a-1}.(1-t)^{b-1}.dt.$$

1/ - $\Gamma(a).\Gamma(b) = \left(\int_0^{+\infty} e^{-t}.t^{a-1}\,dt\right)\left(\int_0^{+\infty} e^{-x}x^{b-1}\,dx\right).$

Fubini-Tonelli $\Rightarrow$ $$= \int_{(\mathbb{R}_+^*)^2} e^{-(t+x)}.t^{a-1}.x^{b-1}\,dt\,dx.$$

$$\varphi : (\mathbb{R}_+^{*2}) \longrightarrow (\mathbb{R}_+^*)^2.$$

$$(u,v) \longmapsto (\underbrace{u^2}_{\varphi_1}, \underbrace{v^2}_{\varphi_2}).$$

- φ est injective sur $(\mathbb{R}_+^*)^2$.

. φ est $\mathcal{C}^1$ sur $(\mathbb{R}_+^*)^2$.

. $J = \begin{vmatrix} \frac{\partial \varphi_1}{\partial u} & \frac{\partial \varphi_2}{\partial u} \\ \frac{\partial \varphi_1}{\partial v} & \frac{\partial \varphi_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & 0 \\ 0 & 2v \end{vmatrix} = 4uv \neq 0.$

$(t,x) = (u^2, v^2)$. $J = \begin{vmatrix} \frac{\partial t}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial t}{\partial v} & \frac{\partial x}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & 0 \\ 0 & 2v \end{vmatrix} = 4uv.$

$\Rightarrow dt.dx = |J| . du dv = 4uv\, du dv.$

$$\Gamma(a).\Gamma(b) = \int_{(\mathbb{R}_+^*)^2} e^{-(u^2+v^2)} . u^{2a-2} . v^{2b-2}\, 4uv\, du dv.$$

$$= 4 \int_{(\mathbb{R}_+^*)^2} e^{-(u^2+v^2)} . u^{2a-1} . v^{2b-1}\, du dv.$$

21 - Mq $B(a,b) = \dfrac{\Gamma(a).\Gamma(b)}{\Gamma(a+b)}$.

passage aux coordonnées polaires ds l'expression précédente.

$$\Gamma(a).\Gamma(b) = 4\int_0^{+\infty}\int_0^{\frac{\pi}{2}} e^{-r^2}\, r^{2a+2b-2} (\cos\theta)^{2a-1} . (\sin\theta)^{2b-1}\, r\,dr\,d\theta \quad \begin{array}{|l} u = r\cos\theta \\ v = r\sin\theta \end{array}$$

Fubini

$$= \left(\int_0^{+\infty} 2r e^{-r^2} (r^2)^{a+b-1}\, dr\right)\left(\int_0^{\pi/2} (\cos^2\theta)^{a-1} . (\sin^2\theta)^{b-1} . 2\cos\theta\sin\theta\, d\theta\right).$$

$t = r^2$ $\qquad$ $t = \cos^2(\theta)$

$$= \underbrace{\left(\int_0^{+\infty} e^{-t} . t^{a+b-1}\, dt\right)}_{\Gamma(a+b).} . \underbrace{\left(+\int_0^1 t^{a-1} . (1-t)^{b-1} (+dt)\right)}_{B(a,b).}$$

Exo 4 : $I = \int_D \sqrt{x^2+y^2}\, dx\, dy$, $D = \{(x,y) \in \mathbb{R}^2 / x^2+y^2-2x \leq 0\}$

1/- $(x,y) \in D \Leftrightarrow (x-1)^2 + y^2 \leq 1$.

$\Leftrightarrow (x,y) \in D(\Omega(1,0), r = 1)$.

2/- Soit $(x,y) \in D$, on pose $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$.

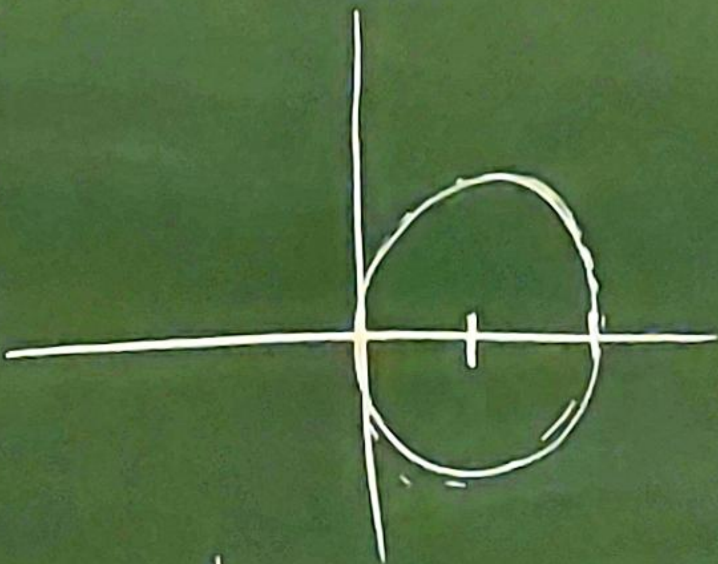
avec $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$.

Or $x^2+y^2-2x \leq 0 \Leftrightarrow r^2 - 2r\cos\theta \leq 0$.

$\Leftrightarrow (r \geq 0)$ et $r \leq 2\cos\theta$.

Donc $D = \{(r,\theta) / 0 \leq r \leq 2\cos\theta,\ \theta \in]-\pi/2, \pi/2[\}$.

$\alpha = 2$, $\beta = \pi/2$.



$b<1$

$$(\sin\theta)^{2b-1} \sim \theta.$$

$$\frac{1}{\theta^{1-2b}}$$

Exo3 :

$$3|-\ I = \int_D \sqrt{x^2+y^2}\, dx\, dy$$

$$= \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=2\cos\theta} r \cdot r\, dr\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8}{3} \cdot \underbrace{\cos^3(\theta)}_{\frac{1}{4}\cos(3\theta) + \frac{3}{4}\cos(\theta)}\, d\theta .$$

$$= 2 \cdot \frac{8}{3} \cdot \frac{1}{4} \cdot \left[\frac{1}{3} \sin(3\theta) + 3 \sin\theta \right]_0^{\pi/2}$$

$$= \frac{4}{3}\left(-1/3 + 3\right) = \frac{32}{9}.$$

Exo 5 :

$$\int_{[0,1]^2} \frac{dx\,dy}{1-xy} = \sum_{n\geq 1} \frac{1}{n^2}$$

$$(x,y) = (\cos\theta - t, \cos\theta + t).$$

$$x+y = 2\cos\theta \Rightarrow 0 \leq \cos\theta \leq 1, \quad \theta \in [0, \tfrac{\pi}{2}].$$

$$0 \leq y = \cos\theta + t \leq 1 \iff -\cos\theta \leq t \leq 1-\cos\theta$$

$$0 \leq x = \cos\theta - t \leq 1 \iff \cos\theta - 1 \leq t \leq \cos\theta. \qquad \tfrac{1}{2} \leq \cos\theta \leq 1$$

Donc $\max(-\cos\theta, \cos\theta - 1) \leq t \leq \min(\cos\theta, 1-\cos\theta)$ $\quad 0 \leq 1-\cos\theta \leq 1/2$

$$\theta \in [0, \tfrac{\pi}{3}] \Rightarrow \cos\theta - 1 \leq t \leq 1-\cos\theta$$

$$\theta \in [\tfrac{\pi}{3}, \tfrac{\pi}{2}] \Rightarrow -\cos\theta \leq t \leq \cos\theta$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} -\sin\theta & -\sin\theta \\ -1 & +1 \end{vmatrix} = -2\sin\theta \neq 0.$$

$$I = \int_{\theta=0}^{\theta=\frac{\pi}{3}} \left(\int_{\cos\theta-1}^{1-\cos\theta} \frac{2\sin\theta}{\sin^2\theta + t^2}\, dt \right) d\theta + \int_{\theta=\frac{\pi}{3}}^{\theta=\frac{\pi}{2}} \left(\int_{-\cos\theta}^{\cos\theta} \frac{2\sin\theta}{\sin^2\theta + t^2}\, dt \right) d\theta.$$

$$* \int_{-a}^{a} \frac{2\sin\theta}{\sin^2\theta + t^2}\, dt = 2 \int_{-a}^{a} \frac{1}{\sin\theta} \cdot \frac{1}{1 + \left(t/\sin\theta\right)^2}\, dt.$$

$$= 2 \left[\operatorname{arctg}\left(\frac{t}{\sin\theta} \right) \right]_{-a}^{a}$$

$$= 4 \operatorname{arctg}\left(\frac{a}{\sin\theta} \right).$$

* $a = \cos\theta$.

$$\int_{-\cos\theta}^{\cos\theta} \frac{2\sin\theta}{\sin^2\theta + t^2}\,dt = 4\,\mathrm{arctg}\left(\mathrm{cotg}(\theta)\right).$$

$$= 4\left(\frac{\pi}{2} - \mathrm{arctg}(\mathrm{tg}\,\theta)\right) = 4\left(\frac{\pi}{2} - \theta\right).$$

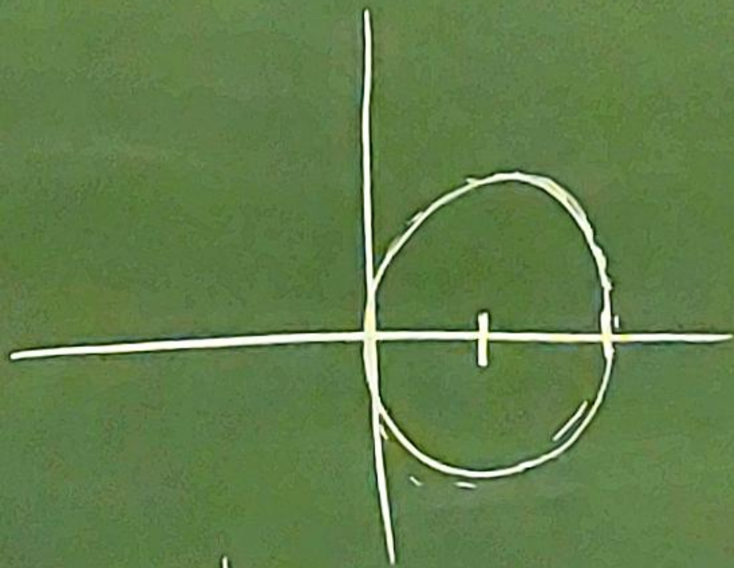
* $a = 1 - \cos\theta$.

$$\cos\theta = 1 - 2\sin^2(\theta/2)$$

$$\sin\theta = 2\cos\frac{\theta}{2}\cdot\sin\frac{\theta}{2}$$

$$\int_{\cos\theta - 1}^{1-\cos\theta} \frac{2\sin\theta}{\sin^2\theta + t^2}\,dt = 4\,\mathrm{arctg}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= 4\,\mathrm{arctg}\left(\frac{2\sin^2(\theta/2)}{2\cos\theta/2\cdot\sin\theta/2}\right) = 2\theta.$$



$b<1$

$$(\sin\theta)^{2b-1} \sim \theta. \qquad \frac{1}{\theta^{1-2b}}$$

$$I = \int_0^{\frac{\pi}{3}} 2\theta + \int_{\pi/3}^{\pi/2} 4\left(\frac{\pi}{2}-\theta\right)d\theta$$

$$= \frac{\pi^2}{9} + 2\pi\left(\frac{\pi}{2}-\pi/3\right) - 2\left(\frac{\pi^2}{4}-\frac{\pi^2}{9}\right).$$

$$= \pi^2/3 + \pi^2/3 - \pi^2/2 = \pi^2/6.$$