$(X, \mathcal{A}, \mu)$

$(Y, \mathcal{B}, \nu)$

$\mathcal{A}\otimes\mathcal{B} = \sigma(A\times B;\ A\in\mathcal{A}, B\in\mathcal{B})$

$C \in \mathcal{A}\otimes\mathcal{B}$.

$C_x = \{y \,;\, (x,y)\in C\} \in \mathcal{B}$

$C^y = \{x\in X;\ (x,y)\in C\} \in \mathcal{A}$

$\mu\otimes\nu$: $\mu\otimes\nu(A\times B) = \mu(A)\,\nu(B)$.

$$\mu\otimes\nu(C) = \int_X \nu(C_x)\,\mu(dx)$$

$$= \int_Y \mu(C^y)\,\nu(dy).$$

$$\boxed{(X\times Y,\ \mathcal{A}\otimes\mathcal{B},\ \mu\otimes\nu)}$$

$$\int_X f(x)\mu(dx) = \int_0^{+\infty} \mu\{f>t\}\,dt.$$

cf. Th 6.1 $\quad (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$

$\exists!\ \lambda_d$ = mesure de Lebesgue sur $\mathbb{R}^d$

(i) $\lambda_d([0,1]^d) = 1$

(ii) $\forall a \in \mathbb{R}^d, \forall A \in \mathcal{B}(\mathbb{R}^d) \quad \lambda_d(a+A) = \lambda_d(A)$

$$\lambda(a +]x,y[) = y - x = \lambda_1(]x,y[)$$

$$a +]x,y[=]a+x, a+y\}$$

Prop 11.7 $\quad \lambda_d = \lambda_1^{\otimes d}$

En particulier $\lambda_2 = \lambda \otimes \lambda$.

$$\left(\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})\right)$$

Thms de Fubini

$$\int_{X \times Y} f(x,y) \; \begin{array}{l} d\mu\otimes\nu(x,y) \\ \mu\otimes\nu(dx,dy) \end{array} = \int_Y \left\{ \int_X f(x,y)\,\mu(dx) \right\} \nu(dy)$$

notations

$$= \int_X \left\{ \int_Y f(x,y)\,\nu(dy) \right\} \mu(dx) \qquad (***)$$

Thm 11.2 (Fubini-Tonelli)

$f : X \times Y \longrightarrow \overline{\mathbb{R}}^+$ mesurable. μ, ν deux mesures σ-finies

a) $x \longmapsto \int_Y f(x,y)\,\nu(dy)$ $\mathcal{A}$-mesurable

$y \longmapsto \int_X f(x,y)\,\mu(dx)$ $\mathcal{B}$-mesurable

b) $(***)$ est vérifiée

⚠ La valeur commune de $(***)$ peut être $\underline{+\infty}$

Thm 11.3 (Fubini-Lebesgue)

$f : X \times Y \longrightarrow \mathbb{R}$

$f \in \mathcal{L}^1(X \times Y)$

$\int_{X \times Y} |f(x,y)|\, \mu \otimes \nu(dx\,dy) < +\infty$

alors

(a) $x \longmapsto \int_Y f(x,y)\,\nu(dy) \in \mathcal{L}^1(X)$

$y \longmapsto \int_X f(x,y)\,\mu(dx) \in \mathcal{L}^1(Y)$

(b) $(***)$ est vérifiée.

Utilisation classique

1) je veux calculer $\int f(x,y)\, d\mu\otimes\nu(x,y)$

2) je calcule par F-T $\int_{X\times Y} |f|$

→ si $\int |f| < +\infty$ je passe à 3)

⟹ si $\int |f| = +\infty$ je ne peux rien dire...

3) je calcule $\int_{X\times Y} f$ à l'aide de F-L

Thms de Fubini

$$\int_{X\times Y} f(x,y) \quad \begin{array}{|l} d\mu\otimes\nu(x,y) \\ \mu\otimes\nu(dx,dy) \end{array}$$

notations

$$|f(x,y)| \leq g(x) \in \mathcal{L}^1$$

$$\int_{X \times Y} |f(x,y)| \, \mu(dx)\, \nu(dy)$$

$$\leq \int_{X \times Y} g(x)\, \mu(dx)\, \nu(dy)$$

$$\leq \int_Y \overbrace{\left\{ \int_X g(x)\, \mu(dx) \right\}}^{\text{cste}} \nu(dy) \leq \|g\|_1 \, \nu(Y)$$

Appli 11.4 Formule d'i.p.p.

f, g deux fonctions localement intégrables sur $\mathbb{R}$. (pour la mesure de Lebesgue)

$\Updownarrow$

$\forall K$ compact de $\mathbb{R}$ $\quad f \cdot 1_K$ intégrable.

$$\forall x \in \mathbb{R} \quad \left.\begin{array}{l} F(x) = \int_0^x f(t)\,dt \\ G(x) = \int_0^x g(t)\,dt \end{array}\right\} \int_0^x f(t)G(t)\,dt = F(x)G(x) - \int_0^x F(t)g(t)\,dt.$$

Exo 11.3

$$f(x,y) = \begin{cases} \dfrac{x^2 - y^2}{(x^2+y^2)^2} & \text{si } x \neq y \\ f(0,0) = 0 \end{cases}$$

1) $\displaystyle\int_0^1 \left\{ \int_0^1 f(x,y)\,dy \right\} dx = \pi/4$

2) $\displaystyle\int_0^1 \left\{ \int_0^1 f(x,y)\,dx \right\} dy = -\pi/4$

1 $\displaystyle\int_0^1 f(x,y)\,dy = \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dy$

$\displaystyle = \int_0^1 \left(\frac{x^2+y^2}{(x^2+y^2)^2} - \frac{2y^2}{(x^2+y^2)^2} \right) dy$

$\displaystyle = \int_0^1 \frac{dy}{x^2+y^2} + \int_0^1 \frac{-2y}{(x^2+y^2)^2}\, y\, dy$

$\displaystyle = \int_0^1 \frac{dy}{x^2+y^2} + \left[\frac{y}{x^2+y^2} \right]_0^1 - \int_0^1 \frac{1}{x^2+y^2}\,dy$

$\displaystyle = \frac{1}{1+x^2}$

$\displaystyle \int_0^1 \frac{dx}{1+x^2} = \left[\arctan(x) \right]_0^1 = \pi/4$

$$\frac{\partial}{\partial y}\left(\frac{1}{x^2+y^2} \right) = \frac{-2y}{(x^2+y^2)^2}$$

$$(x,y) \longmapsto \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\int_{-\infty}^{+\infty} e^{-a|x|} e^{itx}\, dx = \int_{-\infty}^{0} e^{ax+itx}\, dx + \int_{0}^{+\infty} e^{-ax+itx}\, dx$$

$$\| $$

$$\frac{2a}{a^2+t^2}$$

$$= \lim_{A\to-\infty} \int_A^0 e^{(a+it)x}\, dx + \lim_{A\to+\infty} \int_0^A e^{(-a+it)x}\, dx$$

$$= \lim_{A\to-\infty} \left[\frac{1}{a+it} e^{(a+it)x} \right]_A^0 - \lim_{A\to+\infty} \left[\frac{1}{-a+it} e^{(-a+it)x} \right]_0^A$$

$$= \frac{1}{a+it} - \frac{1}{-a+it} = \frac{a-it}{a^2+t^2} + \frac{a+it}{a^2+t^2} = \frac{2a}{a^2+t^2}$$

$$f_a(t) = \int_{-\infty}^{+\infty} \frac{e^{-itx}\, e^{-a|x|}}{1+x^2}\, dx.$$

? $$f_a(t) = \int_{-\infty}^{+\infty} \frac{a}{a^2+(y+t)^2}\, e^{-|y|}\, dy.$$

$$\frac{a}{a^2+(y+t)^2} = \frac{1}{2}\int_{-\infty}^{+\infty} e^{-a|u|}\, e^{i(t+y)u}\, du$$

$$\int_{-\infty}^{+\infty} e^{-|y|} \left\{ \frac{1}{2}\int_{-\infty}^{+\infty} e^{-a|u|}\, e^{i(t+y)u}\, du \right\} dy$$

$$\int_{-\infty}^{+\infty} \frac{e^{-a|u|+itu}}{2} \underbrace{\left\{ \int_{-\infty}^{+\infty} e^{-|y|}\, e^{iyu}\, dy \right\}}_{\frac{2}{1+u^2}} du = \int_{-\infty}^{+\infty} \frac{e^{-a|u|}\, e^{itu}}{1+u^2}\, du$$

$$(x,y) \longmapsto e^{-|y|}e^{-a|x|}e^{i(t+y)}$$

?

$$\int_{-\infty}^{+\infty} \frac{e^{itx}}{1+x^2}\,dx$$

$$\int_{-\infty}^{+\infty} \frac{e^{itx}\, e^{-a|x|}}{1+x^2}\, dx = \int_{-\infty}^{+\infty} \frac{a}{a^2+(y+t)^2}\, e^{-|y|}\, dy$$

$$\left|\frac{e^{itx}\, e^{-a|x|}}{1+x^2}\right| \leq \frac{1}{1+x^2} \in \mathcal{L}^1$$

$$= \int_{-\infty}^{+\infty} \frac{1}{1+\left(\frac{y+t}{a}\right)^2}\, e^{-|y|}\, \frac{dy}{a}$$

$$u = \frac{y+t}{a}$$

$$= \int_{-\infty}^{+\infty} \frac{1}{1+u^2}\, e^{-|au-t|}\, du$$

$$\int_{-\infty}^{+\infty} \frac{e^{itx}}{1+x^2}\, dx = e^{-|t|} \int_{-\infty}^{+\infty} \frac{du}{1+u^2}$$

$$= \pi e^{-|t|}$$