Exo 1

Corrigé de l'exercice

1)

2) a) $\underline{\underline{\nabla_0}}\vec{u} = \underline{\underline{R}} - \underline{\underline{1}}$

b) $\|\underline{\underline{\nabla_0}}\vec{u}\| \ll 1$; $\|\underline{\underline{\nabla_0}}\vec{u}\|^2 = \underline{\underline{\nabla_0}}\vec{u} : \underline{\underline{\nabla_0}}\vec{u} = tr\left((\underline{\underline{\nabla_0}}\vec{u})^T(\underline{\underline{\nabla_0}}\vec{u})\right) \approx 0$

donc $(\underline{\underline{R}}^T - \underline{\underline{1}})(\underline{\underline{R}} - \underline{\underline{1}}) = \underline{\underline{0}}$ (Belba)

3) $\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\underline{\underline{\nabla_0}}\vec{u} + (\underline{\underline{\nabla_0}}\vec{u})^T\right) = \frac{1}{2}\left(\underline{\underline{R}} - \underline{\underline{1}} + \underline{\underline{R}}^T - \underline{\underline{1}}\right)$

$= \frac{1}{2}\left(R + R^T - 2\underline{\underline{1}}\right)$

$= -\frac{1}{2}\left(\underline{\underline{R}}^T - \underline{\underline{1}}\right)\left(\underline{\underline{R}} - \underline{\underline{1}}\right) = \underline{\underline{0}}$

$\underline{\underline{\nabla_0}}\vec{u} = \underline{\underline{\varepsilon}} + \underline{\underline{\omega}}$; $\omega = \frac{1}{2}\left(\underline{\underline{\nabla_0}}\vec{u} - (\underline{\underline{\nabla_0}}\vec{u})^T\right)$

or $\underline{\underline{\varepsilon}} = 0$ donc ~~$\underline{\underline{\nabla_0}}\vec{u}$~~ $\underline{\underline{\omega}} = \underline{\underline{R}} - \underline{\underline{1}}$

soit $\vec{W}$ tq $\forall \vec{v}$, $\underline{\underline{\omega}}\vec{v} = \vec{W} \wedge \vec{v}$

or $\|\underline{\underline{\nabla_0}}\vec{u}\| \ll 1 \overset{?}{\Rightarrow} \|\vec{W}\| \ll 1$

$$\underline{\underline{\omega}} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \text{ et } \vec{W} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

Donc cette implication est vraie

Donc $\vec{u}(x_0, t) = \vec{c}(t) + \vec{W}(t) \wedge \vec{x_0}$ avec $\|\vec{W}\| \ll 1$.

une transformation homogène $\Rightarrow$ si $\underline{\underline{F}}$ ne dépend pas de $\vec{x_0}$

donc $\underline{\underline{F}}(t)$ dépend du temps uniquement

4) $\underline{\underline{\varepsilon}}(t)$; $\vec{u}(\vec{x_0}, t) \; ?$

$\underline{\underline{\varepsilon}}(t)$ vérifie les conditions de compatibilité car $\underline{\underline{\varepsilon}}$ ne dépend de l'espace.

$$\underline{\underline{\nabla}} \vec{u} = \underline{\underline{\varepsilon}}(t) + \underline{\underline{W}}(t)$$

$$\boxed{\vec{u} = \underline{\underline{\varepsilon}}(t)\,\vec{x_0} + \vec{w} \wedge \vec{x_0} + \vec{c}(t)}$$ → ~~sp~~ $\underline{\underline{\nabla}} \vec{u} = \underline{\underline{\varepsilon}}(t)$

car on a vu que la dérivé de cette partie est nulle

Exercice n°3

1) ~~[illegible]~~

$$\underline{\underline{\sigma}} = 2\mu\, \underline{\underline{\varepsilon}} + \lambda (\mathrm{tr}\, \underline{\underline{\varepsilon}}) \underline{\underline{1}} \quad (*)$$

2) on a

on remplace $(*)$ dans l'équation n°1 : $\Rightarrow$ $2\mu\, \vec{\nabla} \cdot \underline{\underline{\varepsilon}} + \lambda \underbrace{\vec{\nabla} \cdot (\mathrm{tr}(\underline{\underline{\varepsilon}}) \underline{\underline{1}})}_{\vec{\nabla}(\mathrm{tr}\, \underline{\underline{\varepsilon}})} + \rho \vec{g} = \vec{0} \quad (**)$ ②

$$2 \vec{\nabla} \cdot \underline{\underline{\varepsilon}} = \vec{\nabla} \cdot (\underline{\underline{\nabla}} \vec{u}) + \vec{\nabla} \cdot ((\underline{\underline{\nabla}} \vec{u})^T)$$

$$2 (\vec{\nabla} \cdot \underline{\underline{\varepsilon}})_i = \frac{\partial}{\partial x_j} (\underline{\underline{\nabla}} \vec{u})_{ij} + \frac{\partial}{\partial x_j} (\underline{\underline{\nabla}} \vec{u})_{ji}$$

$$\text{or } (\underline{\underline{\nabla}} \vec{u})_{ij} = \frac{\partial u_i}{\partial x_j}$$

$$\text{donc } 2 (\vec{\nabla} \cdot \underline{\underline{\varepsilon}})_i = \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} \right) + \frac{\partial^2 u_j}{\partial x_j \partial x_i}$$

$$= (\Delta u)_i + \frac{\partial}{\partial x_i} \underbrace{\left(\frac{\partial u_j}{\partial x_j} \right)}_{\vec{\nabla} \cdot \vec{u}}$$

$$\boxed{2 (\vec{\nabla} \cdot \underline{\underline{\varepsilon}})_i = \vec{\Delta} \vec{u} + \vec{\nabla} (\vec{\nabla} \cdot \vec{u})} \quad ①$$

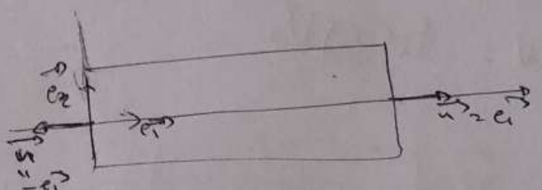
on remplace ① et ② dans $(**)$
et on la trouve.

③ on passe à la Trace.

②

Exercice 4 : Traction / Compression d'une barre cylindrique).

1) $\underline{\underline{\sigma}} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & | & | \\ | & | & | \\ 0 & 0 & 0 \end{pmatrix}_{\{\vec{e_1},\vec{e_2},\vec{e_3}\})}$ $\vec{\nabla} . \underline{\underline{\sigma}} = \vec{0}$ (car $\underline{\underline{\sigma}}$ = cte)



$\vec{T} = \underline{\underline{\sigma}} \vec{n}$

Sur Σ_0 : $\vec{T} . \vec{e_2} = 0 \Rightarrow \underline{\underline{\sigma}} \vec{n} . \vec{e_2} = 0 \Rightarrow -\underline{\underline{\sigma}} \vec{e_1} . \vec{e_2} = \vec{0} = -\vec{e_2} \underline{\underline{\sigma}} \vec{e_1}$

$\sigma_{21} = 0$

$\vec{T} . \vec{e_3} = 0 \Rightarrow -\vec{e_3} . \underline{\underline{\sigma}} \vec{e_1} = 0$

σ_{31}

De même sur Σ_{ℓ_0}.

(2) $\underline{\underline{\sigma}} = 2\mu \underline{\underline{\varepsilon}} + \lambda (\operatorname{tr} \underline{\underline{\varepsilon}}) \underline{\underline{1}}$ (loi de comportement élastique linéaire dans un état non contraint

$\underline{\underline{\varepsilon}} = \frac{1+\nu}{E} \underline{\underline{\sigma}} - \frac{\nu}{E} \operatorname{tr}(\underline{\underline{\sigma}}) \underline{\underline{1}}$

$\varepsilon_{11} = \frac{1+\nu}{E} \sigma - \frac{\nu}{E} \sigma = \frac{\sigma}{E} \Rightarrow \sigma = E \varepsilon_{11}$ car $E = \frac{\sigma}{\varepsilon_{11}}$

E le rapport entre la sollicitation et la réponse

$\varepsilon_{22} = \varepsilon_{33} = -\frac{\nu}{E} \sigma = -\nu \varepsilon_{11}$

$\underline{\underline{\varepsilon}} = \frac{\sigma}{E} \vec{e_1} \otimes \vec{e_1} - \frac{\nu}{E} \sigma (\vec{e_2} \otimes \vec{e_2} + \vec{e_3} \otimes \vec{e_3})$

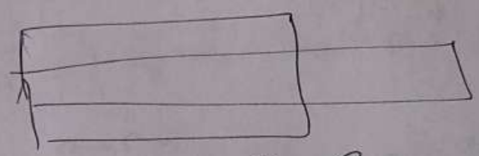
$\underline{\underline{\sigma}} = \sigma \vec{e_1} \otimes \vec{e_1}$

(Astuce).

(5) $$\frac{dV}{dt} = \int_{\Omega} \vec{v} \cdot \vec{n}\, dA$$

$$dV = \int_{\Omega} \vec{u} \cdot \vec{n}\, dA = \int_{\Omega} \vec{\nabla} \cdot \vec{u}\, dV = \int_{\Omega} tr(\underline{\underline{\varepsilon}})\, dV$$

$$V - V_0 \quad dV = tr(\underline{\underline{\varepsilon}}) V_0$$



$$V_0 = A_0 l_0$$

$$V = A l \Rightarrow A = A_0 + a \;,\; l = l_0 + \delta$$

$$V = (A_0 + a)(l_0 + \delta) \simeq A_0 l_0 + A_0 \delta + a l_0$$

Exercice n°: sphère creuse sous pression.

1) $\vec{u}(r, \cancel{\theta}, \cancel{\varphi}) = u(r)\, \vec{e_r}$

2) $\vec{T} = \underline{\underline{\sigma}}\, \vec{n}$ → pression

pour $r = a \Rightarrow \begin{cases} \vec{T} = p_a \vec{e_r} \\ \vec{T} = -\underline{\underline{\sigma}}\, \vec{e_r} \end{cases}$

on a $\underline{\underline{\sigma}}\, \vec{e_r} \cdot \vec{e_r} = -p_a = \sigma_{rr}$

Pour $r = b \Rightarrow \vec{T} = -p_b \vec{e_r} = \underline{\underline{\sigma}}\, \vec{e_r}$

donc $\underline{\underline{\sigma}}\, \vec{e_r} \cdot \vec{e_r} = -p_b = \sigma_{rr}$

3) on a $\vec{u} = u(r)\, \vec{e_r}$.

$$[\underline{\underline{\nabla}}\vec{u}] = \begin{pmatrix} u' & 0 & 0 \\ 0 & \frac{u}{r} & 0 \\ 0 & 0 & \frac{u}{r} \end{pmatrix}$$

$$\underline{\underline{\nabla}}\vec{u} = u' \, \vec{e_r} \otimes \vec{e_r} + \frac{u}{r} \left(\vec{e_\theta} \otimes \vec{e_\theta} + \vec{e_\phi} \otimes \vec{e_\phi} \right).$$

$$\underline{\underline{\varepsilon}} = \underline{\underline{\nabla}}\vec{u} \quad , \quad \underline{\underline{\sigma}} = \underline{\underline{\sigma_0}} + 2\mu \, \underline{\underline{\varepsilon}} + \lambda (tr(\underline{\underline{\varepsilon}})) \underline{\underline{1}} \quad (1)$$

$$\varepsilon_{rr} = u' , \varepsilon_{\theta\theta} = \varepsilon_{\phi\phi} = \frac{u}{r}$$

(4) on écrit l'équa° d'équilibre. (lorsqu'on demande une rel° d'équa° diff Il faut écrire l'équa° d'équilibre).

$$\vec{\nabla} \cdot \underline{\underline{\sigma}} = \vec{0}$$

d'après (1) $\Rightarrow$ $\sigma_{rr} = \; + 2\mu \varepsilon_{rr} + \lambda \, tr(\varepsilon) \, 1_{rr}$

$$= (\lambda + 2\mu) \varepsilon_{rr} + 2\lambda \, \varepsilon_{\theta\theta}$$

$\sigma_{rr} =$

$$\sigma_{\phi\phi} = \sigma_{\theta\theta} = \lambda \varepsilon_{rr} + 2(\mu + \lambda) \varepsilon_{\theta\theta} =$$

$$[\underline{\underline{\sigma}}] = \begin{pmatrix} \sigma_{rr} & & (0) \\ & \sigma_{\theta\theta} & \\ (0) & & \sigma_{\phi\phi} \end{pmatrix}$$

$$\boxed{\vec{\nabla} \cdot \underline{\underline{\sigma}} = \left(\sigma_{rr}' + \frac{2\sigma_{rr} - \sigma_{\theta\theta}}{r} \right| \vec{e_r}} + ($$

$$\vec{\nabla} \cdot \underline{\underline{\sigma}} = \vec{0} \Rightarrow \sigma_{rr}' + \frac{2\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \quad , \quad \sigma_{rr} = (\lambda + 2\mu) u' + 2\lambda \frac{u}{r}$$

$$\sigma_{rr} - \sigma_{\theta\theta} = 2\mu (\varepsilon_{rr} - \varepsilon_{\theta\theta}) = 2\mu \left(u' - \frac{u}{r} \right) ,$$

$$\sigma_{rr}' = (\lambda+2\mu)u'' + 2\lambda\left(\frac{u'}{r} - \frac{u}{r^2}\right)$$

$$(\lambda+2\mu)\left(u'' + 2\frac{u'}{r} - \frac{2u}{r^2}\right) = 0$$

Comme ~~$\lambda+2\mu$~~ $\neq 0$ donc $\Rightarrow$ $u'' + \frac{2u'}{r} - \frac{2u}{r^2} = 0$

que $\lambda > 0$ et $\mu > 0$

$\Rightarrow \lambda + 2\mu \gg 0$

5) or $u'' + \frac{2}{r}u' - \frac{2}{r^2}u = \left(u' + \frac{2u}{r}\right)' = 0$

et $u' + \frac{2u}{r} = \frac{1}{r^2}(r^2u)'$

$$\boxed{\left(\frac{1}{r^2}(r^2u)'\right)' = 0}$$

$$\frac{1}{r^2}(r^2u)' = C_1$$

$$(r^2u)' = C_1 r^2$$

$$r^2u = C_1 r^3 + C_2$$

$$u(r) = C_1 r + \frac{C_2}{r^2}$$

$\sigma_{rr} = (\lambda+2\mu)u' + 2\lambda\frac{u}{r}$ $\Rightarrow$ $\begin{cases} u = C_1 r + \frac{C_2}{r^2} \\ u' = C_1 - \frac{2C_2}{r^3} \end{cases}$

$$\sigma_{rr} = (2\mu+3\lambda)C_1 - \frac{2C_2 2\mu}{r^3}$$

on pose $\boxed{C_1 = (2\mu+3\lambda)c_1}$ $\boxed{C_2 = 2\mu c_2}$

$$\sigma_{\theta\theta} = (2\mu+3\lambda)C_1 + \frac{2C_2}{r^3}\mu$$

Donc $\sigma_{rr} = C_1 - \frac{2C_2}{r^3}$, $\sigma_{\theta\theta} = C_1 + \frac{C_2}{r^3}$

en $r=a$, $\sigma_{rr}(a) = -p_a$

$r=b$: $\sigma_{rr}(b) = -p_b$

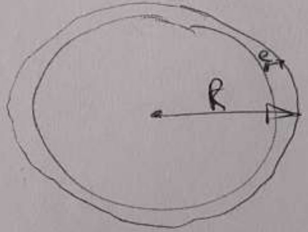
$$C_1 - \frac{2C_2}{a^3} = -p_a$$

$$C_1 + \frac{C_2}{b^3} = -p_b$$

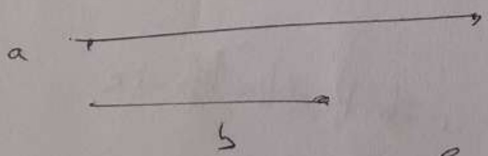
Conclusion : $C_1 = \dfrac{a^3 p_a - b^3 p_b}{b^3 - a^3}$, $\cancel{C_2 = \frac{1}{2}\frac{a^3 - b^3}{b^3 - a^3}}$

$$C_2 = \frac{1}{2}\,\frac{a^3 b^3}{b^3 - a^3}\left(p_a - p_b\right)$$

6



Paroi mince pour $\frac{e}{R} \ll 1$



$a = R - \frac{e}{2}$, $b = R + \frac{e}{2}$ avec $\delta = \frac{e}{2R}$

$= R(1-\delta)$, $b = R(1+\delta)$ et on remplace —.

$a^3 \simeq R^3(1-3\delta)$, $b^3 = R^3(1+3\delta)$

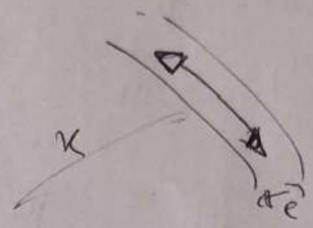
après calcul :

$$\sigma_{rr} = -\frac{p_a + p_b}{2} - (p_a - p_b)\left(\frac{r-R}{e}\right)$$

$$\sigma_{\theta\theta} = -\frac{p_a + p_b}{2} + \frac{(p_a - p_b)R}{2e} - \frac{(p_a - p_b)}{2e}(r-R).$$

$$\sigma_{\theta\theta} = \sigma_{rr} + \frac{p_a - p_b}{2}\,\frac{R}{e}$$

$|\sigma_{\theta\theta}| >> |\sigma_{rr}|$

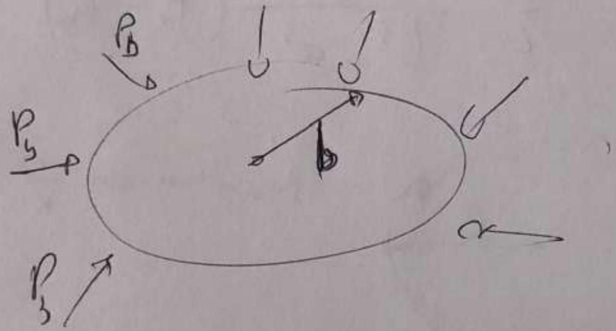


(7) $\frac{b}{a} \longrightarrow \infty$

$u = c_1 r + \frac{c_2}{r^2}$ Comme que $u(r \to \infty)$ est fini alors $c_1 = 0$

donc $u = \frac{c_2}{r^2}$

(8)



$u = c_1 r + \frac{c_2}{r^2}$

si $r^2 \longrightarrow 0$ ~~alors~~ et u doit être finie

donc $c_2 = 0$

$\boxed{u = c_1 r}$

Dans ce cas $\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{\varphi\varphi} = -p_b$

$\sigma = -p_b \underline{\underline{1}}$

$\vec{u} = -\frac{p_b}{2\mu + 3\lambda} r \vec{e_r}$ or $2\mu + 3\lambda = 3K \Rightarrow \vec{u} = -\frac{p_b}{3K} r \vec{e_r}$